International Mathematical Forum, 5, 2010, no. 55, 2731 - 2750

Ideals with an Assigned Initial Ideals[1]

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Abstract

The stratum $\mathcal{S}t(J,\prec)$ (the homogeneous stratum $\mathcal{S}t_h(J,\prec)$ respectively) of a monomial ideal J in a polynomial ring R is the family of all (homogeneous) ideals of R whose initial ideal with respect to the term order $\prec$ is J. $\mathcal{S}t(J,\prec)$ and $\mathcal{S}t_h(J,\prec)$ have a natural structure of affine schemes. Moreover they are homogeneous w.r.t. a non-standard grading called level. This property allows us to draw consequences that are interesting from both a theoretical and a computational point of view. For instance a smooth stratum is always isomorphic to an affine space (Corollary 3.6). As applications, in §5 we prove that strata and homogeneous strata w.r.t. any term ordering $\prec$ of every saturated `Lex`-segment ideal J are smooth. For $\mathcal{S}t_h(J,\mathtt{Lex})$ we also give a formula for the dimension. In the same way in §6 we take in consideration any ideal $\mathcal{R}$ in $k[x_0,\dots,x_n]$ generated by a saturated `RevLex`-segment ideal in $k[x_0,x_1,x_2]$. We also prove that $\mathcal{S}t_h(\mathcal{R},\mathtt{RevLex})$ is smooth and give a formula for its dimension.

Mathematics Subject Classification: 13F20, 14C05

Keywords: Initial ideal, Hilbert scheme

1 Introduction

Let us consider a polynomial ring R over a field k, a term order $\prec$ on R and a monomial ideal $J \subset R$. In this paper we study the family $\mathcal{S}t(J,\prec)$ of all ideals in R having J as initial ideal with respect to $\prec$ in order to provide it with a structure of algebraic scheme, when possible. Sometimes, this family can be too huge to allowe such a structure (see Example 2.2). In order to avoid this situation we either consider the family $\mathcal{S}t_h(J,\prec))$ of homogeneous ideals

[1]Acknowledgements. Written with the support of the University Ministry funds.

or pose some extra condition on the term order: we say that $\prec$ is *reliable* if for every monomial $\mathbf{m}$ the set of monomials $\mathbf{n} \prec \mathbf{m}$ is finite. Under one of the previous conditions, the family of ideals I such that $\mathrm{In}(I) = J$ is in a natural way an affine scheme, that we call *stratum* (*homogeneous stratum* respectively) following the terminology introduced in [8], where homogeneous strata with respect to `RevLex` are related with a suitable stratification of Hilbert schemes.

There is a very natural way for explicitly building on the algebraic structure of strata, making use of Buchberger's algorithm for Gröbner bases. In this way the ideal J corresponds to the origin in both $\mathcal{S}t(J,\prec)$ and $\mathcal{S}t_h(J,\prec)$ and moreover the set-theoretic inclusion $\mathcal{S}t_h(J,\prec) \subset \mathcal{S}t(J,\prec)$ is an algebraic map, because $\mathcal{S}t_h(J,\prec)$ becomes the section of $\mathcal{S}t(J,\prec)$ with a suitable linear space.

A possible objection to our construction is that of giving rise to geometrical objects having a well defined support, but possibly different structures of affine schemes, because the procedure of reduction with respect to polynomials that are not a Gröbner basis is not unique. This is claimed for instance by Robbiano in [10, §3]. Even though this remark can look reasonable at first, however we want to emphasize that it is not correct and that strata have a unique and well defined structure of affine schemes that does not depend on the procedure used to obtain them and that they can be reducible and non-reduced. For a proof of the well definiteness of strata we refer to [5]. Reducible or non-reduced strata are presented in Example 2.7 and in §4.

There are no difficulties to give an implementation of our construction in order to obtain explicitly the ideal of a stratum in a suitable polynomial ring, that is to realize the stratum as a closed subscheme of an affine space $\mathbb{A}^N(k)$. However, this is computationally very heavy, because it requires to introduce a very big number of variables and the ideal of the stratum is in general given by a huge number of generators, so that even a basic study of strata, like for instance dimensions or singularities, requires times of calculation absolutely unsatisfactory or even not affordable from a practical point of view.

In this paper we present a refined method for the construction of strata, that allows to reduce in a sensible way the time of calculation and the amount of memory necessary to compute their equations. Moreover our method leads to some interesting theoretical consequences.

The main idea is the following: the stratum is given by an ideal $\mathcal{A}(J)$ in a suitable polynomial ring, which in general is not homogeneous w.r.t. the usual grading; however we can define a new grading, the *level*, such that $\mathcal{A}(J)$ turns out to be level-homogeneous.

Indeed, the levels allow us to prove the following result (Corollary 3.6):

J is a smooth point of $\mathcal{S}t(J,\prec) \iff \mathcal{S}t(J,\prec) \cong \mathbb{A}^s(k)$.

The same holds for $\mathcal{S}t_h(J,\prec)$. This fact is claimed (but not proved) in [8]. The fact that strata are level-homogeneous has a more general consequence:

every stratum can be isomorphically embedded in its Zariski tangent space at the origin, whose dimension is in fact the embedding dimension of the stratum. The level-homogeneity of strata is the key point of Procedure 3.7 that gives an algorithm for a simplified construction of a set of equations for the strata. A Maple12 implementation of Procedure 3.7 is available at:
`http://www2.dm.unito.it/paginepersonali/roggero/InitialIdeal(Maple)/`

In the last two sections, to the aim of illustrate the potential of levels, we present the direct computation of strata in two important families of ideals.

In §5 we consider the case of the saturated `Lex`-segment ideals J that are of the utmost importance in the theory of Hilbert schemes. As well known, every Hilbert scheme $\mathcal{H}\mathrm{ilb}^n_{p(z)}$ parameterizing subschemes in $\mathbb{P}^n$ contains one and only one point corresponding to such an ideal (see [6]). This point, the `Lex`-point, play a key role in most of the main general results on Hilbert schemes. For instance, in [9] it is proved that J belongs to only one irreducible component of $\mathcal{H}\mathrm{ilb}^n_{p(z)}$ (called `Lex`-component or the Reeves and Stillman component) and that it is a smooth point on it. By the universal properties of Hilbert schemes, the homogeneous stratum of J can be embedded as a locally closed subscheme of the `Lex`-component. In [8] it is claimed that the smoothness of $\mathcal{S}t_h(J, \mathtt{RevLex})$ at J can be deduced from that of the `Lex`-component. This argument clearly holds when $\mathcal{S}t_h(J, \mathtt{RevLex})$ corresponds to an open subset of $\mathcal{H}\mathrm{ilb}^n_{p(z)}$ and this is in general not true (see Remark 5.7).

However, we can give a direct proof of the smoothness of both strata and homogeneous strata w.r.t. any term ordering $\prec$ of every saturated `Lex`-segment ideal J. In particular we give a formula for the dimension of $\mathcal{S}t_h(J, \mathtt{Lex})$ and a condition for $\mathcal{S}t_h(J, \mathtt{Lex})$ to be an open subset of $\mathcal{H}\mathrm{ilb}^n_{p(z)}$.

Using the same technique, in §6 we consider any ideal $\mathcal{R}$ in $k[x_0, \dots, x_n]$ generated by a saturated `RevLex`-segment ideal in $k[x_0, x_1, x_2]$. If $n = 2$, these ideals $\mathcal{R}$ are the generic initial ideals w.r.t. `RevLex` of the ideals of sets of general points in $\mathbb{P}^2$. For $n \geq 3$ the ideals $\mathcal{R}$ defines arithmetically Cohen-Macaulay subschemes of codimension 2 in $\mathbb{P}^n$. Again, we are able to prove that the homogeneous strata $\mathcal{S}t_h(\mathcal{R}, \mathtt{RevLex})$ are smooth, hence isomorphic to affine spaces, and to give a formula for their dimension.

2 General settings and definition of strata

We shall denote by X the set of variables $x_1, \dots, x_n$ and by $k[X]$ the polynomial ring $k[x_1, \dots, x_n]$ over a field k. In the same way, C will be the set of variables $c_{i,\alpha}$ (that will be introduced in the next section) and X, C the union of the two sets of variables. $\mathbb{T}_X$ will be the semigroup of monomials in the variables X; analogous meanings will have $\mathbb{T}_C$ and $\mathbb{T}_{X,C}$. A monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \mathbb{T}_X$ will be written as X^α where $\alpha = [\alpha_1, \dots, \alpha_n]$ is the ordered list of exponents.

If $\prec$ is any term ordering on $\mathbb{T}_X$, we will also denote by $\prec$ the induced total ordering on $\mathbb{Z}^n$; in this way $(\mathbb{Z}^n, +, \prec)$ is an ordered group.

Definition 2.1. *We shall call* tail *and* homogeneous tail *of* $X^\alpha \in \mathbb{T}_X$ *w.r.t.* $\prec$*, the sets* $\mathrm{tail}(X^\alpha) = \{X^\beta \in \mathbb{T}_X / \beta \prec \alpha\}$ *and* $\mathrm{htail}(X^\alpha) = \{X^\beta \in \mathbb{T}_X / \beta \prec \alpha$ *and* $|\beta| = |\alpha|\}$ *respectively. If* τ *is any subset of* $k[X]$, $\mathrm{tail}_\tau(X^\alpha)$ *and* $\mathrm{htail}_\tau(X^\alpha)$ *will be* $\mathrm{tail}(X^\alpha) \setminus \tau$ *and* $\mathrm{htail}(X^\alpha) \setminus \tau$ *respectively .*

Of course, $\mathrm{htail}(X^\alpha)$ is a finite set, while $\mathrm{tail}(X^\alpha)$ may have infinitely many elements.

Example 2.2. *Consider* $k[x, y]$ *equipped by the term order* `Lex` *with* $y \prec x$ *and the monomial ideal* $J = (x)$*. The family of ideals* I *having* J *as initial ideal contains all ideals of the form* $(x + g(y))$ *with* g *varying in* $k[y]$ *and therefore it depends on infinitely many free parameters.*

A *reliable term order* $\prec$ in $\mathbb{T}_X$ is a term order such that every monomial has a finite tail. For instance, every graded term order is reliable. More generally, if $\prec$ is defined using a matrix with integer entries M, we can check if it is reliable through the criterion given by following :

Lemma 2.3. $\prec$ *is reliable* $\Longleftrightarrow$ *the first row of* M *has strictly positive entries.*

Proof. If the first row $[a_1, \ldots, a_n]$ has strictly positive entries, $X^\beta \prec X^\alpha$ only if $\sum \beta_i a_i \leq \sum \alpha_i a_i$, where $\alpha_i a_i \geq 0$, hence each β_i is upper bounded by a/a_i. On the converse, if we have $a_i = 0$ for some i, and of course also $a_j > 0$ for some j, then $\mathrm{tail}(x_j)$ contains all the monomials x_i^r for every $r > 0$. $\square$

When general tails are concerned, we will always assume that the term ordering is reliable.

If F is a polynomial, $\mathrm{LM}_\prec(F)$, or for short $\mathrm{LM}(F)$, is the leading monomial of F with respect to $\prec$. If I is an ideal, $\mathrm{In}_\prec(I)$, or for short $\mathrm{In}(I)$, will be the initial ideal of I w.r.t. $\prec$ and B_I will be its reduced Gröbner basis. If I is monomial, then of course $\mathrm{In}(I) = I$ and B_I is its monomial basis.

In this paper we take into consideration some families of ideals I in $k[X]$ having the same initial ideal J. If $B_J = \{X^{\alpha_1}, \ldots, X^{\alpha_s}\}$, the reduced Gröbner basis B_I of every such ideal I is formed by s polynomials of the type $X^{\alpha_i} + \sum a_{i\beta} X^\beta$, where $a_{i\beta} \in k$ and $X^\beta \in \mathrm{tail}_J(X^{\alpha_i})$. Let us consider the polynomials

$$F_i = X^{\alpha_i} + \sum c_{i\alpha} X^\beta \tag{1}$$

of the previous type, but whose coefficients $c_{i\beta}$ are new variables and let $\mathcal{I}$ be the ideal in $k[X, C]$ generated by the set $\mathcal{B} = \{F_1, \ldots, F_s\}$. The reduced Gröbner basis of I can be obtained from $\mathcal{B}$ by specializing in k the parameters

C. However, not every specialization of $\mathcal{B}$ returns a Gröbner basis and so an ideal I with $\mathrm{In}(I) = J$. In order to characterize the good specializations, we consider the following construction.

Let us fix any term order $\prec_1$ on $\mathbb{T}_C$ and let $\prec'$ the term order on $\mathbb{T}_{X,C}$ which is an elimination order for the variables X and coincides with $\prec$ and $\prec_1$ on $\mathbb{T}_X$ and $\mathbb{T}_C$ respectively. Moreover let us consider some reductions H_{ij} by $\mathcal{B}$ of the S-polynomials of each pair of elements in $\mathcal{B}$: notice that this procedure commutes with the specialization in k of the variables C, because the leading monomials of the F_i's are monomials in the variables X. A specialization of $\mathcal{B}$ is a Gröbner basis if and only if all the H_{ij} specialize to zero. This construction not only gives a characterization of all the ideals I such that $\mathrm{In}(I) = J$, but also leads us to obtain this family as the set of closed points of an affine subscheme of the affine space $\mathbb{A}^N$ over the field k, where $N = \sum_i |\mathrm{tail}_J(X^{\alpha_i})|$.

Definition 2.4. *Let $J \subset k[X]$ be a monomial ideal, $B_J = \{X^{\alpha_1}, \dots, X^{\alpha_s}\}$ its basis. We will call* stratum of J *w.r.t.* $\prec$ *the affine subscheme $\mathcal{S}t(J, \prec)$ of $\mathbb{A}^N$ defined by the ideal $\mathcal{A}(J, \prec)$ in $k[C]$ generated by all the coefficients in $k[C]$ of monomials in the variables X appearing in a set of polynomials $\{H_{ij},\ 1 \le i < j \le s\}$. The* homogeneous stratum $\mathcal{S}t_h(J, \prec)$ *is the affine scheme obtained in the same way but using homogeneous tails instead of tails.*

More generally, if $\tau = [\tau_1, \dots, \tau_s]$ is any list of s sets of monomials, we will denote by $\mathcal{S}t(J/\tau, \prec)$ and by $\mathcal{A}(J/\tau, \prec)$ the subscheme and its ideal obtained as before substituting, for every $i = 1, \dots, s$, $\mathrm{tail}_{\tau_i \cup J}(X^{\alpha_i})$ to $\mathrm{tail}_J(X^{\alpha_i})$. When the term order $\prec$ is clearly fixed, we will omit it in the symbols and write simply $\mathcal{S}t(J)$, $\mathcal{A}(J)$ and so on.

It is easy to see that $\mathcal{S}t(J/\tau)$ is the scheme-theoretical intersection of $\mathcal{S}t(J)$ with the linear space defined by $c_{i\beta} = 0$, where for every $i = 1, \dots, s$ X^β belongs to $\mathrm{tail}_J(X^{\alpha_i}) \cap \tau_i$. Of course all these objects are non-empty, because they contain at least a closed point, the origin, corresponding to the trivial case $I = J$.

Example 2.5. *Let J and J' be monomial ideals in $k[X]$ and $\prec$ any term ordering. We can consider the subscheme of $\mathcal{S}t(J)$ whose closed points correspond to ideals $I \in \mathcal{S}t(J)$ such that their reduced Gröbner bases are also reduced modulo J'. Of course this subscheme is empty if some monomial in J' divides a monomial in the basis of J. If this does not happen, this subscheme is $\mathcal{S}t(J/\tau)$, where $\tau_i = J'$ for every i. We will denote it by $\mathcal{S}t(J/J')$.*

Example 2.6. *Let $B_J = \{X^{\alpha_1}, \dots, X^{\alpha_s}\}$ be the basis of the monomial ideal J in $k[X]$. If τ is the list of subsets $\tau_i = \oplus_{i \ne |\alpha_i|} k[X]_i$, then $\mathcal{S}t(J/\tau) = \mathcal{S}t_h(J)$.*

We do not present at this point many explicit examples of strata, because in most non-trivial cases the ideal $\mathcal{A}(J/\tau)$ lives in the polynomial ring $k[C]$

which has in general “a very big” number of variables. In the next section we will see how to obtain a new ideal $\tilde{\mathcal{A}}(J/\tau)$ for $\mathcal{S}t(J/\tau)$ in a polynomial ring $k[C']$, where C' is the smallest possible subset of C such that $\mathcal{S}t(J/\tau)$ can be embedded in an affine space of dimension $|C'|$. However we prefer to present here at least a couple of (necessarily trivial) examples in order to show that strata can in fact be reducible and non-reduced schemes.

Example 2.7. *Let $J = (xy, y) \subset R = k[x, y]$ and let $\tau = [R \setminus \{x\}, R]$ and $\tau' = [R\setminus\{x\}, R\setminus\{y\}]$. In the first case $\mathcal{B} = (xy+cx, y^2)$, the only S-polynomial is $y(xy + cx) - x(y^2) = cxy$ that reduces to $H_{12} = -c^2x$. Hence $\mathcal{S}t(J/\tau)$ is a non-reduced subscheme of $\mathbb{A}^1$ because it is defined by the ideal $(c^2) \subset k[c]$.*

In the second case $\mathcal{B}' = (xy + c_1x, y^2 + c_2y)$, the only S-polynomial is $y(xy + c_1x) - x(y^2 + c_2y) = (c_1 - c_2)xy$ that reduces to $H_{12} = -c_1(c_1 - c_2)x$. Hence $\mathcal{S}t(J/\tau')$ is a reducible subscheme of $\mathbb{A}^2$ because it is defined by the ideal $(c_1(c_1 - c_2)) \subset k[c_1, c_2]$.

3 Strata are homogeneous varieties

In this section we fix a term ordering $\prec$ on $k[X]$ and a monomial ideal $J \subset k[X]$ with $n = |X|$ and $B_J = \{X^{\alpha_1}, \dots, X^{\alpha_1}\}$. What we are going to prove in this section holds true for every stratum $\mathcal{S}t(J, \prec)$, $\mathcal{S}t_h(J, \prec)$ or more generally $\mathcal{S}t(J/\tau, \prec)$ and any τ. Then we will simply denote by $\mathcal{S}t(J)$ either one of them and by $\mathcal{A}(J)$ its ideal.

Definition 3.1. *We shall denote by Λ the grading of either $k[X, C]$ or $k[C]$ over the totally ordered group $(\mathbb{Z}^n, +, \prec)$ given by $\Lambda(X^\alpha) = \alpha$ and $\Lambda(c_{i\beta}) = \alpha_i - \beta$. We shall call $\Lambda(c_{i\beta})$ the* level *of $c_{i\beta}$.*

As we shall use also the usual grading where all variables have degree 1, we will always explicit the symbol Λ when the above defined grading is concerned (so, Λ-degree λ with $\lambda \in \mathbb{Z}^n$, Λ-homogeneous of degree λ, Λ- component of degree λ etc.) and leave the simple term when the usual grading is involved (so, degree r with $r \in \mathbb{Z}$, homogeneous of degree r, component of degree r, etc.).

Lemma 3.2. *In the above settings:*

i) Λ is a positive grading, that is for every $X^\alpha C^\gamma \in \mathbb{T}_{X,C}$, $\Lambda(X^\alpha C^\gamma) \succ \Lambda(1)$.

ii) the ideal $\mathcal{A}(J)$ of the stratum $\mathcal{S}t(J)$ is Λ-homogeneous.

Proof. The first item immediately follows from the definitions. In fact $\Lambda(x_j) \succ \Lambda(1)$ because $\prec$ is a term ordering and $\Lambda(c_{i\beta}) \succ \Lambda(1)$ because $X^\beta \in \mathrm{tail}(X^{\alpha_i})$ and so $X^{\alpha_i} \succ X^\beta$.

For the second item, it is sufficient to observe that the function Λ is, by definition, constant over all the monomials of $\mathbb{T}_{X,C}$ used in the polynomials F_i that we use in the construction of $\mathcal{A}(J)$. Hence the polynomials F_i, their S-polynomials H_{ij} and their reductions are Λ-homogeneous. Finally if we collect a Λ-homogeneous polynomial in $k[X, C]$ with respect to the monomials of $\mathbb{T}_X$, the coefficients are Λ-homogeneous polynomials in $k[C]$. $\square$

We now present some properties of Λ-homogeneous ideals of a polynomial ring graded over an ordered group $(\mathbb{Z}^n, +, \prec)$. As we will apply these properties to any ideal $\mathcal{A} \subseteq k[C]$, we denote by C the set of variables, but for sake of simplicity $C = \{c_1, \dots, c_s\}$ (instead of $C = \{c_{i\beta}, i = 1, \dots, n, \beta \in \mathrm{tail}(X^{\alpha_i})\}$).

If $F \in k[C]$, $L(F)$ will denote the linear component of F, i.e. the sum of monomials of (usual) degree 1 that appear in F. In the same way if $\mathcal{A} \subset k[C]$ is an ideal $L(\mathcal{A}) = \langle L(F)/\ F \in \mathcal{A}\rangle$ denotes the k-vector space of the linear components of elements in $\mathcal{A}$. Obviously, if F and $\mathcal{A}$ are Λ-homogeneous, then also $L(F)$ and $L(\mathcal{A})$ are. If moreover $\mathcal{A} \neq k[C]$, then no element in $\mathcal{A}$ has a non-zero constant term.

Definition 3.3. *Let $\mathcal{A}$ a Λ-homogeneous ideal in $k[C]$ and let C' be a subset of C and $C'' = C \setminus C'$.*

i) *$\mathcal{A}/C'$ is the image of $\mathcal{A}$ in the quotient ring $k[C]/(C') \cong k[C \setminus C']$; we will say that $\mathcal{A}/C'$ is obtained* specializing *C' to 0.*

ii) *C' is a* set of eliminable variables *if, for every $c' \in C'$, $L(\mathcal{A})$ contains elements of the type $c' + l_{c'}$, with $f_{c'} \in k[C'']$. C' is a* maximal set of eliminable variables *if $|C'| = \dim_k(L(\mathcal{A}))$.*

iv) *for a set C' of eliminable variables, $\mathcal{A} \setminus C'$ denotes the ideal $\mathcal{A} \cap k[C \setminus C'] \subset k[C \setminus C']$; we will say that $\mathcal{A} \setminus C'$ is obtained* eliminating *C'.*

v) the embedding dimension *$ed(\mathcal{A})$ of $\mathcal{A}$ is the difference $s - \dim(L(\mathcal{A}))$ that is the cardinality of the complementary $C'' = C \setminus C'$ of a maximal set of eliminable variables.*

vi) *the* dimension *$\dim(\mathcal{A})$ of $\mathcal{A}$ is the Krull dimension of the quotient ring $k[C]/\mathcal{A}$.*

If $c \in L(\mathcal{A}) \cap C$, then it belongs to every maximal set of eliminable variables. If c does not appear in any element of $L(\mathcal{A})$, then it belongs to the complementary of every set of eliminable variables. Finally if c does not belong to $L(\mathcal{A})$ but appears in some elements of it, then we can find a maximal set of eliminable variables containing c and another one not containing c.

Proposition 3.4. *Let $\mathcal{A} \subsetneq k[C]$ be a Λ-homogeneous ideal and let $C' = \{c'_1, \dots, c'_r\}$ a set of eliminable variables for $\mathcal{A}$ and $C'' = C \setminus C'$. Then $\mathcal{A}$ has a set of generators of the following form:*

$$\mathcal{A} = (c'_1 + g_1, \dots, c'_r + g_r, f_1, \dots, f_d) \tag{2}$$

with $g_i, f_j \in k[C'']$, and there is a natural isomorphism:

$$k[C]/\mathcal{A} \cong k[C'']/(\mathcal{A} \cap k[C'']) = k[C'']/(f_1, \dots, f_d). \tag{3}$$

If moreover $r = dim(L(\mathcal{A}))$, then $f_j \in (C'')^2$.

A set of generators (2) can be obtained as the reduced Gröbner basis of $\mathcal{A}$ with respect to any elimination order of the variables C'.

Proof. We can assume, up to a permutation of indexes, that $C' = \{c_1, \dots, c_r\}$ and that $\Lambda(c_1) \preceq \Lambda(c_2) \preceq \cdots \preceq \Lambda(c_r)$. Now we proceed recursively. First of all let us choose a homogeneous polynomial $c_1 + h_1, \in \mathcal{A}$ whose linear forms are $c_1 + l_1 \in L(\mathcal{A})$ with $l_1 \in k[C'']$ and observe that $h_1 \in k[C'']$: in fact neither c_1, nor the other variables $c_i \in C'$ can divide monomials of degree > 1 that appear in h_1, because all the monomials in h_1 have the same Λ-degree. So, we set $g_1 = h_1$ and apply the same argument to the ideal $\mathcal{A} \cap k[C \setminus \{c_1\}]$ in $k[C \setminus \{c_1\}]$. In this way we obtain the $c_i + g_i$'s. Note that $k[C]/\mathcal{A} \cong k[C \setminus \{c_1\}]/\mathcal{A} \cap k[C \setminus \{c_1\}]$.

We can complete them to a Λ-homogeneous set of generators for $\mathcal{A}$: $c_1 + g_1, \dots, c_r + g_r, h_1, \dots, h_d$ and eliminate $c_1, \dots, c_s$ in the polynomials h_j obtaining the $f_1, \dots, f_d$.

Finally let $\prec'$ an elimination order of the variables $c_1, \dots, c_r$ in $\mathbb{T}_C$. It is quite evident that Buchberger algorithm applied to the set of generators given in (2) only changes (if necessary) $f_1, \dots, f_d$ with a Gröbner basis of the ideal they generate $\mathcal{A} \cap k[C'']$ of $k[C'']$. $\square$

In the following it will be useful the following technical result.

Lemma 3.5. *Let $\mathcal{A} \subset k[C]$ be a Λ-homogeneous ideal and C' a subset of C with cardinality s. Then:*

i) $\dim(\mathcal{A}) - s \leq \dim(\mathcal{A}/C') \leq \dim(\mathcal{A})$ *and* $\mathrm{ed}(\mathcal{A}) - s \leq \mathrm{ed}(\mathcal{A}/C') \leq \mathrm{ed}(\mathcal{A})$.

ii) If C' is a set of eliminable variables, then $\dim(\mathcal{A} \setminus C') = \dim(\mathcal{A})$ *and* $\mathrm{ed}(\mathcal{A} \setminus C') = \mathrm{ed}(\mathcal{A})$.

iii) If no element of C' appears in $L(\mathcal{A})$, then $\mathrm{ed}(\mathcal{A}/C') = \mathrm{ed}(\mathcal{A}) - s$.

Proof. The statements follow from well known facts in linear and commutative algebra and from what proved in Proposition 3.4. $\square$

We can now apply the above results to the ideal $\mathcal{A}$ of the stratum $\mathcal{S}t(J)$.

Corollary 3.6. *Either the stratum $\mathcal{S}t(J)$ of a monomial ideal J is (isomorphic to) an affine space or the origin, that is the point corresponding to J, is singular.*

Moreover $\mathcal{S}t(J) \cong \mathbb{A}^d$ if and only if $d = \mathrm{ed}(\mathcal{A}(J)) = \dim(\mathcal{A}(J))$.

Proof. If C'' is a maximal set of s eliminable variables for $\mathcal{A}(J)$, then $\mathcal{S}t(J)$ can also be seen as the closed subscheme in $\mathbb{A}^d$ defined by the ideal $\tilde{\mathcal{A}}(J) = \mathcal{A}(J) \cap k[C \setminus C']$, where $d = |C| - s$. If $\tilde{\mathcal{A}}(J) = (0)$, then $\mathcal{S}t(J)$ is the affine space $\mathbb{A}^d$, while if $\tilde{\mathcal{A}}(J) \neq (0)$, then the origin is a singular point for $\mathcal{S}t(J)$ as a subscheme of $\mathbb{A}^d$, because the tangent cone is not contained in any hyperplane. □

Using the properties that we have just proved, we are now able to sketch an effective and easily implementable procedure to obtain a set of generators for the ideal $\mathcal{A}(J/\tau)$ that defines the stratum $\mathcal{S}t(J/\tau)$ of a monomial ideal $J \subset k[X]$ with respect to a reliable term order $\prec$.

Procedure 3.7. *Let J be a monomial ideal in $k[X]$ with monomial basis $B_J = \{X^{\alpha_1}, \dots, X^{\alpha_r}\}$ and let τ be a list of subsets of $k[X]$.*

- *construct the polynomials $\{F_1, \dots, F_s\}$ by appending to each monomial X^{α_i} in B_J a linear combination of the monomials in $\mathrm{tail}_{\tau\cup J}(X^{\alpha_i})$ where the coefficients C are variables;*
- *for each pair of indexes i, j consider the S-polynomial S_{ij} of F_i and F_j and reduce it completely to $\tilde{H}_{ij}$ w.r.t. B_J;*
- *write the polynomials $\tilde{H}_{ij}$'s collecting monomials in the variables X and extract their coefficients in $k[C]$: they are a set of generators for $L(\mathcal{A}(J/\tau))$;*
- *choose a set C' of $\dim(L(\mathcal{A}(J/\tau)))$ eliminable parameters (i.e. a set of generators for the initial ideal of $L(\mathcal{A}(J/\tau))$ with respect to any term ordering);*
- *reduce S_{ij} to H_{ij} w.r.t. $\{F_1, \dots, F_s\}$ and to a term order in $\mathbb{T}_{X,C}$ which is an elimination order of the variables X and is given by $\prec$ on $\mathbb{T}_X$;*
- *write the polynomials H_{ij} collecting monomials in the variables X and extract their coefficients in $k[C]$: they generate the ideal $\mathcal{A}(J/\tau)$.*
- *Finally: in order to obtain an ideal defining $\mathcal{S}t(J/\tau)$ as an subscheme of the affine space of minimal dimension $\mathbb{A}^N$ where $N = \mathrm{ed}(\mathcal{A}(J/\tau))$, compute $\mathcal{A}(J/\tau) \cap k[C \setminus C']$ using an elimination term order of the C'.*

Remark 3.8. *Thanks to a well known property of Gröbner bases, we can obtain the ideal of $\mathcal{S}t(J/\tau)$ only considering the S-polynomials H_{ij} corresponding to a set of generators for the first syzygies of B_J (see for instance [3], Theorem 9, page 104).*

Even in quite simple cases, the number of parameters C that appear in the F_i is in general very high and furthermore the ideal $\mathcal{A}(J)$ requires a big amount of generators. For these reasons it is not simple to compute a Gröbner basis for $\mathcal{A}(J)$ with respect to any term order. The main advantage of our procedure is that it enables us to know in advance a maximal set of eliminable parameters, only requiring some linear algebra computations so that we can substantially reduce the number of variables involved in the computations.

In fact, if $f \in k[C]$ is the coefficient in H_{ij} of a monomial $X^\beta \in \mathbb{T}_X$, then every monomial in f has level $\lambda = \Lambda(\mathrm{lcm}(X^{\alpha_i}, X^{\alpha_j})) - \Lambda(X^\beta)$ and every coefficient $c_{i\beta}$ that appears in it has level lower or equal to λ. Then we can obtain a set of generators for $\mathcal{A}(J)$ "level by level"(starting from the lowest one), after specialization to 0 of every parameter of level higher than λ.

In the following sections we will see both explicit examples and theoretical applications of Procedure 3.7.

4 Examples

The explicit computations of strata presented in the following examples have been performed using the Maple12 procedure available at:
`http://www2.dm.unito.it/paginepersonali/roggero/InitialIdeal(Maple)/`

As already observed, we can think of $\mathcal{S}t(J/\tau)$ as a section of $\mathcal{S}t(J)$ with a linear subspace. The following example shows that $\mathcal{S}t(J/\tau)$ can be singular at the origin, even if $\mathcal{S}t(J)$ is isomorphic to an affine space and vice versa.

Example 4.1. *Let us consider the ideals $J = (x^2y, xy^2)$, $J' = (y^3, z^3) \subset k[x, y, z, t]$, and the term order* `RevLex` *with $x \succ y \succ z \succ t$. Explicit computations show that $\mathcal{S}t(J)$ of J is smooth and more precisely is isomorphic to the affine space $\mathbb{A}^{12}$. If we cut $\mathcal{S}t(J)$ with the codimension 4 linear space given by $c_{1,y^3} = c_{1,z^3} = c_{2,y^3} = c_{2,z^3} = 0$, we obtain the stratum $\mathcal{S}t(J/J')$. Again by an explicit computation one can see that $\mathcal{S}t(J/J')$ has two irreducible components, one of dimension 10 and one of dimension 9. The elimination of all the eliminable variables gives an embedding of $\mathcal{S}t(J/J') \subset \mathbb{A}^{11}$ such that the two components become respectively a quadric hypersurface and a codimension 2 linear space, both through the origin.*

Example 4.2. *Let $J = (x^2, xy, xz, y^2)$ be the ideal in $k[x, y, z, t]$ with the term order* `Revlex` *such that $x \succ y \succ z \succ t$. The homogeneous stratum $\mathcal{S}t_h(J)$ is a reducible variety with a component of dimension 11 and a component of dimension 8 both containing the origin. If $J' = (t)$, we obtain $\mathcal{S}t_h(J/J') \cong \mathbb{A}^4$.*

Observe that in a natural way $\mathcal{S}t_h(J) \cong \mathcal{S}t(J_1)$ and $\mathcal{S}t_h(J/J') \cong \mathcal{S}t_h(J_1)$ where $J_1 = (x^2, xy, xz, y^2)$ is an ideal in $k[x, y, z]$. Then this same example also shows that the homogeneous stratum can be smooth, while the stratum is not.

The last example shows that in general, but not always, the homogeneous stratum $\mathcal{S}t(J, \prec)$ depends on the ideal J itself and not only on the projective scheme it defines.

Example 4.3. *Let us consider in the polynomial ring $R = k[x, y, z, t]$ with $x \succ y \succ z \succ t$ the ideal:*

$$J = (x^3, x^2y, x^2z, x^2t, xy^2, xyz, xyt, xz^2, xzt, xt^2, y^3, y^2z, y^2t, yz^2, yzt, yt^2, z^3)$$

and its saturation $J_{sat} = (x, y, z^3)$. Of course the projective schemes $\mathrm{Proj}\,(R/J)$ *and* $\mathrm{Proj}\,(R/J_{sat})$ *are canonically isomorphic, while the homogeneous strata of the two ideals may not be. In fact, if we fix the term order* `Lex` *and compute the strata, we obtain $\mathcal{S}t_h(J, \texttt{Lex}) \cong \mathbb{A}^9$, while $\mathcal{S}t_h(J_{sat}, \texttt{Lex}) \cong \mathbb{A}^7$.*

On the other hand, if we choose the term order `RevLex`*, the homogeneous strata of J and of J_{sat} are both isomorphic to $\mathbb{A}^7$.*

5 Strata of `Lex`-segment ideals

In this section we consider any homogeneous saturated segment ideal with respect to the lexicographic term order `Lex`. Now $R = k[x_0, \dots, x_n]$ and we fix the order on the variables $x_0 \succ x_1 \succ \dots \succ x_n$. Every saturated monomial ideal J which is a segment with respect to `Lex` is completely determined by the monomial in its monomial basis which is minimal with respect to $\prec_{\texttt{Lex}}$: if $\mathbf{m} = x_0^{a_n} x_1^{a_{n-1}} \cdots x_{n-2}^{a_2} x_{n-1}^{a_1}$ is such a monomial, and $r = \sum a_i$, then J_r is the vector space generated by all the monomials in degree s that are `Lex`-higher than $\mathbf{m}$ and J is the saturation of the ideal generated by J_r. Following [9] we then identify every such monomial with the list of exponents $(a_n, \dots, a_1)$ (note that our use of indexes is slightly different from that of [9]). More generally:

Definition 5.1. *Let us consider an integer $q \le n$ and any sequence of q non negative integers $(a_q, ..., a_1)$ with $\sum a_i \ne 0$. Let $s \ge 0$ be the minimal index such that $a_{s+1} \ne 0$. We will denote by $J(a_q, \dots, a_1)$ the ideal in R generated by the set of $q - s$ monomials $\mathbf{m}_1, \dots, \mathbf{m}_{q-s}$ where:*

$$\mathbf{m}_i = x_{n-i} \cdot \prod_{j=i}^{q} x_{n-j}^{a_j} \quad \text{if} \quad 1 \le i < q - s \quad \text{and} \quad \mathbf{m}_{q-s} = \prod_{j=s+1}^{q} x_{n-j}^{a_j} \qquad (4)$$

Example 5.2. *For $q = n$, $J(0, \dots, 0, a_1)$ is the ideal $(x_1, x_2, \dots, x_{n-2}, x_{n-1}^{a_1})$ defining a non-reduced structure of length a_1 over the origin in $\mathbb{P}^n$. For a general q, $J(0, \dots, 0, a_1) = (x_{n-q}, \dots, x_{n-2}, x_{n-1}^{a_1})$ defines a non reduced structure over a linear space of codimension q.*

For every $q > 1$, $J(a_q, 0, \dots, 0) = (x_{n-q}^{a_q})$ defines the hypersurface in $\mathbb{P}^n$ corresponding to the divisor a_qH where H is the hyperplane $x_{n-q} = 0$.

In [9] it is proved that very saturated lex-segment ideal J of R is of the type $J(a_n, ..., a_1)$ and that it defines a subschemes of $\mathbb{P}^n$ which corresponds to smooth points in the Hilbert scheme $\mathcal{H}\mathrm{ilb}^n_{p(z)}$, where $p(z)$ is the Hilbert polynomial of R/J. By the universal property of Hilbert schemes, $\mathcal{S}t_h(J)$ can be embedded as a locally closed subscheme in $\mathcal{H}\mathrm{ilb}^n_{p(z)}$. In [8] it is claimed that the smoothness of $\mathcal{S}t_h(J)$ can be deduced from that of the Hilbert scheme containing it. This argument clearly holds when $\mathcal{S}t_h(J)$ corresponds to an open subset of $\mathcal{H}\mathrm{ilb}^n_{p(z)}$. However, it is not difficult to verify that $\mathcal{S}t_h(J)$ has in general a strictly lower dimension than the component of the Hilbert scheme containing it, hence it cannot correspond to an open subset of $\mathcal{H}\mathrm{ilb}^n_{p(z)}$ (see Remark 5.7 or, for an explicit computation in a special case, Example 4.3).

The two ideals $J = J(a_n, ..., a_1)$ and $I = J(a_n, ..., a_1)_{\geq r}$ (where $r = \sum a_i$) define the same point in the corresponding Hilbert scheme. In [5] it is proved that the homogeneous stratum $\mathcal{S}t_h(I, \mathtt{Lex})$ is scheme-theoretically isomorphic to an open subset of the Hilbert scheme and that it is also isomorphic to an affine space $\mathbb{A}^d$ of a suitable dimension $d = d(a_n, ..., a_1)$. On the other hand, the homogeneous stratum $\mathcal{S}t_h(J, \prec)$, with respect to any term ordering $\prec$, is the locally closed subscheme of $\mathcal{S}t_h(I, \mathtt{Lex})$ cut by a suitable linear space, because it can be realized as a stratum of the type $\mathcal{S}t_h(I/\tau)$. Even if $\mathcal{S}t_h(I, \prec)$ is smooth and so isomorphic to an affine space, nothing can be said in general about the smoothness of a stratum of the type $\mathcal{S}t_h(I/\tau)$, because it can be singular or even reducible and non-reduced (see Example 2.5).

However, in the case of the ideals $J(a_q, ..., a_1)$ we will prove that their strata w.r.t. any reliable term ordering and their homogeneous strata w.r.t. any term ordering are always isomorphic to affine spaces. As a consequence this holds for the strata of any lexicographic saturated ideal. If the fixed term ordering is $\mathtt{Lex}$, we also give a formula for the dimension of $\mathcal{S}t_h(J(a_n, ..., a_1), \mathtt{Lex})$.

The starting point of our proof is the nice "inductive structure" of the ideals of the type $J(a_q, ..., a_1)$, underlined in [9]:

$$J(a_q, ..., a_1) = x_{n-q}^{a_q} \cdot ((x_{n-q}) + J(a_{q-1}, ..., a_1)) \tag{5}$$

More generally, we will consider ideals having a structure of the type (5) and study their strata. For the meaning of $\mathcal{S}t(J_0/(\mathbf{n}_0))$ we refer to Example 2.5.

Theorem 5.3. *Let X, Y be two disjoint sets of variables and let $\prec$ be any reliable term order in $\mathbb{T}_{X,Y}$. Let us consider in $k[X, Y]$ a monomial ideal J_0 with basis $\mathbf{n}_1, \ldots, \mathbf{n}_r \in \mathbb{T}_X$, two monomials $\mathbf{m}, \mathbf{n}_0 \in \mathbb{T}_Y$ with and the monomial ideals $J_1 = ((\mathbf{n}_0) + J_1)$. If $\mathcal{S}t(J_0/(\mathbf{n}_0)) \cong \mathbb{A}^{N_0}$ and either $\mathbf{n}_0$ has degree 1 or $\prec$ is an elimination order of the set of variables Y, then the stratum of the ideal:*

$$J = \mathbf{m}((\mathbf{n}_0) + J_0)$$

is isomorphic to an affine space. More precisely:

$$\mathcal{S}t(J) \cong \mathbb{A}^{N_0+N_1+N_2}$$

where $N_1 = |\mathrm{tail}(\mathbf{m})|$ *and* $N_2 = |\mathrm{tail}(\mathbf{n}_0) \setminus J_0|$.

The same holds for $\mathcal{S}t_h(J)$ *assuming that* $\mathcal{S}t_h(J_0/(\mathbf{n}_0)) \cong \mathbb{A}^{N_0}$, $N_1 = |\mathrm{htail}(\mathbf{m})|$, $N_2 = |\mathrm{htail}(\mathbf{n}_0) \setminus J_0|$.

Proof. We prove the statement only for what concerns the stratum, the homogeneous case being analogous.

In order to compute the stratum of J_1 (and respectively the stratum of J) we follow Procedure 3.7 starting from polynomials $G_0, G_1 \dots, G_r$ (respectively $F_0, F_1, \dots, F_r$) as in (1). If:

$$G_0 = \mathbf{n}_0 + \sum D_s \mathbf{v}_{0s}$$
$$G_i = \mathbf{n}_i + \sum C_{is} \mathbf{v}_{is} \quad \text{if} \quad i \geq 1$$

where $\mathbf{v}_{is} \in \mathrm{tail}_{J_1}(\mathbf{n}_i)$, then we can write the F_i collecting first all the monomials multiple of $\mathbf{m}$ and, among the remaining ones, all the monomials multiple of $\mathbf{n}_i$, in the following way:

$$F_0 = \mathbf{m}G_0 + \mathbf{n}_0 \left(\sum C'_{0h}\mathbf{p}_h\right) + \sum C''_{0t}\mathbf{q}_{0t}$$
$$F_i = \mathbf{m}G_i + \mathbf{n}_i \left(\sum C'_{ih}\mathbf{p}_h\right) + \sum C''_{it}\mathbf{q}_{it} \quad \text{if} \quad i \geq 1$$

where $\mathbf{p}_h \in \mathrm{tail}(\mathbf{m})$ (if $i = 0$ also $\mathbf{n}_0\mathbf{p}_h \notin (\mathbf{m})$) and $\mathbf{q}_{it} \in \mathrm{tail}(\mathbf{m}\mathbf{n}_i) \setminus (\mathbf{n}_i, \mathbf{m})$.

First of all we observe that there is a $1-1$ set-theoretic correspondence between $\mathcal{S}t(J)$ and the product $\mathcal{S}t(J_0/\mathbf{n}_0) \times \mathcal{S}t((\mathbf{m})) \times \mathcal{S}t((\mathbf{n}_0)/J_0) \cong \mathbb{A}^{N_0} \times \mathbb{A}^{N_1} \times \mathbb{A}^{N_2}$. If $I \in \mathcal{S}t(J)$, that is if $\mathrm{In}(I) = J$, then I and J share the same Hilbert polynomial and then $\mathrm{ht}(I) = \mathrm{ht}(J) = 1$. If $I = F \cdot I_1$ and $\mathrm{ht}(I_1) \geq 2$, then $\mathrm{LM}(F) = \mathbf{m}$ and $\mathrm{In}(I_1) = J_1$. On the other hand if $\mathrm{LM}(F) = \mathbf{m}$ and I_1 is an ideal such that $\mathrm{In}(I_1) = J_1$, then $\mathrm{In}(F \cdot I_1)$ has the same Hilbert function than J and $\mathrm{In}(F \cdot I_1) \supseteq J$, hence they coincide. Hence the points in $\mathcal{S}t(J)$ correspond $1-1$ to points in $\mathcal{S}t(\mathbf{m}) \times \mathcal{S}t(J_1) \cong \mathbb{A}^{N_1} \times \mathcal{S}t(J_1)$. By hypothesis, $\mathbf{n}_0$ is prime to each $\mathbf{n}_i$ so that the reductions of the S-polynomials $S(G_0, G_i)$ w.r.t. $G_0, \dots, G_r$ vanish and so they do not contribute to the construction of $\mathcal{A}(J_1)$. Moreover our assumptions on $\mathbf{n}_0 = y$ insure that it cannot appear in any step of reduction of $S(G_i, G_j)$ for every $i, j > 1$. Hence $\mathcal{S}t(J_1) \cong \mathcal{S}t((\mathbf{n}_0)/J_0) \times \mathcal{S}t(J_0/(\mathbf{n}_0)) \cong \mathbb{A}^{N_2+N_0}$.

Now we shall prove that this correspondence is in fact a scheme-theoretical isomorphism, proving that the embedding dimension of $\mathcal{S}t(J)$ is lower than or equal to $N_0 + N_1 + N_2$ computing a suitable set of elements of $L(\mathcal{A}(J))$ (for the meaning of $L(\mathcal{A}(J))$ see §3). More precisely we will prove that all the parameters are eliminable for $\mathcal{A}(J)$ except:

- N_0 among the C_{is} and the N_2 parameters D;

- N_1 parameters C', for instance the C'_{rh}.

The first step is to prove that $\mathcal{A}(J_0/(\mathbf{n}_0)) \subseteq \mathcal{A}(J)$, where $\mathcal{A}(J_0/(\mathbf{n}_0))$ is the ideal in $k[C]$ that we obtain applying Procedure 3.7 to the polynomials $G_1, \dots, G_r$. We can write $S(F_i, F_j)$ as $\mathbf{m}S(G_i, G_j) + M_{ij}$ where M_{ij} does not contain monomials multiple of $\mathbf{m}$. By hypothesis, $\mathbf{m}\mathbf{n}_0$ does not divide any monomial in $\mathbf{m}S(G_i, G_j)$ and no monomial in M_{ij} belongs to J. Hence the reduction of $S(F_i, F_j)$ with respect to J can be written as $H(F_i, F_j) = \mathbf{m}H(G_i, G_j) + \tilde{M}_{ij} + M_{ij}$, where $H(G_i, G_j)$ is the reduction of $S(G_i, G_j)$ with respect to $G_1, \dots G_r$ and no monomial in $\tilde{M}_{ij}$ is multiple of $\mathbf{m}$. Then, collecting the coefficients of monomials that are multiple of $\mathbf{m}$ we obtain the wanted inclusion $\mathcal{A}(J_0/(\mathbf{n}_0)) \subset \mathcal{A}(J)$.

Now we fix a maximal set of eliminable variables for $\mathcal{A}(J_0/(\mathbf{n}_0))$ and specialize to 0 the complementary set of N_0 variables C. Thanks to what just proved, this is equivalent to specialize to 0 all the variables C. By Lemma 3.5 we then have $\mathrm{ed}(\mathcal{A}(J)) \leq \mathrm{ed}(\mathcal{A}(J)/C) + N_0$.

As $S(F_0, F_i) = \mathbf{n}_i F_0 - \mathbf{n}_0 F_i$ and $\mathbf{m}\mathbf{n}_i \mathbf{v}_{0s} \in J$, no variable D appears in $L(\mathcal{A}(J))$ and so we can specialize to 0 also the N_2 variables D obtaining that $\mathrm{ed}(\mathcal{A}(J)) \leq \mathrm{ed}(\mathcal{A}(J)/C \cup D) + N_0 + N_1$ and that the procedure can be continued using the polynomials:

$$\tilde{F}_0 = \mathbf{m}\mathbf{n}_0 + \mathbf{n}_0 \left(\sum C'_{0h}\mathbf{p}_h\right) + \sum C''_{0t}\mathbf{q}_{0t}$$
$$\tilde{F}_i = \mathbf{m}\mathbf{n}_i + \mathbf{n}_i \left(\sum C'_{ih}\mathbf{p}_h\right) + \sum C''_{it}\mathbf{q}_{it} \qquad \text{if} \quad i \geq 1$$

Let us observe that two parameters of the same level are both in C' or both in C'', because the hypothesis on the $\mathbf{q}_{jt}$ does not allow equality between $\mathbf{m}\mathbf{n}_i/\mathbf{q}_{it}$ and $\mathbf{m}\mathbf{n}_j/\mathbf{n}_j\mathbf{p}_h$. Moreover if $j > 0$ there are no parameters in $\tilde{F}_j$ of the same level of some C''_{0t}.

Computing the reduction of $S(\tilde{F}_0, \tilde{F}_r)$ with respect to J, we can see that for every t the parameter C''_{0t} belongs to $L(\mathcal{A}(J))$ and for every h either C'_{0h} or the difference $C'_{0h} - C'_{rh}$ belongs to $L(\mathcal{A}(J))$: in fact $\mathbf{n}_0\mathbf{n}_r\mathbf{p}_h \notin J$ and $\mathbf{n}_r\mathbf{q}_{0t} \notin J$. Moreover using the S-polynomials $S(\tilde{F}_i, \tilde{F}_r)$ with $i > 0$ we see that also $C'_{ih} - C'_{rh} \in L(\mathcal{A}(J))$.

Finally, we use the S-polynomials $S(\tilde{F}_i, \tilde{F}_j)$ with $i, j > 0$ to prove that also the other parameters C'' belong to $L(\mathcal{A}(J))$. We may assume by simplicity that $i = 1$. If there is an index $j > 0$ such that no parameter of the same level as C''_{1t_1} appears in $\tilde{F}_j$, we can see that $C''_{1t_1} \in L(\mathcal{A}(J))$ using the reduction of $S(\tilde{F}_1, \tilde{F}_j)$ with respect to J. If, on the contrary, for every $j > 0$ there is a parameter C''_{jt_j} of the same level as C''_{1t_1} (so that $\mathbf{n}_1\mathbf{q}_{jt_j} = \mathbf{n}_j\mathbf{q}_{1t_1}$), then using $S(\tilde{F}_1, \tilde{F}_j)$ we see that $C''_{1t_1} - C''_{jt_j} \in L(\mathcal{A}(J))$ for every $j > 0$. Moreover there is at least an index $j_0 > 0$ such that $\mathbf{n}_0\mathbf{q}_{j_0t_{j_0}} \notin J$; in fact, $\mathbf{n}_0\mathbf{q}_{jt_j}$ is not a multiple of $\mathbf{n}_0\mathbf{m}$ and the inclusion $\mathbf{n}_0(\mathbf{q}_{1t_1}, \dots, \mathbf{q}_{rt_r}) \subseteq \mathbf{m}(\mathbf{n}_1, \dots, \mathbf{n}_r)$ would imply $\mathbf{n}_0\mathbf{n}_1(\mathbf{q}_{1t_1}, \dots, \mathbf{q}_{rt_r}) = \mathbf{n}_0\mathbf{q}_{1t_1}(\mathbf{n}_1, \dots, \mathbf{n}_r) \subseteq \mathbf{m}\mathbf{n}_1(\mathbf{n}_1, \dots, \mathbf{n}_r)$ and then $\mathbf{q}_{1t_1} \in (\mathbf{n}_1)$ against the hypothesis on $\mathbf{q}_{1t_1}$. So, using $S(\tilde{F}_0, \tilde{F}_{j_0})$, we see that $C'''_{j_0t_{j_0}} \in L(\mathcal{A}(J))$, hence also $C'''_{1t_1} \in L(\mathcal{A}(J))$.

Thus $\mathrm{ed}(\mathcal{A}(J)) \leq N_0 + N_1 + N_2$ and on the other hand $\mathrm{ed}(\mathcal{A}(J)) \geq \dim(\mathcal{A}(J)) = N_0 + N_1 + N_2$ and this allows us to conclude. $\square$

The following example shows that Theorem 5.3 does not hold if we weaken the hypotheses on both $\mathbf{n}_0$ and $\mathcal{S}t((\mathbf{n}_0)/J_0)$.

Example 5.4. *Let us consider the ideal $J = (y^2, x_1^2, x_2x_3, x_1x_2^2, x_1x_3^2)$ in $k[y, x_1, x_2, x_3]$ and the term ordering `Lex` with $x_1 \succ y \succ x_2 \succ x_3$. This ideal is of the type $\mathbf{m}((\mathbf{n}_0) + J_0)$ with $\mathbf{m} = 1$, $\mathbf{n}_0 = y^2$ and $J_0 = (x_1^2, x_1x_2^2, x_1x_3^2, x_2x_3)$. Then $N_1 = 0$ and $N_2 = |\mathrm{htail}(\mathbf{n}_0) \setminus J_0| = 4$. By a direct computation we can prove that $\mathcal{S}t_h(J_0/(\mathbf{n}_0))$ is isomorphic to the union of 4 distinct linear spaces of dimension 6 and also that $\mathcal{S}t_h(J)$ is isomorphic to $\mathbb{A}^{12}$. Then $\mathcal{S}t_h(J)$ is not isomorphic to $\mathcal{S}t_h(J_0/(\mathbf{n}_0)) \times \mathbb{A}^4$.*

As a consequence of the previous general result we can prove that the strata and the homogeneous strata of ideals of the type $J(a_q, \dots, a_1)$, and so especially those of the `Lex` segment ideals, are isomorphic to affine spaces. We point out that Theorem 5.3 does not require any special condition on the order induced by the chosen term ordering on the set of variables. In the special case of the `Lex` segments ideals, if we fix the term ordering `Lex` that orders in the usual way the set of variables, we can also give a formula for the dimensions of the homogeneous strata.

Corollary 5.5. *Let us fix in $k[x_0, \dots, x_n]$ any term order and any sequence of $q \leq n$ non-negative integers $a_q, \dots, a_1$. Then the homogeneous stratum of $J(a_q, ..., a_1)$ is isomorphic to an affine space. The same holds for the stratum $\mathcal{S}t(J(a_q, ..., a_1))$ if it is defined, i.e. if the term ordering is reliable.*

Proof. We prove the statement by induction on q for the homogeneous stratum, the non homogeneous case being analogous. If $q = 1$ the ideal $J(a_1)$ is $(x_{n-1}^{a_1})$ and its homogeneous stratum is an affine space.

Let us assume that the claim holds for every $q' < q$, $(q \geq 2)$ and let us apply Theorem 5.3 to $J_0 = J(a_{q-1}, \dots, a_1)$, $\mathbf{n}_0 = x_{n-q}$ and $\mathbf{m} = x_{n-q}^{a_q}$. Observe that in this case $\mathcal{S}t_h(J(a_{q-1}, \dots, a_1), (x_q)))$ is isomorphic to the stratum of $J(a_{q-1}, \dots, a_1)$ as an ideal of $k[x_0, \dots, x_{n-q-1}, x_{n-q+1}, \dots, x_n]$, and so by inductive hypothesis it is isomorphic to an affine space. $\square$

We stress that $J(a_n, ..., a_1)$ is a `Lex`-segment ideal with respect to the `Lex` term ordering with $x_0 \succ x_1 \succ \cdots \succ x_n$. However, our result on $\mathcal{S}t_h\,(J(a_n, ..., a_1), \prec)$

and $\mathcal{S}t(J(a_n, ..., a_1), \prec)$ holds for any term ordering, even if it changes the order on the set of variables. When the term order is `Lex` and $x_0 \succ x_1 \succ \cdots \succ x_n$, we give a formula for the dimension of the homogeneous stratum.

Corollary 5.6. *Let us fix in $k[x_0, \dots, x_n]$ the term ordering* ***`Lex`*** *with $x_0 \succ x_1 \succ \cdots \succ x_n$ and any sequence of $q \leq n$ non-negative integers $a_q, \dots, a_1$. Assume that $\sum a_i \neq 0$ and let $s \geq 0$ be the minimal index such that $a_{s+1} \neq 0$.*

Then $\mathcal{S}t_h(J(a_q, \dots, a_1), \textbf{Lex})$ is isomorphic to an affine space of dimension $M = M(a_q, \dots, a_1)$ given by:

$$M(a_q, \dots, a_1) = \frac{(q-s)^2 - (q-s) - 2}{2} + \sum_{j=s+1}^{q} \binom{a_j + j}{j} - \sum_{j=s+1}^{q} \nu_j$$

where, for every $j > s+1$, ν_j is such that $a_{j-1} = \cdots = a_{j-\nu_j} = 0$ and $a_{j-\nu_j-1} \neq 0$.

Proof. We prove the statement by induction on $q - s$. If $q - s = 1$, then the ideal $J(a_q, 0, \dots, 0)$ is $(x_{n-q}^{a_q})$ and its homogeneous stratum is an affine space whose dimension is the number $\binom{a_q+q}{q} - 1$ of the elements in the homogeneous tail of $x_{n-q}^{a_q}$. This number coincides with $M(a_q, 0, \dots, 0)$.

Now let us assume $q - s \geq 2$ and that the thesis holds for lower cases of the difference $q - s$. We can apply Theorem 5.3 to $J_0 = J(a_{q-1}, \dots, a_1)$, $\mathbf{n}_0 = x_{n-q}$ and $\mathbf{m} = x_{n-q}^{a_q}$: note that the number s is the same for the two ideals $J(a_q, \dots, a_1)$ and $J(a_{q-1}, \dots, a_1)$. Now we observe that x_q cannot appear in the tail w.r.t. `Lex` of any monomial in the basis of $J(a_{q-1}, \dots, a_1)$ because $x_0 \succ x_1 \succ \cdots \succ x_n$. Thus $\mathcal{S}t_h(J(a_{q-1}, \dots, a_1), (x_q)) = \mathcal{S}t_h(J(a_{q-1}, \dots, a_1))$. The difference between $\dim(\mathcal{S}t_h(J(a_q, \dots, a_1)))$ and $\dim(\mathcal{S}t_h(J(a_{q-1}, \dots, a_1)))$ is given by the sum of two numbers. The first one is the number of elements in the tail of $\mathbf{m} = x_{n-q}^{a_q}$, that is $\binom{a_q+q}{q} - 1$. The second one is the number of monomials in the tail of x_q non contained in $J(a_{q-1}, \dots, a_1)$, that is $q - \nu_q$. This is precisely the difference $M(a_q, \dots, a_1) - M(a_{q-1}, \dots, a_1)$. □

Remark 5.7. *For every admissible Hilbert polynomial $p(z)$ in $\mathbb{P}^n$, the Hilbert scheme $\mathcal{H}\mathrm{ilb}^n_{p(z)}$ contains a point, usually called* ***`Lex`****-point, corresponding to* $\mathrm{Proj}\,(k[x_0, \dots, x_n]/J(a_n, \dots, a_1))$ *for a suitable sequence $(a_n, \dots, a_1)$. In [9] it is proved that the* ***`Lex`****-point is a smooth point of $\mathcal{H}\mathrm{ilb}^n_{p(z)}$ and an explicit formula is given for the dimension of the* ***`Lex`****-component (also called the Reeves and Stillman component), that is the only component of $\mathcal{H}\mathrm{ilb}^n_{p(z)}$ containing the* ***`Lex`****-point. A direct comparison clearly shows that this dimension and that of $\mathcal{S}t_h(J(a_n, \dots, a_2, 0), \textbf{Lex})$ given in the previous result are in general different.*

As an evidence we can consider $J(0, \dots, 0, a_1) = (x_0, \dots, x_{n-2}, x_{n-1}^{a_1})$ with constant Hilbert polynomial $p(z) = a_1$: the dimension of the homogeneous stratum $\mathcal{S}t_h(J(0, \dots, 0, a_1), \textbf{Lex})$ is $2n-2+a_1$, while the dimension of the ***`Lex`****-component of $\mathcal{H}\mathrm{ilb}^n_{a_1}$ is na_0 (see [9], especially the remark after Lemma 4.2).*

Of course, the ideals corresponding to points of $\mathcal{S}t_h(J(a_1,\dots,a_r),\texttt{Lex})$ also correspond to points in the Lex*-component of $\mathcal{H}\mathrm{ilb}^n_{p(z)}$. Then, if the dimensions are different, the dimension of $\mathcal{S}t_h(J(0,\dots,0,a_1),\texttt{Lex})$ must be lower than that of the* Lex*-component. This means that in the stratification of the Hilbert scheme through strata given in [8], the stratum of the lexicographic point is in general a "slice" in its component, that is a locally closed subscheme of it which is not in general an open subset. The following result describes an important case in which the two dimensions agree.*

Corollary 5.8. *Let us fix in $k[x_0,\dots,x_n]$ the term ordering* Lex *with $x_0 \succ x_1 \succ \dots \succ x_n$ and consider any ideal of the type $J(a_n,\dots,a_2,0)$.*

Then $\mathcal{S}t_h(J(a_n,\dots,a_2,0),\texttt{Lex})$ is isomorphic to an open subset of the Lex*-component of the corresponding Hilbert scheme.*

Proof. Every ideal $J(a_n,\dots,a_1)$ is saturated and strongly stable and so the lower variable x_n does not appear in any monomial of its monomial basis. If moreover, as in the present case, $a_1 = 0$ then also x_{n-1} does not appear in those monomials. We can then apply [5, Theorem 4.7] obtaining that $\mathcal{S}t(J(a_n,\dots,a_2,0),\texttt{Lex}) \cong \mathcal{S}t_h(J(a_n,\dots,a_2,0)_{\geq r},,\texttt{Lex})$, where r is the Gotzmann number of $p(z)$, that is the regularity of $J(a_n,\dots,a_2,0)$. Moreover $J(a_n,\dots,a_2,0) \geq r$ is generated b the maximal monomials of degree r w.r.t. Lex and so, by [5, Corollary 6.9], $\mathcal{S}t_h J(a_n,\dots,a_2,0)_{\geq r},,\texttt{Lex})$ is isomorphic to an open subset of the Hilbert scheme $\mathcal{H}\mathrm{ilb}^n_{p(z)}$. □

6 Strata of RevLex-segment ideals

Let Z be a general set of μ points in $\mathbb{P}^2$ in general position, $I_Z \subseteq k[x,y,z]$ be its ideal. In [2] it is proved that the initial ideal of I_Z with respect to the the term order RevLex with $x \succ y \succ z$ is the ideal $\mathcal{R}(\mu) = (x^r,\dots x^{r-i}y^{i+\delta_i},\dots y^{r+\delta_r})$, for some positive integers r and t and $\delta_i = 0$ if $i < t$ and $\delta_i = 1$ otherwise. In every degree s the monomials contained in $\mathcal{R}(\mu)_s$ are the maximal ones w.r.t. RevLex, that is they form a RevLex-segment. Especially, in degree μ, the ideal $\mathcal{R}(\mu)$ contains the $\binom{\mu+2}{2} - \mu$ maximal monomials w.r.t. RevLex and for this reason, using the terminology introduced in [1], $\mathcal{R}(\mu)$ is a *Hilb-segment ideal* and corresponds to the RevLex-*point* of $\mathcal{H}\mathrm{ilb}^n_{p(z)}$ where $n = 2$ and $p(z)$ is the constant polynomial μ (see for instance [7]).

In [1] this construction is generalized to the case of μ points in $\mathbb{P}^n$, $n \geq 3$ proving that $\mathcal{H}\mathrm{ilb}^n_N$ contains a RevLex-point and that it is a singular point if $\mu > n$ (see [1, Theorem 6.4]).

In this last section we consider another type of generalization of $\mathcal{R}(\mu)$, that is the ideals $\mathcal{R}(\mu,n)$ that are generated by the same monomials $x^r,\dots x^{r-i}y^{i+\delta_i}$, $\dots, y^{r+\delta_r}$ as $\mathcal{R}(\mu)$ but in $k[x,y,z_1,\dots,z_{n-1}]$. Using Procedure 3.7 we are able to prove that the homogeneous stratum $\mathcal{S}t_h(\mathcal{R}(\mu, n), \texttt{RevLex})$ is isomorphic to an affine space and to give a formula for its dimension. We underline that even though the case $n = 2$ of our statement is a well known fact, we need to prove it explicitly because the proof itself, not only the statement, is the starting point for the proof of the general case. Finally, we note that the ideals $\mathcal{R}(\mu, n)$ define arithmetically Cohen-Macaulay subschemes in $\mathbb{P}^n$ and so the image of their strata in the corresponding Hilbert schemes are contained in the smooth open subset studied in [4].

Theorem 6.1. *Let $\prec$ be the term order **RevLex** in $k[x, y, z_1, \ldots, z_{n-1}]$ with $x \succ y \succ z_1 \succ \cdots \succ z_{n-1}$. For any positive integer μ let r and t be the only integers such that $1 \leq t \leq r+1$ and $\mu = \frac{(r+2)(r+1)}{2} - t$ and let us consider the ideal in $k[x, y, z_1, \ldots, z_{n-1}]$:*

$$\mathcal{R} = \mathcal{R}(\mu, n) = (x^r, \ldots x^{r-i}y^{i+\delta_i}, \ldots y^{r+\delta_r})$$

where $\delta_i = 0$ if $i < t$ and $\delta_i = 1$ otherwise. Then the homogeneous stratum $\mathcal{S}t_h(\mathcal{R}(\mu, n), \textbf{RevLex})$ is isomorphic to an affine space $\mathbb{A}^N$ of dimension

$$N = N(n, \mu) = 2(n-1)\mu + t(r+1-t)\binom{n-2}{2}. \tag{6}$$

Proof. As the term ordering is fixed, in the following we avoid to indicate it. We first prove the assertion in a special case.

Step 1: the case $n = 2$. As above said, the homogeneous stratum $\mathcal{S}t_h(\mathcal{R})$ of $\mathcal{R} = \mathcal{R}(\mu)$ contains an algebraic family of dimension 2μ corresponding to the sets of μ general points in $\mathbb{P}^2$. Hence in order to prove that $\mathcal{S}t_h(\mathcal{R}) \cong \mathbb{A}^{2\mu}$ it is sufficient to prove that $\mathrm{ed}(\mathcal{A}(\mathcal{R})) \leq 2\mu$ (see Corollary 3.6).

Following Procedure 3.7, let $F_i = x^{r-i}y^{i+\delta_i} + \sum c_{ijk}x^jy^kz^h$, where $j+k+h = r+\delta_i$ and $x^jy^k \notin \mathcal{R}$. We want to show that all the variables C are eliminable for $\mathcal{A}$ except 2μ of them and for this we compute $L(\mathcal{A})$, where $\mathcal{A} = \mathcal{A}(\mathcal{R})$. Now we recall that we can obtain $L(\mathcal{A})$ only using the S-polynomials $S_i = S(F_i, F_{i+1})$, because they correspond to a set of generators for the syzygies of $\mathcal{R}$ (see Remark 3.8) and their reduction $\tilde{H}_i$ w.r.t. $\mathcal{R}$. Moreover, $L(\mathcal{A})$ has a set of generators that are linear combinations of coefficients C of the same level. Now observe that any two coefficients $c_{i,j,k} \in C$ of the same level must share the same number $h = r + \delta_i - j - k$ and that there are no parameters of the same level in both F_0 and F_r because no monomials $\mathbf{n}$ and $\mathbf{m} \in k[x, y] \setminus \mathcal{R}$ satisfy $\frac{x^r}{\mathbf{n}} = \frac{y^r}{\mathbf{m}}$. thanks to these remarks, we can consider a fixed h at a time, that is think of polynomials F_i of the type $F_i = x^{r-i}y^i + z^h \sum_{j+k=r+\delta_i-h} c_{ijk}x^jy^k$ with $x^jy^k \notin \mathcal{R}$.

First we take in consideration the special case $2\mu = r(r+1)$, so that $t = r+1$ and $\mathcal{R} = (x^r, x^{r-1}y, \ldots, y^r)$ and prove that all coefficients C are eliminable, except those with $h = 1$, that are $(r+1)r = 2\mu$ (in this case $x^j y^k \notin \mathcal{R}$ requires $h > 0$).

So, let us fix any $h \geq 2$ and let l be the level of c_{ijk}, with $j+k = r-h$ and let p and q be respectively the minimum and the maximum i such that in F_i there is a parameter of level l that we will denote simply $\overline{c}_i$. If $p \neq 0$, using $\tilde{H}_{p-1}$, $\tilde{H}_p, \dots, \tilde{H}_{q-1}$ we find $-\overline{c}_p, \overline{c}_p - \overline{c}_{p+1}, \dots, \overline{c}_{q-1} - \overline{c}_q \in L(A)$, so that $\overline{c}_i \in L(\mathcal{A})$. If $p = 0$ and $q \neq r$, we argue in the same way using $\tilde{H}_0 \dots, \tilde{H}_{q-1}, \tilde{H}_q$. This allows us to conclude.

Now we consider the case $t \leq r$ so that $S_i = yF_i - xF_{i+1}$ if $i \neq t-1$, while $S_{t-1} = y^2 F_{t-1} - xF_t$. We assume by induction that the statement holds for $r' = r-1$, $t' = t$ and $\mu' = \frac{(r+1)(r)}{2} - t$ (the initial case $r = 1$ being obvious).

As above, all the variables c_{ijk} such that either $j+k \leq r+\delta_i-2$ or $i \neq t-1$ and $j+k = r+\delta_i-1$ are eliminable. For every $i \leq r-1$ and every monomial of the type y^b, such that $b \leq r-1$ and also $b \neq r-1$ for $i = t-1$, using $\tilde{H}_i$ we can see that $c_{i0b} \in L(\mathcal{A})$. Now we specialize to 0 all the parameters c_{i0b} for every $i \neq r$: in this way the embedding dimension drops at most by $r+1$ because those coefficients are all in $L(\mathcal{A})$ except at most $r+1$ of them, namely c_{i0r}, $i = 0, \dots, r-1$ and $c_{(t-1)0(r-1)}$.

Now the polynomials F_i take the form: $F'_0 = xG_0, \dots, F'_{r-1} = xG_{r-1}$, $F'_r = xG_r$ where $G_0, \dots, G_{r-1}$ are the polynomials that we use in order to construct the homogeneous stratum of $\mathcal{R}(\mu')$. Now we observe that y^{r+1} does not divide any monomial in $S(G_i, G_{i+1})$ if $i \leq r-2$, so that the reduction of $S(F_i, F_{i+1})$ w.r.t. $\mathcal{R} = \mathcal{R}(\mu)$ and that of $S(G_i, G_{i+1})$ w.r.t. $\mathcal{R}(\mu')$ give the same list of coefficients. Thus, by the inductive hypothesis, we know that all the parameters that appear in F'_i, $i < r$, are eliminable, except (at most) $(r+1)r - 2t$ of them.

Finally we can see that all the coefficients appearing in F_r is eliminable, except at most the $(r+1)$ coefficients c_{rjk} such that $x^{j+1}y^k \in \mathcal{R}$.

Thus we can conclude that $\mathrm{ed}(\mathcal{R}) \leq (r+1) + (r+1)r - 2t + (r+1) = 2\mu$.

Step 2: the case $n \geq 3$. To prove this general case, we first observe that, as `RevLex` is a graded term ordering, in the case $n = 2$ the homogeneous stratum $\mathcal{S}t(\mathcal{R}(\mu))$ is canonically isomorphic to the (non-homogeneous) stratum $\mathcal{S}t(\mathcal{R}_{af}(\mu))$ of the "affine" ideal in $k[x, y]$ having the same monomial basis as $\mathcal{R}(\mu)$. Thus, in order to construct $L(\mathcal{R}(\mu, n))$ we can consider polynomials $F_i = x^{r-i}y^{i+\delta_i} + \sum C_{ijk} x^j y^k$ where $C_{ijk} = x^j y^j \sum_\gamma c_{ijk\gamma} Z^\gamma$, Z^γ being any monomial in the variables $z_1, \dots, z_{n-1}$ of degree $r + \delta_i - j - k$. Applying Procedure 3.7, we can consider each C_{ijk} as a variable. Thanks to what proved in Step 1 we know that all the variables C_{ijk} are eliminable except 2μ of them and that there are no relations among the remaining ones. Finally we come back to the variables $c_{ijk\gamma}$: the non-eliminable among them are those appearing in the non-eliminable C_{ijk}. We can compute their number following the same line of the proof of Step 1, where the non eliminable variables are explicitly listed.

We observe for instance that the formula of the dimension is quadratic in the number n because for every non-eliminable C_{ijk} the difference between the degree of $x^{r-i}y^{i+\delta_i}$ and that of $x^j y^k$, that is the degree of every Z^γ, is at most 2. □

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Received: April, 2010