Invariant variational problems and Cartan connections on gauge-natural bundles

M. Palese and E. Winterroth

Department of Mathematics, University of Torino

via C. Alberto 10, I-10123 Torino, Italy

E-MAIL: MARCELLA.PALESE@UNITO.IT

Abstract

A principal Cartan connection is canonically defined by gauge-natural invariant variational problems of finite order due to the existence of a reductive split structure associated with canonical Lagrangian conserved quantities on gauge-natural bundles.

02.40.Ma; 02.40.Vh; 03.50.-z

Key words: jet space; variational sequence; reductive structure; Cartan connection

1 Introduction

In the following classical physical fields are assumed to be sections of bundles functorially associated with gauge-natural prolongations of principal bundles [2, 5]. We consider finite order Lagrangian variational problems in terms of exterior differentials of forms modulo contact forms as framed in the context of finite order variational sequences [7].

As well known, following Noether's theory [8], from invariance properties of the Lagrangian the existence of suitable conserved currents and identities can be deduced. Within such a picture *generalized Bergmann–Bianchi identities* are conditions for a Noether conserved current to be not only closed but also the global divergence of a tensor density called a superpotential [10]. Recently, we proposed an approach to deal with the problem of *canonical* covariance and uniqueness of conserved quantities which uses *variational* *derivatives* taken with respect to the class of (generalized) variation vector fields being Lie derivatives of sections of bundles by gauge-natural lifts of infinitesimal principal automorphisms [9, 11, 12, 13, 3].

In this note, we shortly review some of the outcomes and in particular we recall how the kernel of the gauge-natural Jacobi morphism (coinciding with generalized Bergmann–Bianchi identities) defines a split reductive structure on the relevant underlying principal bundle. As a consequence, we prove that a principal Cartan connection is canonically defined by gauge-natural invariant variational problems of finite order.

Let us recall some useful concepts of prolongations, mainly with the aim of fixing the notation; for details see *e.g.* [5, 14]. Let $\pi : \boldsymbol{Y} \to \boldsymbol{X}$ be a fibered manifold, with $\dim \boldsymbol{X} = n$ and $\dim \boldsymbol{Y} = n + m$. For $s \geq q \geq 0$ integers we deal with the s–jet space $J_s\boldsymbol{Y}$ of s–jet prolongations of (local) sections of π; in particular, we set $J_0\boldsymbol{Y} \equiv \boldsymbol{Y}$. We recall that there are the natural fiberings $\pi^s_q : J_s\boldsymbol{Y} \to J_q\boldsymbol{Y}$, $s \geq q$, $\pi^s : J_s\boldsymbol{Y} \to \boldsymbol{X}$, and, among these, the *affine* fiberings π^s_{s-1}. We denote by $V\boldsymbol{Y}$ the vector subbundle of the tangent bundle $T\boldsymbol{Y}$ of vectors on $\boldsymbol{Y}$ which are vertical with respect to the fibering π.

For $s \geq 1$, we consider the following natural splitting induced by the natural contact structure on jets bundles (see *e.g.* [6, 7]): $J_s\boldsymbol{Y} \underset{J_{s-1}\boldsymbol{Y}}{\times} T^*J_{s-1}\boldsymbol{Y} = J_s\boldsymbol{Y} \underset{J_{s-1}\boldsymbol{Y}}{\times} (T^*\boldsymbol{X} \oplus V^*J_{s-1}\boldsymbol{Y})$.

A vector field ξ on $\boldsymbol{Y}$ is said to be *vertical* if it takes values in $V\boldsymbol{Y}$. A vertical vector field can be prolonged to a vertical vector field $j_s\xi$ characterized by the fact that its flow is the natural prolongation of the flow of ξ. Given a vector field $\Xi : J_s\boldsymbol{Y} \to TJ_s\boldsymbol{Y}$, the above splitting yields $\Xi \circ \pi^{s+1}_s = \Xi_H + \Xi_V$, where Ξ_H and Ξ_V are the horizontal and the vertical part of Ξ, respectively. As well known, the above splitting induces also a decomposition of the exterior differential on $\boldsymbol{Y}$, $(\pi^{r+1}_r)^* \circ d = d_H + d_V$, where d_H and d_V are called the *horizontal* and *vertical differential*, respectively. Such decompositions always rise the order of the objects.

Let $\boldsymbol{P} \to \boldsymbol{X}$ be a principal bundle with structure group $\boldsymbol{G}$. For $r \leq k$ integers consider the *gauge-natural prolongation of* $\boldsymbol{P}$ given by $\boldsymbol{W}^{(r,k)}\boldsymbol{P} \doteq J_r\boldsymbol{P} \underset{\boldsymbol{X}}{\times} L_k(\boldsymbol{X})$, where $L_k(\boldsymbol{X})$ is the bundle of k–frames in $\boldsymbol{X}$ [2, 5]; $\boldsymbol{W}^{(r,k)}\boldsymbol{P}$ is a principal bundle over $\boldsymbol{X}$ with structure group $\boldsymbol{W}^{(r,k)}_n\boldsymbol{G}$ which is the *semidirect* product with respect to the action of $GL_k(n)$ on $\boldsymbol{G}^r_n$ given by jet composition and $GL_k(n)$ is the group of k–frames in $I\!R^n$. Here we denote by $\boldsymbol{G}^r_n$ the space of (r,n)-velocities on $\boldsymbol{G}$.

Let $\boldsymbol{F}$ be a manifold and $\zeta : \boldsymbol{W}^{(r,k)}_n\boldsymbol{G}\times\boldsymbol{F} \to \boldsymbol{F}$ be a left action of $\boldsymbol{W}^{(r,k)}_n\boldsymbol{G}$ on $\boldsymbol{F}$. There is a naturally defined right action of $\boldsymbol{W}^{(r,k)}_n\boldsymbol{G}$ on $\boldsymbol{W}^{(r,k)}\boldsymbol{P}\times\boldsymbol{F}$ so that we have in the standard way the associated *gauge-natural bundle* of order (r,k): $\boldsymbol{Y}_\zeta \doteq \boldsymbol{W}^{(r,k)}\boldsymbol{P}\times_\zeta \boldsymbol{F}$. All our considerations shall refer to $\boldsymbol{Y}$ as a gauge-natural bundle as just defined.

Denote now by $\mathcal{A}^{(r,k)}$ the sheaf of right invariant vector fields on $\boldsymbol{W}^{(r,k)}\boldsymbol{P}$. The *gauge-natural lift* is defined as the functorial map $\mathfrak{G} : \boldsymbol{Y}_\zeta \underset{\boldsymbol{X}}{\times} \mathcal{A}^{(r,k)} \to T\boldsymbol{Y}_\zeta : (\boldsymbol{y},\bar{\Xi}) \mapsto \hat{\Xi}(\boldsymbol{y})$, where, for any $\boldsymbol{y}\in\boldsymbol{Y}_\zeta$, one sets: $\hat{\Xi}(\boldsymbol{y}) = \frac{d}{dt}[(\Phi_{\zeta\,t})(\boldsymbol{y})]_{t=0}$, and $\Phi_{\zeta\,t}$ denotes the (local) flow corresponding to the gauge-natural lift of Φ_t. Such a functor defines a class of parametrized contact transformations.

This mapping fulfils the following properties (see [5]): $\mathfrak{G}$ is linear over $id_{\boldsymbol{Y}_\zeta}$; we have $T\pi_\zeta\circ\mathfrak{G} = id_{T\boldsymbol{X}}\circ\bar{\pi}^{(r,k)}$, where $\bar{\pi}^{(r,k)}$ is the natural projection $\boldsymbol{Y}_\zeta \underset{\boldsymbol{X}}{\times} \mathcal{A}^{(r,k)} \to T\boldsymbol{X}$; for any pair $(\bar{\Lambda},\bar{\Xi})\in\mathcal{A}^{(r,k)}$, we have $\mathfrak{G}([\bar{\Lambda},\bar{\Xi}]) = [\mathfrak{G}(\bar{\Lambda}),\mathfrak{G}(\bar{\Xi})]$. In the following, by an abuse of notation we denote by $V\mathcal{A}^{(r,k)}$ the bundle of vertical parts of sections of $\mathcal{A}^{(r,k)}\to\boldsymbol{X}$.

Let γ be a (local) section of $\boldsymbol{Y}_\zeta$, $\bar{\Xi}\in\mathcal{A}^{(r,k)}$ and $\hat{\Xi}$ its gauge-natural lift. Following [5] we define the *generalized Lie derivative* of γ along the vector field $\hat{\Xi}$ to be the (local) section $£_{\bar{\Xi}}\gamma : \boldsymbol{X}\to V\boldsymbol{Y}_\zeta$, given by $£_{\bar{\Xi}}\gamma = T\gamma\circ\xi - \hat{\Xi}\circ\gamma$.

The Lie derivative operator acting on sections of gauge-natural bundles is an homomorphism of Lie algebras; furthermore, for any gauge-natural lift, the fundamental relation holds true: $\hat{\Xi}_V = -£_{\bar{\Xi}}$.

2 Gauge-natural Jacobi equations and Cartan connections

The fibered splitting induced by the contact structure on finite order jets yields the *sheaf splitting* $\mathcal{H}^p_{(s+1,s)} = \bigoplus_{t=0}^p \mathcal{C}^{p-t}_{(s+1,s)} \wedge\mathcal{H}^t_{s+1}$, where the sheaves $\mathcal{H}^p_{(s,q)}$ and $\mathcal{H}^p_s$ of *horizontal forms* with respect to the projections π^s_q and π^s_0, respectively, while $\mathcal{C}^p_{(s,q)}\subset\mathcal{H}^p_{(s,q)}$ and $\mathcal{C}^p{}_s\subset\mathcal{C}^p_{(s+1,s)}$ are *contact forms*, *i.e.* horizontal forms valued into $\mathcal{C}^*_s[\boldsymbol{Y}]$ (they have the property of vanishing along any section of the gauge-natural bundle). We put $\mathcal{H}^{p,h}{}_{s+1}\doteq h(\Lambda^p_s)$ for $0<p\leq n$ and the map h is *the horizontalization*, *i.e.* the projection on the summand of lesser contact degree. Let $\eta\in\mathcal{C}^1_s\wedge\mathcal{C}^1_{(s,0)}\wedge\mathcal{H}^{n,h}{}_{s+1}$; then there is a unique morphism $K_\eta\in\mathcal{C}^1_{(2s,s)}\otimes\mathcal{C}^1_{(2s,0)}\wedge\mathcal{H}^{n,h}{}_{2s+1}$ such that, for all $\Xi:\boldsymbol{Y}\to V\boldsymbol{Y}$, $C^1_1(j_{2s}\Xi\otimes K_\eta) = E_{j_s\Xi\rfloor\eta}$, where C^1_1 stands for tensor contraction on the first

factor and $\rfloor$ denotes inner product and $E_{j_s\Xi\rfloor\eta} = (\pi^{2s+1}_{s+1})^* j_s\Xi\rfloor\eta + F_{j_s\Xi\rfloor\eta}$ (with $F_{j_s\Xi\rfloor\eta}$ a local divergence) is a uniquely defined global section of $\mathcal{C}^1_{(2s,0)} \wedge \mathcal{H}^{n,h}_{2s+1}$ (see [15]).

By an abuse of notation, let us denote by $d \ker h$ the sheaf generated by the presheaf $d \ker h$ in the standard way. We set $\Theta^*_s \doteq \ker h + d \ker h$. We have that $0 \to I\!\!R_Y \to \mathcal{V}^*_s$, where $\mathcal{V}^*_s = \Lambda^*_s / \Theta^*_s$, is an exact resolution of the constant sheaf $I\!\!R_Y$ [7]. A section $E_{d\lambda} \doteq \mathcal{E}_n(\lambda) \in \mathcal{V}^{n+1}_s$ is the *generalized higher order Euler–Lagrange type morphism* associated with λ.

The morphism K_η can be integrated by parts to provide a representation of the *generalized Jacobi morphism* associated with λ [10]. Let λ be a Lagrangian and consider $\hat{\Xi}_V$ as a variation vector field. Let us set $\chi(\lambda, \hat{\Xi}_V) \equiv E_{j_s\hat{\Xi}\rfloor hd\mathcal{L}_{j_{2s+1}\bar{\Xi}_V}\lambda}$. Because of linearity properties of $K_{hd\mathcal{L}_{j_{2s}\bar{\Xi}_V}\lambda}$, and by using a global decomposition formula due to Kolář, we can decompose the morphism defined above as $\chi(\lambda, \hat{\Xi}_V) = E_{\chi(\lambda,\hat{\Xi}_V)} + F_{\chi(\lambda,\hat{\Xi}_V)}$, where $F_{\chi(\lambda,\hat{\Xi}_V)}$ is a *local* horizontal differential which can be globalized by fixing of a connection.

Definition 1 We call the morphism $\mathcal{J}(\lambda, \hat{\Xi}_V) \doteq E_{\chi(\lambda,\hat{\Xi}_V)}$ the *gauge-natural generalized Jacobi morphism* associated with the Lagrangian λ and the variation vector field $\hat{\Xi}_V$. We call the morphism $\mathfrak{H}(\lambda, \hat{\Xi}_V) \doteq \hat{\Xi}_V \rfloor \mathcal{E}_n(\hat{\Xi}_V \rfloor \mathcal{E}_n(\lambda))$ the *gauge-natural Hessian morphism* associated with λ. □

The morphism $\mathcal{J}(\lambda, \hat{\Xi}_V)$ is a *linear* morphism with respect to the projection $J_{4s}\boldsymbol{Y}_\zeta \underset{\boldsymbol{X}}{\times} V J_{4s}\mathcal{A}^{(r,k)} \to J_{4s}\boldsymbol{Y}_\zeta$. Notice that, since $\delta^2_{\mathfrak{G}}\lambda \doteq \mathcal{L}_{\hat{\Xi}_V}\mathcal{L}_{\hat{\Xi}_V}\lambda = \hat{\Xi}_V \rfloor \mathcal{E}_n(\hat{\Xi}_V \rfloor \mathcal{E}_n(\lambda))$, we have $\mathfrak{H}(\lambda, \hat{\Xi}_V) = \delta^2_{\mathfrak{G}}\lambda$; furthermore, being also $\delta^2_{\mathfrak{G}}\lambda = \mathcal{E}_n(\hat{\Xi}_V \rfloor h(d\delta\lambda))$ [10], then $\mathfrak{H}(\lambda, \hat{\Xi}_V)$ is self-adjoint. Furthermore, we have $\mathcal{J}(\lambda, \hat{\Xi}_V) \doteq E_{\chi(\lambda,\hat{\Xi}_V)} = \mathcal{E}_n(\hat{\Xi}_V \rfloor h(d\delta\lambda)) = \mathfrak{H}(\lambda, \hat{\Xi}_V)$, stating that *the Hessian and as a consequence also the Jacobi morphism are symmetric self-adjoint morphisms*. The Jacobi morphism $\mathcal{J}(\lambda, \hat{\Xi}_V)$ can be interpreted as an endomorphism of $J_{4s}V\mathcal{A}^{(r,k)}$.

In the following we concentrate on some geometric aspects of the space $\mathfrak{K} \doteq \ker \mathcal{J}(\lambda, \hat{\Xi}_V)$. Such a kernel defines generalized gauge-natural Jacobi equations [10], the solutions of which we call *generalized Jacobi vector fields.* It characterizes *canonical* covariant conserved quantities. In fact, given $[\alpha] \in \mathcal{V}^n_s$, since the *variational Lie derivative* of classes of forms can be represented the variational sequence, we have the corresponding version of the First Noether Theorem:

$$\mathcal{L}_{j_s\Xi}[\alpha] = \omega(\lambda, \hat{\Xi}_V) + d_H(j_{2s}\hat{\Xi}_V \rfloor p_{d_V h(\alpha)} + \xi \rfloor h(\alpha)) \,, \tag{1}$$

where we put $\omega(\lambda, \hat{\bar{\Xi}}_V) \doteq \hat{\bar{\Xi}}_V \rfloor \mathcal{E}_n(\lambda) \doteq -\pounds_{\bar{\Xi}} \rfloor \mathcal{E}_n(\lambda)$.

As usual, λ is defined a *gauge-natural invariant Lagrangian* if the gauge-natural lift $(\hat{\bar{\Xi}}, \xi)$ of *any* vector field $\bar{\Xi} \in \mathcal{A}^{(r,k)}$ is a symmetry for λ, *i.e.* if $\mathcal{L}_{j_{s+1}\bar{\Xi}} \lambda = 0$. In this case, as an immediate consequence we have that $\omega(\lambda, \hat{\bar{\Xi}}_V) = d_H(-j_s \pounds_{\bar{\Xi}} \rfloor p_{d_V \lambda} + \xi \rfloor \lambda)$.

The generalized *Bergmann–Bianchi morphism* $\beta(\lambda, \hat{\bar{\Xi}}_V) \doteq E_{\omega(\lambda, \hat{\bar{\Xi}}_V)}$ is canonically vanishing along $\mathfrak{K}$. This fact characterizes canonical covariant conserved Noether currents [10, 9]. Furthermore, along the kernel of the gauge-natural generalized gauge-natural Jacobi morphism we have that $\mathcal{L}_{j_{s+1}\bar{\Xi}_H}[\mathcal{L}_{j_{s+1}\bar{\Xi}_V}\lambda] \equiv 0$. Hence Bergmann–Bianchi identities are equivalent to the invariance condition $\mathcal{L}_{j_{s+1}\bar{\Xi}}[\mathcal{L}_{j_{s+1}\bar{\Xi}_V}\lambda] \equiv 0$ and can be suitably interpreted as Noether identities associated with the invariance properties of the Euler–Lagrange morphism $\mathcal{E}_n(\omega)$ [12]. This fact can be used to prove that $\mathfrak{K}$ is characterized as a vector subbundle, being the kernel of a Hamiltonian operator [3, 11].

Proposition 1 *A principal Cartan connection is canonically defined by gauge-natural invariant variational problems of finite order.*

PROOF. Let $\mathfrak{h}$ be the Lie algebra of right-invariant vertical vector fields on $W^{(r+4s,k+4s)}\boldsymbol{P}$ and $\mathfrak{k}$ the Lie subalgebra of generalized Jacobi vector fields defined as solutions of generalized Jacobi equations. Consider now that, since the Jacobi morphism self-adjoint, its cokernel coincides with the cokernel of the adjoint morphism, thus we have that $dim\mathfrak{K} = dimCoker\mathcal{J}$. If we further consider that $\mathfrak{K}$ is of constant rank because it is the kernel of a Hamiltonian operator [11], we are able to define the split structure given by $\mathfrak{h} = \mathfrak{K} \oplus Im\mathcal{J}$. The Lie derivative of a solution of Euler–Lagrange equations *with respect to a Jacobi vector field* is again a solution of Euler–Lagrange equations. However, the Lie derivative with respect to vertical parts of the commutator between the gauge-natural lift of a Jacobi vector field and (the vertical part of) a lift not lying in $\mathfrak{K}$ *is not* a solution of Euler–Lagrange equations. Thus, since $\mathcal{J}$ is a projector and a derivation of $\mathfrak{h}$, it is easy to see that the split structure is also reductive, being $[\mathfrak{k}, Im\mathcal{J}] = Im\mathcal{J}$. We have then proved that *the kernel* $\mathfrak{K}$ *defines a reductive structure on* $W^{(r+4s,k+4s)}\boldsymbol{P}$.

In particular, for each $\boldsymbol{p} \in W^{(r,k)}\boldsymbol{P}$ by denoting $\mathcal{W} \equiv \mathfrak{h}_{\boldsymbol{p}}$, $\mathcal{K} \equiv \mathfrak{k}_{\boldsymbol{p}}$ and $\mathcal{V} \equiv Im\mathcal{J}_{\boldsymbol{p}}$ we have the reductive Lie algebra decomposition $\mathcal{W} = \mathcal{K} \oplus \mathcal{V}$, with $[\mathcal{K}, \mathcal{V}] = \mathcal{V}$. Notice that $\mathcal{W}$ is the Lie algebra of the Lie group $W^{(r,k)}_n\boldsymbol{G}$. Since $\mathfrak{K}$ is a vector subbundle of $\mathcal{A}^{(r,k)} = T\boldsymbol{W}^{(r,k)}\boldsymbol{P}/\boldsymbol{W}^{(r,k)}_n\boldsymbol{G}$ there exists a principal subbundle $\boldsymbol{Q} \subset \boldsymbol{W}^{(r,k)}\boldsymbol{P}$ such that $dim\boldsymbol{Q} = dim\mathcal{W}$, $\mathcal{K} = T\boldsymbol{Q}/\boldsymbol{K}|_{\boldsymbol{q}}$, where $\boldsymbol{K}$ is the (reduced) Lie group of the Lie algebra $\mathcal{K}$ and the embedding $\boldsymbol{Q} \to \boldsymbol{W}^{(r,k)}\boldsymbol{P}$ is a principal bundle homomorphism over the injective group homomorphism $\boldsymbol{K} \to \boldsymbol{W}^{(r,k)}_n\boldsymbol{G}$.

Now, if ω is a principal connection on $\boldsymbol{W}^{(r,k)}\boldsymbol{P}$, the restriction $\omega|_{\boldsymbol{Q}}$ is a Cartan connection of the principal bundle $\boldsymbol{Q} \to \boldsymbol{X}$. In fact, let us consider a principal connection $\bar{\omega}$ on the principal bundle $\boldsymbol{Q}$ *i.e.* a $\mathcal{K}$-invariant horizontal distribution defining the vertical parallelism $\bar{\omega} : V\boldsymbol{Q} \to \mathcal{K}$ by means of the fundamental vector field mapping in the usual and standard way. Since $\mathcal{K}$ is a subalgebra of the Lie algebra $\mathcal{W}$ and $dim\boldsymbol{Q} = dim\mathcal{W}$, it is defined a principal Cartan connection of type $\mathcal{W}/\mathcal{K}$, that is an $\mathcal{W}$-valued absolute parallelism $\hat{\omega} : T\boldsymbol{Q} \to \mathcal{W}$ which is an homomorphism of of Lie algebras, when restricted to $\mathcal{K}$, preserving Lie brackets if one of the arguments is in $\mathcal{K}$, and such that $\hat{\omega}|_{VQ} = \bar{\omega}$, that means that $\hat{\omega}$ is an extension of the natural vertical parallelism.

We have then to show that such a $\hat{\omega}$ exists. We can define $\hat{\omega}$ as the restiction of the natural vertical parallelism defined by a principal connection ω on $W^{(r,k)}\boldsymbol{P}$ by means of the fundamental vector field mapping $\omega : VW^{(r,k)}\boldsymbol{P} \to \mathcal{W}$ to $T\boldsymbol{Q}$. This restiction is, in particular, $\mathcal{K}$-invariant since is by construction $\mathcal{W}$-invariant. Of course, this definition is well done provided that $T\boldsymbol{Q} \subset VW^{(r,k)}\boldsymbol{P}$. In fact, it is easy to see that $T\boldsymbol{Q} \subset VW^{(r,k)}\boldsymbol{P}$ holds true as a consequence of the reductive split structure on $W^{(r+4s,k+4s)}\boldsymbol{P}$. In particular, $\forall \boldsymbol{q} \in \boldsymbol{Q}$, we have $T_{\boldsymbol{q}}\boldsymbol{Q} \cap \mathcal{H}_{\boldsymbol{q}} = 0$, where $\mathcal{H}_{\boldsymbol{q}}$, $\forall \boldsymbol{p} \in \boldsymbol{W}^{(r,k)}\boldsymbol{P}$ is defined by ω as $T_{\boldsymbol{p}}\boldsymbol{W}^{(r,k)}\boldsymbol{P} = V_{\boldsymbol{p}}\boldsymbol{W}^{(r,k)}\boldsymbol{P} \oplus \mathcal{H}_{\boldsymbol{p}}$; furthermore, $dim\boldsymbol{X} = dim\mathcal{W}/\mathcal{K}$.

QED

Let us now explicate some consequences. In fact, let us consider once more the reductive decomposition $\mathcal{W} = \mathcal{K} \oplus \mathcal{V}$, with $[\mathcal{K}, \mathcal{V}] = \mathcal{V}$. The $\mathcal{K}$-component $\eta = pr_{\mathcal{K}} \circ \hat{\omega}$ is a principal connection form on the $\mathcal{K}$-manifold $\boldsymbol{Q}$. A $\mathcal{K}$-invariant horizontal distribution $\mathfrak{H} = \hat{\omega}^{-1}(\mathcal{V}) \subset \boldsymbol{Q}$ complementary to the $\mathcal{K}$-invariant vertical distribution $\zeta_{\mathcal{K}}(\boldsymbol{Q}) \subset \boldsymbol{Q}$ spanned by the $\mathcal{K}$-action and such that $[\zeta_{\mathcal{K}}, \Gamma(\mathfrak{H})] \subset \Gamma(\mathfrak{H})$, with $\Gamma(\mathfrak{H}) \subset \chi(\boldsymbol{Q})$ is the space of section of the bundle $\mathfrak{H}$. The $\mathcal{V}$-component $\theta = pr_{\mathcal{V}} \circ \hat{\omega}$ is a sort of a displacement form and we have $ker\theta = \zeta_{\mathcal{K}}(\boldsymbol{Q})$. In fact, being $\boldsymbol{K}$ a reductive Lie subgroup of $\boldsymbol{W}^{(r,k)}_n\boldsymbol{G}$ the principal Cartan connection could be seen as a $\boldsymbol{K}$-structure equipped with a principal connection form η on $\boldsymbol{Q}$. By considering the reduction of the structure bundle $W^{(r,k)}\boldsymbol{P}$ to a subbundle with structure group a subgroup of the differential group (of a certain order), we see that generalized Jacobi vector fields can be interpreted as a kind of reductive gauge-natural lift in the sense of [4].

acknowledgments

The authors have been supported by the University of Torino through the research project 2008 *Teorie di campo classiche: calcolo delle variazioni, struttura Lagrangiana ed Hamiltoniana, etc.*; and (E.W.) through the research contract *Sequenze variazionali e teoremi di Noether su fibrati gauge-naturali: quantita conservate canoniche in teoria dei campi* 2009-2010, partially by the University of Brno (Czech Republic) thanks to an EČC grant 2008.

References

[1] P.G. Bergmann: Non-Linear Field Theories, *Phys. Rev.* **75** (4)(1949) 680–685.

[2] D.J. Eck: Gauge-natural bundles and generalized gauge theories, *Mem. Amer. Math. Soc.* **247** (1981) 1–48.

[3] M. Ferraris, M. Francaviglia, M. Palese, E. Winterroth: Canonical connections in gauge-natural field theories, *Int. J. Geom. Methods Mod. Phys.* **5** (6) (2008) 973–988.

[4] M. Godina, P. Matteucci: Reductive G-structures and Lie derivatives, *J. Geom. Phys.* **47** (1) (2003) 66–86.

[5] I. Kolář, P.W. Michor, J. Slovák: *Natural Operations in Differential Geometry*, (Springer–Verlag, N.Y., 1993).

[6] D. Krupka: Some Geometric Aspects of Variational Problems in Fibred Manifolds,*Folia Fac. Sci. Nat. UJEP Brunensis* **14**, J. E. Purkyně Univ. (Brno, 1973) 1–65.

[7] D. Krupka: Variational Sequences on Finite Order Jet Spaces, *Proc. Diff. Geom. and its Appl.* (Brno, 1989); J. Janyška, D. Krupka eds.; World Scientific (Singapore, 1990) 236–254.

[8] E. Noether: Invariante Variationsprobleme, *Nachr. Ges. Wiss. Gött., Math. Phys. Kl.* **II** (1918) 235–257.

[9] M. Palese, E. Winterroth: Covariant gauge-natural conservation laws, *Rep. Math. Phys.* **54** (3) (2004) 349–364; *erratum* in *arXiv:math-ph/0406009*.

[10] M. Palese, E. Winterroth: Global Generalized Bianchi Identities for Invariant Variational Problems on Gauge-natural Bundles, *Arch. Math. (Brno)* **41** (3) (2005) 289–310.

[11] M. Palese, E. Winterroth: The relation between the Jacobi morphism and the Hessian in gauge-natural field theories, *Theoret. Math. Phys.* **152** (2) (2007) 1191–1200.

[12] M. Palese, E. Winterroth: Noether identities in Einstein–Dirac theory and the Lie derivative of spinor fields, *Proc. Diff Geom. Appl., Olomouc (2007)*, World Scientific Publ., Hackensack, NJ, (2008) 643–653.

[13] M. Palese, E. Winterroth: Lagrangian reductive structures on gauge-natural bundles, *Rep. Math. Phys.* **62** (2) (2008) 229–239.

[14] D.J. Saunders: *The Geometry of Jet Bundles*, Cambridge Univ. Press (Cambridge, 1989).

[15] R. Vitolo: Finite Order Lagrangian Bicomplexes, *Math. Proc. Camb. Phil. Soc.* **125** (1) (1999) 321–333.