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Inverse cascade in Charney-Hasegawa-Mima turbulence

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Abstract. – The inverse energy cascade in the Charney-Hasegawa-Mima turbulence is investigated. The Kolmogorov law for the third-order velocity structure function is derived and shown to be independent of the parameter λ, at variance with the energy spectrum, as shown by high-resolution direct numerical simulations. In the asymptotic limit of strong rotation, $\lambda \to \infty$, the Kolmogorov constant is found to be $C_\lambda \simeq 11$ while coherent vortices are observed to form at a dynamical scale which slowly grows with time. These vortices form an almost quenched pattern and induce a strong deviation form Gaussianity in the velocity field.

The existence of an inverse cascade is the most remarkable property of two-dimensional turbulence. It was predicted by Kraichnan [1] for Navier-Stokes equation: as a consequence of inviscid enstrophy conservation, energy is forced to flow to large scales. The inverse cascade can be sustained only in the presence of an external forcing injecting energy at a characteristic scale into the system. At scales larger than the forcing the turbulent flow is essentially random with Gaussian velocity difference statistics following Kolmogorov scaling [2]. Thus, in the presence of small-scale forcing, the inverse energy cascade prevents the formation of the large-scale vorticity coherent structures observed in the case of decaying turbulence [3].

Large-scale coherent structures in the presence of forced inverse cascade have been observed only in the presence of a characteristic scale breaking scale invariance. A well-known example is the so-called Bose-Einstein condensation, when the energy accumulates at the largest available scale [4] forming vortices at the system size.

Another example of vortex formation is the quasi-crystallization phenomenon observed in the Charney-Hasegawa-Mima (CHM) turbulence, a paradigm for both geostrophic motion in planetary atmospheres [5] and drift-wave turbulence in a magnetically confined plasma [6]. For the stream function $\psi(\boldsymbol{x},t)$ the CHM equation is written as

$$\frac{\partial}{\partial t}(\nabla^2\psi - \lambda^2\psi) + J(\nabla^2\psi, \psi) = \nu\nabla^4\psi + f\,, \tag{1}$$

where J denotes the Jacobian, ν is a damping coefficient (viscosity in the Charney model) and f represents a forcing term. In this case, vortices have been observed to form, in a quasi-crystal structure, at the intrinsic scale $1/\lambda$, corresponding to the Rossby deformation radius in the atmosphere [7] or to the effective ion Larmor radius in plasma [8].

In this letter we focus on dynamics on scales much larger than λ^{-1}. In this regime there is no intrinsic scale involved in the evolution of the system, which nevertheless exhibits the formation of strong vortices. In this case we observe that the scale of vortices is a dynamical one which increases in time as a consequence of vortex merging, similar to what observed in decaying turbulence [9, 10]. The characteristic time of evolution slows down leading, in the limit of large Reynolds numbers, to a disordered pattern of quenched vortices. Despite the presence of strong vortices, we find that the two-dimensional 3/2 Kolmogorov law for the third-order velocity structure function holds, independently of the value of λ. As a consequence, the kinetic-energy spectrum follows Kolmogorov scaling but with a different constant with respect to the Navier-Stokes turbulence.

The CHM equation (1) has two quadratic inviscid ($\nu = f = 0$) invariants corresponding to total energy

$$E = E_k + E_\lambda = \frac{1}{2}\langle(\nabla\psi)^2 + \lambda^2\psi^2\rangle\,, \tag{2}$$

where $\langle ... \rangle$ denotes spatial average, and total enstrophy

$$Z = Z_k + Z_\lambda = \frac{1}{2}\langle(\nabla^2\psi)^2 + \lambda^2(\nabla\psi)^2\rangle\,. \tag{3}$$

Both the inviscid invariants consist of two terms, the first corresponding to the kinetic contribution and the second to the potential one. The kinetic terms are, by definition, the only ones which survive in the Navier-Stokes limit $\lambda \to 0$.

The range of scales is separated by the characteristic wave number λ. For $k \gg \lambda$ the kinetic contributions dominate in (2)-(3), at very large scales, $k \ll \lambda$, the leading terms are the potential ones. In the following we will assume that the forcing F is limited to a narrow band of wave number around k_f. This will be the other relevant wave number in our problem.

Kolmogorov-like dimensional analysis can be easily extended to the present problem [8, 11–13]. If $k_f \gg \lambda$, we recover the well-known energy spectra for two-dimensional Navier-Stokes turbulence with energy spectrum $E(k) \propto k^{-5/3}$ and $E(k) \propto k^{-3}$ for $\lambda < k < k_f$ and $k > k_f$, respectively. When $k_f \ll \lambda$, one obtains the prediction $E(k) \propto k^{-11/3}$ for $k < k_f$, and $E(k) \propto k^{-5}$ for $k_f > k > \lambda$. Dimensionally predicted spectra have been confirmed by direct numerical simulations [8, 14, 15] but little is known about the structure functions and probability density functions.

The starting point for a statistical analysis of the turbulent cascade is the exact Kolmogorov's law which determines the scaling exponent for the third-order structure function [16]. A standard calculation, reported in the appendix, leads to the result

$$\left\langle \left(\delta u_\parallel(\ell)\right)^3 \right\rangle = \frac{3}{2}\varepsilon\ell\,, \tag{4}$$

where $\delta u_\parallel$ represents the longitudinal increment of the velocity $\boldsymbol{u} = (\partial_y\psi, -\partial_x\psi)$ and ε is the energy input due to the forcing. The "3/2" law (4) is a well-known result for the two-dimensional Navier-Stokes turbulence (see, *e.g.*, [17]). In the appendix we show the remarkable result that (4) is universal with respect to the class of equations (1) parameterized by λ.

From a dimensional point of view, (4) implies a scaling exponent $h = 1/3$ for velocity increments and thus a scaling exponent 4/3 for ψ, as in the Navier-Stokes turbulence. From (2) one obtains the different predictions for the spectrum discussed above. In particular, the kinetic spectrum has the form

$$E_k(k) = C_\lambda \varepsilon^{2/3} k^{-5/3} \tag{5}$$

for any value of λ, but with a Kolmogorov constant C_λ which, in principle, can depend on λ. A simple physical argument for this dependency is as follows. The scaling of the eddy turnover time depends on the scaling exponent. In the kinetic limit $k \gg \lambda$ one has the standard Kolmogorov scaling $\tau(k) \simeq \varepsilon^{-1/3} k^{-2/3}$ [16]. On the other hand, in the potential limit $k \ll \lambda$, (1) gives $\tau(k) \simeq (\lambda/k)^2 \varepsilon^{-1/3} k^{-2/3}$. Thus, for $k \ll \lambda$, the efficiency of energy transfer is reduced and one expects a larger value of the constant in (5).

We have numerically investigated the inverse cascade in the potential-energy regime by direct numerical simulations. In order to avoid complications induced by the crossover from the kinetic domain to the potential domain, we study the system in the limit $\lambda \to \infty$. This is to be seen only as a formal procedure, equivalent to considering wave numbers much smaller than λ, which physically might be the case for magnetized plasma in the presence of a strong magnetic field. Indeed this limit provides us with a model suitable for any $\lambda \gg k_f$: because the energy is transferred to large scale, the dominance of the potential term will be assured in all the inertial range. In the limit $\lambda \gg k_f$, rescaling the time $t \to t/\lambda^2$, one obtains the so-called asymptotic model [12]

$$\frac{\partial \psi}{\partial t} = J(\nabla^2 \psi, \psi) - \nu \nabla^4 \psi - f\,, \tag{6}$$

for which the inviscid conserved quantities become

$$\begin{aligned} E &= \frac{1}{2}\langle \psi^2 \rangle\,, \\ Z &= \frac{1}{2}\langle (\nabla \psi)^2 \rangle\,. \end{aligned} \tag{7}$$

We have integrated (6) with a standard pseudo-spectral code in a double periodic domain of size $L = 2\pi$ at resolution $N = 512$. The forcing is white in time in a narrow band of wave numbers around $k_f = 160$. The dissipative term in (6) has the role of removing potential enstrophy at small scales and, as customary, it is numerically substituted by a hyperviscous term (of order 8 in our simulations). Time evolution is obtained by a standard second-order Runge-Kutta scheme starting from a zero initial condition. The run is stopped at a given time T at which the energy-containing scales are still much smaller than the computational box in order to avoid condensation effects [4] (see fig. 1). All the results discussed in the following are taken after averaging over 14 independent realizations.

The limitation in the resolution ($N = 512$) is due to the discussed scaling of the characteristic time. Even with this moderate resolution, the ratio of the large-scale characteristic time with the forcing scale time is about 2000 and thus time evolution is very expensive (10^6 time steps for each realization). In the case of Navier-Stokes turbulence ($\lambda = 0$) this would correspond to an integration covering about 5 decades of inertial range.

In fig. 1 we plot the potential-energy spectrum $E_\lambda(k)$ at two different times. The scaling exponent $-11/3$ is clearly visible even if some accumulation at the largest mode is evident. This accumulation is not due to condensation as it is still well below the largest mode and it moves in time. We think that the existence of this "bump" is a genuine effect, probably due to the rapid growth of characteristic times and to the presence of intense vortices, as discussed below. The energy flux $\varepsilon \simeq 1.24$ is estimated by the plateau of the energy flux shown in the inset.

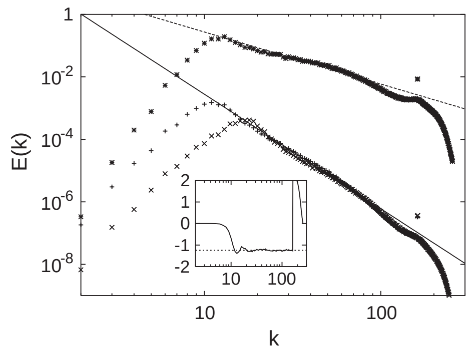

Fig. 1 – Potential-energy spectrum $E_\lambda(k)$ at times $t = 4 \times 10^{-3}$ ($\times$) and $t = 10^{-2}$ ($+$) and kinetic energy spectrum $E_k(k)$ at time $t = 10^{-2}$ ($*$) averaged over 14 realizations of the asymptotic model (6). The continuous line represents the dimensional prediction $k^{-11/3}$, the dashed line is the Kolmogorov spectrum $E_k(k) = C_\lambda \varepsilon^{2/3} k^{-5/3}$ with constant $C_\lambda = 11$. In the inset we plot the average energy flux at time $t = 10^{-2}$ with the line $\epsilon = 1.24$.

The "3/2" law for the longitudinal velocity structure function is shown in fig. 2, also plotted at two different times. The compensation with the theoretical prediction (4) is remarkable, taking into account the limited resolution of our runs. As expected, the extension of the inertial range increases with time without changing the small-scale statistics. The oscillations observed at small scales are due to the contamination of the forcing. A similar effect was observed also in NS simulations.

As discussed above, the fact that the "3/2" law is independent of λ (and thus the velocity scaling exponent has always the Kolmogorov value $h = 1/3$) does not imply that the statistics, and in particular the form of the pdf of velocity differences, is the same as for the Navier-Stokes equation. For example, in fig. 1 we also plot the kinetic energy spectrum $E_k(k)$ at final time $t = 10^{-2}$. The scaling exponent is compatible with the Kolmogorov value 5/3 as predicted by (5), but the Kolmogorov constant $C_\lambda \simeq 11$ is about two times that of Navier-Stokes [2]. A larger constant means a suppression of the energy flux which is a direct consequence of the dilatation of the characteristic times.

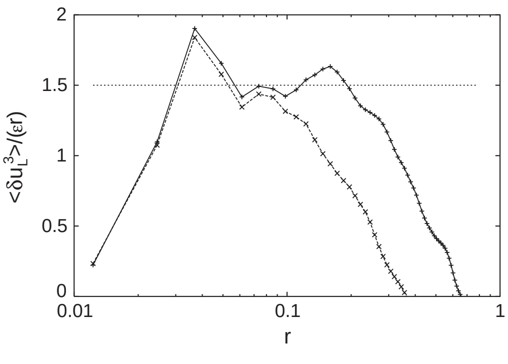

Fig. 2 – Third-order longitudinal structure function $\langle(\delta u_{\rm L}(r))^3\rangle$ compensated with the dimensional prediction εr at times $t = 4 \times 10^{-3}$ ($\times$) and $t = 10^{-2}$ ($+$) averaged over 14 independent realizations. The horizontal line represents the Kolmogorov law (4).

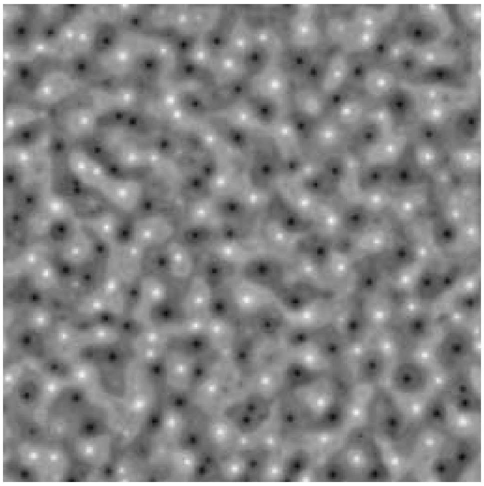

Fig. 3 – Gray-scale plot of the stream function ψ at time $t = 10^{-2}$. The characteristic scale of vortices $\ell_E \sim L/10$ corresponds to the peak of the energy spectrum in fig. 1.

A more significant difference with respect to the Navier-Stokes inverse cascade is the presence of strong vortices, as shown in fig. 3. Vortices in CHM turbulence are injected at the forcing scale and they organize themselves to form a random pattern on the characteristic scale ℓ_E. This dynamical scale is associated to the peak of the spectrum of fig. 1. Vortex dynamics slows down as ℓ_E increases as $\tau(\ell_E) \sim \ell_E^{8/3}$. Thus, in the limit of large Reynolds number the system will end in a disordered pattern of quenched vortices forming a kind of "turbulent glass".

An important consequence of the presence of strong vortices is that the statistics of the

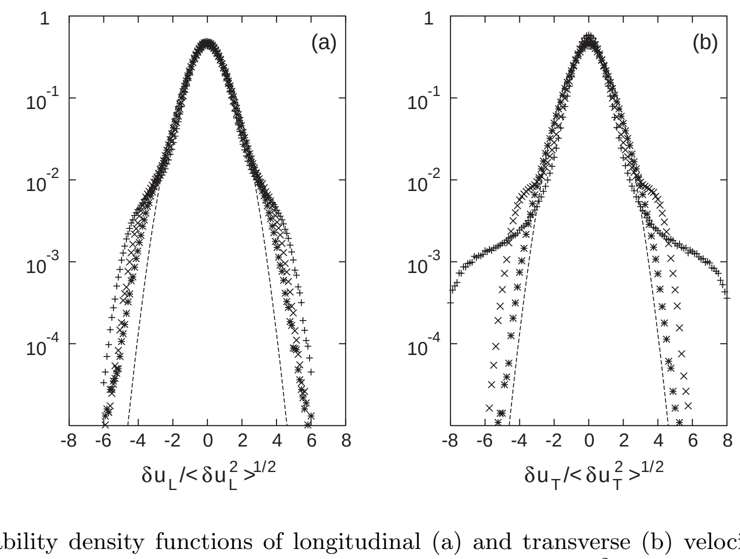

Fig. 4 – Probability density functions of longitudinal (a) and transverse (b) velocity differences at separations $\ell = 0.05$ $(+)$, $\ell = 0.1$ $(\times)$ and $\ell = 0.2$ $(*)$ at time $t = 10^{-2}$. The dashed line represents the Gaussian distribution.

velocity field strongly deviates from Gaussianity. In fig. 4 we plot the pdf of longitudinal and transverse velocity differences at three different scales within the inertial range. The effect of vortices is evident by the presence of large "wings" in the tails, in particular on the transverse velocity differences which are more sensible to a rotating structure.

In conclusion, we have derived the Kolmogorov "3/2" law for a two-dimensional energy cascade in the Charney-Hasegawa-Mima turbulence and shown that it is independent of the value of the intrinsic scale λ. Velocity difference statistics satisfies Kolmogorov scaling with non-universal coefficients. In the asymptotic limit $\lambda \to \infty$ the Kolmogorov constant is found to be about 2 times the Navier-Stokes case. Strong coherent vortices are found to emerge at the forcing scale and aggregate to form a pattern of quenched vortices at large scale [8, 15]. As a consequence of the presence of vortices, strong deviations from Gaussianity are observed in the velocity field.

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Appendix

In this appendix we report, for completeness, the main steps leading to (4). The starting point is the time derivative of the two-point correlator $\langle \nabla\psi \cdot \nabla\psi' + \lambda^2 \psi\psi' \rangle$ (where $\psi = \psi(\boldsymbol{x})$, $\psi' = \psi(\boldsymbol{x} + \ell)$) which, making use of (1), homogeneity and integration by parts, can be written as

$$\begin{aligned} \frac{\partial}{\partial t} \langle \nabla\psi \cdot \nabla\psi' + \lambda^2 \psi\psi' \rangle &= \langle \psi J(\nabla^2\psi', \psi') \rangle + \langle \psi' J(\nabla^2\psi, \psi) \rangle - 2\nu \langle \psi \nabla^2 \psi' \rangle - \\ & - \langle \psi \left[f(\boldsymbol{x} + \ell) + f(\boldsymbol{x} - \ell) \right] \rangle . \end{aligned} \tag{8}$$

The inertial term can be rewritten in terms of velocity differences $\delta\boldsymbol{u}(\ell) = \boldsymbol{u}(\boldsymbol{x} + \ell) - \boldsymbol{u}(\boldsymbol{x})$ (where $u_i \equiv \epsilon_{i,j} \partial_j \psi$) by making use of homogeneity and incompressibility ($\nabla \cdot \boldsymbol{u} = 0$):

$$\langle \psi J(\nabla^2\psi', \psi') \rangle + \langle \psi' J(\nabla^2\psi, \psi) \rangle = \frac{1}{2} \frac{\partial}{\partial \ell_i} \langle |\delta\boldsymbol{u}(\ell)|^2 \delta u_i(\ell) \rangle . \tag{9}$$

The dissipative term in (8) disappears in the limit of $\nu \to 0$. For simplicity, the forcing term is assumed to be isotropic, stationary, Gaussian, of zero mean and with covariance $\langle f(\boldsymbol{x}, t) f(0,0) \rangle = \delta(t) C(x/\ell_f)$ with C constant for $x \ll \ell_f$ and rapidly decaying for $x > \ell_f$. From the energy balance one has $C(0) = 2\varepsilon$, where $\varepsilon = \frac{\mathrm{d}E}{\mathrm{d}t}$ is the energy input. In the inertial range of scales, between the forcing scale and the energy-containing scale ℓ_E, Galilean invariant quantities are stationary [18], thus

$$0 = \frac{\partial}{\partial t} \langle (\delta\nabla\psi)^2 + \lambda^2 (\delta\psi)^2 \rangle = 2 \frac{\mathrm{d}E}{\mathrm{d}t} - 2 \frac{\partial}{\partial t} \langle \nabla\psi \cdot \nabla\psi' + \lambda^2 \psi\psi' \rangle . \tag{10}$$

Taking into account (9) and (10), (8) can be written as

$$\frac{\partial}{\partial \ell_i} \langle |\delta\boldsymbol{u}(\ell)|^2 \delta u_i(\ell) \rangle = 4\varepsilon - C(\ell/\ell_f) . \tag{11}$$

Assuming isotropy, and introducing the longitudinal velocity difference $\delta u_\parallel(\ell) = \delta u_i(\ell) \ell_i / \ell$, for $\ell \ll \ell_f$, one ends with the well-known "3/2" law

$$\left\langle \left(\delta u_\parallel(\ell) \right)^3 \right\rangle = \frac{3}{2} \varepsilon \ell . \tag{12}$$

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