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Nuclear Physics B 843 [FS] (2011) 302–343
NUCLEAR PHYSICS B
www.elsevier.com/locate/nuclphysb

Extended Y-system for the AdS_5/CFT_4 correspondence

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Received 9 July 2010; received in revised form 24 September 2010; accepted 27 September 2010

Available online 8 October 2010

Abstract

We study the analytic properties of the AdS_5/CFT_4 Y functions. It is shown that the TBA equations, including the dressing factor, can be obtained from the Y-system with some additional information on the square-root discontinuities across semi-infinite segments in the complex plane. The Y-system extended by the discontinuity relations constitutes a fundamental set of local functional constraints that can be easily transformed into integral form through Cauchy's theorem.

1. Introduction and summary

The presence of a quantum two-dimensional integrable model (IM) [1] in multicolor reggeised gluon scattering was discovered by Lipatov more than fifteen years ago [2]. However, only rather recently and thanks to the AdS/CFT correspondence [3], the connection between integrable models and supersymmetric gauge theories started to develop faster and faster, as we will very briefly summarize in the rest of this section.

For the purposes of the present paper, the relevant version of the AdS/CFT conjecture relates the free type IIB superstring theory on the $AdS_5 \times S^5$ curved space–time to the planar limit of the $\mathcal{N} = 4$ Super Yang–Mills theory (SYM) in four dimensions living on the boundary of AdS_5 [3]. The nature of the relation is a strong/weak coupling duality and thus powerful to use, but difficult to test.

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0550-3213/$ – see front matter
doi:10.1016/j.nuclphysb.2010.09.015

The 't Hooft planar limit is defined by the scaling of the colour number $N \to \infty$ and the SYM coupling $g_{YM} \to 0$ while keeping the coupling $N g_{YM}^2 = \lambda = 4\pi^2 g^2$ finite.[1] In this limit only planar Feynman diagrams survive [4]. As part of the duality, once the string tension is set proportional to g, the quantum energy of a specific string state corresponds to the conformal dimension of a local single-trace composite operator $\mathcal{O}$ in SYM. On the integrable model side, this dimension is an eigenvalue of the dilatation operator which is believed to be equivalent to an integrable Hamiltonian. The coincidence of the one loop mixing matrix in the purely scalar sector with an integrable $so(6)$ spin chain was first proven [5] and then extended to all the gauge theory sectors and at all loops in a way which shows integrability in a weaker sense, but still provides the investigators with many powerful tools (cf. [6] and references therein). More in detail, any composite operator $\mathcal{O}$ can be thought of as a state of a hypothetical spin chain Hamiltonian, whose degrees of freedom are the operators in the trace.

Although the actual form of the Hamiltonian describing the model at arbitrary values of the coupling g is still unknown, the large quantum number spectrum turned out to be exactly described by certain Bethe Ansatz-like equations, which are thus called asymptotic Bethe Ansatz (ABA) equations (cf. [6,7] and references therein). The ABA leads to the energy $E(g)$ which coincides with the anomalous dimension of a long operator: $\Delta_{\mathcal{O}}(g) = \Delta_{\mathcal{O}}^{bare} + E(g)$.

In integrable system language, the ABA equations (Beisert and Staudacher's [6] with the final dressing phase [7]) are the analogue of the Bethe–Yang equations for scattering models describing the quantisation of momenta for a finite number of interacting particles living on a ring with very large circumference [1], as described in [8].

In a parallel way, integrability in superstring theory was discovered at classical level [9], extended to semiclassical and partially to quantum level in [10]. The string integrability investigations helped and were helped by the aforementioned Bethe Ansatz description, the paradigm of this interplay being the final conjectured form of the dressing phase [7].

As already mentioned, this is only part of the SYM/IM correspondence. In fact, an important limitation emerges as a consequence of the asymptotic character of the Bethe Ansatz: the latter ought to be modified by finite size effects as soon as the site-to-site interaction range in the loop expansion of the dilatation operator becomes greater than the chain length. This wrapping effect [11,18] is particularly relevant in the semiclassical string theory which covers the strong coupling regime though, for special reasons, it may not effect particular families of operators.

A first important step leading to the partial solution of this problem was made by Bajnok and Janik in [14]. Adapting the formulas for the finite-size Lüscher corrections [12,13] to the AdS_5/CFT_4 context, they were able to predict the four loop contribution to the Konishi operator. The result was readily confirmed by the complicated diagrammatic calculations of [15]. The method proposed by Bajnok and Janik can be extended to higher orders in g^2 [16,17] but – as in most of the known perturbative schemes – the technical complication increases very sharply with the loop order and precise non-perturbative predictions are usually out of reach.

In $(1+1)$-dimensional massive relativistic scattering theories there is one well-known way to treat finite-size effects non-perturbatively and exactly: the Thermodynamic Bethe Ansatz (TBA) method. For this purpose, the TBA was first proposed for the ground state energy by Al.B. Zamolodchikov in [19] and adapted to excited states in [20,21].

[1] A warning about notations: another definition for the string coupling g is also widely used, so that in many works is found the relation $\lambda = 16\pi^2 g^2$. Appendix G can be consulted to match the different conventions.

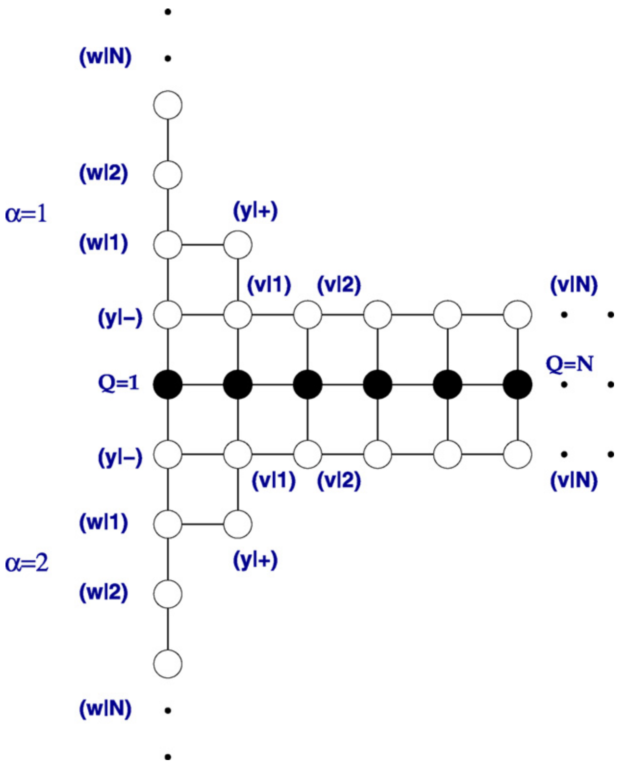

Fig. 1. The Y-system diagram corresponding to the AdS_5/CFT_4 TBA equations.

As a result of the TBA procedure, the exact finite-size energy can be written in terms of the pseudoenergies ε_a: the solutions of a system of non-linear integral equations. Even for relatively simple relativistic systems, such as the sine-Gordon model, an exact and exhaustive study of the TBA equations for excited states is an unfinished business.

An alternative but equivalent approach to excited states was adopted in [22,23]. Under the perspective of a different non-linear integral equation, this idea was applied to some sectors of the asymptotic Beisert–Staudacher equations in [24–26] and to the wrapping effects in the Hubbard model [27]. The latter system is deeply related to the model studied in this paper.

Starting from the mirror version of Beisert–Staudacher equations due to Arutyunov and Frolov [28] (see also [29]), the ground state TBA equations were recently and independently proposed in [30–32]. The associated set of functional relations for the functions $Y_a = e^{\varepsilon_a}$, the Y-system [33–37], was derived confirming an earlier proposal by Gromov, Kazakov and Vieira coming from symmetry arguments [38].

The AdS_5/CFT_4 Y-system conjectured in [38], derived in [30–32] and associated to the 'T-hook' diagram represented in Fig. 1 is:

$$Y_Q\left(u-\frac{i}{g}\right)Y_Q\left(u+\frac{i}{g}\right)=\prod_{Q'}\left(1+Y_{Q'}(u)\right)^{A_{QQ'}}\prod_{\alpha}\frac{\left(1+\frac{1}{Y^{(\alpha)}_{(v|Q-1)}(u)}\right)^{\delta_{Q,1}-1}}{\left(1+\frac{1}{Y^{(\alpha)}_{(y|-)}(u)}\right)^{\delta_{Q,1}}}, \tag{1.1}$$

$$Y^{(\alpha)}_{(y|-)}\left(u+\frac{i}{g}\right)Y^{(\alpha)}_{(y|-)}\left(u-\frac{i}{g}\right)=\frac{\left(1+Y^{(\alpha)}_{(v|1)}(u)\right)}{\left(1+Y^{(\alpha)}_{(w|1)}(u)\right)}\frac{1}{\left(1+\frac{1}{Y_1(u)}\right)}, \tag{1.2}$$

$$Y^{(\alpha)}_{(w|M)}\left(u+\frac{i}{g}\right)Y^{(\alpha)}_{(w|M)}\left(u-\frac{i}{g}\right)=\prod_N\left(1+Y^{(\alpha)}_{(w|N)}(u)\right)^{A_{MN}}\left[\frac{\left(1+\frac{1}{Y^{(\alpha)}_{(y|-)}(u)}\right)}{\left(1+\frac{1}{Y^{(\alpha)}_{(y|+)}(u)}\right)}\right]^{\delta_{M,1}}, \tag{1.3}$$

$$Y^{(\alpha)}_{(v|M)}\left(u+\frac{i}{g}\right)Y^{(\alpha)}_{(v|M)}\left(u-\frac{i}{g}\right)=\frac{\prod_N\left(1+Y^{(\alpha)}_{(v|N)}(u)\right)^{A_{MN}}}{\left(1+\frac{1}{Y_{M+1}(u)}\right)}\left[\frac{\left(1+Y^{(\alpha)}_{(y|-)}(u)\right)}{\left(1+Y^{(\alpha)}_{(y|+)}(u)\right)}\right]^{\delta_{M,1}}, \tag{1.4}$$

where $A_{1,M}=\delta_{2,M}$, $A_{NM}=\delta_{M,N+1}+\delta_{M,N-1}$ and $A_{MN}=A_{NM}$.

In the integrable model framework the Y-systems play a very central rôle. Firstly, a Y-system exhibits a very high degree of universality. Not only the whole set of excited states of a given theory is associated to a single Y-system but also many different models may have identical Y-systems. Two excited states of the same theory or two states of different models sharing a common Y-system differ in the analytic properties of the Y functions inside a fundamental strip of the complex rapidity plane. Given this analytic information the Y-system can be easily transformed to the integral TBA form. Roughly speaking, two different models have different leading asymptotic behaviors, while different states of the same model differ in the number and positions of the $1+Y_a$ zeros in the fundamental strip. In relativistic models the Y functions are in general meromorphic in the rapidity u with zeros and poles both linked to $1+Y_a$ zeros through the Y-system.

For the ground state energy numerics is reliable and the accuracy is in general very high. However, for excited states the situation is complicated by the presence of the finite number of auxiliary equations constraining the positions of this special subset of zeros. Unfortunately, both the number of special zeros and their positions in the complex rapidity plane can change drastically as the coupling constant or the system size interpolate between the infrared and the ultraviolet regimes. Therefore, the situation at moderate L may be substantially different from the infrared distribution described by the asymptotic Bethe–Yang equations [20,21,39].

The situation for the AdS_5/CFT_4-related model is further complicated by the presence of square root branch discontinuities inside and at the border of the fundamental strip $|\mathrm{Im}(u)|\leqslant 1/g$. According to the known Y → TBA transformation procedures this extra information should be independently supplied. However, it was pointed out in [32] (see also [40]) that such discontinuity information is stored into functions which depend non-locally on the TBA pseudoenergies. In other words, they crucially depend on the particular excited state under consideration.

The main objective of this paper is to show that this problem can be overcome and in particular that the discontinuity information is encoded in the Y-system together with the following set of local and state-independent functional relations. Setting

$$\Delta(u)=\left[\ln Y_1(u)\right]_{+1}, \tag{1.5}$$

then Δ is the function introduced in [32] and the local discontinuity relations are:

$$[\Delta]_{\pm 2N}=\mp\sum_{\alpha=1,2}\left(\left[\ln\left(1+\frac{1}{Y^{(\alpha)}_{(y|\mp)}}\right)\right]_{\pm 2N}+\sum_{M=1}^{N}\left[\ln\left(1+\frac{1}{Y^{(\alpha)}_{(v|M)}}\right)\right]_{\pm(2N-M)}+\ln\left(\frac{Y^{(\alpha)}_{(y|-)}}{Y^{(\alpha)}_{(y|+)}}\right)\right), \tag{1.6}$$

$$\left[\ln\left(\frac{Y^{(\alpha)}_{(y|-)}}{Y^{(\alpha)}_{(y|+)}}\right)\right]_{\pm 2N} = -\sum_{Q=1}^{N}\left[\ln\left(1+\frac{1}{Y_Q}\right)\right]_{\pm(2N-Q)}, \tag{1.7}$$

with $N = 1, 2, \ldots, \infty$ and

$$\left[\ln Y^{(\alpha)}_{(w|1)}\right]_{\pm 1} = \ln\left(\frac{1+1/Y^{(\alpha)}_{(y|-)}}{1+1/Y^{(\alpha)}_{(y|+)}}\right), \qquad \left[\ln Y^{(\alpha)}_{(v|1)}\right]_{\pm 1} = \ln\left(\frac{1+Y^{(\alpha)}_{(y|-)}}{1+Y^{(\alpha)}_{(y|+)}}\right), \tag{1.8}$$

where the symbol $[f]_Z$ with $Z \in \mathbb{Z}$ denotes the discontinuity of $f(z)$

$$[f]_Z = \lim_{\epsilon \to 0^+} f(u + iZ/g + i\epsilon) - f(u + iZ/g - i\epsilon), \tag{1.9}$$

on the semi-infinite segments described by $z = u + iZ/g$ with $u \in (-\infty, -2) \cup (2, +\infty)$ and the function $[f(u)]_Z$ is the analytic extension of the discontinuity (1.9) to generic complex values of u. To retrieve the TBA equations, the extended Y-system has to be supplemented with the asymptotics

$$e^{\Delta(u)} \sim u^L \quad \text{for } u \to \infty,\ \mathrm{Im}(u) < 0;$$
$$e^{\Delta(u)} \sim 1/u^L \quad \text{for } u \to \infty,\ \mathrm{Im}(u) > 0, \tag{1.10}$$

which capture all the dependence on the scale L.

In this paper, instead of describing how relations (1.6)–(1.8) can be deduced from the TBA equations, we will show how the ground state TBA equations for both the mirror and the direct AdS_5/CFT_4 theories can be derived from (1.6)–(1.8) and the Y-system using Cauchy's integral theorem. Many other interesting results will emerge along the way on the analytic properties of the Y functions, on the dressing factor and on the quantisation of L.

The rest of this paper is organised as follows. Section 2 contains the TBA equations of [30–32] written in a form more appropriate to the study of their analytic structure. A previously unnoticed link between the quantisation of the total momentum, the dilogarithm trick [19] and the trace condition is discussed in Section 3.

The AdS_5/CFT_4-related Y-system is described in Section 4; its validity as the rapidity parameter u is moved in the complex plane is briefly discussed together with some preliminary comments on the analytic structure of the Y functions. The functional relations versus TBA transformation method for a particular equation involving the fermionic y-particles is described in detail in Section 5. The derivation of the discontinuity function Δ from the local Y-system and the relevant discontinuity relations is given in Section 6. The TBA equations for the w-, v- and Q-particles are derived in Sections 7, 8 and 9, respectively.

As a preliminary application of the method, we have derived the TBA for the direct theory: this result is described in Section 10. Finally Section 11 contains our conclusions. There are also seven appendices. The S-matrix elements involved in the definition of the TBA kernels are given in Appendix A. Appendix B presents some previously known identities with an emphasis on aspects of analytic continuation. Appendices C to F contain the most technical parts of the calculations. In particular, in Appendices C and D we prove the equivalence between the two different explicit expressions for the mirror dressing factor Σ and $\hat{\sigma}$ given in [48] and [31], respectively. Different conventions have been adopted in [30–32] for the string coupling, the labeling and the definition of the Y functions and the TBA kernels: the purpose of Appendix G is to provide the reader with a concise dictionary.

2. The TBA equations

The origin of the non-standard properties of the AdS_5/CFT_4 thermodynamic Bethe Ansatz system rests on the presence, in the S-matrix elements, of a double-valued function $x(u)$ defined as

$$x(u) = \left(\frac{u}{2} - i\sqrt{1-\frac{u^2}{4}}\right). \tag{2.1}$$

In the first Riemann sheet: $\mathrm{Im}(x) < 0$ and x behaves under complex conjugation as $x(u) = 1/x^*(u^*)$. It is convenient to parameterize the TBA pseudoenergies in terms of a common rapidity variable u, analogously the kernels will be defined in terms of a pair of complex variables [30–32]. The relationship between the Hubbard variables [41,42] and u is [29]:

$$\tilde{p}^Q(u) = \frac{ig}{2}\left(\sqrt{4-\left(u+i\frac{Q}{g}\right)^2} - \sqrt{4-\left(u-i\frac{Q}{g}\right)^2}\right),$$
$$ie^{-iq} = y(u), \qquad \lambda = u = 2\sin(q). \tag{2.2}$$

The double-valued function $y(u)$ can be written in terms of $x(u)$ as

$$y(u) = \begin{cases} x(u) & \text{for } \mathrm{Im}(y) < 0; \\ 1/x(u) & \text{for } \mathrm{Im}(y) > 0. \end{cases} \tag{2.3}$$

The thermodynamic Bethe Ansatz equations derived in [30–32] and the associated Y-system can be encoded in the diagram represented in Fig. 1 and originally proposed in [38]. The reader is addressed to [43] for a concise up-to-date summary of the main results on TBA and excited states. The TBA equations, at arbitrary chemical potentials [44–46], are:

$$\varepsilon_Q(u) = \mu_Q + L\tilde{E}_Q(u) - \sum_{Q'} L_{Q'} * \phi^{\Sigma}_{Q'Q}(u)$$
$$+ \sum_{\alpha}\left(\sum_M L^{(\alpha)}_{v|M} * \phi_{(v|M),Q}(u) + L^{(\alpha)}_y *_{\bar{\gamma}_o} \phi_{y,Q}(u)\right), \tag{2.4}$$

$$\varepsilon^{(\alpha)}_y(u) = \mu^{(\alpha)}_y - \sum_Q L_Q * \phi_{Q,y}(u) + \sum_M \left(L^{(\alpha)}_{(v|M)} - L^{(\alpha)}_{(w|M)}\right) * \phi_M(u), \tag{2.5}$$

$$\varepsilon^{(\alpha)}_{(w|K)}(u) = \mu^{(\alpha)}_{(w|K)} + \sum_M L^{(\alpha)}_{(w|M)} * \phi_{MK}(u) + L^{(\alpha)}_y *_{\bar{\gamma}_o} \phi_K(u), \tag{2.6}$$

$$\varepsilon^{(\alpha)}_{(v|K)}(u) = \mu^{(\alpha)}_{(v|K)} - \sum_Q L_Q * \phi_{Q,(v|K)}(u) + \sum_M L^{(\alpha)}_{(v|M)} * \phi_{MK}(u)$$
$$+ L^{(\alpha)}_y *_{\bar{\gamma}_o} \phi_K(u), \tag{2.7}$$

where $\alpha = 1, 2$, $K = 1, 2, \ldots$, the contour $\bar{\gamma}_o$ is presented in Fig. 2,

$$\tilde{E}_Q(u) = \ln\frac{x(u - iQ/g)}{x(u + iQ/g)} \tag{2.8}$$

and the inverse of the temperature $L \in \mathbb{N}$ is a discrete SYM variable.

In the following we will agree that, when not specified otherwise, the sums $\sum_\alpha$, $\sum_\tau$, $\sum_K$ (for any capital letter K) will always stand for $\sum_{\alpha=1,2}$, $\sum_{\tau=\pm1}$ and $\sum_{K=1}^{\infty}$, respectively. Further,

Fig. 2. The contour $\bar{\gamma}_{\mathsf{o}}$.

Fig. 3. The contour $\bar{\gamma}_{\mathsf{x}}$.

$$Y_a(u) = e^{\varepsilon_a(u)}, \qquad L_a(u) = \ln\big(1 + 1/Y_a(u)\big), \qquad \Lambda_a(u) = \ln\big(1 + Y_a(u)\big), \tag{2.9}$$

for any choice of the collective index a and $\{\mu_a\}$ will be the set of chemical potentials. The symbols '$*$' and '$*_\gamma$' denote respectively the convolutions

$$\mathcal{F} * \phi(u) = \int_{\mathbb{R}} dz\, \mathcal{F}(z)\phi(z,u), \qquad \mathcal{F} *_\gamma \phi(u) = \oint_{\gamma} dz\, \mathcal{F}(z)\phi(z,u). \tag{2.10}$$

The kernels are

$$\phi_{ab}(z,u) = \frac{1}{2\pi i}\frac{d}{dz}\ln S_{ab}(z,u), \tag{2.11}$$

where the 'S-matrix' elements S_{ab} are listed in Appendix A.

The kernel $\phi^{\Sigma}_{Q'Q}$ appearing in Eq. (2.4) can be naturally separated into two parts

$$\phi^{\Sigma}_{Q'Q}(z,u) = -\phi_{Q'Q}(z-u) - 2K^{\Sigma}_{Q'Q}(z,u), \tag{2.12}$$

with

$$\phi_{Q'Q}(u) = \frac{1}{2\pi i}\frac{d}{du}\ln S_{Q'Q}(u), \qquad K^{\Sigma}_{Q'Q}(z,u) = \frac{1}{2\pi i}\frac{d}{dz}\ln \Sigma_{Q'Q}(z,u). \tag{2.13}$$

In (2.13), $\Sigma_{Q'Q}$ is the improved dressing factor [47] evaluated in the mirror kinematics [48]. Starting form the explicit expression for $\Sigma_{Q'Q}$ given [48], a main result of Appendices C and D is the following compact expression for the kernel $K^{\Sigma}_{Q'Q}$, valid under convolution with $\sum_{Q'} L_{Q'}$ (see also the end of Section 9):

$$K^{\Sigma}_{Q'Q}(z,u) = \frac{1}{2\pi i}\frac{d}{dz}\ln \Sigma_{Q'Q}(z,u) = \oint_{\bar{\gamma}_{\mathsf{x}}} ds\, \phi_{Q',y}(z,s) \oint_{\bar{\gamma}_{\mathsf{x}}} dt\, K^{[2]}_{\Gamma}(s-t)\phi_{y,Q}(t,u), \tag{2.14}$$

where the contour $\bar{\gamma}_{\mathsf{x}}$ is represented in Fig. 3 and

$$K^{[N]}_{\Gamma}(u-z) = \frac{1}{2\pi i}\frac{d}{du}\ln\frac{\Gamma(N/2 - ig(u-z)/2)}{\Gamma(N/2 + ig(u-z)/2)}. \tag{2.15}$$

The result (2.14) shows that $\Sigma_{Q'Q} \equiv \hat{\sigma}_{Q'Q}$, where $\hat{\sigma}$ is the mirror dressing factor of [31,49,50].

It is interesting that the result (2.14) has precisely the double convolution form predicted by [51] and formally identical to that of the dressing factor in the direct channel [47]. Further, the equilibrium free energy at finite temperature $T = 1/L$ is

$$\tilde{F}(L) = -\frac{1}{L}\sum_{Q}\int_{\mathbb{R}} \frac{du}{2\pi}\frac{d\tilde{p}^{Q}}{du} L_Q(u), \tag{2.16}$$

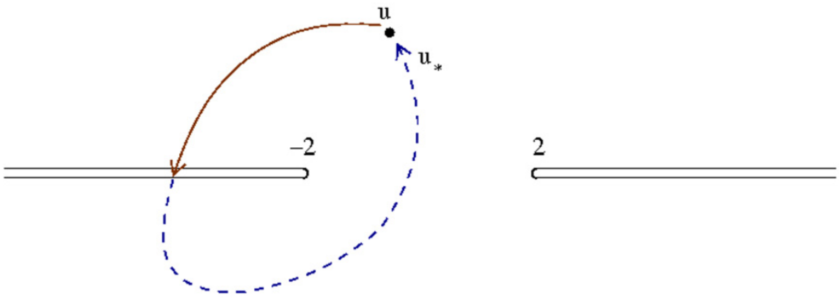

Fig. 4. The second sheet image u_* of u.

with

$$\tilde{p}^Q(u) = gx(u - iQ/g) - gx(u + iQ/g) + iQ. \tag{2.17}$$

We shall restrict the set of allowed chemical potentials by

$$\prod_{Q'} e^{\mu_{Q'} C_{Q'Q}} = e^{2\mu_y^{(\alpha)} - \mu_{(v|1)}^{(\alpha)} + \mu_{(w|1)}^{(\alpha)}} = \prod_M e^{\mu_{(w|M)}^{(\alpha)} C_{MK}} = \prod_M e^{\mu_{(v|M)}^{(\alpha)} C_{MK}} = 1, \tag{2.18}$$

with $C_{MN} = 2\delta_{M,N} - A_{MN}$, $A_{1,M} = \delta_{2,M}$, $A_{NM} = \delta_{M,N+1} + \delta_{M,N-1}$ and $A_{MN} = A_{NM}$. Then in the derivation of the Y-system the chemical potentials cancel and in all the subsequent considerations they can be ignored and eventually restored in the final equations. The case with all vanishing chemical potentials apart for $\mu_y^{(1)} = -\mu_y^{(2)} = i\pi$ is perhaps the most interesting: it corresponds to the calculation of the Witten index [52] and, through the relation

$$E_0(L) = L\tilde{F}(L), \tag{2.19}$$

to the zero energy protected ground state of AdS_5/CFT_4. A vanishing ground state energy, or in general any protected state, should correspond to a singularity in the associated TBA solution. A way to explore the neighborhoods of this point, while preserving the Y-systems structure, is to slightly displace the chemical potentials from these critical values such that (2.18) are still fulfilled.

As we shall see in more details in the following sections, the functions $Y_a(u) = e^{\varepsilon_a(u)}$ solutions of the TBA equations (2.4)–(2.7) live on multi-sheeted coverings of the complex plane with an infinite number of square-root branch points in the set $u \in \{\pm 2 + im/g\}$ with $m \in \mathbb{Z}$. For the mirror-AdS_5/CFT_4 theory under consideration all the cuts are conventionally set parallel to the real axis and external to the strip $|\mathrm{Re}(u)| < 2$.

In particular, as a direct consequence of the mapping between the function $Y_y^{(\alpha)}(q)$ of [30] and $Y_y^{(\alpha)}(u)$ through the relation $ie^{-iq} = y(u)$ and (2.3), $Y_y^{(\alpha)}(u)$ has a pair of square-root branch points on the real axis at $u = \pm 2$. In Eqs. (2.4)–(2.7), the integration contour $\bar{\gamma}_{\mathrm{o}}$, represented in Fig. 2, is a negative oriented closed path surrounding precisely this pair of singularities. In the following sections we shall denote with $Y_{(y|-)}^{(\alpha)}(u)$, or when it will not be possible source of confusion simply by $Y_y^{(\alpha)}(u)$, the first sheet evaluation of $Y_y^{(\alpha)}$ and with $Y_{(y|+)}^{(\alpha)}(u) = Y_y^{(\alpha)}(u_*)$ the evaluation obtained by replacing u with its second sheet image u_* reached by analytic continuation through the branch cut $u \in (-\infty, -2)$ (see, Fig. 4). For the function x defined in (2.1) and appearing in the definition of the S-matrix elements we have

$$x_+(u) = x(u_*) = 1/x(u), \quad \mathrm{Im}\big(x(u_*)\big) > 0. \tag{2.20}$$

3. Total momentum quantisation, the trace condition and the dilogarithm trick

The study of excited states goes beyond the objectives of this paper. Here, we simply remind to the reader that in analogy with the early results for relativistic models [20,21] excited state TBA equations can be obtained from (2.4)–(2.7) by replacing the contour of integration with a more general complex path γ_a enclosing a finite number of poles $\{u_a^{(k)}\}$ of $\frac{dL_a(u)}{du}$. The number of active singularities, or equivalently of extra residues $\ln S_{ab}(u_a^{(k)}, u)$ explicitly appearing in the excited state TBA variants, crucially depends on the particular energy level $E_n(L)$ under consideration, the scale L and the coupling. The first simple examples of active singularity transitions were observed in [20,21] and the complete range of possibilities discussed in [39]. Adopting these ideas, excited state TBA equations for AdS_5/CFT_4 were proposed in [31,53] and partially studied in [53–57].

Finally, we would like to remark that the quantity

$$P^{(n)} = \sum_Q \int_{\gamma_Q^{(n)}} \frac{du}{2\pi} \frac{d\tilde{E}_Q(u)}{du} L_Q(u)$$
$$= \sum_Q \left(\int_{\mathbb{R}} \frac{du}{2\pi} \frac{d\tilde{E}_Q(u)}{du} L_Q(u) + \sum_k \left(r_Q^{(k)} p^Q\left(u_Q^{(k)}\right)\right)\right), \tag{3.1}$$

can be computed exactly adapting the standard dilogarithm trick (see, for example [19]). In Eq. (3.1), $r_Q^{(k)} = +1$ or -1 is the residue of $\frac{dL_Q(u)}{du}$ at $u = u_Q^{(k)}$ and $p^Q \equiv i\tilde{E}_Q$ is the single particle momentum in the direct theory. Considering the possible monodromy properties of the Rogers dilogarithm one can argue that

$$P^{(n)} = \frac{2\pi Z^{(n)}}{L}, \quad Z^{(n)} \in \mathbb{Z}, \tag{3.2}$$

where $Z^{(n)} = 0$ for the ground state and for any zero momentum state. Therefore, we have found in $P^{(n)}$ a quantity which is naturally quantised and it is a very natural candidate to be the total momentum of the direct theory. To our knowledge, even in the context of excited-state TBA equations for relativistic models, this fact was never noticed before. Then, it is natural to expect that the momentum condition (trace condition) on the physical states should correspond to the stronger requirement $Z^{(n)}/L \in \mathbb{Z}$. The latter constraint was first imposed in [31,58] and used to infer the equivalence between the TBA equations in [30,32,53] and [59] with those in [31] and [58], respectively (see also the related discussion in [43], and the end of Appendix G). Although this is a very interesting observation, we would like to underline here that for a generic physical state with non-zero momentum a difference $\Delta\mu_a = i2\pi k_a$ with $k_a \in \mathbb{Z}$ between the two sets of chemical potentials would still remain. As it is well known, the energy levels $E_n(L)$ are in general not $2\pi i$-periodic in the μ_a's and this very small difference between the two sets of TBA equations may be not totally harmless.

4. The Y-system for the AdS/CFT correspondence

A consequence of the non-trivial analytic properties of the AdS_5/CFT_4 Y functions is that the product $Y_a(u - i/g)Y_a(u + i/g)$ requires a prescription on how to pass from $(u - i/g)$ to $(u + i/g)$. If $Y_a(u + i/g)$ can be obtained from $Y_a(u - i/g)$ by shifting vertically $u \to u + i2/g$

for $\mathrm{Re}(u) \in (-2, 2)$, in general this is not true for u outside this region. Only applying the two shifts $\pm i/g$ to $Y_a(u)$ starting from $\mathrm{Re}(u) \in (-2, 2)$ leads to the local Y-system of Eqs. (1.1)–(1.4) above. A very direct way of deriving the Y-system is simply by analytic continuation of the TBA equations. The relevant identities involving the TBA kernels are collected in Appendix B; here is a brief summary of the derivation:

- Q-particles: The Y-system equation (1.1) is obtained from the TBA equation (2.4) using properties (B.10) and (B.11).
- y-particles: The Y-system equation (1.2) for $Y^{(\alpha)}_{(y|-)}(u) \equiv Y^{(\alpha)}_y(u)$ is obtained from the TBA equation (2.5) using properties (B.2) and (B.5).
- w-particles: The Y-system equation (1.3) is obtained from the TBA equation (2.6) using properties (B.2) and (B.5).
- v-particles: Starting from the TBA equation (2.7) and using properties (B.2), (B.5) and (B.6) we have:

$$
\begin{aligned}
&Y^{(\alpha)}_{(v|M)}\left(u+\frac{i}{g}\right)Y^{(\alpha)}_{(v|M)}\left(u-\frac{i}{g}\right)\\
&\quad= \frac{\prod_N \big(1+Y^{(\alpha)}_{(v|N)}(u)\big)^{A_{MN}}}{\big(1+\frac{1}{Y_{M+1}(u)}\big)}\left[\frac{\big(1+\frac{1}{Y^{(\alpha)}_{(y|-)}(u)}\big)}{\big(1+\frac{1}{Y^{(\alpha)}_{(y|+)}(u)}\big)}\right]^{\delta_{M,1}}\\
&\qquad\times \exp\left(\delta_{M,1}\sum_Q \big(L_Q * \phi_{Q,(y|-)}(u) - L_Q * \phi_{Q,(y|+)}(u)\big)\right).
\end{aligned}
\tag{4.1}
$$

  Now, recall that we know a priori that the relevant values of the functions $Y^{(\alpha)}_y$ are from two different sheets, which are connected by encircling one of the branch points at ± 2. The difference between these two determinations of $\ln Y^{(\alpha)}_y$ can be easily read from the TBA equation (2.5) to coincide with the integral expression on the rhs of (4.1):

$$
\ln \frac{Y^{(\alpha)}_{(y|-)}(u)}{Y^{(\alpha)}_{(y|+)}(u)} = \big[\ln Y^{(\alpha)}_y(u)\big]_0 = -\sum_Q \big(L_Q * \phi_{Q,(y|-)}(u) - L_Q * \phi_{Q,(y|+)}(u)\big). \tag{4.2}
$$

  Eqs. (4.2) and (4.1) then lead to (1.4).

Apart for some minor subtleties, Eqs. (1.1)–(1.4) derived in this way match the proposal [38] by Gromov, Kazakov and Vieira.

Finally, we would like to mention that although the Y-system was derived for $|\mathrm{Re}(u)| < 2$, the coupled equations (1.1)–(1.4) are simultaneously valid for any complex value of u. To see this we can adapt to the current situation the argument first used in a similar context in [39]. Subtracting the right-hand sides of (1.1)–(1.4) from the left-hand sides yields a set of analytic functions (depending on the same variable u) which are identically zero. These functions therefore remain zero during any process of analytic continuation, and so the Y-system and its consequences hold equally for all complex values of u.

Table 1 shows the location of the square branch points for the various Y's. For any analytic continuation path connecting the region $|\mathrm{Re}(u)| < 2$ to a generic point u', this pattern of singularities dictates unambiguously which branch of every Y function should be selected in order for the Y-system to hold. Notice that the most natural choice is to avoid crossing any of the

Table 1
Square-root branch points for the Y functions.

Function	Singularity position	
$Y_y^{(\alpha)}(u)$	$u = \pm 2 + i\frac{2J}{g}$,	$J = 0, \pm 1, \pm 2, \ldots$
$Y_{(w\|M)}^{(\alpha)}(u)$ $Y_{(v\|M)}^{(\alpha)}(u)$ $Y_M(u)$	$u = \pm 2 + i\frac{J}{g}$,	$J = \pm M, \pm(M+2), \pm(M+4), \ldots$

semi-infinite segments $(-\infty, -2) \cup (2, +\infty) + im/g$, $m \in \mathbb{Z}$, so that all the Y functions are simultaneously evaluated on the same sheet. Because it contains the physical values of the Y's, this will be referred to as the first Riemann sheet.

A short discussion on the independent analytic continuation of Y_1 around the branch points at $u = -2 \pm i/g$ and the associated non-local variants of the Y-system is postponed to Section 6.

5. TBA ≡ dispersion relation

From the works [30–32] it has emerged that, contrary to the more studied relativistic invariant cases, the transformation from Y-system to TBA equations is not straightforward. The local form of the AdS_5/CFT_4-related Y-system does not contain information on the branch points and in particular it is almost totally insensitive to the precise form of the dressing factor. In this, and in the following sections we shall show that there exists a set of local functional constraints for the discontinuities that can be directly transformed to integral form. To explain the idea let us first consider Eq. (4.2). By setting

$$T(z,u) = \frac{1}{\sqrt{4-z^2}(z-u)} = \frac{2\pi i}{\sqrt{4-u^2}} K(z,u), \tag{5.1}$$

where

$$K(z - iQ/g, u) - K(z + iQ/g, u) = \phi_{Q,(y|-)}(z,u) - \phi_{Q,(y|+)}(z,u), \tag{5.2}$$

Eq. (4.2) can be written in the form

$$\frac{\ln(Y_{(y|-)}^{(\alpha)}(u)/Y_{(y|+)}^{(\alpha)}(u))}{\sqrt{4-u^2}} = -\sum_Q \int_{\mathbb{R}} \frac{dz}{2\pi i} L_Q(z)\big(T(u - iQ/g, z) - T(u + iQ/g, z)\big). \tag{5.3}$$

We shall now prove that the relation on the discontinuities

$$\big[\ln\big(Y_{(y|-)}^{(\alpha)}/Y_{(y|+)}^{(\alpha)}\big)\big]_{\pm 2N} = -\sum_{Q=1}^{N} [L_Q]_{\pm(2N-Q)}, \tag{5.4}$$

combined with some more general analyticity information, is equivalent to (5.3). In (5.4), ln denotes the principal branch of the complex logarithm while the symbol $[f]_Z$ with $Z \in \mathbb{Z}$ stands for the discontinuity of $f(z)$ on the semi-infinite segments described by $z = u + iZ/g$ with $u \in (-\infty, -2) \cup (2, +\infty)$:

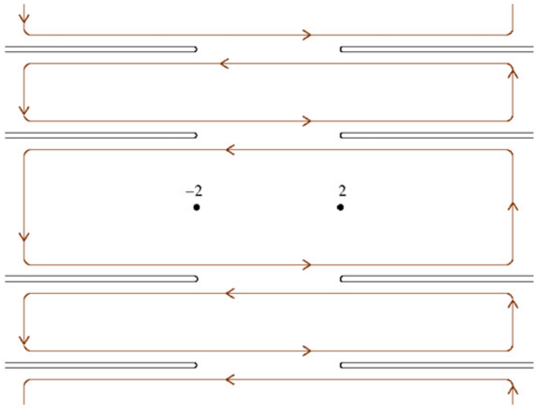

Fig. 5. The deformed contour Γ_{O}.

$$[f]_Z = \lim_{\epsilon \to 0^+} f(u + iZ/g + i\epsilon) - f(u + iZ/g - i\epsilon). \tag{5.5}$$

Thus, the function

$$\big[f(u)\big]_Z = f(u + iZ/g) - f(u_* + iZ/g) \tag{5.6}$$

is the analytic extension of the discontinuity (5.5) to generic complex values of u.

We conjecture that the relevance of Eq. (5.4) and the other discontinuity relations introduced in the following sections is not restricted to the ground state but, provided the analytic properties on the relevant reference sheets are suitably modified, they can be directly transformed into excited state TBA equations.

First notice that the quantity appearing on the lhs of (5.3) is analytic at the points $u = \pm 2$, but it still has an infinite set of branch points at $u = \pm 2 \pm i2N/g$ with $N \in \mathbb{N}$. In the following we shall assume that, for the ground state TBA equations, these are the only singularities of (5.3) on the first Riemann sheet. By applying Cauchy's integral theorem we can first write

$$\frac{\ln(Y^{(\alpha)}_{(y|-)}(u)/Y^{(\alpha)}_{(y|+)}(u))}{\sqrt{4-u^2}} = \oint_\gamma \frac{dz}{2\pi i} \frac{\ln(Y^{(\alpha)}_{(y|-)}(z)/Y^{(\alpha)}_{(y|+)}(z))}{(z-u)\sqrt{4-z^2}}, \tag{5.7}$$

where γ is a positive oriented contour running inside the strip $|\mathrm{Im}(u)| < 1/g$, and then deform γ into the homotopically equivalent contour Γ_{O} represented in Fig. 5 as the union of an infinite number of rectangular contours lying between the branch cuts of (5.3). Since $\ln(Y^{(\alpha)}_{(y|-)}/Y^{(\alpha)}_{(y|+)}) \to O(1)$ uniformly as $|u| \to \infty$ the sum of the vertical segment contributions vanishes as the horizontal size of the rectangles tends to infinity. Using relation (5.4) we can write the rhs of (5.7) as

$$-\sum_{J,\tau}\bigg(\int\limits_{\mathbb{R}-i2\tau J/g+i\epsilon} - \int\limits_{\mathbb{R}-i2\tau J/g-i\epsilon}\bigg)\frac{dz}{2\pi i}\sum_{Q=1}^{J}\frac{L_Q(z+i\tau Q/g)}{\sqrt{4-z^2}(z-u)} \tag{5.8}$$

($\tau = \pm 1$, $J = 1, 2, \ldots$). However, as a consequence of the identity

$$\bigg(\int\limits_{\mathbb{R}+i\epsilon} - \int\limits_{\mathbb{R}-i\epsilon}\bigg) dz \bigg(f\bigg(z + \frac{i2}{g}\bigg) + f(z)\bigg) = \bigg(\int\limits_{\mathbb{R}+\frac{i2}{g}+i\epsilon} - \int\limits_{\mathbb{R}-i\epsilon}\bigg) dz\, f(z), \tag{5.9}$$

which holds for any function $f(u)$ analytic in the strip $0 < \mathrm{Im}(u) < \frac{2}{g}$ and vanishing for $|\mathrm{Re}(u)| \to \infty$ in this region, several cancellations take place in (5.8). The property (5.9) and a change of integration variables $z \to z - i\tau Q/g$ in (5.8) leads to:

$$\begin{aligned}
&\frac{\ln(Y^{(\alpha)}_{(y|-)}(u)/Y^{(\alpha)}_{(y|+)}(u))}{\sqrt{4-u^2}} \\
&\quad = \sum_{Q,\tau} \tau \int_{\mathbb{R}+i\tau(Q/g-\epsilon)} \frac{dz}{2\pi i} \frac{L_Q(z)}{\sqrt{4-(z+i\tau Q/g)^2}(z+i\tau Q/g-u)} \\
&\quad = -\sum_{Q} \int_{\mathbb{R}} \frac{dz}{2\pi i} \biggl(\frac{L_Q(z)}{\sqrt{4-(z-iQ/g)^2}(z-iQ/g-u)} \\
&\qquad - \frac{L_Q(z)}{\sqrt{4-(z+iQ/g)^2}(z+iQ/g-u)} \biggr),
\end{aligned} \tag{5.10}$$

which matches the result (4.2). To get the TBA equation (2.5) we still need to prove that

$$\ln\bigl(Y^{(\alpha)}_{(y|-)}(u)\, Y^{(\alpha)}_{(y|+)}(u)\bigr) = \sum_M \bigl(2L^{(\alpha)}_{(v|M)} - 2L^{(\alpha)}_{(w|M)} - L_M\bigr) * \phi_M(u). \tag{5.11}$$

Let us show that (5.11) can be obtained as a dispersion relation as well. We simply use the fact (see Appendix F) that the functional constraints (5.4), together with the Y-system, imply:

$$\bigl[\ln\bigl(Y^{(\alpha)}_{(y|-)} Y^{(\alpha)}_{(y|+)}\bigr)\bigr]_{\pm 2N} = \sum_{J=1}^{N} \bigl[2L^{(\alpha)}_{(v|J)} - 2L^{(\alpha)}_{(w|J)} - L_J\bigr]_{\pm(2N-J)}. \tag{5.12}$$

Since $\ln(Y^{(\alpha)}_{(y|-)} Y^{(\alpha)}_{(y|+)})$ is regular in the strip $|\mathrm{Im}(u)| < 2/g$, by Cauchy's formula we finally arrive to the desired result (5.11):

$$\begin{aligned}
&\ln\bigl(Y^{(\alpha)}_{(y|-)}(u) Y^{(\alpha)}_{(y|+)}(u)\bigr) \\
&\quad = -\sum_{M,\tau} \tau \int_{\mathbb{R}+i\tau(M/g-\epsilon)} \frac{dz}{2\pi i} \frac{2L^{(\alpha)}_{(v|M)}(z) - 2L^{(\alpha)}_{(w|M)}(z) - L_M(z)}{(z+i\tau M/g-u)} \\
&\quad = \sum_M \int_{\mathbb{R}} \frac{dz}{2\pi i} \bigl(2L^{(\alpha)}_{(v|M)}(z) - 2L^{(\alpha)}_{(w|M)}(z) - L_M(z)\bigr) \\
&\qquad \times \biggl(\frac{1}{(z-iM/g-u)} - \frac{1}{(z+iM/g-u)} \biggr).
\end{aligned} \tag{5.13}$$

All the kernels appearing in the system of TBA equations are basically certain linear combinations of $K(z,u)$ and $\phi_M(u)$. Therefore, the derivation of the rest of the TBA system from functional equations for discontinuities goes under the same spell.

6. The discontinuity function $\Delta(u)$

Following closely the method described in the previous section, we shall now derive a spectral representation for the function

$$\Delta(u) = \big[\ln Y_1(u)\big]_{+1} = \ln \frac{Y_{(1|-)}(u+\frac{i}{g})}{Y_{(1|+)}(u+\frac{i}{g})}, \tag{6.1}$$

where we have set $Y_{(1|-)}(u+\frac{i}{g}) \equiv Y_1(u+\frac{i}{g})$, $Y_{(1|+)}(u+\frac{i}{g}) \equiv Y_1(u_*+\frac{i}{g})$, and the points u and u_* are analytically connected through the path depicted in Fig. 4. In other words, this function encodes the branching properties of Y_1 around $-2+i/g$, and can be related to the analytic behavior around $-2-i/g$ through the Y-system relation (1.1). Using the fact that $Y_1(u-\frac{i}{g})Y_1(u+\frac{i}{g})\prod_\alpha(1+\frac{1}{Y^{(\alpha)}_{(y|-)}(u)}) = (1+Y_2(u))$ is regular on the real axis, we find:

$$\bar{\Delta}(u) = \big[\ln Y_1(u)\big]_{-1} = \ln \frac{Y_1(u-\frac{i}{g})}{Y_1(u_*-\frac{i}{g})} = -\Delta(u) - \sum_\alpha \ln \frac{\big(1+\frac{1}{Y^{(\alpha)}_{(y|-)}(u)}\big)}{\big(1+\frac{1}{Y^{(\alpha)}_{(y|+)}(u)}\big)}. \tag{6.2}$$

It is interesting to recall how the function Δ was first introduced in [32]. A non-local variant of the Y-system equation (1.1) is obtained if the function $Y_1(u)$ is shifted by $\pm i/g$ starting from a point in the region $|\mathrm{Re}(u)| > 2$, $|\mathrm{Im}(u)| < 1/g$. In this way, we are clearly forced to pass either of the branch cuts $u \in (-\infty,-2) \cup (2,\infty) \pm i/g$, so that, for $\mathrm{Im}(u) > 0$, on the lhs we get the product $Y_1(u-\frac{i}{g})Y_1(u_*+\frac{i}{g}) = Y_{(1|-)}(u-\frac{i}{g})Y_{(1|+)}(u+\frac{i}{g})$. Using (1.1) and the definition (6.1), we see that this function satisfies

$$Y_1\left(u-\frac{i}{g}\right)Y_1\left(u_*+\frac{i}{g}\right) = \frac{(1+Y_2(u))}{\prod_\alpha\big(1+\frac{1}{Y^{(\alpha)}_{(y|-)}(u)}\big)}e^{-\Delta(u)}, \tag{6.3}$$

which is non-local, as we will see in a moment. Reconnecting with the discussion of Section 4, we stress that this relation cannot be obtained from (1.1) by following any analytic continuation path. The correct analytic continuation of (1.1) is rather

$$Y_1\left(u_*-\frac{i}{g}\right)Y_1\left(u_*+\frac{i}{g}\right) = \frac{(1+Y_2(u))}{\prod_\alpha\big(1+\frac{1}{Y^{(\alpha)}_{(y|+)}(u)}\big)}, \tag{6.4}$$

and, since $Y_1((u+\frac{2i}{g})_*-\frac{i}{g}) \neq Y_1(u_*+\frac{i}{g})$, the functions on the lhs of (6.4) are now from two different sheets.

In [32] it was shown that, for $0 < \mathrm{Im}(u) < \frac{1}{g}$, $\Delta(u)$ admits the following integral representation:

$$\begin{aligned}\Delta(u) = {} & \sum_{N,\alpha}\int_{\mathbb{R}} dz\, L^{(\alpha)}_{(v|N)}(z)\left(K\left(z+\frac{i}{g}N,u\right)+K\left(z-\frac{i}{g}N,u\right)\right) \\ & + 2L\ln x(u) - \sum_\alpha L^{(\alpha)}_{(y|-)}(u) + \sum_\alpha L^{(\alpha)}_y *_{\bar{\gamma}_\circ} K(u) + \Delta^\Sigma(u).\end{aligned} \tag{6.5}$$

Eq. (6.5) needs to be modified when u is moved outside the region $0 < \mathrm{Im}(u) < \frac{1}{g}$. For example, one has to use the property

$$L^{(\alpha)}_y *_{\bar{\gamma}_\circ} K(u) \to -L^{(\alpha)}_y *_{\bar{\gamma}_\circ} K(u) + L^{(\alpha)}_{(y|+)}(u) + L^{(\alpha)}_{(y|-)}(u) \tag{6.6}$$

under the analytic continuation $u \to u_*$ to check that $\Delta(u_*) = -\Delta(u)$, as expected from the definition (6.1). Finally, in (6.5) we have defined

$$\Delta^{\Sigma}(u) = 2\sum_{Q} L_Q * K_Q^{\Sigma}(u), \tag{6.7}$$

and

$$K_Q^{\Sigma}(z,u) = K_{Q,1}^{\Sigma}(z, u + i/g) - K_{Q,1}^{\Sigma}(z, u_* + i/g). \tag{6.8}$$

An explicit expression for the kernel[2] K_Q^{Σ} requires the knowledge of the dressing factor in the mirror kinematics, which was found in [48]. Stemming from the result contained in [60], in Appendix C we show that Δ^{Σ} admits the following representation (see Eq. (C.12)):

$$\Delta^{\Sigma}(u) = \sum_{M,\alpha} \oint_{\bar{\gamma}_{\mathsf{x}}} dz \, \ln Y_y^{(\alpha)}(z)\big(-K(z + i2M/g, u) + K(z - i2M/g, u)\big). \tag{6.9}$$

The function $\Delta(u)$ has a branch point at every $u = \pm 2 + i2Z/g$, $Z \in \mathbb{Z}$. The discontinuities across the branch cuts lying away from the real axis fulfil the following simple functional equations:

$$\big[\Delta(z)\big]_{\pm 2N} = \mp \sum_{\alpha} \left[L_{(y|\mp)}^{(\alpha)}(z) + \sum_{M=1}^{N} L_{(v|M)}^{(\alpha)}\left(z \mp \frac{iM}{g}\right)\right]_{\pm 2N} \mp \sum_{\alpha} \big[\ln Y_y^{(\alpha)}(z)\big]_0 \tag{6.10}$$

($N = 1, 2, \ldots$). Moreover, assuming that the asymptotic behavior is dominated by the energy term in (6.5), we see that:

$$\begin{aligned} &e^{\Delta(u)} \sim u^L \to \infty \quad \big(\text{for } u \to \infty,\ \mathrm{Im}(u) < 0\big); \\ &e^{\Delta(u)} \sim 1/u^L \to 0 \quad \big(\text{for } u \to \infty,\ \mathrm{Im}(u) > 0\big), \end{aligned} \tag{6.11}$$

and an extra branch cut appears when taking the logarithm. Notice that, in order for (6.10) to be true, we are forced to draw this cut in the interior of the first sheet: we will assume that $\Delta(u)$ has a constant discontinuity running along the imaginary axis:

$$\Delta(iv + \epsilon) - \Delta(iv - \epsilon) = i2L\pi \quad (v \in \mathbb{R}). \tag{6.12}$$

Let us now show that the local information (6.10) and (6.12) is enough to recover Eq. (6.5). Again, we start from Cauchy's integral formula for $\Delta(u)/\sqrt{4-u^2}$, with u lying in the region $|\mathrm{Im}(u)| < \frac{1}{g}$. A positive oriented path γ encircling u can be replaced by the contour Γ_{x} represented in Fig. 6 plus two vertical lines on both sides of the logarithmic cut (6.12):

$$\begin{aligned} \frac{\Delta(u)}{\sqrt{4-u^2}} &= \oint_{\Gamma_{\mathsf{x}}} \frac{dz}{2\pi i} \frac{\Delta(z)}{(z-u)\sqrt{4-z^2}} + \left(\int_{i\mathbb{R}-\epsilon} - \int_{i\mathbb{R}+\epsilon} \right) \frac{dz}{2\pi i} \frac{\Delta(z)}{(z-u)\sqrt{4-z^2}} \\ &= I_{\Gamma_{\mathsf{x}}}(u) - \frac{L}{\sqrt{4-u^2}} \int_{i\mathbb{R}} dz \frac{d}{dz} \ln\left(\frac{x(z) - x(u)}{x(z) - \frac{1}{x(u)}}\right) \\ &= I_{\Gamma_{\mathsf{x}}}(u) + \frac{2L \ln x(u)}{\sqrt{4-u^2}}, \end{aligned} \tag{6.13}$$

[2] To relate this kernel to the following definition given in [32]:

$$\check{K}_Q^{\Sigma}(z,u) = K_{Q,1}^{\Sigma}(z, u_* + i/g) + K_{Q,1}^{\Sigma}(z, u - i/g) - K_{Q,2}^{\Sigma}(z,u),$$

it is sufficient to use property (B.12), yielding: $K_Q^{\Sigma}(z,u) = -\check{K}_Q^{\Sigma}(z,u)$.

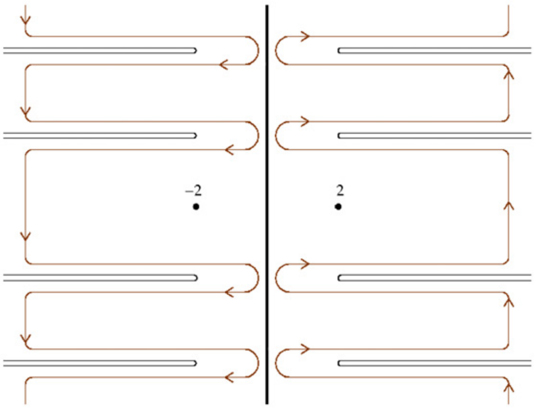

Fig. 6. The contour Γ_{X}: the vertical thick line corresponds to the logarithmic cut of $\Delta(u)$.

and we see that, not only Eq. (6.12) signals that the quantisation of L is deeply related to reflection symmetry of $e^{\Delta(u)}$ about the imaginary axis but it also naturally leads to the term $2L\ln x(u)$ in (6.5) which is directly related to the 'driving' terms $L\tilde{E}_Q(u)$ appearing in the TBA equations. In the same context, but from a slightly different perspective, the quantisation of L has been also discussed in [61]. Next, using (6.10), the integral $I_{\Gamma_{\mathsf{X}}}(u)$ can be rewritten as:

$$I_{\Gamma_{\mathsf{X}}}(u) = -\sum_{N,\alpha,\tau} \tau \oint_{\bar{\gamma}_{\mathsf{x}}} \frac{dz}{2\pi i} \times \frac{L^{(\alpha)}_{(y|-\tau)}(z+i2\tau N/g) + \sum_{M=1}^{N} L^{(\alpha)}_{(v|M)}(z+i\tau(2N-M)/g) + \ln Y^{(\alpha)}_y(z)}{(z+i2\tau N/g-u)\sqrt{4-(z+i2\tau N/g)^2}}. \tag{6.14}$$

For $0 < \mathrm{Im}(u) < \frac{1}{g}$, demanding that $L^{(\alpha)}_{(v|M)}(u)$ is regular on the whole first sheet and that $L^{(\alpha)}_{(y|-)}(u)$ and $L^{(\alpha)}_{(y|+)}(u)$ are regular respectively for $\mathrm{Im}(u) > 0$ and $\mathrm{Im}(u) < 0$, we get:

$$\begin{aligned} I_{\Gamma_{\mathsf{X}}}(u) = \frac{1}{\sqrt{4-u^2}} \sum_\alpha \bigg(& \int_{\mathbb{R}+i(2/g-\epsilon)} dz\, L^{(\alpha)}_{(y|-)}(z) K(z,u) + \int_{\mathbb{R}-i(2/g-\epsilon)} dz\, L^{(\alpha)}_{(y|+)}(z) K(z,u) \\ & + \sum_N \int_{\mathbb{R}} dz\, L^{(\alpha)}_{(v|N)}(z) \big(K(z+iN/g,u) + K(z-iN/g,u) \big) \\ & + \int_{\bar{\gamma}_{\mathsf{x}}} dz\, \ln Y^{(\alpha)}_{(y|-)}(z) \sum_N \big(-K(z+i2N/g,u) + K(z-i2N/g,u) \big) \bigg). \end{aligned} \tag{6.15}$$

We see that the last two lines already reproduce the convolutions with v-related functions and the term Δ^{Σ} in (6.5). To conclude the derivation we use the property $K(z,u) = -K(z_*,u)$, and replace the two integrals over $\mathbb{R} \pm i(2/g - \epsilon)$ in the first line of (6.15) with a single convolution along $\bar{\gamma}_{\mathsf{o}}$, plus a residue due to the pole of $K(z,u)$ at $z = u$ (see Fig. 7):

$$\int_{\mathbb{R}+i(2/g-\epsilon)} dz\, L^{(\alpha)}_{(y|-)}(z) K(z,u) + \int_{\mathbb{R}-i(2/g-\epsilon)} dz\, L^{(\alpha)}_{(y|+)}(z) K(z,u)$$

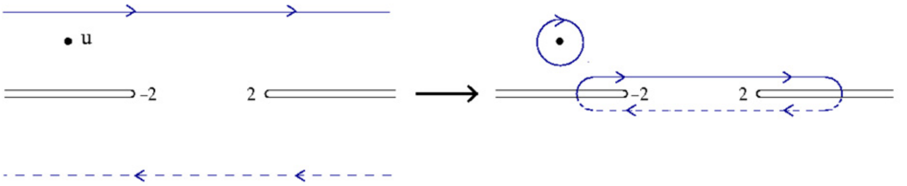

Fig. 7. The contour deformation corresponding to Eq. (6.16).

$$= \int_{\mathbb{R}+i(2/g-\epsilon)} dz\, L_y^{(\alpha)}(z) K(z,u) - \int_{\mathbb{R}-i(2/g-\epsilon)} dz\, L_y^{(\alpha)}(z_*) K(z_*,u)$$

$$= -L_{(y|-)}^{(\alpha)}(u) + L_y^{(\alpha)} *_{\bar{\gamma}_o} K(u). \tag{6.16}$$

Using (6.16) and (6.15) in (6.13) we finally arrive to (6.5).

7. w-particles

As revealed by the TBA equation (2.6), the $Y_{(w|M)}^{(\alpha)}$ functions are free of singularities in the strip $|\mathrm{Im}(u)| < M/g$, with square root branch points at the edge of this region. The discontinuities of $\ln Y_{(w|1)}^{(\alpha)}$ take the very simple form:

$$\big[\ln Y_{(w|1)}^{(\alpha)}\big]_{\pm 1} = \big[L_y^{(\alpha)}\big]_0. \tag{7.1}$$

When supplemented with this simple information, the Y-system equation (1.3) is sufficient to reconstruct the whole set of discontinuities of the w-related functions, and thus to define a dispersion relation. Taking the logarithm of the Y-system equation we have:

$$\ln Y_{(w|M)}^{(\alpha)}(u) = \sum_N A_{MN} \Lambda_{(w|N)}^{(\alpha)}\left(u - \frac{i}{g}\right) + \delta_{M,1}\left(L_{(y|-)}^{(\alpha)}\left(u - \frac{i}{g}\right) - L_{(y|+)}^{(\alpha)}\left(u - \frac{i}{g}\right)\right)$$
$$- \ln Y_{(w|M)}^{(\alpha)}\left(u - \frac{i2}{g}\right). \tag{7.2}$$

First, we notice that from the above stated analytic properties we can deduce that $\ln Y_{(w|M)}^{(\alpha)}(u)$ is regular except possibly at the points $u = -2 \pm i(M+2I)/g$, $u = 2 \pm i(M+2I)/g$, with $I = 0, 1, \ldots$. At these values of u, (7.2) imposes a constraint on the monodromy properties of $\ln Y_{(w|M)}^{(\alpha)}$. A very useful relation is obtained after $M+I$ iterations of (7.2):

$$\ln Y_{(w|M)}^{(\alpha)}\big(u \pm i(M+2I)/g\big)$$
$$= \left(L_{(y|-)}^{(\alpha)}(u \pm i2I/g) - L_{(y|+)}^{(\alpha)}(u \pm i2I/g) + \sum_{N=1}^{M-1} L_{(w|N)}^{(\alpha)}\big(u \pm i(2I+N)/g\big)\right)$$
$$+ \sum_{J=1}^{I}\left(L_{(w|J)}^{(\alpha)}\big(u \pm i(2I-J)/g\big) + \sum_{N=1}^{M-1} 2L_{(w|J+N)}^{(\alpha)}\big(u \pm i(2I-J+N)/g\big)\right.$$
$$\left. + L_{(w|M+J)}^{(\alpha)}\big(u \pm i(M+2I-J)/g\big)\right) + \Lambda_{(w|M+I+1)}^{(\alpha)}\big(u \pm i(M+I-1)/g\big)$$

$$+\sum_{N=1}^{M-1} L^{(\alpha)}_{(w|N+I+1)}\big(u \pm i(I+N-1)/g\big) - \ln Y^{(\alpha)}_{(w|I+1)}\big(u \pm i(I-1)/g\big). \tag{7.3}$$

To simplify the notation, let us rewrite (7.3) through the definition of a family of functions $D^{(w|M)}_{\pm(M+2I)}$ [3]:

$$\begin{aligned}
&\ln Y^{(\alpha)}_{(w|M)}\big(u \pm i(M+2I)/g\big)\\
&= D^{(w|M)}_{\pm(M+2I)}(u) + \Lambda^{(\alpha)}_{(w|M+I+1)}\big(u \pm i(M+I-1)/g\big)\\
&\quad + \sum_{N=1}^{M-1} L^{(\alpha)}_{(w|N+I+1)}\big(u \pm i(I+N-1)/g\big) - \ln Y^{(\alpha)}_{(w|I+1)}\big(u \pm i(I-1)/g\big).
\end{aligned} \tag{7.4}$$

By using (7.1) and the fact that $\ln Y^{(\alpha)}_{(w|N)}(u)$ is regular for $|\mathrm{Im}(u)| < N/g$, we see that they are tied in the following way to the discontinuities of w-related functions:

$$\begin{aligned}
&\big[\ln Y^{(\alpha)}_{(w|M)}\big(u \pm i(M+2I)/g\big)\big]_0\\
&= \big[D^{(w|M)}_{\pm(M+2I)}(u)\big]_0 - \big[\ln Y^{(\alpha)}_{(w|I+1)}\big(u \pm i(I-1)/g\big)\big]_0\\
&\quad + \left[\Lambda^{(\alpha)}_{(w|M+I+1)}\big(u \pm i(M+I-1)/g\big) + \sum_{N=1}^{M-1} L^{(\alpha)}_{(w|N+I+1)}\big(u \pm i(I+N-1)/g\big)\right]_0\\
&= \big[D^{(w|M)}_{\pm(M+2I)}(u)\big]_0 - \delta_{I,0}\big[\ln Y^{(\alpha)}_{(w|1)}(u \mp i/g)\big]_0\\
&= \big[D^{(w|M)}_{\pm(M+2I)}(u)\big]_0 - \delta_{I,0}\big[L^{(\alpha)}_y(u)\big]_0.
\end{aligned} \tag{7.5}$$

The comparison between (7.3) and (7.4) gives an explicit expression for these functions. For example:

$$D^{(w|M)}_{\pm M}(u) = L^{(\alpha)}_{(y|-)}(u) - L^{(\alpha)}_{(y|+)}(u) + \sum_{N=1}^{M-1} L^{(\alpha)}_{(w|N)}(u \pm iN/g). \tag{7.6}$$

All the other functions in the family can be obtained using the following rule, which is also very significant in view of property (5.9):

$$\begin{aligned}
&D^{(w|M)}_{\pm(2J+M)}(u) - D^{(w|M)}_{\pm(2J+M-2)}(u \pm i2/g)\\
&= 2\sum_{N=J+1}^{M+J-1} L^{(\alpha)}_{(w|N)}(u \pm iN/g) + L^{(\alpha)}_{(w|J)}(u \pm iJ/g)\\
&\quad + L^{(\alpha)}_{(w|M+J)}\big(u \pm i(M+J)/g\big)
\end{aligned} \tag{7.7}$$

$(J = 1, 2, \ldots)$. Let us show how to derive the TBA equation from this information. From (7.5) and (7.6) we see that, with $|\mathrm{Im}(u)| < M/g$ on the first Riemann sheet, Cauchy's formula reads:

[3] Here and in Section 8 the label "α" is omitted for the sake of legibility: $D^{(w|M)}_{\pm(M+2I)} \equiv D^{(w|M),(\alpha)}_{\pm(M+2I)}$, $D^{(v|M)}_{\pm(M+2I)} \equiv D^{(v|M),(\alpha)}_{\pm(M+2I)}$.

$$\begin{aligned}
\ln Y^{(\alpha)}_{(w|M)}(u) = &-\oint_{\bar\gamma_{\mathsf{x}}} \frac{dz}{2\pi i}\left(\frac{L^{(\alpha)}_y(z)}{(z-u+iM/g)}+\frac{L^{(\alpha)}_y(z)}{(z-u-iM/g)}\right)\\
&+\sum_{I=0}^{\infty}\sum_{\tau}\oint_{\bar\gamma_{\mathsf{x}}}\frac{dz}{2\pi i}\frac{D^{(w|M)}_{\tau(M+2I)}(z)}{(z-u+i\tau(M+2I)/g)}\\
= &-\oint_{\bar\gamma_{\mathsf{x}}} \frac{dz}{2\pi i}\left(\frac{L^{(\alpha)}_y(z)}{(z-u+iM/g)}+\frac{L^{(\alpha)}_y(z)}{(z-u-iM/g)}\right)\\
&-\left(\int_{\mathbb{R}-i\epsilon}\frac{dz}{2\pi i(z-u+iM/g)}-\int_{\mathbb{R}+i\epsilon}\frac{dz}{2\pi i(z-u-iM/g)}\right)\\
&\times\left(L^{(\alpha)}_{(y|-)}(z)-L^{(\alpha)}_{(y|+)}(z)\right)\\
&-\sum_{\tau}\tau\int_{\mathbb{R}-i\tau\epsilon}\frac{dz}{2\pi i(z-u+i\tau M/g)}\sum_{N=1}^{M-1}L^{(\alpha)}_{(w|N)}(z+i\tau N/g)\\
&-\sum_{N=1}^{\infty}\sum_{\tau}\tau\int_{\mathbb{R}-i\tau\epsilon}\frac{dz}{2\pi i}\frac{\left(D^{(w|M)}_{\tau(M+2N)}(z)-D^{(w|M)}_{\tau(M+2N-2)}(z+i2\tau/g)\right)}{(z-u+i\tau(M+2N)/g)}. \qquad (7.8)
\end{aligned}$$

As anticipated, now we just need to insert (7.7) into the last line. Moreover, we can use the property (C.4) to reassemble all the convolutions with y-related functions into a single integral along $\bar\gamma_{\mathsf{o}}$, and finally we find:

$$\begin{aligned}
\ln Y^{(\alpha)}_{(w|M)}(u) = &\,L^{(\alpha)}_y *_{\bar\gamma_{\mathsf{o}}}\phi_M(u)+\sum_{N=1}^{M-1}L^{(\alpha)}_{(w|N)}*\phi_{M-N}(u)\\
&-\sum_{J,\tau}\tau\int_{\mathbb{R}}\frac{dz}{2\pi i(z-u+i\tau(M+2J)/g)}\\
&\times\Bigg(L^{(\alpha)}_{(w|J)}(z+i\tau J/g)+L^{(\alpha)}_{(w|M+J)}\big(z+i\tau(M+J)/g\big)\\
&+2\sum_{N=J+1}^{M+J-1}L^{(\alpha)}_{(w|N)}(z+i\tau N/g)\Bigg)\\
= &\,L^{(\alpha)}_y *_{\bar\gamma_{\mathsf{o}}}\phi_M(u)+\sum_{N}L^{(\alpha)}_{(w|N)}*\phi_{NM}(u). \qquad (7.9)
\end{aligned}$$

The result (7.9) perfectly matches the TBA equation (2.6).

8. *v*-particles

For v-related functions the situation is very similar to that encountered in the previous section. The essential information is encoded in the statement that $\ln Y^{(\alpha)}_{(v|M)}(u)$ is analytic for $|\mathrm{Im}(u)| < M/g$ and in the value of the discontinuities closest to the real line:

$$\big[\ln Y^{(\alpha)}_{(v|1)}\big]_{\pm 1} = \big[\Lambda^{(\alpha)}_{y}\big]_0. \tag{8.1}$$

Also in this case, the Y-system leads to the introduction of a set of functions $D^{(v|M)}_{\pm(M+2I)}$ ($I = 0, 1, \ldots$), constrained by the following relations

$$\begin{aligned} D^{(v|M)}_{\pm M}(u) &= \Lambda^{(\alpha)}_{(y|-)}(u) - \Lambda^{(\alpha)}_{(y|+)}(u) + \sum_{N=1}^{M-1} L^{(\alpha)}_{(v|N)}(u \pm iN/g), \\ D^{(v|M)}_{\pm(2J+M)}&(u) - D^{(v|M)}_{\pm(2J+M-2)}(u \pm i2/g) \\ &= L^{(\alpha)}_{(v|J)}(u \pm iJ/g) + L^{(\alpha)}_{(v|M+J)}\big(u \pm i(M+J)/g\big) \\ &\quad + 2\sum_{N=J+1}^{M+J-1} L^{(\alpha)}_{(v|N)}(u \pm iN/g) - \sum_{Q=J+1}^{M+J} L_Q(u \pm iQ/g), \end{aligned} \tag{8.2}$$

and satisfying

$$\big[\ln Y^{(\alpha)}_{(v|M)}\big]_{\pm M} = \big[D^{(v|M)}_{\pm M}(u) - \Lambda^{(\alpha)}_{y}\big]_0, \qquad \big[\ln Y^{(\alpha)}_{(v|M)}\big]_{\pm(M+2J)} = \big[D^{(v|M)}_{\pm(M+2J)}\big]_0, \tag{8.3}$$

for ($J = 1, 2, \ldots$). Finally, using Cauchy's theorem:

$$\begin{aligned} \ln Y^{(\alpha)}_{(v|M)}(u) &= \Lambda^{(\alpha)}_{y} *_{\bar{\gamma}_\circ} \phi_M(u) + \sum_N L^{(\alpha)}_{(w|N)} * \phi_{NM}(u) \\ &\qquad + \sum_{J,\tau} \tau \int_{\mathbb{R}-i\tau\epsilon} \frac{dz}{2\pi i(z-u+i\tau(M+2J))} \sum_{Q=J+1}^{M+J} L_Q(z+i\tau Q/g) \\ &= L^{(\alpha)}_{y} *_{\bar{\gamma}_\circ} \phi_M(u) + \sum_N L^{(\alpha)}_{(w|N)} * \phi_{NM}(u) \\ &\qquad + \ln Y^{(\alpha)}_{y} *_{\bar{\gamma}_\circ} \phi_M(u) - \sum_Q \sum_{I=0}^{M-1} L_Q * \phi_{Q-M+2I}(u). \end{aligned} \tag{8.4}$$

Thanks to the identity (E.8), Eq. (8.4) is equivalent to the TBA equation (2.7).

9. Q-particles and the dressing kernel

We already know from (6.1), (6.2) that the discontinuities of $\ln Y_1(u)$ at the border of the analyticity strip around the real axis can be linked via the following relations

$$\begin{aligned} \big[\ln Y_1(u)\big]_{+1} &= \Delta(u) = \frac{1}{2}\big[\Delta(u)\big]_0, \\ \big[\ln Y_1(u)\big]_{-1} &= \bar{\Delta}(u) = -\frac{1}{2}\big[\Delta(u)\big]_0 - \sum_\alpha \big[L^{(\alpha)}_{y}(u)\big]_0, \end{aligned} \tag{9.1}$$

to the function $\Delta(u)$, defined through the integral representation (6.5). It is precisely the non-locality of Δ (as well as of the difference $\ln Y^{(\alpha)}_{(y|-)} - \ln Y^{(\alpha)}_{(y|+)}$, see (4.2)) which makes the Q- and y-related functions drastically different from the other species of excitations, forcing us to lean on an additional infinite set of functional relations which are not obtainable from the Y-system: (5.4) and (6.10) respectively.

In Section 6 we have shown how to retrieve the form of Δ from the discontinuity relations (6.10). The rest of the derivation of the TBA equation (2.4) is based solely on the Y-system and goes along the same way as for w- and v-nodes.

We define the family of D^Q_{Q+2I} functions through the following recursion relation:

$$\begin{aligned} & D^Q_{\pm(Q+2J)}(u) - D^Q_{\pm(Q+2J-2)}(u \pm i2/g) \\ & \quad = L_J(u \pm iJ/g) + 2\sum_{Q'=J+1}^{Q+J-1} L_{Q'}\big(u \pm iQ'/g\big) \\ & \qquad + L_{Q+J}\big(u \pm i(Q+J)/g\big) - \sum_\alpha \sum_{N=J}^{Q+J-1} L^{(\alpha)}_{(v|N)}(u \pm iN/g) \end{aligned} \tag{9.2}$$

$(J = 1, 2, \ldots)$, with initial condition:

$$D^Q_{\pm Q}(u) = \sum_{Q'=1}^{Q-1} L_{Q'}\big(u \pm iQ'/g\big) - \sum_\alpha \left(\sum_{N=1}^{Q-1} L^{(\alpha)}_{(v|N)}(u \pm iN/g) + L^{(\alpha)}_y(u) \right). \tag{9.3}$$

These functions are related to the monodromy properties of the $(Y_Q)'s$ through the following:

$$\begin{aligned} & \big[\ln Y_Q(u)\big]_{+Q} = \left[D^Q_Q(u) + \ln Y_1\left(u + \frac{i}{g}\right) + \sum_\alpha L^{(\alpha)}_y(u) \right]_0, \\ & \big[\ln Y_Q(u)\big]_{-Q} = \left[D^Q_{-Q}(u) - \ln Y_1\left(u + \frac{i}{g}\right) \right]_0, \\ & \big[\ln Y_Q(u)\big]_{\pm(Q+2J)} = \big[D^Q_{\pm(Q+2J)}(u)\big]_0 \quad (J = 1, 2, \ldots). \end{aligned} \tag{9.4}$$

Proceeding in the usual way, we apply Cauchy's method to $\ln Y_Q(u)$, with $|\mathrm{Im}(u)| < Q/g$, and, as a consequence of (9.2), (9.3), (9.4), we obtain:

$$\begin{aligned} \ln Y_Q(u) = & -\oint_{\bar\gamma_x} dz\, \ln Y_1\left(z + \frac{i}{g}\right) \phi_Q(z-u) + \sum_{Q'} L_{Q'} * \phi_{Q'Q}(u) \\ & - \sum_\alpha \sum_{N=1}^{Q-1} L^{(\alpha)}_{(v|N)} * \phi_{Q-N}(u) \\ & + \sum_\alpha \left(\oint_{\bar\gamma_x} \frac{dz}{2\pi i(z-u+iQ/g)} L^{(\alpha)}_{(y|-)}(z) \right. \\ & \left. - \sum_\tau \tau \int_{\mathbb{R}+i\tau\epsilon} \frac{dz}{2\pi i(z-u-i\tau Q/g)} L^{(\alpha)}_{(y|-)}(z) \right) \\ & - \sum_{J,\alpha,\tau} \tau \int_{\mathbb{R}-i\tau\epsilon} \frac{dz}{2\pi i(z-u+i\tau(Q+2J)/g)} \sum_{N=J}^{Q+J-1} L^{(\alpha)}_{(v|N)}(z + i\tau N/g). \end{aligned} \tag{9.5}$$

In the end, reordering the different terms we get

$$\ln Y_Q(u) = -\oint_{\bar{\gamma}_x} dz\, \ln Y_1\left(z + \frac{i}{g}\right)\phi_Q(z-u) + \sum_{Q'} L_{Q'} * \phi_{Q'Q}(u) - \sum_\alpha \int_{\mathbb{R}+i\epsilon} dz\, L^{(\alpha)}_{(y|-)}(z)\phi_Q(z-u) - \sum_{N,\alpha}\sum_{J=1}^{Q} L^{(\alpha)}_{(v|N)} * \phi_{N-Q+2J}(u). \tag{9.6}$$

This is a very peculiar equation. Notice that the only visible kernels are difference-type, while the genuinely non-relativistic elements of (2.4) are hidden inside the first convolution

$$\oint_{\bar{\gamma}_x} dz\, \ln Y_1\left(z + \frac{i}{g}\right)\phi_Q(z-u) = \frac{1}{2}\oint_{\bar{\gamma}_x} dz\, \Delta(z)\phi_Q(z-u), \tag{9.7}$$

and we need to use the result (6.5) to unravel them. For example, the energy term comes naturally to the surface through the following dispersion relation:

$$-\int_{\bar{\gamma}_x} dz\, \ln x(z)\phi_Q(z-u) = \ln \frac{x(u - \frac{iQ}{g})}{x(u + \frac{iQ}{g})} \equiv \tilde{E}_Q(u). \tag{9.8}$$

A number of very similar computations is reported in Appendix E, culminating in (E.15). This result shows that Eq. (9.6) reduces to the original TBA equation (2.4) on the condition that the following identity holds for the dressing-related part:

$$-\frac{1}{2}\Delta^{\Sigma} *_{\bar{\gamma}_x} \phi_Q(u) = 2\sum_{Q'} L_{Q'} * K^{\Sigma}_{Q'Q}(u). \tag{9.9}$$

In Appendix C we evaluate the lhs of this equation. The result is:

$$\sum_{Q'} L_{Q'} * K^{\Sigma}_{Q'Q}(u) = \sum_{Q'} L_{Q'} * \oint_{\bar{\gamma}_x} ds\, \phi_{Q',y}(s) \oint_{\bar{\gamma}_x} dt\, K^{[2]}_{\Gamma}(s-t)\phi_{y,Q}(t,u). \tag{9.10}$$

Further discussion and a direct proof of this relation is given in Appendix D. As discussed there, (9.10) realizes a manifest symmetry between the direct and mirror theories. This is the object of the next section.

10. Application: the TBA equations for the direct theory

A possible question is whether the TBA equations for the direct theory can be obtained by considering the same set of discontinuity relations but with complementary prescriptions for the cuts: running parallel to the real axis but inside the strip $|\mathrm{Re}(u)| < 2$. The result is particularly simple: the TBA equations remain formally unchanged but for the replacement $\bar{\gamma}_o \to \bar{\gamma}_x$:

$$\varepsilon_Q(u) = \mu_Q + RE_Q(u) - \sum_Q L_{Q'} * \phi^{\sigma}_{Q'Q}(u) + \sum_\alpha \left(\sum_M L^{(\alpha)}_{v|M} * \phi^{\mathrm{d}}_{(v|M),Q}(u) + L^{(\alpha)}_y *_{\bar{\gamma}_x} \phi^{\mathrm{d}}_{y,Q}(u)\right), \tag{10.1}$$

$$\varepsilon^{(\alpha)}_y(u) = \mu^{(\alpha)}_y - \sum_Q L_Q * \phi^{\mathrm{d}}_{Q,y}(u) + \sum_M \left(L^{(\alpha)}_{(v|M)} - L^{(\alpha)}_{(w|M)}\right) * \phi_M(u), \tag{10.2}$$

$$\varepsilon^{(\alpha)}_{(w|K)}(u) = \mu^{(\alpha)}_{(w|K)} + \sum_M L^{(\alpha)}_{(w|M)} * \phi_{MK}(u) + L^{(\alpha)}_y *_{\bar{\gamma}_x} \phi_K(u), \tag{10.3}$$

$$\varepsilon^{(\alpha)}_{(v|K)}(u) = \mu^{(\alpha)}_{(v|K)} - \sum_Q L_Q * \phi^{\mathrm{d}}_{Q,(v|K)}(u) + \sum_M L^{(\alpha)}_{(v|M)} * \phi_{MK}(u) + L^{(\alpha)}_y *_{\bar{\gamma}_x} \phi_K(u), \tag{10.4}$$

where $\alpha = 1, 2$, $K = 1, 2, \ldots,$

$$\phi^{\sigma}_{Q'Q}(z,u) = -\phi_{Q'Q}(z-u) + 2K^{\sigma}_{Q'Q}(z,u),$$
$$K^{\sigma}_{Q'Q}(z,u) = \frac{1}{2\pi i}\frac{d}{dz}\ln\sigma_{Q'Q}(z,u), \tag{10.5}$$

and the new set of kernels $\{\phi^{\mathrm{d}}_{y,Q}, \phi^{\mathrm{d}}_{Q,y}, \phi^{\mathrm{d}}_{(v|M),Q}, \phi^{\mathrm{d}}_{Q,(v|M)}\}$ is obtained from the kernels used in (2.4)–(2.7) by replacing the function x with the x_{d} defined in (C.16) and (C.17). Finally, the energy and momentum are respectively

$$E_Q(u) = igx_{\mathrm{d}}\left(u - i\frac{Q}{g}\right) - igx_{\mathrm{d}}\left(u + i\frac{Q}{g}\right) - Q, \tag{10.6}$$

and

$$p^Q(u) = i\ln\left(\frac{x_{\mathrm{d}}(u - iQ/g)}{x_{\mathrm{d}}(u + iQ/g)}\right). \tag{10.7}$$

An interesting observation is related to the driving terms (10.6) appearing in the TBA equations. For the direct theory, a requirement analogous to (6.12)

$$\frac{\Delta(iv+\epsilon) - \Delta(iv-\epsilon)}{2\pi i} \in \mathbb{Z} \quad (v \in \mathbb{R}), \tag{10.8}$$

a part for the trivial case $\Delta(iv+\epsilon) = \Delta(iv - i\epsilon)$ cannot be straightforwardly implemented due to the presence of an infinite number of square-root branch cuts crossing the imaginary axis. The result (10.6) comes instead from the requirement

$$\frac{\Delta(u)}{\sqrt{4-u^2}} \sim R + O\big(1/|u|\big) \tag{10.9}$$

as $|u| \to \infty$. The surprising consequence of this fact is that, contrary to the mirror scale factor L, there is no evident mathematical advantage to quantize R. The steps that lead to (10.1)–(10.4) parallel completely the derivation of the TBA for the mirror theory starting from the Y-system and the discontinuity relations. Therefore, automatically the solutions to (10.1)–(10.4) satisfy the Y-system of Section 4 which, in turn, can easily be re-derived from (10.1)–(10.4) in the region $|\mathrm{Re}(u)| > 2$. The latter simple observation motivates a further, perhaps more interesting, final comment. The TBA equations for the direct AdS_5/CFT_4 theory can be directly derived from the Beisert and Staudacher equations [6] through the string hypothesis and the TBA method. There are various variants of these Bethe Ansatz equations which differ in the way the momentum-related factors $x_{\mathrm{d}}(u - iQ/g)/x_{\mathrm{d}}(u + iQ/g)$ are rearranged inside the equations. From the result (10.5) we see that there in no trace of these factors in the kernels $\phi^{\sigma}_{Q'Q}$. Therefore, contrary to our initial expectations equations (10.1)–(10.4) do not descend, for example, from Eq. (6.4) of [28] where the space length on which the TBA procedure is based is the '**R**-charge' J. More precisely, (6.4) of [28] leads to TBA equations which differ from (10.1)–(10.4) by a scale dependent set of chemical potentials for the Q-particles

$$\mu_Q(R) = -Q \sum_{Q'} \int_{\mathbb{R}} \frac{du}{2\pi} \frac{dp^{Q'}}{du} L_{Q'}(u) = Q \tilde{E}_0(R), \tag{10.10}$$

where $\tilde{E}_0(R)$ is the ground state energy of the mirror theory. Note that, since the chemical potentials (10.10) fulfil the constraints (2.18), both versions of the TBA equations satisfy the Y-system described in Section 4 and the discontinuity relations (1.6)–(1.8). They can be obtained by imposing different asymptotic conditions on the pseudoenergies when Cauchy's theorem is applied. We suspect that the form of these additional R-dependent terms – if really needed – should descend from some independent physical requirement, as for example, from the relation between the AdS_5/CFT_4 Y and T functions [38] and the transfer matrix of a possible underlying lattice system.

11. Conclusions

The analytic properties of the solutions of the AdS_5/CFT_4 thermodynamic Bethe Ansatz equations are very different from those of the relativistic integrable quantum field theories. The Y functions live on a complicated, and almost completely unexplored, Riemann surface with an infinite number of square-root branch points. The AdS_5/CFT_4 Y-system does not contain enough information to allow its transformation into the integral form relevant for AdS_5/CFT_4 and in particular to reproduce the dressing kernel. In this paper the analytic properties of the Y functions are studied in more detail and the needed extra analytic information is identified and encoded in a new universal set of local functional relations for the square-root discontinuities. From these equations and the Y-system the ground state TBA equations for both the mirror and the direct theories were derived using a novel method based on the Cauchy integral theorem and the interpretation of the TBA equations as dispersion relations.

The simplified set of TBA equations obtained in [32,60] can be easily derived from the extended Y-system and Fourier transform techniques can be used to short-cut further some of the steps in the Y $\to$ TBA inversion process. Therefore, the apparently very-complicated structure of the AdS_5/CFT_4 TBA equations is encoded into a fairly simple set of local functional relations.

To get the TBA equations for the ground state we have adopted certain minimality assumptions for the number of logarithmic singularities in the reference Riemann sheet. A main conjecture of this paper is that by the same method, including extra logarithmic singularities, one has direct access to the full spectrum of the theory. Possibly, this will lead to a rigorous proof of the TBA equations for the excited states conjectured in the papers [31,38,53]. Work in this direction is in progress.

Other sets of functional relations – the T-systems – play also a very fundamental rôle in integrable models [36,62]. In general Y- and T-systems are almost totally equivalent, but in the AdS_5/CFT_4-related cases [38], T-systems seem to contain more information compared to the basic Y-systems and they can be considered more fundamental. In this respect, it would be important to find the analog of Eqs. (1.6)–(1.8) for the discontinuities of the T functions.

Our final remark is that much more work is needed to understand the integrable structure of the AdS_5/CFT_4 thermodynamic Bethe Ansatz equations and in particular the Riemann surface associated to their solutions but we think that the results of this paper provide a further important step in this direction.

Acknowledgements

We would like to thank Gleb Arutyunov, Zoltan Bajnok, Sergey Frolov, Nikolay Gromov and Vladimir Kazakov, for suggestions and very useful discussions. We acknowledge the INFN grants IS PI14 "*Topics in non-perturbative gauge dynamics in field and string theory*" and PI11 for travel financial support, and the University PRIN 2007JHLPEZ "Fisica Statistica dei Sistemi Fortemente Correlati all'Equilibrio e Fuori Equilibrio: Risultati Esatti e Metodi di Teoria dei Campi".

Appendix A. The S-matrix elements

Here we report the scalar factors S_{AB} involved in the definition of kernels in the TBA equations (2.4)–(2.7),

$$S_{y,Q}(u,z)=S_{Q,y}(z,u)=\left(\frac{x(z-\frac{i}{g}Q)-x(u)}{x(z+\frac{i}{g}Q)-x(u)}\right)\sqrt{\frac{x(z+\frac{i}{g}Q)}{x(z-\frac{i}{g}Q)}}, \tag{A.1}$$

$$\begin{aligned}S_{(v|M),Q}(u,z)&=S_{Q,(v|M)}(z,u)\\&=\left(\frac{x(z-\frac{i}{g}Q)-x(u+\frac{i}{g}M)}{x(z+\frac{i}{g}Q)-x(u+\frac{i}{g}M)}\right)\left(\frac{x(z+\frac{i}{g}Q)}{x(z-\frac{i}{g}Q)}\right)\\&\quad\times\left(\frac{x(z-\frac{i}{g}Q)-x(u-\frac{i}{g}M)}{x(z+\frac{i}{g}Q)-x(u-\frac{i}{g}M)}\right)\prod_{j=1}^{M-1}\left(\frac{z-u-\frac{i}{g}(Q-M+2j)}{z-u+\frac{i}{g}(Q-M+2j)}\right),\end{aligned} \tag{A.2}$$

$$S_M(u)=\left(\frac{u-\frac{i}{g}M}{u+\frac{i}{g}M}\right), \tag{A.3}$$

$$\begin{aligned}S_{KM}(u)&=\prod_{k=1}^{K}\prod_{l=1}^{M}S_{((K+2-2k)-(M-2l))}(u)\\&=\left(\frac{u-\frac{i}{g}|K-M|}{u+\frac{i}{g}|K-M|}\right)\left(\frac{u-\frac{i}{g}(K+M)}{u+\frac{i}{g}(K+M)}\right)\prod_{k=1}^{\min(K,M)-1}\left(\frac{u-\frac{i}{g}(|K-M|+2k)}{u+\frac{i}{g}(|K-M|+2k)}\right)^2.\end{aligned} \tag{A.4}$$

The elements $S^{\Sigma}_{Q'Q}$ are:

$$S^{\Sigma}_{Q'Q}(z,u)=\big(S_{Q'Q}(z-u)\big)^{-1}\big(\Sigma_{Q'Q}(z,u)\big)^{-2}, \tag{A.5}$$

where $\Sigma_{Q'Q}$ is the improved dressing factor for the mirror bound states

$$\Sigma_{Q'Q}(z,u)=\prod_{k=1}^{Q'}\prod_{l=1}^{Q}\left(\frac{1-\frac{1}{x(z+\frac{i}{g}(Q'+2-2k))x(u+\frac{i}{g}(Q-2l))}}{1-\frac{1}{x(z+\frac{i}{g}(Q'-2k))x(u+\frac{i}{g}(Q+2-2l))}}\right)\sigma_{Q'Q}(z,u), \tag{A.6}$$

with $\sigma_{Q'Q}$ evaluated in the mirror kinematics. A precise analytic expression for the mirror improved dressing factor has been given in [48], and a more compact integral representation in [31]. We show in Appendices C and D the equivalence of the two results.

Appendix B. The kernels and their properties

The functions ϕ_M and ϕ_{MN} are relativistic-like kernels common to many models, from the **xxx** quantum spin chain to the Hubbard model:

$$\phi_M(u)=\frac{M/g}{\pi(u^2+(M/g)^2)},$$
$$\phi_{MN}(u)=\phi_{|M-N|}(u)+2\phi_{|M-N|+2}(u)+\cdots+2\phi_{M+N-2}(u)+\phi_{M+N}(u). \tag{B.1}$$

Their behavior is summarized by the following statement: $\phi_M(u)$ has two poles at $u=\pm iM/g$, with residues ± 1. Consider, for example, the quantity $\mathcal{F}*\phi_M(u)\equiv\int_{\mathbb{R}}dz\,\mathcal{F}(z)\phi_M(z-u)$. In terms of the integration variable z, there are two poles at $z=u\pm iM/g$ and, by analytically continuing $\mathcal{F}*\phi_M(u)$ to $\mathcal{F}*\phi_M(u\pm i/g)$, we find

$$\begin{aligned}\mathcal{F}*\phi_M(u+i/g)+\mathcal{F}*\phi_M(u-i/g)&=\mathcal{F}*\phi_{M+1}(u)+\mathcal{F}*\phi_{M-1}(u)+\mathcal{F}(u)\delta_{M,1}\\&=\mathcal{F}*\phi_{M,1}(u)+\mathcal{F}(u)\delta_{M,1}.\end{aligned} \tag{B.2}$$

A simple way to understand the result (B.2) is by noticing that the contour of integration can be freely shifted slightly below the real axis. Then for $M=1$, while $\mathcal{F}*\phi_1(u)\to\mathcal{F}*\phi_1(u+i/g)$ the pole of $\phi_1(z-u)$ at $z-u=-i/g$ crosses the contour of integration. The corresponding residue is precisely the local term $\mathcal{F}(u)$ in the rhs of (B.2).

Similarly:

$$\mathcal{F}*\phi_{MN}(u+i/g)+\mathcal{F}*\phi_{MN}(u-i/g)=\sum_K A_{MK}\mathcal{F}*\phi_{KN}(u)+A_{MN}\mathcal{F}(u). \tag{B.3}$$

Suppose that $\mathcal{F}(u)$ is now a function with square root branch points at $u=-2$ and $u=2$, as for example $L_y^{(\alpha)}(u)$. For $u\in(-2,2)$ we have:

$$\begin{aligned}&\mathcal{F}*_{\bar{\gamma}_\circ}\phi_N(u+i/g)+\mathcal{F}*_{\bar{\gamma}_\circ}\phi_N(u-i/g)\\&\quad=\big(\mathcal{F}_-(u)-\mathcal{F}_+(u)\big)\delta_{N,1}+\mathcal{F}*_{\bar{\gamma}_\circ}\phi_{N+1}(u)+\mathcal{F}*_{\bar{\gamma}_\circ}\phi_{N-1}(u)\end{aligned} \tag{B.4}$$

where $\mathcal{F}_-(u)$ and $\mathcal{F}_+(u)$ indicate the function $\mathcal{F}$ evaluated respectively on the first and second sheets. Computing the same expression for $u\notin(-2,2)$ leads to the result (B.4) without the term $(\mathcal{F}_-(u)-\mathcal{F}_+(u))\delta_{N,1}$. This simple argument reveals the presence of a pair of branch points for $\mathcal{F}*_{\bar{\gamma}_\circ}\phi_N(u)$ at $u=-2\pm iN/g$ and $u=2\pm iN/g$. The non-relativistic nature of the model is evident from the elements of the scattering matrix $S_{y,Q}(z,u)$ and $S_{(v|M),Q}(z,u)$: they depend separately on the variables z and u. Because also these kernels have isolated poles with residues ±1, the analogue of the shifting properties reported above can be computed in a similar fashion, but require some additional care due to the fact that the singularities can now disappear inside a branch cut instead of hitting the real line. For example, for $u\in(-2,2)$:

$$\mathcal{F}*\phi_{Q,y}(u+i/g)+\mathcal{F}*\phi_{Q,y}(u-i/g)=\mathcal{F}*\phi_{Q,(v|1)}(u)+\delta_{Q,1}\mathcal{F}(u). \tag{B.5}$$

The validity of (B.5) is restricted to $u\in(-2,2)$, and a comment must be made on a possible ambiguity. Obviously, the continuation outside this range is possible, and the rhs of (B.5) keeps the same form. However, as soon as $\mathrm{Re}(u)$ leaves the interval $(-2,2)$, on the lhs a branch cut comes between $\phi_{1,(v|1)}(z,u-i/g)$ and $\phi_{1,(v|1)}(z,u+i/g)$ and these two points are not anymore connected by vertical analytic continuation.

Other useful identities are listed below. For $u\in(-2,2)$:

$$\begin{aligned} &\mathcal{F} * \phi_{Q,(v|N)}(u+i/g) + \mathcal{F} * \phi_{Q,(v|N)}(u-i/g) \\ &\quad = \sum_M A_{NM} \mathcal{F} * \phi_{Q,(v|M)}(u) + \delta_{Q,N+1} \mathcal{F}(u) \\ &\qquad + \delta_{N,1} \big(\mathcal{F} * \phi_{Q,(y|-)}(u) - \mathcal{F} * \phi_{Q,(y|+)}(u) \big). \end{aligned} \tag{B.6}$$

To write (B.6) we have used the properties

$$\begin{aligned} &\frac{1}{2\pi i} \frac{d}{dz} \ln \left(\frac{x(z+iQ/g)}{x(z-iQ/g)} \frac{x(z-iQ/g)-x(u+i2/g)}{x(z+iQ/g)-x(u+i2/g)} \frac{x(z-iQ/g)-x(u-i2/g)}{x(z+iQ/g)-x(u-i2/g)} \right) \\ &\quad = \phi_{Q,(v|2)}(z,u) - \phi_Q(z-u), \end{aligned} \tag{B.7}$$

$$\phi_Q(z-u) = \phi_{Q,(y|-)}(z,u) + \phi_{Q,(y|+)}(z,u), \tag{B.8}$$

where

$$\phi_{Q,(y|-)}(z,u) \equiv \phi_{Q,y}(z,u), \qquad \phi_{Q,(y|+)}(z,u) \equiv \phi_{Q,y}(z,u_*). \tag{B.9}$$

For $u \in \mathbb{R}$ we have:

$$\begin{aligned} &\mathcal{F} * \phi_{(v|N),Q}(u+i/g) + \mathcal{F} * \phi_{(v|N),Q}(u-i/g) \\ &= -\delta_{Q,N+1} \mathcal{F}(u) + \sum_{Q'} A_{Q'Q} \mathcal{F} * \phi_{(v|N),Q'}(u). \end{aligned} \tag{B.10}$$

The equation that expresses the effect of the shifts on the kernel $\phi_{y,Q}$ has some extra subtlety:

$$\begin{aligned} &\mathcal{F} *_{\bar{\gamma}_\circ} \phi_{y,Q}(u+i/g) + \mathcal{F} *_{\bar{\gamma}_\circ} \phi_{y,Q}(u-i/g) \\ &\quad = -\delta_{Q,1} \mathcal{F}(u) + \sum_{Q'} A_{Q'Q} \mathcal{F} *_{\bar{\gamma}_\circ} \phi_{y,Q'}(u). \end{aligned} \tag{B.11}$$

Since $\mathcal{F}(u) \equiv \mathcal{F}_-(u)$, the property (B.11) is at the origin of the asymmetry between $Y^{(\alpha)}_{(y|-)}(u) \equiv Y^{(\alpha)}_y(u)$ and $Y^{(\alpha)}_{(y|+)}(u) \equiv Y^{(\alpha)}_y(u_*)$ on the rhs of (1.1). Finally, another useful relation was proved in [60]:

$$\mathcal{F} * K^{\Sigma}_{Q'Q}(u+i/g) + \mathcal{F} * K^{\Sigma}_{Q'Q}(u-i/g) = \sum_P A_{QP} \mathcal{F} * K^{\Sigma}_{Q'P}(u) \quad \big(|\mathrm{Re}(u)| < 2\big). \tag{B.12}$$

Appendix C. The dressing factor in the direct and mirror theories

In Section 6, the contribution to Δ coming from the improved dressing factor of the mirror theory was denoted by Δ^{Σ}:

$$\Delta^{\Sigma}(u) = 2 \sum_Q L_Q * K^{\Sigma}_Q(u), \tag{C.1}$$

with

$$\begin{aligned} &K^{\Sigma}_Q(z,u) = K^{\Sigma}_{Q,1}(z,u+i/g) - K^{\Sigma}_{Q,1}(z,u_*+i/g), \\ &K^{\Sigma}_{Q,1}(z,u) = \frac{1}{2\pi i} \frac{d}{dz} \ln \Sigma_{Q,1}(z,u), \end{aligned} \tag{C.2}$$

and in [60] the following expression was derived:

$$\begin{aligned}\Delta^{\Sigma}(u) = {} & 2\sum_N \int_{\mathbb{R}} dz \sum_Q L_Q(z)\big(K\big(z + i(2N+Q)/g, u\big) + K\big(z - i(2N+Q)/g, u\big)\big) \\ & - 2\int_{\mathbb{R}} dz \sum_Q L_Q(z) \sum_N \oint_{\bar{\gamma}_\mathrm{o}} ds\, \phi_{Q,y}(z,s) \\ & \times \big(K(s + i2N/g, u) + K(s - i2N/g, u)\big) \\ = {} & 2\sum_N \int_{\mathbb{R}} dz \sum_Q L_Q(u)\big(K\big(z + i(2N+Q)/g, u\big) + K\big(z - i(2N+Q)/g, u\big)\big) \\ & + \sum_{N,\alpha} \oint_{\bar{\gamma}_\mathrm{o}} dz \ln Y_y^{(\alpha)}(z)\big(K(z + i2N/g, u) + K(z - i2N/g, u)\big). \end{aligned} \tag{C.3}$$

The aim of this appendix is to show how to recast (C.3) into a form that emphasizes the symmetries between the direct and the mirror theory. Let us start from the following simple property. For any kernel $f(z,u)$ which is analytic in a neighborhood of the real axis, and for any function $r(z)$ with a square root branch cut along $(-\infty,-2)\cup(2,+\infty)$, we have:

$$\int_{\mathbb{R}\pm i\epsilon} dz\,\big(r(z) - r(z_*)\big) f(z,u) = \oint_{\bar{\gamma}_\mathrm{o}} dz\, r(z) f(z,u) \pm \oint_{\bar{\gamma}_\mathrm{x}} dz\, r(z) f(z,u). \tag{C.4}$$

Now, let us consider the following example:

$$\int_{\mathbb{R}\pm i\epsilon} dz \ln\big(Y_{(y|-)}^{(\alpha)}(z)/Y_{(y|+)}^{(\alpha)}(z)\big) f(z,u) = \ln Y_y^{(\alpha)} *_{\bar{\gamma}_\mathrm{o}} f(u) \pm \ln Y_y^{(\alpha)} *_{\bar{\gamma}_\mathrm{x}} f(u). \tag{C.5}$$

In this case a further step is possible thanks to relation (5.4). In fact, as long as the kernels $f(z,u)$ and $h(z,u)$ are analytic and sufficiently damped at infinity in the upper and lower half plane respectively, the lhs of (C.5) can be rewritten as a pair of dispersion relations:

$$\begin{aligned} \int_{\mathbb{R}+i\epsilon} dz \ln\big(Y_{(y|-)}^{(\alpha)}(z)/Y_{(y|+)}^{(\alpha)}(z)\big) f(z,u) &= -\oint_{\Gamma_\mathrm{O}^+} dz \ln\big(Y_{(y|-)}^{(\alpha)}(z)/Y_{(y|+)}^{(\alpha)}(z)\big) f(z,u); \\ \int_{\mathbb{R}-i\epsilon} dz \ln\big(Y_{(y|-)}^{(\alpha)}(z)/Y_{(y|+)}^{(\alpha)}(z)\big) h(z,u) &= \oint_{\Gamma_\mathrm{O}^-} dz \ln\big(Y_{(y|-)}^{(\alpha)}(z)/Y_{(y|+)}^{(\alpha)}(z)\big) h(z,u), \end{aligned} \tag{C.6}$$

where the integration contours Γ_O^+, Γ_O^- run inside the upper (or lower, respectively) half plane and, as the horizontal size of rectangles goes to infinity, $\Gamma_\mathrm{O} \sim \Gamma_\mathrm{O}^+ \cup \Gamma_\mathrm{O}^-$ (Γ_O is represented in Fig. 5). It is then sufficient to rephrase the derivation of Section 5 to transform (C.5) into:

$$\begin{aligned} \sum_Q \int_{\mathbb{R}} dz\, L_Q(z) f(z + iQ/g, u) &= \sum_\alpha \big(-\ln Y_y^{(\alpha)} *_{\bar{\gamma}_\mathrm{o}} f(u) - \ln Y_y^{(\alpha)} *_{\bar{\gamma}_\mathrm{x}} f(u)\big); \\ \sum_Q \int_{\mathbb{R}} dz\, L_Q(z) h(z - iQ/g, u) &= \sum_\alpha \big(-\ln Y_y^{(\alpha)} *_{\bar{\gamma}_\mathrm{o}} h(u) + \ln Y_y^{(\alpha)} *_{\bar{\gamma}_\mathrm{x}} h(u)\big). \end{aligned} \tag{C.7}$$

These identities are appropriate to simplify (C.3). In fact, taking $f(z,u)=\sum_{N=1}^{\infty}K(z+i2N/g,u)$ and $h(z,u)=\sum_{N=1}^{\infty}K(z-i2N/g,u)$ and summing the respective equations we find:

$$\int_{\mathbb{R}} dz\sum_{Q}L_Q(z)\sum_{N}\big(K\big(z+i(2N+Q)/g,u\big)+K\big(z-i(2N+Q)/g,u\big)\big)$$
$$=-\sum_{\alpha}\oint_{\bar{\gamma}_{\mathrm{o}}}dz\,\ln Y_y^{(\alpha)}(z)\sum_{N}\big(K(z+i2N/g,u)+K(z-i2N/g,u)\big)$$
$$-\sum_{\alpha}\oint_{\bar{\gamma}_{\mathrm{x}}}dz\,\ln Y_y^{(\alpha)}(z)\sum_{N}\big(K(z+i2N/g,u)-K(z-i2N/g,u)\big). \tag{C.8}$$

Therefore, we arrive at the following equation:

$$\Delta^{\Sigma}(u)=\sum_{N,\alpha}\oint_{\bar{\gamma}_{\mathrm{x}}}dz\,\ln Y_y^{(\alpha)}(z)\big(-K(z+i2N/g,u)+K(z-i2N/g,u)\big). \tag{C.9}$$

Introducing the function

$$K_{\Gamma}^{[N]}(z)=\frac{1}{2\pi i}\frac{d}{dz}\ln\frac{\Gamma(N/2-igz/2)}{\Gamma(N/2+igz/2)}, \tag{C.10}$$

which grows at most as $\ln|z|$ as $|z|\to\infty$ and using the property $K(z,u)\sim 1/z^2$ for $|z|\to\infty$ we can write

$$\oint_{\bar{\gamma}_{\mathrm{x}}}dz\,K_{\Gamma}^{[2]}(s-z)K(z,u)=\sum_{N}\big(K(s+i2N/g,u)-K(s-i2N/g,u)\big)+K_{\Gamma}^{[2]}(s-u). \tag{C.11}$$

Eq. (C.11) and the discontinuous parts of the TBA equation (2.5) (independently re-derived in Section 5 and Appendix F) lead to

$$\Delta^{\Sigma}(u)=-\sum_{\alpha}\oint_{\bar{\gamma}_{\mathrm{x}}}ds\,\ln Y_y^{(\alpha)}(s)\oint_{\bar{\gamma}_{\mathrm{x}}}dz\,K_{\Gamma}^{[2]}(s-z)K(z,u)$$
$$+\sum_{\alpha}\oint_{\bar{\gamma}_{\mathrm{x}}}ds\,\ln Y_y^{(\alpha)}(s)K_{\Gamma}^{[2]}(s-u)$$
$$=2\sum_{Q'}L_{Q'}*\oint_{\bar{\gamma}_{\mathrm{x}}}ds\,\phi_{Q,y}(s)\bigg(\oint_{\bar{\gamma}_{\mathrm{x}}}dt\,K_{\Gamma}^{[2]}(s-t)K(t,u)-K_{\Gamma}^{[2]}(s-u)\bigg), \tag{C.12}$$

with $u\notin(-\infty,-2)\cup(2,+\infty)$. Notice that the rhs of Eq. (C.12) coincides with the discontinuity of the function

$$2\sum_{Q'}L_{Q'}*\oint_{\bar{\gamma}_{\mathrm{x}}}ds\,\phi_{Q',y}(s)\oint_{\bar{\gamma}_{\mathrm{x}}}dt\,K_{\Gamma}^{[2]}(s-t)\frac{1}{2\pi i}\frac{d}{dt}\ln\big(x(t)-x(u)\big) \tag{C.13}$$

across $(-\infty,-2)\cup(2,+\infty)$. Therefore we are free to trade (C.13) for $\frac{1}{2}\Delta^{\Sigma}(u)$ every time we need to evaluate convolutions of the kind $\oint_{\bar{\gamma}_{\mathrm{x}}}du\,\Delta^{\Sigma}(u)a(u,v)$. We encountered such an expression when we derived a dispersion relation for the Q-particles, see (9.9). By property (E.1), we can now evaluate it to be:

$$\frac{1}{2}\Delta^{\Sigma} *_{\bar{\gamma}_{\mathrm{x}}} \phi_Q(v) = -2\sum_{Q'} L_{Q'} * \oint_{\bar{\gamma}_{\mathrm{x}}} ds\, \phi_{Q',y}(s) \oint_{\bar{\gamma}_{\mathrm{x}}} dt\, K_{\Gamma}^{[2]}(s-t)\phi_{y,Q}(t,v). \tag{C.14}$$

This form resembles closely the integral representation of the direct theory dressing kernel found in [47] (see also [48]):

$$\begin{aligned}\ln\sigma_{Q'Q}(u,v) = {} & \chi\big(x_{\mathrm{d}}\big(u+iQ'/g\big), x_{\mathrm{d}}(v+iQ/g)\big) + \chi\big(x_{\mathrm{d}}\big(u-iQ'/g\big), x_{\mathrm{d}}(v-iQ/g)\big) \\ & - \chi\big(x_{\mathrm{d}}\big(u+iQ'/g\big), x_{\mathrm{d}}(v-iQ/g)\big) \\ & - \chi\big(x_{\mathrm{d}}\big(u-iQ'/g\big), x_{\mathrm{d}}(v+iQ/g)\big),\end{aligned} \tag{C.15}$$

where

$$x_{\mathrm{d}}(u) = \left(\frac{u}{2} + u\sqrt{\frac{1}{4} - \frac{1}{u^2}}\right), \tag{C.16}$$

with $|x_{\mathrm{d}}| \geqslant 1$ on the first Riemann sheet. Notice that $x(u)$ and $x_{\mathrm{d}}(u)$ are different sections of the same function: $x_{\mathrm{d}}(u)$ has a cut on the segment $u \in (-2, 2)$ while the branch cut of $x(u)$ is on $(-\infty, -2) \cup (2, +\infty)$. More explicitly:

$$x_{\mathrm{d}}(u) = \begin{cases} x(u) & \text{for } \mathrm{Im}(u) < 0; \\ 1/x(u) & \text{for } \mathrm{Im}(u) > 0. \end{cases} \tag{C.17}$$

In (C.15) we have also introduced the function

$$\chi(a,b) = i\oint \frac{dx_1}{2\pi i}\oint \frac{dx_2}{2\pi i}\frac{1}{(x_1-a)}\frac{1}{(x_2-b)}\ln\frac{\Gamma[1+ig(x_1+1/x_1-x_2-1/x_2)/2]}{\Gamma[1-ig(x_1+1/x_1-x_2-1/x_2)/2]}, \tag{C.18}$$

where the integrals run over the unit circles $|x_1| = |x_2| = 1$. After a simple change of variables $x_1 = x_{\mathrm{d}}(s)$, $x_2 = x_{\mathrm{d}}(t)$ and an integration by parts we get

$$\frac{1}{2\pi i}\frac{d}{du}\ln\sigma_{Q'Q}(u,v) = -\oint_{\bar{\gamma}_{\mathrm{o}}} ds\, \phi^{\mathrm{d}}_{Q',y}(u,s) \oint_{\bar{\gamma}_{\mathrm{o}}} dt\, K_{\Gamma}^{[2]}(s-t)\phi^{\mathrm{d}}_{y,Q}(t,v). \tag{C.19}$$

In (C.19) the superscripts indicate that the kernels $\phi^{\mathrm{d}}_{Q,y}$, $\phi^{\mathrm{d}}_{y,Q}$ differ from those defined in Appendix A by the replacement $x \to x_{\mathrm{d}}$.[4] From (C.14) we see that (9.9) is fulfilled iff the mirror theory improved dressing kernel admits an integral representation analogous to (C.19):

$$K^{\Sigma}_{Q'Q}(u,v) = \frac{1}{2\pi i}\frac{d}{du}\ln\Sigma_{Q'Q}(u,v) = \oint_{\bar{\gamma}_{\mathrm{x}}} ds\, \phi_{Q',y}(u,s) \oint_{\bar{\gamma}_{\mathrm{x}}} dt\, K_{\Gamma}^{[2]}(s-t)\phi_{y,Q}(t,v). \tag{C.20}$$

This is precisely the result found in [31]. More specifically, we just need (C.20) to be valid when the dressing kernel is convolved with $\sum_{Q'} L_{Q'}$. This concludes the derivation of the TBA equations for the Q-particle described in Section 9. In the following appendix we have proved (C.20) starting from the formula for $\Sigma_{Q'Q}$ given in [48].

[4] Notice that a (zero-contribution) extra factor $\frac{1}{2\pi i}\frac{d}{du}\ln\sqrt{\frac{x(u+iQ/g)}{x(u-iQ/g)}}$ has been included in (C.19) to emphasize its relationship with the kernels $\phi_{Q,y}$ and $\phi_{y,Q}$.

Notice that (C.20) can be obtained from the Dorey–Hofman–Maldacena formula (C.19) by the following three steps: switching to the mirror kinematics, changing all the contours of integration from $\bar{\gamma}_{\mathsf{o}}$ to $\bar{\gamma}_{\mathsf{x}}$, and imposing an overall minus sign. Therefore, by this duality transformation we have related the original dressing factor $\sigma_{Q'Q}$ in the $su(2)$ sector of the direct theory to the $sl(2)$ improved dressing factor $\Sigma_{Q'Q}$ of the mirror theory.

Appendix D. The improved dressing factor revised

The main objective of this section is to prove Eq. (C.20) starting from the expression for the mirror improved dressing factor obtained in [48]. The following identities are useful to convert the result of [48] in our notation:

$$\frac{d}{du}\Phi\big(x(u),x(v)\big)=i\oint_{\bar{\gamma}_{\mathsf{o}}}\frac{ds}{2\pi i}\frac{d}{du}\ln\big(x(s)-x(u)\big)\oint_{\bar{\gamma}_{\mathsf{o}}}\frac{dt}{2\pi i}K_{\Gamma}^{[2]}(s-t)\frac{d}{dt}\ln\big(x(t)-x(v)\big),$$
$$\frac{d}{du}\Psi\big(x(u),x(v)\big)=i\oint_{\bar{\gamma}_{\mathsf{o}}}\frac{ds}{2\pi i}\frac{d}{du}\ln\big(x(s)-x(v)\big)K_{\Gamma}^{[2]}(u-s). \tag{D.1}$$

Then one can check that the mirror improved dressing kernel takes the form:

$$\begin{aligned}K_{Q'Q}^{\Sigma}(u,v)=&-\oint_{\bar{\gamma}_{\mathsf{o}}}ds\,\phi_{Q',y}(u,s)\oint_{\bar{\gamma}_{\mathsf{o}}}dt\,K_{\Gamma}^{[2]}(s-t)\phi_{y,Q}(t,v)\\&-\frac{1}{2}\oint_{\bar{\gamma}_{\mathsf{o}}}ds\,\phi_{Q',y}(u,s)\big(K_{\Gamma}^{[2]}(s-v+iQ/g)+K_{\Gamma}^{[2]}(s-v-iQ/g)\big)\\&+\frac{1}{2}\oint_{\bar{\gamma}_{\mathsf{o}}}ds\,\big(K_{\Gamma}^{[2]}\big(u+iQ'/g-s\big)+K_{\Gamma}^{[2]}\big(u-iQ'/g-s\big)\big)\phi_{y,Q}(s,v)\\&+K_{\Gamma}^{[Q+Q']}(u-v)+\frac{1}{2\pi i}\frac{d}{du}\ln\frac{\big(1-\frac{1}{x(u+iQ'/g)x(v-iQ/g)}\big)}{\big(1-\frac{1}{x(v+iQ/g)x(u-iQ'/g)}\big)}\\&\times\sqrt{\frac{x(u+iQ'/g)x(v-iQ/g)}{x(v+iQ/g)x(u-iQ'/g)}}.\end{aligned} \tag{D.2}$$

Let us rearrange some terms in this formula. First, by using the identity

$$K_{\Gamma}^{[2]}(u+iQ/g)+K_{\Gamma}^{[2]}(u-iQ/g)=2K_{\Gamma}^{[Q+2]}(u)+\phi_{Q}(u) \tag{D.3}$$

we can rewrite (D.2) as:

$$\begin{aligned}K_{Q'Q}^{\Sigma}(u,v)=&-\oint_{\bar{\gamma}_{\mathsf{o}}}ds\,\phi_{Q',y}(u,s)\oint_{\bar{\gamma}_{\mathsf{o}}}dt\,K_{\Gamma}^{[2]}(s-t)\phi_{y,Q}(t,v)\\&-\oint_{\bar{\gamma}_{\mathsf{o}}}ds\,\phi_{Q',y}(u,s)K_{\Gamma}^{[2+Q]}(s-v)-\frac{1}{2}\oint_{\bar{\gamma}_{\mathsf{o}}}ds\,\phi_{Q',y}(u,s)\phi_{Q}(s-v)\\&+\oint_{\bar{\gamma}_{\mathsf{o}}}ds\,K_{\Gamma}^{[2+Q']}(u-s)\phi_{y,Q}(s,v)+\frac{1}{2}\oint_{\bar{\gamma}_{\mathsf{o}}}ds\,\phi_{Q'}(u-s)\phi_{y,Q}(s,v)\end{aligned}$$

$$+ K_{\Gamma}^{[Q+Q']}(u-v) + \frac{1}{2\pi i}\frac{d}{du}\ln\frac{(1-\frac{1}{x(u+iQ'/g)x(v-iQ/g)})}{(1-\frac{1}{x(v+iQ/g)x(u-iQ'/g)})} \times \sqrt{\frac{x(u+iQ'/g)x(v-iQ/g)}{x(v+iQ/g)x(u-iQ'/g)}}. \tag{D.4}$$

Next, we notice that, subtracting the identities (E.7) and (E.9), we find

$$-\frac{1}{2}\left(\oint_{\bar{\gamma}_o} ds\, \phi_{Q',y}(u,s)\phi_Q(s-v) - \oint_{\bar{\gamma}_o} ds\, \phi_{Q'}(u-s)\phi_{y,Q}(s,v)\right) = -\phi_{Q+Q'}(u-v) - \frac{1}{2\pi i}\frac{d}{du}\ln\frac{(1-\frac{1}{x(u+iQ'/g)x(v-iQ/g)})}{(1-\frac{1}{x(v+iQ/g)x(u-iQ'/g)})} \times \sqrt{\frac{x(u+iQ'/g)x(v-iQ/g)}{x(v+iQ/g)x(u-iQ'/g)}}, \tag{D.5}$$

so that the dressing kernel can be rewritten as:

$$K^{\Sigma}_{Q'Q}(u,v) = -\oint_{\bar{\gamma}_o} ds\, \phi_{Q',y}(u,s)\oint_{\bar{\gamma}_o} dt\, K_{\Gamma}^{[2]}(s-t)\phi_{y,Q}(t,v) - \oint_{\bar{\gamma}_o} ds\, \phi_{Q',y}(u,s)K_{\Gamma}^{[2+Q]}(s-v) + \oint_{\bar{\gamma}_o} ds\, K_{\Gamma}^{[2+Q']}(u-s)\phi_{y,Q}(s,v) + K_{\Gamma}^{[Q+Q'+2]}(u-v). \tag{D.6}$$

We want to show that, under convolution with $\sum_{Q'} L_{Q'}$, this is perfectly equivalent to (C.20):

$$\sum_{Q'} L_{Q'} * K^{\Sigma}_{Q'Q}(v) = \sum_{Q'} L_{Q'} * \oint_{\bar{\gamma}_x} ds\, \phi^{(\alpha)}_{Q',y}(s)\oint_{\bar{\gamma}_x} dt\, K_{\Gamma}^{[2]}(s-t)\phi_{y,Q}(t,v) = -\oint_{\bar{\gamma}_x} ds\, \ln Y_y^{(\alpha)}(s)\oint_{\bar{\gamma}_x} K_{\Gamma}^{[2]}(s-t)\phi_{y,Q}(t,v). \tag{D.7}$$

To begin, let us apply property (C.7) to change the contour of integration under the 's' variable in the last line. Taking into account that $\Gamma(1+i\frac{g}{2}u)$ is free of singularities for u in the lower half plane we find:

$$-\oint_{\bar{\gamma}_x} ds\, \ln Y_y^{(\alpha)}(s)\oint_{\bar{\gamma}_x}\frac{dt}{2\pi i}\frac{d}{ds}\ln\frac{\Gamma(1-i\frac{g}{2}(s-t))}{\Gamma(1+i\frac{g}{2}(s-t))}\phi_{y,Q}(t,v)$$
$$= \oint_{\bar{\gamma}_o} ds\, \ln Y_y^{(\alpha)}(s)\oint_{\bar{\gamma}_x}\frac{dt}{2\pi i}\frac{d}{ds}\ln\left(\Gamma\left(1-i\frac{g}{2}(s-t)\right)\Gamma\left(1+i\frac{g}{2}(s-t)\right)\right)\phi_{y,Q}(t,v)$$
$$+ \sum_{Q'}\int_{\mathbb{R}} ds\, L_{Q'}(s)\oint_{\bar{\gamma}_x}\frac{dt}{2\pi i}\frac{d}{ds}\left(\ln\Gamma\left(1-i\frac{g}{2}\left(s+iQ'/g-t\right)\right)\right.$$

$$\times \Gamma\left(1 + i\frac{g}{2}\left(s - iQ'/g - t\right)\right)\Big)\phi_{y,Q}(t,v). \tag{D.8}$$

The next step is a change of integration contour under the 't' variable by means of (C.4). We arrive at:

$$\begin{aligned}
&-\oint_{\bar{\gamma}_{\mathsf{x}}} ds\, \ln Y_y^{(\alpha)}(s) \oint_{\bar{\gamma}_{\mathsf{x}}} dt K_\Gamma^{[2]}(s-t)\phi_{y,Q}(t,v)\\
&\quad= \oint_{\bar{\gamma}_{\mathsf{o}}} ds\, \ln Y_y^{(\alpha)}(s) \oint_{\bar{\gamma}_{\mathsf{o}}} dt\, K_\Gamma^{[2]}(s-t)\phi_{y,Q}(t,v)\\
&\qquad+ \oint_{\bar{\gamma}_{\mathsf{o}}} ds\, \ln Y_y^{(\alpha)}(s) \frac{d}{ds}\left(-\int_{\mathbb{R}-i\epsilon} \frac{dt}{2\pi i} \ln \Gamma\left(1 - i\frac{g}{2}(s-t)\right)\right.\\
&\qquad+ \left.\int_{\mathbb{R}+i\epsilon} \frac{dt}{2\pi i} \ln \Gamma\left(1 + i\frac{g}{2}(s-t)\right)\right)\left(\phi_{(y|-),Q}(t,v) - \phi_{(y|+),Q}(t,v)\right)\\
&\qquad+ \sum_{Q'} \int_{\mathbb{R}} ds\, L_{Q'}(s) \oint_{\bar{\gamma}_{\mathsf{o}}} dt\, K_\Gamma^{[2+Q']}(s-t)\phi_{y,Q}(t,v)\\
&\qquad+ \sum_{Q'} \int_{\mathbb{R}} ds\, L_{Q'}(s) \frac{d}{ds}\left(-\int_{\mathbb{R}-i\epsilon} \frac{dt}{2\pi i} \ln \Gamma\left(1 - i\frac{g}{2}\left(s + iQ'/g - t\right)\right)\right.\\
&\qquad+ \left.\int_{\mathbb{R}+i\epsilon} \frac{dt}{2\pi i} \ln \Gamma\left(1 + i\frac{g}{2}\left(s - iQ'/g - t\right)\right)\right)\\
&\qquad\times \left(\phi_{(y|-),Q}(t,v) - \phi_{(y|+),Q}(t,v)\right).
\end{aligned} \tag{D.9}$$

Using Jordan's lemma, one can easily evaluate the integrals over $\mathbb{R} \pm i\epsilon$. Closing the contours of integrations with two semicircles of infinite radius in the upper or lower half plane respectively, we find:

$$\begin{aligned}
&\frac{d}{ds}\left(-\int_{\mathbb{R}-i\epsilon} \frac{dt}{2\pi i} \ln \Gamma\left(1 - i\frac{g}{2}(s-t)\right) + \int_{\mathbb{R}+i\epsilon} \frac{dt}{2\pi i} \ln \Gamma\left(1 + i\frac{g}{2}(s-t)\right)\right)\\
&\quad\times \left(\phi_{(y|-),Q}(t,v) - \phi_{(y|+),Q}(t,v)\right) = K_\Gamma^{[2+Q]}(s-t);
\end{aligned} \tag{D.10}$$

$$\begin{aligned}
&\frac{d}{ds}\left(-\int_{\mathbb{R}-i\epsilon} \frac{dt}{2\pi i} \ln \Gamma\left(1 - i\frac{g}{2}\left(s + iQ'/g - t\right)\right)\right.\\
&\quad+ \left.\int_{\mathbb{R}+i\epsilon} \frac{dt}{2\pi i} \ln \Gamma\left(1 + i\frac{g}{2}\left(s - iQ'/g - t\right)\right)\right)\\
&\quad\times \left(\phi_{(y|-),Q}(t,v) - \phi_{(y|+),Q}(t,v)\right) = K_\Gamma^{[2+Q+Q']}(s-t),
\end{aligned} \tag{D.11}$$

and this finally proves (D.7).

Appendix E. Some useful identities

In this appendix we collect some useful identities which fill the gap between (8.4) and (9.6) and the standard form of the TBA equations. All the needed manipulations are straightforward applications of the following lemmas.

Consider a function $f(u)$ with a square root branch cut along $(-\infty,-2)\cup(2,+\infty)$. Let $f(u)$ have isolated poles with residues $+1$ at $u=u_1,\dots,u_n$ on the first sheet and at $u_*=v_1,\dots,v_m$ on the second sheet, but otherwise let it be analytic and asymptotically $o(u)$. Then a simple application of the residue theorem yields:

$$f *_{\bar{\gamma}_{\mathsf{x}}} \phi_Q(u) = f\left(u+i\frac{Q}{g}\right) - f\left(u-i\frac{Q}{g}\right) + \sum_{i=1}^{n}\phi_Q(u_i-u), \tag{E.1}$$

and

$$f *_{\bar{\gamma}_{\mathsf{o}}} \phi_Q(u) = f\left(u+i\frac{Q}{g}\right) - f\left(\left(u-i\frac{Q}{g}\right)_*\right) + \sum_{i/\,\mathrm{Im}(u_i)>0}\phi_Q(u_i-u) + \sum_{i/\,\mathrm{Im}(v_i)<0}\phi_Q(v_i-u). \tag{E.2}$$

Now suppose that $g(u)$ is a generic function and that, for u in the lower half plane, $f[g](u)$ is defined by the following integral representation:

$$f[g](u) = \oint_{\bar{\gamma}_{\mathsf{o}}} \frac{dz}{2\pi i} g(z)\frac{d}{dz}\ln\bigl(x(z)-x(u)\bigr) \quad \bigl(\mathrm{Im}(u)<0\bigr). \tag{E.3}$$

Notice that $f[g](u)$ is still analytic in the specified range $\mathrm{Im}(u)<0$ but that, when u is continued through $(-2,2)$, (E.3) needs to be replaced by another representation:

$$f[g](u) = \oint_{\bar{\gamma}_{\mathsf{o}}} \frac{dz}{2\pi i} g(z)\frac{d}{dz}\ln\bigl(x(z)-x(u)\bigr) - g(u) \quad \bigl(\mathrm{Im}(u)>0\bigr), \tag{E.4}$$

showing that the analytic behavior of $f[g](u)$ in the upper half plane depends on the specific form of $g(u)$ and Eq. (E.1) is in general incorrect.

However, the following relation is true for any $g(u)$:

$$\oint_{\bar{\gamma}_{\mathsf{x}}} dz\, f[g](z)\phi_Q(z-u) = -\oint_{\bar{\gamma}_{\mathsf{o}}} dz\, g(z)\phi_{y,Q}(z,u) - \int_{\mathbb{R}+i\epsilon} dz\, g(z)\phi_Q(z-u). \tag{E.5}$$

As a first example, let us consider the integral:

$$\oint_{\bar{\gamma}_{\mathsf{o}}} dz\, \ln Y_y^{(\alpha)}(z)\phi_M(z-u) = -\sum_Q\int_{\mathbb{R}} dz\, L_Q(z)\left(\oint_{\bar{\gamma}_{\mathsf{o}}} ds\, \phi_{Q,y}(z,s)\phi_M(s-u)\right), \tag{E.6}$$

which we encounter in formula (8.4). Taking into account the position of the poles of $\phi_{Q,y}$ and using (E.2) we find:

$$\oint_{\bar{\gamma}_\mathsf{o}} ds\, \phi_{Q,y}(z,s)\phi_M(s-u)$$
$$= \frac{1}{2\pi i}\frac{d}{dz}\ln\frac{(x(z-iQ/g)-x(u+iM/g))(x(z+iQ/g)-\frac{1}{x(u-iM/g)})}{(x(z+iQ/g)-x(u+iM/g))(x(z-iQ/g)-\frac{1}{x(u-iM/g)})}$$
$$+\phi_M(z-u+iQ/g)$$
$$=\phi_{Q,(v|M)}(z,u)-\phi_Q(z-u+iM/g)-\sum_{J=1}^{M-1}\phi_{Q-M+2J}(z-u)+\phi_M(z-u+iQ/g), \tag{E.7}$$

and, finally

$$\ln Y_y^{(\alpha)} *_{\bar{\gamma}_\mathsf{o}} \phi_M(u) = -\sum_Q\left(L_Q*\phi_{Q,(v|M)}(u)-\sum_{I=0}^{M-1}L_Q*\phi_{Q-M+2I}(u)\right), \tag{E.8}$$

which shows that (8.4) is the TBA equation for the v-particles. We report also the very similar result:

$$\oint_{\bar{\gamma}_\mathsf{o}} ds\, \phi_M(u-s)\phi_{y,Q}(s,v)$$
$$=-\phi_M(u-v-iQ/g)$$
$$+\frac{1}{2\pi i}\frac{d}{du}\ln\frac{(\frac{1}{x(u-iM/g)}-x(v+iQ/g))(x(u+iM/g)-x(v-iQ/g))}{(\frac{1}{x(u-iM/g)}-x(v-iQ/g))(x(u+iM/g)-x(v+iQ/g))}. \tag{E.9}$$

Now let us deal with the convolution

$$\oint_{\bar{\gamma}_\mathsf{x}} dz\, \ln Y_1\left(z+\frac{i}{g}\right)\phi_Q(z-u). \tag{E.10}$$

As a first step we notice that, inserting (6.5), the latter can be broken up in the following terms:

$$\oint_{\bar{\gamma}_\mathsf{x}} dz \ln Y_1\left(z+\frac{i}{g}\right)\phi_Q(z-u)$$
$$=\frac{1}{2}\oint_{\bar{\gamma}_\mathsf{x}} dz\, \Delta(z)\phi_Q(z-u)$$
$$=\oint_{\bar{\gamma}_\mathsf{x}}\Bigg(\int_{\mathbb{R}}\frac{dt}{2\pi i}\sum_{N,\alpha}L^{(\alpha)}_{(v|N)}(t)\frac{d}{dt}\ln\big(x(t+iN/g)-x(z)\big)\big(x(t-iN/g)-x(z)\big)$$
$$+L\ln x(z)+f[g_y](z)+\frac{1}{2}\Delta^{\Sigma}(z)\Bigg)\phi_Q(z-u)\,dz, \tag{E.11}$$

with $g_y(u)\equiv\sum_\alpha L_y^{(\alpha)}(u)$. Now, as an application of (E.1) we have, in the first place:

$$-\int_{\bar{\gamma}_\mathsf{x}} dz\, \ln x(z)\phi_Q(z-u)=\ln\frac{x(u-\frac{iQ}{g})}{x(u+\frac{iQ}{g})}\equiv\tilde{E}_Q(u), \tag{E.12}$$

which shows the appearance of the required energy term. Another easy computation shows that

$$
\begin{aligned}
&-\int_{\bar{\gamma}_{\mathsf{x}}} \frac{dz}{2\pi i}\frac{d}{dt}\ln\bigl(x(t+iN/g)-x(z)\bigr)\bigl(x(t-iN/g)-x(z)\bigr)\phi_Q(z-u)\\
&=\frac{1}{2\pi i}\frac{d}{dt}\ln\frac{(t+i\frac{N}{g}-u-i\frac{Q}{g})(t-i\frac{N}{g}-u-i\frac{Q}{g})}{(t+i\frac{N}{g}-u+i\frac{Q}{g})(t-i\frac{N}{g}-u+i\frac{Q}{g})}\\
&\quad\times\frac{(x(t+i\frac{N}{g})-x(u-i\frac{Q}{g}))(x(t-i\frac{N}{g})-x(u-i\frac{Q}{g}))}{(x(t+i\frac{N}{g})-x(u+i\frac{Q}{g}))(x(t-i\frac{N}{g})-x(u+i\frac{Q}{g}))}\\
&=\sum_{J=0}^{N}\phi_{Q-N+2J}(t-u)+\phi_{(v|N),Q}(t,u)\\
&=\sum_{J=1}^{Q}\phi_{N-Q+2J}(t-u)+\phi_{(v|N),Q}(t,u).
\end{aligned}
\tag{E.13}
$$

Finally, by means of the second lemma (E.5) we find:

$$
-\int_{\bar{\gamma}_{\mathsf{x}}} dz\, f[g_y](z)\phi_Q(z-u)=\int_{\mathbb{R}+i\epsilon} dz\, L_y^{(\alpha)}(z)\phi_Q(z-u)+\oint_{\bar{\gamma}_{\circ}} dz\, L_y^{(\alpha)}(z)\phi_{y,Q}(z,u).
\tag{E.14}
$$

Using also the fundamental results (C.14), (C.20) for the dressing-related part, we have come to the following:

$$
\begin{aligned}
&-\oint_{\bar{\gamma}_{\mathsf{x}}} dz\,\ln Y_1\left(z+\frac{i}{g}\right)\phi_Q(z-u)\\
&=2\sum_{Q'}L_{Q'}*K^{\Sigma}_{Q'Q}(u)+L\tilde{E}_Q(u)+\sum_{\alpha}L_y^{(\alpha)}*_{\bar{\gamma}_{\circ}}\phi_{y,Q}(u)\\
&\quad+\sum_{\alpha}\int_{\mathbb{R}+i\epsilon} dz\, L_y^{(\alpha)}(z)\phi_Q(z-u)+\sum_{N,\alpha}L^{(\alpha)}_{(v|N)}*\left(\sum_{J=1}^{Q}\phi_{N-Q+2J}+\phi_{(v|N),Q}\right)(u).
\end{aligned}
\tag{E.15}
$$

The latter equation shows that the Cauchy dispersion relation (9.6) coincides with the TBA equation (2.4).

Appendix F. Completing the TBA equations for the fermionic nodes

Let us show how to reproduce the missing part of the TBA equations (2.5) and (5.11) using only the functional relations (5.4) and the Y-system equations (1.3), (1.4), (1.2).

First of all, we notice that (1.2) implies the following constraint on the discontinuities of y-related functions:

$$\begin{aligned}&\big[\ln\big(Y^{(\alpha)}_{(y|-)}Y^{(\alpha)}_{(y|+)}\big)+\ln\big(Y^{(\alpha)}_{(y|-)}/Y^{(\alpha)}_{(y|+)}\big)\big]_{2N}+\big[\ln\big(Y^{(\alpha)}_{(y|-)}Y^{(\alpha)}_{(y|+)}\big)+\ln\big(Y^{(\alpha)}_{(y|-)}/Y^{(\alpha)}_{(y|+)}\big)\big]_{2N-2}\\&\quad=2\big[\Lambda^{(\alpha)}_{(v|1)}-\Lambda^{(\alpha)}_{(w|1)}-L_1\big]_{2N-1}=2\big[L^{(\alpha)}_{(v|1)}-L^{(\alpha)}_{(w|1)}+\ln\big(Y^{(\alpha)}_{(v|1)}/Y^{(\alpha)}_{(w|1)}\big)-L_1\big]_{2N-1}\end{aligned}\tag{F.1}$$

($N\in\mathbb{Z}$). Next we use the following identity, which is a consequence of the Y-system equations (1.3) and (1.4):

$$\begin{aligned}&\big[\ln\big(Y^{(\alpha)}_{(v|1)}/Y^{(\alpha)}_{(w|1)}\big)\big]_{\pm(2N-1)}\\&\quad=\big[\ln\big(Y^{(\alpha)}_{(y|-)}/Y^{(\alpha)}_{(y|+)}\big)\big]_{\pm(2N-2)}-\sum_{Q=2}^{N}[L_Q]_{\pm(2N-Q)}\\&\qquad+\sum_{J=1}^{N-1}\sum_{J'}A_{JJ'}\big[L^{(\alpha)}_{(v|J)}-L^{(\alpha)}_{(w|J)}\big]_{\pm(2N-1-J')}+\big[L^{(\alpha)}_{(v|N)}-L^{(\alpha)}_{(w|N)}\big]_{\pm N},\end{aligned}\tag{F.2}$$

with $N=1,2,\ldots$. Using the identity (5.4):

$$\big[\ln\big(Y^{(\alpha)}_{(y|-)}/Y^{(\alpha)}_{(y|+)}\big)\big]_{\pm 2N}=-\sum_{Q=1}^{N}[L_Q]_{\pm(2N-Q)},\tag{F.3}$$

we find:

$$\begin{aligned}&\big[\ln\big(Y^{(\alpha)}_{(y|-)}Y^{(\alpha)}_{(y|+)}\big)\big]_{2N}+\big[\ln\big(Y^{(\alpha)}_{(y|-)}Y^{(\alpha)}_{(y|+)}\big)\big]_{2N-2}\\&\quad=2\sum_{J=1}^{N}\big[L^{(\alpha)}_{(v|J)}-L^{(\alpha)}_{(w|J)}\big]_{\pm(2N-J)}+2\sum_{J=1}^{N-1}\big[L^{(\alpha)}_{(v|J)}-L^{(\alpha)}_{(w|J)}\big]_{\pm(2N-2-J)}\\&\qquad-\sum_{Q=1}^{N}[L_Q]_{\pm(2N-Q)}-\sum_{Q=1}^{N-1}[L_Q]_{\pm(2N-2-Q)}.\end{aligned}\tag{F.4}$$

Using the fact that $\ln(Y^{(\alpha)}_{(y|-)}Y^{(\alpha)}_{(y|+)})$ is regular on the real line we can solve the previous equation and we finally get to:

$$\big[\ln\big(Y^{(\alpha)}_{(y|-)}Y^{(\alpha)}_{(y|+)}\big)\big]_{\pm 2N}=2\sum_{J=1}^{N}\big[L^{(\alpha)}_{(v|J)}-L^{(\alpha)}_{(w|J)}\big]_{\pm(2N-J)}-\sum_{Q=1}^{N}[L_Q]_{\pm(2N-Q)}.\tag{F.5}$$

Appendix G. A Rosetta Stone for AdS_5/CFT_4

The system of TBA equations arising from AdS_5/CFT_4 is intricate enough that one might find a bit nonsense that three different notations are actually being used in [30–32]. Sharing our part of responsibility for this proliferation of languages, we would like to remedy by providing the reader with some clues to help in the translation.

First of all, there are little differences in the definition of the string coupling, which is denoted g by all the authors. In this paper, g is related to the 't Hooft parameter λ by $\lambda=4\pi^2g^2$, and agrees with the g introduced by Arutyunov, Frolov (AF) in [32]. There is a factor of 2 between this definition and the one adopted by Gromov, Kazakov, Kozak and Vieira (GKKV) in [38]. Therefore, in the rest of this section we will represent the coupling g of [38] using its Roman Latex style variant: $\mathrm{g}\equiv g^{\text{Ref. [38]}}$. (See Table 2.)

Table 2
The coupling constants.

This paper	AF	GKKV
g	g	2g

Our definition of Y functions is conventional, in that $Y_a = e^{\varepsilon_a}$, where ε_a is the ratio between the densities of holes and particles of species a. Other choices are possible. In particular, we remind the reader that the double index appearing in the notation of GKKV runs over the nodes of the diagram depicted in Fig. 1, allowing for a compact expression of the Y-system (1.1)–(1.4):

$$\frac{Y_{a,s}(v+i/2)Y_{a,s}(v-i/2)}{Y_{a+1,s}(v)Y_{a-1,s}(v)} = \frac{(1+Y_{a,s+1}(v))(1+Y_{a,s-1}(v))}{(1+Y_{a+1,s}(v))(1+Y_{a-1,s}(v))},$$

and that the $Y_{a,s}$ functions are related through the transformation

$$Y_{a,s}(v) = \frac{T_{a,s+1}(v)T_{a,s-1}(v)}{T_{a+1,s}(v)T_{a-1,s}(v)}$$

to a set of T functions obeying the Hirota bilinear equation (T-system):

$$T_{a,s}(v+i/2)T_{a,s}(v-i/2) = T_{a+1,s}(v)T_{a-1,s}(v) + T_{a,s+1}(v)T_{a,s-1}(v).$$

The same authors have also introduced a more pictorial notation, through the identifications:

$$Y_{1,1s}(v) \equiv Y_{\otimes s}(v),$$
$$Y_{2,2s}(v) \equiv Y_{\oplus s}(v),$$
$$Y_{1,sM}(v) \equiv Y_{\bigcirc sM}(v),$$
$$Y_{M,s}(v) \equiv Y_{\Delta sM}(v),$$
$$Y_{M,0}(v) \equiv Y_{\bullet M}(v),$$

with $s = \pm 1$, $M = 1, 2, \ldots$.

Moreover, GKKV use rescaled variables, following from a different definition of the function x which expresses the Bethe roots in terms of the rapidities.[5] The notation of AF is much more similar to ours, the main difference being the inversion of the Y functions for the Q-particles. The precise relation between Y functions in the different settings is shown in Table 3. In Table 4 we summarize the relations between our kernels and those defined by other authors. Notice that the two variables are swapped in the notation of GKKV, and hence that there is a relation between right convolutions in one system of notations and left convolutions in the other. In the last row,

[5] In fact, compare the definition (2.1) with the following one, adopted in [38]:

$$x^{\text{Ref. [38]}}(v) + \frac{1}{x^{\text{Ref. [38]}}(v)} = \frac{v}{\mathrm{g}}.$$

Table 3
The Y functions.

This paper		AF	GKKV	
$Y^{(\alpha)}_{(y\|-)}(u)$	$\alpha=1,2$	$Y^{(\alpha)}_{-}(u)$	$1/Y_{\otimes_s}(ug)$	$s=-3+2\alpha=-1,+1$
$Y^{(\alpha)}_{(y\|+)}(u)$	$\alpha=1,2$	$Y^{(\alpha)}_{+}(u)$	$Y_{\oplus_s}(ug)$	$s=-3+2\alpha=-1,+1$
$Y^{(\alpha)}_{(w\|M)}(u)$	$\alpha=1,2,\ M=1,2,\dots$	$Y^{(\alpha)}_{M\|w}(u)$	$Y_{\bigcirc_{s(M+1)}}(ug)$	$s=-3+2\alpha=-1,+1$
$Y^{(\alpha)}_{(v\|M)}(u)$	$\alpha=1,2,\ M=1,2,\dots$	$Y^{(\alpha)}_{M\|vw}(u)$	$1/Y_{\Delta_{s(M+1)}}(ug)$	$s=-3+2\alpha=-1,+1$
$Y_M(u)$	$M=1,2,\dots$	$1/Y_M(u)$	$1/Y_{\bullet_M}(ug)$	

Table 4
The TBA kernels.

This paper	AF	GKKV
$\phi_{(y\|-),M}(u,z)$	$-K^{yM}_{-}(u,z)$	$-g\mathcal{R}^{10}_{M1}(zg,ug)$
$\phi_{(y\|+),M}(u,z)$	$K^{yM}_{+}(u,z)$	$-g\mathcal{B}^{10}_{M1}(zg,ug)$
$\phi_{M,(y\|-)}(u,z)$	$K^{My}_{-}(u,z)$	$g\mathcal{R}^{01}_{1M}(zg,ug)$
$\phi_{M,(y\|+)}(u,z)$	$K^{My}_{+}(u,z)$	$g\mathcal{B}^{01}_{1M}(zg,ug)$
$\phi_M(u-z)$	$K_M(u-z)$	$gK_M(zg-ug)$
$\phi_{MN}(u-z)$	$K_{MN}(u-z)$	$gK_{NM}(zg-ug)$
$\phi_{(v\|M),Q}(u,z)$	$-K^{MQ}_{vwx}(u,z)$	$-g\mathcal{B}^{10}_{Q,M-2}(zg,ug)-g\mathcal{R}^{10}_{QM}(zg,ug)$
$\phi_{Q,(v\|M)}(u,z)$	$K^{QM}_{xv}(u,z)$	$g\mathcal{B}^{01}_{M-2,Q}(zg,ug)+g\mathcal{R}^{01}_{MQ}(zg,ug)$
$\frac{1}{2\pi i}\frac{d}{du}\ln\sigma_{MN}(u,z)$	$\frac{1}{2\pi i}\frac{d}{du}\ln\sigma_{MN}(u,z)$	$-g\mathcal{S}_{NM}(zg,ug)$

$\sigma_{MN}(u,z)$ (corresponding to $\sigma_{MN}(gu,gz)$ in [38]) is the dressing factor for the scattering of an M and an N particle in the direct theory, which is understood to be analytically continued to the mirror kinematics in the context of mirror TBA.

For the mirror improved dressing factor we have adhered to the notation of AF:

$$\Sigma_{MN}(u,z)=\prod_{k=1}^{M}\prod_{l=1}^{N}\left(\frac{1-\frac{1}{x(u+\frac{i}{g}(M+2-2k))x(z+\frac{i}{g}(N-2l))}}{1-\frac{1}{x(u+\frac{i}{g}(M-2k))x(z+\frac{i}{g}(N+2-2l))}}\right)\sigma_{MN}(u,z).$$

On the other hand, the double product factor on the right-hand side emerges from the kernels of GKKV through the following identity:

$$\begin{aligned}&-gK_{NM}(zg-ug)+g\mathcal{R}^{(11)}_{NM}(zg,ug)-g\mathcal{B}^{(11)}_{NM}(zg,ug)\\&\quad=\frac{1}{2\pi i}\frac{d}{du}\ln\prod_{k=1}^{M}\prod_{l=1}^{N}\left(\frac{1-\frac{1}{x(u+\frac{i}{g}(M+2-2k))x(z+\frac{i}{g}(N-2l))}}{1-\frac{1}{x(u+\frac{i}{g}(M-2k))x(z+\frac{i}{g}(N+2-2l))}}\right)^2\frac{x(u+\frac{i}{g}(M+2-2k))}{x(u+\frac{i}{g}(M-2k))}\\&\quad=\frac{1}{2\pi i}\frac{d}{du}\ln\prod_{k=1}^{M}\prod_{l=1}^{N}\left(\frac{1-\frac{1}{x(u+\frac{i}{g}(M+2-2k))x(z+\frac{i}{g}(N-2l))}}{1-\frac{1}{x(u+\frac{i}{g}(M-2k))x(z+\frac{i}{g}(N+2-2l))}}\right)^2-N\tilde{p}^M(u).\end{aligned}\tag{G.1}$$

The only difference between the TBA equations as written in [38] and in [30,32] is originated by the last term of the preceding equality. As we have anticipated in Section 3, it results in the introduction of a chemical potential proportional to the total momentum of the state under consideration. By the trace condition, we expect the extra chemical potentials to be integer multiples of $2\pi i$. Therefore, beside possibly the swapping of some pairs of excited levels, the substance of the analysis should be unaffected, and the TBA equations of [30,32,38] are an equivalent starting point to study the spectrum of $\mathrm{AdS}_5/\mathrm{CFT}_4$.

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