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Gauge-natural PVPs, Superpotentials and Trivial Lagrangians

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Abstract

We define the superpotentials for gauge-natural Parametrized Variational Problems. In view of a definition of augmented PVPs, we show that a pure divergence Lagrangian has trivial equations also in the case of PVPs but, as it happens in the free case, it has a non-vanishing superpotential. We compute it.

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1. Introduction

In [1] we defined and studied the so-called *gauge-natural Parametrized Variational Problems* (PVPs) extending the natural case studied by García and coworkers in [2]. They are Variational Problems where variations of the fields are not taken arbitrarily, but along given parametrizations.

In General Relativity different notions of energy were given (see [5] for a review). Two of the authors (L.F. and M.F.) contributed to the debate in particular with a variational pathway to this definition. The major source of difficulty is that the naive definition of conserved quantities as integrals of the superpotential suffers the following ambiguity: adding a divergence to the Lagrangian nothing should change in the Dynamics, nevertheless the conserved quantities are changed by a summand (see [3, 4]). In General Relativity, furthermore, the naive prescription gives anomalous results and a boundary correction for the superpotential was independently found by many authors (see for example [6, 7, 8] and reference quoted therein). In [3] a general setting was proposed in which for a given Field Theory the corrected superpotential can be derived from an *augmented Lagrangian* .

The aim of the present paper is to introduce a general definition of superpotentials for gauge-natural PVPs and to study their behavior with respect to the addition of a divergence to the Lagrangian, in view of the possibility of generalizing the prescriptions for the augmented Lagrangians to the case of PVPs.

Due to the very strict limitations in length we are compelled to skip some details. We remark however that the reader can find in [1, 4] all the preliminary material with an identical notation.

2. PVPs

Let $C \xrightarrow{\pi} M$ be the *configuration bundle* whose (global) sections $\Gamma(C)$ are the fields (by an abuse of language we will often denote bundles with the same label as their total spaces). Let moreover (x^μ, y^i) be a fibered coordinate system on C and $m = \dim M$.

Definition 1 *A first order Lagrangian C is a fibered morphism*

$$L : J^1C \longrightarrow \wedge^m T^*M.$$

In local coordinates we write L as $L(x^\mu, y^i, y^i_\mu)\, ds$.

Definition 2 *A parametrization of order* 1 *and rank* 1 *of the set of constrained variations is a couple $(E, \mathbb{P})$, where E is a vector bundle $E \xrightarrow{\pi_E} C$, while $\mathbb{P}$ is a fibered morphism (section)*

$$\mathbb{P} : J^1C \longrightarrow (J^1E)^* \otimes_{J^1C} VC.$$

If $(x^\mu, y^a, \varepsilon^A)$ are local fibered coordinates on E and $\{\partial_a\}$ is the induced fiberwise natural basis of VC, a parametrization of order 1 and rank 1 associates to any section $y^a(x)$ of C and any section $\varepsilon^A(x, y)$ the section

$$\left[p^a_A \varepsilon^A + p^a_A{}^\mu d_\mu \varepsilon^A\right] \partial_a \tag{1}$$

of VC, where p^a_A and $p^a_A{}^\mu$ are functions of $\big(x^\nu, y^b(x), \partial_\nu y^b(x)\big)$, while ε^A depends on $(x^\nu, y^b(x))$.

Definition 3 *The set $\{C, L, \mathbb{P}\}$, where C is the configuration bundle, L the Lagrangian and $\mathbb{P}$ the parameterization, is called a "Parametrized Variational Problem" (PVP).*

The role of the parametrization is to select a subset of variations in order to define criticality of a section with respect to this set.

Definition 4 *We define critical for $\{C, L, \mathbb{P}\}$ those sections of C for which, for any compact $D \subset M$ and for any section $\varepsilon^A(x,y)$ such that both $\varepsilon^A(x,y(x))$ and $d_\mu \varepsilon^A(x,y(x))$ vanish for all $x \in \partial D$, we have*

$$\int_D \left[\frac{\partial L}{\partial y^a}(p^a_A \varepsilon^A + p^a_A{}^\mu d_\mu \varepsilon^A) + \frac{\partial L}{\partial y^a_\mu} d_\mu (p^a_A \varepsilon^A + p^a_A{}^\nu d_\nu \varepsilon^A) \right] ds = 0. \tag{2}$$

To set up a characterization of critical sections in terms of a set of differential equations let us split the integrand of (2) into a first summand that factorizes ε^A (without any derivative) plus the total derivative of a second term (a general theorem ensures that this splitting is unique; see [1]). To do this, we integrate by parts the derivatives of ε in the integrand of equation (2). What we get is

$$\int_D \left(\mathbb{E}_A \varepsilon^A + d_\mu (\mathbb{F}^\mu_A \varepsilon^A + \mathbb{F}^{\mu\nu}_A d_\nu \varepsilon^A) \right) ds = 0$$

with

$$\begin{aligned} \mathbb{E}_A &= \left(\frac{\partial L}{\partial y^a} - d_\nu \frac{\partial L}{\partial y^a_\nu} \right) p^a_A - d_\mu \left[\left(\frac{\partial L}{\partial y^a} - d_\nu \frac{\partial L}{\partial y^a_\nu} \right) p^a_A{}^\mu \right] \\ \mathbb{F}^\mu_A &= \left(\frac{\partial L}{\partial y^a} - d_\nu \frac{\partial L}{\partial y^a_\nu} \right) p^a_A{}^\mu + \frac{\partial L}{\partial y^a_\mu} p^a_A \quad \text{and} \quad \mathbb{F}^{\mu\nu}_A = \frac{\partial L}{\partial y^a_\mu} p^a_A{}^\nu . \end{aligned} \tag{3}$$

In [1] we have shown that the coefficients $\mathbb{E}_A$, $\mathbb{F}^\mu_A$ and $\mathbb{F}^{\mu\nu}_A$ are the components of two global morphisms

$$\begin{aligned} &\mathbb{E}(L,\mathbb{P}) : J^3 C \longrightarrow E^* \otimes_C \Lambda^m T^* M \\ &\mathbb{F}(L,\mathbb{P},\gamma) : J^2 C \longrightarrow (J^1 E)^* \otimes_{J^1 C} \Lambda^m T^* M \end{aligned}$$

and that a global meaning in terms of variational morphisms and global operations between them can be given to the whole procedure. The same can also be done for higher order Lagrangians and for higher rank and higher order parametrizations.

3. Gauge-natural PVP and Superpotentials

For the definitions of gauge-natural bundles and morphisms we refer the reader to [1, 4] and references quoted therein.

Definition 5 *A gauge-natural variational problem with parametrized variations is defined by the set $(\mathfrak{C}(P), L, \mathfrak{F}, \mathbb{P}, \mathbb{J}, \bar{\omega})$ of the following objects:*

1. *a gauge-natural configuration bundle $\mathfrak{C}(P) \xrightarrow{\pi} M$ of order (r,s) with structure G-bundle $P \xrightarrow{\eta} M$;*
2. *a variationally gauge-natural (local) Lagrangian morphism L of order k;*
3. *a vector bundle $F \xrightarrow{\pi_F} \mathfrak{C}(P)$ such that the composite projection $F \xrightarrow{\pi \circ \pi_F} M$ is a gauge-natural bundle $\mathfrak{F}(P)$ of order (p,f) called* bundle of parameters*;*

4. *a gauge-natural parametrization* $\mathbb{P} : J^1\mathfrak{C}(P) \longrightarrow (J^q\mathfrak{F}(P))^* \otimes_{J^q\mathfrak{C}(P)} V\mathfrak{C}(P)$ *(*$q \in \{0,1\}$*);*
5. *a gauge-natural morphism* $\mathbb{J} : J^1\mathfrak{C} \longrightarrow (J^{r-q}IGA(P))^* \otimes_{J^1\mathfrak{C}} \mathfrak{F}(P)$ *such that we have* $\big\langle \mathbb{P} \mid j^q < \mathbb{J}|j^{r-q}\Xi > \big\rangle \circ j^1\sigma = \pounds_\Xi\sigma$*; here* $IGA(P)$ *denotes the bundle of Infinitesimal Generator of Automorphisms (see [4], Ch. 3);*
6. *a morphism* $\bar{\omega} : J^k\mathfrak{C}(P) \longrightarrow \mathcal{C}(M) \times_M \mathcal{C}(P)$ *that associates to any configuration a couple* (γ, ω) *where* γ *is a linear connection on* M *and* ω *is a principal connection on* P*.*

In [1] we proved that if we have a gauge-natural PVP we can associate to any gauge symmetry Ξ a conserved current

$$\mathcal{E} : J^{2k+l+s-1}\mathfrak{C}(P) \longrightarrow J^{r-q-1}\mathrm{IGA}(P) \otimes \Lambda^{m-1}T^*M$$

such that

$$d\left(< \mathcal{E} \mid j^{r-q-1}\Xi > \circ j^{2k+l+s-1}\sigma\right) = 0.$$

The analytic expression of the current is the following:

$$< \mathcal{E} \mid j^{r-q-1}\Xi > = < \mathbb{F} \mid j^{l+k-1}\mathbb{J} \circ \Xi > - T\eta\, \Xi \lrcorner L \tag{4}$$

The conserved current is thence a variational morphism. General theorems on variational morphisms ensure (see [4], Chapter 5 and Chapter 8) that the current can be uniquely (given $\bar{\omega}$) split into a *reduced current* $\tilde{\mathcal{E}}$ that vanishes on-shell plus the divergence of another variational morphism $\mathcal{U}$ called *superpotential*. For any gauge-natural PVP we have thence

$$< \mathcal{E} \mid j^{r-q-1}\Xi > = < \tilde{\mathcal{E}} \mid j^{r-q-1}\Xi > + \mathrm{Div}\ < \mathcal{U} \mid j^{r-q-2}\Xi > . \tag{5}$$

The integral

$$Q_D(\Xi, \sigma) = \int_{\partial D} < \mathcal{U} \mid j^{r-q-2}\Xi > \circ j^{2k+l+s+r-q-3}\sigma$$

can be naively interpreted as the conserved current associated to the symmetry Ξ, the $(m-1)$-surface D and the solution σ. In ordinary Variational Problems this interpretation is affected by the following ambiguity: if one adds a pure divergence to the Lagrangian of the starting problem the equations (i.e. the true observable things) are left unchanged. One expects that so happens also for the conserved quantities, whereas in fact they gain a summand. A way to avoid this ambiguity was studied for ordinary variational problems in [3]. Let us study what happens with PVPs.

4. Trivial Lagrangians

Let us consider a gauge-natural variational morphisms

$$\alpha : \mathfrak{C}(P) \longrightarrow \Lambda^{m-1}T^*M$$

with coordinate expression $\alpha^\mu ds_\mu$. Let the (first-order) Lagrangian L be $L = d_\mu \alpha^\mu ds$. Constrained variations are a subset of free variations, thus if any section σ is critical for L in the free problem, then so it is also in the parametrized case. Anyway for $k = 1$ one can directly see from expressions (3) that if the free Euler-Lagrange equations for L are satisfied so are the parametrized ones. Thanks to Lemma 17 in [1] (but one can also directly compute it) one has that for any variation δy

$$< \delta L \mid j^1 \delta y >= Div < \delta\alpha \mid \delta y >$$

where $\delta\alpha$ is a variational morphism $\delta\alpha : \mathfrak{C}(P) \longrightarrow (J^{k-1}V\mathfrak{C}(P))^* \otimes_{J^{k-1}\mathfrak{C}} \Lambda^{m-1}T^*M$ that, using the same notation as before, has coordinate expression

$$< \delta\alpha \mid \delta y >= \frac{\partial \alpha^\mu}{\partial y^i} \delta y^i \, ds_\mu.$$

If we now restrict to the admissible variations δy that can be derived from the parametrization $\mathbb{P}$ we find that if ε is a section of the bundle of parameters $\mathfrak{F}(P)$ then the $\mathbb{P}$-Poincaré-Cartan morphism is

$$< \mathbb{F} \mid j^q \varepsilon >:= \Big\langle \delta\alpha \mid \; < \mathbb{P} | j^q \varepsilon > \Big\rangle$$

A gauge transformation Ξ that projects onto a vector field $\xi = \xi^\mu \partial_\mu$ on the base is a symmetry for L. Taking equation (4) and the properties of the morphism $\mathbb{J}$ (see 5. in Definition 5) into account, one has that the conserved current is

$$< \mathcal{E} \mid j^r \Xi >= \Big\langle \delta\alpha \mid \pounds_\Xi y \Big\rangle - \xi \lrcorner L$$

Now by gauge-naturality of α (see again [1]) one means

$$\Big\langle \delta\alpha \mid \pounds_\Xi y \Big\rangle = \pounds_\xi \alpha$$

and thanks to Cartan formula one easily gets

$$< \mathcal{E} \mid j^1 \Xi >= d_\mu(\xi^{[\nu} \alpha^{\mu]}).$$

Being the splitting (5) unique one has (here the order is much lower than in the general case)

$$< \mathcal{U} \mid \Xi >= \xi^{[\nu} \alpha^{\mu]} \, ds_{\nu\mu}$$

that is the same result that one gets in the non parametrized case (see [3]). The analogy between parametrized and free case still holds when if α depends on the derivatives of the field, some more care however is needed in the computations.

5. Conclusions

As a first step toward the definition of Augmented Parametrized Variational Problems we defined the superpotentials for PVPs and studied their behavior under the addition of a pure divergence to the Lagrangian. The result obtained is that the superpotential is modified by a summand that is identical to what one gets in the case of free variations.

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