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Entropy of Self–Gravitating Systems from Holst's Lagrangian*

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Abstract: we shall prove here that conservation laws from Holst's Lagrangian, often used in LQG, do not agree with the corresponding conservation laws in standard GR. Nevertheless, these differences vanish on-shell, i.e. along solutions, so that they eventually define the same classical conserved quantities. Accordingly, they define in particular the same entropy of solutions, and the standard law $S = \frac{1}{4}A$ is reproduced for systems described by Holst's Lagragian.

This provides the classical support to the computation usually done in LQG for the entropy of black holes which is in turn used to fix the Barbero-Immirzi parameter.

1. Introduction

We have been recently investigating (see [**1**]) how conservation laws depend on the variational principle when there are many dynamically equivalent frameworks to describe the same single physical situation.

In GR there exist a number of different formulations able to catch gravitational physics: purely–metric, metric–affine, purely–affine, purely–tetrad and tetrad–affine gravity are just the most common ones; they differ by the choice of the fundamental fields, though in the end they all produce as solution a metric which obeys Einstein field equations (possibly with cosmological constant).

Besides the choice of fundamental fields, one can also modify the Lagrangian though preserving the space of solutions. It is known, e.g., that the non-linear Lagrangian $L = f(R)\sqrt{g}$, in the metric–affine (à la Palatini) formulation and in vacuum, for a generic analytical function $f(R)$ of the scalar curvature R induces field equations which are equivalent to Einstein field equations with a (family of) suitable cosmological constants (see [**2**]).

In [**1**] we proved that in this non-linear first-order $f(R)$ formulation of GR (which in vacuum is *exactly* GR, not a modification of GR) conservation laws are described by a superpotential which makes them to differ from the standard conservation laws in GR by terms which vanish on-shell, i.e. when evaluated along any solution of field equations.

The superpotential was there computed by means of the so–called *augmented variational principle* (see [**3**]). Augmented Lagrangians depend on two sets of fields, one representing the dynamical physical field, the other representing a reference vacuum field. The conserved quantities depend on both the configuration and the reference fields and they are interpreted as the difference of the corresponding conserved quantities (energy, momentum, charge, . . .) between the vacuum field and the dynamical configuration. In the augmented Lagrangian a pure

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divergence (depending on both configuration and vacuum) is chosen in order to improve the superpotential, so that the infinitesimal variation of the conserved quantity Q is

$$\delta_X Q = \int \delta_X \mathcal{U} - i_X \mathbb{F} \tag{1.1}$$

where $\mathcal{U}$ is the superpotential, i_X denotes the contraction along any deformation X and $\mathbb{F}$ is the boundary part of the action functional. There are many motivations to assume (1.1) as physically sound; see, in particular, **[4]** and **[5]**. Moreover, augmented variational principles have been proven to be effective in describing black hole solutions and GR; see **[3]**, **[6]**, **[7]**.

The situation for $f(R)$-theories is quite satisfactory for the physical intuition; conservation laws depend on the Lagrangian in such a way that the corresponding conserved quantities are in fact independent of the Lagrangian. In this way we argue that the conserved quantity is associated to the solution of field equations more than to the specific Lagrangian used to find it. However, terms vanishing on-shell as the ones found in $f(R)$-theories are not the most general corrections which leave the corresponding conserved quantities unchanged. In fact, any correction which reduces on-shell to a pure divergence would leave the conserved quantity unchanged.

We shall study here another dynamically equivalent formulation of GR, the so–called *Holst action*; see **[8]**, **[9]**,**[10]**. The gravitational field is here described by (co)tetrads e^a_μ and a suitable SU(2)-connection ω^{ab}_μ; see **[11]**, **[12]**. The Hilbert Lagrangian is modified by a term which in the end does not affect the solution space. We shall show that Holst action defines a superpotential which differs from the superpotential of standard GR by terms which are pure divergences on-shell. This provides an explicit example of the generic situation expected which leaves the conserved quantities unchanged.

Besides the importance of having an explicit example of this general situation, Holst action is important also for another reason. Holst-Barbero-Immirzi formulation is used in Loop Quantum Gravity (LQG) and depends on a real parameter γ called *Barbero-Immirzi parameter*, which is fixed so that microstate counting of black hole entropy reproduces the standard law $S = \frac{1}{4}A$, i.e. one-quarter of the area of the horizon. However, the result $S = \frac{1}{4}A$ is obtained classically, by relying on Nöther conserved quantities (see **[13]**). Now, if in Holst-Barbero-Immirzi gravity such conserved quantities could be different from the standard GR ones, then the classical prescription $S = \frac{1}{4}A$ would appear to loose its motivation. By proving that the conserved quantities for Holst-Barbero-Immirzi formulation are the usual ones we also provide a solid basis for the fixing of the Barbero-Immirzi parameter.

Let us finally remark that further investigations will be devoted to treat in greater generality the issue of dynamically equivalent field theories and their conservation laws. In this direction, to the best of our knowledge, no general result is available, yet.

2. The Holst's formulation

The Holst Lagrangian (see **[8]**, **[9]**, **[14]**), used in the framework of Loop Quantum Gravity (LQG), is defined as

$$L_h(e^a, \omega^{ab}) = R^{ab} \wedge e^c \wedge e^d \, \epsilon_{abcd} + \frac{2}{\gamma} R^{ab} \wedge e_a \wedge e_b = L_1 + \frac{2}{\gamma} L_2$$

where L_1 is the standard Lagrangian for General Relativity (GR) in the frame-affine formalism. Here L_2 is an additional term which does not affect the solution space and $\gamma \in \mathbb{R} - \{0\}$ is the Barbero-Immirzi parameter.

The augmented Lagrangian is

$$L_h^{Aug} = L_h - \bar{L}_h + \mathrm{Div}(\alpha_1 + \alpha_2) \tag{2.1}$$

where we denote by $\bar{L}_h$ the Holst Lagrangian for the vacuum fields $(\bar{e}^a, \bar{\omega}^{ab})$ and, in this case, the correction terms are defined by

$$\begin{cases} \alpha_1 = (\omega^{ab} - \overline{\omega}^{ab}) \wedge e^c \wedge e^d \, \epsilon_{abcd} \\ \alpha_2 = (\omega^{ab} - \overline{\omega}^{ab}) \wedge e_a \wedge e_b \end{cases}$$

Here the configuration and the vacuum fields are chosen so that $e = \overline{e}$ but $de \neq d\overline{e}$ at the boundary surface.

The superpotential associated to L_1 is $\mathcal{U}_1 = K_1 - i_\xi \alpha_1$, using the definition of Kosmann lift $\xi_{(v)}^{ab} = e_\alpha^a \, e^{b\beta} \nabla_\beta \xi^\alpha$ (see **[15]**), where

$$\begin{aligned} K_1 &= e_\rho^a \, e_\sigma^b \, \xi_{(v)}^{cd} \, \epsilon_{abcd} \, \epsilon^{\mu\nu\rho\sigma} \, ds_{\mu\nu} = 4\sqrt{g} \nabla^\nu \xi^\mu \, ds_{\mu\nu} \\ i_\xi \alpha_1 &= \xi^\lambda \left(\omega_\nu^{ab} - \overline{\omega}_\nu^{ab} \right) e_\rho^c \, e_\sigma^d \, \epsilon_{abcd} \, \epsilon^{\mu\nu\rho\sigma} \, ds_{\mu\lambda} = 4\sqrt{g} \, g^{\alpha\beta} \, w_{\alpha\beta}^\mu \, \xi^\lambda \, ds_{\mu\lambda} + \Delta^{\mu\lambda} \, ds_{\mu\lambda} \end{aligned}$$

where we have used the on-shell relation between the frame and the connection, namely

$$\omega_\mu^{ab} = e_\alpha^a \left(\Gamma_{\beta\mu}^\alpha \, e^{b\beta} + d_\mu e^{b\alpha} \right)$$

and set $ds_{\mu\nu} = \partial_\nu \rfloor \partial_\mu \rfloor ds$, being ds the (local) volume element.

The superpotential associated to L_2 is $\mathcal{U}_2 = K_2 - i_\xi \alpha_2$

$$\begin{aligned} K_2 &= e_{a\rho} \, e_{b\sigma} \, \xi_{(v)}^{ab} \, \epsilon^{\mu\nu\rho\sigma} \, ds_{\mu\nu} \simeq \mathrm{Div}\left(3\xi_\sigma \epsilon^{\mu\nu\rho\sigma} \, ds_{\mu\nu\rho}\right) \\ i_\xi \alpha_2 &= \xi^\lambda \left(\omega_\nu^{ab} - \overline{\omega}_\nu^{ab} \right) e^{a\rho} \, e_{b\sigma} \, \epsilon^{\mu\nu\rho\sigma} \, ds_{\mu\lambda} = \xi^\lambda \left(e_{b\sigma} \, d_\nu e^{b\alpha} - \overline{e}_{b\sigma} \, d_\nu \overline{e}^{b\alpha} \right) \epsilon_\alpha^{\cdot\mu\nu\sigma} \, ds_{\mu\lambda} \end{aligned}$$

Here the symbol $\simeq$ refers to identity modulo boundary conditions.

Calculation is carried out on the boundary surface where $e = \overline{e}$ but $de \neq d\overline{e}$ and we separated the standard GR terms from the corrections Δ, K_2 and $-i_\xi \alpha_2$.

Now using field equations together with boundary conditions and integrating by parts, these corrections can be eventually recasted as

$$\begin{aligned} \Delta^{\mu\lambda} &= 4\sqrt{g} \left(e_b^\nu \, d_\nu e^{b\mu} - e_b^\mu \, d_\nu e^{b\nu} - \overline{e}_b^\nu \, d_\nu \overline{e}^{b\mu} + \overline{e}_b^\mu \, d_\nu \overline{e}^{b\nu} \right) \xi^\lambda \\ &\simeq 4 d_\nu \left[\sqrt{g} \xi^\lambda \left(\overline{e}_b^\nu \, e^{b\mu} - \overline{e}_b^\mu \, e^{b\nu} \right) \right] \end{aligned}$$

i.e. a pure divergence term which does not affect the value of conserved quantities. An analogous result is obtained for α_2 that can be written on-shell as

$$i_\xi \alpha_2 \simeq d_\nu \left(\xi^\lambda \overline{e}_\sigma^b \, e_{b\alpha} \, \epsilon^{\alpha\mu\nu\sigma} \right)$$

while K_2 is already a pure divergence.

Here we have an explicit example of two dynamically equivalent theories with different superpotentials (and conservation laws). The difference, however, reduces to a divergence on-shell so that conserved quantities (as well as the entropy of solutions) are uneffected.

3. Conclusions and perspective

We have shown that the Holst Lagrangian induces conservation laws which only apparently differ from the standard GR ones. The difference, in fact, is just happening under the form of terms which are pure divergences on-shell, hence not affecting the values of conserved quantities.

Further investigations will be devoted to obtain general results about dynamically equivalent theories, of which this specific case is an instructive example.

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