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Numerical integration based on bivariate quadratic spline quasi-interpolants on bounded domains

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Abstract In this paper we generate and study new cubature formulas based on spline quasi-interpolants defined as linear combinations of C^1 bivariate quadratic B-splines on a rectangular domain Ω, endowed with a non-uniform criss-cross triangulation, with discrete linear functionals as coefficients. Such B-splines have their supports contained in Ω and there is no data point outside this domain. Numerical results illustrate the methods.

Keywords Cubatures · bivariate spline approximation · quasi-interpolation

Mathematics Subject Classification (2000) MSC 65D07 · MSC 41A15 · MSC 65D32

1 Introduction

Quasi-interpolants (abbr. QIs) have been studied in the literature [2, 3, 25, 34, 35, 37] (and references therein) in order to be employed in widespread applications in mechanics, engineering and scientific computations and many results have been recently achieved on this subject. In fact QIs possess many desiderable properties, such as locality, boundedness in some relevant norm and reproduction of a polynomial space of a certain order [2]. In particular the latter property can let QIs reach optimal approximation order for smooth functions [7, 8, 15]. Moreover the construction of QIs does not need the solution of any system of equations.

Let $\Omega = [\alpha_1, \beta_1] \times [\alpha_2, \beta_2]$ be a rectangle decomposed into mn subrectangles by the two partitions

$$X_m := \{x_i : 0 \leq i \leq m\}, \quad Y_n := \{y_j : 0 \leq j \leq n\},$$

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of the segments $[\alpha_1, \beta_1] = [x_0, x_m]$ and $[\alpha_2, \beta_2] = [y_0, y_n]$, respectively.

Then the so called criss-cross triangulation $\mathscr{T}_{mn}$ of Ω is defined by drawing the two diagonals in each subrectangle (Fig. 1.1).

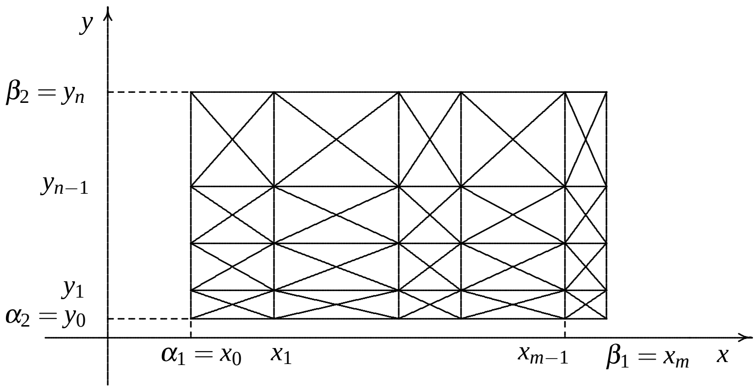


Fig. 1.1 Triangulation $\mathscr{T}_{mn}$ of $\Omega = [\alpha_1, \beta_1] \times [\alpha_2, \beta_2]$.

In [4,38] and [2], p.72, the authors introduce two C^1 bivariate quadratic spline QIs, defined on uniform and nonuniform criss-cross partitions $\mathscr{T}_{mn}$ of the rectangular domain Ω, using C^1 quadratic box-splines with octagonal support and simple knots [5,29]

$$x_{-2} < x_{-1} < \alpha_1 = x_0 < x_1 < \ldots < x_m = \beta_1 < x_{m+1} < x_{m+2},$$
$$y_{-2} < y_{-1} < \alpha_2 = y_0 < y_1 < \ldots < y_n = \beta_2 < y_{n+1} < y_{n+2}.$$

The first QI is defined by the Schoenberg-Marsden operator, reproducing bilinear polynomials. Based on this QI, cubature formulas have been generated in [12,13], also for finite part integrals. These results have been generalized to nonuniform criss-cross triangulations $\mathscr{T}_{mn}$ of Ω in [27,38].

The second one reproduces $\mathbb{P}_2$, where $\mathbb{P}_r$ is the space of all polynomials in two variables of total degree at most r.

In [9–11] some tools are given to construct a more general class of spline QIs, among which the two above mentioned, defined as a linear combination of such C^1 quadratic box-splines, whose coefficients are in turn linear combinations of values of the function to be approximated. In those papers their effective construction is studied and the corresponding Matlab software is given.

In [7,8] some approximation power performances and error analysis on the function and on its partial derivatives of first and second order with local and global upper bounds are given for such a general class of spline QIs.

A possible drawback of using such box-splines is that some of their supports are not completely contained in Ω (Fig. 1.2) and this causes the data sites, involved in the QIs definition, can also lie outside it.

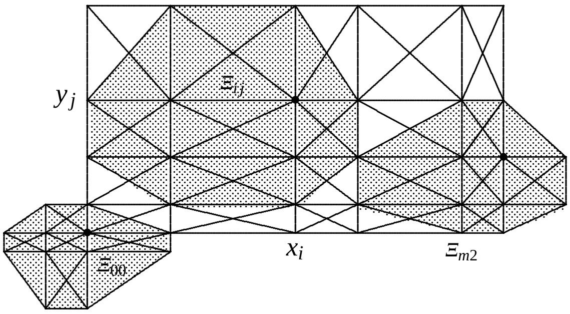


Fig. 1.2 Some B-spline octagonal supports with simple knots.

In [30,32,15,14] such a drawback is removed by defining and studying new QIs of the form

$$Qf := \sum_{ij} \lambda_{ij}^{(Q)}(f) B_{ij}, \tag{1.1}$$

where $f \in C(\Omega)$ and the B_{ij}'s are B-splines with multiple knots

$$\begin{aligned} x_{-2} = x_{-1} = \alpha_1 = x_0 < x_1 < \ldots < x_m = \beta_1 = x_{m+1} = x_{m+2}, \\ y_{-2} = y_{-1} = \alpha_2 = y_0 < y_1 < \ldots < y_n = \beta_2 = y_{n+1} = y_{n+2} \end{aligned}$$

on the boundary $\partial\Omega$ of Ω and with supports Ξ_{ij} all contained in Ω (Fig. 1.3). The $\lambda_{ij}^{(Q)}$'s are linear functionals, defined as linear combinations of function values at some triangular mesh-points $(x_\ell^{(i)}, y_\ell^{(j)})$ either in Ξ_{ij} or close to it, that is

$$\lambda_{ij}^{(Q)}(f) := \sum_{\ell} \nu_\ell^{(ij)} f(x_\ell^{(i)}, y_\ell^{(j)}),$$

with $\nu_\ell^{(ij)}$ non-zero weights, such that $Qf = f$, $\forall f \in \mathbb{P}_r$, for some $0 < r \leq 2$.

In this paper we define and study cubature formulas based on such new QIs. It is organised as follows.

In section 2 we recall some definitions and properties of the above QIs defined by means of multiple knot B-splines. Based on these QIs, new cubature formulas are generated in section 3 and their approximation order is studied for both symmetric and nonuniform partitions of the domain Ω. Finally in section 4 some numerical results are presented.

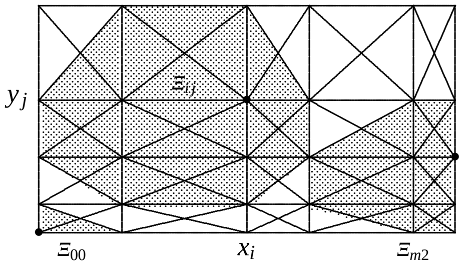


Fig. 1.3 Some B-spline supports with multiple knots.

2 Quadratic spline QIs on a bounded rectangle

2.1 The quadratic spline space $S_2^1(\mathscr{T}_{mn})$

Let $S_2^1(\mathscr{T}_{mn})$ denote the space of C^1 functions whose restrictions on each triangular cell of $\mathscr{T}_{mn}$ are polynomials of total degree 2, i.e. $\sum_{i+j\leq 2} a_{ij}x^iy^j,\ i,j\geq 0,\ a_{ij}\in\mathbb{R}$. Such functions are called bivariate C^1 splines of total degree 2.

In this section we recall some definitions and properties of the above spline space $S_2^1(\mathscr{T}_{mn})$, where QIs of kind (1.1) are defined.

First we need to define the sets of indices $\mathscr{K}_{mn} := \{(i,j) : 0\leq i\leq m+1,\ 0\leq j\leq n+1\}$, $\widehat{\mathscr{K}_{mn}} := \{(i,j) : 1\leq i\leq m,\ 1\leq j\leq n\}$.

Moreover we set $h_i := x_i - x_{i-1}$, $k_j := y_j - y_{j-1}$, $s_i := \frac{1}{2}(x_{i-1}+x_i)$, $t_j := \frac{1}{2}(y_{j-1}+y_j)$, $(i,j)\in\mathscr{K}_{mn}$ with the convention $h_0 = h_{m+1} = k_0 = k_{n+1} = 0$, $s_0 = x_0$, $s_{m+1} = x_m$, $t_0 = y_0$, $t_{n+1} = y_n$, $A_{r,s} = (x_r, y_s)$, for $-1\leq r\leq m+1$ and $-1\leq s\leq n+1$. Let the data sites $M_{i,j} = (s_i,t_j)$, $(i,j)\in\mathscr{K}_{mn}$ be the *mn* intersection points of diagonals in the subrectangles Ω_{ij}, the $2(m+n)$ midpoints of the subintervals on the four boundary edges of Ω and the four vertices of Ω.

Also let $\mathscr{B}_{mn} := \{B_{ij} : (i,j)\in\mathscr{K}_{mn}\}$ be the collection of $(m+2)(n+2)$ B-splines generating the space $\mathscr{S}_2^1(\mathscr{T}_{mn})$ of all C^1 piecewise quadratic functions on the criss-cross triangulation $\mathscr{T}_{mn}$, associated with the partition $X_m\times Y_n$ of the domain Ω. There are *mn inner B-splines* associated with the set of indices $\widehat{\mathscr{K}_{mn}}$, whose restrictions to the boundary Γ of Ω are equal to zero. In particular $(m-2)(n-2)$ B_{ij}'s have octagonal support, simple knots and they vanish on $\partial\Omega$ with C^1 continuity, while $2m+2n-4$ B_{ij}'s have non octagonal support, double knots on $\partial\Omega$, where they vanish with C^0 continuity. To the set $\widehat{\mathscr{K}_{mn}}$, we add $2m+2n+4$ *boundary B-splines* associated with $\widetilde{\mathscr{K}_{mn}} := \{(i,0),(i,n+1) : 0\leq i\leq m+1; (0,j),(m+1,j) : 0\leq j\leq n+1\}$, whose restrictions to $\partial\Omega$ are univariate quadratic B-splines, where they have triple knots. All such B-splines are non negative and form a partition of unity. Boundary B-splines are *linearly independent* as the univariate ones, but the inner B-splines are *linearly dependent*, the only dependence relationship being:

$$\sum_{(i,j)\in\widehat{\mathscr{K}_{mn}}} (-1)^{i+j} h_i k_j B_{ij} = 0.$$

This classification of B-splines is given in [30–32].

Since $\dim S_2^1(\mathscr{T}_{mn}) = (m+2)(n+2)-1$ [30,32], then a basis for $S_2^1(\mathscr{T}_{mn})$ is obtained by deleting any element from $\mathscr{B}_{mn}$. However, although $\mathscr{B}_{mn}$ is not a basis of $\mathscr{S}_2^1(\mathscr{T}_{mn})$, this fact has no influence on the definition and properties of QI operators. We remark that the set $\mathscr{B}_{mn}$, containing B-splines with multiple knots on the boundary $\partial\Omega$, can also be expressed in terms of the $(m+2)(n+2)$ B-splines with octagonal supports and simple knots [15].

2.2 The spline quasi-interpolants

In this section we recall the main properties of three QIs of kind (1.1), which cubature formulas of next section are based on.

In [30,32], the Schoenberg-Marsden operator S_1, defined by putting $\lambda_{ij}^{(S_1)}(f) = f(M_{ij})$ in (1.1) with $(i,j)\in\mathscr{K}_{mn}$, is described in terms of the set $\mathscr{B}_{mn}$. S_1 still reproduces bilinear polynomials, it has infinity norm equal to 1 and the number of requested function evaluation points is $N_S(m,n) = (m+2)(n+2)$ (Fig. 2.1(a)). Moreover its approximation order is equal to 2.

In the same papers the QI

$$S_2 f := \sum_{(i,j)\in\mathscr{K}_{mn}} \lambda_{ij}^{(S_2)}(f) B_{ij}$$

where

$$\lambda_{ij}^{(S_2)}(f) := b_{ij} f(M_{ij}) + a_i f(M_{i-1,j}) + c_i f(M_{i+1,j}) + \overline{a}_j f(M_{i,j-1}) + \overline{c}_j f(M_{i,j+1})$$

with

$$b_{ij} = 1-(a_i+c_i+\overline{a}_j+\overline{c}_j),$$

$$a_i = -\frac{\sigma_i^2\sigma'_{i+1}}{\sigma_i+\sigma'_{i+1}},\ c_i = -\frac{\sigma_i(\sigma'_{i+1})^2}{\sigma_i+\sigma'_{i+1}},\ \overline{a}_j = -\frac{\tau_j^2\tau'_{j+1}}{\tau_j+\tau'_{i+1}},\ \overline{c}_j = -\frac{\tau_j(\tau'_{j+1})^2}{\tau_j+\tau'_{j+1}} \qquad (2.1)$$

and

$$\sigma_i = \frac{h_i}{h_{i-1}+h_i},\ \sigma'_i = \frac{h_{i-1}}{h_{i-1}+h_i} = 1-\sigma_i\,,$$
$$\tau_j = \frac{k_j}{k_{j-1}+k_j},\ \tau'_j = \frac{k_{j-1}}{k_{j-1}+k_j} = 1-\tau_j\,, \qquad (2.2)$$

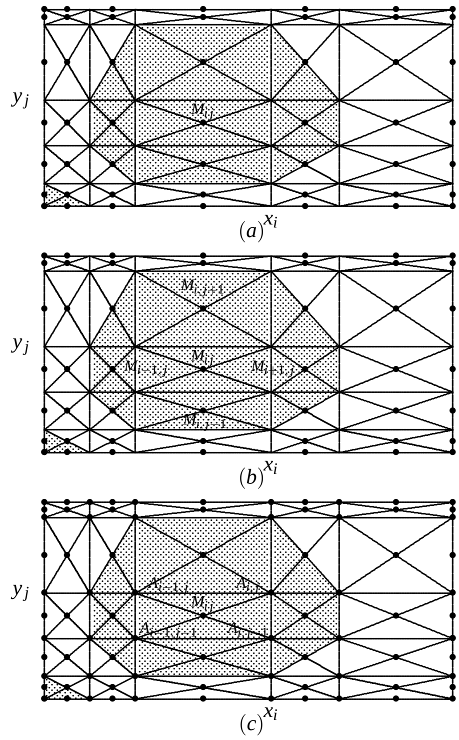


Fig. 2.1 Function evaluation points involved in the definition of (a) $\lambda_{ij}^{(S_1)}(f)$, (b) $\lambda_{ij}^{(S_2)}(f)$ and (c) $\lambda_{ij}^{(W_2)}(f)$, for any (labelled •) and all (•) $(i,j) \in \mathscr{K}_{mn}$.

is defined. S_2 is exact on $\mathbb{P}_2$ and it reaches optimal approximation order by using the same number $N_S(m,n)$ of function evaluations, needed to define S_1 (Fig. 2.1(b)). In terms of fundamental functions defined by

$$\tilde{B}_{ij} := b_{ij}B_{ij} + a_{i+1}B_{i+1,j} + c_{i-1}B_{i-1,j} + \overline{a}_{j+1}B_{i,j+1} + \overline{c}_{j-1}B_{i,j-1}\,, \qquad (2.3)$$

S_2 can be written in the form

$$S_2 f = \sum_{(i,j)\in\mathscr{K}_{mn}} f(M_{ij})\tilde{B}_{ij}.$$

In [15] the modified QI operator W_2, with

$$W_2 f := \sum_{(i,j)\in\mathscr{K}_{mn}} \lambda_{ij}^{(W_2)}(f) B_{ij}$$

and

$$\lambda_{ij}^{(W_2)}(f) := 2f(M_{i,j}) - \frac{1}{4}\sum_{h=-1}^{0}\sum_{k=-1}^{0} f(A_{i+h,j+k}),$$

is introduced and studied (see also [2,4,38]). Here the mesh-points A_{rs} that lie outside Ω should be thought of as if they were projected on $\partial\Omega$, i.e. as multiple knots on $\partial\Omega$:

$$
\begin{aligned}
&M_{i0} = (s_i, \alpha_2),\ \ M_{i,n+1} = (s_i, \beta_2),\ \ M_{0j} = (\alpha_1, t_j),\ \ M_{m+1,j} = (\beta_1, t_j),\\
&A_{-1,0} = A_{-1,-1} = A_{0,-1} = A_{0,0},\\
&A_{m+1,0} = A_{m+1,-1} = A_{m,-1} = A_{m,0},\\
&A_{-1,n} = A_{-1,n+1} = A_{0,n+1} = A_{0,n},\\
&A_{m+1,n} = A_{m+1,n+1} = A_{m,n+1} = A_{m,n},\\
&A_{i,-1} = A_{i,0},\ \ A_{i,n+1} = A_{i,n},\ i = 1,\dots,m-1,\\
&A_{-1,j} = A_{0j},\ \ A_{m+1,j} = A_{m,j},\ j = 1,\dots,n-1.
\end{aligned}
$$

W_2 is also exact on $\mathbb{P}_2$ and it reaches optimal approximation order, but for its definition $N_W(m,n) = 2N_S(m,n) - m - n - 7$ function evaluations are requested (Fig. 2.1(c)).

Moreover, due to multiple knot B-splines, Qf interpolates f at $(x,y) = (\alpha_1, \alpha_2)$, $(\beta_1, \alpha_2),\ (\alpha_1, \beta_2)$ and (β_1, β_2) when Q is one of the QIs $S_1,\ S_2$ and W_2.

Now we recall some results on the approximation power of the above defined spline operators. Let

- $\Delta = \max_{ij}\{h_i, k_j\}$;
- $\|\cdot\|_\Omega$ the supremum norm over Ω;
- $\omega(\psi,\delta) = \max\{|\psi(x,y) - \psi(u,v)|; (u,v),(x,y) \in \Omega\ , ||(x,y)-(u,v)|| \leq \delta\}$ the modulus of continuity of ψ on Ω, with $\psi \in C(\Omega)$ and $||(s,t)|| = (s^2+t^2)^{1/2}$;
- $D^\alpha = D^{(\alpha_1,\alpha_2)} = \frac{\partial^{|\alpha|}}{\partial x^{\alpha_1}\partial y^{\alpha_2}}$ with $|\ \alpha\ | = \alpha_1 + \alpha_2$;
- $\omega(D^s f, \delta) = \max\{\omega(D^\alpha f, \delta), |\ \alpha\ | = s\}$.

An error analysis for both S_2 and W_2 on f and its partial derivatives of first and second order in terms of the smoothness of f and of the triangulation properties is presented in [15]. Based on the results given in [4,38] and in [15], here we can state the following theorem.

Theorem 1 *Let Q be one of the QIs $S_1,\ S_2,\ W_2$. Then there exist positive constants $C_{0,Q}, C_{1,Q}, C_{S_1}, C_{2,Q}, C_{3,Q}$ such that*
(i) if $f \in C(\Omega)$, then

$$\|\ f - Qf\ \|_\Omega \leq C_{0,Q}\,\omega(f,\Delta);$$

(ii) if $f \in C^1(\Omega)$, then

$$\|\ f - Qf\ \|_\Omega \leq C_{1,Q}\Delta\,\omega(Df,\Delta/2);$$

(iii) if $f \in C^2(\Omega)$, then

$$\|\ f - S_1 f\ \|_\Omega \leq C_{S_1}\Delta^2 \max_{|\alpha|=2} \|\ D^\alpha f\ \|_\Omega$$

and, if $Q = S_2$ or W_2, then

$$\|\ f - Qf\ \|_\Omega \leq C_{2,Q}\Delta^2\omega(D^2 f,\Delta/2).$$

(iv) Finally, if $f \in C^3(\Omega)$ and $Q = S_2$ or W_2, then

$$\| f - Qf \|_{\Omega} \leq C_{3,Q} \Delta^3 \max_{|\alpha|=3} \| D^{\alpha} f \|_{\Omega} .$$

Remark 1 *Additional results on the approximation orders of the quasi-interpolant S_2 and its derivatives are given in [20], [21].*

3 Cubature rules based on spline QIs

3.1 General results

For any function $f \in C(\Omega)$, we consider the numerical evaluation of the integral

$$I(f) = I(f;\Omega) := \int_{\Omega} f(x,y) \mathrm{d}x \mathrm{d}y$$

by cubature rules defined by

$$I(Qf) = I(Qf;\Omega) := \sum_{ij} w_{ij}^{(Q)} f(P_{ij}),$$

where Q is any QI among S_1, S_2, W_2, defined on the bounded domain Ω, and the P_{ij}'s are the evaluation points of Fig. 2.1, belonging to the criss-cross triangular mesh $\mathscr{T}_{mn}$ [26].

For such cubatures the weights $w_{ij}^{(Q)}$ are computed as follows. In case of $Q = S_1$ we detail the proof of a result sketched in [17].

Theorem 2 *If $Q = S_1$, then*

$$w_{ij}^{(S_1)} := \int_{\Omega} B_{ij} = \int_{\Xi_{ij}} B_{ij}$$

$$= \frac{1}{12}(h_{i-1} + h_{i+1})(k_{j-1} + k_{j+1}) + \frac{1}{6}[(h_{i-1} + h_{i+1})k_j + h_i(k_{j-1} + k_{j+1})] \tag{3.1}$$

$$= \frac{1}{24}[(h_{i-1} + h_{i+1})(k_{j-1} + 4k_j + k_{j+1}) + (h_{i-1} + 4h_i + h_{i+1})(k_{j-1} + k_{j+1})] \tag{3.2}$$

for $(i, j) \in \mathscr{K}_{mn}$.

Proof. A quadratic polynomial $p \in \mathbb{P}_2$ on a triangle T of the partition $\mathscr{T}_{mn}$ can be represented in the local Bernstein basis as

$$p(\lambda) = \sum_{|\alpha|=2} c(\alpha) b_{\alpha}(\lambda)$$

where $b_{\alpha}(\lambda) = \frac{2}{\alpha!}\lambda^{\alpha}$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ are the barycentric coordinates of T, $\alpha! = \alpha_1!\alpha_2!\alpha_3!$ and $\lambda^{\alpha} = \lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\lambda_3^{\alpha_3}$.

If T is included in the rectangle with edges of length h_i and k_j, its area is equal to $\frac{1}{4}h_ik_j$ and, since [1]

$$\int_T b_\alpha = \frac{1}{24}h_ik_j$$

for all b_α, then

$$\int_T p = \frac{1}{24}h_ik_j \sum_{|\alpha|=2} c(\alpha).$$

The support Ξ_{ij} of B_{ij}, composed by 28 triangles $T_k^{(ij)},\ k=1,\dots,28$ (Fig. 3.1), sometimes degenerates for boundary and some inner B-splines. However, the general expression of B_{ij} is still valid when some h_r or k_s are zero. For the computation of this integral, it is enough to sum up the BB-coefficients $c(\alpha)$, given in [31], and multiply such a sum by $\frac{1}{24}$ times the area of the rectangle containing the triangle.

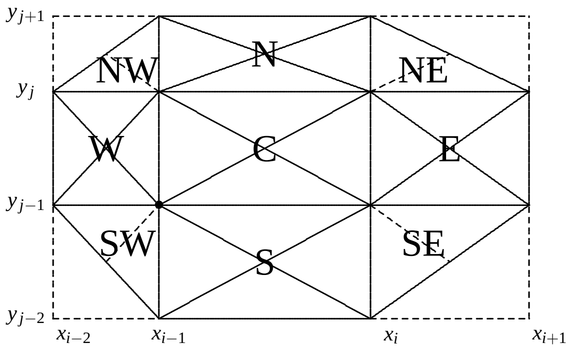


Fig. 3.1 Subregions of Ξ_{ij}, support of B_{ij}.

Then, since $\Xi_{ij} = \cup_{k=1}^{28} T_k^{(ij)}$, by [31] and Fig. 3.1 we deduce

$$\begin{aligned}
\int_C B_{ij} &= \sum_{k=1}^{4}\int_{T_k^{(ij)}} B_{ij} = \frac{1}{12}h_ik_j[(\sigma_i' + \sigma_{i+1})(\tau_j' + \tau_{j+1}) \\
&\quad + 2(\sigma_i' + \sigma_{i+1} + \tau_j' + \tau_{j+1})],\\
\int_E B_{ij} &= \sum_{k=5}^{8}\int_{T_k^{(ij)}} B_{ij} = \frac{1}{12}h_{i+1}k_j\sigma_{i+1}(\tau_j' + \tau_{j+1} + 2),\\
\int_{NE} B_{ij} &= \sum_{k=9}^{10}\int_{T_k^{(ij)}} B_{ij} = \frac{1}{12}h_{i+1}k_{j+1}\sigma_{i+1}\tau_{j+1},\\
\int_N B_{ij} &= \sum_{k=11}^{14}\int_{T_k^{(ij)}} B_{ij} = \frac{1}{12}h_ik_{j+1}\tau_{j+1}(\sigma_i' + \sigma_{i+1} + 2),\\
\int_{NW} B_{ij} &= \sum_{k=15}^{16}\int_{T_k^{(ij)}} B_{ij} = \frac{1}{12}h_{i-1}k_{j+1}\sigma_i'\tau_{j+1},
\end{aligned}$$

$$\int_W B_{ij} = \sum_{k=17}^{20} \int_{T_k^{(ij)}} B_{ij} = \frac{1}{12} h_{i-1} k_j \sigma_i' (\tau_j' + \tau_{j+1} + 2),$$
$$\int_{SW} B_{ij} = \sum_{k=21}^{22} \int_{T_k^{(ij)}} B_{ij} = \frac{1}{12} h_{i-1} k_{j-1} \sigma_i' \tau_j',$$
$$\int_S B_{ij} = \sum_{k=23}^{26} \int_{T_k^{(ij)}} B_{ij} = \frac{1}{12} h_i k_{j-1} \tau_j' (\sigma_i' + \sigma_{i+1} + 2),$$
$$\int_{SE} B_{ij} = \sum_{k=27}^{28} \int_{T_k^{(ij)}} B_{ij} = \frac{1}{12} h_{i+1} k_{j-1} \sigma_{i+1} \tau_j'.$$

Now, after some simple algebraic manipulations, we get

$$\begin{aligned}
\int_{\Xi_{ij}} B_{ij} &= \int_C B_{ij} + \int_{N\cup S\cup E\cup W} B_{ij} + \int_{NE\cup SE\cup NW\cup SW} B_{ij} \\
&= \frac{1}{12} h_i k_j [(\sigma_i' + \sigma_{i+1})(\tau_j' + \tau_{j+1}) + 2(\sigma_i' + \sigma_{i+1} + \tau_j' + \tau_{j+1})] \\
&+ \frac{1}{12} h_i (\sigma_i' + \sigma_{i+1} + 2)(k_{j+1}\tau_{j+1} + k_{j-1}\tau_j') \\
&+ \frac{1}{2} k_j (\tau_j' + \tau_{j+1} + 2)(h_{i+1} k_j \sigma_{i+1} + h_{i-1} k_j \sigma_i') \\
&+ \frac{1}{12} (h_{i-1}\sigma_i' + h_{i+1}\sigma_{i+1})(k_{j-1}\tau_j' + k_{j+1}\tau_{j+1}),
\end{aligned}$$

from which we obtain (3.1) and, by dividing and multiplying by 2, after conveniently collecting terms, we obtain (3.2). ■

We remark that, since the B_{ij}'s form a partition of unity and they are non negative, then all $w_{ij}^{(S_1)}$'s are positive. Therefore

$$\sum_{(i,j)\in\mathscr{K}_{mn}} | w_{ij}^{(S_1)} | = \sum_{(i,j)\in\mathscr{K}_{mn}} w_{ij}^{(S_1)} = (\beta_1 - \alpha_1)(\beta_2 - \alpha_2).$$

Moreover if $h_i = h$ and $k_j = k$, for any i and j, then the weights can be computed once for all as $w_{ij}^{(S_1)} = c_{ij}^{(S_1)} hk$, with the $c_{ij}^{(S_1)}$'s given in Table 3.1. The $w_{ij}^{(S_1)}$'s, $i = 2,\dots,m-1,\ j = 2,\dots,n-1$ coincide with those given in [13] for uniform partitions.

Theorem 3 *If $Q = S_2$, then*

$$\begin{aligned}
w_{ij}^{(S_2)} := \int_\Omega \tilde{B}_{ij} &= b_{ij} w_{ij}^{(S_1)} + a_{i+1} w_{i+1,j}^{(S_1)} + c_{i-1} w_{i-1,j}^{(S_1)} \\
&+ \overline{a}_{j+1} w_{i,j+1}^{(S_1)} + \overline{c}_{j-1} w_{i,j-1}^{(S_1)}\,,
\end{aligned} \tag{3.3}$$

for $(i,j) \in \mathscr{K}_{mn}$ and with the convention $a_{m+2} = \overline{a}_{n+2} = c_{-1} = \overline{c}_{-1} = 0$.

Table 3.1 The coefficients $c_{ij}^{(S_1)}$'s for a uniform partition.

$j\backslash i$	0	1	2	$\cdots$	$m-1$	m	$m+1$
$n+1$	1/12	1/4	1/3	$\cdots$	1/3	1/4	1/12
n	1/4	5/12	2/3	$\cdots$	2/3	5/12	1/4
$n-1$	1/3	2/3	1	$\cdots$	1	2/3	1/3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2	1/3	2/3	1	$\cdots$	1	2/3	1/3
1	1/4	5/12	2/3	$\cdots$	2/3	5/12	1/4
0	1/12	1/4	1/3	$\cdots$	1/3	1/4	1/12

Table 3.2 The coefficients $c_{ij}^{(S_2)}$'s for a uniform partition.

$j\backslash i$	0	1	2	3	$\ldots$	$m-2$	$m-1$	m	$m+1$
$n+1$	$-1/12$	7/36	1/9	1/9	$\cdots$	1/9	1/9	7/36	$-1/12$
n	7/36	2/3	8/9	7/8	$\cdots$	7/8	8/9	2/3	7/36
$n-1$	1/9	8/9	37/36	73/72	$\cdots$	73/72	37/36	8/9	1/9
$n-2$	1/9	7/8	73/72	1	$\cdots$	1	73/72	7/8	1/9
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
3	1/9	7/8	73/72	1	$\cdots$	1	73/72	7/8	1/9
2	1/9	8/9	37/36	73/72	$\cdots$	73/72	37/36	8/9	1/9
1	7/36	2/3	8/9	7/8	$\cdots$	7/8	8/9	2/3	7/36
0	$-1/12$	7/36	1/9	1/9	$\cdots$	1/9	1/9	7/36	$-1/12$

Proof. The proof follows immediately from (2.3) and from Theorem 2. ∎

In this case from (3.3), since $c_{-1} = a_{m+2} = \bar{c}_{-1} = \bar{a}_{n+2} = 0$, $\mid b_{ij} \mid \leq 3$, $\mid a_{i+1} \mid$, $\mid c_{i-1} \mid, \mid \bar{a}_{j+1} \mid, \mid \bar{c}_{j-1} \mid \leq \frac{1}{2}$ and $w_{ij}^{(S_1)} > 0$ for any i and j, we can deduce that

$$\sum_{(i,j)\in\mathscr{K}_{mn}} \mid w_{ij}^{(S_2)} \mid \leq \sum_{ij} (\mid b_{ij} \mid + \mid a_{i+1} \mid + \mid c_{i-1} \mid \\ + \mid \bar{a}_{j+1} \mid + \mid \bar{c}_{j-1} \mid)\, w_{ij}^{(S_1)} \leq 5(\beta_1 - \alpha_1)(\beta_2 - \alpha_2).$$

Again if $h_i = h$ and $k_j = k$, for all i and j, then $w_{ij}^{(S_2)} = c_{ij}^{(S_2)} hk$, with the $c_{ij}^{(S_2)}$'s given in Table 3.2. It is interesting to remark that in case of uniform criss-cross triangulations only $c_{ij}^{(S_2)}$, $(i,j) = (0,0), (m+1,0), (0,n+1), (m+1,n+1)$ are negative.

Theorem 4 *For any function $f \in C(\Omega)$,*

$$I(W_2 f) = \sum_{ij} w_{ij}^{(W_2)} f(P_{ij}) \\ = \sum_{(i,j)\in\mathscr{K}_{mn}} \overline{w}_{ij}^{(W_2)} f(M_{ij}) + \sum_{i=0}^{m}\sum_{j=0}^{n} \overline{\overline{w}}_{ij}^{(W_2)} f(A_{ij}), \tag{3.4}$$

where

$$\overline{w}_{ij}^{(W_2)} = 2 w_{ij}^{(S_1)},$$

Table 3.3 The coefficients $\overline{\overline{c}}_{ij}^{(W_2)}$'s for a uniform partition.

$j\backslash i$	0	1	2	$\cdots$	$m-1$	m	$m+1$
$n+1$	$-7/16$	$-9/16$	$-2/3$	$\cdots$	$-2/3$	$-9/16$	$-7/16$
n	$-9/16$	$-11/16$	$-5/6$	$\cdots$	$-5/6$	$-11/16$	$-9/16$
$n-1$	$-2/3$	$-5/6$	-1	$\cdots$	-1	$-5/6$	$-2/3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2	$-2/3$	$-5/6$	-1	$\cdots$	-1	$-5/6$	$-2/3$
1	$-9/16$	$-11/16$	$-5/6$	$\cdots$	$-5/6$	$-11/16$	$-9/16$
0	$-7/16$	$-9/16$	$-2/3$	$\cdots$	$-2/3$	$-9/16$	$-7/16$

$$\overline{\overline{w}}_{ij}^{(W_2)} = -\frac{1}{4}(\gamma_1 w_{ij}^{(S_1)} + \gamma_2 w_{i,j+1}^{(S_1)} + \gamma_3 w_{i+1,j}^{(S_1)} + \gamma_4 w_{i+1,j+1}^{(S_1)}) \tag{3.5}$$

with

$$\gamma_\ell w_{rs}^{(S_1)} = \begin{cases} 4w_{rs}^{(S_1)} & \text{if } (r,s) = (0,0),(0,n+1),(m+1,0),(m+1,n+1), \\ 2w_{rs}^{(S_1)} & \text{if } (r,s) = (0,j),(i,0),(m+1,j),(i,n+1), \\ & \qquad i = 1,\ldots,m,\ j = 1,\ldots,n, \\ w_{rs}^{(S_1)} & \text{if } r = 1,\ldots,m,\ s = 1,\ldots,n, \end{cases} \tag{3.6}$$

$\ell = 1,2,3,4.$

Proof. From its definition, we can rewrite $W_2 f$ in the following form:

$$W_2 f = 2 \sum_{(i,j)\in\mathscr{K}_{mn}} f(M_{ij})B_{ij} - \frac{1}{4}\sum_{i=0}^{m}\sum_{j=0}^{n} f(A_{ij})\overline{B}_{ij}$$

where

$$\overline{B}_{ij} = \gamma_1 B_{ij} + \gamma_2 B_{i,j+1} + \gamma_3 B_{i+1,j} + \gamma_4 B_{i+1,j+1}.$$

The γ_ℓ's are obtained, after some algebraic manipulations, by taking into account the multiplicity of the A_{ij}'s along the boundary $\partial\Omega$: 4 at the four corners and 2 along the four edges of $\partial\Omega$. ∎

From (3.5) it is easy to verify that

$$\sum_{i=0}^{m}\sum_{j=0}^{n} | \overline{\overline{w}}_{ij}^{(W_2)} | \leq \left(\sum_{\ell=1}^{4}\gamma_\ell\right) \sum_{(i,j)\in\mathscr{K}_{mn}} w_{ij}^{(S_1)} \leq 9(\beta_1-\alpha_1)(\beta_2-\alpha_2).$$

Thus, in conclusion, we have

$$\sum_{ij} | w_{ij}^{(W_2)} | = \sum_{ij} | \overline{w}_{ij}^{(W_2)} | + \sum_{ij} | \overline{\overline{w}}_{ij}^{(W_2)} | \leq 11(\beta_1-\alpha_1)(\beta_2-\alpha_2).$$

If $h_i = h$ and $k_j = k$, for all i and j, then $\overline{w}_{ij}^{(W_2)} = \overline{c}_{ij}^{(W_2)}hk$, where $\overline{c}_{ij}^{(W_2)} = 2c_{ij}^{(S_1)}$ and $\overline{\overline{w}}_{ij}^{(W_2)} = \overline{\overline{c}}_{ij}^{(W_2)}hk$, with the $\overline{\overline{c}}_{ij}^{(W_2)}$'s given in Table 3.3.

According to Rabinowitz and Richter [28], a good quadrature or cubature formula has all points inside the region Ω and all weights positive. Positive weights imply that $I(Q\cdot)$ is also a positive functional. In [22] we can find again such two desiderable properties for a numerical integration formula: (P_1) the points used lie in the integration domain; (P_2) the coefficients be positive. Regarding property (P_1), the function to be integrated may not be defined outside the domain of integration. Therefore, the use of integration rules based on B-splines with multiple knots is plainly justified in that case.

Moreover the sum of the coefficients of any formula of degree zero or higher is equal to $\mid \Omega \mid$, the measure of the integration domain. Thus, property (P_2) means that, if the coefficients are all positive, there is a natural bound to their size. This is not the case of rules based on the operators S_2 and W_2. However, if we define the quantity [22]

$$N_1(I(Qf)) := \frac{1}{\mid \Omega \mid} \sum_{ij} \mid w_{ij}^{(Q)} \mid,$$

which is > 1 if and only if some coefficients are negative, we have $N_1(I(S_1 f)) = 1$, $N_1(I(S_2 f)) \leq 5$ and $N_1(I(W_2 f)) \leq 11$. Therefore the sum of the absolute values of weights, associated with the operators S_2 and W_2, is uniformly bounded, as we proved just above.

Finally, from Theorem 1 we can immediately deduce some results on the convergence of $I(Qf)$ to $I(f)$.

Theorem 5 *Let $f \in C(\Omega)$. Then for $Q = S_1,\ S_2,\ W_2$ we have*

$$I(Qf) \to I(f) \ \text{ as } \Delta \to 0,$$

where $\Delta = \max_{ij}\{h_i, k_j\}$. In particular

$$\mid E(Qf) \mid = \mid I(f) - I(Qf) \mid \leq C_Q \omega(f, \Delta),$$

where C_Q is a positive constant independent on m and n.

Moreover if $f \in C^k(\Omega)$, then

$$E(Qf) = O(\Delta^k)$$

with $k = 1,\ 2$ for $Q = S_1,\ S_2,\ W_2$ and $k = 3$ for $Q = S_2,\ W_2$.

Proof. Let $R(Qf) = f - Qf$. Since

$$\mid E(Qf) \mid \leq I(\mid R(Qf) \mid),$$

then we get

$$\mid E(Qf) \mid \leq \parallel R(Qf) \parallel (\beta_1 - \alpha_1)(\beta_2 - \alpha_2),$$

where $\parallel R(Qf) \parallel$ is bounded in Theorem 1. ∎

3.2 Symmetric criss-cross triangulations

It is well known that if $I(Qf)$ is exact for all bivariate polynomials of total degree p or lower, but it is not exact for some bivariate polynomials of total degree $p+1$, then $I(Qf)$ is said to have "degree of precision" p. For any integrand function f, the accuracy with which $I(Qf)$ approximates the integral $I(f)$ is related to the degree of precision of $I(Qf)$ and to the accuracy with which f itself can be approximated by polynomials, this latter depending on the smoothness of f. A contribution in this direction is given by the following results on the precision degree of the above integration rules for symmetric criss-cross triangulations.

Theorem 6 *Let Ω be any rectangular domain. If the partition $\mathscr{T}_{mn}$ has symmetric knots with respect to the centre of Ω, i.e. $(x_{m-i}, y_{n-j}) = (\alpha_1+\beta_1-x_i, \alpha_2+\beta_2-y_j)$, then*

$$w^{(Q)}_{m-i+1,n-j+1} = w^{(Q)}_{i,j}, \quad \text{for } Q = S_1, S_2 \tag{3.7}$$

and

$$\overline{\overline{w}}^{(W_2)}_{m-i,n-j} = \overline{\overline{w}}^{(W_2)}_{i,j}.$$

Proof. Let us consider the case $Q = S_1$. If $x_{m-i} = \alpha_1+\beta_1-x_i$ and $y_{n-j} = \alpha_2+\beta_2-y_j$, then from the definition of h_i and k_j we immediately deduce $h_{m-i+1} = h_i$ and $k_{n-j+1} = k_j$ and, from Theorem 2, we obtain (3.7).

If $Q = S_2$, then from (2.2), for $i = 2, \ldots, m$,

$$\begin{aligned} \sigma_i &= \tfrac{h_i}{h_{i-1}+h_i} = \tfrac{h_{m-i+1}}{h_{m-i+2}+h_{m-i+1}} = \sigma'_{m-i+2}, \\ \sigma'_i &= \tfrac{h_{i-1}}{h_{i-1}+h_i} = \tfrac{h_{m-i+2}}{h_{m-i+2}+h_{m-i+1}} = \sigma_{m-i+2}\,. \end{aligned} \tag{3.8}$$

Moreover $\sigma_0 = 0$ and $\sigma_1 = 1$ since $h_0 = 0$ and $\sigma_{m+1} = 0$ since $h_{m+1} = 0$. Now, from (2.1) and (3.8) we can write

$$\begin{aligned} a_{m-i+1} &= -\tfrac{\sigma^2_{m-i+1}\sigma'_{m-i+2}}{\sigma_{m-i+1}+\sigma'_{m-i+2}} = -\tfrac{(\sigma'_{i+1})^2\sigma_i}{\sigma'_{i+1}+\sigma_i} = c_i, \\ c_{m-i+1} &= -\tfrac{\sigma_{m-i+1}(\sigma'_{m-i+2})^2}{\sigma_{m-i+1}+\sigma'_{m-i+2}} = -\tfrac{(\sigma_i)^2\sigma'_{i+1}}{\sigma'_{i+1}+\sigma_i} = a_i. \end{aligned}$$

Similarly, from (2.1) and (2.2) we deduce $\tau_j = \tau'_{n-j+2}$, $\tau'_j = \tau_{n-j+2}$, with $\tau_0 = \tau_{n+1} = 0$ and $\tau_1 = 1$. Moreover $\overline{a}_{n-j+1} = \overline{c}_j$, $\overline{c}_{n-j+1} = \overline{a}_j$ and then

$$\begin{aligned} b_{m-i+1,n-j+1} &= 1 - (a_{m-i+1} + c_{m-i+1} + \overline{a}_{n-j+1} + \overline{c}_{n-j+1}) \\ &= 1 - (c_i + a_i + \overline{c}_j + \overline{a}_j) = b_{ij}. \end{aligned}$$

Then, from (3.3), we obtain

$$\begin{aligned} w^{(S_2)}_{m-i+1,n-j+1} = {} & b_{m-i+1,n-j+1}\, w^{(S_1)}_{m-i+1,n-j+1} \\ & + a_{m-i+2}\, w^{(S_1)}_{m-i+2,n-j+1} + c_{m-i}\, w^{(S_1)}_{m-i,n-j+1} \\ & + \overline{a}_{n-j+2}\, w^{(S_1)}_{m-i+1,n-j+2} + \overline{c}_{n-j}\, w^{(S_1)}_{m-i+1,n-j}. \end{aligned}$$

Now, since $a_{m-i+2} = c_{i-1}$, $c_{m-i} = a_{i+1}$, $\overline{a}_{n-j+2} = \overline{c}_{j-1}$ and $\overline{c}_{n-j} = \overline{a}_{j+1}$, we have (3.7).

Finally, if $Q = W_2$, then from (3.5) it results

$$\begin{aligned}\overline{\overline{w}}^{(W_2)}_{m-i,n-j} &= -\frac{1}{4}(\gamma_1 w^{(S_1)}_{m-i,n-j} + \gamma_2 w^{(S_1)}_{m-i,n-j+1} \\ &+ \gamma_3 w^{(S_1)}_{m-i+1,n-j} + \gamma_4 w^{(S_1)}_{m-i+1,n-j+1}) \\ &= -\frac{1}{4}(\gamma_1 w^{(S_1)}_{i+1,j+1} + \gamma_2 w^{(S_1)}_{i+1,j} + \gamma_3 w^{(S_1)}_{i,j+1} + \gamma_4 w^{(S_1)}_{ij}).\end{aligned}$$

Therefore, from (3.6), the theorem is proved. ∎

Thus, by using Theorem 6, in case of symmetric partitions we can state something more on the precision degree of the cubature formulas based on the operators $S_1,\ S_2$ and W_2. We set $m_{r,s}(x,y) = x^r y^s$.

Corollary 1 *Let Ω be any rectangular domain and $\mathscr{T}_{mn}$ have symmetric knots $(x_{m-i}, y_{n-j}) = (\alpha_1 + \beta_1 - x_i, \alpha_2 + \beta_2 - y_j)$, $i = 0,\dots,m,\ j = 0,\dots,n$. If $Q = S_2,\ W_2$, then*

$$I(Qm_{r,s}) = I(m_{r,s}) \tag{3.9}$$

with $0 \le r,s \le 3,\ r+s = 3$ *and* $(r,s) = (3,1),(1,3)$.

Proof. Let Q be either S_2 or W_2. Any integral on a rectangular domain can be converted, via convenient changes of variables, to an integral on the square domain $[-1,1]^2$. Therefore, by setting

$$x = H_1 u + K_1, \quad y = H_2 v + K_2,$$

with

$$H_1 = \frac{\beta_1 - \alpha_1}{2},\ K_1 = \frac{\alpha_1 + \beta_1}{2},\ H_2 = \frac{\beta_2 - \alpha_2}{2},\ K_2 = \frac{\alpha_2 + \beta_2}{2},$$

we can write

$$I(m_{r,s}) = \int_\Omega m_{r,s}(x,y)\mathrm{d}x\mathrm{d}y = H_1 H_2 \int_{[-1,1]^2} \Big(p_2(u,v) + \sum_{ij} \mu_{ij} m_{i,j}(u,v)\Big)\mathrm{d}u\mathrm{d}v,$$

where $p_2 \in \mathbb{P}_2$ and in the sum at least either i or j is odd and $\mu_{ij} \in \mathbb{R}$.

Now, from the hypothesis on $\mathscr{T}_{mn}$, from Theorem 6 and by the polynomial reproduction and linearity properties of the operators S_2 and W_2, it results

$$I(Qm_{r,s}) = H_1 H_2 [I(Qp_2) + \sum_{ij} \mu_{ij} I(Qm_{i,j})] = H_1 H_2 I(p_2).$$

So we get the desired result. ∎

Then, in case of uniform partitions, S_2 and W_2 have a better precision degree, i.e. increased by one, similarly to some integration rules based on tensor product of interpolatory quadrature formulas, giving rise to Newton-Cotes formulas, to be preferred in case of an odd number of interpolation points [23]. Indeed, since uniform partitions are special cases of symmetric ones, the degree of precision of S_2 and W_2 is not 2, but 3. Therefore these new integration rules can be compared, e.g., with the tensor product of Simpson rules.

Remark 2 *If $\mathscr{T}_{mn}$ is any criss-cross triangulation, then Maple symbolic computation gives evidence that*

- *(3.9) holds for $Q = W_2$ with $r,s = 1,2,\ r+s = 3$:*

$$I(W_2 m_{2,1}) = \frac{1}{6}(x_m^3 - x_0^3)(y_n^2 - y_0^2) = I(m_{2,1})$$

and

$$I(W_2 m_{1,2}) = \frac{1}{6}(x_m^2 - x_0^2)(y_n^3 - y_0^3) = I(m_{1,2});$$

- *there exists a constant C_{X_m}, depending only on the knots $x_i,\ i = 0,\dots,m$, such that*

$$I(S_1 m_{2,1}) = C_{X_m}(y_n^2 - y_0^2)$$

and, similarly, there exists a constant C_{Y_n}, depending only on the knots $y_j,\ j = 0,\dots,n$, such that

$$I(S_1 m_{1,2}) = C_{Y_n}(x_m^2 - x_0^2).$$

Then (3.9) holds for $Q = S_1$ with $r,s = 1,2,\ r+s = 3$, if Ω is of type $[\alpha_1,\beta_1]\times[-\alpha_2,\alpha_2]$ and $[-\alpha_1,\alpha_1]\times[\alpha_2,\beta_2]$, respectively.

In case of rules based on the operator S_2 we have only numerical evidence that such precision degree is also attained, but till now we have no proof of this fact.

Corollary 2 *Let $Q = S_1,\ S_2,\ W_2$. If $\Omega = [-\alpha_1,\alpha_1]\times[-\alpha_2,\alpha_2]$ and $\mathscr{T}_{mn}$ has symmetric knots, then*

$$I(Qm_{r,s}) = I(m_{r,s})$$

in the following two cases:

- *$r+s = 2k+1$,*
- *$r+s = 2k$ with r and s odd,*

where $k \in \mathbb{N}$.

Proof. From Theorem 6, by reasoning as in Corollary 1 and taking into account the definition of Ω, we obtain the thesis. ∎

Remark 3 *We recall [22] that a region $\Omega \subset \mathbb{R}^2$ is said to be "fully symmetrical" if, whenever (x,y) is a point of the integration domain, all points of the form $(\pm x,\pm y)$ and $(\pm y,\pm x)$ are also integration points.*

Similarly a numerical integration formula is "fully symmetrical" [6,22] if, whenever it contains the point (x,y) with associated coefficient w, it contains all the points $(\pm x,\pm y)$ and $(\pm y,\pm x)$ with the same associated coefficient.

Therefore, if Ω is a "fully symmetrical" domain, the integration rules based on the operators $S_1,\ S_2$ and W_2 are "fully symmetrical", so that we find again the results of Corollary 2 for the case $r+s$ odd also in [22] (p. 493).

Finally, if $\mathscr{T}_{mn}$ has symmetric knots, then we can provide another result on the convergence of $I(Qf)$ to $I(f)$.

Theorem 7 *Let $Q = S_2$, W_2 and $\mathscr{T}_{mn}$ have symmetric knots $(x_{m-i}, y_{n-j}) = (\alpha_1+\beta_1-x_i, \alpha_2+\beta_2-y_j)$, $i=0,\dots,m$, $j=0,\dots,n$. If $f\in C^3(\Omega)$, then there exists a positive constant $\overline{C}_{3,Q}$ such that*

$$| E(Qf) | \leq \overline{C}_{3,Q}\Delta^3\omega(D^3 f,\Delta).$$

Moreover if $f\in C^4(\Omega)$, then there exists a positive constant $\overline{C}_{4,Q}$ such that

$$| E(Qf) | \leq \overline{C}_{4,Q}\Delta^4 \max_{|\alpha|=4} \| D^\alpha f \|_\Omega .$$

Proof. Due to the symmetry of knots and weights with respect to the centre of Ω [16, 18, 24], from Corollary 1 the cubatures associated with the QIs S_2 and W_2 are exact on the space $\mathbb{P}_3$. Then, similarly to [38] and [15], we have

$$\| f-Qf \|_\Omega = \| f-Qf \|_T \leq (1+\| Q \|) \| f-q_3^* \|_{\cup T} \leq (1+\| Q \|) \| f-q_3 \|_{\cup T}$$

with $\cup T$ the union of the triangular cells of the B-spline supports that are non zero on T, q_3^* the best approximation polinomial of f in $\mathbb{P}_3$ on $\cup T$ and

$$q_3(x,y) = \sum_{i=0}^{3}\frac{1}{i!}\left[(x-\xi)\frac{\partial}{\partial x}+(y-\eta)\frac{\partial}{\partial y}\right]^i f(\xi,\eta), \tag{3.10}$$

where $(\xi,\eta)\in T$ and for partial derivatives in (3.10) the power i stands for their order.

If $f\in C^3(\Omega)$, we set $f=q_3+r_3$ with

$$r_3(x,y) = \frac{1}{3!}\sum_{i=0}^{3}\binom{3}{i} D^{(3-i,i)}[f(u,v)-f(\xi,\eta)](x-\xi)^{3-i}(y-\eta)^i,$$

where the point (u,v) lies somewhere in the segment joining (ξ,η) to (x,y). Then, since $(x-\xi)+(y-\eta)\leq 4\Delta$, from the modulus of continuity properties [36], it results

$$\| f-q_3 \|_{\cup T} = \| r_3 \|_{\cup T} \leq \frac{1}{6}[(x-\xi)+(y-\eta)]^3\omega(D^3 f,\sqrt{10}\Delta) \leq \frac{256}{6}\Delta^3\omega(D^3 f,\Delta).$$

If $f\in C^4(\Omega)$, we set $f=q_3+r_4$ with

$$r_4(x,y) = \frac{1}{4!}\sum_{i=0}^{4}\binom{4}{i} D^{(4-i,i)} f(u,v)(x-\xi)^{4-i}(y-\eta)^i. \tag{3.11}$$

Then, from (3.11), we get

$$\| f-q_4 \|_{\cup T} = \| r_4 \|_{\cup T} \leq \frac{256}{24}\Delta^4 \max_{|\alpha|=4} \| D^\alpha f \|_\Omega .$$

Finally, since $\| S_2 \| \leq 5$ and $\| W_2 \| \leq 3$ [15], we obtain

$$\overline{C}_{3,S_2} = 256(\beta_1-\alpha_1)(\beta_2-\alpha_2),\ \ \overline{C}_{3,W_2} = \frac{512}{3}(\beta_1-\alpha_1)(\beta_2-\alpha_2),$$

$$\overline{C}_{4,S_2} = 64(\beta_1-\alpha_1)(\beta_2-\alpha_2), \;\; \overline{C}_{4,W_2} = \frac{256}{6}(\beta_1-\alpha_1)(\beta_2-\alpha_2). \qquad \blacksquare$$

Of course such constants are not the best ones, since in the proof we considered the polynomial q_3, instead of q_3^*.

We remark that similar results for S_2 in the univariate case can be found in [33].

Approximation order in case of both symmetric/uniform and nonuniform criss-cross triangulations can be observed in the next section.

4 Numerical results

In this section we propose some numerical results obtained by testing cubatures of section 3, when integrating the following functions:

$$\begin{aligned}
&f_1(x,y) := \sqrt{|\,xy\,|},\\
&f_2(x,y) := |\,x^2+y^2-0.25\,|,\\
&f_3(x,y) := \sqrt{|\,x-y\,|},\\
&f_4(x,y) := \frac{1}{9}\sqrt{64-81\left(\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2\right)}-\frac{1}{2},\\
&f_5(x,y) := e^{-\frac{(5-10x)^2}{2}}+.75e^{-\frac{(5-10y)^2}{2}}+.75e^{-\frac{(5-10x)^2}{2}}e^{-\frac{(5-10y)^2}{2}},\\
&f_6(x,y) := \sqrt{|\,y\,|},
\end{aligned}$$

$(x,y)\in[0,1]\times[0,1],[-1,1]\times[-1,1]$. All computations have been performed on a personal computer with a 16-digit arithmetic, by using Matlab.

4.1 Comparison based on about the same relative error

In tables 4.1, 4.2, 4.3 we present the relative errors provided by the application of our rules on uniform criss-cross triangulations and other known ones, given in [13, 27] and [19], p.216, in particular Simpson product rule, with approximation order $O(\Delta^4)$, and the ones based on the Schoenberg-Marsden operators S^{u} and S^{nu}, defined by using simple knot B-splines on uniform and nonuniform criss-cross triangulations, respectively, whose approximation order is the same as the integration rule based on S_1.

Here we want to underline the different number of integrand function evaluations (figures in parentheses) in order to get about the same relative error size.

These results show that the new cubatures are comparable with the ones based on simple knot B-splines, but they do not need any function evaluation points outside the domain Ω.

Table 4.1 Absolute values of relative errors in the computation of $I(f_1;[0,1]\times[0,1]) = 4/9$ by several integration methods.

Romberg [19]	Gauss-Legendre [19]	Clenshaw-Curtis+trapz. rule [19]	S^{u} [13]	S_1	S_2	W_2	Simpson product rule
1.4(−2)	1.3(−2)	1.1(−4)	5.6(−2)	7.8(−2)	8.6(−3)	8.7(−3)	7.7(−3)
(32060)	(4096)	(6786)	(25)	(25)	(25)	(37)	(121)
			4.9(−3)	9.0(−3)	6.3(−4)	1.4(−3)	3.8(−3)
			(276)	(272)	(272)	(477)	(289)
			9.7(−4)	2.2(−3)	1.5(−4)	3.4(−4)	1.0(−3)
			(1476)	(1480)	(1480)	(2880)	(1521)

Table 4.2 Absolute values of relative errors in the computation of $I(f_2;[-1,1]\times[-1,1]) = 5/3+\pi/16$ by several integration methods.

Romberg [19]	Gauss-Legendre [19]	tanh product rule [19]	S^{u} [13]	S_1	S_2	W_2	Simpson product rule
5.9(−7)	1.9(−5)	1.6(−5)	9.5(−3)	9.5(−3)	2.9(−5)	1.5(−4)	1.7(−4)
(36504)	(4096)	(10057)	(540)	(528)	(528)	(1007)	(625)
			9.7(−4)	8.6(−4)	6.8(−6)	2.3(−5)	4.9(−5)
			(4608)	(5184)	(5184)	(10221)	(5041)

Table 4.3 Absolute values of relative errors in the computation of $I(f_3;[0,1]\times[0,1]) = 8/15$ by several integration methods.

Romberg [19]	Gauss-Legendre [19]	tanh product rule [19]	S^{u} [13]	S^{nu} [27]	S_1	S_2	W_2
6.2(−5)	2.1(−3)	9.4(−3)	1.6(−3)	1.0(−3)	7.8(−3)	1.3(−3)	2.6(−3)
(16224)	(4096)	(2751)	(160)	(160)	(160)	(160)	(291)
		4.8(−3)	7.5(−4)	7.3(−4)	7.2(−4)	1.8(−5)	1.5(−5)
		(3441)	(1364)	(1364)	(1376)	(1311)	(2539)
		1.8(−3)	2.6(−5)	2.4(−5)	1.9(−4)	1.5(−5)	2.2(−5)
		(10057)	(5152)	(5152)	(5183)	(5183)	(10219)

4.2 Comparison based both on the same spline space and on about the same number of function evaluation points

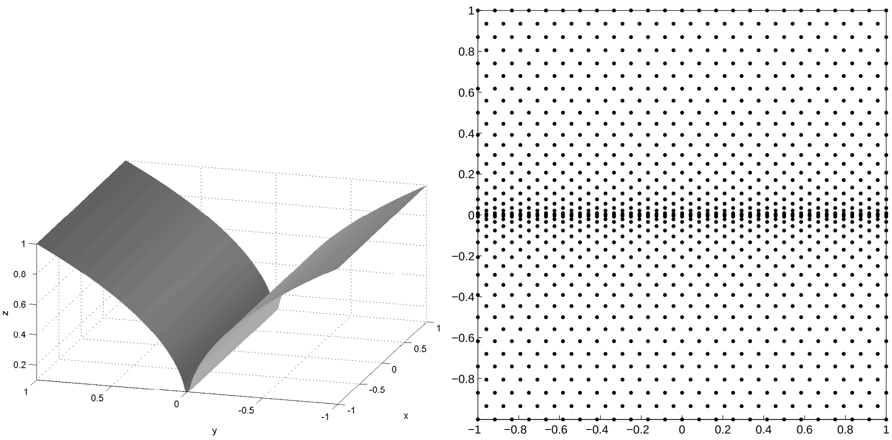
In tables 4.4-4.7 we compare our cubatures with composite Simpson product rule ($I(\Sigma f)$) for some test functions. We denote by $E(Qf)$ the cubature error $I(f)-I(Qf)$. In particular, in tables 4.4 and 4.5 the comparison is carried out both in the same spline space and by using about the same number of function evaluations, in case of uniform criss-cross triangulations. In order to do it, since $N_S(m,n) = (m+2)(n+2)$ and $N_W(m,n) = 2N_S - m - n - 7$, in the seventh column we use $N_W\left(\left\lceil \frac{m}{\sqrt{2}}\right\rceil, \left\lceil \frac{n}{\sqrt{2}}\right\rceil\right) \approx N_S(m,n)$ function evaluations. Moreover, if we denote by $N_\Sigma(m,n) = (m+1)(n+1)$ the number of function evaluation points for Simpson product rule (m and n even),

Table 4.4 Absolute errors in the computation of $I(f_4;[0,1]\times[0,1]) = .2865833317293664$ on uniform triangulations.

$m=n$	$E(S_1 f)$	$E(S_2 f)$	$E(W_2 f)$	$E(\Sigma f)$	$m=n$	$E(W_2 f)$	m,n	$E(\Sigma f)$
4	1.8(−2)	−4.5(−4)	−1.0(−3)	5.0(−4)	3	−2.6(−3)	6,4	3.1(−4)
8	5.1(−3)	−4.2(−5)	−9.1(−5)	3.9(−5)	6	−2.5(−4)	10,8	2.8(−5)
16	1.4(−3)	−3.3(−6)	−7.0(−6)	2.7(−6)	12	−2.1(−5)	18,16	2.2(−6)
32	3.5(−4)	−2.3(−7)	−4.9(−7)	1.7(−7)	23	−1.7(−6)	34,32	1.5(−7)
64	9.0(−5)	−1.5(−8)	−3.2(−8)	1.1(−8)	46	−1.2(−7)	66,64	1.0(−8)

Table 4.5 Absolute errors in the computation of $I(f_5;[0,1]\times[0,1]) = .4857835323466119$ on uniform triangulations.

$m=n$	$E(S_1 f)$	$E(S_2 f)$	$E(W_2 f)$	$E(\Sigma f)$	$m=n$	$E(W_2 f)$	m,n	$E(\Sigma f)$
4	4.5(−2)	3.8(−2)	1.3(−1)	1.1(−1)	3	−6.0(−1)	6,4	1.8(−2)
8	1.5(−5)	−4.8(−5)	−4.5(−5)	1.5(−2)	6	2.2(−3)	10,8	5.2(−3)
16	1.1(−6)	−6.1(−7)	−1.0(−6)	1.2(−6)	12	−4.9(−6)	18,16	5.5(−7)
32	1.6(−7)	−1.6(−8)	−3.3(−8)	7.6(−9)	23	−1.6(−7)	34,32	6.7(−9)
64	3.1(−8)	−6.4(−10)	−1.5(−9)	5.5(−10)	46	−6.3(−9)	66,64	5.1(−10)



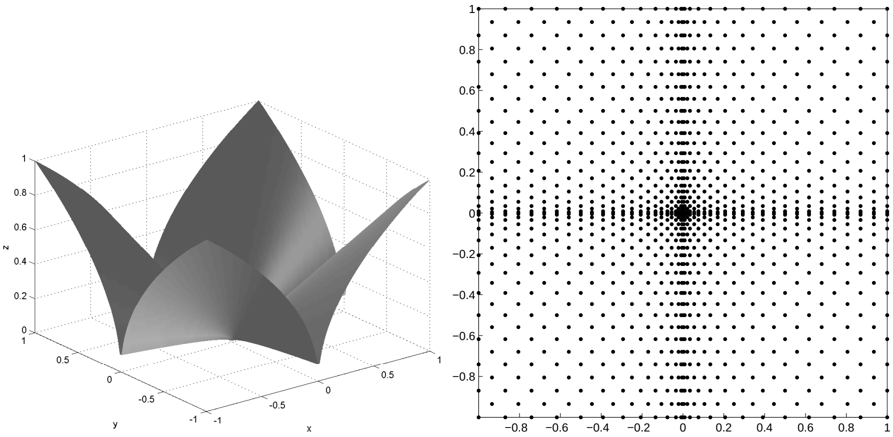
(*a*) (*b*)

Fig. 4.1 (a) $f_6(x,y),\ (x,y)\in[-1,1]\times[-1,1]$; (b) $P^{(1)}_{24,24}$.

then in the ninth column we use $N_\Sigma(m,n+2) = N_\Sigma(m+2,n)$ function evaluation points, that are about $N_S(m,n)$, when $m=n$.

4.3 Comparison between uniform and nonuniform partitions

Finally a comparison between uniform and nonuniform criss-cross triangulations in the same spline space is presented in tables 4.6 and 4.7, where integration rules are applied to the test functions shown in Fig. 4.1(a) and 4.2(a), respectively.



(a) (b)

Fig. 4.2 (a) $f_1(x,y),\ (x,y) \in [-1,1]\times[-1,1]$; (b) $P^{(2)}_{24,24}$.

Table 4.6 Absolute errors in the computation of $I(f_6;[-1,1]\times[-1,1]) = \frac{8}{3}$ on uniform (u) and nonuniform (nu) $P^{(1)}_{mn}$ criss-cross triangulations.

$m=n$	$E(S_1 f)$		$E(S_2 f)$		$E(W_2 f)$		$E(\Sigma f)$
	u	nu	u	nu	u	nu	u
4	$-1.5(-1)$	$-7.9(-3)$	$-8.5(-2)$	$-2.5(-2)$	$-4.6(-1)$	$-2.5(-1)$	$1.1(-1)$
8	$-4.7(-2)$	$2.8(-2)$	$-3.0(-2)$	$-3.9(-3)$	$-1.6(-1)$	$-3.7(-2)$	$4.1(-2)$
16	$-1.5(-2)$	$1.1(-2)$	$-1.1(-2)$	$-5.4(-4)$	$-5.8(-2)$	$-5.0(-3)$	$1.4(-2)$
32	$-4.8(-3)$	$3.4(-3)$	$-3.8(-3)$	$-7.0(-5)$	$-2.1(-2)$	$-6.4(-4)$	$5.1(-3)$
64	$-1.6(-3)$	$9.3(-4)$	$-1.3(-3)$	$-9.0(-6)$	$-7.3(-3)$	$-8.2(-5)$	$1.8(-3)$

In Fig. 4.1(b) and 4.2(b) the following nonuniform symmetric partitions, depending on the features of the integrand functions, are reported, respectively, where for the sake of clarity only the vertices M_{ij} and A_{rs} appear:

$$\begin{aligned} P^{(k)}_{mn} &:= X_m \times Y_n = \{(x_i, y_j)\} \\ &= \{((\beta_1-\alpha_1)\xi^{(k)}_i + \alpha_1, (\beta_2-\alpha_2)\eta^{(k)}_j + \alpha_2)\},\ k=1,2, \end{aligned}$$

with

(1) $\xi^{(1)}_i = \frac{i}{m},\ i=0,\ldots,m$ and $\eta^{(1)}_j = \frac{1}{2}\cos\frac{n/2-j}{n}\pi,\ j=0,\ldots,n/2,$
$\eta^{(1)}_j = 1-\eta^{(1)}_{n-j},\ j=n/2+1,\ldots,n$ (n even);

(2) $\xi^{(2)}_i = \frac{1}{2}\cos\frac{m/2-i}{m}\pi,\ \ \eta^{(2)}_j = \frac{1}{2}\cos\frac{n/2-j}{n}\pi,\ \ i=0,\ldots,m/2,$
$j=0,\ldots,n/2$ and $\xi^{(2)}_i = 1-\xi^{(2)}_{m-i},\ \ \eta^{(2)}_j = 1-\eta^{(2)}_{n-j},$
$i=m/2+1,\ldots,m,\ j=n/2+1,\ldots,n$ ($m,\ n$ even).

Table 4.7 Absolute errors in the computation of $I(f_1;[-1,1]\times[-1,1]) = \frac{16}{9}$ on uniform (u) and nonuniform (nu) $P^{(2)}_{mn}$ criss-cross triangulations.

$m=n$	$E(S_1 f)$		$E(S_2 f)$		$E(W_2 f)$		$E(\Sigma f)$
	u	nu	u	nu	u	nu	u
4	−2.1(−1)	−1.0(−2)	−1.1(−1)	−3.3(−2)	−6.2(−1)	−3.1(−1)	−1.5(−1)
8	−6.3(−2)	3.7(−2)	−4.1(−2)	−4.8(−3)	−2.2(−1)	−4.8(−2)	5.4(−2)
16	−2.0(−2)	1.5(−2)	−1.4(−2)	−6.6(−4)	−7.8(−2)	−6.5(−3)	1.9(−2)
32	−6.4(−3)	4.6(−3)	−5.1(−3)	−8.9(−5)	−2.7(−2)	−8.5(−4)	6.8(−3)
64	−2.1(−3)	1.2(−3)	−1.8(−3)	−1.2(−5)	−9.7(−3)	−1.1(−4)	2.4(−3)

Such examples show that the choice of the partition can be crucial for the accuracy of the results.

We recall that $I(Qf)$ has precision degree $1,\ 2,\ 2$ when $Q = S_1,\ S_2,\ W_2$, respectively. However, due to the symmetry of the weights and knots with respect to the centre of Ω, from Corollary 1, $I(S_2 f)$ and $I(W_2 f)$ have precision degree 3 in case of symmetric criss-cross triangulations. Indeed the above tables confirm the approximation order of the QIs, stated in theorems 5 and 7.

We remark that the signs of cubature error $E(\Sigma f)$ and $E(Qf),\ Q = S_2, W_2$ are opposite, when m and n are large enough, i.e. both two formulas $(I(\Sigma f),\ I(S_2 f))$ and two formulas $(I(\Sigma f), I(W_2 f))$ give lower and upper estimates of the integral value, being the corresponding sequences of integration rules, when m and n increase, convergent to the integral to be calculated from below and from above. P. J. Davis and P. Rabinowitz in their comprehensive monograph on numerical integration refer to this property as ‘bracketing’ property [16], p.54.

5 Final remarks

In this paper we studied cubature formulas based on B-splines with multiple knots, having all supports included into the integration domain Ω [30,32]. This is an advantage with respect to other known ones [13,27,38], always based on B-splines, that need also function evaluation points outside Ω.

Moreover such cubatures turn out to be comparable with others known in literature [19] and in case of symmetry of knots, and if necessary domain, they provide higher precision degree.

We remark that the proposed schemes could be used in the numerical evaluation of 2D singular integrals, defined in the Hadamard finite part sense [12,13].

A deeper and theoretical study of the sign of $E(Qf),\ Q = S_2, W_2$ and of the weights $w^{(S_2)}_{ij}$ is an interesting problem, that we are considering.

Finally a possible suitably weighted means of such integration rules can provide more accurate formulas [33] and this could be a further subject to be investigated.

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