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Existence of Solutions and Asset Pricing Bubbles in General Equilibrium Models

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Abstract

This paper analyses the problem represented by the presence of speculative bubbles on asset prices in general equilibrium models. The main results concerning the existence of solutions in intertemporal general equilibrium models are summarized, then the specific problem of asset pricing is discussed. In particular, the theoretical results concerning the existence of speculative bubbles on asset prices are presented, together with the results that can be obtained through a new approach, based on Euler equations. In the last part a series of examples in which speculative bubbles on asset prices do appear are illustrated, and the corresponding conditions that allow to exclude the presence of such bubbles are derived. Finally, some considerations are developed in order to match the predictions of the theory with the empirical observations concerning the behavior of asset prices.

Keywords: General Equilibrium, Asset Pricing, Speculative Bubbles.

Journal of Economic Literature: C61, C62, D51, G12.

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1 Introduction

The theory of *general equilibrium* is the branch of economic theory that studies the interactions between demand and supply of the different goods in the different markets in order to determine the prices of these goods (while the *partial equilibrium* analysis considers only the relations between demand and supply of a specific good and the price of the same good).

In general equilibrium analysis some simplifications are usually introduced, in particular it is assumed that:

- markets are competitive and individuals are optimizing;
- there is no production (at least in first approximation), agents have fixed endowments of the goods and they must determine only the quantities to exchange (*pure exchange economy*).

One of the central features of modern economics is then the introduction of *time* and *uncertainty*, and the consequent attempt to analyse an environment characterized by the presence of these elements. The main consequence for the behaviour of individuals is that they have only a limited ability to make decisions in such an environment; with reference to the theory of general equilibrium, in particular, this implies that, when agents have limited knowledge and ability to face uncertainty, they trade *sequentially* (i.e. periods by periods) and use a system of contracts which involve only limited commitments into the future.

The traditional Arrow-Debreu general equilibrium model (whose objective is the study of the allocation of resources achievable through a system of markets and whose central result is that, when there are markets and associated prices for all goods and services in the economy, no externalities or public goods, and no informational asymmetries, then competitive markets allocate resources efficiently) can be adapted to take into account the presence of time and uncertainty, assuming the existence, at the initial date, of a *complete set of contingent markets* (i.e. a market for each good produced or consumed in every possible future contingency). Nevertheless, this is an idealization that is not realistic (since the individuals do not have full knowledge of all possible future events and the society cannot costlessy monitor and enforce the commitments of agents), and for this reason it is necessary to consider an extension of this model, introducing a sequence of *spot markets* (for the exchange of goods and services) and a sequence of *financial markets* (in which contracts that allow to transfer resources across time are negotiated); typically, there is a limited number of these markets, i.e. there is a situation of *incomplete markets*.

The equilibrium solution of these models (if it exists) gives the values of prices and quantities (of the goods and of the financial activities) in correspondence of which the individuals solve their optimization problem and the (real and financial) markets clear (i.e. demand equals supply on these markets). A first important problem is therefore represented by the analysis of conditions that guarantee the existence of solutions in this kind of models.

These models can then be used to analyse the issue of asset pricing, and in particular the relation between the equilibrium price of an asset and the stream of future dividends on which the asset represents a claim. What emerges is that, while in the finite-horizon case the equilibrium *price* equals the *fundamental value* of the asset (i.e. the discounted sum of future dividends), in the infinite-horizon case this is not necessarily true (in particular, it is possible for the price to be larger than the fundamental value, and in this case the price of the asset is said to involve a *speculative bubble*). A second important question is therefore represented by the analysis of conditions that allow to exclude the presence of such bubble components, together with the study concerning the fragility of this phenomenon.

The paper is organized as follows. Section 2 analyses the problem of existence of solutions in intertemporal general equilibrium models. Since the literature on this argument is very extensive, and in the light also of the results discussed in the following Section, this analysis is limited to the (relatively) simplest case, the one in which the economy is characterized by discrete time periods and a finite number of states of nature in each period. In such a framework, the model with contingent markets, the model with spot and financial markets with finite horizon and then the model with spot and financial markets with infinite horizon are considered, and the corresponding results in terms of existence of solutions are derived. The main conclusion is that, even if in the passage from an economy with contingent markets to an economy with spot and financial markets and, in this economy, from the finite-horizon case to the infinite-horizon case, something is "lost" in the proof of existence of equilibrium, nevertheless a form of existence is always guaranteed. The model is therefore consistent, and it can be used to explain something of the economy we are dealing with. In particular, this kind of models can be used to investigate the problem of asset pricing, and the relation between the price of the assets and the stream of dividends to which they give rise. This is the question considered in Section 3, with particular reference to the phenomenon of speculative bubbes that can emerge in infinite-horizon incomplete-market economies. More specifically, the results deriving from some models recently proposed are compared with those that can be obtained through a new approach, based on Euler equations. In this case the main conclusion is that, in the models considered, speculative bubbles represent a fragile phenomenon. The occurrence of speculative bubbles as a very special circumstance is shown by means of a series of examples in Section 4, where the use of the Euler equations' approach also allows to show how the presence of bubbles on asset prices is linked to the violation of specific conditions. While, in the light of the theoretical contributions presented, the phenomenon of speculative bubbles can be considered as marginal, the real world is often characterized by the occurrence of *speculative episodes* in which bubbles do appear. For this reason, Section 5 derives some final considerations that try to explain the mechanism that produces these results and that allow to reconcile the conclusions of the theory with the empirical observations.

2 Existence of solutions in general equilibrium models

The questions considered in the analysis of general equilibrium models are three:

- *existence*: it is fundamental to ensure the coherence of the structure of the model and the presence of a solution (otherwise it is useless);
- *optimality*: it allows to evaluate the efficiency of the underlying market structure as a mechanism for the allocation of resources;
- *determinacy (local uniqueness)*: it is essential for comparative statics analysis (how the equilibrium changes when certain parameters of the model vary).

This Section focuses in particular on the question of existence. Furthermore, the analysis considers the simplest case, that of a pure exchange economy, with discrete time periods and a finite number of states of the world at each date. In this context, the first step is represented by the study of a model with complete contingent markets, then the analysis can be extended to the case of a model with spot and financial markets, both in the finite-horizon case and in the infinite-horizon case.

2.1 The model with contingent markets

The standard static model of general equilibrium (Arrow (1951), Debreu (1952), Arrow-Debreu (1954)) assumes that there exists a market for each good and that all goods are traded simultaneously. This model can be generalized to a setting that includes the presence of time and uncertainty (Arrow (1953), Debreu (1959)), defining the notion of *contingent commodity*:

Definition 1 *A contingent commodity is a contract that promises the future delivery of one unit of a particular good if a particular state of nature will occur.*

If there is a complete set of such contingent contracts (one for each good in each possible state of nature) we have the idealyzed situation of *complete contingent markets*. In this way the model with time and uncertainty can be reduced to the model without uncertainty (considering the same good available in different states of nature as different goods), and in this model all the decisions are taken at the initial date ($t = 0$). The simplest case is represented by an exchange economy with two periods; the results can then be generalized to an economy that extends over T periods (with T finite).

2.1.1 Two-period exchange economy

In this case the economy consists of:

- a finite number I of agents ($i = 1, 2, ..., I$);
- a finite number L of goods ($l = 1, 2, ..., L$);
- two periods ($t = 0, 1$);
- S possible states of nature ($s = 1, 2, ..., S$) at time $t = 1$ (if time $t = 0$ is denoted as state $s = 0$ there are in total $S + 1$ states of nature);
- commodity space $\mathbb{R}^n$ with $n = L(S + 1)$;
- initial endowment of the L goods in each state of nature for agent i given by $w^i = (w^i_0, w^i_1, ..., w^i_S)$;
- utility function for agent i given by $u^i : \mathbb{R}^n_+ \rightarrow \mathbb{R}$ defined over consumption bundles $x^i = (x^i_0, x^i_1, ..., x^i_S)$.

The characteristics of agent i are therefore summarized by a utility function and an endowment vector (u^i, w^i), and it is assumed that they satisfy some regularity conditions (*smooth preferences*) (in particular u^i is of class $\mathcal{C}^\infty$ and preferences are strictly monotone).

The collection of I agents with their characteristics $(u, w) = (u^1, ..., u^I, w^1, ..., w^I)$ constitutes the exchange economy $\mathcal{E}(u, w)$, and an allocation of resources for this economy is a vector of consumption bundles $x = (x^1, x^2, ..., x^I) \in \mathbb{R}^{nI}_+$.

A *contingent contract* for good l in state s is a contract that promises to deliver one unit of good l if the state of nature s will occur and nothing otherwise. The corresponding price, payable in $t = 0$, is P_{sl} and if there is a complete set of such contracts each agent i can sell his endowment $w^i = (w^i_0, w^i_1, ..., w^i_S)$ at prices $P = (P_0, P_1, ..., P_S)$ with $P_s = (P_{s1}, P_{s2}, ..., P_{sL})$ to obtain Pw^i, and he can purchase any consumption vector $x^i = (x^i_0, x^i_1, ..., x^i_S)$ that satisfies the constraint:

$$Px^i = \sum_{s=0}^{S} P_s x^i_s \leq \sum_{s=0}^{S} P_s w^i_s = Pw^i$$

Since preferences are monotonic each agent will always fully spend his income, hence the contingent market budget set of each individual is:

$$B(P, w^i) = \left\{x^i \in \mathbb{R}^n_+ \mid P(x^i - w^i) = 0\right\}$$

We now have the following definition of equilibrium for this economy:

Definition 2 *A contingent market equilibrium for the economy $\mathcal{E}(u, w)$ is a pair of allocations and prices $(\overline{x}, \overline{P})$ such that:*

(i) $\overline{x}^i = \arg\max \left\{u^i(x^i) \mid x^i \in B(\overline{P}, w^i)\right\} \quad i = 1, 2, ..., I$

(ii) $\sum_{i=1}^{I}(\overline{x}^i - w^i) = 0$

i.e. at equilibrium each individual solves his maximization problem and, in addition, markets clear (i.e. the total demand of each good in each state of nature is equal to the total supply).

The fundamental result that can be obtained applying the classical existence theorem of general equilibrium theory is that the exchange economy $\mathcal{E}(u, w)$ has at least one equilibrium, and moreover the allocation $\overline{x}$ is Pareto optimal. More precisely, if $E_C(w)$ is the set of contingent market equilibrium allocations corresponding to the parameter value w, we have the following results:

- *existence*: $E_C(w) \neq \emptyset \ \ \forall w$
- *optimality*: $x \in E_C(w) \Rightarrow x$ is Pareto optimal $\forall w$
- *local uniqueness*: $E_C(w)$ is a finite set (hence the equilibrium is locally unique)

2.1.2 Stochastic exchange economy

The previous model can then be extended to an economy with a finite number of periods $t = 1, 2, ..., T$. In this case the uncertainty is modelled with an event-tree and the economy consists of:

- a finite set of states of nature $\mathbf{S} = \{1, 2, ..., S\}$;

- a collection of partitions of $\mathbf{S}$ given by $F = (F_0, F_1, ..., F_T)$ where $F_0 = \mathbf{S}$, $F_T = \{\{1\}, \{2\}, ..., \{S\}\}$ and F_t is finer than F_{t-1} for all $t = 1, 2, ..., T$ (this expresses the idea that information is revealed gradually and increases over time);

- the set of nodes $\mathbf{D} = \cup_{t=0}^{T} F_t$.

Given the setting described by the event tree $\mathbf{D}$ and the associated commodity space $C(\mathbf{D}, \mathbb{R}^L)$ that consists of all functions $f : \mathbf{D} \to \mathbb{R}^L$, the economy can be described as in the two-period model. In particular, each consumer i has an endowment described by the process $w^i = (w^i(\xi), \xi \in \mathbf{D})$ and a utility function $u^i : C_+ \to \mathbb{R}$. Given the information structure F, the associated stochastic exchange economy is denoted by $\mathcal{E}(u, w, F)$ and, defining a process of contingent prices P, the contingent market budget set of each individual is:

$$B(P, w^i) = \left\{ x^i \in C_+ \mid P(x^i - w^i) = 0 \right\}$$

A contingent market equilibrium is defined as for an economy with 2 periods, and also in this case the results of existence and Pareto optimality of equilibrium hold. With a system of contingent markets, therefore, an economy with T periods behaves exactly as an economy with 2 periods.

2.2 The model with spot markets and financial markets with finite horizon

The model with contingent markets presented above is not realistic, hence it is possible to consider a different formulation that takes into account the fact that in reality markets have a sequential structure (Radner (1972)). This is done introducing a system of spot and financial markets and describing trading on these markets as a sequential process of equilibrium over time. Also in this case it is possible to examine first of all an economy with 2 periods and then an economy with T periods (with T finite); finally, the analysis is applied to an economy that extends over an infinite horizon.

2.2.1 Two-period exchange economy

In this model the sequential structure of markets is represented by:

- a system of *spot markets* for each of the L goods at time $t = 0$ and in each state s at time $t = 1$;

- a system of *financial markets* that provide instruments that enable agents to redistribute income across the different states of nature.

In this case $p = (p_0, p_1, ..., p_S) \in \mathbb{R}^n_{++}$ is the vector of spot prices, where $p_s = (p_{s1}, p_{s2}, ..., p_{sL})$ and p_{sl} denotes the price, measured in units of account, payable in state s for one unit of good l. The essential distinction between a *spot market* in state s and a *contingent market* for state s is that is that in the former the payment is made at date 1 in state s (if $s \geq 1$), while in the latter the payment is made at date 0 (as a consequence, in the case of contingent markets there is a single budget constraint, while in the case of spot markets there are $S+1$ budget constraints). We then assume that there are J financial activities (real assets), where the following definition holds:

Definition 3 *A real asset j is a contract which promises to deliver a vector of the L goods:*

$$A^j_s = (A^j_{s1}, A^j_{s2}, ..., A^j_{sL}) \in \mathbb{R}^L \qquad s = 1, 2, ..., S$$

in each state s at date 1*.*

A real asset is therefore characterized by a date-1 commodity vector $A^j = (A^j_1, A^j_2, ..., A^j_S) \in \mathbb{R}^{LS}$ and the revenue (in units of account) that it gives in state s is proportional to the spot price p_s:

$$V^j_s = p_s \cdot A^j_s \qquad s = 1, 2, ..., S$$

The asset j can be purchased at price q_j at time 0 and it gives a return $V^j = (V_1^j, V_2^j, ..., V_S^j)$ at time 1. The (column) vectors V^j can be combined to form the date-1 matrix of returns (of dimension $S \times J$):

$$V = \left[V^1 \cdots V^J\right] = \begin{bmatrix} V_1^1 & \cdots & V_1^J \\ \vdots & \ddots & \vdots \\ V_S^1 & \cdots & V_S^J \end{bmatrix}$$

that can also be expressed in the form:

$$V = V(p_{\mathbf{1}}, A) = \begin{bmatrix} p_1 A_1^1 & \cdots & p_1 A_1^J \\ \vdots & \ddots & \vdots \\ p_S A_S^1 & \cdots & p_S A_S^J \end{bmatrix}$$

where $p_{\mathbf{1}} = (p_1, p_2, ..., p_S) \in \mathbb{R}^{LS}$ is the date-1 vector of spot prices. The matrix V can also be written in the form:

$$V(p_{\mathbf{1}}, A) = \begin{bmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_S \end{bmatrix} \begin{bmatrix} A_1^1 & \cdots & A_1^J \\ \vdots & & \vdots \\ \vdots & & \vdots \\ A_S^1 & \cdots & A_S^J \end{bmatrix}$$

and since real assets are not influenced by the price level (in the sense that, for instance, doubling the spot prices in state s doubles their income), the financial structure of the economy can be summarized by the matrix:

$$A = \left[A^1 \cdots A^J\right] = \begin{bmatrix} A_1^1 & \cdots & A_1^J \\ \vdots & \ddots & \vdots \\ A_S^1 & \cdots & A_S^J \end{bmatrix}$$

of dimension $LS \times J$.

The economy consisting of I individuals with characteristics $(u, w) = (u^1, ..., u^I, w^1, ..., w^I)$ who trade J securities with date-1 payoffs given by the matrix V and with real asset structure given by the matrix A is denoted by $\mathcal{E}(u, w, A)$.

The matrix V generates the subspace of income transfers (or market subspace) $\langle V \rangle$, i.e. the subspace of $\mathbb{R}^S$ spanned by the J columns of V:

$$\langle V \rangle = \left\{\tau \in \mathbb{R}^S \mid \tau = Vz, z \in \mathbb{R}^J\right\}$$

and we have the following definition:

Definition 4 *If $\langle V \rangle = \mathbb{R}^S$ (i.e. the market subspace has maximal dimension), then the asset structure is called complete (i.e. there are complete financial markets), otherwise the asset structure is called incomplete (i.e. there are incomplete financial markets).*

This is a fundamental definition for all subsequent analysis. Since z represents a portfolio of the J assets, the fact that $\langle V \rangle = \mathbb{R}^S$ means that every possible state-dependent income $\tau \in \mathbb{R}^S$ can be obtained by means of an appropriate portfolio in the economy, and in this sense financial markets are complete, otherwise financial markets are incomplete (this is the situation that typically can be observed in reality).

The completeness of the asset strucure requires that dim $\langle V \rangle = S$, in this case the matrix V has full rank and we must have $J \geq S$ (with S assets that have linearly independent payoffs); as a consequence, whenever $J < S$ markets are incomplete. On the other hand, if $J \geq S$ and rank $V = S$, it is not possible to say unambiguously that markets are complete (because completeness is relative to a particular system of spot prices $p_{\mathbf{1}} = (p_1, p_2, ..., p_S)$); what can be proved is that in this case markets are generally (or potentially) complete (i.e. markets are complete for systems of spot prices that belong to an open set of full measure). This turns out to be fundamental in the discussion of the existence of equilibrium for this kind of models.

To introduce this concept of equilibrium it is possible to observe that if $z^i = (z_1^i, z_2^i, ..., z_J^i) \in \mathbb{R}^J$ denotes the number of units of each of the J assets purchased by agent i (where $z_j^i < 0$ represents a short-sale of the asset) there are $S + 1$ budget constraints given by:

$$\begin{cases} p_0(x_0^i - w_0^i) = -qz^i \\ p_s(x_s^i - w_s^i) = V_s z^i \quad s = 1, 2, ...S \end{cases}$$

where $q = (q_1, q_2, ..., q_J)$ and $V_s = (V_s^1, V_s^2, ..., V_s^J)$ is the row s of the matrix V. By defining the full matrix of returns (i.e. date-0 and date-1 returns) as:

$$W = W(q, V) = \begin{bmatrix} -q \\ V \end{bmatrix} = \begin{bmatrix} -q_1 & \cdots & -q_J \\ V_1^1 & \cdots & V_1^J \\ \vdots & \ddots & \vdots \\ V_S^1 & \cdots & V_S^J \end{bmatrix}$$

and for $p \in \mathbb{R}^n, x^i \in \mathbb{R}^n$ (where $n = L(S + 1)$) the box product:

$$p \square x^i = (p_0 x_0^i, p_1 x_1^i, ..., p_S x_S^i)$$

the budget set of individual i can be written in compact form as:

$$\mathcal{B}(p, q, w^i) = \left\{ x^i \in \mathbb{R}^n_+ \mid p\Box(x^i - w^i) = W(q, V)z^i \right\}$$

and in this case the agent chooses a pair (x^i, z^i) consisting of a vector of consumption x^i and a vector of securities (a portfolio) z^i. We now have the following definition of equilibrium:

Definition 5 *A spot-financial market equilibrium for the economy* $\mathcal{E}(u, w, A)$ *is a pair of allocations and prices* $((\overline{x}, \overline{z}), (\overline{p}, \overline{q}))$ *such that:*

(i) $(\overline{x}^i, \overline{z}^i)$ *satisfy* $\overline{x}^i = \arg\max \{u^i(x^i) \mid x^i \in \mathcal{B}(\overline{p}, \overline{q}, w^i)\}$ *and* $\overline{p}\Box(\overline{x}^i - w^i) = W(q, V)\overline{z}^i \quad i = 1, 2, ..., I$

(ii) $\sum_{i=1}^{I}(\overline{x}^i - w^i) = 0$

(iii) $\sum_{i=1}^{I} \overline{z}^i = 0$

Again, this means that at equilibrium individuals solve their optimization problems and markets clear, i.e. demand equals supply (in particular it is assumed that the total net supply of assets is zero, and for this reason the market-clearing condition on financial markets is expressed by *(iii)*).

While an economy $\mathcal{E}(u, w)$ with the characteristics described previously has always a contingent market equilibrium, this is not necessarily true in the case of spot and financial markets, where the possibility of non-existence of equilibrium arises (Hart (1975)). This is due to the fact that the rank of the return matrix may change when the prices (p, q) vary, determining discontinuities in the demand of the individuals that may lead to non-existence of equilibrium (a particular case where this doesn't happen and the return matrix doesn't drop rank is the one in which the only assets in the economy are short-lived numeraire securities, that will be considered in the version of the model with infinite horizon). For this reason, the results obtained in the model with spot and financial markets are weaker than those obtained in the model with contingent markets. In fact, these results (Magill-Shafer (1985), Duffie-Shafer (1985, 1986)) hold *generically* (i.e. in an open set of full measure, and this means that the values of the parameters for which the results don't hold represent a set with zero measure), and they are different considering generically (or potentially) complete markets or incomplete markets.

In the case of *generically complete markets* (the assets with linearly independent returns must be equal in number to the states of nature, and this requires $J \geq S$), if we denote by $E_A(w)$ the set of spot-financial market equilibrium allocations for the parameter value w, we have:

- *existence*: $E_A(w) \neq \emptyset \;\; \forall w \in \Omega'$ (generic set);
- *optimality*: $x \in E_A(w) \Rightarrow x$ is Pareto optimal $\forall w \in \Omega''$ (generic set);
- *local uniqueness*: $E_A(w)$ is a finite set (hence the equilibrium is locally unique).

Hence, when markets are potentially complete the results of classical general equilibrium theory extend generically to an economy with spot and financial markets.

In the case of *incomplete markets*, (i.e. $J < S$, there are less assets than states of nature and therefore there are limited possibilities to redistribute income across the different states of the world) only existence (and local uniqueness) continue to hold (generically), while the optimality of equilibrium allocations doesn't hold any more.

2.2.2 Stochastic exchange economy

As for an economy with contingent markets, also for an economy with spot and financial markets it is possible to extend the previous model to the case with many time periods $t = 1, 2, ..., T$. As before, the uncertainty is modelled by means of an event-tree, furthermore it is possible to consider J assets issued (for simplicity) at time 0, where asset j is characterized by a function $A^j : \mathbf{D} \to \mathbb{R}^L$ and one unit of the asset held at the initial node ξ_0 promises to deliver the commodity vector $A^j(\xi)$ at node ξ.

If $A(\xi) = \left[A^1(\xi), A^2(\xi), ..., A^J(\xi)\right]$, $\xi \in \mathbf{D}$ and $p \in C_+$ is a stochastic spot price process, then:

$$V^j(\xi) = p(\xi)A^j(\xi) \qquad \xi \in \mathbf{D}$$

is the dividend paid by asset j at node ξ. A security price process, then, is a function $q : \mathbf{D} \to \mathbb{R}^J$ and $q(\xi)$ is the vector of prices of the J assets at node ξ (with $q(\xi) = 0$ if ξ is a terminal node). Finally, the trading strategy of agent i is a function $z^i : \mathbf{D} \to \mathbb{R}^J$ and $z^i(\xi)$ is the portfolio of the J assets purchased by individual i at node ξ after the previous portfolio has been liquidated (with $z^i(\xi) = 0$ if ξ is a terminal node).

In this case the budget constraint is given by:

$$\begin{cases} p(\xi_0)(x^i(\xi_0) - w^i(\xi_0)) = -q(\xi_0)z^i(\xi_0) \\ p(\xi)(x^i(\xi) - w^i(\xi)) = \left[p(\xi)A(\xi) + q(\xi)\right] z^i(\xi^-) - q(\xi)z^i(\xi) \ , \forall \xi \in \mathbf{D} \setminus \xi_0 \end{cases}$$

where it is assumed that the portfolio purchased at any node ξ is sold at each immediate successor of ξ and that a new portfolio is then purchased. The payoff obtained on the investment $z_j^i(\xi^-)$ in asset j at the predecessor ξ^- of the node ξ consists of two parts, the dividend $p_j(\xi)A^j(\xi)z_j^i(\xi^-)$ and the capital value $q_j(\xi)z_j^i(\xi^-)$ (that derives from the sale of the portfolio at node ξ). The latter is the new term introduced by extending the model to the multiperiod case (while it is absent at the initial and terminal dates, that are the only dates that appear in the two-period model). We now have the following definition of equilibrium:

Definition 6 *A spot-financial market equilibrium for the economy* $\mathcal{E}(u, w, A)$ *is a pair of allocations and prices* $((\overline{x}, \overline{z}), (\overline{p}, \overline{q}))$ *such that:*

(i) $(\overline{x}^i, \overline{z}^i)$ *solves the maximization problem of individuals subject to the budget constraint;*

(ii) $\sum_{i=1}^{I}(\overline{x}^i - w^i) = 0$

(iii) $\sum_{i=1}^{I}\overline{z}^i = 0$

With reference to the results that can be obtained for this model, as for a two-period economy in the case of generically complete markets we have (generic) existence, optimality and local uniqueness of equilibrium, while in the case of incomplete markets only existence (and local uniqueness) of equilibrium is preserved, while the equilibrium is no longer Pareto optimal.

2.3 The model with spot markets and financial markets with infinite horizon

The previous model can be extended to the infinite horizon case (Magill-Quinzii (1994, 1996)), in order to make it more realistic, since economic activity is a process that has no natural terminal horizon. In this case a new problem is represented by the fact that, if agents are permitted to borrow, they may try to postpone indefinitely the repayment of their debts from one period to the next (the so-called *Ponzi scheme*), and if this happens there is no solution to an agent's decision problem an hence an equilibrium cannot exist. To avoid this problem it is necessary to impose a debt constraint.

Also in this extension of the model the economy is described by means of an event-tree structure (exactly as in the model with finite horizon), in particular we have (considering bounded sequences):

- set of commodities over the event-tree given by $\mathbf{D}\times\mathbf{L} = \{(\xi, l) \mid \xi \in \mathbf{D}, l \in \mathbf{L}\}$;

- initial endowment of individual i given by the process $w^i = (w^i(\xi,l),(\xi,l) \in \mathbf{D} \times \mathbf{L})$;

- consumption of individual i given by the process $x^i = (x^i(\xi,l),(\xi,l) \in \mathbf{D} \times \mathbf{L})$;

- prices of the goods given by the process $p = (p(\xi,l),(\xi,l) \in \mathbf{D} \times \mathbf{L})$;

- set of the assets issued at node ξ given by $J(\xi)$, set of all assets given by $\mathbf{J}$;

- dividends of the assets at nodes after the node of issue given by the process $pA = (p(\xi')A(\xi'),\ \xi'$ successor of $\xi)$, that is the value at spot prices $p(\xi')$ of a bundle $A(\xi')$ of the L goods;

- prices of the assets given by the process $q = (q(\xi,j),(\xi,j) \in \mathbf{D} \times \mathbf{J})$;

- portfolio of individual i given by the process $z^i = (z^i(\xi,j),(\xi,j) \in \mathbf{D} \times \mathbf{J})$.

A distinction that can be made is between *short-lived assets* (they are traded only at the nodes of issue and pay dividends only at the immediate successors of these nodes) and *long-lived assets* (in particular *infinite-lived assets* if they are traded at each node after the node of issue). In the case of short-lived assets the hypotheses necessary to prove the existence of the equilibrium are slightly weaker than the hypotheses in the general case.

If $\succsim = (\succsim_1, \succsim_2, ..., \succsim_I)$ and $w = (w^1, w^2, ..., w^I)$ represent the preference ordering and the initial endowments of the I individuals and A the generic security structure, the economy considered, defined on the event-tree $\mathbf{D}$, is represented by $\mathcal{E}_\infty(\mathbf{D}, \succsim, w, A)$.

The essential hypotheses used to prove the existence of the equilibrium are:

- for each node there is a finite number of immediate successors;

- the aggregate endowment of the economy is bounded above;

- agents are impatient (their preferences are continuous in a particular topology, the Mackey topology), i.e. they prefer early to distant consumption;

- the degree of impatience of agents is bounded below by a positive number (i.e. it doesn't vanish asymptotically) and this lower bound is equal for all nodes;

- there is a borrowing constraint which limits the amount of debt of the agents; this debt constraint can be explicit:

$$q(\xi)z^i(\xi) \geq -M \qquad \forall \xi \in \mathbf{D} \tag{1}$$

(where M is a positive number, in this case at each date the agent's debt cannot exceed the bound M), or it can be implicit:

$$(qz^i) = (q(\xi)z^i(\xi), \xi \in \mathbf{D}) \in l_\infty(\mathbf{D}) \tag{2}$$

(where l_∞ is the space of bounded sequences, hence the debt cannot grow without bound), or it can be introduced through a transversality condition:

$$\lim_{T\to\infty} \sum_{\xi' \in \mathbf{D}_T(\xi)} \pi^i(\xi')q(\xi')z^i(\xi') = 0 \qquad \forall \xi \in \mathbf{D} \tag{3}$$

(where $\mathbf{D}_T(\xi)$ is the set, at time T, of the nodes that are successors of ξ, and π^i are the state prices) - this condition states that the present value of the debt of each individual must be asymptotically zero -.

Each agent, therefore, has a budget set expressed by the condition:

$$p(\xi)(x^i(\xi) - w^i(\xi)) = [p(\xi)A(\xi) + q(\xi)]\, z^i(\xi^-) - q(\xi)z^i(\xi)$$

and, furthermore, by one of conditions (1), (2), (3). The following definition of equilibrium now holds:

Definition 7 *An equilibrium for the economy $\mathcal{E}_\infty(\mathbf{D}, \succsim, w, A)$ is a pair of allocations and prices $((\overline{x}, \overline{z}), (\overline{p}, \overline{q}))$ such that:*

(i) $(\overline{x}^i, \overline{z}^i)$ solves the maximization problem of the individuals subject to the budget constraint;

(ii) $\sum_{i=1}^{I}(\overline{x}^i - w^i) = 0$

(iii) $\sum_{i=1}^{I} \overline{z}^i = 0$

For the economy considered it is possible to prove the existence of the equilibrium for all values of the parameters that characterize it in the case of a particular asset structure, composed only by short-lived numeraire securities (to this end it is sufficient to consider an economy with the same characteristics as $\mathcal{E}_\infty(\mathbf{D}, \succsim, w, A)$ but with finite horizon $\mathcal{E}_T(\mathbf{D}, \succsim, w, A)$, for which it is known that an equilibrium exists for each value of the parameters, then it is possible to consider the limit for T tending to ∞ and to show that the limit of the equilibrium of this economy $\mathcal{E}_T$ is an equilibrium for the limit economy $\mathcal{E}_\infty$).

If the economy is charachterized by the presence of a general asset structure, with long-lived assets, a further problem arises because in this case the return matrix may change rank when the prices (p, q) vary, causing non-existence of equilibrium (this problem doesn't arise if there are only short-lived numeraire securities). What it is possible to show in this situation is that the equilibrium exists for a dense set of economies (i.e. if the economy $\mathcal{E}_\infty(\mathbf{D}, \succsim, w, A)$ doesn't have an equilibrium, then for every $\varepsilon > 0$ there exists a financial structure A' in the ball of radius ε around A such that the economy $\mathcal{E}_\infty(\mathbf{D}, \succsim, w, A')$ has an equilibrium).

In conclusion, the results obtained with reference to the question of existence of the equilibrium in the models considered can be summarized as follows:

- *model with contingent markets*: existence of the equilibrium (both in an economy with 2 periods and in an economy with T periods);
- *model with spot and financial markets with finite horizon*: existence of the equilibrium for a *generic set* of economies (both in an economy with 2 periods and in an economy with T periods);
- *model with spot and financial markets with infinite horizon*: existence of the equilibrium in the case of short-lived numeraire assets, existence of the equilibrium for a *dense set* of economies in the case of a general financial structure.

In the transition from the model with contingent markets to the model with spot and financial markets and, within this model, from the finite horizon case to the infinite horizon case, something is "lost" in terms of existence of solutions. Nevertheless, the existence of the equilibrium (even if in a weaker form) is preserved; the model is therefore *consistent*, and can be used to explain something of the economy under study. In particular, it is possible to obtain indications concerning the equilibrium prices of the assets, and this is the subject of the following Section.

3 Asset pricing and speculative bubbles

The model with spot and financial markets introduced above can be used to investigate the relation between the equilibrium price of an asset and the stream of future dividends to which this asset represents a claim. With reference to this aspect, the *theorem of absence of arbitrage opportunities* on financial markets hold. According to this theorem (in the finite horizon case), the utility maximization problem of individual i considered in the definition of equilibrium has a solution if and only if q is a no-arbitrage asset price, where the following definition holds:

Definition 8 *$q \in \mathbb{R}^J$ is a no-arbitrage asset price if there does not exist a portfolio $z \in \mathbb{R}^J$ such that $W(q,V)z \geq 0$ (and not null), i.e. a portfolio such that:*

$$Vz \geq 0 \qquad qz \leq 0$$

with at least one strict inequality. This means that it does not exist an investment strategy which gives a positive payoff in at least one state with non-negative payoffs in all remaining states.

In the case of a two-period economy the absence of arbitrage opportunities implies the existence of a present value vector $\pi = (\pi_0, \pi_1, ..., \pi_S)$ (whose components are positive state prices, that represent the present value at date 0 of one unit of income in state s) such that $\pi W = 0$, i.e.:

$$\pi_0 q_j = \sum_{s=1}^{S} \pi_s V_s^j \qquad j = 1, 2, ..., J$$

so that (since π_0 can be normalized to 1) the price of each asset is equal to the present value of its future dividend stream. In the case of a T-period economy, similarly, the absence of arbitrage opportunities implies the existence of a present value process $\pi : \mathbf{D} \rightarrow \mathbb{R}_{++}$ (whose components are positive state prices) such that:

$$\pi(\xi)q(\xi) = \sum_{\xi' \in \xi^+} \pi(\xi') \left[p(\xi')A(\xi') + q(\xi') \right] \qquad \forall \xi \in \mathbf{D}^-$$

and therefore the value at date 0 of the price of an asset at node ξ is the present value of its dividends and capital values over the set of immediate successors ξ^+. Solving this system of equations recursively and using the terminal condition $q(\xi) = 0, \forall \xi \notin \mathbf{D}^-$ (where $\mathbf{D}^-$ is the set of non-terminal nodes) we get:

$$q(\xi) = \frac{1}{\pi(\xi)} \sum_{\xi' > \xi} \pi(\xi')p(\xi')A(\xi') \qquad \forall \xi \in \mathbf{D}^- \tag{4}$$

and also in this case the current value of each asset at node ξ is the present value of its future dividend stream over all succeeding nodes $\xi' > \xi$. The right-hand side of expression (4) is the so-called *fundamental value* of the asset, and hence in the finite horizon case the equilibrium price of an asset is equal to this value. In the infinite horizon case, on the contrary, it may happen that the equilibrium price of the asset is larger than its fundamental value: in this case the difference represents a *speculative bubble*.

These are the questions addressed in the present Section. In particular, the issue of speculative bubbles is initially studied by means of the infinite-horizon model introduced in Section 2, then it is considered more specifically along the lines of a recent contribution that tries to give definitive results concerning this question. Finally, the problem is analysed following a new approach, based on Euler equations. This approach is also used, in the next Section, to construct and to study specific examples in which bubbles do appear.

3.1 The model of Magill-Quinzii (1996)

The model, introduced in the previous Section, of Magill-Quinzii (1996) is based on the problem, considered above, of the existence of an equilibrium, but gives also indications on the existence of speculative bubbles. In particular, it is possible to observe that relation (4) remains true, in the infinite horizon case, for finitely-lived securities, for which the terminal condition $q(\xi) = 0$ holds; in the case of infinite-lived securities, on the contrary, this condition doesn't hold, and therefore the equality between the asset price and the fundamental value is not guaranteed.

With reference to an infinite-horizon economy, the following definition can then be introduced:

Definition 9 *Let $((\overline{x}, \overline{z}), (\overline{p}, \overline{q}))$ be an equilibrium for the economy with infinite horizon $\mathcal{E}_\infty(\mathbf{D}, \succeq, w, A)$. The security $j \in J$ is priced at its fundamental value if, for all agents $i \in I$, we have:*

$$\overline{q}(\xi) = \frac{1}{\overline{\pi}^i(\xi)} \sum_{\xi' > \xi} \overline{\pi}^i(\xi') \overline{p}(\xi') A(\xi') \qquad \forall \xi \in \mathbf{D}$$

(where the values $\overline{\pi}^i$ are those corresponding to prices $\overline{q}$). If this relation is not satisfied for some agent $i \in I$, the security j has a speculative bubble.

The central conclusion of the model of Magill-Quinzii in terms of bubbles distinguishes between (infinite-lived) securities in positive net supply and in zero net supply. In particular, the following results hold:

- if an (infinite-lived) security is in *positive net supply* its price (under the hypotheses of the model, in particular the existence of a uniform lower bound to the impatience of the agents) is equal to its fundamental value (hence there cannot be speculative bubbles on the prices of these assets);
- if an (infinite-lived) security is in *zero net supply* it is always possible to add a bubble component to the equilibrium price such that the new price remains an equilibrium price (hence there can be speculative bubbles on the prices of these assets).

Since securities in positive net supply (such as equity contracts) represent an important part of the capital market, the result that on such securities bubbles cannot arise reduces consistently the role of speculation in this class of models, and this is the central conclusion emerging from the analysis of the model of Magill-Quinzii.

3.2 The model of Santos-Woodford (1997)

The problem of bubbles on asset pricing can then be studied following a contribution, due to Santos-Woodford (1997), that examines this problem specifically, trying to make order in the field and to give a definitive theoretical settlement to this controversal question. The scope of the model of Santos-Woodford is the analysis of conditions under which speculative bubbles are possible or not in an intertemporal general equilibrium model with infinite horizon, spot markets for the goods and markets for securities, in which agents have rational expectations. It is similar to the model, with infinite horizon, analysed above, but it is more general because it considers assets that give returns not only in terms of consumption goods but also in terms of other securities, and furthermore it incorporates both the case of infinite-lived agents (as in the model of Magill-Quinzii) and the case of finitely-lived agents but with overlapping generations (not considered by the previous model). For this reason, the model of Santos-Woodford gives results that hold for a very general class of economies.

In this case the economy consists of:

- an event-tree structure with a finite number of nodes ξ at each date $t = 0, 1, 2, ...$ (in particular there is a unique initial node and each node has a unique immediate predecessor and a finite number of immediate successors);

- markets for goods and for securities at each node;

- endowment of consumption goods at each node given by $w(\xi) \geq 0$;

- consumption of goods at each node given by $x(\xi) \geq 0$ with prices $p(\xi)$;

- securities that give a dividend in terms of consumption goods and of other securities;

- net supply of securities at each node given by $z(\xi) \geq 0$ with prices of the securities $q(\xi)$.

The hypotheses that are essential for the results of this model are:

- *monotonicity of preferences* of individuals (the preference relation is non-decreasing on the consumption set and strictly increasing in the consumption of at least one good traded at each node);

- existence of *impatience* of agents with positive lower bound on the degree of impatience that is uniform across the nodes (this hypothesis allows to obtain stronger results).

Given all these elements, the individual $i \in I$ chooses at each node ξ a vector of consumption goods $x^i(\xi)$ and a vector of securities $z^i(\xi)$ subject to the budget constraints:

$$\begin{array}{l} p(\xi)(x^i(\xi) - w^i(\xi)) \leq V(\xi)z^i(\xi^-) - q(\xi)z^i(\xi) \\ q(\xi)z^i(\xi) \geq -M^i(\xi) \end{array}$$

where $V(\xi)$ is the vector of one-period returns in correspondence of node ξ and $M^i(\xi) \geq 0$ represents the borrowing limit of agent i at node ξ. Also in this case, an equilibrium is a process that consists of pairs of allocations and prices such that individuals maximize their utility subject to the budget constraint and markets clear.

The next step is the characterization of the relation between the price of a security and the value of the stream of dividends to which it represents a claim. As noted above, the absence of arbitrage opportunities implies the existence of positive state prices $\pi(\xi)$ such that:

$$\pi(\xi)q(\xi) = \sum_{\xi' \in \xi^+} \pi(\xi')V(\xi') \qquad \forall \xi \in \mathbf{D}^-$$

This equation has a unique solution in the case of complete markets, while in the case of incomplete markets there are different vectors of state prices that satisfy the relation. Iterating this expression we get:

$$\pi(\xi)q(\xi) \geq \sum_{\xi' \in \mathbf{D}^T(\xi)} \pi(\xi')p(\xi')A(\xi')$$

where $\mathbf{D}^T(\xi)$ denotes the nodes between dates $t(\xi)$ and T that belong to the subtree with root ξ. The right-hand side expression is a non-decreasing and upper bounded series, hence it converges as $T \to \infty$ (to a limit no greater than the left-hand side expression) and we have:

$$q(\xi) \geq \frac{1}{\pi(\xi)} \sum_{\xi' > \xi} \pi(\xi')p(\xi')A(\xi') \geq 0$$

where the central term is the fundamental value of the asset. We have therefore:

$$0 \leq f(\xi) \leq q(\xi)$$

and as a consequence, differently from the finite-horizon case, in the infinite-horizon case it is not necessarily true that $q(\xi) = f(\xi)$. It is then possible to define the vector of speculative bubbles:

$$\sigma(\xi) = q(\xi) - f(\xi)$$

that satisfies the bounds:

$$0 \leq \sigma(\xi) \leq q(\xi)$$

from which we get the result of "impossibility of negative bubbles".

In the case of finitely-lived securities the fundamental value is uniquely defined (also with incomplete markets, with different state prices) and $f(\xi) = q(\xi)$, so that there are no bubbles. In the case of infinite-lived securities with incomplete markets, on the contrary, there can be bubbles. Furthermore, with incomplete markets the fundamental value may be different in correspondence of different values of state prices, and in this case the so-called *ambiguous bubbles* can arise.

With reference to this aspect, the fundamental conclusions of the model are the following:

- under the hypothesis of monotonicity of preferences and of finite aggregate endowment of the economy there exists in equilibrium a state-price process such that $q(\xi) = f(\xi)$ for each security that is either finitely-lived or in positive net supply (nevertheless, for securities in positive net supply is not excluded the existence of state prices for which $q(\xi) > f(\xi)$, i.e. it is not excluded the presence of ambiguous bubbles);
- considering also the hypothesis of impatience of the agents, we have in equilibrium (regardless of the state-prices chosen) $q(\xi) = f(\xi)$ for each security that is either finitely lived or in positive net supply (and therefore in this case it is excluded the presence of ambiguous bubbles).

In conclusion, if the agents have monotonic preferences, they are sufficiently impatient and the aggregate endowment of the economy is finite, it is possible to have speculative bubbles only on securities that are both infinite-lived and in zero net supply (this is exactly the same result obtained in the model of Magill-Quinzii). These results hold for a wide class of economies; the conclusion is that, under quite general assumptions, there are not speculative bubbles on asset prices, and the existence of such bubbles is linked to special circumstances, so that it is a fragile phenomenon.

3.3 The approach of Euler equations

The issue of bubbles on asset pricing can also be analysed by means of a different approach, based on the use of Euler equations and inequalities. This approach is more limited than the one considered above (which has led to very general models, used to deal with the problem of existence of solutions and of speculative bubbles in a wide class of intertemporal economies), but it is of some interest because it confirms the results on fragility of bubbles obtained above, and it also provides a method that can be used to obtain examples in which bubbles do appear. In this case it is no longer necessarily true that the number of states of nature at each date is finite, hence it is necessary to introduce in the model also a probabilistic structure.

A first step is represented by the study of a model "a' la Lucas" (Lucas (1978)) that considers consumers which are identical in terms of utilities and endowments, i.e. a *model with homogeneous agents* (Montrucchio-Privileggi (2001)). The economy consists of:

- a set **I** of individuals with utility function $u_t(x)$ defined on a single non-storable consumption good for $t = 0, 1, 2, ...$;

- endowment of the consumption good for each agent at time t given by w_t;

- J infinite-lived assets that give a dividend $d_t \in \mathbb{R}^J_+$ in terms of the consumption good and that are held in quantity $z_t \in \mathbb{R}^J$ by each agent at time t;

- initial endowment of each asset normalized to 1, i.e. $\overline{z}_0 = e = (1, 1, ..., 1) \in \mathbb{R}^J$ (only securities in positive net supply are considered);

- spot markets for the consumption good and for the assets;

- prices of the assets given by $q_t \in \mathbb{R}^J_+$ and price of the consumption good given by $p_t = 1$ (numeraire);

- uncertainty described through a probabilistic space $(\Omega, \mathbf{F}, \mu)$.

Given these elements a plan $(x, z) = (x_t(\omega), z_t(\omega))$ is feasible if it satisfies the constraint:

$$x_t(\omega) + q_t(\omega)[z_{t+1}(\omega) - z_t(\omega)] \leq d_t(\omega)z_t(\omega) + w_t(\omega)$$

according to which the expenditure in each period cannot exceed the endowment of the same period, and in an economy with these characteristics (and in which we also assume that short-selling is prohibited, i.e. individuals cannot borrow) there exists a "no-trade equilibrium" (as a consequence of the fact that individuals are identical in terms of preferences and endowments), that is an equilibrium in which agents hold their assets forever and consume all their available wealth at each date. This equilibrium can be characterized in the following way:

Definition 10 *A no-trade equilibrium for the economy considered is a pair of allocations and prices* $((\overline{x}, \overline{z}), q)$*,* $t = 0, 1, 2, ...$ *such that:*

(i) the price process q *satisfies:*

$$0 \leq q_t < +\infty \qquad \textit{almost surely for all } t$$

(ii) the consumption plan $\overline{x} = \{\overline{x}_t\} = \{d_t \cdot e + w_t\}, \overline{z} = \{e\}$ *is optimal with respect to all feasible consumption plans.*

Is is then possible to prove that at the equilibrium the following relation (that is a stochastic Euler inequality) must be satisfied:

$$u'_{t-1}(\overline{x}_{t-1})q_{t-1} \geq \mathbf{E}_{t-1}[u'_t(\overline{x}_t)(d_t + q_t)] \qquad \text{for all } t \geq 1$$

(in particular it holds as an equality, for instance, when the states of nature are finite at date t). This relation represents a necessary condition of optimality in the short run and can be used to introduce the dinstinction between the fundamental value of an asset and the bubble component, to this end it is possible to write:

$$u'_{t-1}(\overline{x}_{t-1})q_{t-1} = \mathbf{E}_{t-1}[u'_t(\overline{x}_t)(d_t + q_t)] + s_{t-1}$$

where the vectors $s_t \geq 0$ measure the deviation from the equality in the Euler equation. Writing for simplicity $u'_t(\overline{x}_t) = \pi_t$ and iterating the last equation we get:

$$\pi_t q_t = \mathbf{E}_t \sum_{r=1}^{k} \pi_{t+r} d_{t+r} + \mathbf{E}_t \sum_{r=0}^{k-1} s_{t+r} + \mathbf{E}_t[\pi_{t+k} q_{t+k}]$$

and considering the limit for k that goes to ∞:

$$\pi_t q_t = \mathbf{E}_t \sum_{r=1}^{\infty} \pi_{t+r} d_{t+r} + \mathbf{E}_t \sum_{r=0}^{\infty} s_{t+r} + \lim_{k \to +\infty} \mathbf{E}_t[\pi_{t+k} q_{t+k}]$$

and also:

$$q_t = \frac{1}{\pi_t} \mathbf{E}_t \sum_{r=1}^{\infty} \pi_{t+r} d_{t+r} + \frac{1}{\pi_t} \mathbf{E}_t \sum_{r=0}^{\infty} s_{t+r} + \frac{1}{\pi_t} \lim_{k \to +\infty} \mathbf{E}_t[\pi_{t+k} q_{t+k}]$$

The price of the asset can therefore be decomposed into 3 components:

$$q_t = f_t + \widetilde{b}_t + \overline{b}_t$$

where the first term of the right-hand side is the *fundamental value*:

$$f_t = \frac{1}{\pi_t} \mathbf{E}_t \sum_{r=1}^{\infty} \pi_{t+r} d_{t+r}$$

while the other two terms represent the bubble component, in particular the second term is the *bubble due to the violation of the Euler equation*:

$$\widetilde{b}_t = \frac{1}{\pi_t} \mathbf{E}_t \sum_{r=0}^{\infty} s_{t+r}$$

and the third term is the *asymptotic bubble*:

$$\bar{b}_t = \frac{1}{\pi_t} \lim_{k \to +\infty} \mathbf{E}_t[\pi_{t+k} q_{t+k}]$$

The fundamental results that can be obtained in this model concerning the existence of speculative bubbles in the economy are the following:

- the non-existence of bubbles is a generic property (in fact the set of initial endowments for which in equilibrium there is a bubble has zero measure);
- if the agents' preferences exhibit uniformly bounded relative risk aversion, then the pricing equilibrium is unique (without bubbles).

In conclusion, also in this case the existence of bubbles represents a fragile phenomenon.

The next step is the extension of the model to the case of *heterogeneous agents* (Montrucchio (2001)) , in this case the economy consists of:

- a set $\mathbf{I}$ of individuals with utility function $u_t^i(x)$ defined on a single non-storable consumption good for $t = 0, 1, 2, ...$;

- endowment of the consumption good for agent i at time t given by w_t^i;

- J infinite-lived assets that give a dividend $d_t \in \mathbb{R}_+^J$ in terms of the consumption good and that are held in quantity $z_t^i \in \mathbb{R}^J$ by agent i at time t;

- share of the total asset supply received by agent i at time 0 given by $\overline{z}_0^i$, with $\sum_{i \in \mathbf{I}} \overline{z}_0^i = e= (1, 1, ..., 1) \in \mathbb{R}^J$

- spot markets for the consumption good and for the assets;

- prices of the assets given by $q_t \in \mathbb{R}_+^J$ and price of the consumption good given by $p_t = 1$ (numeraire);

- uncertainty described through a probabilistic space $(\Omega, \mathbf{F}, \mu)$.

Each agent i maximizes his utility over time facing, in each period t, the traditional budget constraint:

$$x_t^i(\omega) + q_t(\omega)(z_{t+1}^i(\omega) - z_t^i(\omega)) \leq d_t(\omega) z_t^i(\omega) + w_t^i(\omega)$$

that implies the feasibility of a consumption plan, together with a borrowing constraint given by $z_t^i \geq k_t^i$ (where k_t^i represents a borrowing limit, in particular if $k_t^i = 0$ this means that short-selling is prohibited). For this economy it is then possible to introduce the following definition of equilibrium:

Definition 11 *An equilibrium for the economy considered is a pair of allocations and prices* $((\overline{x}^i, \overline{z}^i), q)$, $i \in \mathbf{I}$, $t = 0, 1, 2, ...$ *such that:*

(i) *the price process* q *satisfies:*

$$0 \leq q_t < +\infty \qquad \textit{almost surely for all } t$$

(ii) *for every* i *the consumption plan* $\overline{x}^i$ *is optimal with respect to all feasible consumption plans;*

(iii) $\sum_{i \in \mathbf{I}} \overline{z}^i_t \leq \mathbf{1}$ *and* $q_t \left(\mathbf{1} - \sum_{i \in \mathbf{I}} \overline{z}^i_t\right) = 0$ *for all* t.

Also in this case it is possible to prove that at the equilibrium the following stochastic Euler inequality must be satisfied:

$$Du^i_{t-1}(\overline{x}^i_{t-1})q_{t-1} \geq \mathbf{E}_{t-1}[Du^i_t(\overline{x}^i_t)(d_t + q_t)] \qquad \text{for all } t \geq 1$$

and under some additional requirements (that are satisfied, for instance, in the case of a finite number of states of nature at each date) also the following condition must hold:

$$\left\{Du^i_{t-1}(\overline{x}^i_{t-1})q_{t-1} - \mathbf{E}_{t-1}[Du^i_t(\overline{x}^i_t)(d_t + q_t)]\right\} \cdot (\overline{z}^i_t - k^i_t) = 0$$

Again it is possible to introduce the distinction between the fundamental value of an asset and the bubble component, in fact using the first relation we can write:

$$Du^i_{t-1}(\overline{x}^i_{t-1})q_{t-1} = \mathbf{E}_{t-1}[Du^i_t(\overline{x}^i_t)(d_t + q_t)] + s^i_{t-1}$$

where the vectors $s_t \geq 0$ measure the deviation from the equality in the Euler equation. Writing for simplicity $Du^i_t(\overline{x}^i_t) = \pi^i_t$ and iterating the last equation we get:

$$\pi^i_t q_t = \mathbf{E}_t \sum_{r=1}^{k} \pi^i_{t+r} d_{t+r} + \mathbf{E}_t \sum_{r=0}^{k-1} s^i_{t+r} + \mathbf{E}_t[\pi^i_{t+k} q_{t+k}]$$

and considering the limit for k that goes to ∞:

$$\pi^i_t q_t = \mathbf{E}_t \sum_{r=1}^{\infty} \pi^i_{t+r} d_{t+r} + \mathbf{E}_t \sum_{r=0}^{\infty} s^i_{t+r} + \lim_{k \to +\infty} \mathbf{E}_t[\pi^i_{t+k} q_{t+k}]$$

and also:

$$q_t = \frac{1}{\pi^i_t} \mathbf{E}_t \sum_{r=1}^{\infty} \pi^i_{t+r} d_{t+r} + \frac{1}{\pi^i_t} \mathbf{E}_t \sum_{r=0}^{\infty} s^i_{t+r} + \frac{1}{\pi^i_t} \lim_{k \to +\infty} \mathbf{E}_t[\pi^i_{t+k} q_{t+k}]$$

and again the price of the asset can be decomposed into 3 components:

$$q_t = f_t^i + \widetilde{b}_t^i + \overline{b}_t^i$$

where the first term of the right-hand side is the *fundamental value*, the second term is the *bubble due to the violation of the Euler equation* and the third term is the *asymptotic bubble.*

At this point it is possible to obtain conditions that allow to rule out bubbles in the model with heterogeneous agents. A first result is the following:

Theorem 12 *Let* $((\overline{x}^i, \overline{z}^i), q)$ *be an equilibrium; if the following conditions are fulfilled:*

(i) $\mathbf{E}\sum_{t=1}^{\infty} Du_t^i(\overline{x}_t^i)(\overline{x}_t^i - w_t^i)^+ < +\infty$

(ii) agent i *exhibits a uniformly bounded relative risk aversion, i.e. there is some scalar* R *such that:*

$$-\frac{D^2 u_t^i(x)x}{Du_t^i(x)} \leq R$$

for all x *and all* $t \geq 1$

(iii) short-selling is prohibited, i.e. $k_t^i = 0$

then:

$$\lim_{t \to +\infty} \mathbf{E}[Du_t^i(\overline{x}_t^i) q_t \cdot \overline{z}_{t+1}^i] = 0$$

Proof. See Montrucchio (2001) ■

The last relation is a transversality condition at infinity, and it can be used to obtain the following fundamental result, that allows to exclude the presence of bubbles at an equilibrium:

Corollary 13 *Under the assumptions of the previous Theorem, if there exists a sequence of times* t_n *and some scalar* $\varepsilon > 0$ *such that:*

$$\overline{z}_{t_n}^i \geq \varepsilon \cdot \mathbf{1}$$

then the asymptotic bubble $\overline{b}_t^i$ *is absent.*

If, in addition, we have:

(i) $\overline{z}_t^i > 0$ *almost surely*

(ii) $(d_t + q_t) \cdot \overline{z}_t^i \leq M_t \overline{x}_t^i$ *for some sequence* M_t

then the bubble component vanishes and $q_t = f_t^i$.

Proof. See Montrucchio (2001) ■

These are the central results concerning non-existence of speculative bubbles in the framework with heterogeneous agents. Furthermore, the transversality condition at infinity obtained above can be used to establish sufficient conditions of optimality, in fact the following result holds:

Proposition 14 *Let $(\overline{x}^i, \overline{z}^i)$ be an allocation for an economy where short-selling is prohibited, and q_t be a price process such that:*

(i)

$$Du^i_{t-1}(\overline{x}^i_{t-1})q_{t-1} \geq \mathbf{E}_{t-1}[Du^i_t(\overline{x}^i_t)(d_t + q_t)]$$
$$\left\{Du^i_{t-1}(\overline{x}^i_{t-1})q_{t-1} - \mathbf{E}_{t-1}[Du^i_t(\overline{x}^i_t)(d_t + q_t)]\right\} \cdot \overline{z}^i_t = 0$$

for agent i and all t

(ii)

$$\lim_{t\to+\infty}\inf \mathbf{E}[Du^i_t(\overline{x}^i_t)q_t \cdot \overline{z}^i_{t+1}] = 0$$

Then the allocation $(\overline{x}^i, \overline{z}^i)$ is weakly optimal.

If the stronger condition:

$$\lim_{t\to+\infty} \mathbf{E}[Du^i_t(\overline{x}^i_t)q_t \cdot \overline{z}^i_{t+1}] = 0$$

is satisfied, then the allocation is strongly optimal.

This condition is useful to check the optimality of solutions when there are bubbles caused by the violation of Euler equations (as it will be clear in the next Section).

In conclusion, also the approach based on Euler equations shows that bubbles are linked to very special situations, and therefore represent a fragile phenomenon. To support this idea of marginality of bubbles it is possible to give a series of examples in which speculative bubbles do appear, and from which it emerges that, in order to obtain such a result, very special conditions are needed. This is the object of the next Section.

4 Examples of bubbles

In this Section some examples in which speculative bubbles do appear are presented, in order to underline the very special conditions that allow the presence of this phenomenon. In particular, the examples considered are the following:

- a monetary model *"a' la Bewley"*;
- a general deterministic model;
- a stochastic model;
- a model with asymptotic bubble.

4.1 A monetary model "a' la Bewley"

The first example, based on the monetary model introduced by Bewley (1980), considers a deterministic economy in which there is a unique asset traded at each date, the so-called *fiat money*, which gives no dividends. There are two individuals ($i = 1, 2$) with the same preferences given by:

$$\sum_{t=0}^{\infty} \beta^t u(x_t^i)$$

where β is the discount factor with $0 < \beta < 1$ and $u(x_t^i)$ is strictly increasing and strictly concave. Since in the Euler equations the functions u_t^i appear, in this situation we have $u_t^i(x_t^i) = \beta^t u(x_t^i)$ and the problem solved by each single agent is:

$$\begin{array}{rc} & \max \sum_{t=0}^{\infty} \beta^t u(x_t^i) \\ s.t. & x_t^i + q_t(z_{t+1}^i - z_t^i) \leq d_t z_t^i + w_t^i \\ & z_0^i = \overline{z}_0^i \\ & x_t^i \geq 0 \\ & z_t^i \geq k_t^i \end{array}$$

The endowments of the good for the two individuals are:

$$w_t^1 = \begin{cases} \overline{w} & \text{for } t \text{ even} \\ \underline{w} & \text{for } t \text{ odd} \end{cases} \qquad w_t^2 = \begin{cases} \underline{w} & \text{for } t \text{ even} \\ \overline{w} & \text{for } t \text{ odd} \end{cases}$$

(with $\underline{w} < \overline{w}$), i.e. they are symmetric across time, while the initial endowments of the asset are:

$$z_0^1 = 0 \qquad z_0^2 = 1$$

In addition, short-selling is prohibited ($k_t^1 = k_t^2 = 0 \ \forall t$) and, since there are no dividends, $d_t = 0 \ \forall t$. Sargent (1987) shows euristically that this model has, together with the "autarchic" equilibrium (where fiat money is never valued, i.e. $q_t = 0$, and each agent consumes his own endowment at each trading date) another equilibrium in which fiat money has a positive price and the consumption of each agent is of the form:

$$\overline{x}_t^1 = \begin{cases} x^* & \text{for } t \text{ even} \\ x^{**} & \text{for } t \text{ odd} \end{cases} \qquad \overline{x}_t^2 = \begin{cases} x^{**} & \text{for } t \text{ even} \\ x^* & \text{for } t \text{ odd} \end{cases}$$

while the asset holding is of the form:

$$\overline{z}_t^1 = \begin{cases} 0 & \text{for } t \text{ even} \\ m & \text{for } t \text{ odd} \end{cases} \qquad \overline{z}_t^2 = \begin{cases} m & \text{for } t \text{ even} \\ 0 & \text{for } t \text{ odd} \end{cases}$$

where m is the initial endowment of the asset.

The same result can be obtained using the approach of Euler equations. In this case the strategy followed consists in looking for an equilibrium in which the asset holding strategies are of a certain form; the budget constraints, then, give the corresponding consumption values, and at this point the solution candidate to be an equilibrium can be determined by means of the Euler equations. Finally, the fact that the values found represent a true equilibrium can be checked with the sufficient condition of optimality presented at the end of Section 3.

In particular, in the model considered it is possible to look for an equilibrium in which the price of the asset is constant ($q_t = q > 0$) and the holding strategies of the two individuals are:

$$\overline{z}_t^1 = \begin{cases} 0 & \text{for } t \text{ even} \\ 1 & \text{for } t \text{ odd} \end{cases} \qquad \overline{z}_t^2 = \begin{cases} 1 & \text{for } t \text{ even} \\ 0 & \text{for } t \text{ odd} \end{cases}$$

while from the budget constraints the equilibrium consumptions turn out to be:

$$\overline{x}_t^1 = \begin{cases} \overline{w} - q & \text{for } t \text{ even} \\ \underline{w} + q & \text{for } t \text{ odd} \end{cases} \qquad \overline{x}_t^2 = \begin{cases} \underline{w} + q & \text{for } t \text{ even} \\ \overline{w} - q & \text{for } t \text{ odd} \end{cases}$$

To determine the equilibrium price it is now possible to use Euler equations; for individual 1 we have:

$$u'_{t-1}(\overline{x}^1_{t-1})q \geq u'_t(\overline{x}^1_t)q$$

$$[u'_{t-1}(\overline{x}^1_{t-1})q - u'_t(\overline{x}^1_t)q] \cdot \overline{z}^1_t = 0$$

from which, substituting the corresponding values for x and z, we get:

$$\text{for } t \text{ even} \begin{cases} u'(\underline{w}+q)q \geq \beta u'(\overline{w}-q)q \\ [u'(\underline{w}+q)q - \beta u'(\overline{w}-q)q] \cdot 0 = 0 \end{cases}$$

$$\text{for } t \text{ odd} \begin{cases} u'(\overline{w}-q)q \geq \beta u'(\underline{w}+q)q \\ [u'(\overline{w}-q)q - \beta u'(\underline{w}+q)q] \cdot 1 = 0 \end{cases}$$

i.e.:

$$\begin{array}{ll} u'(\underline{w}+q) \geq \beta u'(\overline{w}-q) & \text{for } t \text{ even} \\ u'(\overline{w}-q) = \beta u'(\underline{w}+q) & \text{for } t \text{ odd} \end{array}$$

In the same way we get for individual 2:

$$\begin{array}{ll} u'(\overline{w}-q) = \beta u'(\underline{w}+q) & \text{for } t \text{ even} \\ u'(\underline{w}+q) \geq \beta u'(\overline{w}-q) & \text{for } t \text{ odd} \end{array}$$

and, in conclusion, at an equilibrium the following conditions must hold:

$$\begin{array}{l} u'(\overline{w}-q) = \beta u'(\underline{w}+q) \\ u'(\underline{w}+q) \geq \beta u'(\overline{w}-q) \end{array}$$

Substituting the first expression into the second, the latter becomes:

$$u'(\underline{w}+q) \geq \beta u'(\overline{w}-q) = \beta^2 u'(\underline{w}+q)$$

from which $1 \geq \beta^2$, that is always true; in order to have an equilibrium, hence, the first condition must hold, and to check the existence of a positive price q that satisfies this relation we can consider the function:

$$f(q) = u(\overline{w}-q) + \beta u(\underline{w}+q)$$

that is strictly concave and is defined in the interval $[0, \overline{w}]$. The only point of maximum interior to this interval is characterized by the condition $f'(q) = 0$, that is precisely:

$$u'(\overline{w}-q) = \beta u'(\underline{w}+q)$$

together with the conditions $f'(0) > 0$ and $f'(\overline{w}) < 0$ (that are needed to avoid that the maximum is on the boundary of the interval $[0, \overline{w}]$), i.e.:

$$-u'(\overline{w}) + \beta u'(\underline{w}) > 0$$
$$-u'(0) + \beta u'(\underline{w} + \overline{w}) < 0$$

from which:

$$u'(\overline{w}) < \beta u'(\underline{w})$$
$$\beta u'(\underline{w} + \overline{w}) < u'(0)$$

where the second relation is clearly satisfied assuming $u'(0) = +\infty$. As a consequence, the equilibrium value of q is the unique value q^* that satisfies the conditions:

$$u'(\overline{w} - q) = \beta u'(\underline{w} + q)$$
$$u'(\overline{w}) < \beta u'(\underline{w})$$

The fact that the value q^* found in this way is a true equilibrium can be checked using the sufficient condition of optimality presented at the end of Section 3. We have in this case:

$$\lim_{t \to +\infty} \beta^t u'(\overline{x}_t^i) q^* \overline{z}_{t+1}^i = 0 \qquad \text{provided } \beta < 1$$

and $((\overline{x}^i, \overline{z}^i), q^*)$ is an equilibrium. In conclusion, in this model the following result holds:

Proposition 15 *The monetary model "a' la Bewley" considered has an equilibrium with valued fiat money (monetary equilibrium). This equilibrium is characterized by $q_t = q^*$ for each $t \geq 0$, where $q^* > 0$ is the unique quantity such that:*

$$u'(\overline{w} - q^*) = \beta u'(\underline{w} + q^*)$$
$$u'(\overline{w}) < \beta u'(\underline{w})$$

The corresponding equilibrium consumption allocations and portfolio allocations of the two individuals are the following:

$$\overline{x}_t^1 = \begin{cases} \overline{w} - q^* & for\ t\ even \\ \underline{w} + q^* & for\ t\ odd \end{cases} \qquad \overline{z}_t^1 = \begin{cases} 0 & for\ t\ even \\ 1 & for\ t\ odd \end{cases}$$

$$\overline{x}_t^2 = \begin{cases} \underline{w} + q^* & for\ t\ even \\ \overline{w} - q^* & for\ t\ odd \end{cases} \qquad \overline{z}_t^2 = \begin{cases} 1 & for\ t\ even \\ 0 & for\ t\ odd \end{cases}$$

In this case, therefore, at the equilibrium the two individuals exchange one another, in every period, the unit of fiat money the economy is endowed with. Each agent uses a part of his endowment of the consumption good when it is high ($\overline{w}$) to buy the unit of fiat money, and reduces consequently the consumption of that period, while he sells the unit of fiat money when his endowment of the consumption good is low ($\underline{w}$), and in this way he increases the consumption of that period. For this reason, the model is a *consumption smoothing* model.

In this example the fundamental value of the asset is zero (because the dividend at every date is zero), and the fact that at the equilibrium the price of fiat money is positive means that this price involves a bubble component. In particular, we have:

$$\lim_{t \to +\infty} \beta^t u'(\overline{x}_t^i) q^* = 0$$

so that the asymptotic bubble is zero (i.e. $\overline{b}_t^i = 0$), therefore in this case the bubble is entirely due to the violation of the Euler equation (i.e. $\widetilde{b}_t^i > 0$). In fact, at each date one of the two individuals, alternatively, satisfies the corresponding Euler equation as an equality, while the other individual satisfies it as an inequality, and in this way there is "violation" of this equation. In terms of the general analysis presented in Section 3, the presence of the bubble is due to the fact that agents cannot borrow against the value of their future wealth, i.e. there is a borrowing constraint that turns out to be the crucial element at the origin of the bubble.

Using the approach of Euler equations it is also possible to show that the monetary model *"a' la Bewley"* has, together with the "autarchic" equilibrium in which $q_t = 0$ and the equilibrium with positive value of fiat money ($q_t = q^* > 0$), a multiplicity of equilibria. In order to find this result we fix a sequence of prices q_t and we assume that the holding strategies of the two individuals are the same as before, i.e.:

$$\overline{z}_t^1 = \begin{cases} 0 & \text{for } t \text{ even} \\ 1 & \text{for } t \text{ odd} \end{cases} \qquad \overline{z}_t^2 = \begin{cases} 1 & \text{for } t \text{ even} \\ 0 & \text{for } t \text{ odd} \end{cases}$$

From the budget constraints we get the consumptions of the two agents, given by:

$$\overline{x}_t^1 = \begin{cases} \overline{w} - q_t & \text{for } t \text{ even} \\ \underline{w} + q_t & \text{for } t \text{ odd} \end{cases} \qquad \overline{x}_t^2 = \begin{cases} \underline{w} + q_t & \text{for } t \text{ even} \\ \overline{w} - q_t & \text{for } t \text{ odd} \end{cases}$$

and, as before, the Euler equations that must hold at an equilibrium are:

$$\begin{aligned} u'(\overline{w} - q_{t-1})q_{t-1} &= \beta u'(\underline{w} + q_t)q_t \\ u'(\underline{w} + q_{t-1})q_{t-1} &\geq \beta u'(\overline{w} - q_t)q_t \end{aligned} \tag{5}$$

where the first relation holds for $i = 1$ at odd-numbered dates and for $i = 2$ at even-numbered dates, while the second relation holds for $i = 1$ at even-numbered dates and for $i = 2$ at odd-numbered dates. Given an initial value q_0, the first relation determines a sequence of prices; this dynamics has the two fix points $q = 0$ and $q = q^*$ found above, that represent two possible equilibria of the model (the autarchic equilibrium and the equilibrium with valued fiat money), but it is possible to show that there exist sequences of prices which form other equilibria.

To obtain this result it is possible to consider a specific example in which the utility function is:

$$u_t^i(x_t^i) = \beta^t \log x_t^i$$

In this case the stationary price is:

$$q^* = \frac{\beta\overline{w} - \underline{w}}{1 + \beta}$$

while considering a sequence of prices q_t the Euler equations that must hold at equilibrium are:

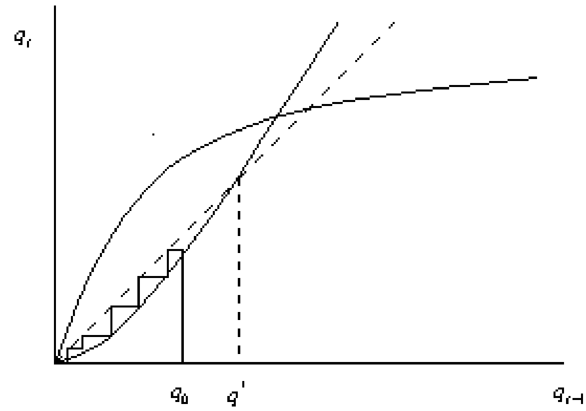
$$\begin{aligned} \frac{1}{\overline{w} - q_{t-1}} q_{t-1} &= \frac{\beta}{\underline{w} + q_t} q_t \\ \frac{1}{\underline{w} + q_{t-1}} q_{t-1} &\geq \frac{\beta}{\overline{w} - q_t} q_t \end{aligned}$$

from which:

$$q_t = \frac{\underline{w} q_{t-1}}{\beta\overline{w} - (1 + \beta)q_{t-1}} \tag{6}$$

$$q_t \leq \frac{\overline{w} q_{t-1}}{\beta\underline{w} + (1 + \beta)q_{t-1}} \tag{7}$$

Considering the graphics of the curves (6) and (7) it is possible to note that both are strictly increasing and in the interval $[0, q^*]$ the second lies above the first, hence for any initial price $q_0 \in (0, q^*)$ there exists a sequence of asset prices q_t, decreasing to 0, which forms an equilibrium. As a consequence, together with the autarchic equilibrium and the equilibrium with valued fiat money, there exist other equilibria, in which there is a bubble on the asset price that disappears when the price converges to 0. Graphically we have:



and the results found can be summarized as follows:

Proposition 16 *The monetary model "a' la Bewley" considered, in which the utility function takes the specific form:*

$$u(x_t^i) = \log x_t^i$$

has, together with the autarchic equilibrium ($q_t = 0$) and the equilibrium with constant positive price of fiat money ($q_t = q^* > 0$), a multiplicity of other equilibria (that consist of sequences of asset prices q_t decreasing to 0), one for each initial value $q_0 \in (0, q^*)$.

The corresponding equilibrium consumption allocations and portfolio allocations of the two individuals are the following:

$$\overline{x}_t^1 = \begin{cases} \overline{w} - q_t & \text{for } t \text{ even} \\ \underline{w} + q_t & \text{for } t \text{ odd} \end{cases} \qquad \overline{z}_t^1 = \begin{cases} 0 & \text{for } t \text{ even} \\ 1 & \text{for } t \text{ odd} \end{cases}$$

$$\overline{x}_t^2 = \begin{cases} \underline{w} + q_t & \text{for } t \text{ even} \\ \overline{w} - q_t & \text{for } t \text{ odd} \end{cases} \qquad \overline{z}_t^2 = \begin{cases} 1 & \text{for } t \text{ even} \\ 0 & \text{for } t \text{ odd} \end{cases}$$

This result can then be extended to the general case (at least locally, around $q = 0$), in fact the derivatives of the curves represented by the equations (5) at the origin are:

$$\left(\frac{dq_t}{dq_{t-1}}\right)_{\mathbf{0}} = \frac{u'(\overline{w})}{\beta u'(\underline{w})} < 1 \qquad \text{and} \qquad \left(\frac{dq_t}{dq_{t-1}}\right)_{\mathbf{0}} = \frac{u'(\underline{w})}{\beta u'(\overline{w})} > \frac{1}{\beta^2} > 1$$

therefore close to the origin the second curve lies above the first, and if the latter rises monotonically we have the same situation described in the case of logarithmic utility, and a multiplicity of equilibria arises. In the general case there is also another possibility, i.e. it is possible for the first curve to be "*backward bending*"; considering the derivative of this curve at $q = q^*$ we have:

$$\left(\frac{dq_t}{dq_{t-1}}\right)_{q*} = \frac{u'(\overline{w} - q^*) - u''(\overline{w} - q^*)q^*}{\beta \left[u'(\underline{w} + q^*) + u''(\underline{w} + q^*)q^*\right]}$$

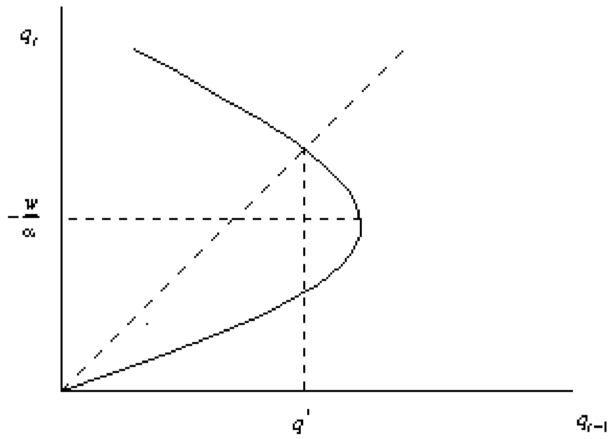
and a sufficient condition for the curve to be *backward bending* is that this derivative is negative, i.e. (since the numerator is always positive):

$$u'(\underline{w} + q^*) + u''(\underline{w} + q^*)q^* < 0 \tag{8}$$

This can happen for instance for the family of utility functions of the form $u(x_t^i) = \frac{(x_t^i)^\alpha}{\alpha}$ with $\alpha < 0$ (isoelastic utilities, for $\alpha \rightarrow 0$ we obtain the logarithmic case). In this example after some computations we get that condition (8) holds for:

$$\alpha < -\frac{\underline{w}(1 + \beta^{\frac{1}{1-\alpha}})}{\overline{w}\beta^{\frac{1}{1-\alpha}} - \underline{w}}$$

and we have a situation of this type:



In conclusion, the analysis based on the approach of Euler equations shows that the monetary model "a' la Bewley" has, together with an autarchic equilibrium (that is a no-trade equilibrium, in which the price of fiat money is $q = 0$) another stationary equilibrium, in which fiat money is valued ($q_t = q^* > 0$). In this case, since the fundamental value of fiat money is always zero, this implies the existence of a speculative bubble, and as long as the discount factor β is less than 1 this bubble is due to the violation of the Euler equation. In addition, in the special case of logarithmic utility (and also in the general case, locally around $q = 0$) a further result is that for every initial value $q_0 \in (0, q^*)$ the model has an equilibrium represented by a decreasing sequence of prices q_t which converges to 0. The conclusion is that a multiplicity of equilibria exists, and this equilibria involve a bubble component (due to the violation of the Euler equation) that then vanishes as $q_t \to 0$.

4.2 A general deterministic model

The second example considered is a generalization, in the deterministic case, of the first one. There are again two individuals with identical preferences:

$$\sum_{t=0}^{\infty} \beta^t u(x_t^i)$$

but in this case the endowments at the beginning of each period are generic, equal respectively to w_t^1 and w_t^2. Also in this example the asset available is fiat money (in fixed net supply of one unit at all dates), in addition short-selling is prohibited and we assume that the strategies of the two individuals concerning the amounts of the asset held in equilibrium are of *"bang-bang" type*, i.e.:

$$\overline{z}_t^1 \in \{0, 1\} \qquad \overline{z}_t^2 = 1 - \overline{z}_t^1$$

At this point, if q_t is a sequence of equilibrium prices Euler equations must hold; for individual 1 these conditions are:

$$u'(\overline{x}_t^1)q_t \geq \beta u'(\overline{x}_{t+1}^1)q_{t+1}$$

$$[u'(\overline{x}_t^1)q_t - \beta u'(\overline{x}_{t+1}^1)q_{t+1}] \cdot \overline{z}_{t+1}^1 = 0$$

and for individual 2 they are:

$$u'(\overline{x}_t^2)q_t \geq \beta u'(\overline{x}_{t+1}^2)q_{t+1}$$

$$[u'(\overline{x}_t^2)q_t - \beta u'(\overline{x}_{t+1}^2)q_{t+1}] \cdot \overline{z}_{t+1}^2 = 0$$

Given this general structure it is possible to obtain an interesting result considering a sequence of "times of switching". First of all the following definition holds:

Definition 17 *A sequence of times of switching is a sequence* $\mathcal{T} = \{0 = T_0, T_1, T_2, ...\}$ *with* $0 < T_1 < T_2 < ...$ *that satisfies the following conditions (for* $i = 1, 2$*):*

$$(i)\ z^i_{T_n} \neq z^i_{T_{n+1}} \quad \forall n$$

$$(ii)\ z^i_t \text{ is constant } \forall t \in [T_n, T_{n+1}) \quad \forall n$$

so that the " times of switching" are the dates at which the individuals exchange one another the unit of fiat money present in the economy. Given the initial values (z^1_0, z^2_0) and a "time switching" $\mathcal{T}$, this determines a strategy (z^1_t, z^2_t), and with reference to this aspect it is possible to prove the following result (where i denotes one individual and $\widehat{i}$ the other one):

Proposition 18 *If* $(\overline{z}^1_t, \overline{z}^2_t)$ *is an optimal strategy and* $T = \{0, T_1, T_2, ...\}$ *is the corresponding time switching, then the following relation must hold:*

$$\frac{u'(w^i_{t+1})}{u'(w^i_t)} \geq \frac{u'(w^{\widehat{i}}_{t+1})}{u'(w^{\widehat{i}}_t)} \qquad \forall t \in [T_n, T_{n+1} - 2] \neq \emptyset \ \ \forall n$$

and furthermore:

$$\overline{z}^i_{t+1} = 1 \qquad \overline{z}^{\widehat{i}}_{t+1} = 0 \qquad \forall t \in [T_n, T_{n+1} - 2]$$

In fact, given an interval $[T_n, T_{n+1} - 2] \neq \emptyset$, on this interval the values $\overline{z}^1_t$ and $\overline{z}^2_t$ are constant, i.e. $\overline{z}^1_{t+1} = \overline{z}^1_t$ and $\overline{z}^2_{t+1} = \overline{z}^2_t$, and as a consequence from the budget constraints of the two individuals it is possible to get the consumptions $\overline{x}^1_t = w^1_t$ and $\overline{x}^2_t = w^2_t$. If we then assume that i is the agent for which $\overline{z}^i_t = 1$ and $\widehat{i}$ is the agent for which $\overline{z}^{\widehat{i}}_t = 0$, then the corresponding Euler equations are:

$$u'(w^i_t)q_t = \beta u'(w^i_{t+1})q_{t+1}$$
$$u'(w^{\widehat{i}}_t)q_t \geq \beta u'(w^{\widehat{i}}_{t+1})q_{t+1}$$

that hold $\forall t \in [T_n, T_{n+1} - 2]$. From these equations (assuming that $q_t \neq 0$ $\forall t$) we get:

$$\frac{q_t}{q_{t+1}} = \frac{\beta u'(w^i_{t+1})}{u'(w^i_t)} \geq \frac{\beta u'(w^{\widehat{i}}_{t+1})}{u'(w^{\widehat{i}}_t)}$$

that is the conclusion.

From this result it is possible to obtain in particular a sufficient condition for the existence of a "switching" in every period, we have in fact:

Proposition 19 *If $\overline{z}_0^1 = 1$ and $\overline{z}_0^2 = 0$, then a sufficient condition for a switching in every period is given by the relations:*

$$\frac{u'(w_{t+1}^1)}{u'(w_t^1)} < \frac{u'(w_{t+1}^2)}{u'(w_t^2)} \qquad \textit{for } t \textit{ even}$$
$$\frac{u'(w_{t+1}^2)}{u'(w_t^2)} < \frac{u'(w_{t+1}^1)}{u'(w_t^1)} \qquad \textit{for } t \textit{ odd}$$

and in this case we have:

$$\overline{z}_t^1 = \begin{cases} 1 & \textit{for } t \textit{ even} \\ 0 & \textit{for } t \textit{ odd} \end{cases} \qquad \overline{z}_t^2 = \begin{cases} 0 & \textit{for } t \textit{ even} \\ 1 & \textit{for } t \textit{ odd} \end{cases}$$

In fact, to have a switching in every period we need $\mathcal{T} = \{0, 1, 2, 3, ...\}$ i.e. $T_1 = 1$, $T_2 = 2$, $T_3 = 3$... and therefore $[T_n, T_{n+1} - 2] = \emptyset \ \ \forall n$. Assuming that $\overline{z}_0^1 = 1$ and $\overline{z}_0^2 = 0$, from the previous Proposition it follows that, in order to have $[0, T_1 - 2] = \emptyset$ (so that $T_1 = 1$), it is sufficient that the relation stated in that Proposition doesn't hold, i.e. it is sufficient to have:

$$\frac{u'(w_1^1)}{u'(w_0^1)} < \frac{u'(w_1^2)}{u'(w_0^2)}$$

In this way at time $t = 1$ we have $\overline{z}_1^1 = 0$ and $\overline{z}_1^2 = 1$, if we now assume:

$$\frac{u'(w_2^2)}{u'(w_1^2)} < \frac{u'(w_2^1)}{u'(w_1^1)}$$

then $[1, T_2 - 2] = \emptyset$ (so that $T_2 = 2$), i.e. there is a new switching and at time $t = 2$ we have $\overline{z}_2^1 = 1$ and $\overline{z}_2^2 = 0$ and so on. By proceeding in this way we get $T_1 = 1$, $T_2 = 2$, $T_3 = 3$ that is the result of the Proposition.

An example of application of this result can be obtained considering the case in which there are two individuals with identical preferences given by:

$$u(x_t^i) = \log x_t^i$$

while their endowments are given by:

$$w_t^1 = \begin{cases} A\rho^t & \text{for } t \text{ even} \\ B\rho^t & \text{for } t \text{ odd} \end{cases} \qquad w_t^2 = \begin{cases} B\rho^t & \text{for } t \text{ even} \\ A\rho^t & \text{for } t \text{ odd} \end{cases}$$

i.e. they grow at the same average rate but fluctuate over time in a deterministic fashion (where ρ is the growth factor and $0 < A < B$). The initial quantities of the asset (again *fiat money*, so that the dividend equals zero in every period) held by the two agents are:

$$z_0^1 = 1 \qquad z_0^2 = 0$$

and hence in this case the individual with the smaller endowment in period 0 (agent 1) has all of the asset available in the economy. In addition, short-selling is prohibited, i.e. the individuals cannot borrow.

First of all it is possible to verify that in this example the sufficient condition that guarantees the existence of a switching in every period is satisfied; this condition becomes (both for t even and for t odd):

$$\frac{A}{B} < \frac{B}{A}$$

that is clearly verified, since $A < B$. As a consequence, in this economy there is "switching" in every period and the portfolio allocations of equilibrium are:

$$\overline{z}_t^1 = \begin{cases} 1 & \text{for } t \text{ even} \\ 0 & \text{for } t \text{ odd} \end{cases} \qquad \overline{z}_t^2 = \begin{cases} 0 & \text{for } t \text{ even} \\ 1 & \text{for } t \text{ odd} \end{cases}$$

while the corresponding consumption allocations are:

$$\overline{x}_t^1 = \begin{cases} A\rho^t + q_t & \text{for } t \text{ even} \\ B\rho^t - q_t & \text{for } t \text{ odd} \end{cases} \qquad \overline{x}_t^2 = \begin{cases} B\rho^t - q_t & \text{for } t \text{ even} \\ A\rho^t + q_t & \text{for } t \text{ odd} \end{cases}$$

In order to determine the equilibrium price we can then consider the Euler equations, that are:

$$\frac{q_t}{B\rho^t - q_t} = \frac{\beta q_{t+1}}{A\rho^{t+1} + q_{t+1}}$$
$$\frac{q_t}{A\rho^t + q_t} \geq \frac{\beta q_{t+1}}{B\rho^{t+1} - q_{t+1}}$$

Looking for a solution of the form:

$$q_t = c\rho^t$$

we get:

$$c = \frac{\beta B - A}{1+\beta}$$

and therefore:

$$q_t = \frac{\beta B - A}{1+\beta}\rho^t$$

provided $A < \beta B$. This is the equilibrium price and also in this case the fact that this is a true equilibrium can be checked by means of the sufficient condition of optimality, we have in fact:

$$\lim_{t\to+\infty} \beta^t u'(\overline{x}_t^i) q_t \overline{z}_{t+1}^i = 0 \qquad \text{provided } \beta < 1$$

and $((\overline{x}^i, \overline{z}^i), q)$ is an equilibrium. The corresponding consumption allocations are:

$$\overline{x}_t^1 = \begin{cases} \frac{\beta(A+B)}{1+\beta}\rho^t & \text{for } t \text{ even} \\ \frac{A+B}{1+\beta}\rho^t & \text{for } t \text{ odd} \end{cases} \qquad \overline{x}_t^2 = \begin{cases} \frac{A+B}{1+\beta}\rho^t & \text{for } t \text{ even} \\ \frac{\beta(A+B)}{1+\beta}\rho^t & \text{for } t \text{ odd} \end{cases}$$

In conclusion, the results obtained can be summarized in the following Proposition:

Proposition 20 *The model considered has an equilibrium with valued fiat money, characterized by:*

$$q_t = \frac{\beta B - A}{1+\beta}\rho^t$$

with $A < \beta B$. The corresponding equilibrium consumption allocations and portfolio allocations of the two individuals are the following:

$$\overline{x}_t^1 = \begin{cases} \frac{\beta(A+B)}{1+\beta}\rho^t & \textit{for } t \textit{ even} \\ \frac{A+B}{1+\beta}\rho^t & \textit{for } t \textit{ odd} \end{cases} \qquad \overline{z}_t^1 = \begin{cases} 1 & \textit{for } t \textit{ even} \\ 0 & \textit{for } t \textit{ odd} \end{cases}$$

$$\overline{x}_t^2 = \begin{cases} \frac{A+B}{1+\beta}\rho^t & \textit{for } t \textit{ even} \\ \frac{\beta(A+B)}{1+\beta}\rho^t & \textit{for } t \textit{ odd} \end{cases} \qquad \overline{z}_t^2 = \begin{cases} 0 & \textit{for } t \textit{ even} \\ 1 & \textit{for } t \textit{ odd} \end{cases}$$

Also in this case the fundamental value of fiat money is zero, while the equilibrium price q_t is positive, hence there is a bubble. In particular, if $\beta < 1$ we have:

$$\lim_{t \to +\infty} \beta^t u'(\overline{x}_t^i) q_t = 0$$

i.e. the asymptotic component is zero and the bubble is due entirely to the violation of the Euler equation.

The origin of this bubble is represented by the short-sales constraint. In fact, in even periods agent 1 is poor because his endowment is relatively low, and he owns all of the asset present in the economy at the beginning of the period, that he then sells to agent 2 (in order to smooth his consumption). The problem is represented by the fact that he would like to sell even more of the asset (and agent 2 would like to buy it), but he cannot because of the short-sales constraint; the same is true for agent 2 in odd periods. In conclusion, no agent can permanently reduce his holdings because of the short-sales constraint, and furthermore agents' endowments grow as fast as the bubble, so that the individuals can always buy the asset when they are wealthy. These elements determine an increase in the price of the asset (whose fundamental value is zero) and turn out to be the bubble producing factors in this economy.

4.3 A stochastic model

The third example presented is an extension of the previous ones (that are deterministic models) to the stochastic setting. In the economy examined, at each date t there is realized a random state $s_t \in \{\xi, \eta\}$, following a Markov process with transition probabilities:

$$\pi(s_{t+1} = \xi \mid s_t = \eta) = \pi(s_{t+1} = \eta \mid s_t = \xi) = \pi$$
$$\pi(s_{t+1} = \xi \mid s_t = \xi) = \pi(s_{t+1} = \eta \mid s_t = \eta) = 1 - \pi$$

where $0 < \pi < 1$, given an initial condition $s_0 \in \{\xi, \eta\}$. In this case the state space is $\Omega = \{\xi, \eta\}^{\mathbb{N}}$ and the generic node s^t of the tree that can be used to describe the uncertainty in this model is identified with a sequence:

$$s^t = \{s_0, s_1, ..., s_t\} \in \{\xi, \eta\}^{t+1}$$

At each node there is a single consumption good and a single asset that is traded, again represented by *fiat money* (in fixed net supply of one unit at all dates). The economy consists of two individuals with identical preferences given by:

$$\mathbf{E}\left[\sum_{t=0}^{\infty}\beta^t u(x_t^i)\right]$$

with $0 < \beta < 1$ and endowments:

$$w^1(s^t) = \begin{cases} \underline{w} & \text{if } s_t = \xi \\ \overline{w} & \text{if } s_t = \eta \end{cases} \qquad w^2(s^t) = \begin{cases} \overline{w} & \text{if } s_t = \xi \\ \underline{w} & \text{if } s_t = \eta \end{cases}$$

with $\underline{w} < \overline{w}$, and in addition one of the two individuals is initially endowed with one unit of fiat money, and the other with zero units (which of them has the asset does not matter). Finally, in this economy borrowing is not possible.

As usual it is possible to look for an equilibrium in which the asset holding strategy of the two agents is of *"bang-bang" type*, more precisely:

$$\overline{z}^1(s^t) = \begin{cases} 0 & \text{if } s_t = \xi \\ 1 & \text{if } s_t = \eta \end{cases} \qquad \overline{z}^2(s^t) = \begin{cases} 1 & \text{if } s_t = \xi \\ 0 & \text{if } s_t = \eta \end{cases}$$

and in this model it is convenient to write the budget constraints in the form:

$$x^1(s^t) + q(s^t)[z^1(s^t) - z^1(s^{t-1})] = w^1(s^t)$$
$$x^2(s^t) + q(s^t)[z^2(s^t) - z^2(s^{t-1})] = w^2(s^t)$$

In this case, then, we assume that equilibrium prices are of the form:

$$q(s^t) = \Phi(\mid z^i(s^t) - z^i(s^{t-1}) \mid)$$

and since $\mid z^i(s^t) - z^i(s^{t-1}) \mid$ can assume only the values 0 and 1 there are two possible prices:

$$q(s^t) \begin{cases} \Phi(0) = q(s^{t-2}, \xi, \xi) = q(s^{t-2}, \eta, \eta) = q^{**} \\ \Phi(1) = q(s^{t-2}, \xi, \eta) = q(s^{t-2}, \eta, \xi) = q^{*} \end{cases}$$

As usual it is possible to write the Euler equations (that in this case are true stochastic Euler equations). For agent 1 we get:

$$u'(x^1(s^{t-1}))q(s^{t-1}) \geq \beta\pi(s_t = \xi \mid s^{t-1})u'(x^1(s^{t-1}, \xi))q(s^{t-1}, \xi) + \\ + \beta\pi(s_t = \eta \mid s^{t-1})u'(x^1(s^{t-1}, \eta))q(s^{t-1}, \eta)$$

$$[u'(x^1(s^{t-1}))q(s^{t-1}) - \beta\pi(s_t = \xi \mid s^{t-1})u'(x^1(s^{t-1}, \xi))q(s^{t-1}, \xi) - \\ - \beta\pi(s_t = \eta \mid s^{t-1})u'(x^1(s^{t-1}, \eta))q(s^{t-1}, \eta)] \cdot z^1(s^{t-1}) = 0$$

and for agent 2 we have:

$$u'(x^2(s^{t-1}))q(s^{t-1}) \geq \beta\pi(s_t = \xi \mid s^{t-1})u'(x^2(s^{t-1}, \xi))q(s^{t-1}, \xi) + \\ + \beta\pi(s_t = \eta \mid s^{t-1})u'(x^2(s^{t-1}, \eta))q(s^{t-1}, \eta)$$

$$[u'(x^2(s^{t-1}))q(s^{t-1}) - \beta\pi(s_t = \xi \mid s^{t-1})u'(x^2(s^{t-1}, \xi))q(s^{t-1}, \xi) - \\ -\beta\pi(s_t = \eta \mid s^{t-1})u'(x^2(s^{t-1}, \eta))q(s^{t-1}, \eta)] \cdot z^2(s^{t-1}) = 0$$

After some computations we get that in every state s_t the following relations must hold:

$$\begin{aligned} u'(\overline{w} - q^*)q^* &= \beta\pi u'(\underline{w} + q^*)q^* + \beta(1-\pi)u'(\overline{w})q^{**} \\ u'(\overline{w})q^{**} &= \beta\pi u'(\underline{w} + q^*)q^* + \beta(1-\pi)u'(\overline{w})q^{**} \\ u'(\underline{w} + q^*)q^* &\geq \beta\pi u'(\overline{w} - q^*)q^* + \beta(1-\pi)u'(\underline{w})q^{**} \\ u'(\underline{w})q^{**} &\geq \beta\pi u'(\overline{w} - q^*)q^* + \beta(1-\pi)u'(\underline{w})q^{**} \end{aligned}$$

Their solution is given by the values (that are unique) q^* and q^{**} that satisfy the relations:

$$\begin{aligned} \frac{u'(\overline{w} - q^*)}{u'(\underline{w} + q^*)} &= \frac{\beta\pi}{1 - \beta(1-\pi)} \\ q^{**} &= \frac{u'(\overline{w} - q^*)}{u'(\overline{w})}q^* \end{aligned}$$

under the limitations:

$$\frac{\beta^2\pi(1-\pi)}{1 - \beta + \beta\pi(1 - \beta\pi)} \leq \frac{u'(\overline{w})}{u'(\underline{w})} < \frac{\beta\pi}{1 - \beta(1-\pi)}$$

It is then possible to determine the equilibrium allocations, and the results can be summarized in the following Proposition:

Proposition 21 *Under the assumption:*

$$\frac{\beta^2\pi(1-\pi)}{1 - \beta + \beta\pi(1 - \beta\pi)} \leq \frac{u'(\overline{w})}{u'(\underline{w})} < \frac{\beta\pi}{1 - \beta(1-\pi)}$$

the stochastic model considered has an equilibrium with valued fiat money given by the following processes:

$$\text{if}\ \left\{\begin{array}{l} s_{t-1}=\xi \\ \\ s_t=\xi \end{array}\right. \quad \text{then} \quad \left\{\begin{array}{ll} \overline{x}^1(s^t)=\underline{w} & \overline{z}^1(s^t)=0 \\ \\ \overline{x}^2(s^t)=\overline{w} & \overline{z}^2(s^t)=1 \end{array}\right. \qquad q(s^t)=q^{**}$$

$$\text{if}\ \left\{\begin{array}{l} s_{t-1}=\xi \\ \\ s_t=\eta \end{array}\right. \quad \text{then} \quad \left\{\begin{array}{ll} \overline{x}^1(s^t)=\overline{w}-q^* & \overline{z}^1(s^t)=1 \\ \\ \overline{x}^2(s^t)=\underline{w}+q^* & \overline{z}^2(s^t)=0 \end{array}\right. \qquad q(s^t)=q^{*}$$

$$\text{if}\ \left\{\begin{array}{l} s_{t-1}=\eta \\ \\ s_t=\xi \end{array}\right. \quad \text{then} \quad \left\{\begin{array}{ll} \overline{x}^1(s^t)=\underline{w}+q^* & \overline{z}^1(s^t)=0 \\ \\ \overline{x}^2(s^t)=\overline{w}-q^* & \overline{z}^2(s^t)=1 \end{array}\right. \qquad q(s^t)=q^{*}$$

$$\text{if}\ \left\{\begin{array}{l} s_{t-1}=\eta \\ \\ s_t=\eta \end{array}\right. \quad \text{then} \quad \left\{\begin{array}{ll} \overline{x}^1(s^t)=\overline{w} & \overline{z}^1(s^t)=1 \\ \\ \overline{x}^2(s^t)=\underline{w} & \overline{z}^2(s^t)=0 \end{array}\right. \qquad q(s^t)=q^{**}$$

where q^* and q^{**} are the unique quantities that satisfy the relations:

$$\frac{u'(\overline{w}-q^*)}{u'(\underline{w}+q^*)} = \frac{\beta\pi}{1-\beta(1-\pi)}$$
$$q^{**} = \frac{u'(\overline{w}-q^*)}{u'(\overline{w})}q^*$$

and $q^{**} > q^*$.

The fact that this is a true equilibrium, once again, can be verified by means of the sufficient condition of optimality presented at the end of Section 3, we have in this situation:

$$\lim_{t\to+\infty} \beta^t E\left[u'(\overline{x}^i(s^t))q(s_t)\overline{z}^i(s^{t+1})\right] = 0$$

and therefore the allocation found is optimal.

In this case at the equilibrium the entire money supply, at the end of trading at each node, is held by the individual that had the higher endowment

($\overline{w}$) at that node, while the individual with the lower endowment ($\underline{w}$) spends during the period any money that he holds at the beginning of the period. The exchange value of fiat money, then, is q^* if the agent that holds the entire money supply at the beginning of the period is the agent with endowment $\underline{w}$, while it is q^{**} if the agent with the money has endowment $\overline{w}$. Again there is a bubble, as the equilibrium price of fiat money is positive while its fundamental value is zero, and since we have:

$$\lim_{t\to+\infty} \beta^t E\left[u'(\overline{x}^i(s^t))q(s_t)\right] = 0 \qquad \text{provided } \beta < 1$$

the asymptotic component of the bubble is zero and the bubble is due to the violation of the Euler equation (by each of the two individuals, alternatively, in each period).

4.4 An example of asymptotic bubble

All the previous examples are characterized by the fact that the presence of a speculative bubble in equilibrium is linked to the violation of the Euler equation. It is therefore interesting to build an example in which, on the contrary, the bubble arises even if Euler equation is not violated.

A simple case is given by the monetary model *"a' la Bewley"* considered in the first example, in which agents don't discount the future, so that $\beta = 1$. In this situation the equilibrium is characterized by the conditions:

$$u'(\overline{w} - q^*) = u'(\underline{w} + q^*)$$
$$u'(\overline{w}) < u'(\underline{w})$$

from which:

$$\overline{w} - q^* = \underline{w} + q^*$$
$$\overline{w} > \underline{w}$$

that is:

$$q^* = \frac{\overline{w} - \underline{w}}{2}$$
$$\overline{w} > \underline{w}$$

and then:

$$\overline{w} - q^* = \frac{\overline{w} + \underline{w}}{2} = \underline{w} + q^*$$

and finally:

$$\overline{x}_t^1 = \overline{x}_t^2 = \frac{\overline{w} + \underline{w}}{2} \qquad \forall t$$

while the holding strategies are the same as in the case in which $\beta < 1$ (the two agents hold alternatively the unit of the asset in even periods and in odd periods). In this case we have:

$$\lim_{t \to +\infty} \inf u'(\overline{x}_t^i) q^* \overline{z}_{t+1}^i = 0$$

so that the allocation $(\overline{x}^i, \overline{z}^i)$ is weakly optimal and the solution found is an equilibrium, nevertheless this allocation is not strongly optimal because:

$$\lim_{t \to +\infty} \sup u'(\overline{x}_t^i) q^* \overline{z}_{t+1}^i > 0$$

In this example the Euler equations are always satisfied as equalities, in fact since $\beta = 1$ they are:

$$u'(\overline{w} - q) = u'(\underline{w} + q)$$
$$u'(\underline{w} + q) \geq u'(\overline{w} - q)$$

and this implies that they reduce to the same relation:

$$u'(\overline{w} - q) = u'(\underline{w} + q)$$

and are satisfied as equalities. In this example, therefore, $\widetilde{b}_t^i = 0$, i.e. the bubble component due to the violation of the Euler equation is zero, and as a consequence the bubble on the asset price is of asymptotic kind, we have in fact:

$$\overline{b}_t^i = \frac{1}{Du_t^i(\overline{x}_t^i)} \lim_{k \to +\infty} Du_{t+k}^i(\overline{x}_{t+k}^i) q^* = q^*$$

The results obtained can be summarized in the following Proposition:

Proposition 22 *The monetary model "a' la Bewley" in which agents don't discount the future (i.e. $\beta = 1$) has an equilibrium with valued fiat money where the price of the asset is:*

$$q^* = \frac{\overline{w} - \underline{w}}{2} \qquad \forall t$$

with $\overline{w} > \underline{w}$. The corresponding equilibrium consumption allocations and portfolio allocations of the two individuals are the following:

$$\overline{x}_t^1 = \frac{\overline{w} + \underline{w}}{2} \quad \forall t \qquad \overline{z}_t^1 = \begin{cases} 0 & \text{for } t \text{ even} \\ 1 & \text{for } t \text{ odd} \end{cases}$$

$$\overline{x}_t^2 = \frac{\overline{w} + \underline{w}}{2} \quad \forall t \qquad \overline{z}_t^2 = \begin{cases} 1 & \text{for } t \text{ even} \\ 0 & \text{for } t \text{ odd} \end{cases}$$

Furthermore, in this case Euler equations are never violated, and the bubble on the price of the asset is entirely of asymptotic kind.

The result concerning the nature of the bubble term extends to the other examples involving fiat money that have been considered, whenever $\beta = 1$. The conclusion is that in models with valued fiat money of the type examined, if $0 < \beta < 1$ the bubble component on the price of the asset is due to the violation of the Euler equation, while if $\beta = 1$ the bubble is of asymptotic kind.

5 Conclusions

The conclusion that emerges from the analysis of the models examined is that *the phenomenon of speculative bubbles* on asset prices in intertemporal general equilibrium models *is negligible*, because it can arise only under quite special circumstances. This is confirmed also by the examples analysed, in which the presence of such bubbles is linked to particular situations (the violation of Euler equations, or of the hypotheses on which the models are based, and since these hypotheses are satisfied by a wide class of economies, their violation represents a rather special circumstance).

In the models presented it is assumed that individuals use all the available information to make their predictions (and, furthermore, that these predictions are based on the correct model of the economy), i.e. that they have *rational expectations.* This assumption, therefore, implies that *the price of a security is equal*, in equilibrium, *to its fundamental value* (the present value of its future dividend stream), and deviations from this value are only occasional.

In reality, there are periods in which this is not true, and the prices of some assets far exceed their fundamental values. These periods are the so-called *speculative booms*, in which speculative bubbles on the prices of the assets arise. There is therefore an evident contrast between these episodes and the conclusions of the theory. A possible explanation is represented by the fact that, during the periods of speculative boom, there is a "breakdown" of rational expectations, on which the theory is based. With reference to this aspect, it is possible to observe that the crucial characteristic of the asset markets is represented by their liquidity. In fact, a long-lived security can be purchased for two reasons: the first is the possibility to receive the future stream of dividends that it offers, the second is the possibility to resale it subsequently and to obtain a capital gain. When an agent buys an asset for the first reason, he will never accept to pay more for it than the present value of its future dividend stream, but when he buys an asset for the second reason what matters in assessing its value is what other agents will be ready to pay for it later. If agents can evaluate the future dividends of the asset with reasonable precision, and believe that all other agents can make a similar evaluation, then they have no reason to believe that the price of the asset at any future date will differ from the value of its remaining dividend stream, and hence they will not accept to pay for the asset more than the present value of its future dividends, even when they buy the asset in order to resale it later. In some circumstances, anyway, there can appear elements (for instance an innovation or a new discovery, significative changes in techniques of production or other circumstances of this kind) whose consequences may be important but whose probability of success is difficult to evaluate. The result is that in this case individuals don't know how to assess future values of the assets linked to these new elements, and they know that the other agents face the same difficulties. The consequence is that it is no longer rational for them to believe that the price of a security should equal its fundamental value, and therefore the hypothesis of rational expectations disappears.

In this context it often happens that new investors enter the market (aware of the possibility of these innovations to create gains in the future) and in this way begin to give up prices; this confirms the expectations of these individuals and we have an initial phase of rising prices. When this process has continued for a while, the information concerning the possibility of consistent gains in the future spreads among a broader segment of investors, that in turn enter the market. The more investors are attracted to the market, the more the prices rise and the more expectations of rising prices become self-fulfilling. At this point agents' expectations of rising prices begin to feed on themselves and cease to be related to the rational valuation made by an agent who buys the asset in order to receive its future dividend stream. It is precisely in this moment that the assumption of rational expectations breaks down, because agents recognize that there are other agents in the market who are not pricing the assets by their fundamental values, but are basing their valuation on the upward intertia of the market. This is the phase in which speculative bubbles reach their maximum level. Finally, after this process has continued for a certain period, individuals begin to have doubts about the possibility of the market to continue this phase of rising prices, therefore they start selling the assets and there is a phase of decreasing prices, until the prices of the assets reach once again a level that corresponds approximately to their fundamental values.

Outside of the periods in which the rational expectations behavior of the individuals breaks down, therefore, the price of an asset is equal to its fundamental value, and the results of the models illustrated are valid. It is in this way that the conclusions of the theory are consistent with the reality.

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