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Mathematical Modeling of the Arterial Blood Flow

J. LABADIN[1], A. AHMADI[2]

Faculty of Computer Science and Information Technology
University Malaysia Sarawak, 94300 Kota Samarahan, Sarawak, Malaysia
[1] ljane@fit.unimas.my, [2]azizi0407@gmail.com

Abstract. Blood flow is a study of measuring the blood pressure and finding the flow through the blood vessel. Blood flow problem has been studied for centuries where one of the motivations was to understand the conditions that contribute to high blood pressure. This occurs when the blood vessel became narrowed from its normal size. This paper presents a mathematical modeling of the arterial blood flow which was derived from the Navier-Stokes equations and some assumptions. A system of nonlinear partial differential equations for blood flow and the cross-sectional area of the artery was obtained. Finite difference method was adopted to solve the equations numerically. The result obtained is very sensitive to the values of the initial conditions and this helps to explain the condition of hypertension.

Keywords: Mathematical modeling, arterial flow, MatLab

1. Introduction

Blood flow is a study of measuring the blood pressure and finding the flow through the blood vessel. This study is important for human health. Most of the researches study the blood flow in the arteries and veins. One of the motivations to study the blood flow was to understand the conditions that may contribute to high blood pressure. Past studies indicated that one of the reasons a person having hypertension is when the blood vessel becomes narrow. This paper will focus on the diastolic hypertension. Blood is non-Newtonian fluid and to model such fluid is very complicated. In this problem, blood is assumed to be a Newtonian fluid. Even though this will make the problem much simpler, it still is valid since blood in a large vessel acting almost like a Newtonian fluid. In order to model this problem, Navier-Stokes equations will be used to derive the governing equations that represent this problem.

2. Formulation of the Governing Equations

We have adopted Yang, Zhang and Asada's [1] local arterial flow model. This includes the assumptions that the arterial vessel is rectilinear, deformable, thick shell of isotropic, incompressible material with circular section and without longitudinal movements. Meanwhile blood is considered as an incompressible Newtonian fluid, and the flow is axially symmetric. The model approach is to use the two-dimensional Navier-Stokes equations and continuity equation for a Newtonian and incompressible fluid in cylindrical coordinate (r, z, t):

$$\frac{\partial u}{\partial t}+w\frac{\partial u}{\partial r}+u\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial z}+v\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}\right), \tag{1}$$

$$\frac{\partial w}{\partial t}+w\frac{\partial w}{\partial r}+u\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial z}+v\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}+\frac{w}{r^2}\right), \tag{2}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rw)+\frac{\partial u}{\partial z}=0, \tag{3}$$

where P = pressure,
ρ = density,

v = kinematic viscosity,
$u\,(r, z, t)$ = the components of velocity in axial (z) directions,
$w\,(r, z, t)$ = the components of velocity in radial (r) directions.

For convenience we define a new variable, which is the radial coordinate, η:

$$\eta = \frac{r}{R(z,t)} \tag{4}$$

where $R(z, t)$ denotes the inner radius of the vessel. Assuming that P is independent of the radial coordinate, η, then the pressure P is uniform within the cross section ($P = P(z, t)$). Hence

$$\frac{\partial^2 u}{\partial z^2} \le 1, \qquad \frac{\partial^2 w}{\partial z^2} \le 1, \qquad \frac{\partial P}{\partial r} \le 1$$

Using simple algebra to change the variable such as

$$\frac{\partial u(r,z,t)}{\partial t} = \frac{\partial u(\eta,t)}{\partial t}\cdot\frac{\partial \eta}{\partial t} + \frac{\partial u(\eta,t)}{\partial t}\cdot\frac{\partial t}{\partial t},$$
$$= -\frac{\eta}{R}\frac{\partial u(\eta,t)}{\partial t}\cdot\frac{\partial R}{\partial t} + \frac{\partial u(\eta,t)}{\partial t},$$

equations (1), (2) and (3) can be written in the new coordinate (η, z, t) as:

$$\frac{\partial u}{\partial t} + \frac{1}{R}\left(\eta\left(u\frac{\partial R}{\partial z} + \frac{\partial R}{\partial t}\right) - w\right)\frac{\partial u}{\partial \eta} + u\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \frac{v}{R^2}\left(\frac{\partial^2 u}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial u}{\partial \eta}\right), \tag{5}$$

$$\frac{\partial w}{\partial t} + \frac{1}{R}\left(\eta\left(u\frac{\partial R}{\partial z} + \frac{\partial R}{\partial t}\right) - w\right)\frac{\partial w}{\partial \eta} + u\frac{\partial w}{\partial z} = \frac{v}{R^2}\left(\frac{\partial^2 w}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial w}{\partial \eta} + \frac{w}{\eta^2}\right), \tag{6}$$

$$\frac{1}{R}\frac{\partial w}{\partial \eta} + \frac{w}{\eta R} + \frac{\partial u}{\partial z} - \frac{\eta}{R}\frac{\partial R}{\partial z}\frac{\partial u}{\partial \eta} = 0\cdot \tag{7}$$

The system of equations above is a hemodynamic type of model. [1] stated that according to Belardinelli and Cavalcanti in 1991, the velocity profile in the axial direction, $u(\eta,z,t)$, is assumed to have the expression in the polynomial form below

$$u(\eta,z,t) = \sum_{k=1}^{N} q_k\left(\eta^{2k} - 1\right)\cdot \tag{8}$$

While the velocity profile in the radial direction is

$$w(\eta,z,t) = \frac{\partial R}{\partial z}\eta w + \frac{\partial R}{\partial t}\eta - \frac{\partial R}{\partial t}\frac{1}{N}\eta\sum_{k=1}^{N}\frac{1}{k}\left(\eta^{2k} - 1\right)\cdot \tag{9}$$

[1] chose $N = 1$ to simplify (8) and (9), so that

$$u(\eta,z,t) = q(z,t)\left(\eta^2 - 1\right), \tag{10}$$

$$w(\eta,z,t) = \frac{\partial R}{\partial z}\eta w + \frac{\partial R}{\partial t}\eta - \frac{\partial R}{\partial t}\eta\left(\eta^2 - 1\right)\cdot \tag{11}$$

Then, when equations (10) and (11) are substituted into equations (5) and (7), we will get the dynamic equations of $q(z, t)$ and $R(z, t)$, which are

$$\frac{\partial q}{\partial t} - \frac{4q}{R}\frac{\partial R}{\partial t} - \frac{2q^2}{R}\frac{\partial R}{\partial z} + \frac{4v}{R^2}q + \frac{1}{\rho}\frac{\partial P}{\partial z} = 0, \tag{12}$$

$$2R\frac{\partial R}{\partial t} + \frac{R^2}{2}\frac{\partial q}{\partial z} + q\frac{\partial R}{\partial z} = 0\cdot \tag{13}$$

Now, the cross-sectional area $S(z, t)$ and blood flow $Q(z, t)$ are defined as

$$S = \pi R^2, \qquad Q = \iint_S u\,\partial\eta = \frac{1}{2}\pi q R^2\cdot$$