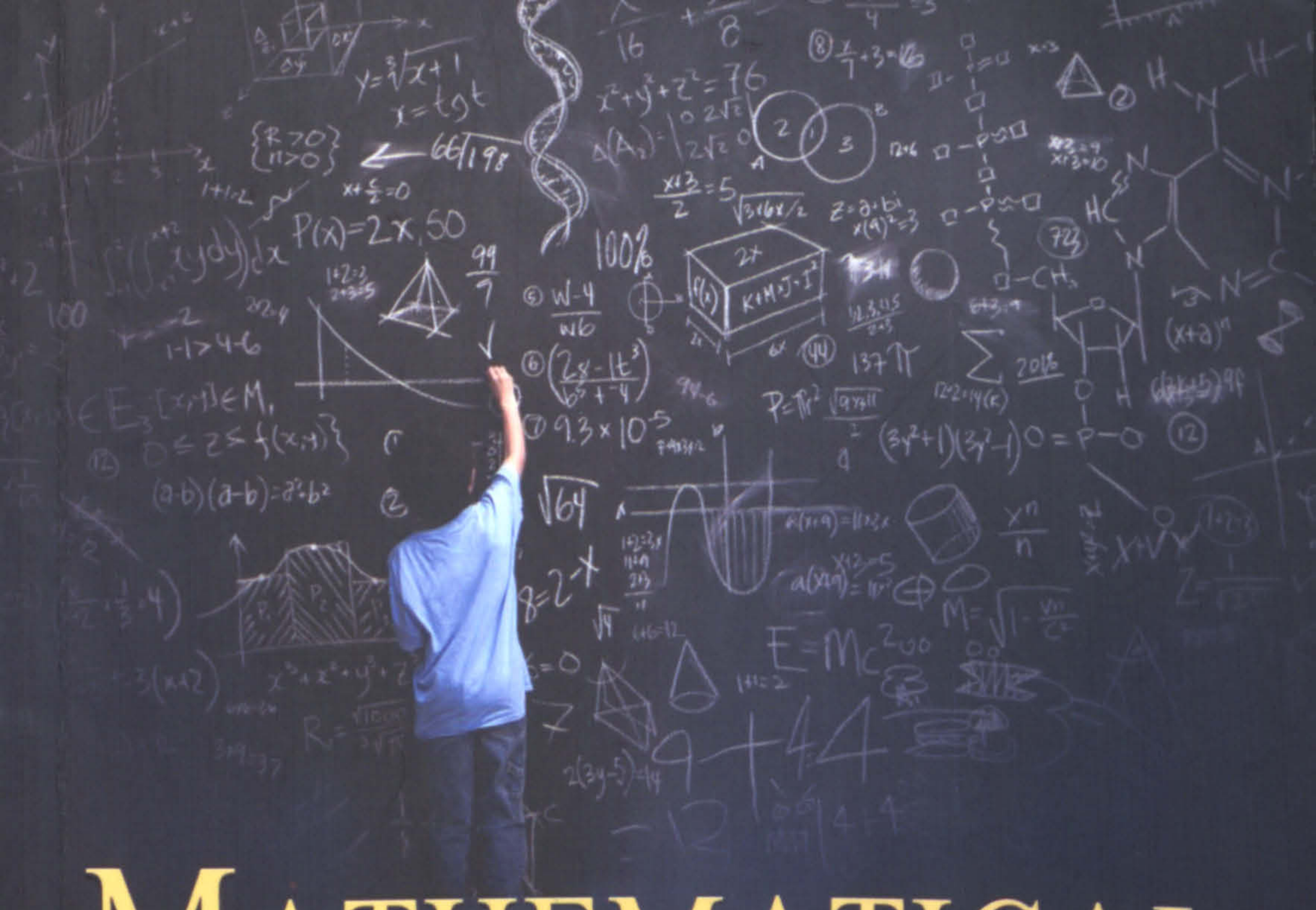

MATHEMATICAL FORMULAE BOOK

ONG, S.N. SZE & K.L. CHIEW

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W.K. TIONG, S.N. SZE & K.L. CHIEW

Universiti Malaysia Sarawak
Kota Samarahan

Published in Malaysia by
UNIMAS Publisher,
Universiti Malaysia Sarawak,
94300 Kota Samarahan,
Sarawak, Malaysia.

Printed in Malaysia by
Percetakan Nasional Malaysia Berhad,
Jalan Tun Abang Haji Openg,
93554 Kuching,
Sarawak, Malaysia.

Perpustakaan Negara Malaysia Cataloguing-in-Publication Data

Tiong, W. K.
MATHEMATICAL FORMULAE BOOK / W. K. TIONG, S.N. SZE & K.L. CHIEW.
ISBN 978-967-2008-11-8
1. Mathematics--Formulae. 2. Mathematic.
I. Sze, S. N. II. Chiew, K. l. II. Judul.
510

Contents

PREFACE

The "*Mathematical Formulae Book*" is a compilation of useful and important mathematical formulae, designed specially for undergraduate students. Detailed explanations of each formula are not included and thus, students should not replace their actual textbook with this book. The mathematical formulae book were selected from various topics in undergraduate mathematics courses available in major textbooks related to:

- Discrete Mathematics
- Linear Algebra
- Statistics
- Calculus
- Multivariable Calculus
- Vector Calculus
- Differential Equations
- Numerical Methods
- Operational Research

Therefore, undergraduate as well as postgraduate students can use this book as a quick reference for formulae at anytime even during examinations.

Though this book has been revised several times, there are obviously many rooms for improvement. We welcome any feedbacks, comments and suggestions in the hope to increase the quality of this book.

W.K. Tiong
S.N. Sze
K.L. Chiew
March 2016

ALGEBRA

Arithmetic Operations

$$a(b+c) = ab + ac, \qquad \frac{a+c}{b} = \frac{a}{b} + \frac{c}{b},$$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}, \qquad \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}.$$

Indices

$$a^0 = 1, \qquad a^m \times a^n = a^{m+n},$$

$$\frac{a^m}{a^n} = a^{m-n}, \qquad (a^m)^n = a^{mn} = (a^n)^m,$$

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}, \qquad a^{-n} = \frac{1}{a^n},$$

$$(ab)^n = a^n b^n, \qquad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n},$$

$$a^{1/n} = \sqrt[n]{a}, \qquad \sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b},$$

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

Identities

$$\begin{aligned}
(a+b)^2 &= a^2 + 2ab + b^2, \\
(a-b)^2 &= a^2 - 2ab + b^2, \\
(a+b)(a-b) &= a^2 - b^2, \\
(a+b)(a^2 - ab + b^2) &= a^3 + b^3, \\
(a-b)(a^2 + ab + b^2) &= a^3 - b^3.
\end{aligned}$$

Completing the Square

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}.$$

Quadratic Formula

If $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Logarithms

For any positive base $a \neq 1$, the expression $\log_a x = y$ means $a^y = x$.

$$\begin{aligned}
\log_e x &= \ln x; \\
\log_a 1 &= 0; \\
\log_a a &= 1; \\
\log_a(xy) &= \log_a x + \log_a y; \\
\log_a x^n &= n \log_a x; \\
\log_a(x/y) &= \log_a x - \log_a y; \\
a^{\log_a x} &= x; \\
\log_a a^x &= x; \\
\log_b x &= \log_a x / \log_a b, \\
&= \log_a x \cdot \log_b a.
\end{aligned}$$

Partial Fraction

The proper rational function $R(x)/Q(x)$ can be expressed as a sum of **partial fractions** as follows:

- **Case 1:** $Q(x)$ **is a product of distinct linear factors**

$$\frac{R(x)}{(a_1x+b_1)(a_2x+b_2)\cdots(a_nx+b_n)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_n}{a_nx+b_n}.$$

- **Case 2:** $Q(x)$ **is a product of linear factors, some being repeated**

$$\frac{R(x)}{(a_1x+b_1)^m} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{(a_1x+b_1)^2} + \cdots + \frac{A_m}{(a_1x+b_1)^m}.$$

- **Case 3:** $Q(x)$ **is a product of distinct irreducible quadratic factors.**

$$\frac{R(x)}{(a_1x^2+b_1x+c_1)\cdots(a_nx^2+b_nx+c_n)} = \frac{A_1x+B_1}{a_1x^2+b_1x+c_1} + \cdots + \frac{A_nx+B_n}{a_nx^2+b_nx+c_n}.$$

- **Case 4:** $Q(x)$ **is a product of irreducible quadratic factors, some being repeated.**

$$\frac{R(x)}{(a_1x^2+b_1x+c_1)^r} = \frac{A_1x+B_1}{a_1x^2+b_1x+c_1} + \frac{A_2x+B_2}{(a_1x^2+b_1x+c_1)^2} + \cdots + \frac{A_nx+B_n}{(a_1x^2+b_1x+c_1)^r}.$$

TRIGONOMETRY

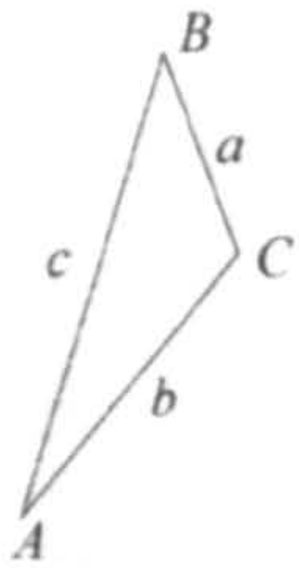
Angle Measurement

$$\pi \text{ radians} = 180^\circ, \qquad 1 \text{ rad} = \frac{180^\circ}{\pi}, \qquad 1^\circ = \frac{\pi}{180} \text{ rad}.$$

Trigonometric Functions of Important Angles

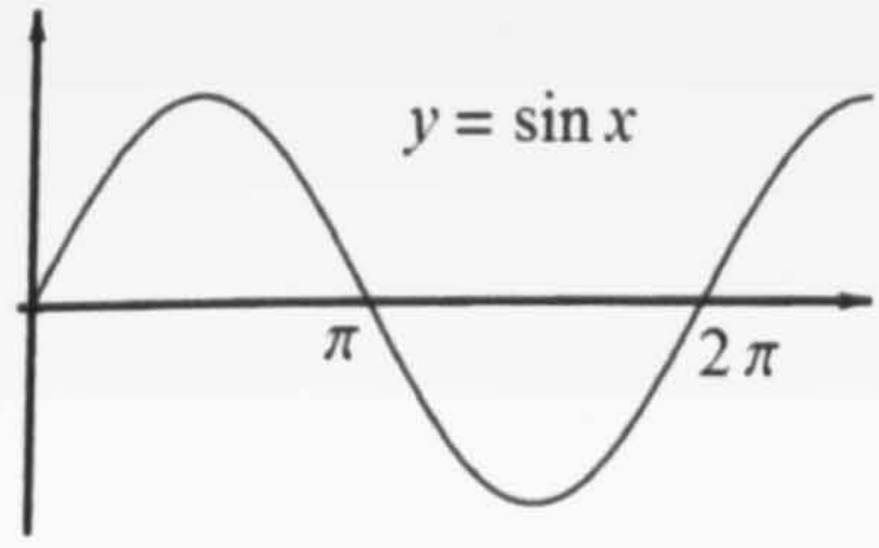
θ	radians	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0	0	1	0
30°	$\pi/6$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$
45°	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
60°	$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
90°	$\pi/2$	1	0	–

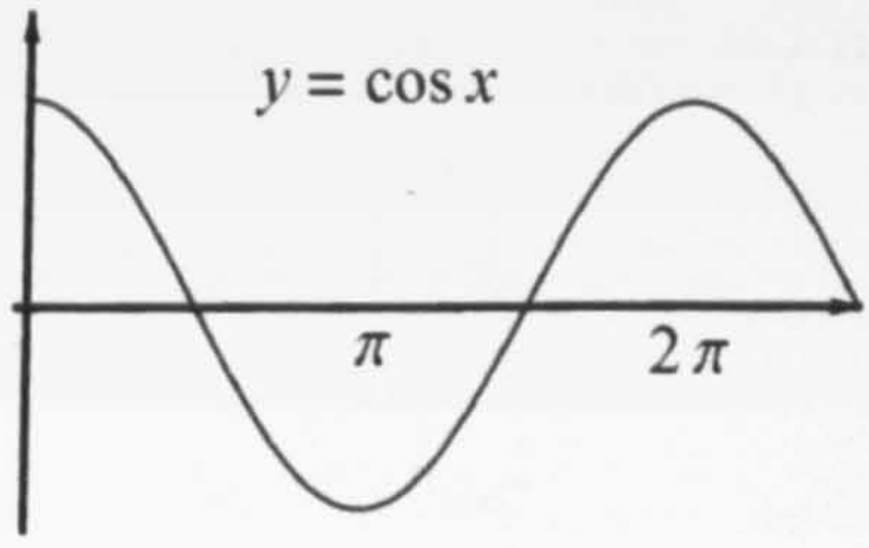
The Laws of Sines and Cosines

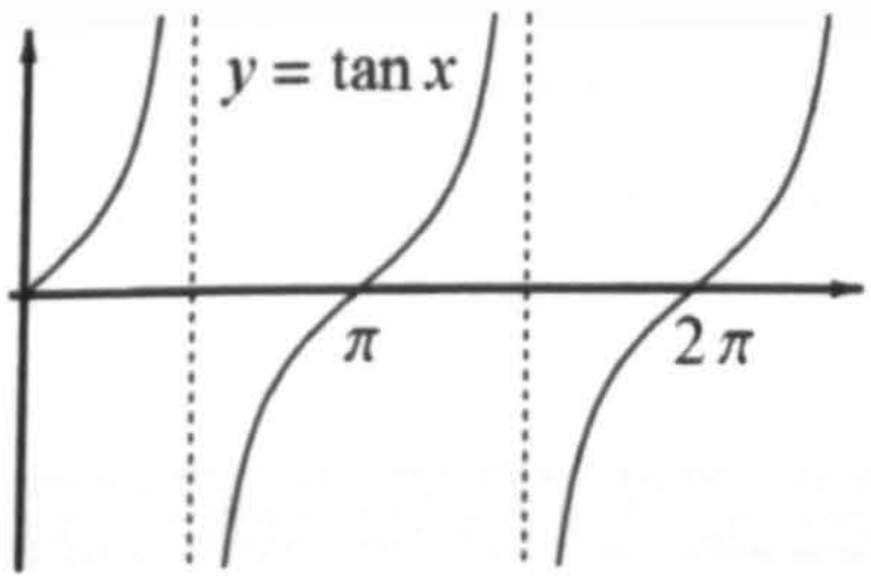


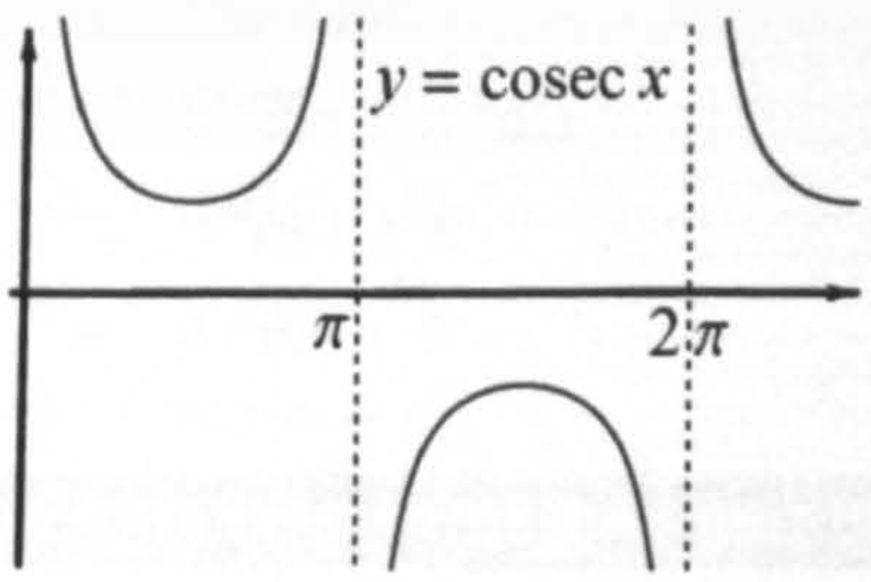
The Law of Sines	The Law of Cosines
$\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$	$a^2 = b^2 + c^2 - 2bc\cos A$ $b^2 = a^2 + c^2 - 2ac\cos B$ $c^2 = a^2 + b^2 - 2ab\cos C$

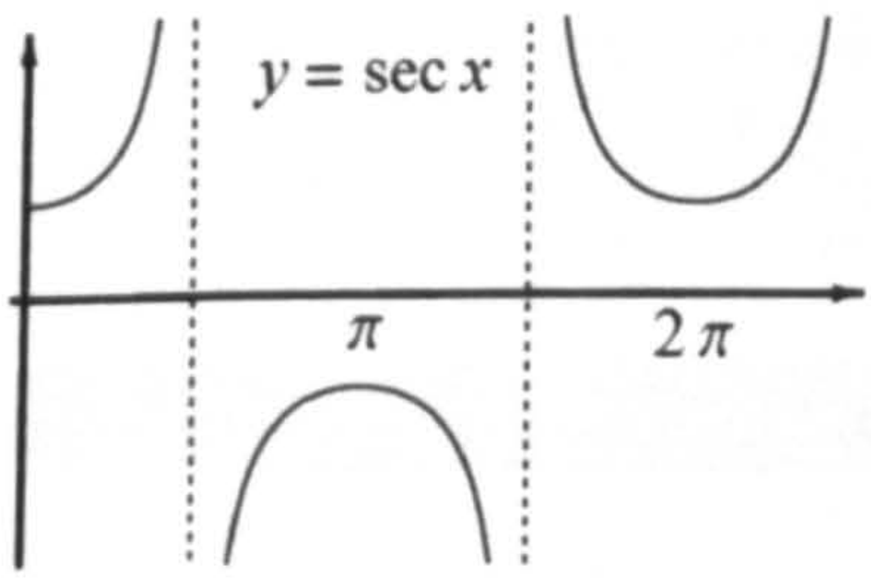
Graphs of Trigonometric Functions

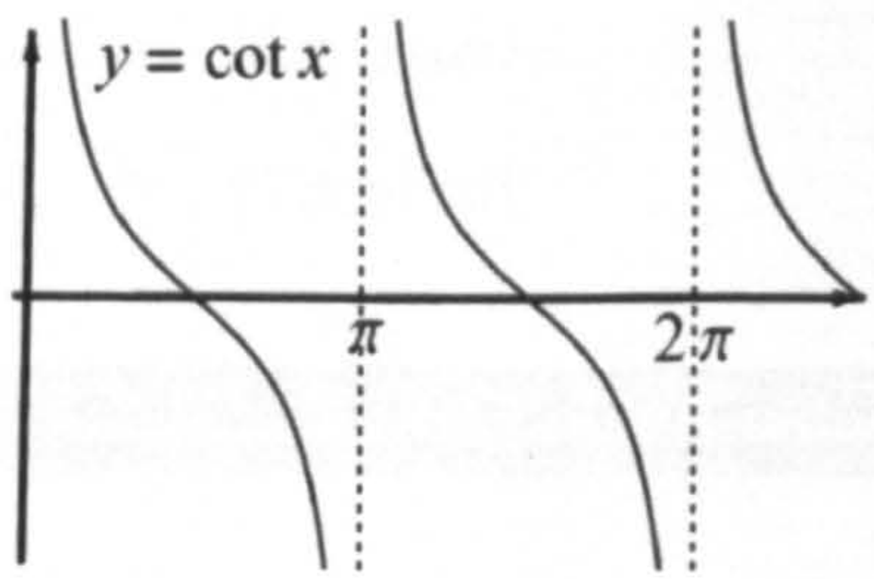












Trigonometric Identities

$$\tan a \equiv \frac{\sin a}{\cos a}, \quad \sec a \equiv \frac{1}{\cos a}, \quad \operatorname{cosec} a \equiv \frac{1}{\sin a}, \quad \cot a \equiv \frac{\cos a}{\sin a},$$

$$\begin{aligned}
\sin(90^\circ - a) &\equiv \cos a \equiv (\sin 90^\circ + a), \\
\cos(90^\circ - a) &\equiv \sin a \equiv -\cos(90^\circ + a), \\
\tan(90^\circ - a) &\equiv \cot a, \\
\sin(-\theta) &\equiv -\sin\theta, \\
\cos(-\theta) &\equiv \cos\theta, \\
\tan(-\theta) &\equiv -\tan\theta, \\
\sin^2 a + \cos^2 a &\equiv 1, \\
1 + \tan^2 a &\equiv \sec^2 a, \\
1 + \cot^2 a &\equiv \operatorname{cosec}^2 a.
\end{aligned}$$

Addition and Subtraction Formulae

$$\begin{aligned}
\sin(a \pm b) &\equiv \sin a \cos b \pm \cos a \sin b, \\
\cos(a \pm b) &\equiv \cos a \cos b \mp \sin a \sin b, \\
\tan(a \pm b) &\equiv \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b}.
\end{aligned}$$

Double-Angle Formulae

$$\begin{aligned}
\sin 2a &\equiv 2 \sin a \cos a, \\
\cos 2a &\equiv \cos^2 a - \sin^2 a, \\
&\equiv 2\cos^2 a - 1, \\
&\equiv 1 - 2\sin^2 a, \\
\tan 2a &\equiv \frac{2\tan a}{1 - \tan^2 a}.
\end{aligned}$$

Sum to Product Formulae

$$\begin{aligned}
\sin a \pm \sin b &\equiv 2\sin\left(\frac{a \pm b}{2}\right)\cos\left(\frac{a \mp b}{2}\right), \\
\cos a + \cos b &\equiv 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right), \\
\cos a - \cos b &\equiv -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right).
\end{aligned}$$

Product to Sum Formulae

$$\begin{aligned}
2\sin a\cos b &\equiv \sin(a+b) + \sin(a-b), \\
2\cos a\cos b &\equiv \cos(a-b) + \cos(a+b), \\
2\cos a\sin b &\equiv \sin(a+b) - \sin(a-b), \\
2\sin a\sin b &\equiv \cos(a-b) - \cos(a+b), \\
\tan a\tan b &\equiv \frac{\cos(a-b) - \cos(a+b)}{\cos(a-b) + \cos(a+b)}.
\end{aligned}$$

HYPERBOLIC FUNCTIONS

Definition

$$\sinh x \equiv \frac{e^x - e^{-x}}{2}, \qquad \cosh x \equiv \frac{e^x + e^{-x}}{2},$$

$$\operatorname{sech} x \equiv \frac{1}{\cosh x}, \qquad \operatorname{cosech} x \equiv \frac{1}{\sinh x},$$

$$\tanh x \equiv \frac{\sinh x}{\cosh x}, \qquad \coth x \equiv \frac{1}{\tanh x}.$$

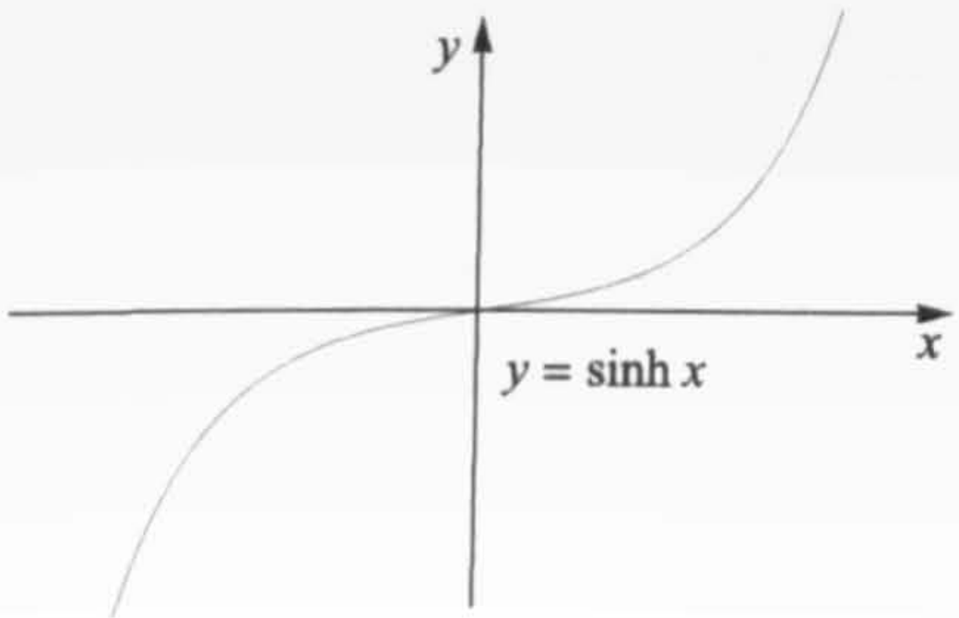
$$\equiv \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

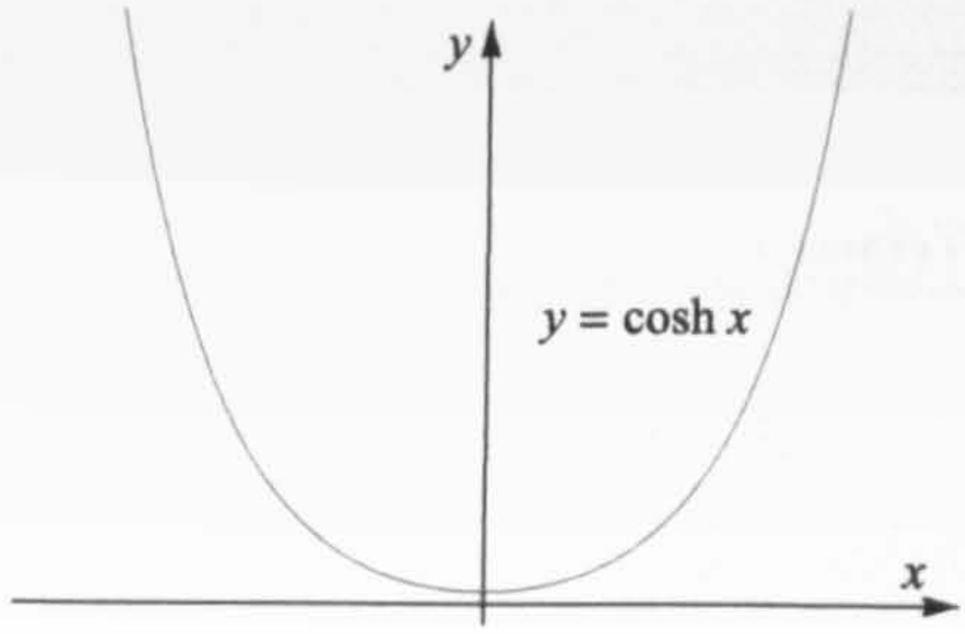
Identities

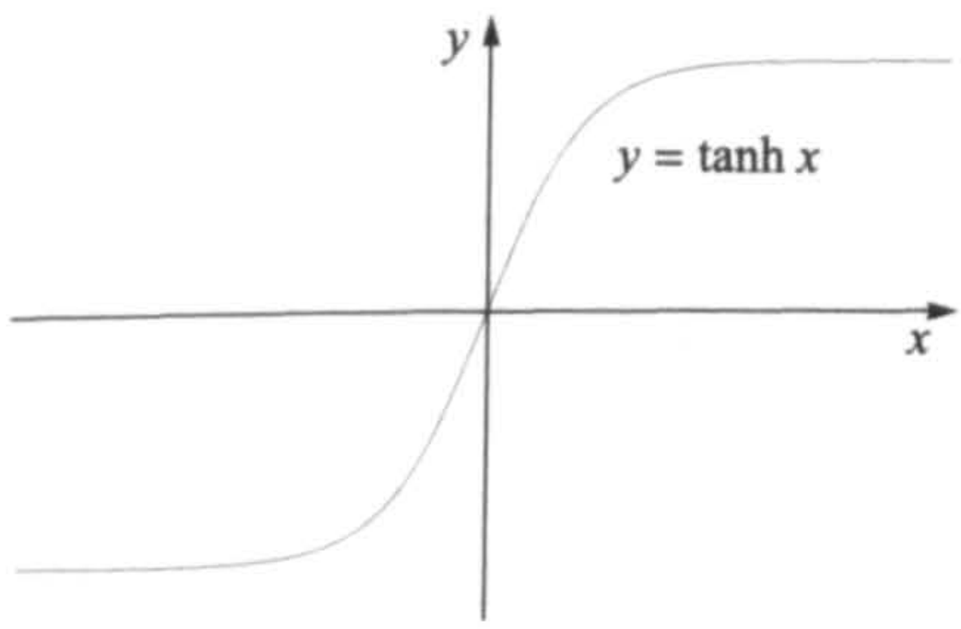
$$\begin{aligned}
\cosh^2 x - \sinh^2 x &\equiv 1, \\
1 - \tanh^2 x &\equiv \operatorname{sech}^2 x, \\
\coth^2 x - 1 &\equiv \operatorname{cosech}^2 x, \\
\sinh 2x &\equiv 2 \sinh x \cosh x, \\
\cosh 2x &\equiv \cosh^2 x + \sinh^2 x, \\
\cosh^2 x &\equiv \frac{\cosh 2x + 1}{2}, \\
\sinh^2 x &\equiv \frac{\cosh 2x - 1}{2}, \\
\sinh^{-1}\left(\frac{x}{a}\right) &\equiv \log_e\left(\frac{x + \sqrt{x^2 + a^2}}{a}\right), \\
\cosh^{-1}\left(\frac{x}{a}\right) &\equiv \log_e\left(\frac{x + \sqrt{x^2 - a^2}}{a}\right), \quad x \geq a, \\
\tanh^{-1}\left(\frac{x}{a}\right) &\equiv \frac{1}{2}\log_e\left(\frac{a + x}{a - x}\right), \quad |x| < a.
\end{aligned}$$

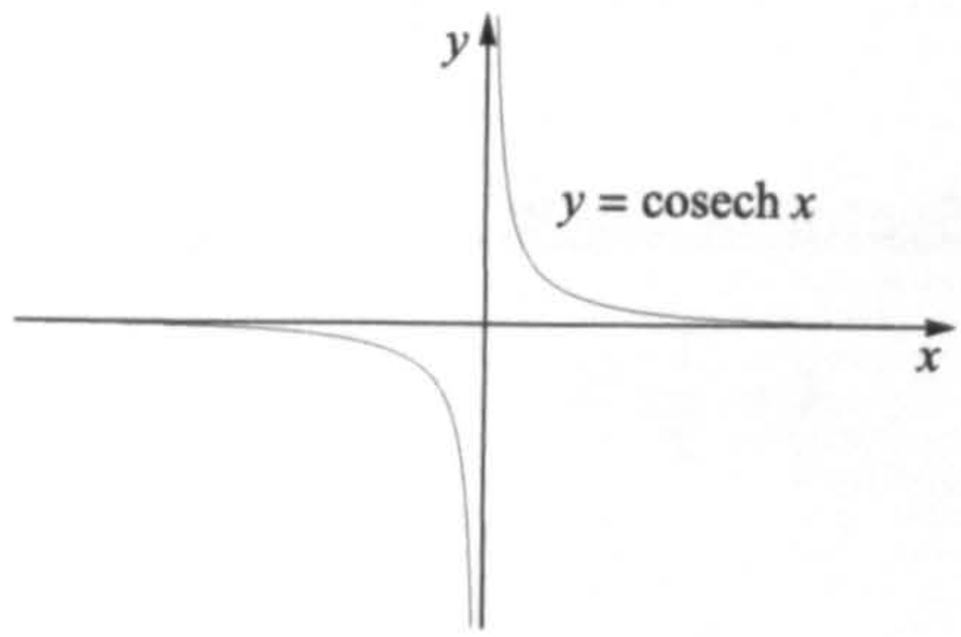
$$\operatorname{sech}^{-1} x = \cosh^{-1}\frac{1}{x}, \quad \operatorname{cosech}^{-1} x = \sinh^{-1}\frac{1}{x}, \quad \coth^{-1} x = \tanh^{-1}\frac{1}{x},$$

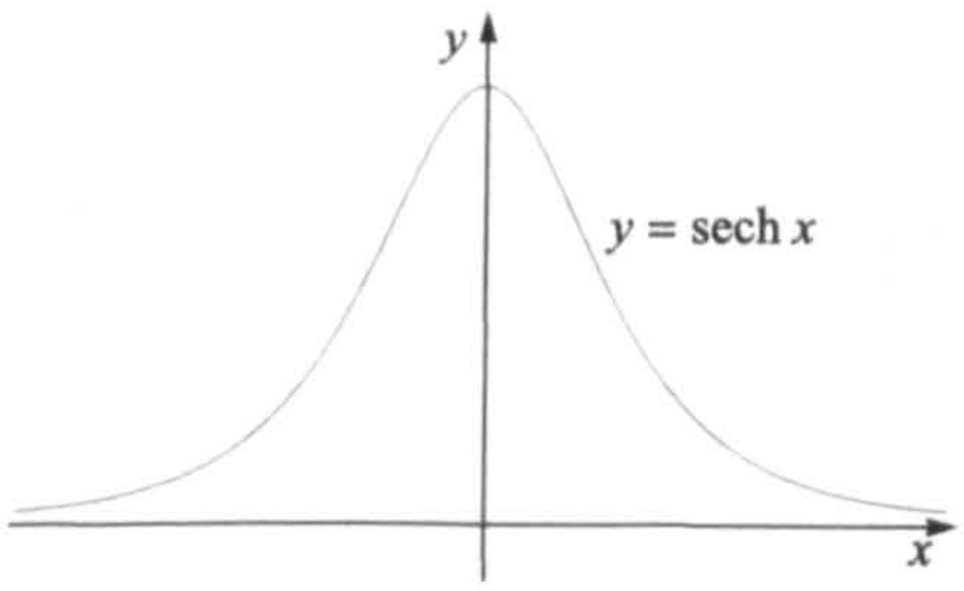
Graphs of Hyperbolic Functions

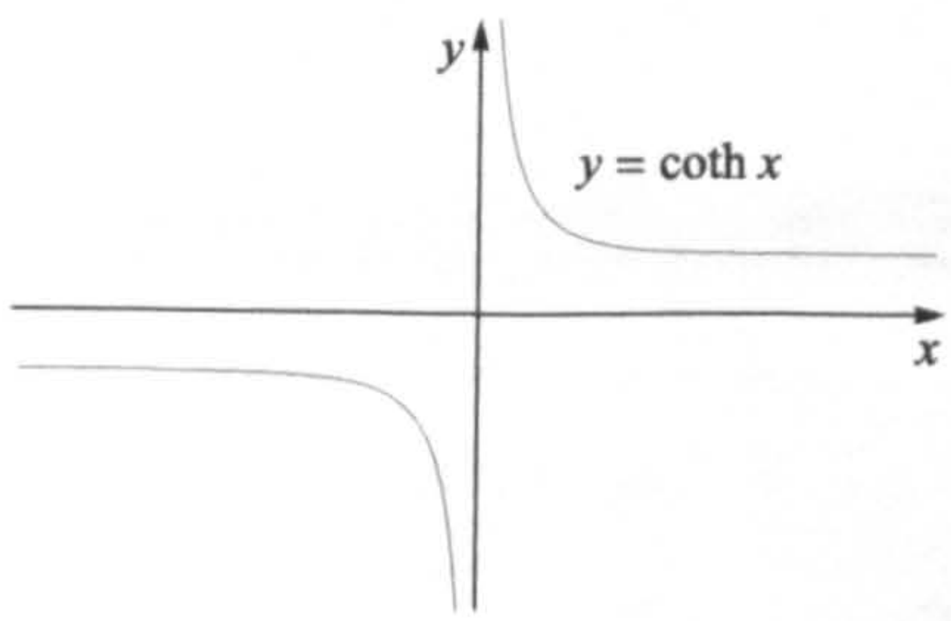










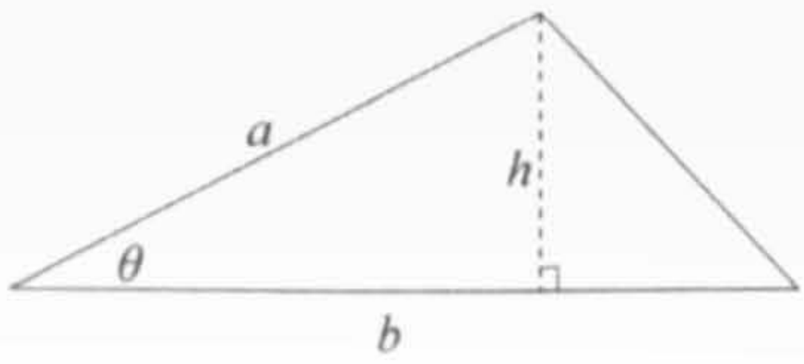


GEOMETRY

Area (A), Circumference (C) & Volume (V)

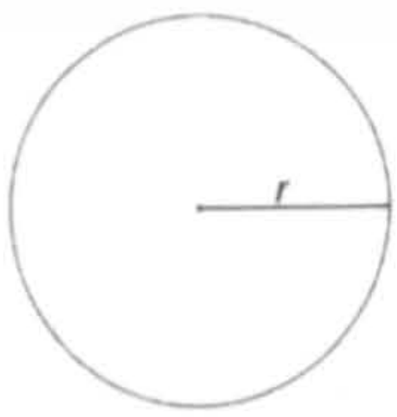
Triangle

$$A = \frac{1}{2}bh = \frac{1}{2}ab\sin\theta$$



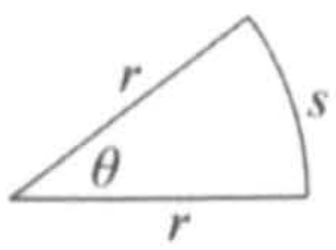
Circle

$$\begin{aligned} A &= \pi r^2 \\ C &= 2\pi r \end{aligned}$$



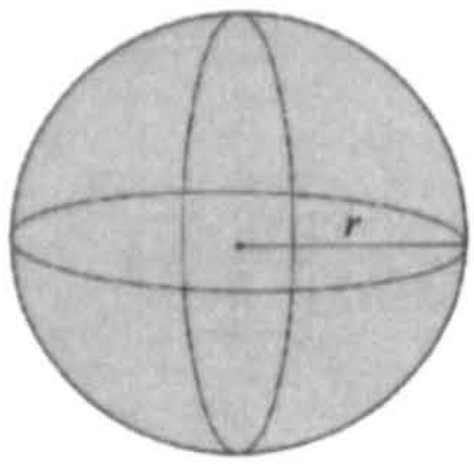
Sector of Circle

$$A = \frac{1}{2}r^2\theta; \quad s = r\theta$$



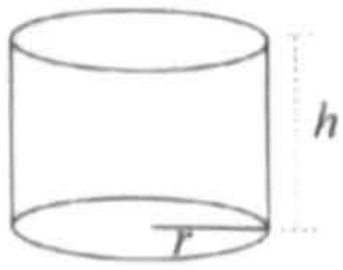
Sphere

$$\begin{aligned} A &= 4\pi r^2 \\ V &= \frac{4}{3}\pi r^3 \end{aligned}$$



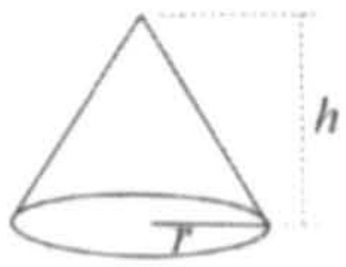
Cylinder

$$\begin{aligned} A &= 2\pi r^2 + 2\pi rh \\ V &= \pi r^2 h \end{aligned}$$



Cone

$$\begin{aligned} A &= \pi r\sqrt{r^2 + h^2} \\ V &= \frac{1}{3}\pi r^2 h \end{aligned}$$



Geometry Coordinates

- The distance between (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

- The area of a triangle with vertices $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) in the (x, y) plane is

$$\frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}.$$

Equation of a Line

General formula : $ax + by + c = 0.$

Slope m and intercept h : $y = mx + h.$

Intercepts g and h : $\frac{x}{g} + \frac{y}{h} = 1.$

Slope m and point (x_1, y_1) : $y - y_1 = m(x - x_1).$

Two points $(x_1, y_1), (x_2, y_2)$: $\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}.$

Angle θ between two lines : $\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}.$

Circle

Standard form : $(x - h)^2 + (y - k)^2 = r^2$

General form : $x^2 + y^2 + 2gx + 2fy + c = 0$

Radius of a circle : $r = \sqrt{f^2 + g^2 - c}$

Parametric form : $x = a + r\cos t, y = b + r\sin t$

Parabola

$$(x-h)^2 = 4p(y-k)$$

$$(y-k)^2 = 4p(x-h)$$

$$F(h+p,k) \quad \text{or} \quad F(h,k+p)$$

Ellipse

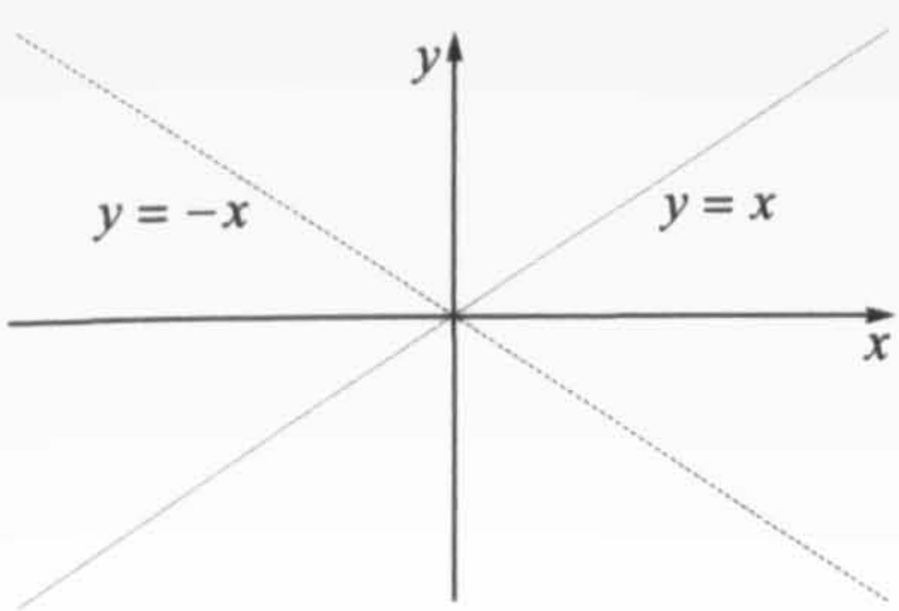
$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, \quad a > b$$

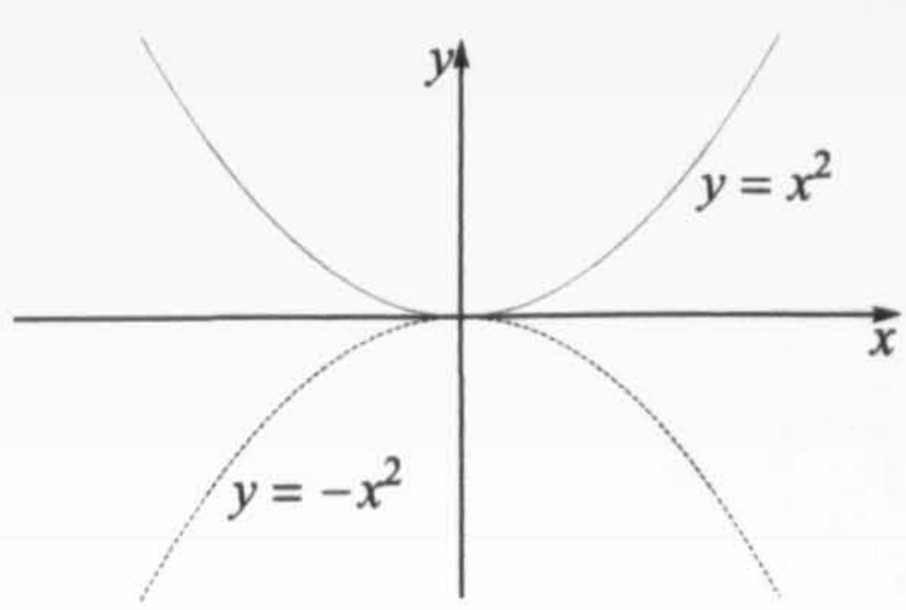
$$F(h \pm c,k) \quad \text{or} \quad F(h,k \pm c)$$

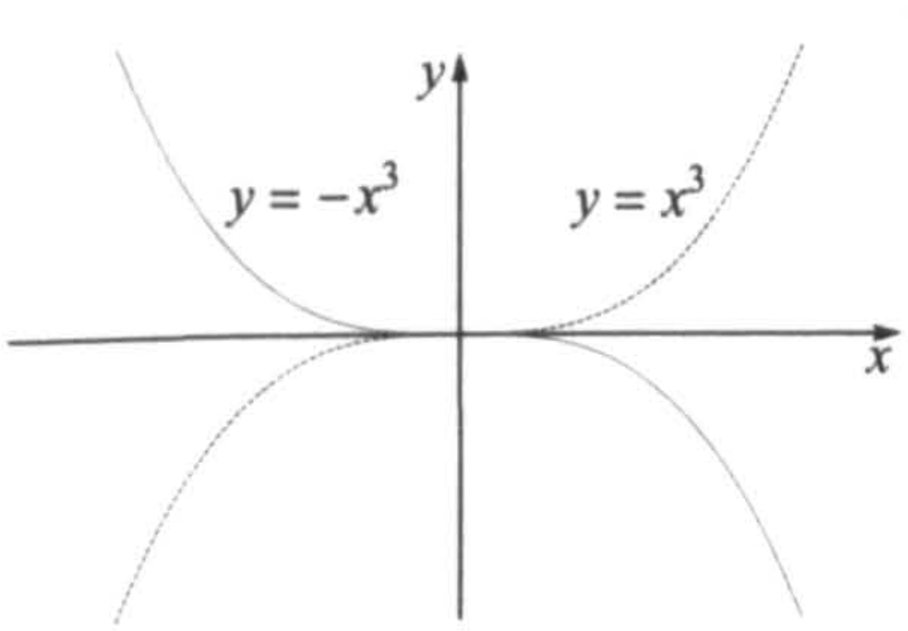
Equation of a Plane

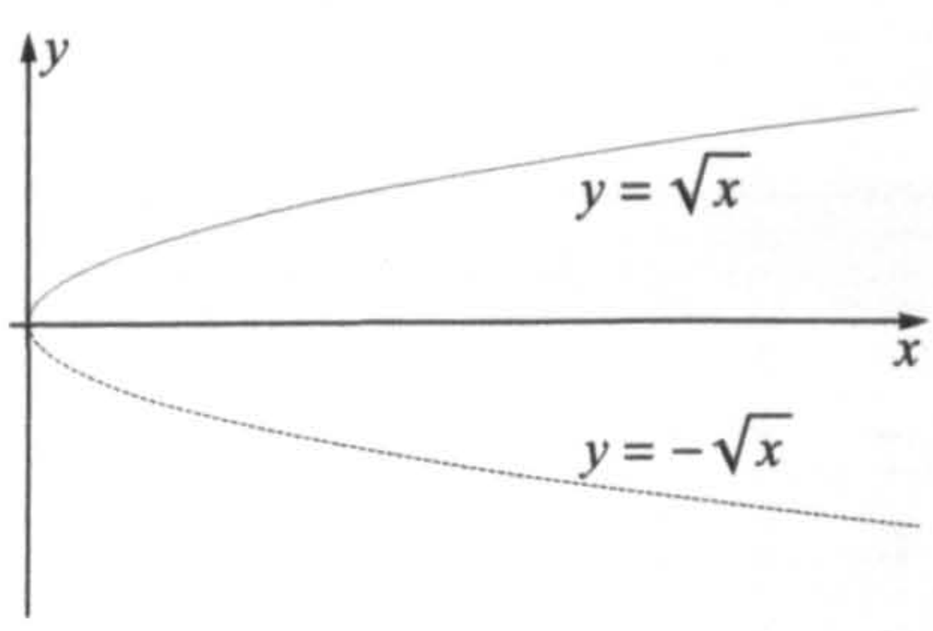
General formula: $ax + by + cz + d = 0$.

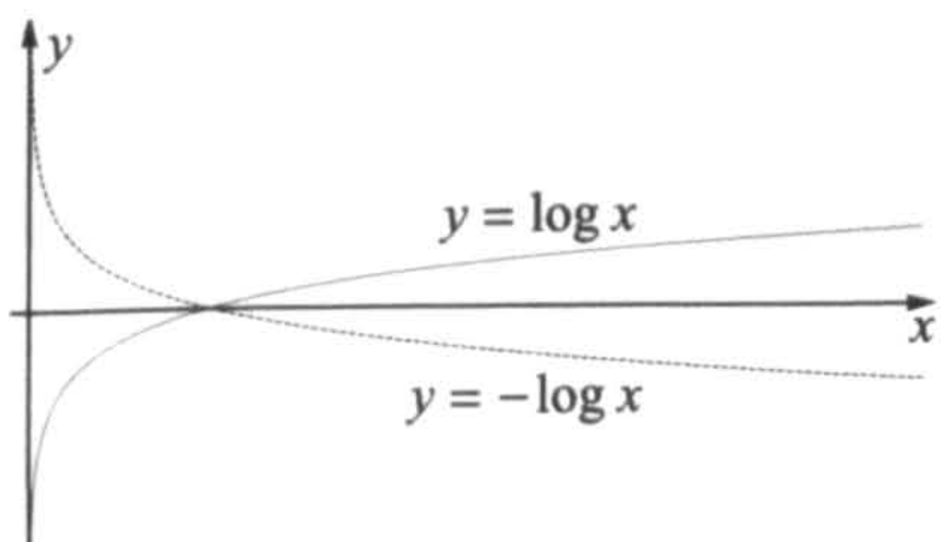
GRAPHS OF COMMON FUNCTIONS

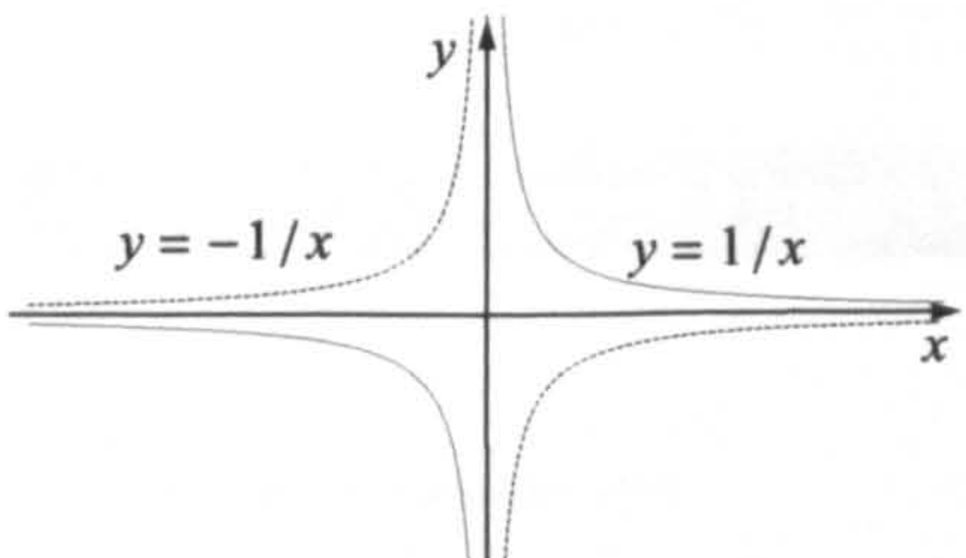


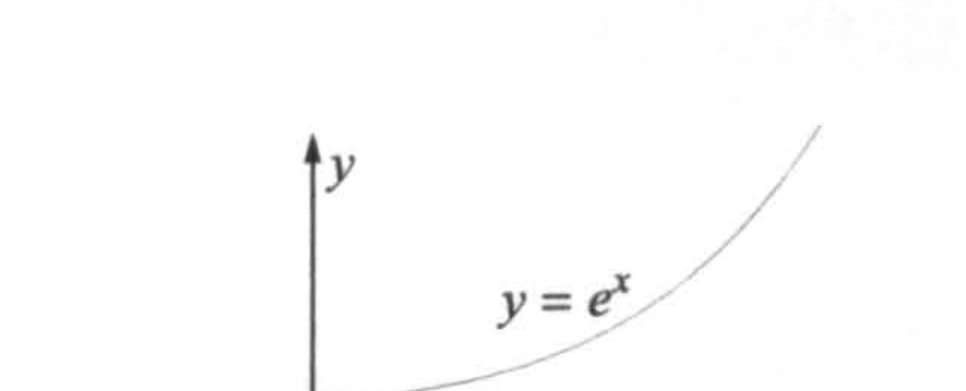












Truth Tables

p	$\neg p$
T	F
F	T

(a) $\neg p$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

(b) $p \wedge q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

(c) $p \vee q$

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

(d) $p \oplus q$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

(e) $p \rightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

(f) $p \leftrightarrow q$

Logical Equivalences Laws

Equivalences Laws	Name
Identity laws	$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$
Domination laws	$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$
Idempotent laws	$p \vee p \equiv p$ $p \wedge p \equiv p$
Commutative laws	$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$

Double negation law	$\neg(\neg p) \equiv p$
Associative laws	$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Distributive laws	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
De Morgan's laws	$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$
Absorption laws	$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$
Negation laws	$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$

Logical Equivalences Involving Conditional Statements

$p \to q \equiv \neg p \vee q$

$p \to q \equiv \neg q \to \neg p$

$p \vee q \equiv \neg p \to q$

$p \wedge q \equiv \neg(p \to \neg q)$

$\neg(p \to q) \equiv p \wedge \neg q)$

$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$

$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$

$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$

$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

Logical Equivalences Involving Biconditional Statements

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$

$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

SET

- The **universal set** is denoted by U. The **empty set** or **null set** is denoted by $\emptyset$, or $\{\}$. The **complement** of A is denoted by A^c or $\overline{A}$. The **union** of A and B is denoted by $A \cup B$. The **intersection** of A and B is denoted by $A \cap B$.

- $A \subseteq B$ means A is a **subset** of B. Two sets A, B are **equal**, $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

- If $A \subseteq B$ and $A \neq B$, then we write $A \subset B$ and A is said to be **proper subset** of A.

Commonly Used Sets

$$
\begin{aligned}
\mathbf{N} &= \text{the set of natural numbers } \{0, 1, 2, 3, \ldots\} \\
\mathbf{Z} &= \text{the set of integers } \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} \\
\mathbf{Q} &= \text{the set of rational numbers } \left\{ \frac{p}{q} \,\middle|\, p, q \in \mathbf{Z}, q \neq 0 \right\} \\
\mathbf{R} &= \text{the set of real numbers, i.e. all numbers expressible} \\
&\quad \text{as finite or infinite decimal expressions} \\
\mathbf{C} &= \text{the set of complex numbers } \{x + \sqrt{-1}y \,|\, x, y \in \mathbf{R}\}
\end{aligned}
$$

Set Cardinality

For any sets A, B, C and X,

$$
\begin{aligned}
|A \cup B| &= |A| + |B| - |A \cap B|, \\
|A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| \\
&\quad + |A \cap B \cap C|, \\
|A \times B| &= |A||B|, \\
|P(X)| &= 2^n \text{ where } n = |X|.
\end{aligned}
$$

Set Identities

Identity laws	$A \cup \emptyset = A$ $A \cap U = A$
Domination laws	$A \cup U = U$ $A \cap \emptyset = \emptyset$
Idempotent laws	$A \cup A = A$ $A \cap A = A$
Commutative laws	$A \cup B = B \cup A$ $A \cap B = B \cap A$
Associative laws	$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$
Distributive laws	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
De Morgan's laws	$(A \cup B)^c = A^c \cap B^c$ $(A \cap B)^c = A^c \cup B^c$
Absorption laws	$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$
Complement laws	$A \cup A^c = U$ $A \cap A^c = \emptyset$ $(A^c)^c = A$