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Almost invariant subspaces and high gain feedback

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H.L. Trentelman

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INTRODUCTION

The theory that will be presented in this monograph falls within a area of research that is commonly known as 'the geometric approach' to linear systems theory. In this area, questions concerning the structural properties of linear time invariant finite dimensional systems and problems involving the synthesis of feedback controllers for these systems are studied in a linear algebraic framework. The basic philosophy underlying the geometric approach is that a system is an entity defined by a number of mappings working on abstract linear spaces (the input space, the state space, the output space) and that several relevant structural features of the system are therefore determined by the way in which these mappings intertwine in their domains and codomains. These structural features can be expressed in terms of the geometrical properties of distinguished subspaces connected with these mappings. Moreover, questions involving the existence and synthesis of feedback controllers with given purposes can be regarded as questions concerning the mutual position of certain subspaces and the existence of mappings with certain properties. The seminal book where this philosophy was introduced is WONHAM (1979).

Indeed, the latter point of view has turned out to be an extremely successful one. Taking a brief glance at the development of the geometric approach over the last sixteen or so years, it can be seen that its framework has provided the tools, not only for a better understanding of the structure of linear systems (MORSE (1974)), but also for solving a large number of very well motivated feedback synthesis problems, such as the problem of disturbance decoupling by state feedback, non interacting control design, the regulator problem (WONHAM (1979)), disturbance decoupling by measurement feedback (SCHUMACHER (1981)), etc. In addition, the conceptual clarity of the geometric approach has contributed a great deal to the birth of research areas outside the realm of finite dimensional linear systems as, for example, tracking and regulation in infinite dimensions (SCHUMACHER (1981)) and the differential geometric approach to nonlinear systems (HIRSCHORN (1981), NIJMEIJER (1983)).

One of the basic concepts in all of the above is the concept of controlled invariance or (A,B)-invariance (BASILE & MARRO (1969), WONHAM & MORSE (1970)). In the context of linear time invariant finite dimensional systems, a controlled invariant subspace is a subspace of the state space with the property that for every initial condition lying in the subspace an appropriate control input can be found such that the resulting state trajectory lies entirely in that subspace. The importance of this concept in the design of feedback controllers for linear systems stems, roughly speaking, from the fact that controlled invariant subspaces can be made invariant using state feedback. Indeed, the standard application of the concept of controlled invariance, the problem of disturbance decoupling by state feedback, almost becomes a triviality once the equivalence between the above 'open loop' definition and its 'closed loop' counterpart in terms of state feedback has been established.

In the present work the main issue will be the notion of *almost* controlled invariance or *almost* (A,B)-invariance (WILLEMS (1980)). Whereas a controlled invariant subspace is defined by the property that one can stay in it by choosing the control input properly, for a subspace to be almost controlled invariant it is only required that one can stay arbitrarily close to it while moving along trajectories of the system. In this form, the definition of almost controlled invariance is again an 'open loop' one: the control inputs that are required to keep the state trajectories close to the subspace are allowed to depend on the initial condition in an arbitrary way. It will appear however that, similar to the case of 'ordinary' controlled invariant subspaces, almost controlled invariant subspaces can be made 'almost invariant' by state feedback: the controls that are needed in order to stay close to the subspace can be chosen to be generated by a state feedback control law. This equivalence between the open loop definition and its state feedback counterpart will make the notion of almost controlled invariance applicable to feedback synthesis problems in which it is required that certain output variables should remain 'small'. An example of such synthesis problem is provided by the almost disturbance decoupling problem by state feedback, the 'approximate' version of the 'exact' disturbance decoupling problem we mentioned before.

The first chapter of this tract is devoted to a careful study of the basic notions of almost controlled invariance and almost controllability subspace. In this chapter a framework will be established that will allow us to obtain equivalences between the open loop definitions on the one hand and characterizations of these notions in terms of 'simpler' subspaces on the other. Whereas in the case of 'ordinary' controlled invariant subspaces setting up such equivalences is relatively easy, in the 'almost' context this task will appear to be quite involved. In order to obtain rigorous proofs of the desired equivalences, the concept of 'factor system modulo S^{∞}' will be introduced (section 1.4).

In the first part of chapter 2 we will consider almost controlled invariant subspaces and almost controllability subspaces in a framework of distributional inputs and distributional state trajectories. It will turn out that almost controlled invariant subspaces can be regarded as ordinary controlled invariant subspaces if we admit distributions as inputs to keep the state trajectories within these subspaces. In fact, due to the linear finite dimensional context, we may restrict ourselves to a very special class of distributions: the class of Bohl distributions or, equivalently, the class of distributions that have a rational Laplace transform. Parallel to our description of almost controlled invariance in terms of Bohl distributions we will establish characterizations involving the particularly useful concept of (ξ,ω)-representation. This device allows us to treat the various geometric concepts by means of elementary algebraic manipulations of polynomials and rational functions.

The purpose of the second part of chapter 2 is to make the concepts of almost controlled invariant subspace and almost controllability subspace instrumental to the application in feedback synthesis problems. It is here that we will show that these subspaces can be made 'almost invariant' using state feedback. A proof of this important property will involve the fact that almost controlled invariant subspaces can be obtained as the limits of sequences of ordinary controlled invariant subspaces (section 2.4).

Measuring the distance of trajectories to subspaces in terms of integrated pointwise distance rather than in terms of pointwise distance itself, will appear in chapter 3 to lead to different classes of almost controlled invariant subspaces and almost controllability subspaces. It will be shown in this chapter that also for these subspaces, called L_p-almost controlled invariant subspaces and L_p-almost controllability subspaces, equivalences between the basic open loop definitions and characterizations in terms of 'almost invariance' using state feedback exist. This will be the subject of section 3.3 and section 3.4. In the former we will exploit the above-mentioned equivalence to establish conditions for solvability of the problem of L_p-almost disturbance decoupling by state feedback. In the latter the spectral assignability properties of L_p-almost controllability subspaces will be used as an instrument to obtain conditions for solvability of the

L_p-almost disturbance decoupling problem with pole placement. In section 3.5 the subspaces that we have introduced will be applied to the classical problem of stabilization by dynamic output feedback. In this section the notion of almost stabilizability subspace will be the spine of a result that says that invertible minimum phase systems can always be stabilized by means of dynamic compensators with dynamic order equal to the system's pole/zero excess minus the number of inputs.

Chapter 4 of this monograph is devoted to the study of a version of the L_p-almost disturbance decoupling problem in which it is required that certain components of the state vector in the closed loop system are bounded functions of the accuracy of approximate decoupling between the disturbances and the to-be-controlled outputs. It turns out that, while requiring the influence of the disturbances on the to-be-controlled output to be small, the feedback gains necessary to achieve this will in general become very large. This phenomenon will obviously result in 'large' state trajectories, which, in certain situations, might be unacceptable. Recognizing this inherent difficulty in the design of high gain feedback control systems motivates the following question: when is it possible to find state feedback control laws such that the approximate decoupling between the disturbance input and a first to-be-controlled output is arbitrarily accurate, while, simultaneously, the closed loop operator between the disturbance input and a second to-be-controlled output is a bounded function of this decoupling accuracy? In section 4.1 we will introduce a feedback synthesis problem that is based on this question.

In the final chapter of this tract we will introduce the concept of almost conditionally invariant subspace. Together with the notion of almost controlled invariance this concept will be applied to study the problem of almost disturbance decoupling by measurement feedback. This will be the topic of section 5.2. In section 5.3 and section 5.4 we will consider this problem with a constraint involving the high gain behaviour of the closed loop transfer matrices between disturbance and control inputs. This constraint leads us to formulate the L_p-almost disturbance decoupling problem by measurement feedback and guaranteed roll-off (section 5.3). In section 5.4 necessary and sufficient conditions for the solvability of this problem will be given. The last three sections of this monograph contain a discussion on the role of almost observability subspaces in the design of reduced and minimal order PID-observers for linear systems.

CHAPTER 1

ALMOST CONTROLLED INVARIANT SUBSPACES

In this chapter we will 'set the scene' for the theory that we will develop in this monograph. On the basis of a problem involving the modelling of a dynamical system, it will be explained how the concepts of invariance and controlled invariance lead in a natural way to the definition of the class of almost controlled invariant subspaces. We will briefly recall the basic properties of controlled invariant subspaces and controllability subspaces, in particular the equivalence between their 'dynamical' characterizations and their 'geometric' characterizations. We will then introduce the notions of almost controlled invariant subspace and almost controllability subspace. These subspaces will be defined in terms of the 'approximate holdability' properties they have with respect to the state trajectories of our linear system. Our purpose is then to establish the equivalence between these 'dynamical' definitions and characterizations in terms of the geometric properties of the mappings defining our linear system. To be able to do this, we will set up a considerable machinery, involving the properties of the state trajectories of our system modulo a subspace obtained as the limit of a recursive algorithm. This machinery will also play a central role in later chapters. Finally, we will apply the machinery developed to establish the desired 'geometric' characterizations.

The chapter is divided into six sections. In the first section we will motivate the introduction of almost controlled invariant subspaces, introduce some notation and recall the basic notion of controlled invariance. In section 2, almost controlled invariant subspaces and almost controllability subspaces are defined. In section 3 we study the almost controllability subspace algorithm and in section 4 we prove an important property of the 'limiting' subspace of the latter algorithm. The results of section 3 and section 4 are applied in section 5 to obtain geometric characterizations of the newly introduced subspaces. Finally, in section 6 we show how supremal almost controlled invariant subspaces may be computed.

1.1 INVARIANCE AND ALMOST INVARIANCE

Assume that we have a dynamical system whose evolution in time is modelled by the linear time invariant flow

$$(1.1) \qquad \dot{x}(t) = Ax(t),$$

where the state variable x takes its values in some finite dimensional linear space X, called the *state space*. In the above, A is a linear mapping from X into X, called the *system mapping*. An important concept in the study of flows is the notion of *invariance*. A subspace V of the state space X will be said to be *dynamically invariant* with respect to the flow (1.1) if every initial condition $x(0) = x_o \in V$ gives rise to a trajectory $x(t)$ that lies entirely in V. We will call V *geometrically invariant* or A-*invariant* if $AV \subset V$. It is clear that the familes of dynamically and geometrically invariant subspaces coincide.

Assume now that in our study of the above system we are interested in particular in the variable $z(t) = Hx(t)$. Here, z is assumed to take its values in some finite dimensional linear space Z and H is a linear mapping from X to Z. Suppose that it is desired to keep the value of $z(t)$ at some nominal value, say $z(t) = 0$. (As an example, for the moment think of (1.1) as a simple model describing the motion of a satellite and $z(t)$ as the variable that measures the deviation from a geostationary orbit in which we want to keep the satellite). If (1.1) would be a correct description of the dynamics of our system and its initial condition were 0, then $z(t)$ would be zero for all times, as required.

Suppose now however that our model is not accurate and that the dynamics of the system are in fact described by

$$(1.2) \qquad \dot{x}(t) = Ax(t) + d(t) \ , \ z(t) = Hx(t),$$

where $d(t) := \dot{x}(t) - Ax(t)$ is unknown and may be any element of some class of functions taking values in X. Under these assumptions, again with $x(0) = 0$, the variable $z(t)$ will in general no longer be held at the desired nominal value zero. However, if we would know at forehand (for example, due to certain physical considerations) that the possible disturbances in (1.2) all take their values in the same subspace G which, in turn, is contained in the kernel of the linear mapping H, then we might ask ourselves the question: does there exist a geometrically invariant subspace V 'between' G and the kernel of H? Indeed, if there is an A-invariant subspace that contains G and is itself contained in the kernel of H, then the variable z(t) in (1.2) would be held at its nominal value zero. To see this, define $X_1 := V$ and let X_2 be a subspace such that $X = X_1 \oplus X_2$. Then, in a basis for X compatible with this decomposition, the matrix of A would be of the form $A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$, H would be of the form $(0 \ H_2)$ and d(t) could be represented as $d(t) = \begin{pmatrix} d_1(t) \\ 0 \end{pmatrix}$. Writing the system (1.2) in this representation, we obtain

$$
\begin{aligned}
&\dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t) + d_1(t), \\
(1.3) \qquad &\dot{x}_2(t) = A_{22}x_2(t), \\
&z(t) = H_2x_2(t).
\end{aligned}
$$

It is obvious from these equations that, whatever the disturbance $d_1(t)$ might be, if the system starts at rest, then z(t) will be at its desired nominal value zero for all future time.

Assume now however, that we dó know that all disturbances in (1.2) take their values in the same subspace G of the kernel of H, but *there is no geometrically invariant subspace lying between* G *and the kernel of* H. It is here that the concept of *controlled invariance* enters into our considerations. Suppose that it is possible to change the dynamic behaviour of the system (1.2) by some external mechanism. In particular, suppose that our system is given by

$$(1.4) \qquad \dot{x}(t) = Ax(t) + Bu(t) + d(t) \;, \quad z(t) = Hx(t),$$

where u(t), the *control input*, may be chosen in some appropriate class of functions taking their values in the finite dimensional linear space U, called the *input space*. B is a linear mapping from U into X, called the *input mapping*. Assume that it is allowed to use for u(t) linear functions of the current state x(t), i.e. assume that we may take u(t) = Fx(t) in (1.4), where F is a linear mapping from X into U, called a *state feedback*.

With this possibility, we may then change the dynamics of (1.2) to

(1.5) $$\dot{x}(t) = (A+BF)x(t) + d(t) \quad , \quad z(t) = Hx(t).$$

Coming back to our problem, the following question then becomes relevant: is it possible to find a subspace V, containing G and contained in the kernel of H, for which there exists a mapping F such that V is (A + BF)-invariant? If the answer to this question is yes, then it follows from the above considerations that if we control the system (1.4) by means of the state feedback control law $u(t) = Fx(t)$, then in the controlled system our variable $z(t)$ will still be kept at its nominal value zero (provided of course the system starts at rest). A subspace V with the property that it can be made (A + BF)-invariant by suitable choice of F will temporarily be called *geometrically controlled invariant*. As in the uncontrolled case, this notion of invariance has its dynamical counterpart. A subspace V will be called *dynamically controlled invariant* if for every initial condition $x(0) = x_0$ in V, there is an input function $u(t)$ such that the solution of $\dot{x} = Ax + Bu(t)$, $x(0) = x_0$ lies entirely in V. It may be proven that, as in the uncontrolled case, the above notions of invariance coincide. Therefore, a subspace V with one of these properties will simply be called *controlled invariant* or (A,B)-*invariant* (BASILE & MARRO (1969a), WONHAM & MORSE (1970)).

Now, assume that the situation is even worse: assume that we do know that all disturbances in (1.4) take their values in the same subspace G contained in the kernel of H, but *there is no controlled invariant subspace lying between G and the kernel of* H. A natural question is then: is it possible to choose F such that $z(t)$ is kept 'close' to its desired nominal value, i.e. such that $z(t)$ is kept 'small'. In the following, if $d(t)$ is a disturbance, let $w(t)$ be such that $d(t) = Gw(t)$. (Assume G is a mapping such that $G = \operatorname{im} G$). It is well known that (1.5) yields:

(1.6) $$z(t) = \int_0^t T_F(t-\tau)w(\tau)d\tau,$$

with $T_F(t) := H\exp[t(A+BF)]G$. For the moment, assume that we restrict ourselves to integrable disturbances $w(t)$. Then, if $T_F(t)$ is a bounded function, we may show that for all points of time $t \geq 0$:

$$(1.7) \qquad \| z(t) \| \leq \sup_{\tau \geq 0} \| T_F(\tau) \| \circ \int_0^\infty \|w(\sigma)\| \, d\sigma.$$

The latter inquality motivates the question: given any small real number $\varepsilon > 0$, is it possible to find a state feedback F such that $\sup_{\tau \geq 0} \| T_F(\tau) \|$ is smaller than ε? If this were indeed the case than we could at least make sure that the variable z(t) would be kept close to the desired value zero in the case that the disturbances 'live' in some ball of fixed radius in the space of integrable functions. Note that $T_F(t)$ is exactly equal to zero if and only if G is contained in some $(A + BF)$-invariant subspace contained in the kernel of H. We might now be tempted to ponder on the following question: is it possible to formalize a concept of 'approximate $(A + BF)$-invariance' or 'geometrical approximate invariance' to treat the latter problem of making $T_F(t)$ approximately zero? Of course, the idea would then be to look for a 'geometrically approximately invariant' subspace lying between G and the kernel of H. Since it is not at all clear how to formalize such a notion in a pure linear algebra context, it seems reasonable (in view of the equivalence between the geometrical and dynamical definition in the 'exact' case) to formalize instead a notion of 'dynamical approximate invariance'. Thus we are led to the following definition: a subspace V will be called *almost controlled invariant* if for every initial condition $x(0) = x_0$ in V and for every positive real number ε, there is an input function u(t) such that the solution of $\dot{x} = Ax + Bu(t)$, $x(0) = x_0$ moves at most at an ε-distance from V. It will turn out that this concept of almost invariance can indeed be fruitfully used to treat the problem we have been considering in this section.

In the rest of this section, we will briefly review the basic notational conventions we will use in the sequel and recall some basic facts on controlled invariant and controllability subspaces.

If we speak about the linear system with system mapping A and input mapping B, we will always mean the system $\dot{x}(t) = Ax(t) + Bu(t)$. Here, x will be assumed to take its values in the linear space X, called the state space and u will be assumed to take its values in the linear space U, called the input space. It will be assumed that X and U are linear spaces over the field $\mathbb{R}$ and X will always have dimension n, U will always have dimension m. A and B will be linear mappings from X into X and U into X respectively.

Spaces and subspaces will always be denoted by light italic capitals. If we use the word 'mapping' we will always mean 'linear mapping'. Mappings will always be denoted by capitals. Elements from linear spaces and time functions taking values in linear spaces will be denoted by lower case letters. Sometimes, if x is a time function taking values in X, we will denote $x(\cdot)$ or $x(t)$ to distinguish x from elements of the linear space X. If $n \in \mathbb{N}$, then $\underline{n}$ will denote the set $\{1,2,\ldots,n\}$. We will denote $\mathbb{R}^+ := \{r \in \mathbb{R} \mid r \geq 0\}$ and $\mathbb{R}^- := \{r \in \mathbb{R} \mid r < 0\}$.

If V and W are subspaces of X, then their sum will be denoted by $V + W$. If $V \cap W = \{0\}$, then this sum will often be denoted by $V \oplus W$. If $x_1,\ldots,x_r$ are vectors in X, then $\mathrm{span}\{x_1,\ldots,x_r\}$ will denote the subspace of linear combinations of the vectors x_i. If $T: X \longrightarrow X$ is a mapping and V is a T-invariant subspace of X, then $T|V$ will denote the restriction of T to V.

If V is a subspace of X, then two vectors x_1,x_2 in X are called equivalent modulo V if $x_1 - x_2 \in V$. The quotient space under this equivalence relation will be denoted by X/V. An element of X/V will typically be denoted by ξ. If $x \in \xi$, then we will also denote the equivalence class ξ by $[x]$. The canonical projection of X onto X/V is the mapping P defined by $Px := [x]$. If W is a second subspace of X and $V \subset W$, then we define $W/V := PW$. If W_1 and W_2 are subspaces of X and $V \subset W_1 \cap W_2$, then it may be verified that $(W_1/V) \cap (W_2/V) = (W_1 \cap W_2)/V$.

If V is T-invariant, then T induces a mapping from X/V into X/V, defined by $[x] \longrightarrow [Tx]$. This mapping will be denoted by $T|X/V$ and will be called the quotient mapping of T modulo V. If $V \subset W$ and V and W are both T-invariant, then $T|W/V$ will denote the quotient mapping of $T|W$ modulo V. All facts in the context of linear algebra that we need here can be found in GANTMACHER (1959).

Finally we will recall some basic facts on controlled invariant and controllability subspaces. Consider the linear system with system mapping A and input mapping B. We will denote by $\Sigma(A,B)$ the linear space of all state trajectories of this system. Formally:

$$\Sigma(A,B) := \{x: \mathbb{R} \longrightarrow X \mid x \text{ is absolutely continuous and } \dot{x}(t) - Ax(t) \in \mathrm{im}B \text{ a.e.}\}. \tag{1.8}$$

As a standing assumption, we will assume that the input mapping B is injective. Its (m-dimensional) image in X will be denoted by B or $\operatorname{im} B$. If $F: X \longrightarrow U$ is a mapping we will often denote $A_F := A + BF$.

DEFINITION 1.1. A subspace V of X will be called a controlled invariant subspace if $\forall x_o \in V$, $\exists x \in \Sigma(A,B)$ such that $x(0) = x_o$ and $x(t) \in V$, $\forall t \in \mathbb{R}$.

A controllability subspace is defined as a subspace of X with the property that it is possible to travel between any two points of the subspace, moving along a trajectory that lies entirely in that subspace:

DEFINITION 1.2. A subspace R of X will be called a controllability subspace if $\forall x_o, x_1 \in R$, $\exists T > 0$ and $x \in \Sigma(A,B)$ such that $x(0) = x_o$, $x(T) = x_1$ and $x(t) \in R$, $\forall t \in \mathbb{R}$.

Controlled invariant subspaces are also called (A,B)-invariant subspaces or A(mod B)-invariant subspaces. We will denote by $\underline{V}$ or $\underline{V}(A,B)$ and $\underline{R}$ or $\underline{R}(A,B)$ the classes of all controlled invariant subspaces and controllability subspaces respectively. It is immediate that $\underline{R} \subset \underline{V}$.

In the following, if B_1 is a subspace of B and $F: X \longrightarrow U$ is a mapping, we will denote $\langle A_F | B_1 \rangle := B_1 + A_F B_1 + \ldots + A_F^{n-1} B_1$. The following geometric characterizations of the classes $\underline{V}$ and $\underline{R}$ are well known:

PROPOSITION 1.3. *The following statements are equivalent:*

(i) $V \in \underline{V}(A,B)$,

(ii) $\exists F: X \longrightarrow U$ *such that* $A_F V \subset V$,

(iii) $AV \subset V + B$.

PROPOSITION 1.4. $R \in \underline{R}(A,B)$ *if and only if there is a subspace* B_1 *of* B *and a mapping* $F: X \longrightarrow U$ *such that* $R = \langle A_F | B_1 \rangle$.

For a proof of the above equivalences, we refer to WONHAM (1979). The latter uses the statement of PROP. 1.3 (ii) as a definition of controlled invariance and the statement of PROP. 1.4 as a definition of controllability subspace.

If $V \in \underline{V}$ then the set of all mapping $F: X \longrightarrow U$ with the property that $A_F V \subset V$, will be denoted by $\underline{F}(V)$. It may be proven readily that the classes

$\underline{V}$ and $\underline{R}$ are closed under the operation of subspace addition. It may therefore immediately be concluded (WONHAM (1979, LEMMA 4.4)) that, with every subspace K of X, there exists a supremal controlled invariant subspace contained in K and a supremal controllability subspace contained in K. These subspaces will be denoted by $V^*(K)$ and $R^*(K)$ respectively.

1.2 ALMOST CONTROLLED INVARIANT AND ALMOST CONTROLLABILITY SUBSPACES

In this section we will introduce the concepts of almost controlled invariant subspace and almost controllability subspace. Consider the linear system with system mapping A and input mapping B. Generalizing the dynamical definition of controlled invariance, we define an *almost controlled invariant subspace* as a subspace of the state space with the property that, beginning in it, one can stay arbitrarily close to it by choosing the input properly. In the same way, an *almost controllability subspace* will be defined as a subspace of the state space with the property that, starting in it, one can steer to an arbitrary point in the same subspace while staying arbitrarily close to that subspace.

The concepts of almost controlled invariant subspace and almost controllability subspace where introduced in WILLEMS (1980).

Of course, in order to be able to measure the distance from a point to a subspace, we should have a notion of distance in the state space X. We will therefore always assume that X is a normed linear space. The norm in X will be denoted by $\|\cdot\|$. If L is a subspace of X and $x \in X$, we will define the distance of x to L by

$$(1.9) \qquad d(x,L) := \inf_{x' \in L} \|x - x'\|.$$

We then have the following definitions:

<u>DEFINITION 1.5</u>. A subspace $V_a \subset X$ is said to be an almost controlled invariant subspace if $\forall x_0 \in V_a$ and $\varepsilon > 0$, $\exists x \in \Sigma(A,B)$ such that $x(0) = x_0$ and $d(x(t), V_a) \leq \varepsilon$, $\forall t \in \mathbb{R}$.

<u>DEFINITION 1.6</u>. A subspace $R_a \subset X$ is said to be an almost controllability

subspace if $\forall x_o, x_1 \in R_a$, $\exists T > 0$ such that $\forall \varepsilon > 0$ $\exists x \in \Sigma(A,B)$ with the properties that $x(0) = x_o$, $x(T) = x_1$ and $d(x(t), R_a) \leq \varepsilon$, $\forall t \in \mathbb{R}$.

Almost controlled invariant subspaces are also called almost A(mod B)-invariant subspaces or almost (A,B)-invariant subspaces. Let $\underline{V}_a$ or $\underline{V}_a(A,B)$ and $\underline{R}_a$ or $\underline{R}_a(A,B)$ denote the classes of all almost controlled invariant subspaces and almost controllability subspaces of the system (A,B). It is a trivial matter to verify that the inclusions $\underline{R} \subset \underline{V} \subset \underline{V}_a$ and $\underline{R} \subset \underline{R}_a \subset \underline{V}_a$ hold. In analogy to $\underline{V}$ and $\underline{R}$, our new classes of subspaces exhibit the property of closedness under addition:

<u>THEOREM 1.7.</u> *$\underline{V}_a$ and $\underline{R}_a$ are closed uder the operation of subspace addition.*

<u>PROOF:</u> We will give a proof of the closedness property of $\underline{R}_a$. An analogous but simpler proof applies to $\underline{V}_a$. Let $R_{a,1}$ and $R_{a,2} \in \underline{R}_a$. Define $R_a = R_{a,1} + R_{a,2}$ and let $x_0 = x_{01} + x_{02}$ and $x_1 = x_{11} + x_{12}$ with x_{0i}, $x_{1i} \in R_{a,i}$. For $i = 1,2$ there are $T_i > 0$ and for all $\varepsilon > 0$ there are $x_i(\cdot) \in \Sigma(A,B)$ such that $x_i(0) = x_{0i}$, $x_i(T_i) = 0$ and $d(x_i(t), R_{a,i}) \leq \frac{\varepsilon}{2}$, $\forall t$. Assume that $T_1 \leq T_2$. Define a new trajectory x_1' by $x_1'(t) := x_1(t)$ for $t \leq T_1$ and $x_1'(t) := 0$ for $t > T_1$. Define $x(t) := x_1'(t) + x_2(t)$. Then $x(\cdot) \in \Sigma(A,B)$, $x(0) = x_0$, $x(T_2) = 0$ and

$$\begin{aligned} d(x(t), R_a) &\leq d(x_1'(t), R_a) + d(x_2(t), R_a) \\ &\leq d(x_1'(t), R_{a,1}) + d(x_2(t), R_{a,2}) \leq \varepsilon. \end{aligned}$$

Now, to complete the proof, it suffices to show that we can move from 0 to x_1 along a trajectory that is at most a distance of ε away from R_a. Proceeding in an analogous way as above, we find $\tilde{T}_1$ and $\tilde{T}_2$ and two trajectories $\tilde{x}_1$ and $\tilde{x}_2$ along which it is possible to move from 0 at $t = 0$ to x_{11} at $t = \tilde{T}_1$ and x_{12} at $t = \tilde{T}_2$. Assume that $\tilde{T}_1 \leq \tilde{T}_2$. Then the idea is to stay some more time in 0 and modify x_1 into x_1' by defining $\tilde{x}_1'(t) = 0$ for $t \leq \tilde{T}_2 - \tilde{T}_1$ and $\tilde{x}_1'(t) = \tilde{x}_1(t - \tilde{T}_2 + \tilde{T}_1)$ for $t > \tilde{T}_2 - \tilde{T}_1$. It may then be verified that $\tilde{x}(t) := \tilde{x}_1'(t) + \tilde{x}_2(t)$ will lead us from 0 at $t = 0$ to $x_1 = x_{11} + x_{12}$ at $t = \tilde{T}_2$ while staying close to R_a.

□

From the above result, it may immediately be concluded that, for every subspace K of X, there exists a supremal almost controlled invariant subspace contained in K. This subspace will be denoted by $V_a^*(K)$. In the same way, $R_a^*(K)$ will denote the supremal almost controllability subspace contained in K.

One of the main purposes of this chapter is to establish a characterization of almost controlled invariant subspaces and almost controllability subspaces in terms of the geometry of the input mapping B and the system mapping A. Recall that such a characterization has indeed been obtained for controlled invariant subspaces and controllability subspaces (PROP. 1.3 and PROP. 1.4). It turns out that also for the 'almost versions' geometric characterizations can be found. In fact, those were the main result of WILLEMS (1980). In the sequel, a sequence of subspaces $\{B_i\}_{i=1}^{k}$ will be called a chain in B if $B \supset B_1 \supset B_2 \supset \ldots \supset B_k$. Denote $A_F := A + BF$. Then the following holds:

PROPOSITION 1.8.
(i) $V_a \in \underline{V}_a$ *if and only if there exists* $V \in \underline{V}$ *and* $R_a \in \underline{R}_a$ *such that* $V_a = V + R_a$.
(ii) $R_a \in \underline{R}_a$ *if and only if there is a mapping* $F: X \longrightarrow U$ *and a chain* $\{B_i\}_{i=1}^{k}$ *in* B *such that* $R_a = B_1 + A_F B_2 + \ldots + A_F^{k-1} B_k$.

Although easy to formulate, a rigorous proof of the above result turns out to be quite complicated. We will however set up in sections 1.3 and 1.4 the necessary mathematical framework to prove proposition 1.8. In section 1.5 the result will then be restated and proven in two seperate corollaries.

We will start of here by showing that every subspace of the form $B_1 + A_F B_2 + \ldots + A_F^{k-1} B_k$, with $\{B_i\}_{i=1}^{k}$ a chain in B, is in $\underline{R}_a$. By TH 1.7, it suffices to prove that every subspace of the form $\mathcal{L} = \mathrm{span}\{b, A_F b, \ldots, A_F^k b\}$, with $b \in B$ and $k \in \mathbb{N}$, is in $\underline{R}_a$.

Let $x_0, x_1 \in \mathcal{L}$. We will show that we can move from x_0 to x_1 along a trajectory that stays arbitrarily close to $\mathcal{L}$. We will first consider the case that $x_1 = 0$. Since $x_0 \in \mathcal{L}$, there are $\lambda_i \in \mathbb{R}$ such that $x_0 = \sum_{i=0}^{k} \lambda_i A_F^i b$. Let $u_0 \in U$ be such that $b = Bu_0$. Now, the idea of the proof is the following: by using an 'input' $u = -\sum_{i=0}^{k} \lambda_i \delta^{(i)} u_0$ for the system $\dot{x} = A_F x + Bu$, $x(0) = x_0$, we would obtain a 'trajectory' x consisting of an 'impulsive part' at $t = 0$ and a 'smooth part' $x(t) = 0$ for $t > 0$. This impulsive part would be lying entirely in $\mathcal{L}$. ($\delta^{(i)}$ denotes the ith distributional derivative of the Dirac distribution. Distributional inputs will be treated in detail in CH.2, SECTION 2.1). We will approximate the distribution u by smooth inputs u_n and thus find smooth trajectories x_n staying close to $\mathcal{L}$ and bringing us close to 0. The final transfer to 0 is then made by moving along a trajectory that stays close to 0.

To make the above precise, let $\{\varphi_n\}_{n \in \mathbb{N}}$ be a sequence of smooth functions $\mathbb{R} \longrightarrow \mathbb{R}$ with (i) supp $\varphi_n \subset [0, \frac{1}{n}]$, (ii) $\varphi_n \geq 0$ (iii) $\int_0^{1/n} \varphi_n = 1$ and (iv) $\varphi_n^{(\ell)}(0) = 0$ for $\ell = 0, 1, \ldots, k$. (Here (ℓ) denotes the ℓth derivative). Define a sequence of smooth inputs u_n by

$$(1.10) \qquad u_n(t) := -\sum_{i=0}^{k} \lambda_i \varphi_n^{(i)}(t) u_0.$$

(It may be seen that u_n converges to u as $n \longrightarrow \infty$ in the sense of distributions). Using u_n as input for the system (A_F, B) with $x(0) = x_0$, we obtain

$$(1.11) \qquad x_n(t) = e^{A_F t} x_0 - \sum_{i=0}^{k} \lambda_i \int_0^t e^{A_F(t-\tau)} \varphi_n^{(i)}(\tau) d\tau \circ Bu_0$$

which, by repeated partial integration, yields

$$(1.12) \qquad x_n(t) = e^{A_F t} x_0 - \int_0^t e^{A_F(t-\tau)} \varphi_n(\tau) d\tau \circ x_0 - \sum_{j=1}^{k} \sum_{i=j}^{k} \varphi_n^{(i-j)}(t) \lambda_i A_F^{j-1} Bu_0.$$

Now, we claim that for every $T > 0$, $d(x_n(t), \mathcal{L}) \longrightarrow 0$ uniformly on $[0,T]$. To see this, note that the third term of 1.12 is entirely contained in $\mathcal{L}$. In the following, let $K(t) := e^{A_F t}$ for $t \in \mathbb{R}^+$ and $K(t) := 0$ for $t \in \mathbb{R}^-$. Then for all $t \in \mathbb{R}^+$ we have:

$$(1.13)\qquad d(x_n(t),\mathcal{L}) = d(K(t)x_0 - \int_0^{1/n} K(t-\tau)x_0\varphi_n(\tau)d\tau,\mathcal{L})$$

$$= d(\int_0^{1/n} (K(t)x_0 - K(t-\tau)x_0)\varphi_n(\tau)d\tau,\mathcal{L})$$

$$\leq \int_0^{1/n} d(K(t)x_0 - K(t-\tau)x_0,\mathcal{L})\varphi_n(\tau)d\tau$$

$$\leq \sup_{0\leq\tau\leq\frac{1}{n}} d(K(t)x_0 - K(t-\tau)x_0,\mathcal{L}).$$

Using the facts that $d(x_n(0),\mathcal{L}) = 0$ and that the function $t \longrightarrow d(K(t)x_0,\mathcal{L})$ is uniformly continuous on $[0,T]$, it may be shown that the latter converges to 0 as $n \longrightarrow \infty$ uniformly with respect to $t \in [0,T]$. Next we will show that by taking n sufficiently large, the trajectory x_n will bring us arbitrarily close to 0 at $t = T$. In fact, it may be proven that $\lim_{n\longrightarrow\infty} x_n(T) = 0$. To see this, evaluate (1.12) at $t = T$. Note that the third term vanishes for n sufficiently large since $\operatorname{supp}\varphi_n \subset [0,\frac{1}{n}]$. The statement then follows by estimating $\| x_n(T) \|$ in a similar way as (1.13).

Finally, we will show that it is possible to move from an initial point x_0 to 0 while staying arbitrarily close to 0, provided that $\| x_0 \|$ is sufficiently small. In fact, we have:

LEMMA 1.9 *Consider the system* (A,B). *Let* $T > 0$. *For* $x_0 \in \langle A|B\rangle$ *and* $\varepsilon > 0$, *define*

$$(1.14)\qquad \Omega(x_0,\varepsilon) := \{x \in \Sigma(A,B) \,|\, x(0) = x_0,\ x(T) = 0 \text{ and } \| x(t) \| \leq \varepsilon, \forall t \in [0,T]\}.$$

Then $\forall\varepsilon > 0\ \exists\,\delta > 0$ *such that if* $\| x_0 \| < \delta$ *then* $\Omega(x_0,\varepsilon) \neq \emptyset$.

PROOF: Fix $T > 0$. Let $W_T := \int_0^T e^{As}BB^Te^{A^Ts}ds$. It is well known (see e.g. WONHAM (1979, p.37)) that $\operatorname{im} W_T = \langle A|B\rangle$ and that $\ker W_T = \langle A|B\rangle^{\perp}$. Note that $\langle A|B\rangle$ is W_T-invariant. Denote $\overline{W}_T := W_T \,|\, \langle A|B\rangle$. It may then be verified that $\overline{W}_T$ is invertible. Let $z \in \langle A|\mathcal{B}\rangle$ be such that $\overline{W}_T z = -e^{AT}x_0$. Define a smooth input u on [0,T] by

$$u(t) := B^T e^{A^T(T-t)}z.$$

Then we have

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\sigma)}BB^T e^{A^T(T-\sigma)}d\sigma \circ z .$$

From this, it is obvious that $\|x(t)\|$ can be kept small on [0,T] by making $\|x_0\|$ (and consequently $\|z\| = \|\overline{W}_T^{-1} e^{AT}x_0\|$) sufficiently small. Moreover, it may be verified that $x(0) = x_0$ and $x(T) = e^{AT}x_0 + \overline{W}_T z = 0$.

□

Now, to complete the proof of the assertion that $\mathcal{L}$ is an almost controllability subspace, we should prove that it is also possible to move from 0 to a point $x_1 \in \mathcal{L}$ while staying arbitrarily close to $\mathcal{L}$. This can be proven by repeating the above arguments for $T < 0$. Assume that $x_1 = \sum_{i=0}^{k} \mu_i A_F^i B u_0$. Define a smooth $u_n(\cdot)$ by $u_n(t) := -\sum_{i=0}^{k} \mu_i \varphi_n^{(i)}(-t)u_0$ (defined on [T,0]). It may then be verified that if we solve $\dot{x} = A_F x + Bu_n$; $x(0) = x_0$ on [T,0], we obtain a trajectory $x_n(t)$ with $x_n(0) = x_1$, $d(x_n(t),\mathcal{L}) \longrightarrow 0$ $(n \longrightarrow \infty)$ uniformly on [T,0] and $x_n(T) \longrightarrow 0$ $(n \longrightarrow \infty)$. The result then follows after verifying that LEMMA 1.9 also holds for $T < 0$.

Putting things together now, we can construct a trajectory through $x_0, x_1 \in \mathcal{L}$ staying close to $\mathcal{L}$ as follows. First, follow the zero-trajectory. At $t = 0$, move from 0 to x_0 at $t = t_0 > 0$, staying close to $\mathcal{L}$. Then, move from x_0 to 0 at $t = t_1 > t_0$ and from 0 to x_1 at $t = t_2 > t_1$. Finally, move from x_1 back to 0 at $t = t_3 > t_2$ and follow the zero-trajectory.

Thus, we have proven that if $b \in \mathcal{B}$, $F: \mathcal{X} \longrightarrow \mathcal{U}$ is a mapping and $k \in \mathbb{N}$, then $\mathcal{L} := \text{span}\{b, A_F b, \ldots, A_F^k b\}$ is an almost controllability subspace. In the sequel, subspaces of this form will play an important role. We will call $\mathcal{L}$ a *singly generated almost controllability subspace*. Since also every sum of such subspaces is in $\underline{R}_a$, we may indeed conclude that every subspace of the form

(1.15) $\qquad R_a = \mathcal{B}_1 + A_F\mathcal{B}_2 + \ldots + A_F^{k-1}\mathcal{B}_k,$

with $\{B_i\}_{i=1}^k$ a chain in B and $F: X \longrightarrow U$ a mapping, is an almost controllability subspace.

From the fact that the inclusions $\underline{V} \subset \underline{V}_a$ and $\underline{R}_a \subset \underline{V}_a$ hold, we may also immediately conclude that every subspace of the form $V + R_a$ with $V \in \underline{V}$ and $R_a \in \underline{R}_a$ is an almost controlled invariant subspace.

1.3 THE ALMOST CONTROLLABILITY SUBSPACE ALGORITHM

In the present section we will study the properties of the sequence of subspaces defined by the following recursive algorithm:

$$(1.16) \qquad S^0 := \{0\}; \; S^{\mu+1} := K \cap (AS^\mu + B), \; \mu \in \mathbb{N}.$$

Here, K is an arbitrary, fixed subspace of X. The above algorithm also appears in WONHAM (1979, §§ 5.3, 5.4), where it is called the 'controllability subspace algorithm'. For reasons that will become clear in the sequel, we will call it the 'almost controllability subspace algorithm' (WILLEMS (1980)). In this section, we will prove the following result:

THEOREM 1.10. *Let K be a subspace of X and, for $\mu \in \mathbb{N} \cup \{0\}$, let S^μ be defined by* (1.16). *Then the following holds:*

(i) *The sequence $\{S^\mu\}_{\mu=0}^\infty$ is monotonically nondecreasing. Moreover, if $S^\mu = S^{\mu+1}$, then $S^\mu = S^{\mu+\nu}$ for all $\nu \in \mathbb{N}$.*

(ii) *For each $\mu \in \mathbb{N}$ there is a chain $\{B_i\}_{i=1}^\mu$ in B and a mapping $F: X \longrightarrow U$ such that*

$$(1.17) \qquad S^\mu = B_1 \oplus A_F B_2 \oplus \ldots \oplus A_F^{\mu-1} B_\mu,$$

with

$$(1.18) \qquad B_1 = B \cap K$$

and

$$(1.19) \qquad \dim B_i = \dim A_F^{i-1} B_i = \dim S^i - \dim S^{i-1}, \; i \in \underline{\mu}.$$

<u>PROOF</u>: (i) The proof of the monotone property can be found in WONHAM (1979, p. 107). Suppose now that $S^{\mu} = S^{\mu+1}$ for some μ. Then $S^{\mu+2} = K \cap (AS^{\mu+1} + B) = K \cap (AS^{\mu} + B) = S^{\mu+1} = S^{\mu}$. In the same way it follows that $S^{\mu+\nu} = S^{\mu}$ for all $\nu \in \mathbb{N}$. (ii) The proof of this property will be given by induction. The claim is obviously true for $\mu = 1$. Now assume it is true up to μ. Let $\{B_i\}_{i=1}^{\mu}$ be a chain in B and $F\colon X \longrightarrow U$ a mapping such that (1.17) to (1.19) hold. We will show that $S^{\mu+1}$ can also be represented in this way. This will be done by constructing *an extra term* $B_{\mu+1}$ in addition to the 'old' chain $\{B_i\}_{i=1}^{\mu}$ and by defining *a new feedback mapping* $F_{new}\colon X \longrightarrow U$. First, let B_i' $(i \in \underline{\mu})$ be subspaces such that $B_i' \oplus B_i = B_{i-1}$. (define $B_0 := B$). We then have:

$$
\begin{aligned}
S^{\mu+1} &= K \cap (A_F S^{\mu} + B) \\
&= K \cap (A_F[B_1 + A_F B_2 + \ldots + A_F^{\mu-1} B_{\mu}] + B) \\
&= K \cap (S^{\mu} + G),
\end{aligned}
$$

where we define

$$
G := B_1' + A_F B_2' + \ldots + A_F^{\mu-1} B_{\mu}' + A_F^{\mu} B_{\mu}.
$$

Using the modular distributive rule (WONHAM (1979, p.4)), together with the fact that $S^{\mu} \subset K$, we obtain

$$
S^{\mu+1} = S^{\mu} + (G \cap K). \tag{1.20}
$$

Let $\hat{G} \subset G \cap K$ be a subspace such that $S^{\mu+1} = S^{\mu} \oplus \hat{G}$. and let $\{v_1, \ldots, v_r\}$ be a basis for $\hat{G}$. By the definition of G, each v_j can be represented as

$$
v_j = b_{1,j}' + A_F b_{2,j}' + \ldots + A_F^{\mu-1} b_{\mu,j}' + A_F^{\mu} b_j,
$$

with $b_{i,j}' \in B_i'$ $(i \in \underline{\mu})$ and $b_j \in B_{\mu}$. By the assumption that $\hat{G} \cap S^{\mu} = \{0\}$, it may be verified directly that for fixed $i \in \underline{\mu}$ the system $\{A_F^i b_1, \ldots, A_F^i b_r\}$ is linearly independent. Define now

$$
B_{\mu+1} := \operatorname{span}\{b_1, \ldots, b_r\}.
$$

Note that $\mathcal{B}_{\mu+1} \subset \mathcal{B}_{\mu}$. For $j \in \underline{r}$ and $i \in \underline{\mu}$, define vectors $x_{j,i}$ by

$$
\begin{aligned}
x_{j,1} &:= b_j, \\
x_{j,2} &:= A_F b_j + b'_{\mu,j}, \\
x_{j,3} &:= A_F^2 b_j + A_F b'_{\mu,j} + b'_{\mu-1,j}, \\
&\;\vdots \\
x_{j,\mu} &:= A_F^{\mu-1} b_j + A_F^{\mu-2} b'_{\mu,j} + \dots + A_F b'_{3,j} + b'_{2,j}.
\end{aligned}
$$

It follows immediately that $x_{j,i} = A_F x_{j,i-1} + b'_{\mu-i+2,j}$ and that $v_j = A_F x_{j,\mu} + b'_{1,j}$. Moreover, by the independency of the systems $\{A_F^i b_j \; ; \; j \in \underline{r}\}$ and by the fact that the spaces $A_F^{i-1}\mathcal{B}_i$ $(i \in \underline{\mu})$ are independent (recall from (1.17) that their sum is a direct sum), it may be shown that the system $\{x_{j,i} \; ; \; j \in \underline{r},\; i \in \underline{\mu}\}$ is linearly independent. Now extend this system to a basis for X. Let $u_{j,k}$ $(j \in \underline{r},\; k \in \underline{\mu})$ be vectors in U such that $b'_{\mu-k+1,j} = Bu_{j,k}$. Define a mapping $F'' : X \longrightarrow U$ as follows:

$$F'' x_{j,i} := u_{j,i},$$

F'' arbitrary on the extension.

Then it is immediate that for $i \in \underline{\mu}$

$$\operatorname{span}\{x_{1,i},\dots,x_{r,i}\} = (A_F + BF'')^{i-1}\mathcal{B}_{\mu+1}$$

and

$$\hat{G} = \operatorname{span}\{v_1,\dots,v_r\} = (A_F + BF'')^{\mu}\mathcal{B}_{\mu+1}.$$

Moreover, for $i \in \{1,\dots,\mu-1\}$ we have

$$BF''(A_F + BF'')^{i-1}\mathcal{B}_{\mu+1} \subset \mathcal{B}'_{\mu+1-i} \subset \mathcal{B}_1. \tag{1.21}$$

Let $\{\mathcal{B}''_i\}_{i=1}^{\mu}$ be a chain in $\mathcal{B}$ such that for $i \in \underline{\mu}$ $\mathcal{B}_{\mu+1} \oplus \mathcal{B}''_i = \mathcal{B}_i$. Since $S^{\mu+1} = S^{\mu} \oplus \hat{G}$, we then obtain

$$S^{\mu+1} = (B_1 + A_F B_2 + \dots + A_F^{\mu-1} B_\mu) + (A_F + BF'')^\mu B_{\mu+1}$$
$$= B_{\mu+1} + A_F B_{\mu+1} + \dots + A_F^{\mu-1} B_{\mu+1} + (A_F + BF'')^\mu B_{\mu+1}$$
$$+ B_1'' + A_F B_2'' + \dots + A_F^{\mu-1} B_\mu''.$$

Thus, by (1. 21) we find that

$$S^{\mu+1} = B_{\mu+1} + (A_F + BF'')B_{\mu+1} + \dots + (A_F + BF'')^\mu B_{\mu+1} \tag{1.22}$$
$$+ B_1'' + A_F B_2'' + \dots + A_F^{\mu-1} B_\mu'' .$$

We contend that all sums in the expression (1.22) are, in fact, direct sums. To see this, assume the contrary. Then the following strict inequality must hold:

$$\dim S^{\mu+1} < \sum_{i=1}^{\mu+1} \dim(A_F + BF'')^{i-1} B_{\mu+1} + \sum_{i=1}^{\mu} \dim A_F^{i-1} B_i''$$
$$\leq \sum_{i=1}^{\mu} \dim B_{\mu+1} + \dim \hat{G} + \sum_{i=1}^{\mu} \dim B_i''.$$

Since however, by definition, $B_{\mu+1} \oplus B_i'' = B_i$ $(i \in \underline{\mu})$, it follows that

$$\dim S^{\mu+1} < \sum_{i=1}^{\mu} \dim B_i + \dim \hat{G}.$$

On the other hand however, $\dim S^{\mu+1} = \dim S^\mu + \dim \hat{G}$, which, by assumption that (1.17) to (1.19) are valid for μ, equals $\sum_{i=1}^{\mu} \dim B_\mu + \dim \hat{G}$. Thus we have obtained a contradiction.

Define now

$$V := B_{\mu+1} \oplus (A_F + BF'')B_{\mu+1} \oplus \dots \oplus (A_F + BF'')^{\mu-1} B_{\mu+1}$$

and

$$W := B_1'' \oplus A_F B_2'' \oplus \dots \oplus A_F^{\mu-1} B_\mu''.$$

We have $V \cap W = \{0\}$. Let R be a subspace of X such that $V \oplus W \oplus R = X$. Define

$F_{new}: X \longrightarrow U$ as follows:

$$F_{new}|V: = (F + F'')|V,$$
$$F_{new}|W: = F|W,$$
$$F_{new}|R: = 0|R.$$

Then it follows immediately from (1.22) that

$$S^{\mu+1} = B_1 \oplus (A_F + BF_{new})B_2 \oplus \ldots \oplus (A_F + BF_{new})^{\mu} B_{\mu+1}.$$

Finally we will prove (1.19). We claim that for $i \in \{1,\ldots,\mu+1\}$ $\dim(A + BF_{new})^{i-1} B_i = \dim B_i$. Suppose the contrary. Then there must be inequality for at least one $i \in \underline{\mu}$ and therefore

$$\sum_{i=1}^{\mu} \dim B_i + \dim \hat{G} = \dim S^{\mu+1}$$
$$= \sum_{i=1}^{\mu+1} \dim(A + BF_{new})^{i-1} B_i = \sum_{i=1}^{\mu} \dim(A + BF_{new})^{i-1} B_i + \dim \hat{G}$$
$$< \sum_{i=1}^{\mu} \dim B_i + \dim \hat{G},$$

which, obviously, is a contradiction. The equality (1.19) now follows for $i \in \{1,2,\ldots\mu+1\}$ upon noting that $\dim(A + BF_{new})^{\mu} B_{\mu+1} = \dim B_{\mu+1} = \dim \hat{G}$ $= \dim S^{\mu+1} - \dim S^{\mu}$.

□

As an immediate consequence of the above theorem, we obtain that there exists $k \in \mathbb{N} \cup \{0\}$ with $k \leq \dim K$, such that $S^{\mu} = S^{k}$ for $\mu \geq k$. Moreover, we must have $S^{\mu} = S^{\dim K}$ for $\mu \geq \dim K$. In the sequel, denote

$$S^{\infty}: = S^{\dim K}.$$

Define a family $\underline{G}$ of subspaces $L \subset K$ according to

$$\underline{G}: = \{L \mid L = K \cap (AL + B)\}.$$

Then we can obtain the following characterization and decomposition of the 'limiting' subspace S^{∞}:

COROLLARY 1.11.

(i) *S^∞ is the unique element of $\underline{G}$ with the property that $S^\infty \subset L$ for every $L \in \underline{G}$.*

(ii) *There is a* $k \in \mathbb{N} \cup \{0\}$, $k \leq \dim K$, *a chain* $\{B_i\}_{i=1}^{k}$ *in* B *and a mapping* $F\colon X \longrightarrow U$ *such that*

$$S^\infty = B_1 \oplus A_F B_2 \oplus \ldots \oplus A_F^{k-1} B_k, \tag{1.23}$$

$$B_1 = B \cap K \tag{1.24}$$

and

$$\dim B_i = \dim A_F^{i-1} B_i = \dim S^i - \dim S^{i-1} \quad (i \in \underline{k}). \tag{1.25}$$

PROOF (i) This proof can be found in WONHAM (1979 §5.3). (ii) This follows immediately from TH. 1.10.

□

REMARK 1.12 It follows from the above corollary and from the construction in section 1.2 that for every subspace K the limiting subspace $S^\infty(K)$ is an almost controllability subspace. It will in fact turn out in section 1.6 that for every subspace $K \subset X$ the limiting subspace $S^\infty(K)$ is equal to $R_a^*(K)$, the supremal almost controllability subspace in K.

REMARK 1.13 It was proven in WONHAM (1979) that if $K \subset X$ and $V^* := V^*(K)$ is the supremal (A,B)-invariant subspace in K, then $S^\infty(V^*) = R^*(K)$, the supremal controllability subspace in K. The above corollary can thus be applied to obtain the existence of a chain $B_2 \supset B_3 \supset \ldots \supset B_k$ in $V^* \cap B$ such that $\dim B_i = \dim S^i(V^*) - \dim S^{i-1}(V^*)$ and $R^*(K) = (V^* \cap B) \oplus A_F B_2 \oplus \ldots \oplus A_F^{k-1} B_k$.

1.4 THE FACTOR SYSTEM MODULO S^∞

In this section we will proceed by studying a property of the subspace $S^\infty(K)$ which will be of crucial importance in the further development. First, let us introduce the following notation. As usual, let $\Sigma(A,B)$

denote the linear space of all state trajectories of the system with system mapping A, input mapping B and state space X, as defined by (1.8). For every subspace L of X, define a space of time functions with values in X/L by

$$\Sigma(A,B)/L := \{\xi\colon \mathbb{R} \longrightarrow X/L \mid \exists x \in \Sigma(A,B) \text{ such that } \xi(t) = [x(t)],\ \forall t\}. \tag{1.26}$$

Here, the element $[x] \in X/L$ will denote the equivalence class of vectors $x' \in X$ with the property that $x - x' \in L$. Thus, the space $\Sigma(A,B)/L$ is the space of time functions obtained by projecting each integral curve $x(\cdot)$ onto X/L. This section will be centered around the following questions: (i) if K is a subspace of X and if we take $L = S^{\infty}(K)$ in the above, does there exist an auxiliary system with system mapping $\overline{A}$, input mapping $\overline{B}$ and state space X/L such that each element of $\Sigma(A,B)/L$ is a trajectory of this auxiliary system? Equivalently: does the inclusion $\Sigma(A,B)/L \subset \Sigma(\overline{A},\overline{B})$ hold for some system $(\overline{A},\overline{B})$? (ii) Can we find such system $(\overline{A},\overline{B})$ such that, in addition, each trajectory of this system can be obtained as the projection of a trajectory of the original system (A,B)? Equivalently: does the converse inclusion $\Sigma(\overline{A},\overline{B}) \subset \Sigma(A,B)/L$ hold for this system $(\overline{A},\overline{B})$? It will turn out that the first question may be answered by: yes, we can indeed find such system $(\overline{A},\overline{B})$. It will also turn out that for this system $(\overline{A},\overline{B})$ the converse inclusion holds, provided that we restrict ourselves to smooth trajectories.

Before embarking on the details, let us illustrate the above problem by considering the case that the subspace S^{∞} happens to be controlled invariant. In this case the answers to the above questions are both: yes. This is easily seen by taking a mapping $F \in \underline{F}(S^{\infty})$, by taking for $\overline{A}$ the quotient mapping of $A + BF$ modulo S^{∞} and by defining $\overline{B} := PB$. (P is the canonical projection $X \longrightarrow X/S^{\infty}$). The latter construction however hinges on the fact that S^{∞} is A_F-invariant and can therefore only be applied if S^{∞} is controlled invariant. Since this will in general not be true, we have to take a different approach.

In the following, let $C^{\infty}(\mathbb{R},X)$ denote the space of all X-valued functions on $\mathbb{R}$ that have derivatives of arbitrary high order (henceforth simply called 'smooth'). Denote $\tilde{\Sigma}(A,B) := \Sigma(A,B) \cap C^{\infty}(\mathbb{R},X)$ and let $\tilde{\Sigma}(A,B)/L$ be defined by (1.26) with Σ replaced by $\tilde{\Sigma}$. Then the following important theorem provides an answer to the questions we have posed above:

<u>THEOREM 1.14</u>. *Let K be a subspace of X and denote $S^{\infty} := S^{\infty}(K)$. Then there exist a finite dimensional linear space $\overline{U}$ and mappings $\overline{A}: X/S^{\infty} \longrightarrow X/S^{\infty}$ and $\overline{B} : \overline{U} \longrightarrow X/S^{\infty}$ such that*

(i) $\Sigma(A,B)/S^{\infty} \subset \Sigma(\overline{A},\overline{B})$,

(ii) $\tilde{\Sigma}(A,B)/S^{\infty} = \tilde{\Sigma}(\overline{A},\overline{B})$,

and

(iii) $\mathrm{im}\,\overline{B} = (AS^{\infty} + B)/S^{\infty}$.

Before we turn to a proof of the above result, let us sketch the route we will take in this proof.
We will use the decomposition of S^{∞} as established in COR 1.11 to find a mapping F, a decomposition $X = X_1 \oplus X_2 \oplus X_3$ and a decomposition $U = U_1 \oplus U_2 \oplus U_3$ such that $X_1 \oplus X_2 = S^{\infty}$, $A_F X_1 \subset X_1 \oplus X_2$ and $BU_i \subset X_i$, $i = 1,2,3$. For the moment, assume that this is possible. If we write down the matrices of A_F and B while employing the above decompositions, we obtain

$$(1.27) \qquad A_F = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ 0 & A_{32} & A_{33} \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 & 0 & 0 \\ 0 & B_2 & 0 \\ 0 & 0 & B_3 \end{pmatrix}.$$

Now, if $x = (x_1^T, x_2^T, x_3^T)^T$ is a trajectory of the system (A,B), then the component of this trajectory in X_3 will satisfy the differential equation.

$$(1.28) \qquad \frac{d}{dt}\begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix} = A_0 \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix} + B_0 \begin{pmatrix} x_2 \\ u_3 \end{pmatrix}.$$

Here, we have written $A_0 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A_{33} \end{pmatrix}$ and $B_0 := \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ A_{32} & B_3 \end{pmatrix}$.

Moreover, there are x_1 and u_2 such that the x_2 appearing in (1.28) satisfies

$$(1.29) \qquad \dot{x}_2 = A_{22}x_2 + A_{21}x_1 + A_{23}x_3 + B_2u_2,$$

with x_1 satisfying

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + B_1u_1 \tag{1.30}$$

for some u_1. Let $P: X \longrightarrow X/S^{\infty}$ be the canonical projection and $Q: = P|X_3$ denote its restriction to X_3. Then Q defines an isomorphism between X_3 and X/S^{∞} (just note that $S^{\infty} = X_1 \oplus X_2$). We also have $[x(t)] = Q(0^T, 0^T, x_3^T(t))^T$, $\forall t$ and thus the differential equation (1.28) yields a differential equation for $\xi(t): = [x(t)]$ in the factor space X/S^{∞}, 'driven' by the input $\begin{pmatrix} x_2 \\ u_3 \end{pmatrix}$:

$$\dot{\xi} = \bar{A}\xi + \bar{B}\begin{pmatrix} x_2 \\ u_3 \end{pmatrix}, \tag{1.31}$$

where $\bar{A}: = QA_0Q^{-1}$ and $\bar{B} = QB_0$. Hence, we immediately obtain (i) of the theorem we want to prove.

The difficult part of the proof is however to prove (ii). Let $\xi \in \tilde{\Sigma}(\bar{A},\bar{B})$ and let $(0^T, 0^T, x_3^T(t))^T: = Q^{-1}\xi(t)$. Since ξ satisfies (1.31) for certain smooth functions x_2 and u_3, we may see that the vector $(0^T, 0^T, x_3^T(t))^T$ satisfies (1.28). Thus, the question: does there exist a smooth trajectory x for the system (A,B) such that $[x(t)] = \xi(t)$ $\forall t$, may be formulated as: given x_2 and x_3, do there exist smooth u_1, u_2 and x_1 such that the equations (1.29) and (1.30) are satisfied. Indeed, if these exist, we may take $x(t): = (x_1^T(t), x_2^T(t), x_3^T(t))^T$ and conclude that $x(\cdot) \in \tilde{\Sigma}(A,B)$ and that $[x(t)] = \xi(t)$. In the proof of TH. 1.14 that we will give in the sequel, it will indeed be shown that X_1 and X_2 can be taken in such a way that, for given smooth x_2 and u_3, the dynamic contraints (1.29) and (1.30) yield unique smooth solutions x_1, u_1 and u_2.

In the following, if we have a decomposition $X = X_1 \oplus X_2 \oplus X_3$, then π_i will denote the projection onto X_i along $\bigoplus_{j \neq i} X_j$. Also, we will denote by I_r and $0_{r \times s}$ the $r \times r$ identity matrix and the $r \times s$ zero matrix respectively. We will first prove the following:

<u>LEMMA 1.15.</u> *Let K be a subspace of X. There are subspaces X_1, X_2 and X_3 of X and U_1, U_2 and U_3 of U, a linear mapping $F: X \longrightarrow U$, an integer $k \leq \dim K$ and integers $r_0, r_1, \ldots, r_{k-1}$ such that*

(i) $S^{\infty}(K) = X_1 \oplus X_2$,

(ii) $X = X_1 \oplus X_2 \oplus X_3$,

(iii) $A_F X_1 \subset X_1 \oplus X_2$,

(iv) $BU_i \subset X_i \quad (i = 1,2,3)$,

(v) *the matrix of* $\pi_1 A_F | X_1$ is $A_{11} := \mathrm{diag}\ (N_1, \ldots, N_{k-1})$, *where* $N_1 := 0_{r_1}$ *and*

$$N_i := \begin{pmatrix} 0_{r_i} & \cdots & & 0_{r_i} \\ I_{r_i} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0_{r_i} & \cdots & I_{r_i} & 0_{r_i} \end{pmatrix} \quad , \quad i \geq 2.$$

Here, I_{r_i} *appears* $i-1$ *times.*

(vi) *the matrix of* $\pi_1 B | U_1$ *is* $B_1 := \mathrm{diag}\ (M_1, \ldots, M_{k-1})$, *where* $M_1 := I_{r_1}$ *and*

$$M_i := \begin{pmatrix} I_{r_i} \\ 0_{r_i} \\ \vdots \\ 0_{r_i} \end{pmatrix} , \quad i \geq 2.$$

Here, the block 0_{r_i} *appears* $i-1$ *times.*

(vii) *the matrix of* $\pi_2 A_F | X_1$ *is*

$$A_{21} := \begin{pmatrix} 0 \\ \mathrm{diag}\ (L_1, \ldots, L_{k-1}) \end{pmatrix} ,$$

where $L_1 := I_{r_i}$, *and* $L_i := (0_{r_i}, \ldots, 0_{r_i} I_{r_i})$, $i \geq 2$. *Here, the block* 0_{r_i} *appears* $i-1$ *times. The zero block appearing in* A_{21} *has* r_0 *rows.*

(viii) *the matrix of* $\pi_2 B|U_2$ *is* $B_2 := \begin{pmatrix} I_{r_o} \\ 0_{r_1 \times r_o} \\ \vdots \\ 0_{r_{k-1} \times r_o} \end{pmatrix}$.

PROOF. By COR 1.11, there exists an integer k, a chain $\{B_i\}_{i=1}^k$ in B and a mapping F: $X \longrightarrow U$ such that (1.23), (1.24) and (1.25) hold. For $i \in \{2,3,\dots,k\}$, let B_i' be subspaces such that $B_i' \oplus B_i = B_{i-1}$. Define

(1.32) $\qquad X_1 := B_2 \oplus A_F B_3 \oplus \dots \oplus A_F^{k-2} B_k.$

Then we have $A_F X_1 \subset S$. Define a subspace X_2 by

(1.33) $\qquad X_2 := B_2' \oplus A_F B_3' \oplus \dots \oplus A_F^{k-2} B_k' \oplus A_F^{k-1} B_k.$

We then obtain $S^\infty = X_1 + X_2$ and it is claimed that the latter is, in fact, a direct sum. To prove this, it suffices to show that $\dim X_1 + \dim X_2 = \dim S^\infty$.

Now, we have $\dim X_1 = \sum_{i=2}^{k} \dim A_F^{i-2} B_i$, which, by (1.25) and the fact that $B_i \subset B_{i-1}$ must equal $\sum_{i=2}^{k} \dim B_i$. In the same way, $\dim X_2 = \sum_{i=2}^{k} \dim B_i' + \dim B_k$.

It therefore follows that $\dim X_1 + \dim X_2 = \sum_{i=1}^{k} \dim B_i$, which again by (1.23) and (1.25) equals $\dim S$.

Since $B_1 = B \cap K$, we also have $B_1 = B \cap S^\infty$. Let B_1' be such that $B_1 \oplus B_1' = B$. Note that $B_1' \cap S^\infty = \{0\}$. Choose a subspace $X_3 \subset X$ such that $B_1' \subset X_3$ and $S^\infty \oplus X_3 = X$. Since also $S^\infty = X_1 \oplus X_2$, this yields a decomposition $X = X_1 \oplus X_2 \oplus X_3$. Moreover, $B = B_2 \oplus B_2' \oplus B_1'$, with $B_2 \subset X_1$, $B_2' \subset X_2$ and $B_1' \subset X$. Define $U_1 := B^{-1} B_2$, $U_2 := B^{-1} B_2'$ and $U_3 := B^{-1} B_1'$. Then it may be verified that $U = U_1 \oplus U_2 \oplus U_3$ (recall that we assume throughout that B is injective).

Until so far, we have proven the formulas (i) to (iv) of the lemma. To prove the assertions (v) to (viii), note that it follows from the definition of the subspaces B_i' that

$$(1.34) \qquad B = B_k \oplus \bigoplus_{i=1}^{k} B_i',$$

where for $\ell \in \{1,2,\ldots,k-1\}$, $B_\ell = B_k \oplus \bigoplus_{i=\ell+1}^{k} B_i'$. Now, it may be verified that this yields through (1.32) a decomposition of X_1 into

$$(1.35) \qquad \begin{aligned} X_1 &= B_3' \oplus \{B_4' \oplus A_F B_4'\} \oplus \ldots \\ &\ldots \oplus \{\bigoplus_{i=0}^{k-3} A_F^i B_k'\} \oplus \{\bigoplus_{i=0}^{k-2} A_F^i B_{k+1}'\} \end{aligned}$$

(denote $B_{k+1}' := B_k$). Define $r_i := \dim B_{i+2}'$ $(i = 0,1,\ldots,k-1)$. The claims (v) and (vii) of the lemma then follow by writing down the matrices of $\pi_1 A_F | X_1$ and $\pi_2 A_F | X_1$ in the decompositions (1.33) and (1.35). Finally, note that $B_2 = B_3' \oplus \ldots \oplus B_{k+1}'$. Since $U_1 = B^{-1} B_2$ this yields a decomposition $U_1 = U_{1,3} \oplus \ldots \oplus U_{1,k+1}$ with $BU_{1,i} = B_i'$. Moreover, since $BU_2 = B_2'$, the claims (vi) and (viii) follow by writing down the matrices of $\pi_1 B | U_1$ and $\pi_2 B | U_2$ in the decompositions employed.

□

<u>PROOF OF TH. 1.14.</u> Let X and U be decomposed as in the above lemma. Define a mapping $A_0 : X_3 \longrightarrow X_3$ by $A_0 x := \pi_3 A_F x$ and a mapping $B_0 : X_2 \times U_3 \longrightarrow X_3$ by $B_0(x,u) = \pi_3(A_F x + Bu)$. As before, let $Q: X_3 \longrightarrow X/S^\infty$ denote the restriction of the canonical projection P to X_3. Define $\bar{A} := QA_0Q^{-1}$, $\bar{U} := X_2 \times U_3$ and $\bar{B} := QB_0$. It was already shown that $\Sigma(A,B)/S^\infty \subset \Sigma(\bar{A},\bar{B})$.

$$\begin{tikzcd} & X/S^\infty \arrow[r, "\bar{A}"] & X/S^\infty \\ \bar{U} \arrow[ur, "\bar{B}"] \arrow[dr, "B_0"'] & & \\ & X_3 \arrow[uu, "\cong Q"'] \arrow[r, "A_0"'] & X_3 \arrow[uu, "\cong Q"'] \end{tikzcd}$$

We will proceed by proving (iii). Let $(x,u) \in X_2 \times U_3$. Then we have $\overline{B}(x,u) = Q\pi_3(A_F x + Bu) = P(A_F x + Bu)$. Since $x \in X_2 \subset S^\infty$, the latter is contained in $(A_F S^\infty + B)/S^\infty$. Conversely, let $\xi \in (A_F S^\infty + B)/S^\infty$. Then there is $x \in S^\infty$ and $u \in U$ such that $\xi = P(A_F x + Bu)$. There are unique representations $x = x_1 \oplus x_2$ and $u = u_1 \oplus u_2 \oplus u_3$. It may be verified immediately that $\xi = Q\pi_3(A_F x_2 + Bu_3)$, which is equal to $\overline{B}(x_2,u_3) \in \text{im } \overline{B}$.

Finally, we will prove (ii). Note that the inclusion $\widetilde{\Sigma}(A,B)/S^\infty \subset \widetilde{\Sigma}(\overline{A},\overline{B})$ is immediate from the above considerations. We will prove the converse inclusion. Let $\xi(\cdot) \in \widetilde{\Sigma}(\overline{A},\overline{B})$ and let $(0^T,0^T,x_3(t)^T)^T = Q^{-1}\xi(t)$, $\forall t$. It was already noted that this vector satisfies the equation (1.28) for certain smooth functions $x_2(\cdot)$ and $u_3(\cdot)$. The question is: are there smooth functions $u_1(\cdot)$, $u_2(\cdot)$ and $x_1(\cdot)$ such that

$$(1.36) \qquad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} A_{13} \\ A_{23} \end{pmatrix} x_3 \ ?$$

Here, we may assume that the matrices A_{11}, A_{21}, B_1 and B_2 are as in LEMMA 1.15, (v) to (viii). Now, using the special structure of these matrices, it may be verified by inspection that the dynamic constraint (1.36) yields for given x_2 and x_3 a unique (smooth) solution (u_1,u_2,x_1). (This follows simply by writing down (1.36) in components for x_1 and x_2 compatible with the decomposition in LEMMA 1.15, (v) to (viii)).

Take now $x(t) := (x_1^T(t),x_2^T(t),x_3^T(t))^T$. Then $x \in \widetilde{\Sigma}(A,B)$ and $[x(t)] = Q((0^T,0^T,x_3^T(t))^T) = \xi(t)$, $\forall t$. This completes the proof of TH. 1.14.

□

<u>REMARK 1.16</u>. The system $(\overline{A},\overline{B})$ as defined above will in the following be referred to as *the factor system modulo* S^∞. Several of its properties will turn out to play an important role in obtaining geometric characterizations of almost invariant subspaces.

<u>REMARK 1.17</u>. The fact that for given x_2 and x_3 the equation 1.36 yields a unique solution (u_1,u_2,x_1) is connected with the fact that the system $(\widetilde{A},\widetilde{B},\widetilde{C})$ with $\widetilde{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, $\widetilde{B} = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$ and $\widetilde{C} = (0 \ I)$ is an invertible

system. Indeed, by the special structure of A_{11}, A_{21}, B_1 and B_2, it may be verified that the system matrix

$$M(s) = \begin{pmatrix} Is-\tilde{A} & \tilde{B} \\ \tilde{C} & 0 \end{pmatrix}$$

has full rank for every $s \in \mathbb{C}$. Thus it may be concluded that the transfer matrix $\tilde{C}(Is-\tilde{A})^{-1}\tilde{B}$ has an inverse as a matrix over the field of rational functions (see KAILATH (1980, ex. 6.5.13)).

It will turn out that the structure of the matrices A_{11}, A_{21}, B_1 and B_2 may be exploited to derive even more relationships between the trajectories of (A,B) and those of $(\bar{A},\bar{B})$. As an example, let $\tilde{\Sigma}^s(A,B)$ denote the subspace of elements in $\tilde{\Sigma}(A,B)$ that have compact support. It may then in addition to TH. 1.14 (ii) be seen that $\tilde{\Sigma}^s(A,B)/S^\infty = \tilde{\Sigma}^s(\bar{A},\bar{B})$. (See also CH.2, SECTION 2.7 and 2.8).

1.5 GEOMETRIC CHARACTERIZATIONS OF ALMOST INVARIANCE

In SECTION 1.2 we proved that subspaces of the form (1.15) are almost controllability subspaces and that subspaces of the form $V+R_a$ with $V \in \underline{V}$ and $R_a \in \underline{R}_a$ are almost controlled invariant. In the present section, we will employ the mathematical framework set up in the preceding two sections to show that, conversely, every almost controllability subspace is of the form (1.15) and that every almost controlled invariant subspace is the sum of a controlled invariant subspace and an almost controllability subspace. As usual, assume that we have a linear system with system mapping A and input mapping B. Assume that K is a subspace of X and let $R_a \in \underline{R}_a$. In the sequel, we will first consider the question: is $R_a/S^\infty(K)$ an almost controllability subspace with respect to the factor system modulo $S^\infty(K)$? In a similar way, we will ask ourselves the question: if $V_a \in \underline{V}_a(A,B)$, is $V_a/S^\infty(K)$ an almost controlled invariant subspace with respect to the factor system modulo $S^\infty(K)$? Of course, the same question can be posed for $V \in \underline{V}(A,B)$.

In order to be able to speak about almost invariance in the quotient space X/S^∞, we should endow this space with a norm. If L is a subspace of X, define a norm on X/L by

$$\text{(1.37)} \qquad \|\xi-\xi'\|_m := \inf_{x\in\xi, x'\in\xi'} \|x-x'\|, \quad (\xi,\xi'\in X/L).$$

Here, of course $\|\cdot\|$ denotes the norm on X. In the following, for any subspace $L\subset X$, define

$$\underline{V}_a/L := \{Z\subset X/L \mid \exists V_a\in\underline{V}_a \text{ such that } Z=V_a/L\}$$

and let $\underline{R}_a/L$, $\underline{V}/L$, etc. be defined in an analogous way. We will now prove the following result:

LEMMA 1.18. *Let K be a subspace of X. Denote $S^\infty := S^\infty(K)$ and let $(\overline{A},\overline{B})$ denote the factor system modulo S^∞. Then:*

(i) $\underline{V}_a/S^\infty \subset \underline{V}_a(\overline{A},\overline{B})$,

(ii) $\underline{R}_a/S^\infty \subset \underline{R}_a(\overline{A},\overline{B})$,

(iii) $\underline{V}/S^\infty = \underline{V}(\overline{A},\overline{B})$.

PROOF: (i) Let $V_a\in\underline{V}_a$. Assume that $\xi_0\in V_a/S^\infty$. There is a $x_0\in V_a$ such that $[x_0]=\xi_0$. Let $\varepsilon>0$. There is a trajectory $x(\cdot)\in\Sigma(A,B)$ with the properties that $x(0)=x_0$ and $d(x(t),V_a)\le\varepsilon$ $\forall t$. Define $\xi(t):=[x(t)]$. By TH. 1.14, $\xi(\cdot)\in\Sigma(\overline{A},\overline{B})$. Moreover, $\xi(0)=[x(0)]=\xi_0$. We will show that $d(\xi(t),V_a/S^\infty)\le\varepsilon$ $\forall t$. To see this, first assume that $x'\in V_a$. It follows immediately from (1.37) that

$$\text{(1.38)} \qquad \begin{aligned} d(\xi(t),V_a/S^\infty) &\le \|\xi(t)-[x']\|_m \\ &\le \inf_{x\in[x']} \|x(t)-x\| \le \|x(t)-x'\| \end{aligned}$$

However, (1.38) holds for all $x'\in V_a$. Thus, we obtain

$$d(\xi(t),V_a/S^\infty) \le \inf_{x'\in V_a} \|x(t)-x'\| = d(x(t),V_a) \le \varepsilon.$$

(ii) The proof of this assertion is entirely analogous to the above proof and will be omitted.

(iii)The inclusion $\underline{V}/S^{\infty} \subset \underline{V}(\overline{A},\overline{B})$ may be proved in an analogous fashion as above. We will prove the converse inclusion. Let $\overline{V} \in \underline{V}(\overline{A},\overline{B})$. It is required to find an (A,B)-invariant subspace V such that $\overline{V} = V/S^{\infty}$. Let V_1 be an arbitrary subspace with the properties that $V_1/S^{\infty} = \overline{V}$ and $S^{\infty} \subset V_1$. We assert that $\overline{V} = V^*(V_1)/S^{\infty}$. To prove this, let $\xi_0 \in \overline{V}$. There is a feedback mapping $\overline{F}: X/S^{\infty} \longrightarrow \overline{U}$ such that $\xi(t) := e^{(\overline{A}+\overline{B}\,\overline{F})t}\xi_0 \in \overline{V}, \forall t$. Note that $\xi(\cdot) \in \tilde{\Sigma}(\overline{A},\overline{B})$. By TH 1.14 (ii), there is a (smooth) trajectory $x(\cdot)$ of the system (A,B) such that $[x(t)] = \xi(t), \forall t$. Obviously, $x(t) \in V_1 + S^{\infty} = V_1$, $\forall t$ and consequently, $x(t) \in V^*(V_1)$, $\forall t$. It follows that $x(0) \in V^*(V_1)$ and thus $\xi_0 = [x(0)] \in V^*(V_1)/S^{\infty}$. The converse inclusion is a triviality.

□

<u>REMARK 1.19.</u> In LEMMA 1.18 (i) and (ii) it is also possible to prove the converse inclusions. A proof of this would however take us too far at the moment and will, since we do not use the result here, be omitted.

In order to prove that every almost controllability subspace has a representation of the form (1.15), we will prove the following useful result:

<u>LEMMA 1.20.</u> *Let K be a subspace of X such that $K \cap B = \{0\}$. Define:*

$$N(K) := \{x_0 \in K \mid \exists T > 0 \text{ and } \forall \varepsilon > 0\ \exists x \in \Sigma(A,B) \text{ such that } x(0) = x_0,\ x(T) = 0 \text{ and } d(x(t),K) \le \varepsilon,\ \forall t \ge 0\}.$$

Then $N(K) = \{0\}$.

<u>PROOF</u>: Define $X_1 := B$ and let X_2 be a subspace of X such that $K \subset X_2$ and $X_1 \oplus X_2 = X$. In this decomposition, let the matrix of A be given by $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$. Assume $N(K) \neq \{0\}$. Then there is $x_0 = \begin{pmatrix} 0 \\ x_{02} \end{pmatrix} \in K$, $x_{02} \neq 0$, with the property that there is $T > 0$ and for all $\varepsilon > 0$ a $x \in \Sigma(A,B)$ with $x(0) = x_0$, $x(T) = 0$ and $d(x(t),K) \le \varepsilon$, $\forall t \ge 0$. Define a real number $C \ge 0$ by

$$C := \int_0^T \| e^{A_{22}(T-t)} A_{21} \| \, dt.$$

Let $K \in \mathbb{R}$ be such that $C+K>0$. Take $\varepsilon = \frac{1}{2(C+K)} \| e^{A_{22}T} x_{02} \|$ and let $x \in \Sigma(A,B)$ be such that $x(0)=x_0$, $x(T)=0$ and $d(x(t),K) \le \varepsilon$, $\forall t \ge 0$.

In the decomposition employed, let $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$. Obviously, $\| x_1(t) \| \le \varepsilon$, $\forall t \ge 0$. Moreover, $x_2(t)$ satisfies $\dot{x}_2 = A_{22}x_2 + A_{21}x_1$ (note that $B \cap K = \{0\}$). Thus, we have

$$(1.39) \qquad 0 = x_2(T) = e^{A_{22}T} x_{02} + \int_0^T e^{A_{22}(T-t)} A_{21}x_1(t)dt.$$

Also, $\| \int_0^T e^{A_{22}(T-t)} A_{21}x_1(t)dt \| \le \varepsilon C \le \frac{C}{2(C+K)} \| e^{A_{22}T} x_{02} \| \le \frac{1}{2} \| e^{A_{22}T} x_{02} \|$.

and therefore, by (1.39), $\| x_2(T) \| \ge \frac{1}{2} \| e^{A_{22}T} x_{02} \| > 0$. This however contradicts (1.39). The assumption $N(K) \ne \{0\}$ must consequently be false.

□

The above lemma immediately yields the following:

LEMMA 1.21. *$R_a \in \underline{R}_a(A,B)$ and $R_a \cap B = \{0\} \Rightarrow R_a = \{0\}$.*

PROOF: If $R_a \in \underline{R}_a$ then, by definition, $R_a \subset N(R_a)$.

□

Putting things together now, we are in a position to prove the following result, which states that a subspace of X is an almost controllability subspace if and only if it is a fixed point of the mapping that assigns to each subspace K of X the subspace $S^\infty(K)$ as defined by the algorithm (1.16):

THEOREM 1.22. $R_a \in \underline{R}_a(A,B) \Leftrightarrow R_a = S^\infty(R_a)$.

PROOF: ($\Leftarrow$) If $R_a = S^\infty$, then by COR. 1.11, $R_a = B_1 \oplus \ldots \oplus A_F^{k-1} B_k$. Consequently, R_a is an almost controllability subspace. ($\Rightarrow$) If $R_a \in \underline{R}_a$, then $\overline{R}_a := R_a / S^\infty(R_a)$ is an almost controllability subspace with respect to the factor system $(\overline{A},\overline{B})$ modulo $S^\infty(R_a)$. By TH. 1.14 (iii) we have

$$\overline{R}_a \cap \mathrm{im}\ \overline{B} = (R_a/S^\infty) \cap ((AS^\infty + B)/S^\infty)$$
$$= (R_a \cap (AS^\infty + B))/S^\infty.$$

By COR. 1.11 however, this must be equal to S^∞/S^∞, which is equal to the zero subspace of X/S^∞. By applying LEMMA 1.21, we therefore obtain $\overline{R}_a = \{[0]\}$. It then immediately follows that $R_a = S^\infty(R_a)$.

□

As a direct consequence of the above, we arrive at the following corollary, which gives a characterization of almost controllability subspaces in terms of state feedback and chains of subspaces contained in the image of the input mapping:

COROLLARY 1.23. *A subspace R_a of X is an almost controllability subspace if and only if there is a mapping* $F: X \longrightarrow U$ *and a chain* $\{B_i\}_{i=1}^k$ *in* B *such that* $R_a = B_1 + A_F B_2 + \ldots + A_F^{k-1} B_k$. *Moreover, if this is the case then there is a* $k \in \mathbb{N} \cup \{0\}$, $k \leq \dim R_a$, *a chain* $\{B_i\}_{i=1}^k$ *in* B *and a mapping* $F: X \longrightarrow U$ *such that*

$$(1.40) \qquad R_a = B_1 \oplus A_F B_2 \oplus \ldots \oplus A_F^{k-1} B_k,$$

$$(1.41) \qquad B_1 = R_a \cap B$$

$$(1.42) \qquad \dim B_i = \dim A_F^{i-1} B_i = \dim S^i - \dim S^{i-1}, (i \in \underline{k}).$$

Here, we have denoted $S^i := S^i(R_a)$.

PROOF: This follows immediately from TH 1.22 and COR. 1.11.

□

For another interesting result on the representation of almost controllability subspaces, we refer to MALABRE (1983) (see also SCHUMACHER (1983b)).

Using LEMMA 1.20, we may now also establish a dynamic characterization of $R_a^*(K)$, the supremal almost controllability subspace contained in a given subspace K of X, *purely in terms of distance to the subspace K for* *points of time* $t \geq 0$. By definition, it is immediate that $R^*_a(K) \subset N(R^*_a(K))$. We may however even prove that $R^*_a(K) = N(K)$:

THEOREM 1.24. *Let K be a subspace of X. Then*

$$R^*_a(K) = \{x_0 \in K \mid \exists T > 0 \text{ and } \forall \varepsilon > 0\ \exists x \in \Sigma(A,B) \text{ such that } x(0) = x_0,\ x(T) = 0 \text{ and } d(x(t),K) \leq \varepsilon,\quad t \geq 0\}.$$

PROOF: The inclusion $R^*_a(K) \subset N(K)$ is trivial. For the case that $K \cap B = \{0\}$, the converse inclusion follows from LEMMA 1.20. Consider now the general case. Let $x_0 \in N(K)$. Define $\xi_0 = [x_0]$, the equivalence class modulo $S^\infty(K)$. Let $\varepsilon > 0$. There is $T > 0$ and $x \in \Sigma(A,B)$ such that $x(0) = x_0$, $x(T) = 0$ and $d(x(t),K) \leq \varepsilon, t \geq 0$. Define $\xi(t) := [x(t)]$. By TH. 1.14, $\xi \in \Sigma(\overline{A},\overline{B})$. Moreover, $\xi(0) = \xi_0$, $\xi(T) = 0$ and, in the same way as in (1.38), $d(x(t),K/S^\infty) \leq \varepsilon$, $\forall t \geq 0$. Thus, ξ_0 is an element of the space $N(K/S^\infty)$, associated with the factor system $(\overline{A},\overline{B})$. Since $\operatorname{im} \overline{B} \cap K/S^\infty = \{[0]\}$ (see the proof of TH. 1.22), it follows from LEMMA 1.20 that $\xi_0 = [0]$. Hence, $x_0 \in S^\infty(K)$. Since $S^\infty(K) \in \underline{R}_a$ and contained in K, it follows that $x_0 \in R^*_a(K)$.

□

To obtain a geometric characterization of the class of almost controlled invariant subspaces, we will proceed in a way analogous to the development above. The following lemma is the analogue of LEMMA 1.20:

LEMMA 1.25. *Let K be a subspace of X such that* $K \cap B = \{0\}$ *Define:*

$$M(K) := \{x_0 \in K \mid \forall \varepsilon > 0\ \exists x \in \Sigma(A,B) \text{ such that } x(0) = x_0 \text{ and } d(x(t),K) \leq \varepsilon,\ \forall t \geq 0\}.$$

Then $M(K) = V^*(K)$.

PROOF: Define $X_1 := K$, $X_3 := B$ and let X_2 be a subspace of X such that $X = X_1 \oplus X_2 \oplus X_3$. There is a mapping $F\colon X \longrightarrow U$ such that the matrices of A_F and B in the decomposition employed are given by

$$A_F = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ B_3 \end{pmatrix}.$$

Here, B_3 is a nonsingular matrix. Now, let $x_0 \in M(K)$. In the above decomposition, let $x_0 = (x_{01}^T, 0^T, 0^T)^T$. By definition of $M(K)$ we have that for all $\varepsilon > 0$ there is $x \in \Sigma(A,B)$ such that $x(0) = x_0$ and $d(x(t),K) \leq \varepsilon$, $\forall t \geq 0$. Thus, there is a sequence of trajectories $x_n(t) = (x_{1n}^T(t), x_{2n}^T(t), x_{3n}^T(t))^T$ with $x_{1n}(0) = x_{01}$, $x_{2n}(0) = 0$ $x_{3n}(0) = 0$ and $x_{2n}(t) \longrightarrow 0$, $x_{3n}(t) \longrightarrow 0$ $(n \longrightarrow \infty)$ uniformly on $[0,\infty)$. Since

$$x_{1n}(t) = e^{A_{11}t} x_{01} + \int_0^t e^{A_{11}(t-\tau)} [A_{12}x_{2n}(\tau) + A_{13}x_{3n}(\tau)]d\tau,$$

it follows that $x_{1n}(t) \longrightarrow e^{A_{11}t} x_{01} =: x_1(t)$ uniformly on $[0,T]$ for each $T > 0$. Define $x(t) := (x_1^T(t), 0^T, 0^T)^T$. We contend that $\dot{x}(t) = A_F x(t)$ for $t \geq 0$. To prove this, note that

$$(1.43) \qquad x_n(t) = A_F \int_0^t x_n(\tau)d\tau + B \int_0^t u_n(\tau)d\tau + x_0, \ \forall t \in \mathbb{R},$$

for some input function $u_n(\cdot)$. Since $\int_0^t u_n(\tau)d\tau = B_3^{-1}x_{3n}(t)$, we have that $\int_0^t u_n(\tau)d\tau \longrightarrow 0$, $\forall t \geq 0$. It then follows by letting $n \longrightarrow \infty$ in (1.43), that

$$x(t) = A_F \int_0^t x(\tau)d\tau + x_0, \quad t \geq 0.$$

Thus, we may conclude that for $t \geq 0, \dot{x}(t) = A_F x(t)$, that $x(0) = x_0$ and, a fortiori, that $x(t) \in K$ $\forall t \geq 0$. It follows that, in fact $x(t) \in V^*(K)$, $\forall t \geq 0$. (see e.g. HAUTUS (1980)) and thus that $x_0 \in V^*(K)$. The converse inclusion is trivial □

We now directly obtain the following analogue of LEMMA 1.21:

<u>LEMMA 1.26.</u> $V_a \in \underline{V}_a(A,B)$ *and* $V_a \cap B = \{0\} \Rightarrow V_a \in \underline{V}(A,B)$.

□

<u>PROOF</u>: If $V_a \in \underline{V}_a$, then by definition, $V_a \in M(V_a)$. Hence, if $V_a \cap B = \{0\}$, then $V_a = V^*(V_a) \in \underline{V}$.

The following characterization of the class of almost controlled invariant subspaces may now be proven:

<u>THEOREM 1.27.</u> $V_a \in \underline{V}_a(A,B) \Leftrightarrow V_a = V^*(V_a) + S^\infty(V_a)$.

<u>PROOF</u>: ($\Leftarrow$) Is immediately clear from section 1.2 and COR. 1.11. ($\Rightarrow$). Suppose that $V_a \in \underline{V}_a$. Denote $S^\infty := S^\infty(V_a)$. It follows from LEMMA 1.18 (i) that $\overline{V}_a := V_a/S^\infty$ is almost controlled invariant with respect to the factor system modulo S^∞. Moreover, by TH. 1.14 (iii),

$$\overline{V}_a \cap \operatorname{im} \overline{B} = (V_a \cap (AS^\infty + B))/S^\infty.$$

By COR. 1.11 (i), this must however be equal to the zero subspace in X/S^∞. Therefore, by LEMMA 1.26, we find that $\overline{V}_a \in \underline{V}(\overline{A},\overline{B})$. Apply now LEMMA 1.18 (iii) to find a subspace $V \in \underline{V}(A,B)$ such that $\overline{V}_a = V/S^\infty$. This yields $V_a = V + S^\infty$. To complete the proof, note that $V \subset V^*(V_a)$ and thus that $V_a \subset V^*(V_a) + S^\infty$. The converse inclusion is a triviality.

□

We may immediately apply the above result to obtain the following:

<u>COROLLARY 1.28.</u> *A subspace V_a of X is an almost controlled invariant subspace if and only if there is a controlled invariant subspace V and an almost controllability subspace R_a such that $V_a = V + R_a$.*

<u>PROOF</u>: ($\Rightarrow$) This follows from TH. 1.27, together with the fact that $S^\infty(V_a) \in \underline{R}_a$. ($\Leftarrow$) This was already shown in SECTION 1.2.

□

Finally, we will give a dynamical characterization of the supremal almost controlled invariant subspace $V_a^*(K)$ contained in a given subspace K of X. Again, it follows directly from the definition that $V_a^*(K) \subset M(K)$.

However, the converse inclusion is also valid:

THEOREM 1.29. *Let K be a subspace of X. Then*

$$V_a^*(K) = \{x_0 \in K \mid \forall \varepsilon > 0\ \exists x \in \Sigma(A,B) \text{ such that } x(0) = x_0 \text{ and } d(x(t),K) \leq \varepsilon,\ \forall t \geq 0\}.$$

PROOF: The inclusion $V_a^*(K) \subset M(K)$ is trivial. Conversely, if $K \cap B = \{0\}$, then $M(K) = V^*(K) \subset V_a^*(K)$. Consider now the general situation. Let $x_0 \in M(K)$ and define $\xi_0 = [x_0]$. By an argument similar to the proof of TH. 1.24, we find that $\xi_0 \in M(K/S^\infty)$ (w.r.t. the factor system $(\bar{A},\bar{B})$ modulo $S^\infty(K)$). Since $\operatorname{im} \bar{B} \cap K/S^\infty = \{[0]\}$, we find that $M(K/S^\infty) = V^*(K/S^\infty)$. Now, we claim that the latter equals $V^*(K)/S^\infty$. This may be seen as follows. First, by LEMMA 1.18, $V^*(K/S^\infty) = V/S^\infty$ for some $V \in \underline{V}$. Of course, $V/S^\infty \subset K/S^\infty$ and hence $V \subset K$ (note that $S^\infty(K) \subset K$). Thus, $V \subset V^*(K)$. Conversely, $V^*(K)/S^\infty \subset K/S^\infty$. Thus, by LEMMA 1.18, $V^*(K)/S^\infty \subset V^*(K/S^\infty)$. We may conclude that $[x_0] \in V^*(K)/S^\infty$ and hence that $x_0 \in V^*(K) + S^\infty$. This is however contained in $V_a^*(K)$.

□

1.6 COMPUTATION OF SUPREMAL ALMOST INVARIANT SUBSPACES

The aim of this section is to study algorithms that can be used to compute the supremal almost controlled invariant and almost controllability subspaces whose existence we established in this chapter. It turns out that only two recursive algorithms are needed to compute these subspaces. In the following, let K be an arbitrary but fixed subspace of X. Consider the following algorithms:

(1.44) $\qquad V^0 = X \ ; \ V^{\mu+1} = K \cap A^{-1}(V^\mu + B),$

(1.45) $\qquad S^0 = \{0\} \ ; \ S^{\mu+1} = K \cap (AS^\mu + B).$

Both algorithms already appear in WONHAM (1979), where they were shown to be relevant for the computation of $V^*(K)$ and $R^*(K)$. The algorithm (1.45) has already been studied in SECTION 1.3 of this tract. The connection between almost controlled invariant subspaces and these algorithms was laid in WILLEMS (1980). We will call (1.44) *the invariant subspace algorithm* or ISA and (1.45) *the almost controllability subspace algorithm* or ACSA. The following results summarize the properties of the sequence V^μ as generated by (1.44):

PROPOSITION 1.30.

(i) *The sequence* $\{V^\mu\}_{\mu=0}$ *is monotonically nonincreasing. Moreover, if* $V^\mu = V^{\mu+1}$, *then* $V^\mu = V^{\mu+\nu}$ *for all* $\nu \in \mathbb{N}$.

(ii) *There is* $k \in \mathbb{N} \cup \{0\}$, $k \leq \dim K + 1$, *such that* $V^k = V^{k+\nu}$ *for all* $\nu \in \mathbb{N}$.

PROOF: For a proof, we refer to WONHAM (1979, TH. 4.3).

□

In the following, let $V^\infty(K) := V^{\dim K+1}$. By the above result we then have $V^\infty(K) = V^\mu$ for all $\mu \geq \dim K + 1$.

Properties of (1.45) were already established in SECTION 1.3 and we refer to TH. 1.10 and COR. 1.11. Here, we want to prove one more result concerning the sequence of subspaces generated by ACSA. In the sequel, let $k := \dim K$. For each $\mu \in \mathbb{N}$, define a family of subspaces of X by

$$\underline{H}(\mu) := \{L \subset K \mid \exists F: X \longrightarrow U \text{ and a chain } \{B_i\}_{i=1}^{\mu} \text{ in } B \text{ such that } L = B_1 + A_F B_2 + \ldots + A_F^{\mu-1} B_\mu\}.$$

Let

$$\underline{H} := \bigcup_{\mu=1}^{\infty} \underline{H}(\mu).$$

Then we have the following:

LEMMA 1.31.

(i) *For each* $\mu \in \mathbb{N}$, S^μ *is the unique element of* $\underline{H}(\mu)$ *with the property that* $L \subset S^\mu$ *for every* $L \in \underline{H}(\mu)$.

(ii) S^∞ *is the unique element of* $\underline{H}$ *with the property that* $L \subset S^\infty$ *for every* $L \in \underline{H}$.

PROOF (i) The proof is by induction. For $\mu = 1$, the statement is clearly true. Suppose now it is true for μ. By TH. 1.10, $S^{\mu+1} \in \underline{H}(\mu+1)$. Let $L \in \underline{H}(\mu+1)$, say $L = B_1 + A_F B_2 + \ldots + A_F^\mu B_{\mu+1}$. Define $T := B_2 + A_F B_3 + \ldots + A_F^{\mu-1} B_{\mu+1}$. Since $T \in \underline{H}(\mu)$, it follows by induction hypothesis that $T \subset S^\mu$. Consequently, $L = B_1 + A_F T \subset (B + AS^\mu) \cap K = S^{\mu+1}$.

□

(ii) It follows from COR. 1.11 that $S^{\infty} \in \underline{H}(k) \subset \underline{H}$. Let $L \in \underline{H}$, say $L = B_1 + A_F B_2 + \dots + A_F^{\mu-1} B_\mu$. Then $L \in \underline{H}(\mu)$ and thus $L \subset S^{\mu} \subset S^{\infty}$.

□

Using the above result we then obtain the following theorem, indicating how to compute the subspaces at hand:

THEOREM 1.32.

(i) $R_a^*(K) = S^{\infty}(K)$,

(ii) $V^*(K) = V^{\infty}(K)$,

(iii) $V_a^*(K) = V^{\infty}(K) + S^{\infty}(K)$,

(iv) $R^*(K) = V^{\infty}(K) \cap S^{\infty}(K) = S^{\infty}(V^{\infty}(K)) = V^{\infty}(S^{\infty}(K))$.

PROOF: (i) By COR. 1.23, $\underline{H}$ is exactly the subfamily of elements of $\underline{R}_a$ contained in K. Thus, $R_a^*(K)$ is the unique supremal element of $\underline{H}$. The result then follows from LEMMA 1.31 (ii).
(ii) A proof of this can be found in WONHAM (1979, TH. 4.3).
(iii) By COR. 1.28, there are $V \in \underline{V}$ and $R_a \in \underline{R}_a$ such that $V_a^*(K) = V + R_a$. It is immediate that $V \subset V^*(K)$ and that $R_a \subset R_a^*(K)$. Therefore, by (i) and (ii), $V_a^*(K) \subset V^{\infty}(K) + S^{\infty}(K)$. The converse follows from the inclusions $\underline{V} \subset \underline{V}_a$ and $\underline{R}_a \subset \underline{V}_a$.
(iv) It follows from WONHAM (1979, ex 5.17) that $V^*(K) \cap (B + AS^{\infty}(K)) = R^*(K)$. Consequently, we must have $V^{\infty}(K) \cap S^{\infty}(K) = V^{\infty}(K) \cap K \cap (AS^{\infty}(K) + B) = R^*(K)$. The third equality was proven in WONHAM (1979, TH. 6.5). To prove the last equality, note that $V^{\infty}(S^{\infty}(K)) \subset V^{\infty}(K)$ and that $V^{\infty}(S^{\infty}(K)) \subset S^{\infty}(K)$. The converse inclusion follows from the facts that $R^*(K) \subset S^{\infty}(K)$ and that $V^{\infty}(R^*(K)) = R^*(K)$.

□

REMARK 1.33. It was already proven in an earlier section that a subspace R_a is an almost controllability subspace if and only if $R_a = S^{\infty}(R_a)$ (see TH. 1.22). If we denote the mapping $K \longmapsto S^{\infty}(K)$ by S^{∞}, this may be formulated as: $R_a \in \underline{R}_a \Leftrightarrow R_a$ is a fixed point of the mapping S^{∞}. In the same way, if we denote the mapping $K \longmapsto V^{\infty}(K)$ by V^{∞}, it follows from the above theorem that $V \in \underline{V}$ if and only if $V = V^{\infty}(V)$ or, equivalently, if and only if V is a fixed point of the mapping V^{∞}. Finally, if $R \in \underline{R}$, then it follows from the above that both $V^{\infty}(R) = R$ and $S^{\infty}(R) = R$. Conversely, if the latter two equalities hold, then $R = S^{\infty}(V^{\infty}(R)) = R^*(R)$, so $R \in \underline{R}$. Thus, we may state: $R \in \underline{R}$ if and only if R is a fixed point of both V^{∞} and S^{∞}.

CHAPTER 2

DISTRIBUTIONAL INPUTS AND HIGH GAIN FEEDBACK

In the present chapter we will first show how the families of almost controlled invariant subspaces and almost controllability subspaces may be viewed as 'exact' invariant subspaces when we allow the class of state trajectories to include not only absolutely continuous functions, but also distributions. In order to be able to speak about distributional state trajectories, we will introduce a convenient class of admissible distributional inputs and give a definition of state trajectory satisfying a certain initial condition, when the input is taken from this class. It will turn out that in order to give characterizations of the families of almost controlled invariant subspaces and almost controllability subspaces, it is sufficient to consider an even smaller class of inputs: the class of inputs that are Bohl distributions. All this will in section 2 lead to several equivalent characterizations of almost invariant subspaces in terms of their holdability properties with respect to distributional trajectories. In the same section we will consider the particularly useful characterizations of these subspaces in the frequency domain. In section 3, we will consider two special kinds of almost controlled invariant subspaces: coasting subspaces and sliding subspaces. We will characterize these in terms of distributional trajectories and frequency domain descriptions. It will turn out that these subspaces are fundamental in our theory, in the sense that each almost controlled invariant subspace admits a decomposition into the direct sum of a controllability subspace, a coasting subspace and a sliding subspace. We will also show the relevance of 'zeros at infinity' in this context.

The material in the second part of the chapter is mainly motivated by the applicability of almost invariant subspaces in feedback synthesis problems for linear systems. The main purpose is here, to establish the equivalence between the 'open loop' definitions of chapter 1 and characterizations in terms of sequences of feedback mappings. It will be shown that it is possible to stay arbitrarily close to almost controlled invariant subspaces moving along trajectories that are generated by state feedback. Since 'staying close to a subspace' may be formulated as 'making small the component of the state trajectory modulo that subspace', the equivalence between the 'open loop' and 'closed loop' characterizations will turn out to make the concept of almost controlled invariant subspace applicable to problems of approximate disturbance decoupling. However, to be able to obtain this important equivalence, we will need some rather special results on the approximation of almost controlled invariant subspaces by controlled invariant subspaces. This will be the subject of section 4. Also, we will need a result on the existence of controlled invariant subspaces complementary to a given almost controllability subspace. This will be treated in section 5. In section 6, the mathematical framework we established will be applied to give conditions for the solvability of the first of a series of feedback synthesis problems we will treat: the L_p/L_q almost disturbance decoupling problem. In section 7 and section 8, the pole assignability and stabilizability aspects of almost invariant subspaces will be studied. In particular, it will be shown that it is possible to stay close to an almost controllability subspace moving along feedback generated trajectories in such a way that the spectra of the closed loop mappings are located arbitrarily in the complex plane. This will lead in section 7 to necessary and sufficient conditions for the solvability of the L_p/L_q almost disturbance decoupling problem under the constraint of pole placement. Finally, in section 8, we will introduce the family of almost stabilizability subspaces.

2.1 DISTRIBUTIONALLY CONTROLLED INVARIANT SUBSPACES

As already suggested in the construction outlined in section 1.2, where we approximated a distributional 'input' by smooth inputs, the theory of almost invariant subspaces finds its natural mathematical framework in the theory of distributions or generalized functions as developed by L. Schwartz. Whereas is section 1.1 controlled invariant subspaces were defined in terms of trajectories that had to be absolutely continuous functions, we could of course also have defined them in terms of a broader class of trajectories. In this section, we will take for this broader class of trajectories a class of distributions. This idea, elaborated in WILLEMS (1981), leads to a larger family of controlled invariant subspaces. This new class of controlled invariant subspaces will be called the class of *distributionally controlled invariant subspaces*. Of course, the same idea can be applied to broaden the family of controllability subspaces and obtain a family of *distributionally controllability subspaces*. In the present section, we will first introduce a class of *admissible distributional inputs*. We will define the family of distributionally controlled invariant subspaces in terms of this class of inputs. Then we will consider an important subclass of the class of admissible inputs: the class of Bohl distributional inputs. It will turn out that for the holdability properties of distributionally controlled invariant subspaces it suffices to consider Bohl distributions only.

For the reader who is not familiar with the basic concepts of distribution theory, we refer to the APPENDIX.

Consider the linear system $\dot{x}(t) = Ax(t) + Bu(t)$. We will first formalize what we mean by distributional inputs and distributional trajectories. In particular, we will define what is meant by 'the trajectory through x_o' when u is a distribution. This is by no means a trivial matter, as is illustrated by the following example:

<u>EXAMPLE 2.1.</u> Let $x_o = Bu_o$ ($u_o \in U$) and take $u = -\delta u_o$ as 'input' for the system $\dot{x}(t) = Ax(t) + Bu(t)$. An obvious candidate for the trajectory would be the distribution x defined by $x(t) = 0$ ($t \neq 0$) and $x(0) = x_o$. However, in the sense of distributions this x equals zero. Thus, in this sense the condition $x(0) = x_o$ does not mean anything.

Therefore, we have to proceed carefully. We will take the following space of distributions as the space of admissible inputs:

$$U_D := \{u \in D'^n \mid u = u^- + u^+ \text{ with } u^- \in L_{1,loc}(\mathbb{R}, U) \text{ and } \operatorname{supp} u^- \subset (-\infty, 0] \text{ and } u^+ \in D'^m_+\}.$$

In effect, we restrict ourselves to distributional inputs that are functions for $t < 0$ and whose 'distributional part' has support in $[0,\infty)$. In the following, for $\Omega \subset \mathbb{R}$, let $1_\Omega(t)$ denote the indicator function of Ω. Define

$$K(t) := e^{At}B\, 1_{\mathbb{R}^+}(t) ,$$

$$d^-(t) := e^{At}\, 1_{\mathbb{R}^-}(t) ,$$

$$d^+(t) := e^{At}\, 1_{\mathbb{R}^+}(t) .$$

DEFINITION 2.2. If $u \in U_D$, $u = u^- + u^+$, then the state trajectory of the system (A,B) with $x(0) = x_o$ and input u is defined as the distribution $x^- + x^+ =: x \in D'^m$ with:

(i) $x^- : \mathbb{R} \to X$ defined by $x^-(t) := d^-(t)x_o - \int_t^o e^{A(t-\tau)}Bu^-(\tau)d\tau$

(ii) $x^+ := d^+x_o + K * u^+$.

The convolution appearing in (ii) denotes the convolution of distributions with support in $[0,\infty)$ (see APPENDIX).

In the sequel, the state trajectory with $x(0) = x_o$ and input $u \in U_D$ in the sense defined above will be denoted by $x(x_o,u)$. The distribution x^+ will be called its *restriction* to $[0,\infty)$ and denoted by $x^+(x_o,u)$. Note that $x^+ \in D'^n_+$. Observe that a trajectory in the sense of DEF.2.2 is the sum of a regular function with support in $(-\infty,0]$ and a distribution with support in $[0,\infty)$. The class of all state trajectories generated by inputs $u \in U_D$ is denoted by

$$\Sigma_D(A,B) := \{x \in D'^n \mid \exists\ u \in U_D \text{ and } x_o \in X \text{ such that } x = x(x_o,u)\}\ .$$

For every $T \geq 0$, define the following subspace of $\Sigma_D(A,B)$:

$$\Sigma_D^T(A,B) := \{x \in \Sigma_D(A,B) \mid x^+ = x^{[0,T]} + x^{T+}, \text{ with } x^{[0,T]} \in D'^n_{[0,T]} \text{ and } x^{T+} \text{ a function with supp } x^{T+} \subset [T,\infty)\}.$$

Thus, $\Sigma_D^T(A,B)$ consists of those state trajectories in $\Sigma_D(A,B)$ that have the property that their restriction x^+ can be written as the sum of a distribution with support in $[0,T]$ and a function with support in $[T,\infty)$. If $x \in \Sigma_D^T(A,B)$, we will denote $x(T^+) := \lim_{t \downarrow T} x^{T+}(t)$.

We will now define the notions of controlled invariance and controllability subspace in the above distributional context. If L is a subspace of X and $x \in D'^n$, then we will say that x *lies in* L if for every test-function $\varphi \in D(\mathbb{R})$, $\langle x,\varphi\rangle \in L$.

DEFINITION 2.3. A subspace V_D of X will be called a *distributionally controlled invariant subspace* if $\forall x_o \in V_D$, there exists $x \in \Sigma_D(A,B)$ with $x(0) = x_o$ and whose restriction x^+ lies in V_D.

A subspace R_D of X will be called a *distributionally controllability subspace* if $\forall x_o, x_1 \in R_D$, there exists a $T \geq 0$ and $x \in \Sigma_D^T(A,B)$, such that $x(0) = x_o$, $x^{[0,T]}$ lies in R_D and $x(T^+) = x_1$.

We will denote by $\underline{V}_D$ or $\underline{V}_D(A,B)$ and $\underline{R}_D$ or $\underline{R}_D(A,B)$ the families of all distributionally controlled invariant subspaces and distributionally controllability subspaces. It may be verified immediately that $\underline{R}_D \subset \underline{V}_D$ (take $x_1 = 0$ in the definition of distributionally controllability subspaces).

A very important role in this work will be played by the class of *Bohl distributions*:

DEFINITION 2.4. A distribution $u \in D_+^{'m}$ will be called a *Bohl distribution* if there are vectors $u_i \in U$ and mappings F,G and H such that $u = u_{imp} + u_{reg}$, with $u_{imp} := \sum_{i=0}^{k} u_i \delta^{(i)}$ and $u_{reg}(t) := He^{Ft} G \mathbb{1}_{\mathbb{R}^+}(t)$. The class of Bohl distributions in $D_+^{'m}$ will be denoted by $D_B^{'m}$. An element $u \in D_B^{'m}$ will be called *impulsive* if $u_{reg} = 0$ and *regular* if $u_{imp} = 0$. A function of the form $u(t) = He^{Ft} G \mathbb{1}_{\mathbb{R}^+}(t)$ is called a *Bohl function*.

Note that $D_B^{'m} \subset U_D$. It may be seen that the class of scalar Bohl distributions $D_B^{'}$ forms a *field* (with convolution as multiplication). This may e.g. be seen by noting that $u \in D_B^{'}$ if and only if its Laplace transform is a rational function. The conclusion then follows from the fact that the space of rational functions with coefficients in $\mathbb{R}$ forms a field.

In the following proposition we will see that inputs $u \in D_B^{'m}$ yield state trajectories with restriction $x^+ \in D_B^{'n}$:

PROPOSITION 2.5. *Let* $u = \sum_{i=0}^{k} u_i \delta^{(i)} + u_{reg}$ *be a Bohl distributional input and let* $x_o \in X$. *Then the state trajectory of* (A,B) *with* $x(0) = x_o$ *and input* u *is given by* $x(x_o,u) = x^- + x^+(x_o,u)$, *where*

(i) $\quad x^-(t) = d^-(t)x_o$

(ii) $\quad x^+(x_o,u) = \sum_{i=1}^{k} \sum_{j=1}^{i} A^{i-j}Bu_i \delta^{(j-1)} + x_{reg}$, *where*

$$x_{reg}(t) = d^+(t)(x_o + \sum_{i=0}^{k} A^i Bu_i) + \int_0^t e^{A(t-\tau)} Bu_{reg}(\tau)d\tau$$

PROOF: This follows immediately from DEF. 2.2 using the properties of the distributions δ, $\delta^{(1)}$, etc.

□

REMARK 2.6. It is important to note that if $u \in D_B^{'m}$, then for every $x_o \in X$, the state trajectory with $x(0) = x_o$ and input u is an element of $\Sigma_D^o(A,B)$ (as defined by (2.1), with T = 0), i.e. the restriction $x^+(x_o,u)$ is the sum of a distributional part with support in {0} and a function with support in $[0,\infty)$. If $u = \sum_{i=o}^{k} u_i\delta^{(i)} + u_{reg}$, then $x(x_o,u)(0^+) = x_o + \sum_{i=o}^{k} A^iBu_i$. Moreover, it may be seen immediately from PROP.2.5 that *if* $u \in D_B^{'m}$ *then* $x^+(x_o,u)$ *is impulsive if and only if* u *is impulsive and* $x(0^+) = 0$. Note also that if $u \in D_B^{'m}$ is regular, then $\forall x_o \in X$, $x^+(x_o,u)$ is regular. Finally, note that $x_{reg} = x^+(x(0^+), u_{reg})$.

It has been shown in WILLEMS (1981), that the class $D_B^{'m}$ is sufficiently rich to maintain the holdability properties of the class of distributionally controlled invariant subspaces:

LEMMA 2.7. *If* $V_D \in \underline{V}_D$, *then for all* $x_o \in V_D$ *there exists an input* $u \in D_B^{'m}$ *such that* $x^+(x_o,u)$ *lies in* V_D.

PROOF: The proof uses the fact that the class of scalar Bohl type distributions forms a field. The details of the proof can be found in WILLEMS (1981), LEMMA 3.1).

□

In section 2.2 the above lemma will be applied to show that the families of distributionally controlled invariant subspaces and controllability subspaces are in fact equal to the families of almost controlled invariant and almost controllability subspaces.

2.2 FREQUENCY AND TIME DOMAIN DESCRIPTIONS

Many of the concepts that have appeared so far can be described and studied very conveniently using descriptions in terms of rational functions and polynomials. The use of these *frequency domain descriptions* of concepts appearing in the geometric approach to linear systems was initiated in EMRE & HAUTUS (1980) and HAUTUS (1980). In the latter, frequency domain descriptions were given of controlled invariant subspaces and stabilizability subspaces. In SCHUMACHER (1983 a) and (1984) these results were generalized to almost controlled invariant subspaces. In this section, we will establish frequency domain descriptions of the subspaces S^μ and V^μ generated by the algorithms ACSA and ISA (see CH.1, SECTION 1.6). We will apply these results to obtain frequency domain descriptions of their 'limiting' subspaces $V^*(K)$, $R^*_a(K)$, etc. It will be explained how frequency domain descriptions should be interpreted in the time domain. Finally, we will then translate the results obtained back to the time domain and give characterizations of almost controlled invariant subspaces in terms of Bohl distributions.

In the sequel, let $X[s]$ (resp. $X(s)$, $X_+(s)$) denote the space of all n-vectors whose components are polynomials (resp. rational functions, strictly proper rational functions) with coefficients in $\mathbb{R}$. In a similar way, let $U[s]$, etc. denote the corresponding spaces of m-vectors. If K is a subspace of X, then $K[s]$, (resp. $K(s)$, $K_+(s)$) will denote the space of all elements $\xi(s) \in X[s]$(resp. $X(s)$, $X_+(s)$) with the property that $\xi(s) \in K$, $\forall s$. If $\xi(s) \in X(s)$, it can be uniquely represented as $\xi(s) = \xi_-(s) + \xi_+(s)$, with $\xi_-(s) \in X[s]$ and $\xi_+(s) \in X_+(s)$. If $\xi(s) \in X(s)$, we will denote $[\xi(s)]_- := \xi_-(s)$ and $[\xi(s)]_+ := \xi_+(s)$.

Slightly generalizing a definition in HAUTUS (1980 , DEF. 2.6), we define:

<u>DEFINITION 2.8.</u> If $x_o \in X$, $\xi(s) \in X(s)$ and $\omega(s) \in U(s)$, then the expression $x_o = (Is-A)\,\xi(s) + B\omega(s)$ is called *a* (ξ,ω)*-representation of* x_o.

<u>REMARK 2.9.</u> Of course, the idea is that a (ξ,ω)-representation of x_o corresponds to an equation $\xi(s) = (Is-A)^{-1}(x_o - B\omega(s))$. If $\xi(s) \in X(s)$ and $\omega(s) \in U(s)$ then their inverse Laplace transforms $L^{-1}\xi$ and $L^{-1}\omega$ are distributions in $D_B'^{n}$ and $D_B'^{m}$ respectively (see DEF. 2.4). By taking the inverse Laplace transform in the above equation, we obtain the convolution equation $L^{-1}\xi = d^+x_o - K * L^{-1}\omega$ (here, $d^+(t) := e^{At}\, 1_{\mathbb{R}^+}(t)$ and $K(t) = e^{At} B\, 1_{\mathbb{R}^+}(t)$). By DEF. 2.2, this is equivalent to saying that $L^{-1}\xi$ is the restriction to $[0,\infty)$ of the state trajectory with $x(0) = x_o$ and input $u = -L^{-1}\omega$. We conclude that the statement '$x_o = (Is-A)\xi(s) + B\omega(s)$' is equivalent to the statement: '$L^{-1}\xi = x^+(x_o, -L^{-1}\omega)$' (according to the notation introduced in the remarks following DEF.2.2). In this sense, every result obtained in terms of frequency domain descriptions can immediately be translated in terms of Bohl distributional trajectories and inputs and vise versa.

We will now give a characterization of the subspaces $S^\mu(K)$ and $V^\mu(K)$ in terms of (ξ,ω)-representations:

<u>THEOREM 2.10</u>. *Let K be a subspace of X. Then for every $\mu \in \mathbb{N}$:*

(i) $S^{\mu}(K) = \{x_o \in K | x_o$ *has a* (ξ,ω)*-representation with* $\xi(s) \in K[s]$, $\omega(s) \in U[s]$ *and* $[s^{1-\mu}\xi(s)]_- = 0\}$.

(ii) $V^{\mu}(K) = \{x_o \in K | x_o$ *has a* (ξ,ω)*-representation with* $\xi(s) \in X_+(s)$, $\omega(s) \in U_+(s)$ *and* $[s^{\mu}\xi(s)]_- \in K[s]\}$.

<u>PROOF</u>: (i) The proof is by induction. Assume $\mu = 1$. Since $S^1 = K \cap B$, every $x_o \in S^1$ can be represented as $x_o = Bu_o$. The latter is a (ξ,ω)-representation with $\xi(s) := 0$ and $\omega(s) := u_o$. Conversely, if $x_o \in K$ has a (ξ,ω)-representation with $\xi(s) = 0$, then $x_o = B\omega(s) \in B$. Assume now the statement is true for μ. Let $x_o \in S^{\mu+1}$. Then $x_o \in K$ and there is $\tilde{x}_o \in S^{\mu}$ with $x_o = A\tilde{x}_o + Bu_o$. By induction hypothesis, $\tilde{x}_o = (Is - A)\tilde{\xi}(s) - B\tilde{\omega}(s)$ with $[s^{1-\mu}\tilde{\xi}(s)]_- = 0$. It follows that $x_o = A(Is - A)\tilde{\xi}(s) - AB\tilde{\omega}(s) + Bu_o = (Is - A)\xi(s) + B\omega(s)$, with $\xi(s) := A\tilde{\xi}(s) + B\tilde{\omega}(s)$ and $\omega(s) := u_o - s\tilde{\omega}(s)$. Also, since $[s^{-\mu} A\tilde{\xi}(s)]_- = 0$,

$$[s^{1-(\mu+1)}\xi(s)]_- = [s^{-\mu}B\tilde{\omega}(s)]_- = [s^{-\mu}(Is-A)\tilde{\xi}(s) - s^{-\mu}x_o]_- = 0.$$

Conversely, let $x_o = (Is-A)\xi(s) + B\omega(s) \in K$ with $[s^{-\mu}\xi(s)]_- = 0$. Define $\tilde{\omega}(s) := s^{-1}(\omega(0)-\omega(s))$ and $\tilde{\xi}(s) := s^{-1}(\xi(s)-\xi(0))$. We first contend that $A\tilde{\xi}(s) + B\tilde{\omega}(s) = \xi(s)$. To see this, note that $x_o = -A\xi(0) + B\omega(0)$. Hence, $s(A\tilde{\xi}(s) + B\tilde{\omega}(s)) = A\xi(s) - B\omega(s) + x_o = s\xi(s)$. It then follows that $(Is-A)\tilde{\xi}(s) - B\tilde{\omega}(s) = s\tilde{\xi}(s) - \xi(s) = -\xi(0)$. Now, from the definition of $\tilde{\xi}(s)$ we have $[s^{1-\mu}\tilde{\xi}(s)]_- = 0$. Since also $\xi(0) \in K$, it follows from the induction hypothesis that $\xi(0) \in S^{\mu}$. We may therefore conclude that $x_o = -A\xi(0) + B\omega(0) \in S^{\mu+1}$.

(ii) Assume $\mu = 1$. Note that $V^1 = K$. Let $x_o \in K$ and take an arbitrary (ξ,ω)-representation of x_o with $\omega(s) \in U_+(s)$. Then $[s\,\xi(s)]_- = [x_o + A\xi(0) - B\omega(s)]_- = x_o \in K[s]$. The converse inclusion follows by definition. Assume now the assertion is true for μ. Let $x_o \in V^{\mu+1}$. Then $x_o \in K$ and $Ax_o = -\tilde{x}_o - Bu_o$ with $\tilde{x}_o \in V^{\mu}$. Thus, by hypothesis, $\tilde{x}_o = (Is-A)\tilde{\xi}(s) + B\tilde{\omega}(s)$ with $[s^{\mu}\tilde{\xi}(s)]_- \in K[s]$. Since $(Is-A)x_o = sx_o + \tilde{x}_o + Bu_o$, it follows that $x_o = (Is-A)\xi(s) + B\omega(s)$, with $\xi(s) := s^{-1}(x_o - \tilde{\xi}(s))$ and $\omega(s) = -s^{-1}(u_o + \tilde{\omega}(s))$. Also, $[s^{\mu+1}\xi(s)]_- = s^{\mu}x_o - [s^{\mu}\tilde{\xi}(s)]_- \in K[s]$.

Conversely, let $x_o = (Is-A)\xi(s) + B\omega(s) \in K$ with $\xi(s)$ and $\omega(s)$ strictly proper and $[s^{\mu+1}\xi(s)]_- \in K[s]$. Define $\tilde{\xi}(s) := [s\xi(s)]_+$ and $\tilde{\omega}(s) = [s\omega(s)]_+$. Expand $\xi(s) = \sum_{i=1}^{\infty} \xi_i s^{-i}$ and $\omega(s) = \sum_{i=1}^{\infty} u_i s^{-i}$. It follows by equating powers of s in $x_o = (Is-A)\xi(s) + B\omega(s)$ that $x_o = \xi_1$. Moreover, $s\xi(s) = \tilde{\xi}(s) + x_o$ and $s\omega(s) = \tilde{\omega}(s) + u_1$. We thus obtain the equality $sx_o = (Is-A)s\xi(s) + Bs\omega(s) = (Is-A)\tilde{\xi}(s) + B\tilde{\omega}(s) + sx_o - Ax_o + Bu_1$. It follows that $Ax_o - Bu_1 = (Is - A)\tilde{\xi}(s) + B\tilde{\omega}(s)$. Since also $[s^{\mu}\tilde{\xi}(s)]_- = [s^{\mu+1}\xi(s)]_- - s^{\mu}x_o \in K[s]$, we obtain that $Ax_o - Bu_1 \in V^{\mu}$ and hence that $x_o \in K \cap A^{-1}(V^{\mu} + B) = V^{\mu+1}$.

□

If $\xi(s) \in X[s]$ and $\xi(s) = \sum_{i=o}^{k} \xi_i s^i$, then its *degree* is defined as $\deg \xi(s) := \max \{i | \xi_i \neq 0\}$. In effect, the above result states that $S^{\mu}(K)$ consists exactly of those points in K that have a polynomial (ξ,ω) - representation with $\xi(s) \in K$, $\forall s$ and $\deg \xi(s) \leq \mu-2$ (If $\xi(s)$ is the zero polynomial we define $\deg \xi(s) := -1$). In the same way, $V^{\mu}(K)$ consists exactly of those points in K that have a strictly proper rational (ξ,ω)-representation with, if $\xi(s) = \sum_{i=1}^{\infty} \xi_i s^{-i}$, $\xi_1, \xi_2, \ldots, \xi_{\mu} \in K$. We will now look what this result means in the time domain. If $x \in D_B'^n$, $x = \sum_{i=o}^{k} x_i \delta^{(i)}$, let its *order* be defined as $\text{ord}\, x := \max \{i | x_i \neq 0\}$. If $x = 0$, define $\text{ord}\, x := -1$. Of course, if $\xi(s)$ is the Laplace transform of x, then $\deg \xi(s) = \text{ord}\, x$. Also note that $x \in D_B'^n$ is regular if and only if $\xi(s) \in X_+(s)$ and impulsive if and only if $\xi(s) \in X[s]$. Finally, it may be verified that $\xi(s) \in K(s)$ if and only if x lies in K. (In fact, if $x \in D_B'^n$ and $x = \sum_{i=o}^{k} x_i \delta^{(i)} + x_{reg}$, then x lies in K if and only if $x_i \in K$ $(i = 0,\ldots,k)$ and $x_{reg}(t) \in K$, $\forall t$). We then obtain the following time domain characterizations:

<u>COROLLARY 2.11</u>. *Let K be a subspace of X. Then for $\mu \in \mathbb{N}$:*

(i) $S^{\mu}(K) = \{x_o \in K \,|\, \exists\, u \in D_B'^m$, *impulsive, such that* $x^+(x_o,u)$ *lies in* K, $x(x_o,u)(0^+) = 0$ *and* $\text{ord}\, x^+ \leq \mu-2\}$.

(ii) $V^{\mu}(K) = \{x_o \in K \,|\, \exists\, u \in D_B'^m$, *regular, such that* $\frac{d^i x^+}{dt^i}(0) \in K$ *for* $i = 0,\ldots,\mu-1\}$.

<u>PROOF:</u> (i) This follows immediately from TH. 2.10 and the REMARKS 2.6 and 2.9. (ii) This follows by noting that the i^{th} derivative of a regular Bohl function evaluated in $t = 0$ is equal to the coefficient corresponding to the term of s^{-i-1} in its Laurent expansion around infinity.

□

Combining TH. 2.10 with the results of CH. 1 , SECTION 1.6, we obtain the following frequency domain descriptions of the supremal almost controlled invariant subspaces associated with a given subspace of the state space:

COROLLARY 2.12. *Let K be a subspace of X. Then we have*

(i) $R_a^*(K) = \{x_o \in K \mid x_o$ *has a* (ξ,ω)*-representation with* $\xi(s) \in K[s], \omega(s) \in U[s]\}$.

(ii) $V^*(K) = \{x_o \in K \mid x_o$ *has a* (ξ,ω)*-representation with* $\xi(s) \in K_+(s), \omega(s) \in U_+(s)\}$.

(iii) $V_a^*(K) = \{x_o \in K \mid x_o$ *has a* (ξ,ω)*-representation with* $\xi(s) \in K(s), \omega(s) \in U(s)\}$.

(iv) $R^*(K) = \{x_o \in K \mid x_o$ *has a* (ξ,ω)*-representation with* $\xi(s) \in K[s]$ *and* $\omega(s) \in U[s]$ *and also a* (ξ,ω)*-representation with* $\xi(s) \in K_+(s)$ *and* $\omega(s) \in U_+(s)\}$.

PROOF: (i) This follows immediately from TH. 1.32 and TH. 2.10.
(ii) This follows from TH.1.32 and TH.2.10 by noting that $V^\infty(K) = \bigcap_{\mu=1}^{\infty} V^\mu(K)$.
(iii) If $x_o \in V_a^*(K)$ then, by TH.1.32, $x_o = x_{01} + x_{02}$ with $x_{01} \in V^*(K)$ and $x_{02} \in R_a^*(K)$. Apply then (i) and (ii) to obtain a (ξ,ω)-representation for x_o with $\xi(s) \in K(s)$ and $\omega(s) \in U(s)$. Conversely, let $x_o = (Is - A)\xi(s) + B\omega(s)$. Define $\xi_1(s) := [\xi(s)]_+$, $\omega(s) = [\omega(s)]_+$, $\xi_2(s) := [\xi(s)]_-$ and $\omega_2(s) := [\omega(s)]_-$. Then $x_o - (Is - A)\xi_1(s) - B\omega_1(s) = (Is - A)\xi_2(s) + B\omega_2(s)$. In this equality, the left hand side is proper and the right hand side a polynomial. Consequently, both sides must be equal to the same constant x_{02}. Define $x_{01} := x_o - x_{02}$. Then $x_{02} \in R^*{}_a(K)$ and $x_{01} \in V^*(K)$. This proves statement (iii). Finally, (iv) follows immediately from (i), (ii) and TH. 1.32 (iv).

□

Now, translating these results back to the time domain, we obtain the following characterizations in terms of Bohl type inputs:

COROLLARY 2.13. *Let K be a subspace of X. Then we have:*

(i) $R_a^*(K) = \{x_o \in K \mid \exists u \in D_B'^m$, *impulsive, such that* $x^+(x_o,u)$ *lies in* K *and* $x(x_o,u)(0^+) = 0\}$.

(ii) $V^*(K) = \{x_o \in K \mid \exists u \in D_B'^m$, *regular, such that* $x^+(x_o,u)$ *lies in* $K\}$.

(iii) $V_a^*(K) = \{x_o \in K \mid \exists u \in D_B'^m$, *such that* $x^+(x_o,u)$ *lies in* $K\}$.

(iv) $R^*(K) = \{x_o \in K \mid \exists u_1 \in D_B'^m$, *impulsive, such that* $x^+(x_o,u_1)$ *lies in* K *and* $x(x_o,u_1)(0^+) = 0$ *and* $\exists u_2 \in D_B'^m$, *regular, such that* $x^+(x_o,u_2)$ *lies in* $K\}$.

PROOF: The claims of this corollary follow immediately by translating the results of COR. 2.12 back to the time domain.

□

REMARK 2.14. We note that the statement ' $x^+(x_o,u)$ lies in K ' in the characterization of $V^*(K)$ may, since the x^+ here is regular, be stated as ' $x^+(t) \in K$ for all $t \geq 0$ '.

We also note that if $x_o \in V_a^*(K)$ and $u \in D_B'^m$ is such that $x^+(x_o,u)$ lies in K, then the regular part $x_{reg}(t)$ of $x^+(x_o,u)$ lies in $V^*(K)$ for all $t \geq 0$. Indeed, if $x^+(x_o,u)$ lies in K, then both its impulsive part and its regular part lie in K. Hence, $x_{reg}(t) \in K$, $t \geq 0$ and therefore $x_{reg}(t) \in V^*(K)$, $t \geq 0$. Since then also $x_{reg}(0) = x(x_o,u)(0^+)$ is a vector in $V^*(K)$, we may conclude that the input u first causes an impulsive movement that brings us to $x(x_o,u)(0^+) \in V^*(K)$ and then a regular movement which takes place in $V^*(K)$ for $t \geq 0$.

We may now give the following characterizations of the class of almost controlled invariant subspaces:

<u>THEOREM 2.15</u>. *The following statements are equivalent:*

(i) $V_a \in \underline{V}_a(A,B)$,

(ii) *Every* $x_o \in V_a$ *has a* (ξ,ω)*-representation with* $\xi(s) \in V_a(s)$ *and* $\omega(s) \in U(s)$,

(iii) *For every* $x_o \in V_a$ *there is a* $u \in D_B'^m$ *such that* $x^+(x_o,u)$ *lies in* K,

(iv) $V_a \in \underline{V}_D(A,B)$.

<u>PROOF</u>: The equivalence of (i), (ii) and (iii) follow immediately from COR. 2.12 (iii) and COR. 2.13 (iii). Finally, it follows from the definition of $\underline{V}_D$ that the implication (iii) ⇒ (iv) holds. The converse implication is a consequence of LEMMA 2.7.

□

We conclude this section with the analogous characterizations of the class of almost controllability subspaces:

<u>THEOREM. 2.16</u>. *The following statements are equivalent:*

(i) $R_a \in \underline{R}_a(A,B)$,

(ii) *Every* $x_o \in R_a$ *has a* (ξ,ω)*-representation with* $\xi(s) \in R_a[s]$ *and* $\omega(s) \in U[s]$,

(iii) *For every* $x_o \in R_a$ *there is an impulsive* $u \in D_B'^m$ *such that* $x^+(x_o,u)$ *lies in* R_a *and* $x(x_o,u)(0^+) = 0$,

(iv) $R_a \in \underline{R}_D(A,B)$,

(v) *For all* $x_o,x_1 \in R_a$ *and for all* $T \geq 0$ *there is* $x \in \Sigma_D^T(A,B)$ *such that* $x(0) = x_o$, $x(T^+) = x_1$ *and* $x^{[0,T]}$ *lies in* R_a.

PROOF: The equivalence of statements (i), (ii) and (iii) follows immediately from COR. 2.12 (i) and COR. 2.13 (i).
(iii) $\Rightarrow$ (iv). Let $x_o, x_1 \in R_a$. We have to establish the existence of a $T \geq 0$ and $x \in \Sigma_D^T(A,B)$ such that $x(0) = x_o$, $x(T^+) = x_1$ and $x^{[0,T]}$ lies in R_a. We contend that this may be achieved for $T = 0$. Since $x_o - x_1 \in R_a$, by (iii) there exists an impulsive $u \in D_B'^m$ such that $x^+(x_o - x_1, u)$ lies in R_a and $x(x_o - x_1, u)(0^+) = 0$. Now, the impulsive part of $x^+(x_o - x_1, u)$ is equal to the impulsive part of $x^+(x_o, u)$ (see PROP. 2.5). Thus, the restriction of $x^+(x_o, u)$ to the set $\{0\}$ lies in K. Moreover, $x(x_o - x_1, u)(0^+) = x(x_o, u)(0^+) - x_1$ and thus $x(x_o, u)(0^+) = x_1$.
(iii) $\Rightarrow$ (v). Let $x_o, x_1 \in R_a$ and let $T \geq 0$ arbitrary. There is an impulsive $u_1 \in D_B'^m$ such that $x^+(x_o, u_1)$ lies in K and $x(x_o, u_1)(0^+) = 0$. There is an impulsive $u_2 \in D_B'^m$ such that $x^+(-x_1, u_2)$ lies in K and $x(-x_1, u_2)(0^+) = 0$. It may then be shown that $x(0, u_2)(0^+) = x_1$. Define now a distribution $\sigma^{-T}u_2$ by $<\sigma^{-T}u_2, \varphi> := <u_2, \sigma^T\varphi>, \varphi \in D(\mathbb{R})$, where the time-shift σ^T is defined by $(\sigma^T\varphi)(t) = \varphi(t + T)$. Finally, define $u := u_1 + \sigma^{-T}u_2$. It may then be verified that $x(x_o, u)(0^+) = 0$, that $x^{[0,T]}$ lies in R_a and that $x(x_o, u)(T^+) = x_1$.
(v) $\Rightarrow$ (iv). This implication follows immediately from the definition
(iv) $\Rightarrow$ (iii). For this implication, we refer to WILLEMS (1981 , TH. 5).

□

To conclude, we would like to report the characterizations of the classes $\underline{V}_a$ and $\underline{R}_a$ involving *matrix pencils* by JAFFE & KARCANIAS (1981) and the *'hybrid' characterizations* obtained in SCHUMACHER (1983 a).

2.3 COASTING AND SLIDING SUBSPACES

In this section we will introduce two special types of almost controlled invariant subspaces: coasting subspaces and sliding subspaces. A *coasting subspace* will be defined to be a controlled invariant subspace with the property that for every point in that subspace there is *exactly one* trajectory through that point that lies entirely in the subspace. A *sliding subspace* will be defined as an almost controlled invariant subspace with the property that the only regular trajectory that lies in that subspace for all points of time is the zero-trajectory.

In the sequel we will establish several equivalent characterizations of the above concepts. It will be shown that every almost controlled invariant subspace can be represented as the direct sum of a controllability subspace, a coasting subspace and a sliding subspace.

<u>DEFINITION 2.17</u>. A subspace $C \in V(A,B)$ will be called a *coasting subspace* if for all $x_o \in C$, there is one and only one $x \in \Sigma(A,B)$ auch that $x(0) = x_o$ and $x(t) \in C$, $\forall t \in \mathbb{R}$.

Thus, if C is a coasting subspace, then for each initial condition $x_o \in C$ it is necessary to follow a unique trajectory in order to remain in C: one has to 'coast' along a certain fixed path to keep the movement inside the subspace C. The following theorem gives alternative characterizations of this concept:

<u>THEOREM 2.18</u>. *The following statements are equivalent:*

(i) *C is a coasting subspace,*

(ii) *$C \in \underline{V}$ and $R^*(C) = \{0\}$,*

(iii) *For every $x_o \in C$, there is one and only one (ξ,ω)-representation with $\xi(s) \in C_+(s)$ and $\omega(s) \in U_+(s)$,*

(iv) *$\underline{F}(C) \neq \emptyset$ and if $F_1, F_2 \in \underline{F}(C)$ then $(A + BF_1) \mid C = (A + BF_2) \mid C$.*

<u>PROOF</u>: (i) $\Rightarrow$ (ii). Suppose that $R^*(C) \neq \{0\}$. Let $0 \neq x_1 \in R^*(C)$. Let G be a mapping such that $\operatorname{im} BG = B \cap R^*(C)$ and let $F \in \underline{F}(R^*(C))$. By WONHAM (1979, PROP. 5.2), the system $(A_F|R^*(C), BG)$ is controllable. Thus, there is a $x \in \Sigma(A,B)$ with $x(0) = 0$, $x(1) = x_1$ and such that $x(t) \in R^*(C)$, $\forall t$. Since this trajectory is not identically zero, we have a contradiction with the fact that C is a coasting subspace.

(ii) $\Rightarrow$ (iii). By COR. 2.12, every $x_o \in C$ has a (ξ,ω)-representation with $\xi(s) \in C_+(s)$. Let D be a mapping such that $C = \ker D$. Since $R^*(C) = \{0\}$, it follows that the transfer matrix $D(Is-A)^{-1}B$ is injective (see WONHAM (1979, ex. 4.4) It may then be seen immediately that the above (ξ,ω)-representation is unique.

(iii) $\Rightarrow$ (ii). If $R^*(C) \neq \{0\}$, there is a nonzero $\omega(s) \in U_+(s)$ such that $D(Is-A)^{-1}B\omega(s) = 0$. Define $\xi(s) := (Is-A)^{-1}B\omega(s)$. This pair $\xi(s)$, $\omega(s)$ then yields a (ξ,ω)-representation of $x_o = 0$ with $\xi(s) \in C_+(s)$. Since also $\xi'(s) := 0$ and $\omega'(s) := 0$ yields a (ξ,ω)-representation of 0, this contradicts (iii).

(ii) $\Rightarrow$ (i). If C is not a coasting subspace, there is a $x \in \Sigma(A,B)$, not identically zero, such that $x(0) = 0$ and $x(t) \in C$, $\forall t$. Define a subspace $V \subset C$ by $V := \underset{t \in \mathbb{R}}{\operatorname{span}}\{x(t)\}$. Since $x(t) \in V$, $\forall t$, also $\dot{x}(t) \in V$, $\forall t$. Thus, $Ax(t) \in V + B$, $\forall t$ so V is controlled invariant. We contend that $V \cap B \neq \{0\}$.

Supppose it is not. Let $Z \subset X$ be such that $V \subset Z$ and $Z \oplus B = X$. Let π be the projection along B onto Z. Then we obtain $\dot{x}(t) = \pi\dot{x}(t) = \pi A\pi x(t)$, $\forall t$. Since $x(0) = 0$, this yields $x(t) = 0, \forall t$. This yields a contradiction. We may now conclude from WONHAM (1979, TH. 5.5) that $R^*(V)$ is a nonzero controllability subspace contained in C.
(ii)$\Longleftrightarrow$(iv). This follows immediately from WONHAM (1979, TH. 5.7 and COR 5.2). □

<u>REMARK 2.19</u>. Let C be a coasting subspace and D a mapping such that $C = \ker D$. It is well known (see e.g. WONHAM (1979, section 5.5), that if (A,B) is controllable and (D,A) observable then the fixed spectrum $\sigma(A_F|C)$, is exactly equal to the list of zeros of the nonzero numerator polynomials in the Smith-McMillan form of the transfer matrix $D(Is - A)^{-1}B$. These zeros are also called the *transmission zeros* of the system (A,B,D). Note that it follows from TH. 2.18 that if $x_o \in C$ and $x_o = (Is - A)\xi(s) + B\omega(s)$, then every pole of $\xi(s)$ and $\omega(s)$ must be an element of the fixed spectrum $\sigma(A_F|C)$.

Next, we will give a definition of the notion of sliding subspace:

<u>DEFINITION 2.20</u>. A subspace $S \in \underline{V}_a(A,B)$ will be called a *sliding subspace* if $x \in \Sigma(A,B)$ and $x(t) \in S$, $\forall t \in \mathbb{R}$ imply $x(t) = 0$, $\forall t \in \mathbb{R}$.

The following line up of equivalent characterizations may be given:

<u>THEOREM 2.21</u>. *The following statements are equivalent:*

(i) S *is a sliding subspace*,

(ii) $S \in \underline{V}_a$ *and* $V^*(S) = \{0\}$,

(iii) $S \in \underline{R}_a$ *and* $R^*(S) = \{0\}$,

(iv) $S \in \underline{R}_a$ *and the following holds*: $x_1, x_2 \in \Sigma_D(A,B)$, $x_1(0) = x_2(0)$ *and* x_1^+, x_2^+ *lie in* S *imply* $x_1^+ = x_2^+$;

(v) $S \in \underline{R}_a$ *and every* $x_o \in S$ *has one and only one* (ξ,ω)*-representation with* $\xi(s) \in S(s)$ *and* $\omega(s) \in U(s)$

(vi) $S \in \underline{R}_a$ *and for every* $x_o \in S$ *there is one and only one* $u \in D_B^{'m}$ *such that* $x^+(x_o,u)$ *lies in* S.

(vii) *For every* $x_o, x_1 \in S$ *and* $T \geq 0$, *there* is $x \in \Sigma_D^T(A,B)$ *such that* $x(0) = x_o$, $x(T^+) = x_1$ *and* $x^{[0,T]}$ *lies in* S. *Moreover, if* $x' \in \Sigma_D^T(A,B)$ *is such that* $x'(0) = x_o$, $x'(T^+) = x_1$ *and* $x'^{[0,T]}$ *lies in* S, *then* $x^{[0,T]} = x'^{[0,T]}$,

(viii) *Every* $x_o \in S$ *has one and only one* (ξ,ω)*-representation with* $\xi(s) \in S[s]$ *and* $\omega(s) \in U[s]$,

(ix) *For every* $x_o \in S$, *there is one and only one* $u \in D_B^{'m}$ *such that* $x(x_o,u)(0^+) = 0$ *and* $x^+(x_o,u)$ *lies in* S.

PROOF : (i) $\Leftrightarrow$ (ii). If $V^*(S) \neq \{0\}$, then it is clearly possible to find a nonzero trajectory that remains in S. Conversely, if S is not a sliding subspace, there exists a nonzero trajectory x with $x(t) \in S$, $\forall t$. $V = \operatorname{span}_{t\in\mathbb{R}} \{x(t)\}$. Then $V \neq \{0\}$ and is controlled invariant. Since also $V \subset V^*(S)$, this contradicts (ii).

(ii) $\Leftrightarrow$ (iii). This follows immediately from TH. 1.27.

(iii) $\Leftrightarrow$ (viii). This follows from COR. 2.12 and an argument analogous to the proof of TH. 2.18, implication (iii) $\Leftrightarrow$ (ii).

(viii) $\Leftrightarrow$ (ix). This equivalence follows immediately from REMARK 2.9.

The equivalences (iii) $\Leftrightarrow$ (v) and (v) $\Leftrightarrow$ (vi) follow in the same way.

We will now prove (ii) $\Leftrightarrow$ (iv). Assume that there are x_1, $x_2 \in \Sigma_D(A,B)$, $x_1(0) = x_2(0)$, with x_1^+, x_2^+ lying in S and $x_1^+ \neq x_2^+$. Denote $z := x_1^+ - x_2^+$. Define a subspace $V \subset S$ by $V := \operatorname{span}_{\varphi\in D(\mathbb{R})} \{<z,\varphi>\}$. We claim that $\dot{z} \in V$. Indeed, this is easy since $<\dot{z},\varphi> = <z, \dot{\varphi}> \in V$. Since also $\dot{z} = Az + Bv$ for some $v \in U_D$ is follows that for every $\varphi \in D(\mathbb{R})$, $A<z,\varphi> \in V + B$. Hence, $V \in \underline{V}$. Thus we obtain $\{0\} \neq V \subset V^*(S)$, which contradicts (ii).

(iv) $\Rightarrow$ (vi). This implication is immediate.

We will conclude the proof by proving the equivalence (iii) $\Leftrightarrow$ (vii) :

If $S \in \underline{R}_a$, then by TH. 2.16, for all $x_o, x_1 \in S$ and $T \geq 0$, there is $x \in \Sigma_D^T(A,B)$ such that $x(0) = x_o$, $x(T^+) = x_1$ and $x^{[0,T]}$ lies in S. Assume x' is a second trajectory with these properties and $0 \neq z := x^{[0,T]} - x'^{[0,T]}$. Define then a subspace $V := \operatorname{span}_{\varphi\in D(\mathbb{R})} \{<z,\varphi>\}$. This yields a controlled invariant subspace $\{0\} \neq V \subset V^*(S) \cap R_a^*(S) = R^*(S)$, which contradicts (iii). Finally, if (vii) holds, then $S \in \underline{R}_a$. Moreover, the assumption $R^*(S) \neq 0$ would lead to distinct trajectories between two fixed points x_o and x_1.

□

REMARK 2.22. If S is a sliding subspace, then there is no 'regular motion' possible in S: the only possible trajectories in S are distributional trajectories. Since a sliding subspace is always an almost controllability subspace, it is possible to travel between any two points of the subspace along a regular trajectory, while staying arbitrarily close to it. It follows from (iv) that, in a sense, a sliding subspace is a distributional coasting subspace: if two trajectories in $\Sigma_D(A,B)$ have the same initial condition $x_o \in S$ and their restrictions to $[0,\infty)$ both lie in S, then these restrictions must in fact coincide. From (vii), note that for any two points in S and for every $T \geq 0$, there is *exactly one* (distributional) trajectory along which it is possible to travel from one point at $t = 0$ to the other at $t = T^+$ while remaining in S.

REMARK 2.23. It is also possible to give characterizations of coasting and sliding subspaces in terms of *matrix pencils*. For this, we refer to JAFFE & KARCANIAS (1981).

In the remainder of this section we will show that every almost controlled invariant subspace may be written as the direct sum of a controllability subspace, a coasting subspace and a sliding subspace.

LEMMA 2.24. *If $V \in \underline{V}(A,B)$ then there exists a coasting subspace C such that $V = R^*(V) \oplus C$.*

PROOF: It is well known that $\underline{F}(V) \subset \underline{F}(R^*(V))$. Choose $F \in \underline{F}(V)$ such that $\sigma(A_F|R^*(V)) \cap \sigma(A_F|V/R^*(V)) = \emptyset$. Define C to be the sum of the generalized eigenspaces of the mapping $A_F|V$ corresponding to those eigenvalues that are contained in $\sigma(A_F|V/R^*(V))$. Obviously, $V = R^*(V) \oplus C$ and also $A_F C \subset C$. Moreover, since $C \cap B \subset V \cap B \cap C \subset R^*(V) \cap C = \{0\}$, $R^*(C) = \{0\}$. The result then follows from TH. 2.21.

□

In the following, let K be a subspace of X. Let $\tilde{B} \subset B$ be such that $(B \cap V^*(K)) \oplus \tilde{B} = B$. Let W be a mapping such that im BW $= \tilde{B}$. Of course, (A,BW) defines a linear system with input space, say $U_1 \subset U$. The supremal almost controlled invariant subspace and supremal almost controllability subspace contained in a given subspace K', associated with this new system, will be denoted by $\tilde{V}^*_a(K')$ and $\tilde{R}^*_a(K')$. Also, for a given subspace K' of X, we have an almost controllability subspace algorithm associated with the system (A,BW):

$$(2.1) \qquad \tilde{S}^0(K') = \{0\} \ ; \quad \tilde{S}^{\mu+1}(K') = K' \cap (A\tilde{S}^{\mu}(K') + B).$$

It follows from COR. 1.11 that there is an integer $k \leq \dim K'$ such that $\tilde{S}^k = \tilde{S}^{k+\nu}$, $\forall\ \nu \in \mathbb{N}$. Moreover, if this is the case then $\tilde{S}^k = \tilde{R}^*_a(K')$. By taking

$K = K'$ in the above considerations, we obtain the following result (COMMAULT & DION (1981)) :

LEMMA 2.25. *$\tilde{R}^*_a(K) \cap V^*(K) = \{0\}$ and $R^*_a(K) = R^*(K) \oplus \tilde{R}^*_a(K)$.*

PROOF. The proof is an adaption of the proof of MORSE (1973 , LEMMA 4.1). We will first show that $\tilde{S}^\mu(K) \cap V^*(K) = \{0\}$, $\forall\mu$. For $\mu = 0$ this is obvious. Assume it is true for μ. Let $x \in \tilde{S}^{\mu+1} \cap V^*$. Then $x = A\bar{x}+b$, with $b \in \tilde{B}$ and $\bar{x} \in \tilde{S}^\mu$. Thus, $\bar{x} \in K \cap A^{-1}(V^* + B)$ which, by TH. 1.32, equals V^*. Hence, by induction hypothesis, $\bar{x} = 0$. It follows that $x \in \tilde{B}$. Since also $x \in V^*$ and since by definition $B \cap V^* = \{0\}$, it follows that $x = 0$. To prove the second assertion, by TH. 1.32 it suffices to prove that $S^\mu(K) = S^\mu(V^*)+\tilde{S}^\mu(K)$. For $\mu = 0$, this clearly holds. Suppose now it holds for μ. Then we have

$$\begin{aligned}
S^{\mu+1}(K) &= (AS^\mu(K) + B) \cap K \\
&= (AS^\mu(V^*) + B + AS^\mu(K)) \cap K \\
&= ([AS^\mu(V^*) + B] \cap [V^* + \tilde{B}] + A\tilde{S}^\mu(K)) \cap K \\
&= ([AS^\mu(V^*) + B] \cap V^* + \tilde{B} + A\tilde{S}^\mu(K)) \cap K \\
&= S^{\mu+1}(V^*) + \tilde{S}^{\mu+1}(K).
\end{aligned}$$

Here, we used that fact that $AS^\mu(V^*) + B \subset V^* + B$ and the fact that if $V_1 \subset W$, then $(V_1 + V_2) \cap W = V_1 + V_2 \cap W$.

□

The above leads to the following:

LEMMA 2.26. *Let K be a subspace of X. Then $V^*_a(K) = V^*(K) \oplus \tilde{R}^*_a(K)$. Moreover, $\tilde{R}^*_a(K)$ is a sliding subspace.*

PROOF. From TH. 1.32, $V^*_a(K) = V^*(K) + R^*_a(K)$. Thus, by LEMMA 2.25, $V^*_a(K) = V^*(K) \oplus \tilde{R}^*_a(K)$. To show that $\tilde{R}^*_a(K)$ is a sliding subspace, note that it is clearly in $\underline{R}_a(A,B)$. Moreover, it is immediate that

$$R^*(\tilde{R}^*_a(K)) \subset R^*(K) \cap \tilde{R}^*_a(K) = \{0\}.$$

□

We thus obtain the decomposition result as announced in the introduction to this section:

<u>THEOREM 2.27</u>. *Let* $V_a \in \underline{V}_a(A,B)$. *There exists a coasting subspace* C *and a sliding subspace* S *such that* $V_a = R^*(V_a) \oplus C \oplus S$, *with* $R^*(V_a) \oplus C = V^*(V_a)$ *and* $R^*(V_a) \oplus S = R^*_a(V_a)$.

<u>PROOF</u>: In LEMMA 2.26, take $K = V_a$. Define $S := \tilde{R}^*_a(V_a)$. We then find directly that $V_a = V^*_a(V_a) = V^*(V_a) \oplus S$. By LEMMA 2.24, there is a coasting subspace C such that $V^*(V_a) = R^*(V^*(V_a)) \oplus C$. Since $R^*(V^*(V_a)) = R^*(V_a)$, the result follows.

□

<u>REMARK 2.28</u>. Whereas coasting subspaces are associated with transmission zeros, sliding subspaces are associated with 'zeros at infinity'. Let (A,B,D) be a system with $p\times m$ transfer matrix $G(s) := D(Is-A)^{-1}B$. A square matrix of rational functions will be called a *bicausal isomorphism* if it is proper and invertible and if its inverse is again proper. It may be proven (see e.g. COMMAULT & DION (1982) that there exist bicausal isomorphism $B_1(s)$ and $B_2(s)$ such that $G(s) = B_1(s)\Lambda(s)B_2(s)$, where $\Lambda(s)$ is $p\times m$ and given by

$$\Lambda(s) = \begin{pmatrix} \Lambda_1(s) & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.2}$$

with $\Lambda_1(s) = \text{diag}(s^{-n_1},\ldots, s^{-n_r})$, $n_1 \geq n_2 \geq \ldots \geq n_r$, $r = \text{rank } G(s)$. The form (2.2) is called the *Smith-McMillan form at infinity*. The integers $n_1,\ldots, n_r$ will be called *the orders of the zeros at infinity of* (A,B,D).

Let $K := \ker D$. It follows from LEMMA 2.26, that $V^*_a(K)=V^*(K) \oplus R^*_a(K)$, with $\tilde{R}^*_a(K) = \tilde{S}^k(K)$ for some integer $k \leq \dim K$. Define integers ρ_i by:

$$\begin{aligned} &\rho_1 := \dim \tilde{B} \ (= \dim(V^*(K) + B) - \dim V^*(K)) \\ &\rho_i := \dim \tilde{S}^{i-1}(K) - \dim \tilde{S}^{i-2}(K) \quad (i = 2,\ldots, k+1) \end{aligned} \tag{2.3}$$

It has been shown in COMMAULT & DION (1981) , that the infinite zero orders of the system (A,B,D) are related to the integers $\rho_1,\ldots, \rho_{k+1}$ by the relation

$$n_i = \text{number of integers (counting multiplicity) in the set } \{\rho_1,\ldots, \rho_{k+1}\} \text{ that are larger than or equal to } i. \tag{2.4}$$

On the other hand, it has been proven in COR. 1.11. that there exist a mapping F and a chain $\{B_i\}_{i=1}^k$ in $\tilde{B}$, such that $\tilde{R}^*_a(K) = B_1 \oplus A_F B_2 \oplus \ldots \oplus A_F^{k-1} B_k$, with $\dim B_i = \dim A_F^{i-1} B_i = \rho_{i+1}$.

Now, choose a basic for $\widetilde{R}^*_a(K)$ as follows: first choose a basis $b_1, \ldots, b_{\rho_{k+1}}$ of B_k. Extend this to a basis for B_{k-1} by adding vectors $b_{\rho_{k+1}+1}, \ldots, b_{\rho_k}$. Next, extend this to a basis for B_{k-2}, etc. Preceding this way, we ultimately find a basis for B_1. However, by the equality $\dim B_i = \dim A_F^{i-1} B_i$, this immediately yields the following basis for $\widetilde{R}^*_a(K)$:

$$
\begin{array}{l}
A_F^{k-1} b_1, \ldots, A_F^{k-1} b_{\rho_{k+1}}, \\
A_F^{k-2} b_1, \ldots, A_F^{k-2} b_{\rho_{k+1}}, A_F^{k-2} b_{\rho_{k+1}+1}, \ldots, A_F^{k-2} b_{\rho_k}, \\
\qquad \vdots \qquad\qquad\qquad \vdots \\
A_F b_1, \ldots, A_F b_{\rho_{k+1}}, A_F b_{\rho_{k+1}+1}, \ldots, A_F b_{\rho_k}, \ldots, \ldots, A_F b_{\rho_3}, \\
b_1, \ldots, b_{\rho_{k+1}}, b_{\rho_{k+1}+1}, \ldots, b_{\rho_k}, \ldots, \ldots, b_{\rho_3}, b_{\rho_3+1}, \ldots, b_{\rho_2}.
\end{array} \tag{2.5}
$$

It may be shown that $\rho_1 = r$ ($=$ rank $G(s)$) and it can be verified by inspection that the list of vectors (2.5) may be rearanged to obtain r singly generated almost controllability subspaces $\mathcal{L}_i$, where we define

$$\mathcal{L}_i := \text{span } \{ b_i, A_F b_i, \ldots, A_F^{n_i-2} b_i \} \text{ if } n_i > 1,$$
$$\mathcal{L}_i := \{0\} \text{ if } n_i = 1.$$

We may thus conclude that if the system (A,B,D) has r zeros at infinity of respective orders $n_1, \ldots, n_r$, then the almost controlled invariant subspace $V^*_a(\ker D)$ has a direct sum decomposition

$$V^*_a(\ker D) = V^*(\ker D) \oplus \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \ldots \oplus \mathcal{L}_r,$$

where $\mathcal{L}_i$ is a singly generated almost controllability subspace of dimension $n_i - 1$ and where the sum $\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \ldots \oplus \mathcal{L}_r$ is a sliding subspace.

2.4 APPROXIMATION OF ALMOST CONTROLLED INVARIANT SUBSPACES

It turns out that every almost controlled invariant subspace may be approximated by controlled invariant subspaces. A proof of this property can be given as follows: first decompose the given almost controlled invariant subspace into the direct sum of a controlled invariant subspace and an almost controllability subspace. It was shown in section 2.3 that this can be done. Next, decompose the latter almost controllability subspace into the direct sum of a number of singly generated almost controllability subspaces. It may be shown in fairly simple way that every singly generated almost controllability subspace may be approximated by controlled invariant subspaces. Finally, form the direct sum of the controlled invariant subspace appearing in the first decomposition and the approximants appearing in the latter. The resulting subspaces then form an approximation of the original subspace.

In this section we will make the above precise. For each singly generated almost controllability subspace $\mathcal{L}$, we will define a canonical sequence $\{\mathcal{L}(n)\}_{n\in\mathbb{N}}$ converging to it. On each of the terms $\mathcal{L}(n)$ of this sequence we will then define a mapping F_n with values in the input space U and prove some properties of the sequence $\{F_n\}_{n\in\mathbb{N}}$ that will be useful in the sequel.

When we talk about the convergence of subspaces, it will always be understood that this takes place in the usual Grassmannian sense. Let $\underline{G}(q,X)$ denote the set of all q-dimensional subspaces of the real n-dimensional linear space X. After a basis choice in X, every element $V \in \underline{G}(q,X)$ is determined by a real $n\times q$ matrix M, called a representative of V. Moreover, for every V, there exist integers $1 \leq \alpha_1 \leq \alpha_2 < \dots \leq \alpha_q \leq n$ (depending on V) such that in each representative the rows α_1 to α_q are linearly independent. Thus, V admits exactly one representative in which these rows form the $q\times q$ indentity matrix. Let $Z(V)$ denote the $(n-q)\times q$ matrix formed by the remaining rows and consider $Z(V)$ as an element of $\mathbb{R}^{q(n-q)}$. This yields a bijection of a set $S \subset \underline{G}(q,X)$ containing V onto $\mathbb{R}^{q(n-q)}$, defined by φ_V: $V' \mapsto Z(V')$. (This bijection is defined for all subspaces V' with the property that the rows $\alpha_1,\dots,\alpha_q$ in any of their representatives are independent). Now, define a topology on $\underline{G}(q,X)$ by taking the topology that has the family of subspaces $\{\varphi_V^{-1}(U) \mid U$ is a neighbourhood of 0 in $\mathbb{R}^{q(n-q)}\}$ as its basis at V.

It may be shown that the bijections φ_V form a C^∞-atlas for $\underline{G}(q,X)$. This atlas turns $\underline{G}(q,X)$ into a compact differentiable manifold called a Grassmannian manifold (see also BRICKELL & CLARK (1970)).

The following lemma provides a useful criterion for convergence in the above topology and may be proven by standard means:

<u>LEMMA 2.29.</u> *Let* $\{V_n\}_{n\in\mathbb{N}}$ *and* V *be subspaces of* X *of a given dimension. Then* $\lim_{n\to\infty} V_n = V$ *if and only if there is a basis* $\{v_1,\dots,v_q\}$ *for* V *and bases* $\{v_1(n),\dots, v_q(n)\}$ *for* V_n *such that* $\lim_{n\to\infty} v_i(n) = v_i$ $(i = 1,\dots,q)$.

□

We will also use the following useful fact:

<u>LEMMA 2.30</u>. *Suppose* $v_1,\dots v_q$ *are linearly independent vectors in* X *and* $\lim_{n\to\infty} v_i(n) = v_i$ $(i=1,\dots,q)$. Then there exists an integer $K \in \mathbb{N}$ *such that for* $n \geq K$ *the vectors* $v_1(n),\dots,v_q(n)$ *are linearly independent.*

□

It follows immediately from the above properties that if $\lim_{n\to\infty} V_n = V$ and $\lim_{n\to\infty} W_n = W$, where V and W are subspaces such that $V \cap W = \{0\}$, then for n sufficiently large, $V_n \cap W_n = \{0\}$ and $\lim_{n\to\infty} V_n \oplus W_n = V \oplus W$.

We will now show that for each singly generated almost controllability subspace a sequence of controlled invariant subspaces may be found converging to it. Let $\mathcal{L} = \text{span}\,\{b, A_F b,\dots, A_F^{k-1}b\}$. Assume that the vectors $b, A_F b,\dots, A_F^{k-1}b$ are linearly independent. (It may be seen that the latter can always be achieved by removing the vectors corresponding to the highest powers of A_F: if $\mathcal{L} = \text{span}\,\{b, A_F b,\dots, A_F^{\ell-1}b\}$ has dimension $k < \ell$, then, in fact, $\mathcal{L} = \text{span}\,\{b, A_F b,\dots, A_F^{k-1}b\}$.) Assume that $b = Bu$. The vector u will be called the *generator* of $\mathcal{L}$ and the mapping F its *feedback*. In the sequel, we will denote the k-dimensional singly generated almost controllability subspace with generator u and feedback F by $\mathcal{L}(u,F,k)$.

It will now be shown how $\mathcal{L}(u,F,k)$ can be approximated by controlled invariant subspaces. For $n \in \mathbb{N}$, consider the mapping $I + \frac{1}{n}A_F$. Obviously, for $n \in \mathbb{N}$ sufficiently large, this mapping is non-singular. (In fact, if $\sigma := \max\{|\lambda| : \lambda \in \sigma(A_F)\}$, then $n > \sigma$ will do.) For n sufficiently large, define sequences of vectors $x_i(n)$ $(i \in \underline{k})$ recursively by

$$(2.6)\qquad \begin{cases} x_1(n) := (I + \frac{1}{n}A_F)^{-1}\, Bu \\ x_{i+1}(n) := (I + \frac{1}{n}A_F)^{-1}\, A_F x_i(n) \end{cases}$$

Define $\mathcal{L}(n) := \text{span}\,\{x_1(n),\ldots, x_k(n)\}$. Then the following holds:

LEMMA 2.31. *For* $i \in \underline{k}$, $\lim_{n\to\infty} x_i(n) = A_F^{i-1}Bu$ *and* $\lim_{n\to\infty} \mathcal{L}(n) = \mathcal{L}(u,F,k)$. *Moreover,* $\mathcal{L}(n)$ *is controlled invariant and* $\mathcal{L}(n) \subset \langle A|B\rangle, \forall n$.

PROOF: Of course, $\lim_{n\to\infty} x_1(n) = Bu$. If $\lim_{n\to\infty} x_i(n) = A_F^{i-1}Bu$, then it immediately follows from (2.6) that $\lim_{n\to\infty} x_{i+1}(n) = A_F^{i}Bu$. It follows from LEMMA 2.30 that the vectors $x_1(n),\ldots, x_k(n)$ are linearly independent for n sufficiently large. Hence, by LEMMA 2.29, $\lim_{n\to\infty} \mathcal{L}(n) = \mathcal{L}(u,F,k)$. To show that $\mathcal{L}(n) \in \underline{V}$, note that $A_F x_1(n) = -n\, x_1(n) + n\,Bu \in \mathcal{L}(n) + B$. Suppose now that $A_F x_i(n) \in \mathcal{L}(n) + B$. Then $A_F x_{i+1}(n) = -n\,x_{i+1}(n) + n\,A_F x_i(n) \in \mathcal{L}(n) + B$. The last assertion follows from the fact that $(I + \frac{1}{n} A_F)^{-1} = \sum_{n=0}^{\infty}(-1)^m \frac{A_F^m}{n^m}$.

□

Combining the results of section 2.3 with the above lemma yields

THEOREM 2.32. *Let* $V_a \in \underline{V}_a(A,B)$. *Then there exists a sequence* $\{V_n\}_{n\in\mathbb{N}}$ *with* $V_n \in \underline{V}(A,B)$, *such that* $\lim_{n\to\infty} V_n = V$.

PROOF: By TH. 2.27, every $V_a \in \underline{V}_a$ may be decomposed into $V_a = V \oplus R_a$ with $V = V^*(V_a)$ and $R_a \in \underline{R}_a$. (In fact, R_a may be chosen to be a sliding subspace). Since $R_a \in \underline{R}_a$, by COR. 1.23 there is a F and a chain $\{B_i\}_{i=1}^k$ in B such that $R_a = B_1 \oplus A_F B_2 \oplus\ldots\oplus A_F^{k-1}B_k$ with $\dim B_i = \dim A_F^{i-1} B_i$ $(i \in \underline{k})$. Choose now a basis of B_k, extend this to a basis of B_{k-1} and continue this extension procedure until we have a basis for B_1. In the same way as in REMARK 2.28 this produces a basis for R_a which can be arranged such that $R_a = \bigoplus_{i=1}^{s} \mathcal{L}(u_i, F, r_i)$ for integers r_i,s and vectors $u_i \in U$. Now, let $\mathcal{L}_i(n) \in \underline{V}$ converge to $\mathcal{L}(u_i, F, r_i)$. It may then immediately be seen that $\lim_{n\to\infty} [V \oplus \bigoplus_{i=1}^{s}\mathcal{L}_i(n)] = V_a$.

□

REMARK 2.33. It may be shown that if in the above we choose R_a to be a sliding subspace and if D' is a mapping such that $R_a = \ker D'$, then the integers $r_1,\ldots, r_s$ are the infinite zero orders of the system (A,B,D') and $s = m (= \dim U)$.
We could also directly have applied REMARK 2.28 to the subspace $K = V_a$. In that case, we would have found a decomposition of V_a into $V_a = V^*(V_a) \oplus \bigoplus_{i=1}^{r} \mathcal{L}(\bar{u}_i, \bar{F}, n_i-1)$, where the integers $n_1,\ldots,n_r$ are the infinite zero orders of the system (A,B,D), with D a mapping such that

$\ker D = V_a$.

In the sequel, the subspace $\mathcal{L}(n)$ as defined by (2.6) will also be denoted as $\mathcal{L}_n(u,F,k)$ and the vectors $x_i(n)$ will also be denoted as $x_i(n,u)$, $(i \in \underline{k})$. Given a singly generated almost controllability subspace $\mathcal{L}(u,F,k)$, define a sequence of mappings $F_n : \mathcal{L}_n(u,F,k) \to U$ by:

(2.7) $\qquad F_n x_i(n,u) := -n^i u, \quad i \in \underline{k}.$

The mapping F_n then turns out to make $\mathcal{L}_n(u,F,k)$ invariant under $A + BF_n$:

LEMMA 2.34. $F_n \in \underline{F}(\mathcal{L}(n))$ *and the matrix of* $(A_F + BF_n)|\mathcal{L}(n)$ *with respect to the basis* $x_1(n),\dots, x_k(n)$ *is given by*

$$M(n) = -\begin{pmatrix} n & n^2 & \dots & n^k \\ 0 & n & \ddots & \vdots \\ \vdots & & \ddots & n^2 \\ 0 & \dots & 0 & n \end{pmatrix}.$$

PROOF: It follows by straightforward calculation using (2.6) that $(A_F + BF_n)x_i(n) = -\sum_{j=1}^{i} n^{1+i-j}\, x_j(n)$.

□

We will now prove an important property of the sequence of mappings $\{F_n\}$ as defined by (2.7). In the following, if $f : \mathbb{R}^+ \to \mathbb{R}$ is a measurable function, we will denote its L_p-norm by:

$$\|f\|_p := \begin{cases} \operatorname*{ess\,sup}_{t\in\mathbb{R}^+} |f(t)| & \text{if} \quad p = \infty, \\ \left(\int_0^\infty |f(t)|^p\, dt\right)^{\frac{1}{p}} & \text{if} \quad 1 \le p < \infty. \end{cases}$$

Consider the subspace $\mathcal{L} = \mathcal{L}(u,F,k)$ and let $\mathcal{L}(n) = \mathcal{L}_n(u,F,k) = \operatorname{span}\{x_1(n),\dots, x_k(n)\}$ be its approximating sequence. Let $F_n : X \to U$ be a mapping, on $\mathcal{L}(n)$ defined by (2.7) and extended arbitrarily to X. Consider the sytsem $\dot{x}(t) = A_F x(t) + Bu(t)$ with initial condition $x(0) = x_i(n)$. Apply the feedback law $u = F_n x$. Then, by LEMMA 2.34, the resulting trajectory $x_n(t) := e^{(A+BF_n)t}x_1(n)$ is entirely contained in $\mathcal{L}(n)$. We now ask ourselves the question: given $\varepsilon > 0$, is it possible by chosing n sufficiently large to keep the trajectory $x_n(t)$ within an ε-distance from the original subspace

$\mathcal{L}$ for all $t \geq 0$? At first thought, one might be tempted to think that the only thing one needs to do to achieve this, is to simply choose n such that $\mathcal{L}$ and $\mathcal{L}(n)$ are close. However, the fact that subspaces are close in Grassmannian sense does of course not imply that every two points in these subspaces are close in the original norm on X. It is possible that as $n \to \infty$, $\sup_{t \in \mathbb{R}^+} \|x_n(t)\| \to \infty$. In such a case this behaviour of the trajectories x_n could counteract the effect obtained by decreasing the subspace distance between $\mathcal{L}$ and $\mathcal{L}(n)$.

Still, due to the very special structure of the vectors $x_i(n)$, our question may be answered in an affimative way. In fact, not only the supremum norm of the distance $d(x_n(t),\mathcal{L})$ can be made arbitrarily small by choosing n sufficiently large, but even the L_p-norm of the distance function $d(x_n(t),\mathcal{L})$ $(t \geq 0)$ for all $1 \leq p \leq \infty$ simultaneously:

THEOREM 2.35. *Let* $\mathcal{L} = \mathcal{L}(u,F,k)$ *and let* F_n *be defined by* (2.7). *Then* $\forall \varepsilon > 0 \;\; \exists \;\; K \in \mathbb{N}$ *such that for all* $i \in \underline{k}$ *and all* $1 \leq p \leq \infty$

$$\|d(\mathcal{L}, e^{(A_F + BF_n)t} x_i(n,u))\|_p \leq \varepsilon, \textit{ for all } n \geq K.$$

We will use this result in section 2.6 to construct feedback mappings in the almost disturbance decoupling problem. The proof of the theorem goes through a series of smaller results. Let $\mathcal{L}(n) = \mathcal{L}_n(u,F,k)$ and define a mapping $D_n\colon \mathcal{L}(n) \to \mathcal{L}(n)$ by $D_n\, x_i(n) := -n\, x_i(n)$, $(i \in \underline{k})$. Define a nilpotent mapping $N_n\colon \mathcal{L}(n) \to \mathcal{L}(n)$ by $N_n := (A_F + BF_n) \mid \mathcal{L}(n) - D_n$. Clearly the matrix of N_n in the basis $\{x_1(n),\ldots, x_k(n)\}$ is given by

$$\text{(2.8)} \qquad \operatorname{mat} N_n = \begin{pmatrix} 0 & n^2 & \cdots & n^k \\ \vdots & 0 & n^2 & \vdots \\ \vdots & & \ddots & n^2 \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$$

LEMMA 2.36. *Let* $i \in \underline{k}$. *Then for* $j = i, i+1,\ldots, k$ *we have* $N_n^j x_i(n) = 0$. *On the other hand, for* $j = 1, 2,\ldots, i-1$,

$$\text{(2.9)} \qquad N_n^j\, x_i(n) = \sum_{\ell_1=1}^{i-1} \; \sum_{\ell_2=1}^{\ell_1-1} \cdots \sum_{\ell_j=1}^{\ell_{j-1}-1} (-1)^j \, n^{j+i-\ell_j} \, x_{\ell_j}(n).$$

PROOF: Use (2.8) to find an expression for $N_n x_i(n)$. Apply N_n to this result, etc. (note: for consistency, define $\ell_0 := i$). □

Another technical ingredient that we will need in our proof is:

LEMMA 2.37. *Let* $i \in \underline{k}$. *Then* $x_i(n) = A_F^{i-1}\, Bu - \frac{1}{n} \sum_{\ell=1}^{i} A_F^{i-\ell+1}\, x_\ell(n)$.

PROOF: This follows immediately by induction, using (2.6).

□

Finally, we will need the following result which tells us how 'fast' $x_i(n)$ converges to $\mathcal{L}$:

LEMMA 2.38. *Let* $i \in \underline{k}$. *Then* $\lim_{n\to\infty} n^{k-i+1} d(\mathcal{L}, x_i(n)) < \infty$.

PROOF: Using LEMMA 2.37, iterating the formula for $x_i(n)$, it may be verified immediately that

$$x_i(n) = A_F^{i-1}Bu - \frac{1}{n}\Big(\sum_{\ell_1=1}^{i} A_F^{i}Bu\Big) + \frac{1}{n^2}\Big(\sum_{\ell_1=1}^{i}\sum_{\ell_2=1}^{\ell_1} A_F^{i+1}Bu\Big) + \ldots$$

$$(2.10) \qquad \ldots + (-1)^{k-i}\frac{1}{n^{k-i}}\Big(\sum_{\ell_1=1}^{i}\sum_{\ell_2=1}^{\ell_1}\cdots\sum_{\ell_{k-i}=1}^{\ell_{k-i-1}} A_F^{k-1}\,Bu\Big)$$

$$+ (-1)^{k-i+1}\frac{1}{n^{k-i+1}}\Big(\sum_{\ell_1=1}^{i}\sum_{\ell_2=1}^{\ell_1}\cdots\sum_{\ell_{k-i+1}=1}^{\ell_{k-i}} A_F^{k-\ell_{k-i+1}}\, x_{\ell_{k-i+1}}(n)\Big)$$

Now, in this expansion, all terms but the last one between brackets are contained in $\mathcal{L}$. Denote the last term between brackets by $v(n)$. Obviously, $\lim_{n\to\infty} v(n)$ exists, since $\lim_{n\to\infty} x_\ell(n)$ exists for all $\ell \in \underline{k}$. Thus we obtain

$$d(\mathcal{L}, x_i(n)) = d\Big(\mathcal{L}, \frac{(-1)^{k-i+1}}{n^{k-i+1}}\, v(n)\Big) = \frac{1}{n^{k-i+1}}\; d(\mathcal{L}, v(n))$$

The result then follows.

□

PROOF OF THEOREM 2.35: By the nilpotency of the mapping N_n, for $i \in \underline{k}$ we have

$$e^{(A_F+BF_n)t}\, x_i(n) = e^{N_n t}\, e^{D_n t}\, x_i(n) = \sum_{j=0}^{k-1} \frac{t^j N_n^{\,j}}{j\,!}\, e^{-nt} x_i(n).$$

By the triangular inequality it therefore suffices to show that for

$j = 0,1,\ldots, k-1$, $\lim_{n\to\infty} \|d(\mathcal{L}, t^j e^{-nt} N_n^j x_i(n)\|_p = 0$ and that this limit is achieved uniformly for $1 \le p \le \infty$. Apply now LEMMA 2.36 to find an expression for $N_n^j x_i(n)$. Again by the triangular inequality, it is sufficient to prove that $\lim_{n\to\infty} \|d(\mathcal{L}, t^j e^{-nt} n^{j+i-\ell} x_\ell(n))\|_p = 0$, uniformly in p for all i, j, $\ell \in \underline{k}$. Now, note that

$$\|d(\mathcal{L}, t^j e^{-nt} n^{j+i-\ell} x_\ell(n)\|_p = n^{j+i-\ell}\|t^j e^{-nt}\|_p d(x_\ell(n),\mathcal{L}).$$

It may be verified that for $1 \le p < \infty$, $\|t^j e^{-nt}\|_p = (\frac{1}{pn})^{j+\frac{1}{p}} \Gamma(pj+1)^{\frac{1}{p}}$ and that $\|t^j e^{-nt}\|_\infty = (\frac{j}{n})^j e^{-j}$. Here Γ denotes the gamma function. Using Stirling's formula for the gamma function (HILLE (1959), p. 235) it may be seen that there is a constant c such that for all $1 \le p < \infty$

$$\|t^j e^{-nt}\|_p \le c\, n^{-j-\frac{1}{p}}.$$

Hence, there is a constant c_1 such that for all $1 \le p \le \infty$

$$\begin{aligned}\|d(\mathcal{L}, t^j e^{-nt} n^{j+i-\ell} x_\ell(n)\|_p &\le c_1 n^{i-\ell-\frac{1}{p}} d(x_\ell(n),\mathcal{L}) \\ &\le c_1 n^{i-\ell} d(x_\ell(n),\mathcal{L}).\end{aligned}$$

The fact that the latter expression tends to 0 as $n \to \infty$ then follows from LEMMA 2.38. This concludes the proof of TH. 2.35.

□

2.5 CONTROLLED INVARIANT COMPLEMENTS

In the present section, we will establish a result that will lie at the basis of the application of almost controlled invariant subspaces to synthesis problems involving high gain feedback. In the sequel, a subset Λ of the field of complex numbers $\mathbb{C}$ will be called *symmetric* if $\Lambda \cap \mathbb{R} \neq \emptyset$ and if $\lambda \in \Lambda$ if and only if $\bar{\lambda} \in \Lambda$. Here $\bar{\lambda}$ denotes the complex conjugate of λ. Consider the system with system mapping A and input mapping B. Then we have:

<u>THEOREM 2.39</u>. *Let* $R_a \in \underline{R}_a(A,B)$ *and suppose* Λ *is a symmetric set of* $\dim \langle A|B\rangle - \dim R_a$ *complex numbers. Then there exist a subspace* $V \in \underline{V}(A,B)$ *and a mapping* $F \in \underline{F}(V)$ *such that*

(2.11) $\quad V \oplus R_a = \langle A|B\rangle$,

(2.12) $\quad \sigma(A_F|V) = \Lambda$.

REMARK 2.40. The above result was originally proven in TRENTELMAN (1983) under the assumption that (A,B) is controllable. The special case $R_a = B$ has been treated in WONHAM (1979, LEMMA 3.5). The latter result was used in its dual form to obtain the existence of a reduced order 'Luenberger' observer. In a similar fashion, TH. 2.39 can be dualized to obtain results in the context of reduced order 'PID' observers. This aspect of TH. 2.39 will be elaborated in detail in CH. 5, SECTION 5.5.

At the present, our main motivation to prove the above theorem is that it will enable us to give characterizations of almost controlled invariant subspaces in terms of high gain state feedback. This will be done in sections 2.6, 2.7 and 2.8.

In the remainder of this section we will give a proof of TH. 2.39:

PROOF OF THEOREM 2.39 : By COR. 1.23, $R_a = B_1 \oplus A_F B_2 \oplus \dots \oplus A_F^{k-1} B_k$, where $\{B_i\}_{i=1}^k$ is a chain in B, $B_1 = B \cap R_a$ and $\dim B_i = \dim A_F^{i-1} B_i$ ($i \in \underline{k}$). Let B_1' be a subspace such that $B_1' \oplus B_1 = B$. For i = 2,..., k, let B_i' be such that $B_i' \oplus B_i = B_{i-1}$. Denote $B_{k+1}' := B_k$. We then have the following direct sum decomposition of R_a:

(2.13) $$R_a = B_2' \oplus S_1 \oplus B_3' \oplus S_2 \oplus \dots \oplus B_k' \oplus S_{k-1} \oplus B_{k+1}',$$

where $S_i := A_F B_{i+2}' \oplus \dots \oplus A_F^{k-i} B_{k+1}'$. Define $S_k := \{0\}$. For i = 1,..., k − 1, we have $A_F(B_{i+2}' \oplus S_{i+1}) = S_i$. Let $G_i : S_i \to B_{i+2}' \oplus S_{i+1}$ be a mapping such that $A_F G_i = I_{S_i}$, the identity mapping of S_i.

We have a decomposition $B = B_1' \oplus B_2' \oplus \dots \oplus B_{k+1}'$. This decomposition yields a decomposition $U = U_1' \oplus U_2' \oplus \dots \oplus U_{k+1}'$, where $U_i' := B^{-1} B_i'$ (i = 1,..., k + 1).

Since $B(U_2' \oplus U_3' \oplus \dots \oplus U_{k+1}') = B_2' \oplus B_3' \oplus \dots \oplus B_{k+1}'$, there is a mapping G such that $BG = I_{B'}$, where we have written $I_{B'}$ for the identity mapping of $B_2' \oplus B_3' \oplus \dots \oplus B_{k+1}'$.

Let us now first prove the following lemma:

LEMMA 2.41. *Let* F *be as in the proof of* TH. 2.39 *above. There exists a* *subspace* $D_o \subset \langle A|B\rangle$ *and a mapping* $F_o : X \to U$ *such that* $D_o \oplus R_a = \langle A|B\rangle$ *and* $\sigma(P_{D_o}(A_F + BF_o)|D_o) = \Lambda$. *Here,* P_{D_o} *denotes the projection of* $\langle A|B\rangle$ *onto* D_o *along* R_a.

PROOF OF LEMMA 2.41: Take a subspace $D \subset \langle A|B\rangle$ containing B_1' such that $D \oplus R_a = \langle A|B\rangle$. Define a mapping $Q : R_a \times U_1' \to D$ by $(r,u) \mapsto P_D A_F r + Bu$ (P_D denotes the projection of $\langle A|B\rangle$ onto D along R_a. Note that, since $u \in U_1'$, $Bu \in B_1' \subset D$). Consider the system with system mapping $P_D A_F|D$ and input mapping Q. We contend that this system is controllable. If it were not, there would be a subspace $L \subset D$, $L \neq D$ with $P_D A_F L \subset L$ and $\operatorname{im} Q \subset L$. This however implies that $A_F(\mathcal{L} \oplus R_a) \subset L \oplus R_a$ and $B \subset L \oplus R_a$. Since $L \oplus R_a \neq \langle A|B\rangle$, this contradicts the fact that the system with system mapping $A_F|\langle A|B\rangle$ and input mapping B is controllable.

Now, let $K: D \to R_a \times U_1'$ be such that $\sigma(P_D A_F|D + QK) = \Lambda$. Let $P_{U_1'}$ and P_{R_a} be mappings on $R_a \times U_1'$, defined by $(r,u) \mapsto u$ and $(r,u) \mapsto r$ respectively. Define $F_o : X \to U$ by $F_o|D := P_{U_1'} K$, $F_o|R_a := 0$ and F_o arbitrary on a complement of $\langle A|B\rangle$. Define a mapping $S : \langle A|B\rangle \to \langle A|B\rangle$ by $S|D := I_D + P_{R_a}K$, $S|R_a := I_{R_a}$. Finally, define $D_o := SD$. It may be verified that $\langle A|B\rangle = D_o \oplus R_a$. Moreover, using that fact that $P_D P_{R_a} = 0$ and $P_{D_o} = SP_D$, a direct computation shows that the following diagram commutes:

$$\begin{tikzcd}
D \arrow[rr, "P_D A_F|D + QK"] \arrow[d, "S \simeq"'] & & D \arrow[d, "\simeq S"] \\
D_o \arrow[rr, "P_{D_o}(A_F + BF_o)|D_o"'] & & D_o
\end{tikzcd}$$

Since S is an isomorphism of D and D_o, the lemma follows. □

PROOF OF TH. 2.39. (CONTINUED): Let D_o and F_o be as in the previous lemma. Make a decomposition

$$\langle A|B\rangle = D_0 \oplus B_2' \oplus S_1 \oplus B_3' \oplus S_2 \oplus \ldots \oplus B_k' \oplus S_{k-1} \oplus B_{k+1}' . \tag{2.14}$$

Let P_i be the projection of $\langle A|B\rangle$ onto S_i along the other members of this decomposition. Let $P_{B'}$ be the projection onto $B' := B_2' \oplus \ldots \oplus B_{k+1}'$ along

$\mathcal{D}_o \oplus \mathcal{S}_1 \oplus \ldots \oplus \mathcal{S}_{k-1}$. Denote $A_o := A_F + BF_o$. Define a mapping $T_o: \langle A|\mathcal{B}\rangle \to \langle A|\mathcal{B}\rangle$ by: $T_o|\mathcal{D}_o := I_{\mathcal{D}_o} - G_1P_1A_o|\mathcal{D}_o$, $T_o|\mathcal{R}_a := I_{\mathcal{R}_a}$. Define then mappings T_i and A_{i+1} inductively by $T_i|\mathcal{D}_o := I_{\mathcal{D}_o} - G_{i+1}P_{i+1}A_i|\mathcal{D}_o$, $T_i|\mathcal{R}_a := I_{\mathcal{R}_a}$ and $A_{i+1} := T_i^{-1}A_iT_i$ $(i = 0, \ldots, k-2)$.
Let $T := T_oT_1 \ . \ . \ . \ T_{k-2}$. It may then be verified that

$$(2.15) \qquad T|\mathcal{D}_o = I_{\mathcal{D}_o} - \sum_{j=1}^{k-1} G_jP_jA_{j-1}|\mathcal{D}_o \ , \ T|\mathcal{R}_a = I_{\mathcal{R}_a} \ .$$

Define $F_1 : \mathcal{X} \to \mathcal{U}$ by $F_1|\mathcal{D}_o := -GP_{\mathcal{B}'}A_{k-1}|\mathcal{D}_o$, $F_1|\mathcal{R}_a := 0$ and F_1 arbitrary on a complement of $\langle A|\mathcal{B}\rangle$. Take $\mathcal{V} := T\mathcal{D}_o$ and define $F_{new} := F + F_o + F_1$. We assert that this $\mathcal{V}$ and F_{new} satisfy the conditions (2.11) and (2.12) of the theorem. First, it follows immediately from (2.15), together with the facts that $\operatorname{im} G_j \subset \mathcal{R}_a$ and $\mathcal{D}_o \oplus \mathcal{R}_a = \langle A|\mathcal{B}\rangle$, that $\mathcal{V} \oplus \mathcal{R}_a = \langle A|\mathcal{B}\rangle$. To prove (2.12), let $P_{\mathcal{V}}$ be the projection of $\langle A|\mathcal{B}\rangle$ onto $\mathcal{V}$ along $\mathcal{R}_a$. We will show that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{D}_o \arrow[r, "P_{\mathcal{D}_o}A_o|\mathcal{D}_o"] \arrow[d, "T \simeq"'] & \mathcal{D}_o \arrow[d, "\simeq T"] \\
\mathcal{V} \arrow[r, "P_{\mathcal{V}}(A + BF_{new})|\mathcal{V}"'] & \mathcal{V}
\end{tikzcd}$$

Let $x_o \in \mathcal{D}_o$. Using the facts that $F_1G_j = 0$, $F_oG_j = 0$, $P_{\mathcal{V}}P_{\mathcal{B}'} = 0$, $P_{\mathcal{V}}P_j = 0$, $A_FG_j = I_{\mathcal{S}_j}$ and $BG = T_{\mathcal{B}'}$, the following equality may be verified:
$P_{\mathcal{V}}(A + BF_{new})Tx_o = P_{\mathcal{V}}(A+BF_{new})(x_o - \sum_{j=1}^{k-1} G_jP_jA_{j-1}x_o) = P_{\mathcal{V}}A_ox_o$. Now, since $P_{\mathcal{V}} = TP_{\mathcal{D}_o}$, this equals $TP_{\mathcal{D}_o}A_ox_o$. It follows that $\sigma(P_{\mathcal{V}}|A+BF_{new})|\mathcal{V}) = \sigma(P_{\mathcal{D}_o}A_o|\mathcal{D}_o) = \Lambda$.
To complete the proof, it suffices to show that, in fact, $(A + BF_{new})\mathcal{V} \subset \mathcal{V}$
To show this, we will show that $P_jT^{-1}(A + BF_{new})\mathcal{V} = \{0\}$, $\forall j$ and that $P_{\mathcal{B}'}T^{-1}(A+BF_{new})\mathcal{V} = \{0\}$. If these assertions are true, then it follows immediately that $T^{-1}(A + BF_{new}) \subset \mathcal{D}_o$.
Let $x_1 = Tx_o \in \mathcal{V}$ with $x_o \in \mathcal{D}_o$. Then, using the fact that $T^{-1}|\mathcal{R}_a = I_{\mathcal{R}_a}$, it may be seen that

$$(2.16) \qquad T^{-1}(A + BF_{new})x_1 = T^{-1}A_ox_o - \sum_{j=1}^{k-1} P_jA_{j-1}x_o - P_{\mathcal{B}'}A_{k-1}x_o .$$

Now, writing $A_o x_o = P_{D_o} A_o x_o + P_{R_a} A_o x_o$, and applying T^{-1}, it follows that (2.16) is equal to

$$(2.17) \qquad A_o x_o + \sum_{j=1}^{k-1} G_j P_j A_{j-1} P_{D_o} A_o x_o - \sum_{j=1}^{k-1} P_j A_{j-1} x_o - P_{B'} A_{k-1} x_o.$$

We will now calculate the projections of the expression (2.17) onto S_j and B'. Onto S_1: $P_1 \; T^{-1}(A + BF_{new})x_1 = P_1 A_o x_o = P_1 A_o x_o = 0$. Onto $S_j : P_j T^{-1}(A + BF_{new})x_1 = P_j A_o x_o + P_j G_{j-1} P_{j-1} A_{j-2} P_{D_o} A_o x_o - P_j A_{j-1} x_o$, which, by straightforward calculation, is equal to the null vector of S_j. In the same way it may be seen that $P_{B'} T^{-1}(A + BF_{new})x_1 = 0$.

□

The above result may be extended directly to obtain a result on the existence of controlled invariant subspaces *complementary to almost controllability subspaces in the stabilizable subspace of the system* (A,B). Let $\mathbb{C}_g$ be a symmetric subset of the complex plane $\mathbb{C}$. Let the sum of the generalized eigenspaces of the mapping A associated with its eigenvalues in $\mathbb{C}_g$ be denoted by $X_g(A)$. In the same way, let $X_b(A)$ correspond to the eigenvalues in $\mathbb{C}_b := \mathbb{C} \setminus \mathbb{C}_g$. Let $X_{stab} := \langle A|B\rangle + X_g(A)$. This subspace will be called the *stabilizable subspace* of (A,B) (see HAUTUS (1980) and SCHUMACHER (1981, p. 26).) It is easy to see that X_{stab} is feedback invariant, i.e. that the stabilizable subspace of the system (A,B) and the system (A_F,B) coincide for every mapping $F: X \to U$. It may in fact be shown that X_{stab} is the largest subspace of X with the property that there exists a mapping $F: X \to U$ such that $\sigma(A_F | X_{stab}) \subset \mathbb{C}_g$. Also, the system (A,B) is stabilizable if and only if $X_{stab} = X$.

We may now prove the following:

<u>COROLLARY 2.42</u>. *Let* $R_a \in \underline{R}_a(A,B)$. *There exists a subspace* $V \in \underline{V}(A,B)$ *and a mapping* $F \in \underline{F}(V)$ *such that* $V \oplus R_a = X_{stab}$ *and* $\sigma(A_F | V) \subset \mathbb{C}_g$.

<u>PROOF</u>: Let $F_o: X \to U$ be such that $\sigma(A + BF_o | \langle A|B\rangle) \subset \mathbb{C}_b$. Denote $A_o := A + BF_o$. It may be seen directly that $X_g(A_o) \cap \langle A|B\rangle = \{0\}$. Since X_{stab} is feedback invariant, we thus obtain $X_{stab} = \langle A|B\rangle \oplus X_g(A_o)$. By TH. 2.39, there is a subspace $V_1 \in \underline{V}$ and a mapping $F_1 \in \underline{F}(V_1)$ such that $V_1 \oplus R_a = \langle A|B\rangle$ and $\sigma(A + BF_1 | V_1) \subset \mathbb{C}_g$. Define now $V := V_1 \oplus X_g(A_o)$ and a mapping F by $F|\langle A|B\rangle := F_1|\langle A|B\rangle$, $F|X_g(A_o) := F_o|X_g(A_o)$ and F arbitrary on a complement of X_{stab}. Then obviously $X_{stab} = V \oplus R_a$, $F \in \underline{F}(V)$

and $\sigma(A_F|V) = \sigma(A + BF_1|V_1) \cup \sigma(A_o|X_g(A_o)) \subset \mathbb{C}_g$.

□

REMARK 2.43. The special case that, in the above corollary, the almost controllability subspace R_a is equal to B and the system (A,B) is stabilizable, was proven in its dual formulation in SCHUMACHER (1981, LEMMA 2.9). Again, for this we refer to CH. 5, SECTION 5.5.

2.6 AN APPLICATION: L_p/L_q ALMOST DISTURBANCE DECOUPLING

We will now apply the framework we have set up so far to consider the first of a series of feedback synthesis problems we will study. Consider the linear system given by:

$$\text{(2.18)} \qquad \begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Gd(t), \\ z(t) &= Hx(t). \end{aligned}$$

Here, as usual, x and u are assumed to take their values in $X \simeq \mathbb{R}^n$ and $U \simeq \mathbb{R}^m$ respectively. The variable z is a *to-be-controlled output*. It will be assumed that z takes its values in the ℓ-dimensional linear space Z. The term d represents an *unknown disturbance*, which is assumed to take its values in the r-dimensional linear space D. G and H are mappings from D into X and X into Z respectively. It will be assumed that the state variable x can be measured and may be used as input for a linear time invariant memoryless system

$$\text{(2.19)} \qquad u(t) = Fx(t).$$

The system (2.18) will be called *the plant* and the system (2.19) will be called *the feedback processor*. F is a linear mapping from X to U. Connecting plant and processor yields the closed loop system

$$\text{(2.20)} \qquad \begin{aligned} \dot{x}(t) &= (A + BF)x(t) + Gd(t), \\ z(t) &= Hx(t). \end{aligned}$$

For each feedback processor u(t) = Fx(t) and each initial condition $x(0) = x_o$, the closed loop system (2.20) defines a convolution operator from the space of D-valued measurable functions on $\mathbb{R}^+$ to the space of Z-valued measurable functions on $\mathbb{R}^+$. This convolution operator is defined by

$$(2.21) \qquad z(t) = T_F(t)x_o + \int_o^t W_F(t-\tau)d(\tau)d\tau \ , \qquad (t \geq 0),$$

where the closed loop initial condition response matrix $T_F(t)$ and the closed loop impulse response matrix $W_F(t)$ are defined respectively by $T_F(t) := He^{A_F t}$ and $W_F(t) = He^{A_F t}G$ $(t \geq 0)$.

The question whether there exists a feedback processor (2.19) such that in the closed loop system (2.20) the disturbances d do not influence the output z, i.e. such that $W_F(t) = 0$, $\forall t$, is called the *disturbance decoupling problem* or DDP. It was shown in WONHAM (1979) that such feedback processor exists if and only if im $G \subset V^*(\ker H)$. (See also BASILE & MARRO (1969b) for a dual formulation). More recently, the above was extended to the situation that, instead of using the entire state vector for feedback, one is only allowed to use a linear function $y = Cx$ as input for a possibly non-memoryless feedback processor. This problem is called the disturbance decoupling problem with measurement feedback or DDPM, and was studied in SCHUMACHER (1980), WILLEMS & COMMAULT (1981) and IMAI & AKASHI (1979).

In the present section, we will still restrict ourselves to the case of memoryless state feedback. We will ask ourselves: if DDP is not solvable, is it then possible to choose the feedback processor (2.19) such that the influence of d on the output z is 'small'? Thus, we will study a problem of *'almost' disturbance decoupling*. There are several ways to quantify this 'almost' decoupling, leading to different versions of the almost disturbance decoupling problem. Here, we will consider the L_p/L_q-almost disturbance decoupling problem (WILLEMS (1981)). In the following, if $x : \mathbb{R}^+ \to X$ is a measurable function with values in the n-dimensional normed linear space X, we will say that $x \in L_p(\mathbb{R}^+, X)$ if and only if $\|x\|_p < \infty$. Here, the L_p-norm $\| \ \|_p$ is defined as $\|x\|_\infty := \operatorname{esssup}_{t \in \mathbb{R}^+} \|x(t)\|$ and $\|x\|_p := (\int_o^\infty \|x(t)\|^p dt)^{1/p}$ if $1 \leq p < \infty$.

<u>DEFINITION 2.44.</u> (ADDP)', *the* L_p/L_q*-almost disturbance decoupling problem*, is said to be solvable if the following holds: $\forall \varepsilon > 0 \ \exists F: X \to U$ such that in the closed loop system with $x(0) = 0$, $\|z\|_q \leq \varepsilon \|d\|_p$ for all $d \in L(\mathbb{R}^+, D)$ and for all $1 \leq p \leq q \leq \infty$.

Thus, it is required that the closed loop system defines a convolution operator from $L_p(\mathbb{R}^+,\mathcal{D})$ to $L_q(\mathbb{R}^+,\mathcal{Z})$ *for each pair* $1 \leq p \leq q \leq \infty$ and that each $L_p - L_q$ induced norm of this operator is arbitrarily small. The following translates this requirement in terms of the closed loop impulse response:

LEMMA 2.45. (ADDP)' is *solvable if and only if* $\forall \varepsilon > 0 \ \exists\ F : X \to U$ *with* $\|W_F\|_p \leq \varepsilon$ *for* $p = 1$ *and* $p = \infty$.

PROOF: ($\Rightarrow$) If (ADDP)' is solvable, then both the $L_\infty - L_\infty$ induced norm and the $L_1 - L_\infty$ induced norm of the convolution operator can be made arbitrarily small using the same F. Since the former equals $\|W_F\|_1$ and the latter $\|W_F\|_\infty$, the result follows (see e.g. DESOER & VIDYASAGAR (1975)). ($\Leftarrow$). This follows from the fact that $\|z\|_q \leq \|W_F\|_\infty^{1/p-1/q}\|W_F\|_1^{1-1/p+1/q}\ \|d\|_p$. (EDWARDS (1967,p.150)). □

Now, combining LEMMA 2.45 with TH. 1.29 immediately yields the following necessary condition for (ADDP)' to be solvable:

COROLLARY 2.46. *If* (ADDP)' *is solvable, then* im $G \subset \mathcal{V}_a^*(\ker H)$.

PROOF: The solvability of (ADDP)' implies that for all $\varepsilon > 0$ there is a feedback mapping F such that, for all $x_o \in \text{im}\ G$, $\sup_{t\in\mathbb{R}^+}\|He^{A_Ft}x_o\| \leq \varepsilon$. For $x_o \in \text{im}\ G$, define a trajectory x through x_o by $x(t) := e^{A_Ft}x_o$. This trajectory has the property that $d(x(t), \ker H) \leq \varepsilon, \forall t \geq 0$. By TH. 1.29 this implies that $x_o \in \mathcal{V}_a^*(\ker H)$.

In the sequel, we will prove that the subspace inclusion of the above corollary is also a *sufficient* condition for the solvability of (ADDP)'. This assertion follows from the following property of $\mathcal{V}_a^*(K)$, which states that not only in each $x_o \in \mathcal{V}_a^*(K)$ a trajectory starts which stays arbitrarily close to K, but that starting in a point x_o contained in the intersection of $\mathcal{V}_a^*(K)$ with the unit ball one can even stay close to K moving along trajectories that are all generated by the same feedback law:

THEOREM 2.47. *Let K be a subspace of X. Then* $\forall\ \varepsilon > 0\ \exists\ F: X \to U$ *such that* $\|d(e^{A_Ft}x_o,K)\|_p \leq \varepsilon$ *for all* $x_o \in \mathcal{V}_a^*(K)$ *with* $\|x_o\| \leq 1$ *and for all* $1 \leq p \leq \infty$.

The proof of this result requires the construction of a sequence of feedback mappings $\{F_n\}$. The idea is first to decompose $\mathcal{V}_a^*(K)$ into a direct sum of $V^*(K)$ and an almost controllability subspace R_a. We will then decompose R_a into the direct sum of singly generated controllability subspaces. Each of these subspaces will be approximated by a sequence of controlled invariant subspaces as in SECTION 2.4. On each of these approximants we will define a feedback mapping by (2.7). Finally, these mappings will be used to define a sequence of feedback mappings $\{F_n\}$ on X which will turn out to have the desired properties. In all of this, TH. 2.35 will play a central role. First, we need the following lemma, which is a consequence of TH. 2.39. In the sequel, denote $V^* := V^*(K)$, $V^*_a := V^*_a(K)$.

<u>LEMMA 2.48</u>. *Consider the system* (A,B). *Let* Λ *be a symmetric set of* $\dim[\langle A|B\rangle + V^*] - \dim V^*_a$ *complex numbers. Then there exists a subspace* W *and, for each mapping* $F_o \in \underline{F}(V^*)$, *a mapping* $F_1: X \to U$ *such that:*

$$(2.22)\qquad F_1|V^* = F_o|V^* ,$$

$$(2.23)\qquad V^*_a \oplus W = V^* + \langle A|B\rangle ,$$

$$(2.24)\qquad A_{F_1}(V^* \oplus W) \subset V^* \oplus W ,$$

$$(2.25)\qquad \sigma(A_{F_1}|(V^* \oplus W)/V^*) = \Lambda .$$

<u>PROOF</u>: Let $P: X \to X/V^*$ denote the canonical projection and let $F_o \in \underline{F}(V^*)$. Let $\tilde{B} := PB$ and let $\tilde{A}$ denote the quotient mapping induced by $A + BF_o$ in X/V^*. Let $R_a \in \underline{R}_a$ be such that $V^* \oplus R_a = V^*_a$. Then $PV^*_a = PR_a$. It may be verified that $PR_a \in \underline{R}_a(\tilde{A},\tilde{B})$. Also, $P\langle A|B\rangle = \langle\tilde{A}|\operatorname{im}\tilde{B}\rangle$. Let Λ be as above. Obviously, Λ contains $\dim\langle\tilde{A}|\operatorname{im}\tilde{B}\rangle - \dim PR_a$ complex numbers and thus we may apply TH. 2.39 to find an $(\tilde{A},\tilde{B})$-invariant subspace $\tilde{W} \subset X/V^*$ and a mapping $\tilde{F}$ such that $PR_a \oplus \tilde{W} = \langle\tilde{A}|\operatorname{im}\tilde{B}\rangle$, $(\tilde{A} + \tilde{B}\tilde{F})\tilde{W} \subset \tilde{W}$ and $\sigma(\tilde{A} + \tilde{B}\tilde{F}|\tilde{W}) = \Lambda$. Now, let $W \subset X$ be any subspace such that $PW = \tilde{W}$ and $W \cap V^* = \{0\}$. Define $F_1 := F_o + \tilde{F}P$. It may then be verified that (2.22) to (2.25) hold. □

Next, we need the following technical ingredient:

<u>LEMMA 2.49</u>. *For* $n \in \mathbb{N}$, *let* $v_1(n),\ldots,v_k(n)$ *be vectors in* X. *Assume that for* $i \in \underline{k}$, $\lim_{n\to\infty} v_i(n) = v_i$, *where* $v_1,\ldots,v_k$ *are linearly independent. Also, assume that for* $i \in \underline{k}$ *there are real numbers* $\xi_{ij}(n)$ *such that* $v_j = \xi_{1j}(n)v_1(n) + \ldots + \xi_{kj}(n)v_k(n)$. *Then for all* $i,j \in \underline{k}$, $\lim_{n\to\infty} \xi_{ij}(n) = 0$

$(i \neq j)$ *and* $\lim_{n\to\infty} \xi_{ii}(n) = 1$.

PROOF: Let V and V(n) be the $k\times k$ matrices formed by the column vectors $v_1,\dots, v_k$ and $v_1(n),\dots, v_k(n)$, respectively. By LEMMA 2.30, V(n) is nonsingular for n sufficiently large. Moreover, $V(n)^{-1} \to V^{-1}$ as $n \to \infty$. Let $\Xi(n) := (\xi_{ij}(n))$. Then $V = V(n)\,\Xi(n)$. Thus, $\Xi(n) = V(n)^{-1}V \to I_{k\times k}$.

□

PROOF OF THEOREM 2.47: Decompose $V_a^* = V^* \oplus R_a$, with $R_a \in \underline{R}_a$. Decompose $R_a = \mathcal{L}^1 \oplus\dots\oplus \mathcal{L}^s$ with $\mathcal{L}^i = \mathcal{L}(u_i,F,r_i)$. (Recall that such decomposition is always possible, see e.g. REMARK 2.28).
Denote $\mathcal{L}^i(n) := \mathcal{L}_n(u_i,F,r_i) = \text{span}\,\{x_1(n,u_i),\dots, x_{r_i}(n,u_i)\}$ for the sequence in $\underline{V}$ converging to $\mathcal{L}^i$. On each $\mathcal{L}^i(n)$, define a mapping F_n^i by $F_n^i\, x_j(n,u_i) := -n^j u_i$ $(j = 1,\dots, r_i)$. Define $V(n) := \mathcal{L}^1(n) \oplus\dots\oplus \mathcal{L}^s(n)$. Note that $V(n) \to R_a$ $(n \to \infty)$.

Let Λ be a in LEMMA 2.48 and assume $\Lambda \subset \mathbb{C}^- := \{s \in \mathbb{C} \mid \text{Re } s < 0\}$. According to LEMMA 2.48, there exists a subspace W and a mapping F_1 such that (2.22) to (2.25) are satisfied. Moreover, on V^*, F_1 may be chosen arbitrarily in $\underline{F}(V^*)$. We now contend that for all n sufficiently large, $V^* \oplus R_a \oplus W = V^* \oplus V(n) \oplus W$. To show this, note that by LEMMA 2.31 $V(n) \subset \langle A|B\rangle$. On the other hand, since $V(n) \to R_a$, for n sufficiently large $V(n) \cap [V^* \oplus W] = \{0\}$. The claim then follows by noting that $\dim V(n) = \dim R_a$.

Define now a sequence of mappings $F_n : X \to U$ as follows. On $V^* \oplus W$, define F_n to be equal to F_1. (We stress that on V^* this mapping may be any mapping from $\underline{F}(V^*)$, possibly depending on n.) On $\mathcal{L}^i(n)$, define F_n to be equal to $F + F_n^i$ $(i = 1,\dots,s)$. Extend F_n arbitrarily to a mapping on X. We claim that for all $x_o \in V_a^*$, $\lim_{n\to\infty} \| d(e^{(A+BF_n)t}x_o, K)\|_p = 0$, uniformly for $1 \leq p \leq \infty$. First, note that for $x_o \in V^*$ this is immediate. To complete the proof it is sufficient to prove the claim for each vector $x_o = A_F^{j-1}Bu_i$ $(i = 1,\dots, s\,;\ j = 1,\dots, r_i)$. Now, the crucial point is that $A_F^{j-1}Bu_i \in R_a \subset V^* \oplus V(n) \oplus W$. Thus, for n sufficiently large $A_F^{j-1}Bu_i$ may be expanded as

$$(2.26) \qquad A_F^{j-1}Bu_i = v(n) + \sum_{\ell=1}^{s} \sum_{k=1}^{r_\ell} \xi_{k\ell}(n)\, x_k(n, u_\ell)\,,$$

with $v(n) \in V^* \oplus W$ and $\xi_{k\ell}(n) \in \mathbb{R}$. Since, by LEMMA 2.31, $x_k(n,u_\ell) \to A_F^{k-1}Bu_\ell\ (n\to\infty)$, it follows from LEMMA 2.49 that $\lim_{n\to\infty} v(n) = 0$, $\lim_{n\to\infty} \xi_{k\ell}(n) = 0$, $(k,\ell) \neq (j,i)$, and $\lim_{n\to\infty} \xi_{ji}(n) = 1$.

It follows from (2.26) that

$$\|d(e^{(A+BF_n)t} A_F^{j-1}Bu_i, K)\|_p \le$$

$$\|d(e^{(A+BF_1)t} v(n), K)\|_p + \sum_{\ell=1}^{s} \sum_{k=1}^{r_\ell} |\xi_{k\ell}(n)| \, \|d(e^{(A+BF_n^\ell)t} x_k(n,u_\ell), K)\|_p .$$

By TH. 2.35, all terms in the composite sum on the right in this expression tend to 0 as $n\to\infty$, uniformly for $1 \le p \le \infty$.

Let $P : X \to X/V^*$ denote the canonical projection and let $\bar{A}_{F_1}$ be the quotient mapping of $(A+BF_1)|V^* \oplus W$ modulo V^*. By (2.25), $\sigma(\bar{A}_{F_1}) = \Lambda$. Denote $\bar{K} := PK$, $\bar{v}(n) := Pv(n)$. By the fact that $V^* \subset K$, it may be verified (using DEF. 1.37 of distance in quotient spaces), that

$$d(e^{A_{F_1} t} v(n), K) = d(e^{\bar{A}_{F_1} t} \bar{v}(n), \bar{K}).$$

Hence, we obtain

$$\|d(e^{A_{F_1} t} v(n), K)\|_p \le \|d(e^{\bar{A}_{F_1} t} \bar{v}(n), \bar{K})\|_p$$
$$\le \|e^{\bar{A}_{F_1} t} \bar{v}(n)\|_p \le \|e^{\bar{A}_{F_1} t}\|_p \, \|\bar{v}(n)\| .$$

Now, by the fact that $\sigma(\bar{A}_{F_1}) \subset \mathbb{C}^-$, there is a constant M such that, for all $1 \le p \le \infty$, $\|e^{\bar{A}_{F_1} t}\|_p \le M$. The claim of the theorem then follows from the fact that $\lim_{n\to\infty} \bar{v}(n) = 0$.

□

<u>REMARK 2.50</u>. Note that the sequence $\{F_n\}$ as defined above, makes the subspace $\langle A|B\rangle + V^*$ $(A + BF_n)$-invariant for all n. Moreover, this subspace has a direct sum decomposition $V^* \oplus V(n) \oplus W$, which is valid for all n sufficiently large. Here, $V^* \oplus W$ is $(A + BF_n)$-invariant and also V^* is $(A + BF_n)$-invariant. On V^*, the mapping F_n may be chosen arbitrarily from $\underline{F}(V^*)$. Thus, we have complete freedom of pole assignment on R^*, while the spectrum on V^*/R^* is fixed. The spectrum on $(V^* \oplus W)/V^*$ is arbitrary but *should be contained in* $\mathbb{C}^-$ *and is independent of* n. Finally, $V(n)$ is $(A+BF_n)$-invariant and it may be seen that $\sigma(A + BF_n|V(n)) = \{-n, \ldots, -n\}$ ($\dim V_a^* - \dim V^*$ times, see also LEMMA 2.34). We may summarize this in the following lattice diagram:

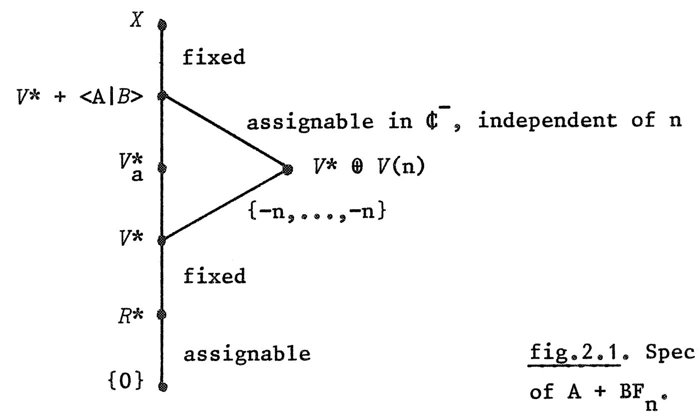


fig.2.1. Spectrum of $A + BF_n$.

Observe that as $n \to \infty$, i.e. as the closed loop trajectories starting in $V_a^*(K)$ stay closer and closer to K, then the spectrum $\sigma(A + BF_n | (V^* \oplus V(n))/V^*)$ tends to *'minus infinity'*.

Thus, we have completed the proof of TH. 2.47. As noted before, it follows immediately that the subspace inclusion of COR. 2.46 is also sufficient for the solvability of (ADDP)'. We record this in a separate corollary:

COROLLARY 2.51. (ADDP)' *is solvable if and only if* $\text{im } G \subset V_a^*(\ker H)$.

□

To conclude this section, we note that the following feedback characterizations of the class of almost controlled invariant subspaces are now valid.

COROLLARY 2.52. *The following statements are equivalent:*

(i) $V_a \in \underline{V}_a(A,B)$,

(ii) $\forall\, \varepsilon > 0\ \exists\, F: X \to U$ *such that* $d(e^{A_F t} x_o, V_a) \leq \varepsilon$, $\forall\, t \geq 0$ *and* $\forall\, x_o \in V_a$ *with* $\| x_o \| \leq 1$,

(iii) $\forall\, \varepsilon > 0\ \exists\, F: X \to U$ *such that* $\| d(e^{A_F t} x_o, V_a) \|_p \leq \varepsilon$, $\forall\, x_o \in V_a$ *with* $\| x_o \| \leq 1$ *and* $\forall\, 1 \leq p \leq \infty$.

PROOF: This follows by combining TH. 1.29 and TH. 2.47.

□

2.7 SPECTRAL ASSIGNABILITY IN ALMOST CONTROLLABILITY SUBSPACES

In section 1.2, almost controllability subspaces were defined as subspaces with the property that one can travel between any two points in the subspace in a given finite time, moving along trajectories that stay arbitrarily close to the subspace for all points of time. This definition generalized the definition of controllability subspace, where the same was required under the restriction that one should be able to move along trajectories lying inside the subspace. It is well known that the latter definition leads to the property which says that in a controllability subspace it is possible to travel along feedback generated trajectories with arbitrary spectrum. In the present section, we will show that an analogous property holds for almost controllability subspaces. It will turn out that, starting in an almost controllability subspace, it is possible to travel along feedback generated trajectories with arbitrary spectrum while staying arbitrarily close to the subspace.

In the following, we will again use the concept of Bohl function (see DEF. 2.4). Let $f(t) = He^{Ft}G\, 1_{\mathbb{R}^+}(t)$ be Bohl. If in this representation the triple (F,G,H) is minimal, then we define the *spectrum* of f as $\sigma(f) := \sigma(F)$. Elements of this spectrum are called the *characteristic values of* f. If F is an $r \times r$ matrix, then we say that f has *McMillan degree* r and denote $\deg f = r$.

Assume that we have a linear system with system mapping A and input mapping B. Then $\Sigma^B(A,B)$ will denote the space of all trajectories $x \in \Sigma(A,B)$ such that $x(t)\cdot 1_{\mathbb{R}^+}(t)$ is Bohl. Also, we denote $\Sigma^{B(r)}(A,B) := \{x \in \Sigma^B(A,B) \mid \deg x \leq r\}$ and $\Sigma_g^{B(r)}(A,B) := \{x \in \Sigma^{B(r)}(A,B) \mid \sigma(f) \subset \mathbb{C}_g\}$. Here, $\mathbb{C}_g$ is a subset of $\mathbb{C}$.

Now, what we would like to show is that for all $\varepsilon > 0$ and all $\mathbb{C}_g \subset \mathbb{C}$, there exists a mapping F such that, for all $x_0 \in \mathcal{R}_a^*(K)$, $d(e^{A_F t}x_0, K) \leq \varepsilon$ $(t \geq 0)$ and $\sigma(A_F \mid \langle A \mid \mathcal{B}\rangle) \subset \mathbb{C}_g$. To show this, one would like to use a construction similar to the one in the foregoing section and construct a suitable sequence of mapping $\{F_n\}_{n\in\mathbb{N}}$. As outlined in REMARK 2.50, such sequence would lead to a part of the closed loop spectrum $\sigma(A + BF_n \mid \langle A \mid \mathcal{B}\rangle)$ that runs off to 'minus infinity' as $n \to \infty$. Thus, the requirement that this spectrum should be contained in $\mathbb{C}_g$ *for all* n can only be met if $\mathbb{C}_g$ includes 'a point at minus infinity'. Also, a part of the closed loop spectrum is necessarily contained in $\mathbb{C}^-$. Therefore, apart from the usual requirement concerning symmetry of the stability sets $\mathbb{C}_g$, we should also make sure that it is possible to 'reach minus infinity' staying in $\mathbb{C}_g$ and that $\mathbb{C}_g$ contains at least one point in $\mathbb{C}^-$. Therefore, unless otherwise stated, we will assume that

(2.27) $\quad \lambda \in \mathbb{C}_g \Leftrightarrow \bar{\lambda} \in \mathbb{C}_g$ ($\bar{\lambda}$ denotes complex conjugate),

(2.28) $\quad \exists\, c \in \mathbb{R}$ such that $(-\infty, c] \subset \mathbb{C}_g$.

For every subspace K of X, define a subspace $P(K)$ by

(2.29) $$P(K) := \{x_o \in K \mid \exists\, r \in \mathbb{N} \text{ such that } \forall \varepsilon > 0 \text{ and } \forall \mathbb{C}_g \; \exists\, x \in \Sigma_g^{B(r)}(A,B) \text{ with } x(0) = x_o \text{ and } d(x(t),K) \leq \varepsilon \;\; \forall t \geq 0\}.$$

Thus, starting in $x_o \in P(K)$, one can move along Bohl trajectories with arbitrary characteristic values, staying arbitrarily close to K. Moreover, for fixed $x_o \in P(K)$ there is an a priori upper bound (depending on x_o) to the McMillan degrees of these trajectories. (Note that this is automatically satisfied for state feedback generated trajectories.) It may be verified immediately that $R^*(K) \subset P(K)$. One of the main purposes of this section is to show that $P(K)$ is equal to $R_a^*(K)$. In the sequel, we will need the following continuity property of the spectra of Bohl functions.

<u>LEMMA 2.53.</u> *For* $n \in \mathbb{N}$, *let* $f_n(t)$ *and* $f(t)$ *be scalar values Bohl functions. Suppose* $\exists\, r \in \mathbb{N}$ *such that* $\deg f_n \leq r, \forall n \in \mathbb{N}$. *Let* $\hat{f}_n(s)$, $\hat{f}(s)$ *denote the Laplace transforms of* $f_n(t)$ *and* $f(t)$ *(continued analytically over* $\mathbb{C}$*). Assume that* $\lim_{n\to\infty} \hat{f}_n(s) = \hat{f}(s)$ *for infinitely many* $s \in \mathbb{C}$. *Then every element of* $\sigma(f)$ *is a limit point of* $\bigcup_{n=1}^{\infty} \sigma(f_n)$.

<u>PROOF</u>: Using the fact that $\deg f_n \leq r$, $\forall n$ and that $\lim_{n\to\infty} \hat{f}_n(s)$ exists for infinitely many s, it may be shown that there exists a subsequence $\hat{f}_{n_m}(s)$ and coprime polynomials $p_m(s) = \xi_r(m)s^r + \ldots + \xi_1(m)s + \xi_o(m)$ and $q_m(s) = \eta_{r-1}(m)s^{r-1} + \ldots + \eta_1(m)s + \eta_o(m)$ such that $\hat{f}_{n_m}(s) = q_m(s)/p_m(s)$ and such that the coefficients $\xi_i(m)$ and $\eta_i(m)$ converge to real numbers ξ_i and η_i. Moreover, it may be assumed that at least one $\xi_i \neq 0$. For a proof of this assertion, we refer to HAZEWINKEL (1980, pp. 169-171). Define polynomials $p(s) := \xi_r s^r + \ldots + \xi_o$, $q(s) := \eta_{r-1} s^{r-1} + \ldots + \eta_o$.

It follows immediately that $\hat{f}(s) = q(s)/p(s)$. Although q and p need not be coprime, if $\lambda \in \sigma(f)$, then $p(\lambda) = 0$. Since $\xi_i(m) \to \xi_i$, $\forall i$, it may be shown that $p_m \to p$ uniformly on compact subsets. Now, let $\lambda \in \sigma(f)$. Let $\varepsilon_o > o$ such that the only zero of p in $|s - \lambda| < \varepsilon_o$ is λ. Let $0 < \varepsilon < \varepsilon_o$. We will show that the disc $|s - \lambda| < \varepsilon$ contains an element of $\bigcup_{n=1}^{\infty} \sigma(f_n)$.

Define $\alpha := \min\{|p(s)|, |s - \lambda| = \varepsilon\}$. Then $\alpha > 0$. Since $p_m \to p$ uniformly on $|s - \lambda| = \varepsilon$, $\exists$ N such that $m \geq N$ implies $|p_m(s) - p(s)| < \alpha$, $\forall s$ with $|s - \lambda| = \varepsilon$. Define now $g := p_N - p$. Then $|g(s)| = |p_N(s) - p(s)| < \alpha < |p(s)|$ on the cirkel $|s - \lambda| = \varepsilon$. Hence, by Rouché's theorem (see CARTAN (1961, p.116)), $g + p = p_N$ has a zero in $|s - \lambda| < \varepsilon$. Since q_N and p_N are coprime, this zero is a pole of $\hat{f}_{n_N}(s)$ and thus an element of $\sigma(f_{n_N})$.

□

REMARK 2.54. The above result is not valid without the assumption on the uniform bound of the McMillan degrees.

Using the above lemma, we may now prove the following:

LEMMA 2.55. *Let K be a subspace of X with the property that $K \cap B = \{0\}$. Then $P(K) = \{0\}$.*

PROOF: In this proof, denote $V^* := V^*(K)$ and $R^* := R^*(K)$. Define $X_1 := K$, $X_3 := B$ and let $X_2 \subset X$ be such that $X = X_1 \oplus X_2 \oplus X_3$. Since $K \cap B = \{0\}$, we have $R^* = \{0\}$. Denote the fixed spectrum $\sigma(A_F|V^*)$ by σ^* (WONHAM (1979, TH. 5.7)). Choose $F \in \underline{F}(V^*)$ such that in the decomposition employed:

$$(2.30) \qquad A_F = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ B_3 \end{pmatrix}.$$

Now let $x_o \in P(K)$, $x_o = (x_{01}^T, 0^T, 0^T)^T$. Let $\mathbb{C}_g$ be such that $\overline{\mathbb{C}}_g \cap \sigma^* = \emptyset$. There is $r \in \mathbb{N}$ and a sequence $x_n \in \Sigma_{g_T}^{B(r)}(A,B)$ (i.e. x_n has its characteriswich values in $\mathbb{C}_g$) with $x_n = (x_{1n}^T, x_{2n}^T, x_{3n}^T)^T$ and $\|A_{12}x_{2n}(t) + A_{13}x_{3n}(t)\| \leq \frac{1}{n}$, $\forall\, t \geq 0$. Denote $z_n := A_{12}x_{2n} + A_{13}x_{3n}$. We have

$$x_{1n}(t) = e^{A_{11}t}x_{01} + \int_o^t e^{A_{11}(t-\tau)} z_n(\tau)\, d\tau$$

and hence $x_{1n}(t) \to e^{A_{11}t}x_{01} =: x_1(t)$ uniformly on $[0,T]$ for each $T \geq 0$. Define $x(t) := (x_1^T(t), 0^T, 0^T)^T$. In exactly the same way as in the proof of LEMMA 1.25 we may show that $x(t) = e^{A_F t}x_o$, $(t \geq 0)$. Thus, since $x(t) \in K$ for $t \geq 0$, we must have $x_o \in V^*$.

Let $\hat{z}_n(s)$ and $\hat{x}_{1n}(s)$ denote the Laplace transforms of z_n and x_{1n}. By our earlier estimate of z_n:

$$\|\hat{z}_n(s)\| \leq \int_o^\infty e^{-\sigma t}\|z_n(t)\|dt \leq \frac{1}{n\sigma}, \quad \sigma := \operatorname{Re} s > 0.$$

Consequently, $\lim_{n\to\infty} \hat{z}_n(s) = 0$ for all s with $\operatorname{Re} s > 0$. Also

$$\hat{x}_{1n}(s) = (Is - A_{11})^{-1} x_{01} + (Is - A_{11})^{-1} \hat{z}_n(s),$$

which is valid for every s with $\operatorname{Re} s > 0$ that is not an element of $\sigma(A_{11})$. We may conclude that for all those s, $\lim_{n\to\infty} \hat{x}_{1n}(s) = (Is - A_{11})^{-1}x_{01}$ and thus, from LEMMA 2.53, that the spectrum of $e^{A_{11}t}x_{01}$ is contained in the closure $\bar{\mathbb{C}}_g$ (since $\sigma(x_{1n}) \subset \mathbb{C}_g$, $\forall n$). On the other hand however, $A_F V^* \subset V^*$, $x_o = (x_{01}^T, 0^T, 0^T)^T \in V^*$ and $\sigma(A_F|V^*) = \sigma^*$, whence $\sigma(e^{A_{11}t}x_{01}) \subset \sigma^*$. Since $\sigma^* \cap \bar{\mathbb{C}}_g = \emptyset$, it follows that $x_{01} = 0$ and hence $x_o = 0$. This concludes the proof of the lemma.

□

Our next step it to generalize the inclusion (i) of TH. 1.14 to Bohl trajectories that have their characteristic values in some prespecified subset $\mathbb{C}_g$ of $\mathbb{C}$ and that have an a priori upper bound to their McMillan degrees. In the following, let $(\bar{A},\bar{B})$ denote the factor system modulo S^∞.

<u>LEMMA 2.56.</u> *Let $\mathbb{C}_g \subset \mathbb{C}$ and $r \in \mathbb{N}$. Let K be a subspace of X. Then*

$$\Sigma_g^{B(r)}(A,B)/S^\infty(K) \subset \Sigma_g^{B(r)}(\bar{A}, \bar{B}).$$

<u>PROOF</u>: If $\xi \in \Sigma_g^{B(r)}(A,B)/S^\infty(K)$, then $\xi(t) = Px(t)$ for all $t \in \mathbb{R}$, with $x \in \Sigma_g^{B(r)}(A,B)$ and $P: X \to X/S^\infty$ the canonical projection. By TH. 1.14, $\xi \in \Sigma(\bar{A}, \bar{B})$. Since x is Bohl, $x(t) = He^{Ft}G$ $(t \geq 0)$ for a suitable triple (F, G, H). It follows that $\xi(t) = PHe^{Ft}G$ for all $t \geq 0$ and thus that ξ is Bohl. From this it is also immediate that $\sigma(\xi) \subset \sigma(x) \subset \mathbb{C}_g$ and that $\deg \xi \leq \deg x \leq r$.

□

Combining the foregoing two lemmas now yields the following result:

LEMMA 2.57. *Let K be a subspace of X. Then $P(K) \subset R_a^*(K)$.*

PROOF: Let $x_o \in P(K)$. Let $r \in \mathbb{N}$ be the McMillan degree bound corresponding to this x_o and let $\varepsilon > 0$ and $\mathbb{C}_g$ be arbitrary. There is $x \in \Sigma_g^{B(r)}(A,B)$ with $x(0) = x_o$ and $d(x(t),K) \leq \varepsilon$ for all $t \geq 0$. Define $\xi_o := [x_o]$ and $\xi(t) := [x(t)]$ (equivalence classe modulo $S^\infty(K)$). By LEMMA 2.56, $\xi \in \Sigma_g^{B(r)}(\bar{A},\bar{B})$. Also $\xi(0) = \xi_o$ and $d(\xi(t), K/S^\infty) \leq \varepsilon$ for $t \geq 0$ (see (1.38)). It follows that $\xi_o \in P(K/S^\infty)$, with respect to $(\bar{A},\bar{B})$. Since, by TH. 1.14 and COR 1.11 (i), $\operatorname{im} \bar{B} \cap (K/S^\infty) = \{[0]\}$ it may be concluded from LEMMA 2.55 that $\xi_o = [0]$. It follows that $x_o \in S^\infty(K) = R_a^*(K)$.

It will now be shown that the converse inclusion in the above lemma is also valid. In fact, we will prove something stronger: it will be shown that, starting in $x_o \in R_a^*(K)$, one can stay arbitrary close to K, moving along trajectories generated by state feedback. At the same time, the closed loop spectrum may be chosen in an arbitrary subset $\mathbb{C}_g$ of $\mathbb{C}$. Moreover, not only the supnorm of the distance of these trajectories to K can be made arbitrarily small, but all L_p-norms:

THEOREM 2.58. *Let K be a subspace of X. Then $\forall \varepsilon > 0$ and $\forall \mathbb{C}_g \subset \mathbb{C}$ (satisfying (2.27) and (2.28)) there is a mapping $F : X \to U$ such that*

$$\| d(e^{A_F t} x_o, K)\|_p \leq \varepsilon \text{ for all } x_o \in R_a^*(K) \text{ with } \|x_o\| \leq 1 \text{ and all } 1 \leq p \leq \infty$$

and

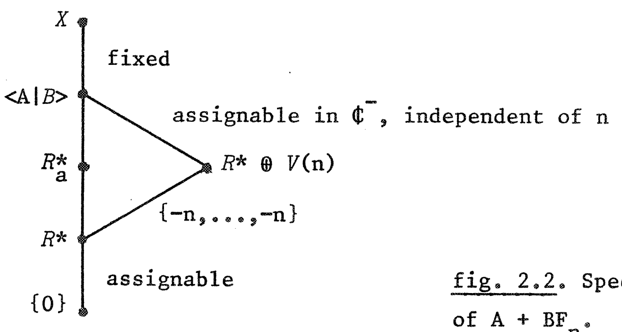
$$\sigma(A_F | \langle A|B\rangle) \subset \mathbb{C}_g .$$

PROOF: Let Λ be a symmetric set of $\dim \langle A|B\rangle - \dim R_a^*(K)$ complex numbers such that $\Lambda \subset \mathbb{C}^- \cap \mathbb{C}_g$. According to TH. 2.39, there is a subspace W and a mapping F_1 such that $A_{F_1}W \subset W$, $\sigma(A_{F_1}|W) = \Lambda$ and $R_a^*(K) \oplus W = \langle A|B\rangle$. Decompose $R_a^*(K) = R^*(K) \oplus R_a$ (see TH. 2.27), with $R_a \in \underline{R}_a$. Also, decompose $R_a = \mathcal{L}^1 \oplus \ldots \oplus \mathcal{L}^s$, with $\mathcal{L}^i = \mathcal{L}(u_i,F,r_i)$ (see SECTION 2.4). Again, approximate $\mathcal{L}^i$ by $\mathcal{L}^i(n) := \mathcal{L}_n(u_i,F,r_i)$ where the approximants $\mathcal{L}_n(u_i,F,r_i)$ are spanned by the vectors $x_j(n,u_i)$ $(j = 1,\ldots, r_i)$. Define mappings $F_n^i : \mathcal{L}^i(n) \to U$ by $F_n^i x_j(n,u_i) = -n^j u_i$. Let $V(n) := \mathcal{L}(n) \oplus \ldots \oplus \mathcal{L}^s(n)$.

Now, for n sufficiently large, $\langle A|B\rangle = R^*(K) \oplus V(n) \oplus W$. Define $F_n : X \to U$ on $R^*(K)$ such that $\sigma(A + BF_n|R^*(K)) \subset \mathbb{C}_g$, on $\mathcal{L}^i(n)$ by $F + F_n^i$ and on W by F_1. Extend this to a mapping on X. In the same way as in the proof of TH. 2.47, it may be shown that, for all $x_o \in R_a^*(K)$ $\|d(e^{(A+BF_n)t}x_o,K)\|_p$ converges to 0, uniformly for $1 \leq p \leq \infty$. Finally, the claim on the spectrum follows from the fact that $\sigma(A + BF_n|\langle A|B\rangle) = \sigma(A_{F_n}|R^*(K)) \cup \sigma(A_{F_n}|\ V(n)) \cup \sigma(A_{F_n}|W)$. Since $\sigma(A_{F_n}|V(n)) = \{-n,\dots,-n\}$, for n sufficiently large this spectrum is contained in $\mathbb{C}_g$ (due to the assumption 2.28).

□

<u>REMARK 2.59.</u> Note that in the above for each $n \in \mathbb{N}$ we have a direct sum decomposition of $\langle A|B\rangle$ into three subspaces, $R^*(K)$, $V(n)$ and W. For each $n \in \mathbb{N}$ sufficiently large these subspaces are $(A + BF_n)$-invariant. The situation with the spectrum is as follows:



<u>fig. 2.2.</u> Spectrum of $A + BF_n$.

We are now in a position to conclude the following:

<u>THEOREM 2.60.</u> *Let K be subspace of X. Then*

$$R_a^*(K) = \{x_o \in K | \exists\ r \in \mathbb{N} \text{ such that } \forall \varepsilon > 0 \text{ and } \forall\, \mathbb{C}_g\ \ \exists\ x \in \Sigma_g^{B(r)}(A,B) \text{ with } x(0) = x_o \text{ and } d(x(t),K) \leq \varepsilon\ \ \forall\, t \geq 0\}.$$

<u>PROOF</u>: One inclusion follows from LEMMA 2.57. The converse follows immediately from TH. 2.58: for each $x_o \in R_a^*(K)$, take $r = n$ (= dim X). For $\varepsilon > 0$ and $\mathbb{C}_g \subset \mathbb{C}$, let F be such that the conditions of TH. 2.58 are satisfied. Define $x(t) := e^{A_F t}x_o$. Then $d(x(t),K) \leq \varepsilon$, $\forall\ t \in \mathbb{R}^+$ and, since $x_o \in \langle A|B\rangle$, $\sigma(x) \subset \mathbb{C}_g$.
□

We also have the following analoque of COR. 2.52, giving a feedback characterization of the class of almost controllability subspaces:

COROLLARY 2.61. *The following statements are equivalent:*

(i) $\quad R_a \in \underline{R}_a(A,B)$,

(ii) $\quad \forall\, \varepsilon > 0$ and $\forall\, \mathbb{C}_g$ *there is a mapping* $F: X \to U$ *such that* $d(e^{A_F t}x_o, R_a) \leq \varepsilon, \forall\, t \geq 0, \forall\, x_o \in R_a$ *with* $\|x_o\| \leq 1$ *and* $\sigma(A_F | \langle A|B\rangle) \subset \mathbb{C}_g$,

(iii) $\quad \forall\, \varepsilon > 0$ and $\forall\, \mathbb{C}_g$ *there is a mapping* $F: X \to U$ *such that* $\| d(e^{A_F t}x_o, R_a)\|_p \leq \varepsilon, \forall\, x_o \in R_a$ *with* $\|x_o\| \leq 1$, $\forall\, 1 \leq p \leq \infty$ *and* $\sigma(A_F | \langle A|B\rangle) \subset \mathbb{C}_g$.

PROOF: This follows immediately from the foregoing results. □

To conclude this section, we will study an extension of the L_p/L_q almost disturbance decoupling problem in which, apart from approximate decoupling, we require that the closed loop spectrum may be located arbitrarily in the complex plane. Again, consider the plant (2.18). Consider the following definition:

DEFINITION 2.62. (ADDPPP)', *the* L_p/L_q *almost disturbance decoupling problem with pole placement*, is said to be solvable if the following holds: $\forall\, \varepsilon > 0$ and $\forall\, \mathbb{C}_g$ there exists a mapping $F: X \to U$ such that in the closed loop system with $x(0) = x_0$, $\|z\|_q \leq \varepsilon \|d\|_p$ for all $d \in L_p(\mathbb{R}^+, D)$ and for all $1 \leq p \leq q \leq \infty$ and $\sigma(A + BF) \subset \mathbb{C}_g$.

We stress that in the above we restrict ourselves to stability sets $\mathbb{C}_g$ that satisfy the conditions (2.27) and (2.28). The results of this section immediately lead to the following necessary and sufficient conditions for the solvability of the above problem:

THEOREM 2.63. (ADDPPP)' *is solvable if and only if the system* (A,B) *is controllable and* im $G \subset R_a^*(\ker H)$.

PROOF: The solvability of (ADDPPP)' can be formulated in terms of the L_1-norm and L_∞-norm of the closed loop impulse response and the closed loop spectrum. The result then follows immediately from TH. 2.58 and TH.2.60. Of course, controllability of (A,B) is a necessary condition in order to be able to locate the closed loop spectrum $\sigma(A_F)$ in an arbitrary subset $\mathbb{C}_g$ of $\mathbb{C}$.

□

2.8 ALMOST STABILIZABILITY SUBSPACES

To conclude this chapter, we will introduce the family of almost stabilizability subspaces. In the definition of the latter family, we first specify a stability set $\mathbb{C}_g$. We then define a subspace to be an almost stabilizability subspace with respect to $\mathbb{C}_g$ if, roughly speaking, starting in it one can stay arbitrarily close to it following trajectories having their characteristic values in $\mathbb{C}_g$. The notion of almost stabilizability subspace is meant to generalize that of stabilizability subspace (HAUTUS (1980)). A stabilizability subspace is a subspace with the property that, starting in it, one can stay in it, moving along trajectories with their spectrum contained in $\mathbb{C}_g$.

Although the term 'almost stabilizability subspace' was not mentioned there explicitely, the idea stems from WILLEMS (1981, COMMENT 7). The term was introduced in SCHUMACHER(1984). Before we introduce the 'almost' version, we will first consider 'exact' stabilizability subspaces. Our treatment of these will differ slightly from the one in HAUTUS (1980). Unless otherwise stated, in this section we will assume that the stability set $\mathbb{C}_g$ is symmetric (see SECTION 4.5). In the following, consider the system with system mapping A and input mapping B.

DEFINITION 2.64. A subspace V_g of X is said to be a *stabilizability subspace* if $\forall x_o \in V_g \;\; \exists\, x \in \Sigma^B(A,B)$ such that $x(0) = x_o$, $x(t) \in V_g$, $\forall\, t \in \mathbb{R}$ and $\sigma(x) \subset \mathbb{C}_g$.

Thus, contrary to HAUTUS (1980), we introduce the class of stabilizability subspaces in terms of the trajectories of the system. However, using a reasoning similar to the one in HAUTUS (1980), we may obtain the original defining property as the following proposition:

PROPOSITION 2.65. *A subspace V_g of X is a stabilizability subspace if and only if there is a mapping* $F: X \to U$ *such that* $A_F V_g \subset V_g$ *and* $\sigma(A_F \,|\, V_g) \subset \mathbb{C}_g$.

□

We will denote by $\underline{V}_g$ or $\underline{V}_g(A,B)$ the class of all stabilizability subspaced associated with a given stability set $\mathbb{C}_g$. It is a trivial matter to verify that $\underline{V}_g$ is closed under subspace addition. For a given subspace $K \subset X$, $V_g^*(K)$ will denote the supremal stabilizability subspace in K. Let us now define the class of almost stabilizability subspaces:

DEFINITION 2.66. A subspace V_a of X is said to be an *almost stabilizability subspace* if $\forall x_o \in V_a$, there is an integer $r \in \mathbb{N}$ and a closed subset D of $\mathbb{C}_g$ such that the following is true: $\forall \varepsilon > 0 \;\; \exists\, x \in \Sigma^{B(r)}(A,B)$ with $x(0) = x_o$, $d(x(t), V_a) \leq \varepsilon$, $\forall\, t \in \mathbb{R}$ and $\sigma(x) \subset D$.

The definition requires that, starting in $x_o \in V_a$, it is possible to move along Bohl trajectories, staying closer and closer to V_a. There should however be an integer r, depending on x_o, that constitutes an upper bound to the McMillan degree of these trajectories. Moreover, the joint characteristic values of these trajectories should not only lie in $\mathbb{C}_g$, but should lie in a *closed subset* of $\mathbb{C}_g$. In effect, this will prevent these characteristic values to accumulate on the boundary of the set $\mathbb{C}_g$. It should be noted that *if* $\mathbb{C}_g$ *is closed itself*, then the above definition is equivalent to: $\forall x_o \in V_a$ there is an integer $r \in \mathbb{N}$ such that $\forall \varepsilon > 0$ $\exists\; x \in \Sigma_g^{B(r)}(A,B)$ with $x(0) = x_o$ and $d(x(t), V_a) \leq \varepsilon$, $\forall t \in \mathbb{R}$.

We will denote by $\underline{V}_{a,g}$ or $\underline{V}_{a,g}(A,B)$ the class of all almost stabilizability subspaces associated with a given set $\mathbb{C}_g$. It may be verified immediately that $\underline{V}_{a,g}$ is closed under subspace addition. The supremal almost stabilizability subspace contained in a given subspace $K \subset X$ will be denoted by $V_{a,g}^*(K)$. We also note that the inclusion $\underline{V}_g \subset \underline{V}_{a,g} \subset \underline{V}_a$ holds. Moreover, if we assume that the stability set $\mathbb{C}_g$ satisfies (2.27) and (2.28), then by TH. 2.60 it follows that $\underline{R}_a \subset \underline{V}_{a,g}$. In that case we may therefore immediately conclude that every subspace of the form $V_g + R_a$, with $V_g \in \underline{V}_g$ and $R_a \in \underline{R}_a$, is an almost stabilizability subspace. In this section we will prove that also the converse of this statement holds, i.e. that every almost stabilizability subspace $V_{a,g}$ can be written as a sum $V_g + R_a$ with V_g a stabilizability subspace and R_a an almost controllability subspace. For every subspace K of X define:

$$A_g(K) := \{x_o \in K \mid \exists\; r \in \mathbb{N} \text{ and a closed subset } D \subset \mathbb{C}_g \text{ such that } \forall\; \varepsilon > 0 \;\; \exists\, x \in \Sigma^{B(r)}(A,B) \text{ with } x(0) = x_o, \; \sigma(x) \subset D \text{ and } d(x(t), K) \leq \varepsilon, \;\; \forall\, t \geq 0\}.$$

<u>LEMMA 2.67</u>. *Let K be a subspace of X with the property that $K \cap B = \{0\}$. Then $A_g(K) = V_g^*(K)$.*

<u>PROOF:</u> This may be proven in a similar fashion as LEMMA 2.55. Decompose $X = X_1 \oplus X_2 \oplus X_3$ with $X_1 := K$ and $X_3 := B$. Again, choose $F \in \underline{F}(V^*)$ such that A_F and B have matrices of the form (2.30). Let C_g and C_b be the sums of the generalized eigenspaces of the mapping $A_F|V^*$ corresponding to its eigenvalues in $\sigma^* \cap ¢_g$ and $\sigma^* \cap (¢\setminus¢_g)$ respectively. It is well known (see WONHAM (1979, p. 114)) that $V_g^*(K) = C_g$ and $V_g^*(K) \oplus C_b = V^*$.

Let $x_o = (x_{01}^T, 0^T, 0^T)^T \in A_g(K)$. There is a closed subset $D \subset ¢_g$, an integer $r \in \mathbb{N}$ and a sequence $x_n \in \Sigma_g^{B(r)}(A,B)$, $x_n = (x_{1n}^T, x_{2n}^T, x_{3n}^T)^T$, with $\sigma(x_n) \subset D$, $x_n(0) = x_o$ and $x_{2n} \to 0$, $x_{3n} \to 0$, uniformly on $[0,\infty)$. As in the proof of LEMMA 2.55, we may show that $x_o \in V^*$ and that $\hat{x}_{1n}(s)$ converges to $(Is - A_{11})^{-1}x_{01}$ for infinitely many $s \in ¢$. Denote $x_1(t) := e^{A_{11}t}x_{01}$. Again by LEMMA 2.53, it follows that $\sigma(x_1) \subset D \subset ¢_g$ (since $\sigma(x_{1n}) \subset D$ for all n and hence $\sigma(x_1) \subset \bar{D} \subset ¢_g$). Consequently, $e^{A_Ft}x_o$ has its spectrum in $¢_g$.

Now, decompose $x_o = x_o^1 + x_o^2$ with $x_o^1 \in V_g^*$ and $x_o^2 \in C_b$. Since $\sigma(e^{A_Ft}x_o^1) \subset ¢_g$, it follows that $\sigma(e^{A_Ft}\, x_o^2) \subset ¢_g$. However, C_b is the sum of the generalized eigenspaces with eigenvalues in $¢\setminus¢_g$ and hence $x_o^2 = 0$. We conclude that $x_o \in V_g^*$.

The converse inclusion follows from the fact that for $F \in \underline{F}(V_g^*)$, $\sigma(A_F|V_g^*) = \sigma^* \cap ¢_g$. The latter is a finite and consequently closed subset of $¢_g$.

□

<u>REMARK 2.68</u>. Note that the assumption on the existence for each x_o of a *closed* subset $D \subset ¢_g$ plays a fundamental role in the above argument.

As an immediate consequence of the above, we obtain the following result:

<u>LEMMA 2.69</u>. *Let $V_a \in \underline{V}_{a,g}(A,B)$ be such that $V_a \cap B = \{0\}$. Then we have $V_a \in \underline{V}_g(A,B)$.*

<u>PROOF:</u> If $V_a \in \underline{V}_{a,g}$ then, by definition, $V_a \subset A_g(V_a)$. Hence, $V_a \subset V_g^*(V_a)$

□

To proceed, we need the following sharpening of TH. 1.14:

<u>LEMMA 2.70.</u> *Let K be a subspace of X. Denote $S^\infty := S^\infty(K)$. Then*

$\Sigma_g^B(A,B)/S^\infty = \Sigma_g^B(\bar{A},\bar{B})$.

PROOF: ($\subset$) This inclusion is an immediate consequence of LEMMA 2.56. ($\supset$). Let $\xi \in \Sigma_g^B(A,B)$. Let X and U be decomposed as in LEMMA 1.15 (i) to (iv), let F be as in LEMMA 1.15 and let the matrices of A_F and B in the latter decomposition be given by (1.27). We have to establish the existence of a $x \in \Sigma_g^B(A,B)$ such that $[x(t)] = \xi(t)$, $\forall t$. As in the proof of TH. 1.14, let $Q^{-1}\xi(t) := (0^T, 0^T, x_3^T(t))^T$. By definition of $(\bar{A},\bar{B})$ we have

$$(2.31) \qquad \frac{d}{dt}\begin{pmatrix}0\\0\\x_3\end{pmatrix} = A_o\begin{pmatrix}0\\0\\x_3\end{pmatrix} + B_o\begin{pmatrix}x_2\\u_3\end{pmatrix} ,$$

where A_o and B_o are given by (1.28). Again, it suffices to prove that there is $x_1(\cdot)$ such that $x(t) := (x_1^T(t), x_2^T(t), x_3^T(t))^T$ is a Bohl trajectory with $\sigma(x) \subset \mathbb{C}_g$. (Since this would yield $\xi(t) = Px(t) = [x(t)], \forall t$)). Now, first note that x_3 is Bohl and $\sigma(x_3) \subset \mathbb{C}_g$. Moreover, we may assume that x_2 is Bohl and $\sigma(x_2) \subset \mathbb{C}_g$. We want to find a Bohl function $x_1(\cdot)$ with $\sigma(x_1) \subset \mathbb{C}_g$ such that the following equation hold:

$$(2.32) \qquad \begin{pmatrix}\dot{x}_1\\\dot{x}_2\end{pmatrix} = \begin{pmatrix}A_{11} & A_{12}\\A_{21} & A_{22}\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} + \begin{pmatrix}A_{13}\\A_{23}\end{pmatrix}x_3 + \begin{pmatrix}B_1 & 0\\0 & B_2\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix} .$$

Here, x_2 and x_3 are as in (2.31) and u_1 an u_2 may be chosen. Furthermore, we may assume that A_{11}, A_{21}, B_1 and B_2 are as in LEMMA 1.15, (v) to (viii). It was already shown in the proof of TH. 1.14 that, indeed, u_1, u_2 and x_1 exist such that (2.32) is satisfied. Moreover, using the very special structure of the matrices in (2.32), it may be verified by inspection that under the dynamic constraint (2.32), x_1 is a linear combination of derivatives of $\begin{pmatrix}x_2(t)\\x_3(t)\end{pmatrix}$ up to some order N. Thus, x_1 is Bohl and, since differentiation does not introduce new characteristic values, $\sigma(x_1) \subset \mathbb{C}_g$.

□

The above lemma yields the following useful result:

LEMMA 2.71. *Let K be a subspace of X. Denote $S^\infty := S^\infty(K)$. Then the following relations hold:*

(i) $\underline{V}_{a,g}(A,B)/S^{\infty} \subset \underline{V}_{a,g}(\bar{A},\bar{B})$,

(ii) $\underline{V}_{g}(A,B)/S^{\infty} \subset \underline{V}_{g}(\bar{A},\bar{B})$.

PROOF: The proof of this lemma uses LEMMA 2.70 and is completely analogous to the proof of LEMMA 1.18. □

Putting all ingredients together now, we obtain the following geometric characterization of the class of almost stabilizability subspaces. We find that a subspace is an almost stabilizability subspace with respect to $\mathbb{C}_g$ if and only if it is the sum of a stabilizability subspace with respect to $\mathbb{C}_g$ and an almost controllability subspace:

THEOREM 2.72. *Assume that* $\mathbb{C}_g$ *satisfies* (2.27) *and* (2.28). *Then* $V_a \in \underline{V}_{a,g}(A,B)$ *if and only if* $V_a = V_g^*(V_a) + S^{\infty}(V_a)$. *Consequently,* $V_a \in \underline{V}_{a,g}$ *if and only if there exists* $V_g \in \underline{V}_g$ *and* $R_a \in \underline{R}_a$ *such that* $V_a = V_g + R_a$.

PROOF: (⇐) This follows from the facts that $\underline{V}_{a,g}$ is closed under subspace addition and that $\underline{V}_g \subset V_{a,g}$ and $S^{\infty} \in \underline{R}_a \subset \underline{V}_{a,g}$.
(⇒) The proof of this implication uses LEMMA 2.71 and is completely analogous to the corresponding proof of TH. 1.27.

The second assertion then follows immediately. □

REMARK 2.73. In SCHUMACHER (1984), almost stabilizability subspaces are *defined* as the sums of stabilizability subspaces and almost controllability subspaces. Starting from this definition, the author establishes characterizations of almost stabilizability subspaces in terms of the trajectories of the system (A,B). However, only the special cases that $\mathbb{C}_g = \{s \in \mathbb{C}\ \mathrm{Re}\, s < 0\}$ and $\mathbb{C}_g = \{s \in \mathbb{C} | \mathrm{Re}\, s \leq 0\}$ are characterized in this way. In our treatment we have established a dynamic characterization for the situation that C_g is *any* stability set (provided of course that it satisfies (2.27) and (2.28)).

We may now also prove the following characterization of $V^*_{a,g}(K)$, purely in terms of distance to K for $t \geq 0$ (compare this with TH. 2.60):

THEOREM 2.74. *Assume that* $\mathbb{C}_g$ *satisfies* (2.27) *and* (2.28). *Let* K *be a subspace of* X. *Then the following equality holds:*

$$V^*_{a,g}(K) = \{x_o \in K \mid \exists\, r \in \mathbb{N} \text{ and a closed subset } D \subset \mathbb{C}_g \text{ such that } \forall \varepsilon > 0 \;\exists\, x \in \Sigma^{B(r)}(A,B) \text{ with } x(0) = x_o,\ \sigma(x) \subset D \text{ and } d(x(t),K) \le \varepsilon,\ \forall t \ge 0\}.$$

PROOF ($\subset$) It may be seen immediately from TH. 2.72 that $V^*_{a,g}(K) = V^*_g(K) + R^*_a(K)$. The fact that $V^*_g(K) \subset A_g(K)$ follows from DEF. 2.64. The inclusion $R^*_a(K) \subset A_g(K)$ follows from TH. 2.60.
($\supset$) The proof of this inclusion may be given using LEMMA 2.71 and is completely analogous to the proof of TH. 1.29.

□

Using the geometric characterization obtained in TH. 2.72, it is also possible to derive frequency domain characterizations of almost stabilizability subspaces and characterizations in terms of Bohl trajectories, in a similar way as in SECTION 2.2. First, we will formulate the frequency domain characterizations of $V^*_g(K)$ and $V^*_{a,g}(K)$. A vector $\xi(s)$ of rational functions will be called *stable* if all its poles lie in $\mathbb{C}_g$.

THEOREM 2.75. *Let K be a subspace of X. Then*

(i) $V^*_g(K) = \{x_o \in K \mid x_o$ *has a (ξ,ω)-representation with* $\xi(s) \in K_+(s)$ *stable and* $\omega(s) \in U_+(s)\}$,

(ii) $V^*_{a,g}(K) = \{x_o \in K \mid x_o$ *has a (ξ,ω) representation with* $\xi(s) \in K(s)$ *stable and* $\omega(s) \in U(s)\}$.

PROOF (i) A proof of this can be found in HAUTUS (1980).
(ii) This may be proven in a similar way as COR. 2.12 (iii), using the characterizations of $V^*_g(K)$ and $R^*_a(K)$.

□

Using the above result, the following characterizations may be obtained:

COROLLARY 2.76. *The following statements are equivalent:*

(i) $V_a \in \underline{V}_{a,g}(A,B)$,

(ii) *Every* $x_o \in V_a$ *has a* (ξ,ω)*-representation with* $\xi(s) \in V_a(s)$ *stable and* $\omega(s) \in U(s)$,

(iii) *For every* $x_o \in V_a$ *there is a* $u \in D_B'^{\,m}$ *such that* $x^+(x_o,u)$ *lies in* K *and the regular part* $x_{reg}(t)$ of $x^+(x_o,u)$ *satisfies* $\sigma(x_{reg}) \subset \mathbb{C}_g$.

PROOF: The equivalence of (i) and (ii) follows immediately from TH. 2.74. The equivalence of (ii) and (iii) follows from the Laplace transform correspondence between the frequency and the time domain (REMARK 2.9). Note that the stability of x_{reg} follows from the fact that x_{reg} is just the inverse Laplace transform of the strictly proper part $[\xi(s)]_+$ of $\xi(s)$. This strictly proper part has its poles in $\mathbb{C}_g$.

□

It turns out that almost stabilizability subspaces have a decomposition into the direct sum of a controllability subspace, a coasting subspace whose associated fixed spectrum is contained in $\mathbb{C}_g$, and a sliding subspace:

THEOREM 2.77. *Let $V_a \in \underline{V}_{a,g}(A,B)$. There exists a sliding subspace S and a coasting subspace C such that*

(2.33) $V_a = R^*(V_a) \oplus C \oplus S$,

(2.34) $V_g^*(V_a) = R^*(V_a) \oplus C$,

(2.35) $R_a^*(V_a) = R^*(V_a) \oplus S$,

(2.36) $\sigma(A_F|C) \subset \mathbb{C}_g$ for every $F \in \underline{F}(C)$.

PROOF: Denote $V_g^* := V_g^*(V_a)$ etc. . For $F \in \underline{F}(V_g^*)$, denote $\sigma^* := \sigma(A_F|V_g^*/R^*)$. It may be shown that this spectrum is independent of F and is contained in $\mathbb{C}_g$. Choose F such that $\sigma(A_F|R^*) \cap \sigma^* = \emptyset$. Let C be the sum of the generalized eigenspaces of the mapping $A_F|V_g^*$ corresponding to the eigenvalues in σ^*. Then $V_g^* = R^* \oplus C$, C is a coasting subspace and, for every $F \in \underline{F}(C)$, $\sigma(A_F|C) = \sigma^* \subset \mathbb{C}_g$. Finally, to obtain a sliding subspace, the same construction as in TH. 2.27 may be applied.

□

The above decomposition result may be applied to obtain the following theorem which states that for every almost stabilizability subspace there exists a sequence of stabilizability subspaces converging to it (SCHUMACHER (1984)):

THEOREM 2.78. *Assume that $\mathbb{C}_g$ satisfies* (2.27) *and* (2.28). *Let $V_a \in \underline{V}_{a,g}(A,B)$. Then there exists a sequence $\{V_n\}_{n\in\mathbb{N}}$ with $V_n \in \underline{V}_g(A,B)$ and $\lim_{n\to\infty} V_n = V_a$.*

PROOF: By TH. 2.77, $V_a = V_g^*(V_a) \oplus R_a$, with $R_a \in \underline{R}_a$. Since R_a may be decomposed into a direct sum of singly generated almost controllability subspaces, it therefore suffices to show that every subspace $\mathcal{L}(u,F,k)$ may be approximated by stabilizability subspaces. However, this was in fact already shown in SECTION 2.4: take $\mathcal{L}(n) := \mathcal{L}_n(u,F,k)$, the canonical sequence as defined by (2.6). Then $\mathcal{L}(n)$ is $(A_F + BF_n)$-invariant (with F_n defined by (2.7)) and $\sigma(A_F + BF_n|\mathcal{L}(n)) = \{-n,\ldots,-n\}$. For n sufficiently large the latter spectrum is contained in $\mathbb{C}_g$ (due to (2.28)).

□

To conclude this chapter we will give feedback characterizations of the class of almost stabilizability subspaces. Apart from (2.27) and (2.28), in order to establish these we shall assume that the stability set $\mathbb{C}_g$ is contained in the open left half complex plane $\mathbb{C}^-$. Given such stability set $\mathbb{C}_g$ let X_{stab} be the stabilizable subspace associated with it. We then have the following analogue of TH. 2.58:

THEOREM 2.79. *Assume that* $\mathbb{C}_g \subset \mathbb{C}^-$ *and satisfies* (2.27) *and* (2.28). *Let* K *be a subspace of* X. *Then* $\forall\ \varepsilon > 0$ *there is a mapping* $F : X \to U$ *such that*

$$\|d(e^{A_F t}x_o, K)\|_p \leq \varepsilon,\ \forall\ x_o \in V^*_{a,g}(K) \textit{ with } \|x_o\| \leq 1,\ \forall\ 1 \leq p \leq \infty$$

and

$$\sigma(A_F|X_{stab}) \subset \mathbb{C}_g.$$

PROOF: This may be proven along the same lines as TH. 2.58. The ingredients of the proof are COR. 2.42 and the fact that the subspace $V^*_{a,g}(K)$ admits a direct sum decomposition into $V^*_g(K)$ and an almost controllability subspace R_a. In the obvious way, one constructs a sequence of mappings $\{F_n\}_{n \in \mathbb{N}}$ such that the closed loop system mapping $A + BF_n$ has $R^*(K)$, $V^*_g(K)$, $V^*_g(K) \oplus V(n)$ and X_{stab} as invariant subspaces. (with $V(n)$ converging to R_a). The situation with the spectrum of $A + BF_n$ is described in the lattice diagram fig. 2.3. on page 91.

□

It follows from this diagram that, in fact, the spectra $\sigma(A + BF_n|X_{stab})$ may be chosen in a *closed subset* D of $\mathbb{C}_g$ for all n. This observation yields the following feeedback characterizations of the class of almost stabilizability subspaces:

<u>THEOREM 2.80</u>. *Assume that* $\mathbb{C}_g \subset \mathbb{C}^-$ *and satisfies* (2.27) *and* (2.28). *Then the following statements are equivalent:*

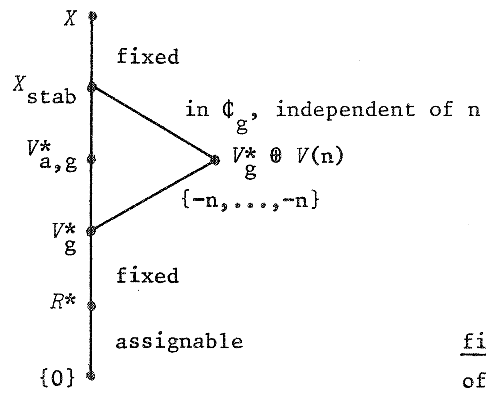
(i) $V_a \in \underline{V}_{a,g}(A,B)$,

(ii) *There is a closed subset* $D \subset \mathbb{C}_g$ *and* $\forall \varepsilon > 0 \;\; \exists F : X \to U$ *such that* $d(e^{A_F t}x_o, V_a) \leq \varepsilon$, $\forall t \geq 0$, $\forall x_o \in V_a$ *with* $\|x_o\| \leq 1$ *and* $\sigma(A_F | X_{stab}) \subset D$,

(iii) *There is a closed subset* $D \subset \mathbb{C}_g$ *and* $\forall \varepsilon > 0 \;\; \exists F : X \to U$ *such that* $\| d(e^{A_F t}x_o, V_a) \|_p \leq \varepsilon$, $\forall x_o \in V_a$ *with* $\|x_o\| \leq 1$, $\forall 1 \leq p \leq \infty$ *and* $\sigma(A_F | X_{stab}) \subset D$.

<u>PROOF:</u> This follows immediately from TH. 2.74 and TH. 2.79, together with the observation made above.

□



<u>fig. 2.3.</u> Spectrum of $A + BF_n$.

CHAPTER 3

L_p-ALMOST CONTROLLED INVARIANT SUBSPACES

In the previous chapters we have considered the class of almost controlled invariant subspaces and set up a framework in which it turned out to be possible to obtain several equivalent characterizations of these subspaces. A particularly useful characterization turned out to be the one in terms of the approximate holdability properties under the use of (high gain) state feedback. Indeed, the latter characterization was shown to make the class of almost controlled invariant subspaces applicable to problems of approximate disturbance rejection. Also for the classes of almost controllability subspaces and almost stabilizability subspaces, we have established the equivalences between open loop descriptions, feedback descriptions, geometric characterizations and holdability properties under the use of Bohl distributional inputs.

In this chapter we will continue our investigations by introducing the notions of supremal L_p-almost controlled invariant subspace and L_p-almost controllability subspace. Compared to the 'ordinary' supremal almost controlled invariant and almost controllability subspaces, the main distinction of these new subspaces will be that the distance of trajectories to subspaces is measured in terms of the L_p-norm of the pointwise distance rather than in terms of the supremum norm. The subspaces introduced will turn out to be useful in the study of several feedback synthesis problems. This will be the topic of the second part of this chapter. Here, it will turn out that the open loop holdability properties in terms of which we will introduce the subspaces considered in this chapter, have their closed loop counterparts. As in the previous chapter, this fact will make these subspaces applicable in problems of approximate disturbance rejection. A completely different type of application will be provided by the property that each almost stabilizability subspace can be considered as the limit of a sequence of stabilizability subspaces. This property will be shown to make the subspaces we discuss in this chapter applicable to the classical problem of stabilization by dynamic output feedback.

The outline of this chapter is as follows. In section 1 we will give definitions of the subspaces to be considered and show how these can be expressed in terms of the subspaces we already know. Section 2 collects some material on the characterization of the new subspaces in terms of frequency domain descriptions, on algorithms to compute them and on their role in the invertibility of systems. In section 3 and section 4 we will establish feedback characterizations of supremal L_p-almost controlled invariant and L_p-almost controllability subspaces and apply these to the L_p-almost disturbance decoupling problem. Finally, in section 5 we will study the problem of stabilization by dynamic output feedback.

3.1 L_p-ALMOST CONTROLLED INVARIANT SUBSPACES

So far, in our development of the theory of almost invariant subspaces we have measured the distance of trajectories to subspaces mainly in terms of the supremum norm of the function formed by calculating at each point of time the distance of the trajectory to the subspace. In particular, we have defined an almost controlled invariant subspace to be a subspace V_a of the state space with the property that for every point in that subspace and for every real number $\varepsilon > 0$, there is a trajectory x through that point such that $\sup_{t\in\mathbb{R}} d(x(t),V_a) \leq \varepsilon$. We have also seen that for every subspace K of X there is a supremal almost controlled invariant subspace $V_a^*(K)$ contained in K. This supremal subspace was characterized as the subspace of K with the property that for every point in that subspace and for every real number $\varepsilon > 0$, there is a trajectory x through that point such that $\sup_{t\in\mathbb{R}^+} d(x(t),K) \leq \varepsilon$. Thus, the subspace $V_a^*(K)$ was characterized in terms of the L_∞-norm of the function obtained by calculating at each $t \in \mathbb{R}^+$ the distance between the trajectories of the system and the subspace K. In a similar way, we have characterized the subspaces $R_a^*(K)$ and $V_{a,g}^*(K)$.

In this section we will consider the following question: which subspaces do we obtain if, instead of using the L_∞-norm of the distance functions, we use their L_p-norms for $1 \leq p < \infty$. Thus, given a subspace K of X, we will, for every p, define a subspace $V_p^*(K)$ by the requirement that for each point in that subspace and for every real number $\varepsilon > 0$, there is a trajectory x through that point such that

$$\int_0^\infty d(x(t),K)^p \, dt \leq \varepsilon .$$

It will turn out that all subspaces $V_p^*(K)$ are for $p \in [1,\infty)$ in fact equal to one and the same subspace which only depends on K. It will be shown that this subspace may be expressed in a simple way in terms of the subspaces $V^*(K)$, $V^*_a(K)$ and $R^*_a(K)$ and the mappings A and B defining our system.

In the sequel, again we will denote $\|f\|_\infty := \operatorname{esssup}_{t\in\mathbb{R}^+} |f(t)|$ and $\|f\|_p := (\int_{\mathbb{R}^+} |f(t)|^p \, dt)^{1/p}$ for $1 \leq p < \infty$.

DEFINITION 3.1. Let K be a subspace of X and let $1 \leq p < \infty$. Then we define *the supremal* L_p*-almost controlled invariant subspace of* K by

$$V^*_p(K) := \{x_0 \in X \mid \forall \varepsilon > 0 \ \exists x \in \Sigma(A,B) \text{ such that } x(0) = x_0 \text{ and } \|d(x(\cdot),K)\|_p \leq \varepsilon\} .$$

We define *the supremal* L_p*-almost controllability subspace of* K by

$$R^*_p(K) := \{x_0 \in X \mid \exists T > 0 \text{ and } \forall \varepsilon > 0 \ \exists x \in \Sigma(A,B) \text{ with } x(0) = x_0, \ x(T) = 0 \text{ and } \|d(x(\cdot),K)\|_p \leq \varepsilon\} .$$

Thus, a point x_0 lies in $R^*_p(K)$ if it is possible to travel from this point to the origin in finite time along trajectories in such a way that the L_p-norms of the pointwise distance for $t \in \mathbb{R}^+$ from these trajectories to the subspace K can be made arbitrarily small. The subspace $V^*_p(K)$ is sometimes called the 'L_p-almost output nulling subspace of K'. This terminology stems from the interpretation that if $K = \ker H$, then for initial conditions $x_0 \in V^*_p(K)$ the output $z(t) = Hx(t)$ can be made arbitrarily small by choosing the input properly.

Let us first consider the case that $p = \infty$. It follows from TH. 1.24 and TH. 1.29 that $R^*_\infty(K) = R^*_a(K)$ and $V^*_\infty(K) = V^*_a(K)$, respectively. Indeed, if $x_0 \in X$ has the property that for all $\varepsilon > 0$ there exists $x \in \Sigma(A,B)$ such that for all $t \in \mathbb{R}^+$, $d(x(t),K) \leq \varepsilon$, then consequently $d(x_0,K) \leq \varepsilon$, $\forall \varepsilon$, and hence $x_0 \in K$. We conclude that the supremal L_∞-almost controlled invariant subspace of K is equal to the ordinary supremal almost controlled invariant subspace contained in K. A similar conclusion holds for the supremal L_∞-almost controllability subspace of K and the ordinary supremal almost controllability subspace contained in K. Whereas $V^*_\infty(K)$ and $R^*_\infty(K)$ are always contained in K, this will in general not be true for $V^*_p(K)$ and $R^*_p(K)$ if we take $1 \leq p < \infty$:

EXAMPLE 3.2. For the linear system $\dot{x} = -x + u$, with state space $X := \mathbb{R}$, consider the sequence of inputs u_n defined by $u_n(t) := (1-n)e^{-nt}\, 1_{\mathbb{R}^+}(t)$. Taking $x(0) = x_0$, for $t \geq 0$ the resulting trajectories are calculated to be $x_n(t) = e^{-nt}\, x_0$. Thus, $\|x_n\|_p = x_0(\sqrt[p]{pn})^{-1}$ $(1 \leq p < \infty)$ and we may conclude that $V_p^*(\{0\}) = X$.

In the following, let $1 \leq p < \infty$ and let K be a subspace of X. We contend that $R_a^*(K) \subset R_p^*(K)$. This may be shown as follows. Let $x_0 \in R_a^*(K)$. Since $R_a^* \in \underline{R}_a$, there is $T > 0$ and for all $\varepsilon > 0$ there exists a trajectory x such that $x(0) = x_0$, $x(T) = 0$ and $d(x(t), R_a^*) \leq \varepsilon$, $\forall t \in \mathbb{R}$. Thus, we may find $T > 0$ and a sequence $x_n \in \Sigma(A,B)$ such that $x_n(0) = x_0$, $x_n(t) = 0$ for $t \geq T$ and $\lim_{n\to\infty} \|d(x_n(\cdot),K)\|_\infty = 0$. It is then immediate that also $\lim_{n\to\infty} \|d(x_n(\cdot),K)\|_p = 0$ for $1 \leq p < \infty$. This proves our assertion.

Consider now the subspace $B + AR_a^*(K)$. We claim that this subspace is contained in $R_p^*(K)$. To see this, recall that R_a^* admits a decomposition into a (direct) sum of singly generated almost controllability subspaces $\mathcal{L}_i = \mathcal{L}(u_i,F,r_i) = \text{span}\,\{Bu_i,\ldots,A_F^{r_i-1}\,Bu_i\}$, $i \in \underline{s}$. Moreover, we may assume that $\text{span}\,\{Bu_1,\ldots,Bu_s\} = B \cap R_a^*$ (see COR. 1.23). Extending the latter span to a basis for B by adding vectors $Bu_{s+1},\ldots,Bu_m$ yields

$$(3.1) \qquad B + AR_a^* = B + A_F\mathcal{L}_1 + \ldots + A_F\mathcal{L}_s = \mathcal{L}_1^* + \ldots + \mathcal{L}_m^* .$$

Here, $m = \dim B$ and the $\mathcal{L}_i^*$ are subspaces defined by $\mathcal{L}_i^* := \text{span}\,\{Bu_i,\ldots,A_F^{r_i}Bu_i\}$ for $i = 1,2,\ldots,s$ and $\mathcal{L}_i^* := \text{span}\,\{Bu_i\}$ for $i = s+1,\ldots,m$. In this way we find a decomposition of $B + AR_a^*$ into singly generated almost controllability subspaces of the form $\text{span}\,\{Bu,\ldots,A_F^k\,Bu\}$, with the first k vectors $Bu, A_FBu, \ldots, A_K^{k-1}Bu$ contained in R_a^*. To prove our claim it therefore suffices to show that the vectors $A_F^k\,Bu$ are contained in $R_p^*(K)$. (Since we already proved that $R_a^* \subset R_p^*(K)$.). Denote $x_0 = A_F^k\,Bu$. Let φ_n be a smooth approximation of the Dirac delta distribution with $\varphi_n \geq 0$, $\text{supp}\,\varphi_n \subset [0,\frac{1}{n}]$, ${}_{\mathbb{R}}\!\int \varphi_n = 1$ and $\varphi_n^{(\ell)}(0) = 0$ for $\ell = 0,\ldots,k$. Define a sequence of inputs by

$$u_n(t) := -\varphi_n^{(k)}(t)u \,, \qquad (t \in \mathbb{R}) \,.$$

Let x_n denote the resulting state trajectory with $x_n(0) = x_0$. Let $\mathcal{L} := \text{span}\,\{Bu,\ldots,A_F^{k-1}Bu\}$ if $k \geq 1$ and $\mathcal{L} := \{0\}$ if $k = 0$. Fix $T > 0$. It was already shown in SECTION 1.1 that for $t \geq 0$

$$x_n(t) = e^{A_Ft}\, x_0 - \int_0^t e^{A_F(t-\tau)}\, x_0\varphi_n(\tau)\,d\tau - \sum_{j=1}^{k} \varphi_n^{(k-j)}(t)A_F^{j-1}Bu \,.$$

Denote $K(t) := e^{A_F t} 1_{\mathbb{R}^+}(t)$. Then, according to (1.13)

$$d(x_n(t),\mathcal{L}) \leq \sup_{0 \leq \tau \leq \frac{1}{n}} d(K(t)x_0 - K(t-\tau)x_0,\mathcal{L}) \ ,$$

for all $t \in [0,T]$. It follows that $d(x_n(t),\mathcal{L}) \to 0 \quad (n \to \infty)$ uniformly on $[\delta,T]$ for each $\delta > 0$. (This may be shown using the uniform continuity of $t \to d(K(t)x_0,\mathcal{L})$ on $[0,T]$.) It is also immediate that there exists a constant M such that for all n and for all $t \in [0,T]$, $d(x_n(t),\mathcal{L}) \leq M$. Now, let $\varepsilon > 0$. Let δ be such that ${}_0\int^{\delta} d(x_n(t),\mathcal{L})^p \, dt \leq \frac{1}{2}(\frac{\varepsilon}{2})^p$, $\forall n$, and let N be such that ${}_{\delta}\int^{T} d(x_n(t),\mathcal{L})^p \, dt \leq \frac{1}{2}(\frac{\varepsilon}{2})^p$ for $n \geq N$. For all $n \geq N$ we then have

$$\| d(x_n(\cdot),\mathcal{L}) 1_{[0,T]} \|_p \leq \tfrac{1}{2}\varepsilon \ .$$

Also, $x_n(T) \to 0 \quad (n \to \infty)$. We may thus apply LEMMA 1.9 to obtain a trajectory x with $x(0) = x_0$, having compact support and $\|d(x(\cdot),\mathcal{L})\|_p \leq \varepsilon$. Since $\mathcal{L} \subset K$ we then also have $\|d(x(\cdot),K)\|_p \leq \varepsilon$. This completes out proof of the claim that $B + AR_a^*(K) \subset R_p^*(K)$.

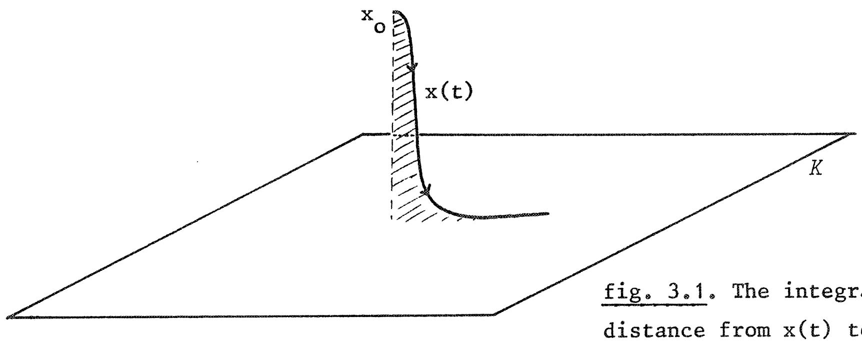


fig. 3.1. The integrated distance from x(t) to K.

Let us now consider the subspace $V_p^*(K)$. It is immediate that $V^*(K) \subset V_p^*(K)$. Therefore, using the same argument as above, we find that $V^*(K) + B + AR_a^*(K) \subset V_p^*(K)$. In the remainder of this section we will prove that the inclusions we have established are, in fact, equalities. We will first prove the following lemma:

LEMMA 3.3. *Let K be a subspace of X and* $1 \leq p < \infty$. *Assume that* $K \cap B = \{0\}$. *Then* $R_p^*(K) = B$.

PROOF: The inclusion $R_p^*(K) \supset B$ follows from our earlier considerations. For the other inclusion, decompose $X = X_1 \oplus X_2$, with $X_2 := B$ and X_1 a subspace such that $K \subset X_1$. After a suitable preliminary feedback F, the matrices of A_F and B in this decomposition are given by

$$A_F = \begin{pmatrix} A_{11} & A_{12} \\ 0 & 0 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 0 \\ B_2 \end{pmatrix} .$$

Let $x_0 \in R_p^*(K)$ and assume $x_0 = (x_{01}^T, x_{02}^T)^T$. Let $T > 0$ be as in the definition of $R_p^*(K)$. There is a sequence $x_n \in \Sigma(A,B)$, $x_n = (x_{1n}^T, x_{2n}^T)^T$, with $\|x_{2n}\|_p \to 0$ $(n \to \infty)$, $x_{1n}(0) = x_{01}$, $x_{2n}(0) = x_{02}$, $x_{1n}(T) = 0$ and $x_{2n}(T) = 0$. Now, in the obvious way we obtain

$$\|x_{1n}(T) - e^{A_{11}T} x_{01}\| \leq \int_0^T \|e^{A_{11}(T-\tau)} A_{12}\| \|x_{2n}(\tau)\| d\tau .$$

Since $x_{1n}(T) = 0$, it therefore follows from Hölder's inequality that

$$\|e^{A_{11}T} x_{01}\| \leq C \|x_{2n}\|_p$$

for some constant C. Since $\|x_{2n}\|_p \to 0$ $(n \to \infty)$, it follows that $x_{01} = 0$. We conclude that $x_0 \in B$, and hence $R_p^*(K) = B$.

□

To prove the geometric characterization of $R_p^*(K)$ that we are looking for, we will again apply the results from SECTION 1.4. Thus, in the following, if K is a subspace of X, let $(\bar{A},\bar{B})$ denote the factor system modulo $S^\infty(K)$ whose existence we established in TH. 1.14. We have the following result:

THEOREM 3.4. *Let K be a subspace of X and $1 \leq p < \infty$. Then*

$$R_p^*(K) = B + AR_a^*(K) .$$

PROOF: Let $x_0 \in R_p^*(K)$. Denote $S^\infty := S^\infty(K)$. Let $\xi_0 := [x_0]$, the equivalence class of x_0 modulo S^∞. Let $T > 0$ be associated with x_0 as in DEF 3.1 and let $\varepsilon > 0$. There is a $x \in \Sigma(A,B)$ with $x(0) = x_0$, $x(T) = 0$ and $\|d(x(\cdot),K)\|_p \leq \varepsilon$. Define $\xi(t) := [x(t)]$. Then, by TH. 1.14, $\xi \in \Sigma(\bar{A},\bar{B})$, $\xi(0) = \xi_0$, $\xi(T) = 0$ and $\|d(\xi(\cdot),K/S^\infty)\|_p \leq \varepsilon$ (see also the proof of LEMMA 1.18). Thus we find that $\xi_0 \in R_p^*(K/S^\infty)$, the L_p-almost controllability subspace of K/S^∞ associated with the system $(\bar{A},\bar{B})$. Since im $\bar{B} \cap (K/S^{\infty}) = \{0\}$, it follows from LEMMA 3.3 that $\xi_0 \in$ im $\bar{B}$. By TH. 1.14 however, im $\bar{B}$ is equal to $(AS^{\infty} + B)/S^{\infty}$. Since $S^{\infty} = R_a^*(K)$, it then follows that $x_0 \in B + AR_a^*(K) + R_a^*(K) = B + AR_a^*(K)$. □

In a similar way it is possible to obtain a representation of the supremal L_p-almost controlled invariant subspace $V_p^*(K)$. Again, let us first consider the case that $K \cap B = \{0\}$:

LEMMA 3.5. *Let K be a subspace of X and $1 \leq p < \infty$. Assume that $K \cap B = \{0\}$. Then $V_p^*(K) = B + V^*(K)$.*

PROOF: Again, the inclusion $B + V^*(K) \subset V_p^*(K)$ has already been proven. For the reverse inclusion, let $X_1 := K$, $X_3 := B$ and let X_2 be a subspace such that $X = X_1 \oplus X_2 \oplus X_3$. After a suitable preliminary feedback F, we have

$$A_F = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ 0 & 0 & 0 \end{pmatrix} , \begin{pmatrix} 0 \\ 0 \\ B_3 \end{pmatrix} .$$

Now, let $x_0 = (x_{01}^T, x_{02}^T, x_{03}^T)^T \in V_p^*(K)$. There is a sequence of trajectories $x_n = (x_{1n}^T, x_{2n}^T, x_{3n}^T)^T$ with $x_n(0) = x_0$, $\|x_{2n}\|_p \to 0$ and $\|x_{3n}\|_p \to 0$. Denote $z_n := A_{12}x_{2n} + A_{13}x_{3n}$. Using the fact that $\|z_n\|_p \to 0$, it may be shown, using Hölder's inequality, that for each $T > 0$, $x_{1n} \to e^{A_{11}t}\, x_{01}$ $(n \to \infty)$ uniformly on $[0,T]$. Fix $T > 0$. Denote $w_n := A_{22}x_{2n} + A_{23}x_{3n}$. Then the following equality holds:

$$(3.2) \qquad x_{2n}(t) = x_{02} + A_{21} \int_0^t x_{1n}(\tau)d\tau + \int_0^t w_n(\tau)d\tau \ .$$

Let $\varepsilon_1 > 0$. Since $\|w_n\|_p \to 0$, it follows from Hölder's inequality that for n sufficiently large $\|\int_0^t w_n(\tau)d\tau\| \leq \varepsilon_1$ for all $t \in [0,T]$. Also, since x_{1n} is uniformly convergent on $[0,T]$, there is a constant M such that $\|x_{1n}(t)\| \leq M$, $\forall\, t \in [0,T]$, $\forall n$.

Let ε_2 be any positive number. It follows from (3.2) that $\|x_{2n}(t)\| \geq$ $\geq \|x_{02}\| - \|A_{21}\| M\varepsilon_2 - \varepsilon_1$ for all $t \in [0,\varepsilon_2]$. Thus, by taking ε_1 and ε_2 sufficiently small, we find that $\|x_{2n}(t)\| \geq \frac{1}{2}\|x_{02}\|$ for all $t \in [0,\varepsilon_2]$ and for all n. Since however $\|x_{2n}\|_p \to 0$, it follows that $x_{02} = 0$. Hence we may conclude that $x_0 \in K \oplus B$. We contend that the vector $\tilde{x}_0 := (x_{01}^T, 0^T, 0^T)^T$ is in $V^*(K)$. To see this, define a vector function $\tilde{x}_n(t) := (x_{1n}^T(t), x_{2n}^T(t), 0^T)^T$ and define $\tilde{x}(t) := (x_1^T(t), 0^T, 0^T)^T$, where $x_1(t) := e^{A_{11}t}\, x_{01}$. We already showed that $x_{1n} \to x_1$ uniformly on $[0,T]$. Now, it may be verified directly that

$$\tilde{x}_n(t) = A_F \int_0^t \tilde{x}_n(\tau)d\tau + \begin{pmatrix} A_{13} \\ A_{23} \\ 0 \end{pmatrix} \int_0^t x_{3n}(\tau)d\tau + \tilde{x}_0 \ .$$

Since $x_{2n} \to 0$ and $x_{3n} \to 0$ in $L_p(\mathbb{R}^+)$, it may be seen that $\tilde{x}_n(t) \to \tilde{x}(t)$ pointwise in t and that

$$\tilde{x}(t) = A_F \int_0^t \tilde{x}(\tau)d\tau + \tilde{x}_0 \ .$$

It follows that $\tilde{x}(t) = e^{A_F t}\, \tilde{x}_0$. Since also $\tilde{x}(0) = \tilde{x}_0$ and since $\tilde{x}(t) \in K$, $\forall t$, we may conclude that $\tilde{x}_0 \in V^*(K)$. This proves that $x_0 \in V^*(K) + B$.

□

The above 'hard' analysis now immediately leads to the following result:

THEOREM 3.6. *Let K be a subspace of X and let $1 \leq p < \infty$. Then*

$$V_p^*(K) = V^*(K) + B + AR_a^*(K) \ .$$

PROOF: This may be proven along the same lines as TH. 3.4.

□

We have thus obtained geometric characterizations of the subspaces $V_p^*(K)$ and $R_p^*(K)$ for all $1 \leq p \leq \infty$. In the remainder of this tract, the following notation will be used:

(3.3) $\quad R_b^*(K) := B + AR_a^*(K)$

(3.4) $\quad V_b^*(K) := V^*(K) + R_b^*(K) \ .$

Because of the results of this section, $R_b^*(K)$ will be referred to as the supremal L_p-almost controllability subspace of K and $V_b^*(K)$ will be referred to as the supremal L_p-almost controlled invariant subspace of K. Note that $R_a^*(K) \subset R_b^*(K)$ and that $V_a^*(K) \subset V_b^*(K)$. It follows from (3.1) that $R_b^*(K) \in \underline{R}_a$ and that $V_b^*(K) \in \underline{V}_a$. Thus, both $V_b^*(K)$ and $R_b^*(K)$ are almost controlled invariant subspaces in the sense of DEF. 1.5.

We conclude this section with the following lattice diagram summarizing the inclusion relations we have established so far:

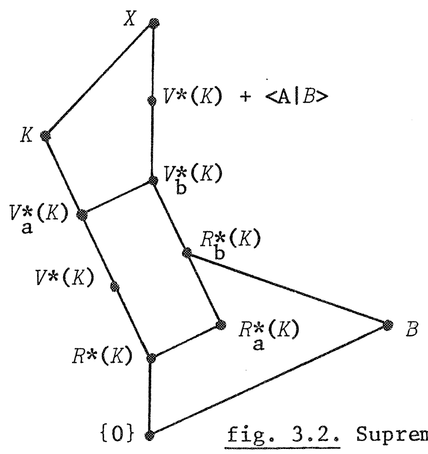


fig. 3.2. Supremal L_p-almost and L_∞-almost controlled invariant subspaces.

3.2 COMPUTATION, FREQUENCY DOMAIN DESCRIPTION AND INVERTIBILITY

In the present section we will show that the supremal L_p-almost controlled invariant subspace and the supremal L_p-almost controllability subspace associated with a given subspace of X may be computed in terms of the limiting subspaces of certain recursive algorithms. It will turn out that most of the work necessary to obtain this result has already been done in SECTION 1.6. Indeed, since the supremal L_p-almost controllability subspace $R_b^*(K)$ may be calculated directly from $R_a^*(K)$ (see (3.3)), setting up a recursive algorithm to compute the former will only involve an adaption of the almost controllability subspace algorithm ACSA. Of course, once we have a recursive algorithm to compute $R_b^*(K)$, this algorithm may be combined with the invariant subspace algorithm ISA to obtain an algorithm to compute the supremal L_p-almost controlled invariant subspace $V_b^*(K)$.

Also, in this section we will extend the results of SECTION 2.2 and show that the subspaces $V_b^*(K)$ and $R_b^*(K)$ may be characterized in terms of (ξ,ω)-representations and in terms of Bohl distributional inputs.

Finally, we will consider the connection between these subspaces and the notion of invertibility of linear systems.

In the following, let K be a subspace of X. Recall that $R_b^*(K) = B + AR_a^*(K)$. Let $S^\mu(K)$ be the sequence of subspaces generated by the almost controllability subspace algorithm ACSA (see SECTION 1.3). Let $k := \dim K$. It was shown that $S^k(K) = R_a^*(K)$. Therefore, to obtain a recursive algorithm to compute the supremal L_p-almost controllability subspace $R_b^*(K)$, we will consider the sequence of subspaces $R^\mu(K)$, $\mu \in \mathbb{N} \cup \{0\}$, defined by

$$(3.5) \qquad R^0(K) := \{0\} \ ; \quad R^\mu(K) := B + AS^{\mu-1}(K) \ , \qquad \mu \in \mathbb{N} \ .$$

Indeed, it may be seen that these subspaces are generated recursively by

$$(3.6) \qquad R^0(K) := \{0\} \ ; \quad R^{\mu+1}(K) := B + A(K \cap R^\mu(K)) \ , \qquad \mu \in \mathbb{N} \ .$$

In the sequel, the above recursive algorithm will be referred to as (ACSA)'. This algorithm inherits its properties from the algorithm ACSA:

<u>THEOREM 3.7</u>. *Let K be a subspace of X and for $\mu \in \mathbb{N} \cup \{0\}$, let $R^\mu := R^\mu(K)$ be defined by* (3.6). *Then*

(i) *The sequence R^μ is monotonically nondecreasing. Moreover, if $R^\mu = R^{\mu+1}$, then $R^\mu = R^{\mu+\nu}$ for all $\nu \in \mathbb{N}$.*

(ii) *There is $k \in \mathbb{N}$, $k \leq \dim K + 1$, such that for all $\nu \in \mathbb{N}$, $R^k = R^{k+\nu}$.*

<u>PROOF</u>: This follows immediately from the fact that $R^\mu \cap K = S^\mu$ and from TH. 1.10.

□

Define now the limiting subspace of (ACSA)' by $R^\infty(K) := R^{\dim K+1}$. From the above we have $R^\infty = R^\mu$ for all $\mu \geq \dim K + 1$. Let $V^\mu(K)$ be the sequence generated by the invariant subspace algorithm ISA (see (1.44)) and let $V^\infty(K)$ be the corresponding limiting subspace. Then the following result is an easy consequence of the foregoing:

<u>THEOREM 3.8</u>. *Let K be a subspace of X. Then*

(i) $\qquad R_b^*(K) = R^\infty(K)$

(ii) $\qquad V_b^*(K) = V^\infty(K) + R^\infty(K)$.

PROOF: (i) $R_b^*(K) = B + AR_a^*(K) = B + AS^\infty(K) = R^\infty(K)$. Of course, (ii) follows immediately from the definition of $V_b^*(K)$ using the fact that $V^\infty(K) = V^*(K)$. □

Next, we will establish characterizations of the above subspaces in terms of (ξ,ω)-representations (see DEF. 2.8). After that, these frequency domain descriptions will immediately be translated back to the time domain in order to obtain characterizations in terms of Bohl distributional inputs.

THEOREM 3.9. *Let K be a subspace of X. Then for all $\mu \in \mathbb{N}$:*

(i) $R^\mu(K) = \{x_0 \in X \mid x_0$ *has a* (ξ,ω)*-representation with* $\xi(s) \in K[s]$, $\omega(s) \in U[s]$ *and* $[s^{1-\mu}\,\xi(s)]_- = 0\}$,

(ii) $R_b^*(K) = \{x_0 \in X \mid x_0$ *has a* (ξ,ω)*-representation with* $\xi(s) \in K[s]$ *and* $\omega(s) \in U[s]\}$,

(iii) $V_b^*(K) = \{x_0 \in X \mid x_0$ *has a* (ξ,ω)*-representation with* $\xi(s) \in K(s)$ *and* $\omega(s) \in U(s)\}$.

PROOF: (i) For $\mu = 1$ the claim obviously holds. Assume $\mu \geq 2$. Let $x_0 \in R^\mu(K)$. By (3.5), there is $\bar{x}_0 \in S^{\mu-1}(K)$ and $u_0 \in U$ such that $x_0 = -A\bar{x}_0 + Bu_0$. Also, by TH. 2.10, there is $\xi_1(s) \in K[s]$ and $\omega_1(s) \in U[s]$ with $[s^{2-\mu}\,\xi_1(s)]_- = 0$ and $-\bar{x}_0 = (Is-A)\xi_1(s) + B\omega_1(s)$. Note that $\bar{x}_0 \in K$. Define now $\xi(s) := \bar{x}_0 + s\xi_1(s)$ and $\omega(s) := u_0 + s\omega_1(s)$. It may then be verified that $x_0 = (Is-A)\xi(s) + B\omega(s)$. Moreover, $\xi(s) \in K[s]$, $\omega(s) \in U[s]$ and $[s^{1-\mu}\,\xi(s)]_- = [s^{1-\mu}\,\bar{x}_0]_- + [s^{2-\mu}\,\xi_1(s)]_- = 0$.

Conversely, let $x_0 = (Is-A)\xi(s) + B\omega(s)$ with $\xi(s) \in K[s]$, $\omega(s) \in U[s]$ and $[s^{1-\mu}\,\xi(s)]_- = 0$. Let $\xi(s) = \sum_{i=0}^{N} \bar{x}_i s^i$ and $\omega(s) = \sum_{i=0}^{N+1} u_i s^i$. Obviously, $\xi(s) = \bar{x}_0 + s\xi_1(s)$ and $\omega(s) = u_0 + s\omega_1(s)$, with $\bar{x}_0 \in K$, $\xi_1(s) \in K[s]$ and $\omega_1(s) \in U[s]$. Hence, $x_0 = Bu_0 - A\bar{x}_0 + s\bar{x}_0 + s^2\xi_1(s) - As\xi_1(s) + Bs\omega_1(s)$. By equating powers of s, the latter equality yields

$$x_0 = -A\bar{x}_0 + Bu_0 \tag{3.7}$$

$$(3.8) \qquad -\bar{x}_0 = (Is-A)\xi_1(s) + B\omega_1(s) \; .$$

Since $[s^{2-\mu}\,\xi_1(s)]_- = [s^{1-\mu}\,\xi(s)]_- = 0$, it follows from (3.8) and TH. 2.10 that $\bar{x}_0 \in S^{\mu-1}(K)$. Therefore, combining (3.5) and (3.7), $x_0 \in R^{\mu}(K)$.

(ii) This now follows immediately by combining (i) and TH. 3.7 (i).

(iii) This can be proven in the same way as COR. 2.12 (iii), using (ii) in the above and COR. 2.12 (ii).

□

As in SECTION 2.2, we may immediately translate the above frequency domain descriptions and obtain the following time domain characterizations in terms of Bohl distributions:

<u>COROLLARY 3.10</u>. *Let K be a subspace of X. Then for all $\mu \in \mathbb{N}$:*

(i) $R^{\mu}(K) = \{x_0 \in X \mid \exists\, u \in D_B'^{m}$, *impulsive, such that* $x^+(x_0,u)(0^+) = 0$, $x^+(x_0,u)$ *lies in K and* $\mathrm{ord}\, x^+ \le \mu-2\}$,

(ii) $R_b^*(K) = \{x_0 \in X \mid \exists\, u \in D_B'^{m}$, *impulsive, such that* $x^+(x_0,u)(0^+) = 0$ *and* $x^+(x_0,u)$ *lies in* $K\}$,

(iii) $V_b^*(K) = \{x_0 \in X \mid \exists\, u \in D_B'^{m}$ *such that* $x^+(x_0,u)$ *lies in* $K\}$.

□

<u>REMARK 3.11</u>. The only difference between on the one hand the characterizations of the subspaces $S^{\mu}(K)$, $R_a^*(K)$ and $V_a^*(K)$ as given in SECTION 2.2 and the characterizations of $R^{\mu}(K)$, $R_b^*(K)$ and $V_b^*(K)$ on the other hand is that in the latter characterizations the expression '$x_0 \in K$' is replaced by '$x_0 \in X$'. The above subspaces have in common that they consist of points that can serve as initial conditions for Bohl distributional trajectories whose restriction x^+ to $\mathbb{R}^+$ lies in K. We contend that if $x_0 \in V_b^*(K)$ and if $x^+(x_0,u)$ is a Bohl distributional trajectory that lies in K, then, in fact, $x^+(x_0,u)$ lies in $V_a^*(K)$. To prove this, recall from PROP. 2.5 that $x^+(x_0,u)$ is the sum of an impulsive part and a regular part: $x^+(x_0,u) = x_{imp} + x_{reg}$. If $x^+(x_0,u)$ lies in K then of course both x_{imp} and x_{reg} lie in K. Thus, $x_{reg}(t) \in K$, $\forall t \ge 0$, and hence $x_{reg}(t) \in V^*(K)$, $\forall t \ge 0$ (see also REMARK 2.14). Therefore, in order to show that $x^+(x_0,u)$ lies in $V_a^*(K)$ it is sufficient to prove that x_{imp} lies in $R_a^*(K)$. To prove this, it is convenient to work in the frequency domain.

Let L denote de Laplace transform and define $\xi(s) := (Lx^+(x_0,u)(s)$ and $\omega(s) := -(Lu)(s)$. Decompose $\xi(s) = \xi_-(s) + \xi_+(s)$ and $\omega(s) = \omega_-(s) + \omega_+(s)$, where $\xi_-(s)$ and $\omega_-(s)$ are polynomials and $\xi_+(s)$ and $\omega_+(s)$ are strictly proper. Obviously, $\xi_-(s) = (Lx_{imp})(s)$ and $\xi_+(s) = (Lx_{reg})(s)$. Since $x_0 = (Is-A)\xi(s) + B\omega(s)$, we obtain

$$x_0 - (Is-A)\xi_-(s) - B\omega_-(s) = (Is-A)\xi_+(s) + B\omega_+(s) \ . \tag{3.9}$$

Since the left hand side in (3.9) is a polynomial and the right hand side a proper rational function, both sides must in fact be equal to a constant vector. This vector may be shown to be $x(0^+) := x(x_0,u)(0^+)$. Hence, $x_0 - x(0^+) = (Is-A)\xi_-(s) + B\omega_-(s)$. We will now show that the polynomial $\xi_-(s)$ has its coefficient in $R_a^*(K)$. In order to show this, assume that $\xi_-(s) = \Sigma_{i=0}^{N} \bar{x}_i s^i$ and $\omega_-(s) = \Sigma_{i=0}^{N+1} u_i s^i$. Using the above established fact that $(Is-A)\xi_-(s) + B\omega_-(s)$ is constant, it follows by equating powers that

$$\begin{aligned} -A\bar{x}_0 + Bu_0 &= (Is-A)\xi_-(s) + B\omega_-(s) \ , \\ -A\bar{x}_1 + Bu_1 &= -\bar{x}_0 \ , \\ &\vdots \\ -A\bar{x}_N + Bu_N &= -\bar{x}_{N-1} \ , \\ Bu_{N+1} &= -\bar{x}_N \ . \end{aligned}$$

It may then be verified by straightforward calculation that

$$-\bar{x}_i = (Is-A)\xi_i(s) + B\omega_i(s) \ , \tag{3.10}$$

where we have defined

$$\xi_i(s) = s^{-i-1}\left(\xi_-(s) - \sum_{j=0}^{i} \bar{x}_j s^j\right)$$

and

$$\omega_i(s) = s^{-i-1}\left(\omega_-(s) - \sum_{j=0}^{i} u_j s^j\right) .$$

Since, for $i = 0,1,\ldots,N$, the polynomials $\xi_i(s)$ are in $K[s]$, we conclude that $\bar{x}_i \in R_a^*(K)$. It follows that $x_{imp} = L^{-1}\xi_-$ lies in $R_a^*(K)$.

Summarizing the above considerations, we see that, if $x_0 \in V_b^*(K)$ and if $x^+(x_0,u)$ lies in K, then the distribution $x^+(x_0,u)$ consists of *an initial jump from* x_0 *to the subspace* $V_a^*(K)$. This initial jump is *followed by an im-*

pulsive motion that takes place in $V_a^*(K)$, *in the direction of* $R_a^*(K)$. After this impulsive motion, *we end up in* $x(0^+) \in V^*(K)$. The rest of the motion of the trajectory $x^+(x_0,u)$ is *regular and takes place in the subspace* $V^*(K)$.

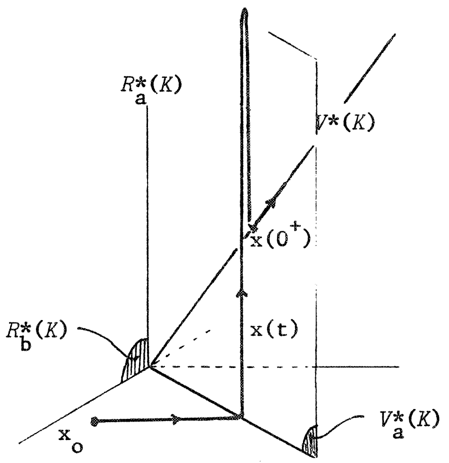


fig. 3.3. Distributional trajectory starting in x_o: initial jump to the subspace $V_a^*(K)$, followed by impulsive motion within $V_a^*(K)$ in the direction of $R_a^*(K)$, ending up in $x(0^+)$ in $V^*(K)$. Finally, regular motion within $V^*(K)$, starting in $x(0^+)$.

The following result follows easily from the foregoing frequency domain descriptions:

COROLLARY 3.12. *Let K be a subspace of X. Then*

(i) $R_a^*(K) = R_b^*(K) \cap K$,

(ii) $V_a^*(K) = V_b^*(K) \cap K$,

(iii) $V^*(K) \cap R_b^*(K) = R^*(K)$.

PROOF: (i) and (ii) can immediately be obtained by combining COR 2.12 and TH. 3.8, (iii) follows from the fact that $V^*(K) \cap R_b^*(K) = V^*(K) \cap K \cap R_b^*(K)$ and from TH. 1.32 (iv).

□

REMARK 3.13. Let Y be a finite dimensional linear space and let C be a mapping from X to Y. Consider the system (A,B) and for each $x_0 \in X$ and $u \in U_D$ (see SECTION 2.1) denote $y(x_0,u) := Cx(x_0,u)$. Here, $x(x_0,u)$ denotes the state trajectory of the system (A,B) with initial consition $x(0) = x_0$ and input u. We will call $y(x_0,u)$ the output corresponding to the input u and initial condition x_0. We will denote its restriction to $\mathbb{R}^+$ by $y^+(x_0,u) := Cx^+(x_0,u)$.

COR. 3.9 says that for every $x_0 \in V_b^*(\ker C)$ there exists a $u \in U_D$ such that $y^+(x_0,u) = 0$. Because of this property the subspace $V_b^*(\ker C)$ is sometimes called the space of distributionally weakly unobservable states (HAUTUS & SILVERMAN (1983)) or the distributional output nulling subspace (WILLEMS, KITAPCI & SILVERMAN (1984)). In this context, the space $V^*(\ker C)$ is sometimes called the space of weakly unobservable states or the output nulling subspace of the system (A,B,C) (ANDERSON (1975)). In HAUTUS & SILVERMAN (1983), $R_b^*(\ker C)$ is called the space of strongly controllable states. In view of COR. 2.13 (iv) and COR. 3.9 (ii) it seems however more appropriate to call $R^*(\ker C)$ the space of strongly controllable states and $R_b^*(\ker C)$ the space of distributionally strongly controllable states.

In the remainder of this section we will outline the connection between the concept of invertibility of linear systems and some of the subspaces we have studied so far. Again, consider the system (A,B) together with the output mapping $C: X \to Y$, in the sequel referred to as the system (A,B,C). We will assume that $Y \simeq \mathbb{R}^p$ and that C is surjective. (Recall also that, as a standing assumption, B is assumed to be injective.)

DEFINITION 3.14. The system (A,B,C) will be called *right-invertible* if for every distribution $y \in D_+'^p$, there exists a distribution $u \in D_+'^m$ such that $y = y^+(0,u)$. The system (A,B,C) will be called *left-invertible* if $u \in D_+'^m$ and $u \neq 0$ imply $y^+(0,u) \neq 0$.

In HAUTUS & SILVERMAN (1983) it is assumed in the definition of invertibility that the outputs u appearing in the above are regular. In WILLEMS, KITAPÇI & SILVERMAN (1984) it is assumed that the outputs y appearing in the above definition belong to a class of distributions that can be written as the sum of an L_2-function and an impulsive distribution. Thus, our definition is more general and more natural.

In the following, let $G(t) := Ce^{At} B\, 1_{\mathbb{R}^+}(t)$ denote the impulse response matrix and let $G(s) := C(Is-A)^{-1} B$ denote the transfer matrix of the system (A,B,C). Note that G(s) is a matrix over the field $\mathbb{R}(s)$ of real rational functions. In this sense, we will say that G(s) is $\mathbb{R}(s)$-surjective if it represents a surjective linear mapping from $U(s)$ to $Y(s)$, where the latter are considered as linear spaces over the field $\mathbb{R}(s)$. In a similar way, G(s) will be called $\mathbb{R}(s)$-injective if it represents an injective linear mapping from $U(s)$ to $Y(s)$.

The following result relates right-invertibility to the supremal L_p-almost controlled invariant subspace of ker C:

THEOREM 3.15. *The following statements are equivalent:*

(i) (A,B,C) *is right-invertible,*

(ii) $V_b^*(\ker C) = X$,

(iii) G(s) *is* $\mathbb{R}(s)$*-surjective.*

PROOF: (i) ⇒ (ii) Let $x_0 \in X$. Define $y(t) := Ce^{At} x_0 1_{\mathbb{R}^+}(t)$. Then there exists $u \in D_+^{\prime m}$ such that $y = y^+(0,u)$. This however is equivalent to saying that the equation

$$Ce^{At} x_0 1_{\mathbb{R}^+}(t) = (G * u)(t) \tag{3.11}$$

has a solution $u \in D_+^{\prime m}$. Since the convolution equation (3.11) has its coefficients in the field of scalar Bohl distributions, this implies that, a fortiori, (3.11) is solvable with $u \in D_B^{\prime m}$, i.e. with u a Bohl distribution. This however yields $y^+(x_0,0) = y = y^+(0,u)$ or, equivalently, $y^+(x_0,-u) = 0$. Hence, $x^+(x_0,-u)$ lies in ker C and, by COR. 3.9, $x_0 \in V_b^*(\ker C)$.

(ii) ⇒ (iii) Suppose that G(s) is not $\mathbb{R}(s)$-surjective. Then there exists a rational vector $\eta(s) \neq 0$ such that $\eta(s)^T G(s) = 0$. Let $x_0 \in X$. By TH. 3.8, x_0 has a (ξ,ω)-representation with $C\xi(s) = 0$. Thus, $C(Is-A)^{-1} x_0 = G(s)\omega(s)$ and $\eta^T(s)C(Is-A)^{-1} x_0 = 0$. Since this holds for all $x_0 \in X$ this yields $\eta(s) = 0$ which is a contradiction.

(iii) ⇒ (i) If (iii) holds, then there is a rational matrix $G^T(s)$ such that $G(s)G^T(s) = I$. Let G^T denote the inverse Laplace transform of $G^T(s)$. G^T is then a matrix over the field D_B' of scalar Bohl distributions. Moreover, the convolution operator of $D_+^{\prime p}$ into $D_+^{\prime m}$ with kernel G^T is a right-inverse of the convolution operator with kernel G(t). Let $y \in D_+^{\prime p}$. Define $u := G^T * y$. Then clearly $y = G * u = y^+(0,u)$.

□

To conclude this section, we state the following result concerning left-invertibility of linear systems:

THEOREM 3.16. *The following statements are equivalent:*

(i) (A,B,C) *is left-invertible,*

(ii) $R^*(\ker C) = \{0\}$,

(iii) $G(s)$ *is* $\mathbb{R}(s)$*-injective.*

PROOF: For a proof of this theorem we refer to HAUTUS & SILVERMAN (1983, TH. 3.26). (See also MORSE & WONHAM (1971, TH. 5).)

□

REMARK 3.17. Several other invertibility properties may be formulated in terms of subspace equalities. It may for example be shown that the transfer matrix $G(s)$ has a right-inverse which is a polynomial matrix if and only if $R_b^*(\ker C) = X$. In the definition of left-invertibility given above, knowledge of y^+ with $x(0) = 0$ implies knowledge of u. We could also define the following version of left-invertibility: (A,B,C) is called strongly left-invertible if for all $u \in L_{1,loc}(\mathbb{R},U)$ and $x_0 \in X$ the following implication holds: $\{y(x_0,u) = 0\} \Rightarrow \{u = 0\}$. It may be shown that (A,B,C) is strongly left-invertible if and only if $V^*(\ker C) = \{0\}$ (see also WILLEMS (1983)).

3.3 L_p-ALMOST DISTURBANCE DECOUPLING

In SECTION 2.6 we have introduced the L_p/L_q-almost disturbance decoupling problem and formulated necessary and sufficient conditions for its solvability. We have noted that in this problem (and in the subsequent version of the same problem including the requirement of freedom of pole assignability, see SECTION 2.7), we have followed only one possible way to quantify the notion of 'approximate decoupling'. The aim of this section is to study a different kind of almost disturbance decoupling problem. Again, consider the linear system (2.18). With the feedback control law (2.19) and initial condition $x(0) = x_0$, let the closed loop system be given by (2.21). Recall that with $x_0 = 0$, (2.21) defines a convolution operator mapping D-valued measurable functions on $\mathbb{R}^+$ to Z-valued measurable functions on $\mathbb{R}^+$. Also recall (see DEF. 2.43) that the L_p/L_q-almost disturbance decoupling problem was said to be solvable if for $1 \le p \le q \le \infty$ *all* L_p-L_q induced norms simultaneously can be made arbitrarily small. A perhaps more natural way to quantify approximate decoupling is to *fix one* p and to require that the L_p-L_p induced norm of the operator mapping disturbances to to-be-controlled outputs can be made arbitrarily small (WILLEMS (1981)):

DEFINITION 3.18. Let $1 \leq p \leq \infty$. $(\text{ADDP})_p$, *the* L_p*-almost disturbance decoupling problem*, is said to be solvable if the following holds: $\forall \varepsilon > 0$ there exists a mapping $F: X \to U$ such that in the closed loop system with $x(0) = 0$, $\|z\|_p \leq \varepsilon \|d\|_p$ for all $d \in L_p(\mathbb{R}^+, D)$.

In effect, it is required that the closed loop system defines an operator from $L_p(\mathbb{R}^+, D)$ to $L_p(\mathbb{R}^+, Z)$ and that the norm of this operator can be made arbitrarily small by suitable choice of a state feedback control law.

It turns out that for $p = 1$ and $p = \infty$ the solvability of $(\text{ADDP})_p$ can be translated directly in terms of the closed loop impulse response matrix $W_F(t) := He^{A_F t} G$:

LEMMA 3.19. *Let* $p \in \{1, \infty\}$. *Then* $(\text{ADDP})_p$ *is solvable if and only if* $\forall \varepsilon > 0\ \exists F: X \to U$ *such that* $\|W_F\|_1 \leq \varepsilon$.

PROOF: The result of this lemma follows immediately from the fact that for $p = 1$ and $p = \infty$ the L_p-L_p induced norm of a convolution operator is equal to the L_1-norm of its kernel (DESOER & VIDYASAGAR (1975)).

□

Combining the above lemma with DEF. 3.1, we thus obtain the following necessary condition for the solvability of the L_p-almost disturbance decoupling problem for the case that $p = 1$ or $p = \infty$:

LEMMA 3.20. *Let* $p \in \{1, \infty\}$. *If* $(\text{ADDP})_p$ *is solvable then* $\text{im } G \subset V_b^*(\ker H)$.

PROOF: For $p = 1$ or $p = \infty$, if $(\text{ADDP})_p$ is solvable, then for all $\varepsilon > 0$ there is a mapping $F: X \to U$ such that $\|He^{A_F t} x_0\|_1 \leq \varepsilon$ for all $x_0 \in \text{im } G$. Hence, for all $x_0 \in \text{im } G$ we may find for all $\varepsilon > 0$ a trajectory x through x_0 such that $\|d(x, \ker H)\|_1 \leq \varepsilon$ (just take $x(t) := e^{A_F t} x_0$). It follows that $x_0 \in V_1^*(\ker H)$. The statement of the lemma then follows from TH. 3.6.

□

In the sequel we will show that also for $p = 2$ the subspace inclusion $\text{im } G \subset V_b^*(\ker H)$ is a necessary condition for the solvability of $(\text{ADDP})_p$. To prove this, the solvability of $(\text{ADDP})_p$ will be expressed in terms of the H^∞-norm of the closed loop transfer matrix from d to z. We will give a characterization of $V_b^*(K)$ involving (ξ, ω)-representations with the property that the H^∞-norm of the distance function $d(\xi(s),K)$ $(s \in \mathbb{C})$ can be made arbitrarily small.

Finally, it will be shown that the subspace inclusion $\operatorname{im} G \subset V_b^*(\ker H)$ is a sufficient condition for the solvability of $(\mathrm{ADDP})_p$ *for all* $1 \leq p \leq \infty$. Our proof of this will be entirely constructive and will provide a scheme by which in principle it is possible to compute the required state feedback mappings.

In the following, let H^∞ denote the *Hardy space* with respect to the open right half plane $\mathbb{C}_o^+ := \{s \in \mathbb{C} \mid \operatorname{Re} s > 0\}$, defined by

$$H^\infty := \{f: \mathbb{C}_o^+ \to \mathbb{C} \mid f \text{ is analytic in } \mathbb{C}_o^+ \text{ and } \sup_{s\in\mathbb{C}_o^+} |f(s)| < \infty\}$$

(see DUREN (1970)). H^∞ is a Banach space with norm

$$\|f\|_\infty := \sup_{s\in\mathbb{C}_o^+} |f(s)| \; .$$

It is well-known that for every function $f \in H^\infty$ the limit $f(i\omega) := \lim_{\sigma\to 0} f(\sigma+i\omega)$ exists for almost every $\omega \in \mathbb{R}$. Also, in this way f may be extended to a function which is essentially bounded on the imaginary axis. It turns out that the H^∞-norm of the original f may be calculated by taking the essential supremum on the imaginary axis of the above extension:

$$\|f\|_\infty = \operatorname*{ess\,sup}_{\omega\in\mathbb{R}} |f(i\omega)| \; .$$

If F is a mapping from X into U, let $\hat{W}_F(s) := H(Is-A_F)^{-1}G$ denote the closed loop transfer matrix from d to z. $\hat{W}_F(s)$ will be called asymptotically stable if all its poles lie in $\mathbb{C}^-$. If F is such that $\hat{W}_F(s)$ is asymptotically stable, then the closed loop system with $x(0) = 0$ defines a convolution operator from $L_2(\mathbb{R}^+,D)$ into $L_2(\mathbb{R}^+,Z)$ and the induced norm of this operator is equal to $\sup_{\omega\in\mathbb{R}} \|\hat{W}_F(i\omega)\|$ (DESOER & VIDYASAGAR (1975)). Here, for $s \in \mathbb{C}$, $\|\hat{W}_F(s)\|$ denotes the induced norm of the matrix $\hat{W}_F(s)$ considered as a mapping from (the complexification of) D into (the complexification of) Z. Thus, the L_2-L_2 induced norm of the closed loop operator is equal to the H^∞-norm of the function $s \mapsto \|\hat{W}_F(s)\|$. This induced norm will be denoted by $\|\hat{W}_F\|_\infty$. We may now state:

<u>LEMMA 3.21</u>. $(\mathrm{ADDP})_2$ *is solvable if and only if* $\forall \varepsilon > 0 \; \exists F: X \to U$ *such that* $\|\hat{W}_F\|_\infty \leq \varepsilon$.

□

We will now show that the subspace inclusion im G $\subset V_b^*(\ker H)$ is a necessary condition for solvability of $(ADDP)_2$. If $x_0 \in X$ and $x_0 = (Is-A)\xi(s) + B\omega(s)$ is a (ξ,ω)-representation of x_0, we will for every $s \in \mathbb{C}$ which is not a pole of ξ interpret $\xi(s)$ as a vector in the complexification of X. If K is a subspace of X, the distance of $\xi(s)$ to K is denoted by $d(\xi(s),K)$. If the complex function $s \mapsto d(\xi(s),K)$ is an element of H^∞, its H^∞-norm will be denoted by $\|d(\xi,K)\|_\infty$. For every subspace K of X, define a subspace $H(K)$ by

$$H(K) := \{x_0 \in X \mid \forall \varepsilon > 0 \ \exists\, (\xi,\omega)\text{-representation of } x_0 \text{ with}$$
$$\xi(s) \in X_+(s),\ \omega(s) \in U_+(s) \text{ and } \|d(\xi,K)\|_\infty \leq \varepsilon\} \ .$$

Obviously, if $(ADDP)_2$ is solvable, then im G $\subset H(\ker H)$. Indeed, solvability of $(ADDP)_2$ implies that for all $\varepsilon > 0$ there is a mapping F such that $\|H(Is-A_F)^{-1} x_0\|_\infty \leq \varepsilon$ for all $x_0 \in$ im G. For $x_0 \in$ im G and $\varepsilon > 0$, take $\xi(s) := (Is-A_F)^{-1} x_0$ and $\omega(s) = F(Is-A_F)^{-1} x_0$. This yields a (ξ,ω)-representation of x_0 and $\|d(\xi,\ker H)\|_\infty = \|H(Is-A_F)^{-1} x_0\|_\infty \leq \varepsilon$. Necessity of the subspace inclusion im G $\subset V_b^*(\ker H)$ for the solvability of $(ADDP)_2$ thus follows from the following:

LEMMA 3.22. *Let K be a subspace of X. Then $H(K) \subset V_b^*(K)$.*

PROOF: First assume that $K \cap B = \{0\}$. Let $X_1 := K$, $X_3 := B$ and let X_2 be such that $X = X_1 \oplus X_2 \oplus X_3$. After a preliminary feedback mapping F we have

$$A_F = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ 0 & 0 & 0 \end{pmatrix} , \quad B = \begin{pmatrix} 0 \\ 0 \\ B_3 \end{pmatrix} .$$

Let $x_0 = (x_{01}^T, x_{02}^T, x_{03}^T)^T \in H(K)$. There are sequences of strictly proper rational functions $\xi_n(s) = (\xi_{1n}^T(s), \xi_{2n}^T(s), \xi_{3n}^T(s))^T$ and $\omega_n(s)$ such that, for all n, $x_0 = (Is-A_F)\xi_n(s) + B\omega_n(s)$ and such that $\|\xi_{2n}\|_\infty \to 0$ and $\|\xi_{3n}\|_\infty \to 0$. Thus, in particular, $\xi_{2n}(s) \to 0$ and $\xi_{3n}(s) \to 0$ pointwise in $\mathbb{C}^+$. Since

$$x_{01} = (Is-A_{11})\xi_{1n}(s) - A_{12}\xi_{2n}(s) - A_{13}\xi_{3n}(s) ,$$

we obtain that $\xi_{1n}(s) \to (Is-A_{11})^{-1} x_{01} =: \xi_1(s) \quad (n \to \infty)$, for all $s \in \mathbb{C}^+$ with $s \notin \sigma(A_{11})$. Since also

$$x_{02} = (Is-A_{22})\xi_{2n}(s) - A_{21}\xi_{1n}(s) - A_{23}\xi_{3n}(s) \ ,$$

it follows that $x_{02} = -A_{21}\xi_1(s)$ for all $s \in \mathbb{C}^+$, $s \notin \sigma(A_{11})$. However, $\xi_1(s)$ is strictly proper and therefore we find, by letting $|s| \to \infty$, that $x_{02} = 0$. We may conclude that $x_0 \in X_1 \oplus X_3 = K \oplus B$. Define now a vector $\tilde{x}_0$ by $\tilde{x}_0 := (x_{01}^T, 0^T, 0^T)^T$. We contend that $\tilde{x}_0 \in V^*(K)$. To show this, let $\tilde{\xi}(s) := (\xi_1^T(s), 0^T, 0^T)^T$ and $\tilde{\xi}_n(s) := (\xi_{1n}^T(s), \xi_{2n}^T(s), 0^T)^T$. We have already proven that $\tilde{\xi}_n(s) \to \tilde{\xi}(s)$ for almost every $s \in \mathbb{C}^+$. It may be verified by inspection that

$$\tilde{x}_0 = (Is-A_F)\tilde{\xi}_n(s) + \begin{pmatrix} A_{13} \\ A_{23} \\ 0 \end{pmatrix} \xi_{3n}(s)$$

and thus we find $\tilde{x}_0 = (Is-A_F)\tilde{\xi}(s)$. Since however $\tilde{\xi}(s) \in K_+(s)$, it follows from COR. 2.12 (ii) that $\tilde{x}_0 \in V^*(K)$. We conclude that $x_0 \in V^*(K) \oplus B$.

Let us now consider the general case. Denote $S^\infty := S^\infty(K)$, with $S^\infty(K)$ defined as in SECTION 1.3. Let $(\bar{A},\bar{B})$ denote the factor system modulo S^∞ (see SECTION 1.4). Now, let $x_0 \in H(K)$. Let $\varepsilon > 0$ and let $x_0 = (Is-A)\xi(s) + B\omega(s)$ with $\|d(\xi,K)\|_\infty \leq \varepsilon$. Define $[\xi](s) := [\xi(s)]$, the equivalence class of $\xi(s)$ modulo S^∞. Also let $[x_0]$ denote the equivalence class of x_0 modulo S^∞. Taking inverse Laplace transforms of $\xi(s)$ and $\omega(s)$ and applying LEMMA 2.56, it may be shown that there is a strictly proper rational $\bar{\omega}(s)$, such that $[x_0] = (Is-\bar{A})[\xi](s) + \bar{B}\bar{\omega}(s)$. Since also $d([\xi](s),K/S^\infty) \leq d(\xi(s),K)$, we find that

$$\|d([\xi],K/S^\infty)\|_\infty \leq \|d(\xi,K)\|_\infty \leq \varepsilon$$

and hence that $[x_0] \in H(K/S^\infty)$, defined with respect to the factor system modulo S^∞. Since $\mathrm{im}\ \bar{B} \cap K/S^\infty = \{[0]\}$ (see the proof of TH. 1.22), we may conclude that $[x_0] \in \mathrm{im}\ \bar{B} + V^*(K/S^\infty)$. The conclusion then follows from the facts that $\mathrm{im}\ \bar{B} = (B + AS^\infty)/S^\infty$ and $V^*(K/S^\infty) = V^*(K)/S^\infty$.

□

<u>REMARK 3.23</u>. In the sequel, it will be shown that the subspace inclusion in LEMMA 3.22 is in fact an equality.

<u>REMARK 3.24</u>. We will restrict ourselves here to the case that p takes one of the values 1, 2 or ∞. It may however be proven that, in fact, the subspace inclusion $\mathrm{im}\ G \subset V_b^*(\ker H)$ is a necessary condition for the solvability of $(\mathrm{ADDP})_p$ *for all* $1 \leq p \leq \infty$.

Our next goal is to show that the subspace inclusion im $G \subset V_b^*(\ker H)$ also provides a sufficient condition for the solvability of $(\mathrm{ADDP})_p$ for $p = 1$, $p = 2$, and $p = \infty$. We will show that it is in fact sufficient for all $1 \leq p \leq \infty$. Let K be a subspace of X. From the definition of $V_b^*(K)$ we have that for each $1 \leq p < \infty$, for each $x_0 \in V_b^*(K)$ and for each $\varepsilon > 0$, there is a trajectory x starting in x_0 such that the L_p-norm of the distance function $d(x(t),K)$ on $\mathbb{R}^+$ is smaller than ε. The following result states that this may be achieved with trajectories generated by state feedback. In fact, for a given $1 \leq p_0 < \infty$ and $\varepsilon > 0$, the same feedback mapping may be used to make the L_p-norm of the distance function smaller than ε for all $x_0 \in V_b^*(K)$ with $\| x_o \| \leq 1$ and for all p in the interval $[1,p_0]$:

<u>THEOREM 3.25</u>. *Let K be a subspace of X. Fix $1 \leq p_0 < \infty$. Then* $\forall \varepsilon > 0$ $\exists F$: $X \to U$ *such that* $\| d(e^{A_F t} x_0, K)\|_p \leq \varepsilon$ *for all* $x_0 \in V_b^*(K)$ *with* $\|x_0\| \leq 1$ *and for all* $1 \leq p \leq p_0$.

<u>REMARK 3.26</u>. It is interesting to compare TH. 3.25 with TH. 2.47. The latter states that the L_p-norm of the distance function can be made arbitrarily small using the same feedback mapping for all $x_0 \in V_a^*(K)$ with $\|x_0\| \leq 1$ and for all $1 \leq p \leq \infty$.

Indeed, once we have proven TH. 3.25 we are done: Suppose that im $G \subset V_b^*(\ker H)$ and let $\varepsilon > 0$. From TH. 3.25 it follows that, in particular, there is a F: $X \to U$ such that $\|d(e^{A_F t} G, \ker H)\|_1 \leq \varepsilon$. This may be restated to obtain $\|He^{A_F t} G\|_1 \leq \varepsilon$. It is a well-known fact (DESOER & VIDYASAGAR (1975)) that for each $1 \leq p \leq \infty$ the L_p-L_p induced norm of a convolution operator is bounded from above by the L_1-norm of its kernel. Hence, from TH. 3.25 it follows that if im $G \subset V_b^*(\ker H)$, then for each $1 \leq p \leq \infty$, $(\mathrm{ADDP})_p$ is solvable.

To prove TH. 3.25 we need a couple of introductory results. In the following, we will be concerned with the k-dimensional singly generated almost controllability subspace $\mathcal{L}(u,F,k) = \mathrm{span}\ \{Bu, A_F Bu, \ldots, A_F^{k-1}Bu\}$. For $i \in \underline{k}$, let vectors $x_i(n)$ be defined by (2.6). These vectors are sometimes denoted by $x_i(n,u)$. Recall that we denoted $\mathcal{L}_n(u,F,k) := \mathrm{span}\ \{x_1(n), \ldots, x_k(n)\}$ and that $\mathcal{L}_n(u,F,k)$ converges to $\mathcal{L}(u,F,k)$ in the Grassmannian topology. Given $\mathcal{L}(u,F,k)$, let $\mathcal{L}' := \mathcal{L}(u,F,k-1)$ denote the subspace of $\mathcal{L}(u,F,k)$ spanned by the vectors $Bu, A_F Bu, \ldots, A_F^{k-2}Bu$ (if $k = 1$, define $\mathcal{L}' := \{0\}$). Then we have:

<u>THEOREM 3.27</u>. *Let* $F_n: \mathcal{L}_n(u,F,k) \to U$ *be defined by* $F_n x_i(n,u) = -n^i u$ $(i \in \underline{k})$. *Fix* $1 \le p_0 < \infty$. *Then for all* $\varepsilon > 0$ *there is* $K \in \mathbb{N}$ *such that for all* $i \in \underline{k}$ *and* $1 \le p \le p_0$

$$\|d(\mathcal{L}', e^{(A_F+BF_n)t} x_i(n,u)\|_p \le \varepsilon , \quad \forall n \ge K .$$

We note that the above theorem is analogous to TH. 2.35, where a comparable result was proven with $\mathcal{L}'$ replaced by the entire subspace $\mathcal{L}(u,F,k)$. However, the latter result was valid for all $1 \le p \le \infty$ simultaneously. To prove TH. 3.27, we need the following analogue of LEMMA 2.38:

<u>LEMMA 3.28</u>. *Let* $i \in \underline{k}$. *Then* $\lim\limits_{n\to\infty} n^{k-i}\, d(\mathcal{L}', x_i(n,u)) < \infty$.

<u>PROOF</u>: Recall the expansion (2.10) of $x_i(n)$. In this expansion all terms but the last two composite sums between brackets are contained in $\mathcal{L}'$. Again denote the last composite sum between brackets by $v(n)$ and denote

$$a := \sum_{\ell_1=1}^{i} \sum_{\ell_2=2}^{\ell_1} \cdots \sum_{\ell_{k-i}=1}^{\ell_{k-i-1}} A_F^{k-1} Bu .$$

We then obtain

$$d(\mathcal{L}', x_i(n)) = d\left(\mathcal{L}', \frac{(-1)^{k-i}}{n^{k-i}} a + \frac{(-1)^{k-i+1}}{n^{k-i+1}} v(n)\right)$$

and hence

$$d(\mathcal{L}', x_i(n)) \le \frac{1}{n^{k-i}} d(\mathcal{L}', a) + \frac{1}{n^{k-i+1}} d(\mathcal{L}', v(n)) .$$

Since $\lim\limits_{n\to\infty} v(n)$ exists, the result follows.

□

<u>PROOF OF THEOREM 3.27</u>: From the proof of TH. 2.35, recall that for all $i \in \underline{k}$

$$e^{(A_F+BF_n)t} x_i(n) = \left(\sum_{j=0}^{k-1} \frac{t^j N_n^j}{j!}\right) e^{-nt} x_i(n) ,$$

where N_n is defined by (2.8). By the triangular inequality it suffices to show that for $j = 0,\dots,k-1$,

$$\lim_{n\to\infty} \|d(\mathcal{L}',t^j e^{-nt} N_n^j x_i(n)\|_p = 0 ,$$

uniformly for $p \in [1,p_0]$. By LEMMA 2.36 and the triangular inequality, it is sufficient to show that

$$\lim_{n\to\infty} \|d(\mathcal{L}',t^j e^{-nt} n^{j+i-\ell} x_\ell(n)\|_p = 0 ,$$

uniformly for $p \in [1,p_0]$, for all $i,j,\ell \in \underline{k}$. Note that

$$\|d(\mathcal{L}',t^j e^{-nt} n^{j+i-\ell} x_\ell(n)\|_p = n^{j+i-\ell}\|t^j e^{-nt}\|_p \; d(\mathcal{L}',x_\ell(n)) .$$

Again,

$$\|t^j e^{-nt}\|_p = (\frac{1}{pn})^{j+\frac{1}{p}} \Gamma(pj+1)^{\frac{1}{p}}$$

for all finite p, where Γ denotes the gamma function. By Stirling's formula (HILLE (1959), p. 235) there exists a constant c such that $\|t^j e^{-nt}\|_p \leq$ $\leq cn^{-j-1/p}$. Hence

$$\|d(\mathcal{L}',t^j e^{-nt} n^{j+i-\ell} x_\ell(n)\|_p \leq cn^{i-\ell-\frac{1}{p}} d(\mathcal{L}',x_\ell(n)) .$$

By applying LEMMA 3.28 it is readily verified that the latter tends to 0 as $n \to \infty$, uniformly for $p \in [1,p_0]$.

□

The idea of the proof of TH. 3.25 is, similar as in TH. 2.47, to decompose $V_b^*(K)$ into the direct sum of $V^*(K)$ and a finite number of singly generated almost controllability subspaces $\mathcal{L}(u_i,F,r_i)$. These will be chosen in such a way that $\mathcal{L}(u_i,F,r_i-1) \subset K$. Next, the $\mathcal{L}(u_i,F,r_i)$ will be approximated by the sequences $\mathcal{L}_n(u_i,F,r_i)$. On each of these approximants we will define a feedback mapping by (2.7). These mappings will then be used to construct a sequence of mappings $\{F_n\}$ on X. We will now first show that indeed $V_b^*(K)$ allows a direct sum decomposition of the form above:

<u>LEMMA 3.29</u>. *Let K be a subspace of X. There exist* $r \in \mathbb{N}$ *and, for* $i \in \underline{r}$, *integers* $r_i \in \mathbb{N}$, *vectors* $u_i \in U$ *and a mapping* $F\colon X \to U$ *such that*

$$V_b^*(K) = V^*(K) \oplus \bigoplus_{i=1}^{r} \mathcal{L}(u_i,F,r_i) ,$$

with $\mathcal{L}(u_i,F,r_{i-1}) \subset K$, $(i \in \underline{r})$.

PROOF: As in SECTION 2.3, let $\tilde{B} \subset B$ be such that $\tilde{B} \oplus (B \cap V^*(K)) = B$. Let W be a mapping such that $\tilde{B} = \text{im } BW$ and let $\tilde{R}_a^*(K)$ be the supremal almost controllability subspace in K with respect to the system (A,BW). In the following, denote $\tilde{R}_a^* := \tilde{R}_a^*(K)$, $V^* := V^*(K)$, etc. From LEMMA 2.25 we have:

$$(3.12) \qquad R_a^* = R^* \oplus \tilde{R}_a^*$$

and $\tilde{R}_a^* \cap V^* = \{0\}$. By COR. 1.23 there is a chain $\{B_i\}_{i=1}^k$ and a mapping F such that

$$\tilde{R}_a^* = B_1 \oplus A_F B_2 \oplus \dots \oplus A_F^{k-1} B_k \ .$$

Define now a subspace $\tilde{R}_b^* := \tilde{B} + A_F \tilde{R}_a^*$. We contend that $V_b^* = V^* \oplus \tilde{R}_b^*$. Assume that $x \in V^* \cap \tilde{R}_b^*$. Then there is $r \in \tilde{R}_a^*$ and $b \in \tilde{B}$ such that $x = A_F r + b$. Thus, $r \in A_F^{-1}(V^* + B) \cap K = A^{-1}(V^* + B) \cap K$. By PROP. 1.30 the latter is equal to V^*. From (3.12) we may then conclude that $r = 0$ and hence that $x = b \in \tilde{B}$. Since $\tilde{B} \cap V^* = \{0\}$, we obtain that $x = 0$. Our assertion is now a consequence of the following equalities:

$$\begin{aligned} V_b^* &= V^* + B + AR_a^* \\ &= V^* + B + A(R^* + \tilde{R}_a^*) \\ &= V^* + B + R^* + A\tilde{R}_a^* \\ &= V^* + \tilde{B} + A\tilde{R}_a^* \ . \end{aligned}$$

Next, we will establish a direct sum decomposition of $\tilde{R}_b^*$ into singly generated almost controllability subspaces. From the definition of $\tilde{R}_b^*$ we have $\dim \tilde{R}_b^* \leq$ $\leq \dim \tilde{B} + \dim \tilde{R}_a^*$, with equality if and only if $\tilde{B} \cap A_F R_a^* = \{0\}$ and $\ker A_F \cap \tilde{R}_a^* = \{0\}$. It is claimed that, indeed, equality holds. Firstly, assume that there exists $r \in \tilde{R}_a^*$ and $b \in \tilde{B}$ with $A_F r = b$. Define $R := \text{span}\,\{r\}$. Then R is controlled invariant. Since $R \subset K$ we find that $r \in R \subset V^*$. Formula (3.12) then yields $r = 0$. Secondly, assume there is $r \in \tilde{R}_a^*$ such that $A_F r = 0$. Then again span $\{r\}$ is controlled invariant, whence $r = 0$. We conclude that

$$(3.13) \qquad \dim \tilde{R}_b^* = \dim \tilde{B} + \dim \tilde{R}_a^* \ .$$

Now, obviously, $\tilde{R}_b^* = \tilde{B} + A_F B_1 + \dots + A_F^k B_k$. Since $\ker A_F \cap R_a^* = \{0\}$, we must have $\dim B_i = \dim A_F^{i-1} B_i = \dim A_F^i B_i$ for all $i \in \underline{k}$. Moreover, we claim that in fact

(3.14) $\qquad \tilde{R}_b^* = \tilde{B} \oplus A_F B_1 \oplus \ldots \oplus A_F^k B_k$.

Assume that (3.14) does not hold. Then we must have

$$\dim \tilde{R}_b^* < \dim \tilde{B} + \sum_{i=1}^{k} A_F^i B_i = \dim \tilde{B} + \sum_{i=1}^{k} A_F^{i-1} B_i = \dim \tilde{B} + \dim \tilde{R}_a^* .$$

This contradicts (3.13). To conclude the proof, use (3.14) to obtain a basis for $\tilde{R}_b^*$ in a similar way as in REMARK 2.28. This basis may be rearranged into singly generated almost controllability subspaces in such a way that, for some $r \in \mathbb{N}$, $r_i \in \mathbb{N}$ and vectors $u_i \in U$ ($i \in \underline{r}$),

$$\tilde{R}_b^* = \bigoplus_{i=1}^{r} \mathcal{L}(u_i, F, r_i)$$

with

$$\bigoplus_{i=1}^{r} \mathcal{L}(u_i, F, r_i - 1) = \tilde{R}_a^* \subset K .$$

(Here we define $\mathcal{L}(u_i, F, 0) := \{0\}$.) This completes the proof of the lemma. □

As a final ingredient in the proof of TH. 3.25, we need the following analogue of LEMMA 2.48. Let K be a subspace of X and denote $V^* := V^*(K)$, etc.

<u>LEMMA 3.30</u>. *Consider the system* (A,B). *Let* Λ *be a symmetric set of* $\dim [\langle A|B\rangle + V^*] - \dim V_b^*$ *complex numbers. Then there exists a subspace* W *and, for each mapping* $F_0 \in \underline{F}(V^*)$, *a mapping* $F_1 : X \to U$ *such that:*

(3.15) $\qquad F_1 | V^* = F_0 | V^*$

(3.16) $\qquad V_b^* \oplus W = V^* + \langle A|B\rangle$

(3.17) $\qquad (A + BF_1)(V^* \oplus W) \subset V^* \oplus W$

(3.18) $\qquad \sigma(A + BF_1) | (V^* \oplus W)/V^*) = \Lambda$.

<u>PROOF</u>: The proof of this result is completely analogous to the proof of LEMMA 2.48. It uses the fact that $V_b^* = V^* \oplus R_a$ for some subspace $R_a \in \underline{R}_a$. This follows from the foregoing lemma.

We are now in a position to prove our main theorem, TH. 3.25. The proof is analogous to the proof of TH. 2.47 and therefore we will only sketch the main steps. We will present these in such a way as to provide a conceptual algorithm for the actual computation of the required feedback control laws:

PROOF OF THEOREM 3.25: In this proof we will construct a sequence of mappings $\{F_n\}$ from X to U. Let $p_0 \in [1,\infty)$.

1. decomposition. Decompose $V_b^* = V^* \oplus \tilde{R}_b^*$, with

$$\tilde{R}_b^* = \bigoplus_{i=1}^{r} \mathcal{L}(u_i,F,r_i) ,$$

in such a way that $\mathcal{L}(u_i,F,r_i-1) \subset K$ for $i \in \underline{r}$.

2. approximation. Approximate $\tilde{R}_b^*$ by controlled invariant subspaces: define

$$x_1(n,u_i) := (I + \tfrac{1}{n} A_F)^{-1} Bu_i \quad \text{and} \quad x_j(n,u_i) := (I + \tfrac{1}{n} A_F)^{-1} A_F x_{j-1}(n,u_i) ,$$

for $j \in \underline{r_i}$. Define $\mathcal{L}_n(u_i,F,r_i) := \text{span}\,\{x_1(n,u_i),\dots,x_{r_i}(n,u_i)\}$. Let

$$V(n) := \bigoplus_{i=1}^{r} \mathcal{L}_n(u_i,F,r_i) .$$

Then $V(n) \to \tilde{R}_b^*$ as $n \to \infty$.

3. feedback mappings on $V(n)$. Define F_n on $V(n)$ by:

$$F_n x_j(n,u_i) := (-n^j I + F)u_i , \qquad j \in r_i,\ i \in \underline{r} .$$

4. feedback mappings outside $V(n)$. Let Λ be as in LEMMA 3.30 and assume that $\Lambda \subset \mathbb{C}^-$. There is a mapping $F_1 \colon X \to U$ and a subspace W such that (3.15) to (3.18) are valid. (On V^*, F_1 may be chosen arbitrarily from $\underline{F}(V^*)$.) It may be verified that for n sufficiently large $V^* \oplus \tilde{R}_b^* \oplus W = V^* \oplus V(n) \oplus W$. Define now

$$F_n \mid V^* \oplus W := F_1 \mid V^* \oplus W$$

and extend F_n arbitrarily to a mapping on X. As in the proof of TH. 2.47 it may be verified that for all $x_0 \in V_b^*$

$$\lim_{n\to\infty} \| d(e^{(A+BF_n)t} x_0, K) \|_p = 0 ,$$

uniformly for $p \in [1,p_0]$. The proof makes use of TH. 3.27 in an essential way: the fact that $\mathcal{L}(u_i,F,r_i-1) \subset K$ assures that $\| d(e^{(A+BF_n)t} x_j(n,u_i),K) \|_p \to 0$ $(n \to \infty)$ uniformly for $p \in [1,p_0]$.

□

REMARK 3.31. The mappings F_n constructed above make the following subspaces $(A+BF_n)$-invariant: $V^* + \langle A|B\rangle$, $V^* \oplus V(n)$, V^* and R^*. Note that $V^* \oplus V(n) \to V_b^*$. The situation with the closed loop spectrum is described in the following lattice diagram:

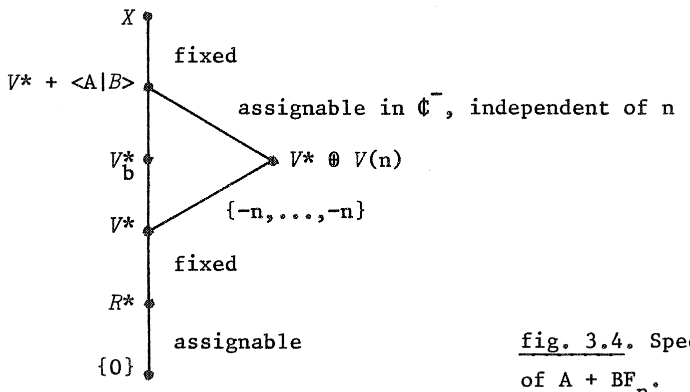


fig. 3.4. Spectrum of $A + BF_n$.

Again observe that $\sigma(A + BF_n \mid (V^* \oplus V(n))/V^*) = \{-n,\dots,-n\}$. All these eigenvalues tend to 'minus infinity' as $n \to \infty$.

As noted before, once we have established TH. 3.25, the sufficiency of the subspace inclusion $\mathrm{im}\, G \subset V_b^*(\ker H)$ for the solvability of $(\mathrm{ADDP})_p$ is immediate. Together with the necessity for the cases $p = 1$, $p = 2$ or $p = \infty$ as established before, this yields:

COROLLARY 3.32. *Let* $p \in \{1,2,\infty\}$. *Then* $(\mathrm{ADDP})_p$ *is solvable if and only if* $\mathrm{im}\, G \subset V_b^*(\ker H)$.

□

Also, we may now establish the following H^∞ frequency domain characterization of $V_b^*(K)$:

COROLLARY 3.33. *Let K be a subspace of X. Then*

$$V_b^*(K) = \{x_0 \in X \mid \forall\, \varepsilon > 0 \;\; \exists\; (\xi,\omega)\text{-representation of } x_0 \text{ with}$$
$$\xi(s) \in X_+(s),\ \omega(s) \in U_+(s) \text{ and } \|d(\xi,K)\|_\infty \leq \varepsilon\}\ .$$

PROOF: One inclusion has been proven in LEMMA 3.22. For the converse, let $x_0 \in V_b^*(K)$ and $\varepsilon > 0$. By TH. 3.25, there is a mapping F such that $\|d(e^{A_F t} x_0,K)\|_1 \leq \varepsilon$. Define $\xi(s) := (Is-A_F)^{-1}x_0$ and $\omega(s) = -F\xi(s)$. This yields a strictly proper (ξ,ω)-representation. The result follows from the fact that, for all $\omega \in \mathbb{R}$,

$$d((Ii\omega-A_F)^{-1}x_0,K) = d\left(\int_0^\infty e^{-i\omega t}e^{A_F t}x_0\,dt,K\right) \leq \int_0^\infty d(e^{A_F t}x_0,K)dt \leq \varepsilon\ . \tag{3.19}$$
□

3.4 SPECTRAL ASSIGNABILITY IN L_p-ALMOST CONTROLLABILITY SUBSPACES

In this section we will extend the results of SECTION 2.7 to supremal L_p-almost controllability subspaces. Recall that for a given subspace K of X, the supremal L_p-almost controllability subspace $R_b^*(K)$ of K consists exactly of those points in X with the property that for each $1 \leq p < \infty$ the following holds: starting in a point of this subspace one may travel to the origin in a given finite time along trajectories of the system with the property that the L_p-norm of their pointwise distance to K for $t \in \mathbb{R}^+$ is arbitrarily small. In the present section we will show that this constrained controllability property is equivalent to a constrained pole assignability property: we will show that $R_b^*(K)$ is exactly that subspace of X with the property that starting in it, one may travel along Bohl trajectories with the property that the L_p-norm of their distance to K is arbitrarily small and with the property that their characteristic values are located in an arbitrary subset of $\mathbb{C}$. Moreover, we will show that in the above statement 'Bohl trajectories' may be replaced by 'state feedback generated trajectories'.

The results obtained will be applied to obtain necessary and sufficient conditions for the solvability of the L_p-almost disturbance decoupling problem with pole placement.

Again, like in SECTION 2.7, we will assume that all stability sets $\mathbb{C}_g$ appearing in this section satisfy (2.27) and (2.28), i.e. all stability sets $\mathbb{C}_g$ are symmetric with respect to the real axis and contain a negative semi infinite interval in $\mathbb{R}$. If $\xi(s)$ is a strictly proper rational vector, then $\sigma(\xi)$ will denote its set of poles. Let deg ξ denote its McMillan degree (i.e. the dimension of every minimal state space realization of $\xi(s)$). Let K be a subspace of X and let $1 \leq p \leq \infty$. We will consider the following subspaces of X:

$$\begin{aligned} I_p(K) := \{x_0 \in X \mid\ & \exists\, r \in \mathbb{N} \text{ such that } \forall \varepsilon > 0 \text{ and } \forall \mathbb{C}_g \\ & \exists\, x \in \Sigma_g^{B(r)}(A,B) \text{ with } x(0) = x_0 \text{ and} \\ & \|d(x,K)\|_p \leq \varepsilon\} , \end{aligned}$$

$$\begin{aligned} J(K) := \{x_0 \in X \mid\ & \exists\, r \in \mathbb{N} \text{ such that } \forall \varepsilon > 0 \text{ and } \forall \mathbb{C}_g \ \exists\ (\xi,\omega)\text{-repre-} \\ & \text{sentation of } x_0 \text{ with } \xi(s) \in X_+(s),\ \omega(s) \in U_+(s), \\ & \deg \xi \leq r,\ \sigma(\xi) \subset \mathbb{C}_g \text{ and } \|d(\xi,K)\|_\infty \leq \varepsilon\} . \end{aligned}$$

In the latter definition, $\|d(\xi,K)\|_\infty$ denotes the H^∞-norm of $d(\xi(s),K)$. Recall (see TH. 2.60) that $I_\infty(K) = R_a^*(K)$. In the sequel we will show that for all $p \in [1,\infty)$, $I_p(K) = J(K) = R_b^*(K)$. First, we have the following:

<u>LEMMA 3.34</u>. *Let K be a subspace of X such that $K \cap B = \{0\}$. Then for all $p \in [1,\infty)$, $I_p(K) \subset B$ and $J(K) \subset B$.*

<u>PROOF</u>: As in the proof of LEMMA 3.5, decompose $X = X_1 \oplus X_2 \oplus X_3$ with $X_1 = K$ and $X_3 = B$. Using the same arguments as in the latter proof, if $x_0 = (x_{01}^T, x_{02}^T, x_{03}^T)^T \in I_p(K)$, it may be shown that $x_{02} = 0$ and that $(x_{01}^T, 0^T, 0^T)^T \in V^*(K)$. Finally, it may be shown along the lines of the proof of LEMMA 2.55 that, in fact, $(x_{01}^T, 0^T, 0^T)^T = 0$ and hence that $x_0 \in B$. (The proof makes use of the continuity property of the spectra of Bohl functions, see LEMMA 2.53.)

The second inclusion follows in an analogous way by combining the proofs of LEMMA 3.22 and LEMMA 2.55.

□

Using the foregoing lemmas, the following result may now be obtained for the general case that K and B are not necessarily independent:

LEMMA 3.35. *Let K be a subspace of X. Then for all* $p \in [1,\infty)$, $I_p(K) \subset R_b^*(K)$ *and* $J(K) \subset R_b^*(K)$.

PROOF: The first inclusion may be proven in a similar way as the first part of the proof of TH. 3.4. The second inclusion may be proven along de lines of the second part of the proof of LEMMA 3.22 (use LEMMA 2.56).

□

Now, the fact that in the previous lemma the converse inclusions also hold, follows from the next theorem. It will be shown that, for all finite $p \geq 1$, starting in $x_0 \in R_b^*(K)$ one may travel along trajectories that are generated by state feedback in such a way that the L_p-norm of the distance for $t \in \mathbb{R}^+$ from these trajectories to K is arbitrarily small and such that the closed loop system mapping restricted to the reachable subspace has its spectrum in any arbitrary subset $\mathbb{C}_g \subset \mathbb{C}$ (provided of course that $\mathbb{C}_g$ satisfies (2.27) and (2.28)):

THEOREM 3.36. *Let K be a subspace of X and let* $p_0 \in [1,\infty)$. *Then for all* $\varepsilon > 0$ *and for all* $\mathbb{C}_g \subset \mathbb{C}$ *there is a mapping* $F: X \to U$ *such that*

$$\|d(e^{A_F t}x_0,K)\|_p \leq \varepsilon \quad \textit{for all } x_0 \in R_b^*(K) \textit{ with } \|x_0\| \leq 1 \quad \textit{and } p \in [1,p_0]$$

and

$$\sigma(A_F|\langle A|B\rangle) \subset \mathbb{C}_g .$$

PROOF: This may be proven analogously to TH. 2.58 and TH. 3.25. In the proof, one uses the existence of a controlled invariant complement of $R_b^*(K)$ in the reachable space $\langle A|B\rangle$ (Apply TH. 2.39 with $R_a = R_b^*(K)$.) The proof also uses the fact that

$$R_b^*(K) = R^*(K) \oplus \bigoplus_{i=1}^{r} \mathcal{L}(u_i,F,r_i)$$

for given singly generated controllability subspaces $\mathcal{L}(u_i,F,r_i)$ with the property that $\mathcal{L}(u_i,F,r_i-1) \subset K$. The proof of the latter assertion is completely similar to the proof of LEMMA 3.29. Ultimately, one finds a sequence of mappings F_n from X to U such that the subspaces $\langle A|B\rangle$, $R^* \oplus V(n)$ and R^* are $(A+BF_n)$-invariant. Here, $V(n)$ is the canonical approximation of $\bigoplus_{i=1}^{r} \mathcal{L}(u_i,F,r_i)$. The situation with the closed loop spectrum is as follows:

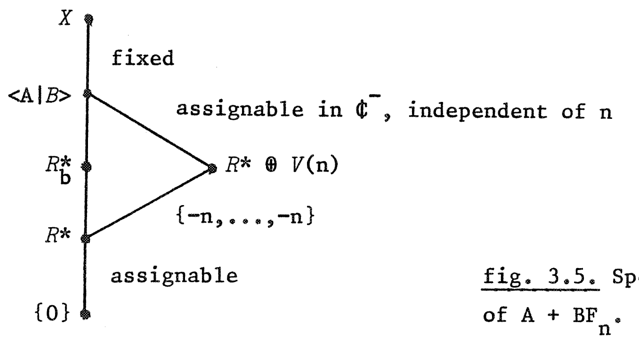


fig. 3.5. Spectrum of $A + BF_n$.

COROLLARY 3.37. *Let K be a subspace of X. Then for all $p \in [1,\infty)$ we have $R_b^*(K) = J(K) = I_p(K)$.*

PROOF: This follows immediately from LEMMA 3.35, LEMMA 3.36 and the estimation in the proof of COR. 3.33.

□

Next, we will apply the results obtained in this section to establish conditions for the solvability of the L_p-almost disturbance decoupling problem with the constraint of spectrum assignability. As usual, consider the system (2.18).

DEFINITION 3.38. Let $1 \leq p \leq \infty$. $(\text{ADDPPP})_p$, *the L_p-almost disturbance decoupling problem with pole placement*, is said to be solvable if $\forall\, \varepsilon > 0$ and $\forall\, \mathbb{C}_g$, there is a mapping $F\colon X \to U$ such that in the closed loop system with $x(0) = 0$, $\|z\|_p \leq \varepsilon \|d\|_p$ for all $d \in L_p(\mathbb{R}^+, D)$ and $\sigma(A+BF) \subset \mathbb{C}_g$.

The following is readily verified:

LEMMA 3.39. *Let $p \in \{1,\infty\}$. Then $(\text{ADDPPP})_p$ is solvable if and only if $\forall\, \varepsilon > 0$ and $\forall\, \mathbb{C}_g$ $\exists\, F\colon X \to U$ such that $\|W_F\|_1 \leq \varepsilon$ and $\sigma(A+BF) \subset \mathbb{C}_g$. Moreover, $(\text{ADDPPP})_2$ is solvable if and only if $\forall\, \varepsilon > 0$ and $\forall\, \mathbb{C}_g$ $\exists\, F\colon X \to U$ such that $\|\hat{W}_F\|_\infty \leq \varepsilon$ and $\sigma(A+BF) \subset \mathbb{C}_g$.*

□

This yields the following necessary and sufficient conditions for solvability in case that $p \in \{1,2,\infty\}$:

THEOREM 3.40. *Let* $p \in \{1,2,\infty\}$. *Then* $(\mathrm{ADDPPP})_p$ *is solvable if and only if* $\operatorname{im} G \subset R_b^*(\ker H)$ *and* (A,B) *is controllable.*

PROOF: For $p \in \{1,\infty\}$, if the problem is solvable, then we immediately obtain $\operatorname{im} G \subset I_1(\ker H) = R_b^*(\ker H)$. If the problem with $p = 2$ is solvable, then $\operatorname{im} G \subset J(\ker H) = R_b^*(\ker H)$. Of course, (A,B) controllable is a necessary condition. The converse implication is immediate from TH. 3.36.

□

REMARK 3.41. In WILLEMS (1981), a different formulation of an L_p-almost disturbance decoupling problem with spectral assignability constraint is considered. This problem is called the L_p-almost disturbance decoupling problem with *strong stabilization* or $(\mathrm{ADDPSS})_p$. It is said to be solvable if $\forall\ \varepsilon > 0$ and $\forall\ r \in \mathbb{R}$, $\exists\ F: X \to U$ such that in the closed loop system with $x(0) = 0$, $\|z\|_p \leq \varepsilon\|d\|_p$ for all $d \in L_p(\mathbb{R}^+, D)$ and $\sigma(A_F) \subset \{s \in \mathbb{C} \mid \operatorname{Re} s \leq r\}$. Thus, in effect it is required that simultaneously the induced norm of the closed loop operator can be made arbitrarily small and the closed loop spectrum can be located to the left of an arbitrary vertical in the complex plane. It may be shown that also this problem is solvable if and only if $\operatorname{im} G \subset R_b^*$ and (A,B) is controllable. Thus, the above problem leads to the same solvability conditions as $(\mathrm{ADDPPP})_p$.

3.5 STABILIZATION BY DYNAMIC HIGH GAIN OUTPUT FEEDBACK

In the present section we will consider an application of almost controlled invariant subspaces of a completely different type. We will be looking at the problem of stabilization by output feedback of the system

$$(3.20)\qquad \begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t)\ , \\ y(t) &= Cx(t)\ . \end{aligned}$$

In these equations, x, u, A and B are as usual (see SECTION 1.1). The vector $y(t)$ will be interpreted as an observed output and will be assumed to take its values in the p-dimensional linear space Y. It will also be assumed that the mapping C (the *output mapping*) is surjective. A *dynamic compensator* is defined as a finite dimensional linear time invariant system, which takes the variable $y(t)$ as its input and has $u(t)$ as its output:

$$(3.21)\qquad \begin{aligned} \dot{w}(t) &= Nw(t) + My(t) \ , \\ u(t) &= Lw(t) + Ky(t) \ . \end{aligned}$$

It is assumed that the state variable w(t) takes its values in the linear space W. The dimension of W will be called the *dynamic order* of the compensator. Combining (3.20) and (3.21) yields the *closed loop system*, which is described by the flow

$$(3.22)\qquad \begin{pmatrix} \dot{x}(t) \\ \dot{w}(t) \end{pmatrix} = \begin{pmatrix} A+BKC & BL \\ MC & N \end{pmatrix} \begin{pmatrix} x(t) \\ w(t) \end{pmatrix} \ .$$

The spectrum of the system mapping of (3.22) will be called the *closed loop spectrum*. Given a symmetric subset $\mathbb{C}_g$ of $\mathbb{C}$, we will say that (3.21) is a *stabilizing compensator* for the system (3.20) if the closed loop spectrum is contained in $\mathbb{C}_g$. A well known basic result is that if (A,B) is stabilizable and (C,A) is detectable, then there always exists a stabilizing compensator of dynamic order n (the dimension of the state space X) (see e.g. KWAKERNAAK & SIVAN (1972)). A much more difficult and as yet unsolved problem is to find for (3.20) a stabilizing compensator of *minimal* dynamic order. Trying to solve this problem motivates the search for stabilizing compensators of low dynamic order. Among the attempts that have been made in this direction we mention the one in LUENBERGER (1964), giving a dynamic order $n - \max\{p,m\}$ and the one in BRASCH & PEARSON (1970) which yields a dynamic order $\min\{\mu_1 - 1, \lambda_1 - 1\}$, where μ_1 is the controllability index of (A,B) and λ_1 the observability index of (C,A) (see WONHAM (1979)). Alternative results were established for example in KIMURA (1975), WANG & DAVISON (1975), KIMURA (1977) and HERMANN & MARTIN (1977).

In this section we will show that if a system is detectable, right-invertible and minimum phase, then it may be stabilized by means of a compensator of dynamic order equal to $\dim R_b^*(\ker C) - p$, the dimension of the supremal L_p-almost controllability subspace of ker C minus the number of outputs. Dualizing this result, it will turn out that if a system is stabilizable, left-invertible and minimum phase, then a stabilizing compensator of dynamic order $n - m - \dim V^*(\ker C)$ exists for it. In particular this implies that if a system is invertible and minimum phase, then it may always be stabilized by a compensator of dynamic order equal to the system's excess of poles over zeros minus its number of inputs. The material of this section is based on SCHUMACHER (1984) and TRENTELMAN (1985).

In the sequel, we will distinguish between the definition of *transmission zero* and that of *invariant zero*. As noted in REMARK 2.19, if the system (A,B,C) is minimal, then the set of its transmission zeros (defined in terms of the numerator polynomials in the Smith-McMillan form of $C(Is-A)^{-1}B$, see ROSENBROCK (1970)), coincides with the fixed spectrum $\sigma(A+BF|V^*(\ker C)/R^*(\ker C))$ (see ANDERSON (1976), HOSOE (1975), WONHAM (1979, p. 113)). In general, however, the set of transmission zeros is only contained in this fixed spectrum (ALING & SCHUMACHER (1984)). The fixed spectrum $\sigma(A+BF|V^*(\ker C)/R^*(\ker C))$ will be called the set of *invariant zeros* of (A,B,C). The notion of 'minimum-phase' will be defined in terms of these invariant zeros:

DEFINITION 3.42. Given a symmetric subset $\mathbb{C}_g$ of $\mathbb{C}$, the system (A,B,C) will be called *minimum phase* if its invariant zeros are contained in $\mathbb{C}_g$.

Next, we define the concept of 'minimum phase input subspace':

DEFINITION 3.43. Let $\mathbb{C}_g$ be a symmetric subset of $\mathbb{C}$ and consider the system (A,B,C). A subspace T will be called a *minimum phase input subspace* if there exists a mapping $T: Y \to X$ such that $T = \text{im } T$, the mapping CT is nonsingular and the system (A,T,C) is minimum phase.

Thus, in particular for a subspace im T to be a minimum phase input subspace, it is required that the system with system mapping A, input mapping T and output mapping C has its invariant zeros in $\mathbb{C}_g$. Note that a minimum phase input subspace always has dimension p. The above concept was originally introduced in SCHUMACHER (1984), where it was defined in terms of the transfer matrix $C(Is-A)^{-1}T$. The fact that our definition coincides with the latter one follows from the following result, which also appears in SCHUMACHER (1984, LEMMA 4.3):

LEMMA 3.44. *T is a minimum phase input subspace if and only if* $\ker C \oplus T = X$ *and the spectrum of the mapping* $PA|\ker C$ *is contained in* $\mathbb{C}_g$*. Here,* P *denotes the projection onto* ker C *along* T*.*

PROOF: In this proof, let V_T^* and R_T^* denote the supremal controlled invariant subspace and controllability subspace with respect to the system (A,T) contained in ker C. Then the set of invariant zeros of the system (A,T,C) is given by the fixed spectrum $\sigma_T^* := \sigma(A+TL|V_T^*/R_T^*)$ (independent of L as long as $(A+TL)V_T^* \subset V_T^*$). Assume now that T is a minimum phase input subspace. Let T be such that $T = \text{im } T$, CT is nonsingular and $\sigma_T^* \subset \mathbb{C}_g$. Obviously, $\ker C \oplus T = X$. From this it is immediate that ker C is (A,T)-invariant, whence $\ker C = V_T^*$. Since therefore $V_T^* \cap T = \{0\}$, it follows that $R_T^* = \{0\}$ (WONHAM (1979, TH. 5.5)). Define a mapping L by $L := -(CT)^{-1}CA$. Then it may be seen that ker C is (A+TL)-invariant. Moreover, since the projection onto ker C along T equals $P = I - T(CT)^{-1}C$, we have $PA = A + TL$. It follows that $\sigma(PA|\ker C) =$ $= \sigma(A+TL|\ker C) = \sigma_T^* \subset \mathbb{C}_g$.

Conversely, if $\ker C \oplus T = X$ then for every mapping $T: Y \to X$ such that $\text{im } T = T$, CT is nonsingular. Again, $V_T^*/R_T^* = \ker C$ and with $L := -(CT)^{-1}CA$ we find $\sigma_T^* = \sigma(A+TL|\ker C) = \sigma(PA|\ker C) \subset \mathbb{C}_g$.

□

The relevance of minimum phase input subspaces in the context of design of stabilizing compensators was established in SCHUMACHER (1981) and manifests itself in the following:

THEOREM 3.45. *Suppose that for the system* (A,B,C) *we have a minimum phase input subspace which is contained in a stabilizability subspace* V. *Then there exists a stabilizing compensator of dynamic order* $\dim V - p$.

PROOF: A proof of this result can be found in SCHUMACHER (1981, LEMMA 2.12).

□

The main instrument in the development of this section will be the observation in SCHUMACHER (1984) that the statement of the above theorem remains valid if 'stabilizability subspace' is replaced by 'almost stabilizability subspace'. A proof of this fact uses the following lemma, which states that if $\mathbb{C}_g$ is open then the set of minimum phase input subspaces is open in $\underline{G}(p,X)$:

LEMMA 3.46. *Suppose that* $\mathbb{C}_g$ *is open. Let* T *be a minimum phase input subspace and assume that* $T_n \to T$ $(n \to \infty)$. *Then there exists* $K \in \mathbb{N}$ *such that* T_n *is a minimum phase input subspace for all* $n \geq K$.

PROOF: This was proven in SCHUMACHER (1984, LEMMA 4.5).

□

Now recall the definition of almost stabilizability subspace, DEF. 2.66. It was shown that, provided that the stability set $\mathbb{C}_g$ satisfies (2.27) and

(2.28), for every almost stabilizability subspace there exists a sequence of stabilizability subspaces converging to it. This fact leads to the following result:

THEOREM 3.47. *Assume that* $\mathbb{C}_g$ *is open and assume that* (2.27) *and* (2.28) *hold. Suppose that for the system* (A,B,C) *we have a minimum phase input subspace contained in an almost stabilizability subspace* V_a. *Then there exists a stabilizing compensator of dynamic order* $\dim V_a - p$.

PROOF: Let $T \subset V_a$ with $V_a \in \underline{V}_{a,g}(A,B)$ and T a minimum phase input subspace. Let $V_n \in \underline{V}_g(A,B)$ be such that $V_n \to V_a$ $(n \to \infty)$. For every n, choose a subspace $T_n \subset V_n$ such that $T_n \to T$ $(n \to \infty)$. By LEMMA 3.46, for n sufficiently large T_n is a minimum phase input subspace. Apply now TH. 3.45 to the pair $T_n \subset V_n$ for any of these sufficiently large n.

□

Note that the procedure in the above proof in fact gives us a whole sequence of stabilizing compensators. Obviously, once we have the above result, the problem is to find a suitable pair (T,V_a) with T a minimum phase input subspace, V_a an almost stabilizability subspace and $T \subset V_a$. In the sequel we will establish the existence of such pairs for certain classes of systems. Before we continue, we have to introduce the notions of conditionally invariant subspace and detectability subspace. Consider the observed linear flow $\dot{x}(t) = Ax(t)$, $y = Cx(t)$. Conditionally invariant subspaces and detectability subspaces may be introduced in terms of observers (WILLEMS & COMMAULT (1981), WILLEMS (1982a)). Here we prefer to introduce them in terms of their invariance properties under output injection mappings (SCHUMACHER (1981)).

DEFINITION 3.48. A subspace S of X will be called *conditionally invariant* if there is a mapping G: $Y \to X$ such that $(A+GC)S \subset S$. Given a symmetric subset $\mathbb{C}_g$ of $\mathbb{C}$, a subspace S_g of X will be called a *detectability subspace* if there is a mapping G: $Y \to X$ such that $(A+GC)S_g \subset S_g$ and $\sigma(A+GC|X/S_g) \subset \mathbb{C}_g$.

The above concepts are the duals of the concepts of controlled invariant subspace and stabilizability subspace. Conditionally invariant subspaces are also called (C,A)-invariant subspaces. A subspace S is conditionally invariant if and only if $A(S \cap \ker C) \subset S$.

Now, assume that V_a is an almost stabilizability subspace and assume that we want to find a minimum phase input subspace T contained in V_a. Obviously, a necessary condition for the existence of such T is that $V_a + \ker C = X$. In SCHUMACHER (1984, REMARK 2) it is shown that given V_a with the latter property, the problem of finding a minimum phase input subspace contained in it is equivalent to a problem of stabilization by static output feedback. As is well known, the latter is, in general, an unsolved problem. Suppose however that *instead of an almost stabilizability subspace, we have a detectability subspace S with the property that $S + \ker C = X$.* It turns out that in this case a minimum phase input subspace contained in S cán be found:

<u>THEOREM 3.49</u>. *Let $\mathbb{C}_g \subset \mathbb{C}$ be symmetric. Assume that* (C,A) *is detectable. Let S be a detectability subspace such that $S + \ker C = X$. Then there exists a minimum phase input subspace T such that $T \subset S$.*

<u>PROOF</u>: Decompose $X = X_1 \oplus X_2 \oplus X_3$ with $X_2 := S \cap \ker C$, X_3 such that $X_2 \oplus X_3 = S$ and X_1 such that $X_1 \oplus X_2 = \ker C$. Since S is conditionally invariant, $AX_2 = A(S \cap \ker C) \subset S = X_2 \oplus X_3$. Therefore, in the above decomposition

$$A = \begin{pmatrix} A_{11} & 0 & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \quad \text{and} \quad C = (0 \quad 0 \quad C_3) .$$

Let $G: Y \to X$ be such that $(A+GC)S \subset S$. Then obviously $\sigma(A+GC|X/S) = \sigma(A_{11})$ and hence, by the fact that S is a detectability subspace, $\sigma(A_{11}) \subset \mathbb{C}_g$. We claim that the pair (A_{32},A_{22}) is detectable. Suppose it is not. Then there is a $\lambda \in \mathbb{C} \setminus \mathbb{C}_g$ and a vector $x_2 \neq 0$ such that $A_{22}x_2 = \lambda x_2$ and $A_{32}x_2 = 0$ (HAUTUS (1970)). It may then be seen that if $x := (0^T, x_2^T, 0^T)^T$, then $Ax = \lambda x$ and $Cx = 0$, which contradicts the detectability of (C,A). Using the detectability of (A_{32},A_{22}), let W be a mapping from X_3 into X_2 such that $\sigma(A_{22} - WA_{32}) \subset \mathbb{C}_g$. Define a subspace $T \subset S$ by

$$T := \left\{ \begin{pmatrix} 0 \\ Wx_3 \\ x_3 \end{pmatrix} ; \; x_3 \in X_3 \right\} .$$

Note that $X_2 \oplus T = X_2 \oplus X_3 = S$. It follows that $T \oplus \ker C = X$. Let $P\colon X \to \ker C$ be the projection along T. It may be verified that in the decomposition of X employed:

$$P = \begin{pmatrix} I & 0 & 0 \\ 0 & I & -W \end{pmatrix}.$$

Hence, the matrix of $PA|\ker C$ is given by

$$PA|\ker C = \begin{pmatrix} A_{11} & 0 \\ A_{21}-WA_{31} & A_{22}-WA_{32} \end{pmatrix}$$

and we may conclude from LEMMA 3.44 that T is a minimum phase input subspace.
□

The above result motivates the following question: when is it possible to find for the system (A,B,C) an almost stabilizability subspace V_a such that $V_a + \ker C = X$ which, at the same time, is a detectability subspace? We will show that under certain assumptions, the subspace $R_b^*(\ker C)$ is such a subspace. First, since $R_b^*(\ker C)$ is an almost controllability subspace (see SECTION 3.1), it follows from TH. 2.72 that it is an almost stabilizability subspace (provided that the stability set $\mathbb{C}_g$ satisfies (2.27) and (2.28). In order for $R_b^*(\ker C)$ to be a detectability subspace and to satisfy the equality $R_b^*(\ker C) + \ker C = X$, we have to restrict our class of systems:

<u>LEMMA 3.50</u>. *Assume that* $\mathbb{C}_g$ *is symmetric. If* (A,B,C) *is right-invertible then* $R_b^*(\ker C) + \ker C = X$. *Moreover, under the above assumption,* $R_b^*(\ker C)$ *is a detectability subspace if and only if* (A,B,C) *is minumum phase.*

<u>PROOF</u>: In this proof, denote $V_b^* := V_b^*(\ker C)$, etc. If (A,B,C) is right-invertible, then by TH. 3.15, $V_b^* = X$. Hence, by COR. 3.12, $V_a^* = V_b^* \cap \ker C = \ker C$. It follows that $X = V_a^* + R_b^* = \ker C + R_b^*$.

Now, let $F \in \underline{F}(V^*)$. It is well known (WONHAM (1979)) that $F \in \underline{F}(R^*)$ and that $\sigma(A_F|R^*)$ may be chosen arbitrarily. Denote the fixed spectrum $\sigma(A_F|V^*/R^*)$ by σ^* (the invariant zeros). Choose F such that $\sigma(A_F|R^*) \cap \sigma^* = \emptyset$. Let V be the sum of the generalized eigenspaces of $A_F|V^*$ corresponding to its eigenvalues in σ^*. Then $V^* = R^* \oplus V$ and $\sigma(A_F|V) = \sigma^*$. Define $X_1 := V$ and $X_2 := R_b^*$. By the fact that $V^* \cap R_b^* = R^*$ and $X = V^* + R_b^*$, we thus obtain a decomposition $X = X_1 \oplus X_2$. Since $V \subset \ker C$ and $\operatorname{im} B \subset R_b^*$, we find that in this decomposition:

$$(3.23) \qquad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ B_2 \end{pmatrix} \quad \text{and} \quad C = (0 \quad C_2) .$$

Now, we contend that R_b^* is conditionally invariant. To show this, note from COR. 3.12 that $R_b^* \cap \ker C = R_a^*$. Thus, from (3.3), $A(R_b^* \cap \ker C) = AR_a^* \subset R_b^*$, which proves our claim. Let G be such that $(A+GC)R_b^* \subset R_b^*$. From (3.23) it follows that $\sigma(A+GC|X/R_b^*) = \sigma(A_{11})$ for all such G. Therefore, to complete the proof it suffices to prove that $\sigma(A_{11}) = \sigma^*$. This can be seen by taking F as above and by again noting from (3.23) that the matrix of $A_F|V$ is given by A_{11}.

□

<u>REMARK 3.51</u>. Using TH. 3.16, together with a duality argument, it may in fact be proven that the condition $R_b^*(\ker C) + \ker C$ is also sufficient for (A,B,C) to be right-invertible.

Putting things together now, we obtain the following result on the existence of stabilizing compensators:

<u>COROLLARY 3.52</u>. *Let* $\mathbb{C}_g$ *be open and assume that* (2.27) *and* (2.28) *hold. Assume that* (A,B,C) *is right-invertible, detectable and minimum phase. Then there is a stabilizing compensator of the form* (3.21) *of dynamic order* $\dim R_b^* - p$.

<u>PROOF</u>: Under the above hypotheses, R_b^* is an almost stabilizability subspace, a detectability subspace and $R_b^* + \ker C = X$. By TH. 3.49 there exists a minimum phase input subspace contained in R_b^*. The result follows from TH. 3.47.

□

We will now show that if (A,B,C) is right-invertible and minimum phase, then (A,B) is stabilizable. The proof of this is as follows. First note that if (A,B,C) is right-invertible, then from the fact that $R_b^* \subset \langle A|B\rangle$, it follows that $X = V^* + \langle A|B\rangle$. Now, let $\sigma_u := \sigma(A|X/\langle A|B\rangle)$ denote the uncontrollable modes of $\langle A|B\rangle$. Let $F \in \underline{F}(V^*)$. Then

$$\begin{aligned}\sigma_u &= \sigma(A_F \mid (V^* + \langle A|B\rangle)/\langle A|B\rangle)\\ &= \sigma(A_F \mid (V^*/(V^* \cap \langle A|B\rangle))\\ &\subset \sigma(A_F \mid V^*/R^*) \ .\end{aligned}$$

If (A,B,C) is minimum phase, the latter spectrum is contained in $\mathbb{C}_g$ (see DEF. 3.42). Thus, the uncontrollable modes are stable and therefore (A,B) is stabilizable.

Next, we contend that if (A,B,C) is left-invertible and minimum phase, then (C,A) is detectable. To show this, note that (A,B,C) is left-invertible if and only if (A^T,C^T,B^T) is right-invertible. Now, it may be proven using MALABRE (1982, TH. 4.1) that (A,B,C) is minimum phase if and only if (A^T,C^T,B^T) is minimum phase. It thus follows from the above that (A^T,C^T) is stabilizable, whence (C,A) is detectable.

In the following, a system will be called *invertible* if it is both left and right-invertible. The foregoing considerations immediately lead to the following result, stating that invertible minimum phase systems can always be stabilized using a dynamic order equal to the excess of poles over zeros minus the number of inputs:

<u>COROLLARY 3.53</u>. *Suppose* $\mathbb{C}_g$ *is open and satisfies* (2.27) *and* (2.28). *Assume that* (A,B,C) *is invertible and minimum phase. Then there exists a stabilizing compensator of dynamic order* d, *with*

$$d = n - [\textit{number of invariant zeros}] - [\textit{number of inputs}] \ .$$

<u>PROOF</u>: If (A,B,C) is invertible then $X = V^* \oplus R_b^*$. Therefore, the number of invariant zeros is equal to $n - \dim R_b^*$. Since, by the above remarks, (C,A) is detectable, the result follows from COR. 3.52.

□

As a special case of the latter corollary we obtain the following:

<u>COROLLARY 3.54</u>. *Suppose that* $\mathbb{C}_g$ *is open and satisfies* (2.27) *and* (2.28). *Assume that* (A,B,C) *is minimum phase and* CB *is nonsingular. Then the system can be stabilized by static output feedback.*

<u>PROOF</u>: Under these assumptions, $X = \ker C \oplus B$. Hence $V^* = \ker C$ and $R_b^* = B$. Thus the system is right-invertible and the number of invariant zeros is equal to $n - m$.

□

REMARK 3.55. The result of COR. 3.54 can also be obtained using results from root locus theory (see e.g. KOUVARITAKIS & SHAKED (1976), OWENS (1978a), OWENS (1978b)). In our context however, the result is obtained in a fairly clean 'geometric' way and no asymptotic analysis is required.

REMARK 3.56. Once we have found a minimum phase input subspace T contained in R_b^*, we may apply the idea of the proof of TH. 3.45 to find a sequence of stabilizing compensators. One has to approximate R_b^* by stabilizability subspaces V_n. This can for example be done in the following way. Since $R_b^* \in \underline{R}_a$, there is an integer r, integers r_i ($i \in \underline{r}$), vectors $u_i \in U$ and a mapping F such that $R_b^* = \bigoplus_{i=1}^{r} \mathcal{L}(u_i,F,r_i)$. Define $V_n := \bigoplus_{i=1}^{r} \mathcal{L}_n(u_i,F,r_i)$, where $\mathcal{L}_n(u_i,F,r_i)$ is defined as in SECTION 2.4. Then $V_n \to R_b^*$ as $n \to \infty$. Next, for each n, choose $T_n \subset V_n$ such that $T_n \to T$. For n sufficiently large, T_n is a minimum phase input subspace. One may then apply TH. 3.45 to each pair (T_n,V_n) to find a stabilizing compensator $\dot{w} = N_n w + M_n y$, $u = L_n w + K_n y$. In general, we will have $\|N_n\|$, $\|M_n\|$, $\|L_n\|$ and $\|K_n\| \to \infty$ as $n \to \infty$. Assume now that (A,B,C) is minimum phase and invertible. It may be verified (going through the proof of TH. 3.45 given in SCHUMACHER (1981, LEMMA 2.12)) that the closed loop spectrum in this case consists of (1) dim R_b^* (= number of infinite zeros) eigenvalues $\lambda(n)$ that run off to infinity along the negative real axis as $n \to \infty$, (2) dim V^* (= number of invariant zeros) eigenvalues $\lambda(n)$ that tend to the invariant zeros of (A,B,C) and (3) dim $R_b^* - m$ (= dynamic order of the compensator) eigenvalues $\lambda(n)$ that tend to the eigenvalues of the mapping $A_{22} - WA_{32}$ appearing in the proof of TH. 3.49 (note that these limiting eigenvalues may be chosen arbitrarily if (C,A) is observable).

We conclude this section by pointing out that the result of COR. 3.52 may be dualized to obtain an existence result on stabilizing compensators for a different class of systems. In the following, given a system (A,B,C) let $V^*(\ker C,A,B)$ denote the supremal (A,B)-invariant subspace in ker C. Let $R_b^*(\ker C,A,B)$ denote the supremal L_p-almost controllability subspace of ker C (with respect to (A,B)) and let $S^*(\mathrm{im}\, B,C,A)$ denote the infimal (C,A)-invariant subspace containing im B (WILLEMS & COMMAULT (1981)). It was proven in MALABRE (1982) that $R_b^*(\ker C,A,B) = S^*(\mathrm{im}\, B,C,A)$. Also, by the duality between conditionally and controlled invariance, $[V^*(\ker C,A,B)]^{\perp} =$ $= S^*(\mathrm{im}\, C^T,B^T,A^T)$. Assume now that (A,B,C) is left-invertible and minimum phase and that (A,B) is stabilizable. Then (A^T,C^T,B^T) is right-invertible and minimum phase, while (B^T,A^T) is detectable. Thus, by COR. 3.52, there is a stabilizing compensator for the system (A^T,C^T,B^T) of dynamic order equal to

$$\dim \mathcal{R}_b^*(\ker B^T,A^T,C^T) - m =$$

$$\dim \mathcal{S}^*(\operatorname{im} C^T,B^T,A^T) - m =$$

$$\dim [\mathcal{V}^*(\ker C,A,B)]^\perp - m .$$

Thus we obtain the following:

COROLLARY 3.57. *Suppose that* $\mathbb{C}_g$ *is open and satisfies* (2.27) *and* (2.28). *Assume that* (A,B,C) *is left-invertible, stabilizable and minimum phase. Then there exists a stabilizing compensator of dynamic order* $n - m - \dim \mathcal{V}^*(\ker C)$.

PROOF: If (N,M,L,K) yields a stabilizing compensator for (A^T,C^T,B^T), then (N^T,L^T,M^T,K^T) yields a stabilizing compensator for (A,B,C).

□

CHAPTER 4

ALMOST DISTURBANCE DECOUPLING WITH BOUNDED PEAKING

In this chapter we will discuss a synthesis problem that will be called the problem of almost disturbance decoupling with bounded peaking. This problem is an extended version of the almost disturbance decoupling problem that we considered in the previous chapter. The idea is that, due to the large feedback gains that will in general be required to achieve the desired approximate decoupling between the disturbances and to-be-controlled outputs, certain components of the closed loop state trajectories may become unacceptably large. In view of this, it is important to have a design procedure that enables us to achieve approximate disturbance decoupling up to any desired degree of accuracy, while simultaneously given components of the state trajectories are bounded functions of this accuracy. Formalizing this idea leads to the formulation of an almost disturbance decoupling problem with boundedness constraint.

The chapter is divided into seven sections. In section 1, we will illustrate in an example the phenomenon of unbounded state trajectories we mentioned above and formulate the main synthesis problems that will be considered in this chapter. In section 2, we will discuss the problem of exact disturbance decoupling under the constraint that a given linear function of the state should be stable. The material in this section will be needed in the development of this chapter, but is also interesting in its own right. In section 3 and section 4, we will derive a necessary and sufficient condition for solvability of the L_p-almost disturbance decoupling problem with bounded peaking. This condition will consist of a subspace inclusion in the spirit of the earlier results described in this tract. In section 5, we will establish conditions for the solvability of our problem under the additional requirement of pole assignability. To illustrate the concepts and ideas introduced in this chapter, section 6 contains a worked example. Finally, in section 7 we will briefly discuss how the material of this chapter may be extended to study the problem of perfect regulation with bounded peaking.

4.1 PROBLEM FORMULATION

In the previous chapters we have considered the problem of approximate disturbance decoupling. We discussed two different versions of this problem, differing in the way we measured the influence of the disturbances on the to-be-controlled output. Both versions were extended to include a spectral assignability constraint. In all these problems, the accuracy of decoupling was measured in terms of induced norms.

We also saw that the feedback mappings necessary to achieve approximate decoupling within a certain degree of accuracy, are in general unbounded functions of the required accuracy: in order to obtain a 'great' accuracy one has to use large feedback gains. This fact brings us to an inherent difficulty in the design of high gain feedback systems: certain variables in the closed loop system may become 'too large' while increasing the feedback gain. In particular, in the L_p-almost disturbance decoupling problem it might happen that some of the state variables become undesirably large, while using a large feedback gain with the purpose of increasing the accuracy of approximate decoupling between certain external disturbance and output channels. As an example to illustrate this behaviour in a 'high gain controlled' feedback system, consider the system $\dot{x}(t) = Ax(t) + Bu(t) + Gd(t)$, $z(t) = Hx(t)$, with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad G = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad H = (1 \quad 0 \quad 0).$$

For $n \in \mathbb{N}$, define a state feedback mapping F_n by

$$F_n := (-27n^3, \ -27n^2, \ -9n).$$

If in the above we use as a control law $u = F_n x$, it may be calculated that the impulse response from the disturbance d to the output z is given by

$$W_{1,n}(t) = He^{(A+BF_n)t}G = e^{-3nt}\left(1 + 3nt + \frac{9}{2}n^2t^2\right).$$

Moreover, $\|W_{1,n}\|_1 = \frac{1}{n}$. Since the latter quantity dominates the induced norm of the closed loop operator mapping disturbances $d \in L_p(\mathbb{R}^+,\mathcal{D})$, to outputs $z \in L_p(\mathbb{R}^+,\mathcal{Z})$ for every $1 \leq p \leq \infty$, we see that the L_p-almost disturbance decoupling problem is solvable for all these p. Indeed, by increasing the feedback gain we may increase the accuracy of decoupling to any desired degree of accuracy.

On the other hand, the impulse response from the disturbance d to the state x is calculated to be

$$W_{2,n}(t) = e^{(A+BF_n)t}\,G = e^{-3nt}\begin{pmatrix} 1 + 9nt + \frac{9}{2}n^2t^2 \\ -\frac{27}{2}n^3t^2 \\ -27n^3t + \frac{81}{2}n^4t^2 \end{pmatrix}$$

and it may be verified that $\|W_{2,n}\|_1 \to \infty$ as $n \to \infty$. This means that in particular both the $L_1 - L_1$ induced norm and the $L_\infty - L_\infty$ induced norm of the closed loop operator mapping d to x grow unboundedly as the feedback gain increases. Hence, for certain disturbances it may happen that some of the components of the state trajectories generated by these disturbances become undesirably large. In other words, we see that an increase in feedback gain increases the disturbance rejection features of the closed loop system, but may cause unacceptably large state trajectories.

In certain situations it might be possible that, instead of all components of the state trajectory, we are concerned only about the magnitude of some of these components: we do not care what happens with the other components as the feedback gain increases. In this way we are led to formulate the following extension of the L_p-almost disturbance decoupling problem. As before, we consider the linear system

$$\text{(4.1)} \qquad \dot{x}(t) = Ax(t) + Bu(t) + Gd(t).$$

However, instead of specifying only one to-be-controlled output, we specify two of these outputs. For the first output we want to solve the usual L_p-almost disturbance decoupling problem. The feedback control laws should, however, be chosen in such a way that the induced norm of the closed loop operator from the disturbance channel to the second output remains bounded as a function of the accuracy of decoupling. More concretely, we consider (4.1) together with the output equations

$$\text{(4.2)} \qquad \begin{aligned} z_1(t) &= H_1x(t) \ , \\ z_2(t) &= H_2x(t) \end{aligned}$$

and we ask whether there exists a constant C and for all $\varepsilon > 0$ a feedback law $u = Fx$ such that, with $x(0) = 0$,

$$\|z_1\|_p \leq \varepsilon ,$$

and

$$\|z_2\|_p \leq C ,$$

for all disturbances d in the unit ball of $L_p(\mathbb{R}^+, \mathcal{D})$. In the above, H_1 and H_2 are assumed to be surjective mappings from $\mathcal{X}$ to finite dimensional linear spaces $\mathcal{Z}_1$ and $\mathcal{Z}_2$ respectively. Furthermore, without loss of generality we will assume that the output z_2 is an *enlargment* of z_1, i.e. there is a mapping $M : \mathcal{Z}_2 \to \mathcal{Z}_1$ such that $H_1 = MH_2$ or, equivalently,

$$\ker H_2 =: \mathcal{K}_2 \subset \mathcal{K}_1 := \ker H_1 . \tag{4.3}$$

We can always make sure that z_2 is an enlargment of z_1 by redefining the mappings H_1 and H_2 as follows: if $\tilde{z}_1 = \tilde{H}_1 x$, $\tilde{z}_2 = \tilde{H}_2 x$, then define $H_1 := \tilde{H}_1$, $H_2 := \begin{pmatrix} \tilde{H}_1 \\ \tilde{H}_2 \end{pmatrix}$, $z_1 := H_1 x$ and $z_2 := H_2 x$. Of course, the existence of feed back laws such that $\tilde{z}_1$ is arbitrarily small while $\tilde{z}_2$ remains bounded, is equivalent to the existence of feedback laws such that z_1 is arbitrarily small while z_2 remains bounded. Moreover, if H_1 and H_2 are defined from $\tilde{H}_1$ and $\tilde{H}_2$ in this way, then the inclusion (4.3) holds. In this chapter, from now on (4.3) will be a standing assumption. Let us now summarize the above considerations in the following definition:

<u>DEFINITION 4.1</u>. Let $1 \leq p \leq \infty$. $(\mathrm{ADDPBP})_p$, the L_p*-almost disturbance decoupling problem with bounded peaking*, is said to be solvable if there exists a constant C and for all $\varepsilon > 0$ a mapping $F : \mathcal{X} \to \mathcal{U}$ such that in the closed loop system with $x(0) = 0$,

$$\|z_1\|_p \leq \varepsilon \|d\|_p \quad \text{and} \tag{4.4}$$

$$\|z_2\|_p \leq C \|d\|_p , \tag{4.5}$$

for all $d \in L_p(\mathbb{R}^+, \mathcal{D})$.

If in the above we take $H_1 = H_2 = H$, then the original L_p-almost disturbance decoupling problem (see DEF. 3.18) is recovered as a special case. If we are interested in boundedness of the entire state vector x, we should take for the second output space $\mathcal{Z}_2$ the state space $\mathcal{X}$ and for H_2 the identity mapping on $\mathcal{X}$.

Of course, we may also define an extension of the above to include a constraint of spectral assignability. In the following definition it will be assumed that the stability sets $\mathbb{C}_g$ under consideration are symmetric and contain a point at minus infinity, i.e. satisfy the conditions (2.27) and (2.28).

DEFINITION 4.2. Let $1 \leq p \leq \infty$ $(\text{ADDPBPPP})_p$, the L_p-*almost disturbance decoupling problem with bounded peaking and pole placement*, is said to be solvable if there exists a constant C and for all $\varepsilon > 0$ and all $\mathbb{C}_g$ a mapping $F : X \to U$ such that in the closed loop system with $x(0) = 0$, both (4.4) and (4.5) hold for all $d \in L_p(\mathbb{R}^+, D)$ and $\sigma(A + BF) \subset \mathbb{C}_g$.

4.2 EXACT DISTURBANCE DECOUPLING WITH OUTPUT STABILITY

Prior to discussions involving the boundedness properties of the enlarged output z_2, we should make sure that in the closed loop system every disturbance $d \in L_p(\mathbb{R}^+, D)$ generates an enlarged output that has at least a finite L_p-norm. Of course, if we control the system by means of a state feedback control law to a achieve approximate decoupling from the disturbances to the first output, this does not imply that the closed loop operator from the disturbances to the second output defines an operator from $L_p(\mathbb{R}^+, D)$ to $L_p(\mathbb{R}^+, Z_2)$. As a subproblem of the problem introduced in the previous section we will therefore begin with studying an extension of the well known (exact) disturbance decoupling problem with state feedback (WONHAM (1979)). We will consider the linear system with two to-be-controlled outputs defined by (4.1) and (4.2) and ask whether there exists a state feedback control law such that in the closed loop system the disturbances do not influence the first output, while simultaneously the transfer matrix from the disturbance to the enlarged output is stable.

In this section we will work with a fixed stability set $\mathbb{C}_g$, which will be assumed to be symmetric (see SECTION 2.5). Asymptotic stability is thus obtained by taking $\mathbb{C}_g = \mathbb{C}^-$. In the sequel, if L is a subspace of X and $x \in \Sigma(A,B)$, then x/L will denote the projection of x on the factor space X/L, i.e. $(x/L)(t) := [x(t)]$, the equivalence class modulo L. Recall that $\Sigma^B(A,B)$ denotes the subset of $\Sigma(A,B)$ of all trajectories whose restriction to $\mathbb{R}^+$ is Bohl (DEF. 2.4). Again, if f is Bohl, then

$\sigma(f)$ will denotes its spectrum (see SECTION 2.7). We will consider the following problem:

DEFINITION 4.3. DDPOS, *the disturbance decoupling problem with output stability*, will be said to be solvable if there is a mapping $F : X \to U$ such that

$$H_1 \, e^{(A+BF)t} \, G = 0 \quad \text{for all} \quad t \in \mathbb{R}, \tag{4.6}$$

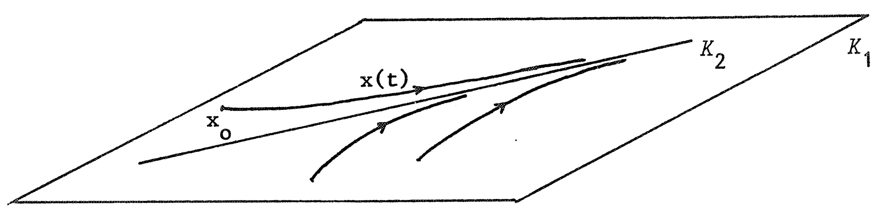
$$\sigma(H_2 \, e^{(A+BF)t} \, G) \subset \mathbb{C}_g. \tag{4.7}$$

In order to obtain conditions for the solvability of this problem, introduce the following subspace:

DEFINITION 4.4. Given a pair of subspaces $K_2 \subset K_1$, we define

$$V_g(K_1,K_2) := \{x_o \in K_1 \mid \exists \; x \in \Sigma^B(A,B) \text{ such that } x(0) = x_o,$$
$$x(t) \in K_1, \forall\, t \in \mathbb{R}^+ \text{ and } \sigma(x/K_2) \subset \mathbb{C}_g\}.$$

Thus, $V_g(K_1,K_2)$ consists of all points in which a regular Bohl trajectory starts that lies entirely in K_1, while the components of this trajectory modulo K_2 are stable.



It follows immediately from the definition that $V_g(K_1,K_2)$ is contained in $V^*(K_1)$. Since $K_2 \subset K_1$, if a trajectory lies in K_2, then it obviously also lies in K_1. Consequently, also the inclusion $V^*(K_2) \subset V_g(K_1, K_2)$ is valid. In fact, it may be shown that:

THEOREM 4.5.

$$V_g(K_1, K_2) = V_g^*(K_1) + V^*(K_2).$$

PROOF: The proof of this theorem is analogous to the proof of HAUTUS (1980, TH. 4.3). Let $x_o \in V_g(K_1,K_2)$ and let $x \in \Sigma^B(A,B)$ be such that $x(0) = x_o$, $x(t) \in K_1, \forall t \in \mathbb{R}^+$ and $\sigma(x/K_2) \subset \mathbb{C}_g$. Assume that x is generated by the input u. Since B is injective, u must be Bohl. Since x and u are Bohl, they have unique decompositions $x = x_b + x_g$ and $u = u_b + u_g$ with $\sigma(x_b)$ and $\sigma(u_b)$ in $\mathbb{C} \setminus \mathbb{C}_g$ and $\sigma(x_g)$ and $\sigma(u_g)$ in $\mathbb{C}_g$. Also, $\dot{x}_g(t) - Ax_g(t) - Bu_g(t) = -\dot{x}_b(t) + Ax_b(t) + Bu_b(t)$. In this equality, the left hand side has its spectrum in $\mathbb{C}_g$, the right hand side in $\mathbb{C} \setminus \mathbb{C}_g$. Since both sides are regular Bohl functions, we may conclude that $\dot{x}_g(t) = Ax_g(t) + Bu_g(t)$ and $\dot{x}_b(t) = Ax_b(t) + Bu_b(t)$. Now, since $x_b/K_2 = x/K_2 - x_g/K_2$, we have $\sigma(x_b/K_2) \subset \mathbb{C}_g$. Thus, necessarily, $x_b(t) \in K_2$, $\forall t$. It follows that x_b is a trajectory in K_2 and hence that $x_b(0) \in V^*(K_2)$. Since $K_2 \subset K_1$, also $x_b(t) \in K_1, \forall t$, and hence $x_g/K_1 = x/K_1$. Consequently, by the fact that $x(t) \in K_1, \forall t$ we find that $x_g(t) \in K_1, \forall t$ and hence that $x_g(0) \in V_g^*(K_1)$ (see e.g. TH. 2.75). Since $x_o = x_b(0) + x_g(0)$ we may conclude that $x_o \in V^*(K_2) + V_g^*(K_1)$. The converse inclusion may be verified immediately.

□

Note that it follows from the above theorem that $V_g(K_1,K_2)$ is controlled invariant. From the proof of TH. 4.5 we see that if we start in a point in $V_g(K_1,K_2)$ and move along a trajectory x that lies in K_1 such that x/K_2 is stable, then x lies in fact in $V_g(K_1,K_2)$. Moreover, not only x/K_2 but even $x/V^*(K_2)$ is stable.

REMARK 4.6. Of course, we could also have defined $V_g(K_1,K_2)$ in terms of (ξ,ω)-representations. In fact, in this way the above subspace was originally introduced in TRENTELMAN (1984). Let H_2 be a mapping such that $\ker H_2 = K_2$. Then $V_g(K_1,K_2)$ is the subspace of all points $x \in X$ for which there is a (ξ,ω)-representation with $\xi(s) \in K_{1,+}(s)$, $\omega(s) \in U_+(s)$ and $H_2\xi(s)$ is stable.

The following theorem provides the key step of this section. It will also be crucial in the further development that will lead to the construction of sequences of feedback mappings satisfying the requirements of the L_p-almost disturbance decoupling problem with bounded peaking. The result states that, starting in $V_g(K_1,K_2)$, it is in fact possible to stay in K_1 moving along state feedback generated trajectories such that the components modulo K_2 are stable. Moreover, in the subspace $V^*(K_2)$ we keep the usual freedom of spectral assignability on the controllability subspace $R^*(K_2)$:

THEOREM 4.7. *For every mapping* $F_o \in \underline{F}(V^*(K_2))$ *there exists a mapping* $F_1 \in \underline{F}(V_g(K_1,K_2))$ *such that*

$$(4.8) \qquad F_1|V^*(K_2) = F_o|V^*(K_2) \quad ,$$

$$(4.9) \qquad \sigma(A + BF_1|V_g(K_1,K_2)/V^*(K_2)) \subset \mathbb{C}_g.$$

PROOF: In this proof, denote $V_g := V_g(K_1,K_2)$. Let $F_o \in \underline{F}(V^*(K_2))$. It is easy to see that there is a mapping $F : X \to U$ such that $F|V^*(K_2) = F_o|V^*(K_2)$ and such that $(A + BF)V_g \subset V_g$. Define $\bar{B} := B \cap V_g$ and let V be a mapping such that $\bar{B} = \text{im } BV$. Consider the controllability subspace $R_o := \langle A + BF|\bar{B}\rangle$. Due to the facts that $\bar{B} \subset V_g$ and $(A + BF)V_g \subset V_g$, we have $R_o \subset K_1$. Since $\underline{R}(A,B) \subset \underline{V}_g(A,B)$, R_o must be contained in the supremal stabilizability subspaces in K_1. Consequently, we must have $\bar{B} \subset V_g^*(K_1)$ and hence

$$(4.10) \qquad B \cap V_g^*(K_1) = \bar{B}.$$

We contend that $V_g^*(K_1)$ is $(A + BF)$-invariant. First, since it is controlled invariant, $(A + BF)V_g^*(K_1) \subset V_g^*(K_1) + B$. On the other hand, $(A + BF)V_g^*(K_1) \subset (A + BF)V_g \subset V_g$. Hence we obtain

$$(A + BF)V_g^*(K_1) \subset (V_g^*(K_1) + B) \cap V_g \subset V_g^*(K_1) + \bar{B} = V_g^*(K_1).$$

Using (4.10) and HAUTUS (1981, PROP. 2.16), we deduce that the system with system mapping $(A + BF)|V_g^*(K_1)$ and input mapping BV is stabilizable. Let $P : V_g \to V_g/V^*(K_2)$ denote the canonical projection. Also, let $\bar{A}_o$ denote the quotient mapping of $(A + BF)|V_g$ modulo $V^*(K_2)$ and let $\bar{B}_o := PBV$. It may be verified using a rank test (see HAUTUS (1970) or HAUTUS (1981, TH. 2.13)) that the system $(\bar{A}_o, \bar{B}_o)$ is stabilizable. Hence, there is a mapping $\bar{F}_2$ on $V_g/V^*(K_2)$ such that $\sigma(\bar{A}_o + \bar{B}_o\bar{F}_2) \subset \mathbb{C}_g$. Let F_2 be a mapping on V_g such that $F_2 = \bar{F}_2P$ and extend F_2 arbitrarily to a mapping on X. Define now $F_1 := F + VF_2$. Then we have $F_1|V^*(K_2) = F_o|V^*(K_2)$ and consequently $(A + BF_1)V^*(K_2) \subset V^*(K_2)$. Moreover, it may be verified that the following diagram commutes:

$$\begin{tikzcd}
V_g \arrow[r, "A + BF_1"] \arrow[d, "P"'] & V_g \arrow[d, "P"] \\
V_g/V^*(K_2) \arrow[r, "\bar{A} + \bar{B}_o\bar{F}_2"'] & V_g/V^*(K_2)
\end{tikzcd}$$

□

We are now in a position to prove the main result of this section. Let $K_i := \ker H_i$. Then we have:

<u>THEOREM 4.8</u>. DDPOS *is solvable if and only if* $\operatorname{im} G \subset V_g(K_1,K_2)$.

<u>PROOF:</u> If DDPOS is solvable, then it follows immediately from DEF. 4.4 That $\operatorname{im} G \subset V_g$. Conversely, assume that this subspace inclusion holds. Let F be a mapping in $\underline{F}(V^*(K_2)) \cap \underline{F}(V_g)$ such that (4.9) holds. Again, let $P : V_g \to V_g/V^*(K_2)$ be the canonical projection. Denote by $\bar{A}_F$ the quotient mapping of $(A + BF)|V_g$ modulo $V^*(K_2)$ and let $\bar{H}_2$ be a mapping such that $\bar{H}_2P = H_2|V_g$. Then we have $\langle A + BF|\operatorname{im} G\rangle \subset V_g \subset K_1$, which yields the decoupling from d to z_1. Moreover, since $\sigma(\bar{A}_F) \subset \mathbb{C}_g$ and

$$H_2\, e^{(A+BF)t}\, G = \bar{H}_2\, e^{\bar{A}_F t}\, PG \quad ,$$

also (4.7) is satisfied. This completes the proof of the theorem.

□

<u>REMARK 4.9</u>. The above results are closely related to those in HAUTUS (1981). In the latter, for a given subspace K a subspace S^-_Σ is defined as the subspace of all points x_o in which a regular Bohl trajectory starts such that its components modulo K are stable. This space can be recovered in our context by taking $K_1 = X$ and $K_2 = K$, i.e. $S^-_\Sigma = V_g(X,K)$. The space

S^-_Σ was was used in HAUTUS (1981) to study the problem OSDP : given $\dot{x}(t) = Ax(t) + Bu(t) + Gd(t)$, $z(t) = Hx(t)$, find a mapping $F : X \to U$ such that $H(Is - A - BF)^{-1}G$ is stable. Of course, this problem is a special case of the problem DDPOS defined in this section and can be recovered from the latter by taking $H_1 = 0$ and $H_2 = H$. A necessary and sufficient condition for the solvability of OSDP is therefore found to be that $\mathrm{im}\, G \subset V_g(X, \ker H) = X_{stab} + V^*(\ker H)$. Here, X_{stab} is the stabilizable subspace of (A,B) (see SECTION 2.5). If we take $H_1 = 0$, $H_2 = H$ and $\mathrm{im}\, G = X$, then the output stabilization problem OSP (BHATTACHARRYA,PEARSON & WONHAM (1972), WONHAM (1979, p. 92)) is recovered from DDPOS. Finally, we note that also the ordinary disturbance decoupling problem DDP can be recovered from DEF. 4.3, by taking $H_1 = H_2 = H$.

4.3 CONSTRAINED L_p-ALMOST INVARIANCE

In this section, we will obtain a necessary condition for the solvability of the L_p-almost disturbance decoupling problem with bounded peaking. Consider the system described by (4.1) and (4.2). If F is a mapping from X to U, then the closed loop impulse response matrix from d to z_i will be denoted by $W_{i,F}(t) := H_i e^{A_F t}G$. Its Laplace transform will be denoted by $\hat{W}_{i,F}(s) := H_i(Is - A_F)^{-1}G$. Again, $\|W_{i,F}\|_1$ will denote the L_1 norm and $\|\hat{W}_{i,F}\|_\infty$ the H^∞-norm (which is equal to the L_2-L_2 induced norm of the closed loop operator). As before, for $p \in \{1,2,\infty\}$ solvability of $(\mathrm{ADDPBP})_p$ can be reformulated in terms of the above quantities:

<u>LEMMA 4.10</u>. *Let* $p \in \{1,\infty\}$. *Then* $(\mathrm{ADDPBP})_p$ *is solvable if and only if there exists* $C \in \mathbb{R}$ *and for all* $\varepsilon > 0$ *a mapping* $F : X \to U$ *such that* $\|W_{1,F}\|_1 \leq \varepsilon$ *and* $\|W_{2,F}\|_1 \leq C$. $(\mathrm{ADDPBP})_2$ *is solvable if and only if there exists* $C \in \mathbb{R}$ *and for all* $\varepsilon > 0$ *a mapping* $F : X \to U$ *such that* $\|\hat{W}_{1,F}\|_\infty \leq \varepsilon$ *and* $\|\hat{W}_{2,F}\|_\infty \leq C$.

<u>PROOF</u>: The proof of this lemma is immediate.

□

From the above lemma we see that solvability of the L_p-almost disturbance decoupling problem with bounded peaking for $p = 1$ or $p = \infty$ implies that in every point x_0 in im G trajectories start such that the L_1-norm of the distance of these trajectories to $K_1 = \ker H_1$ is arbitrarily small, while the L_1-norm of the distance to $K_2 = \ker H_2$ remains bounded. Here, 'bounded' should be interpreted as follows: while decreasing the L_1-norm of the distance to K_1, in general it will happen that the L_1-norm of the distance to K_2 is forced to increase. Now, for $p \in \{1,\infty\}$ solvability of $(\mathrm{ADDPBP})_p$ means that there is an upper bound to this increasing integrated distance to K_2. From these considerations it is clear that, in order to obtain a necessary condition for the solvability of $(\mathrm{ADDPBP})_p$, we should consider the subspace of $V_b^*(K_1)$ consisting of all points x_o with the property that there exists a constant C and for all $\varepsilon > 0$ a trajectory x such that $\|d(x,K_1)\|_1 \leq \varepsilon$ while simultaneously $\|d(x,K_2)\|_1 \leq C$. Here,we stress that C should be independent of ε. In the following, as usual let $\Sigma^{B(r)}(A,B)$ denote the subset of $\Sigma(A,B)$ consisting of all Bohl trajectories x with $\deg x \leq r$ (see SECTION 2.7). Instead of looking at arbitrarily small L_1-norms, we will be slightly more general and consider arbitrarily small L_p-norms:

<u>DEFINTION 4.11</u>. Given a pair of subspaces $K_2 \subset K_1$ and $p \in [1,\infty)$, define:

$$V_p(K_1,K_2) := \{x_o \in X \mid \exists\, C \in \mathbb{R} \text{ and } r \in \mathbb{N} \text{ such that for all } \varepsilon > 0\ \exists\, x \in \Sigma^{B(r)}(A,B) \text{ with } x(0) = x_o,\ \|d(x,K_1)\|_p \leq \varepsilon \text{ and } \|d(x,K_2)\|_1 \leq C\}.$$

Thus, a point x_o lies in $V_p(K_1,K_2)$ if, starting in x_o, we can make the L_p-norm of the distance to K_1 arbitrarily small, while moving along Bohl trajectories with an upper bound to their McMillan degree. In this course, the L_1-norms of the distances to K_2 should remain bounded. It will be shown that, in fact, $V_p(K_1,K_2)$ is independent of p for $1 \leq p \leq \infty$. At this point we note however that the L_1-norm appearing in the boundedness constraint is essential. We will show in section 4.7 that a constraint $\|d(x,K_2)\|_2 \leq C$ yields a subspace different from $V_p(K_1,K_2)$. This may be seen immediately from the following considerations: imposing the constraint that the L_1-norms of the distances to K_2 should remain bounded is equivalent to requiring that all components of the trajectories modulo K_2 should remain bounded in L_1-norm. This however means that these components are still allowed to converge in distributional sense to *zero*-order impulsive distributions (but not to higher order distributions). If, however, we require that the components modulo K_2 remain bounded in L_2-norm this is no longer allowed (since, if a sequence of smooth functions $\{\varphi_n\}$ converges to δ, then $\|\varphi_n\|_2 \to \infty$). From this discussion, it is clear that the subspace with L_2-boundedness constraint will in general not coincide with the subspace defined in DEF. 4.11.

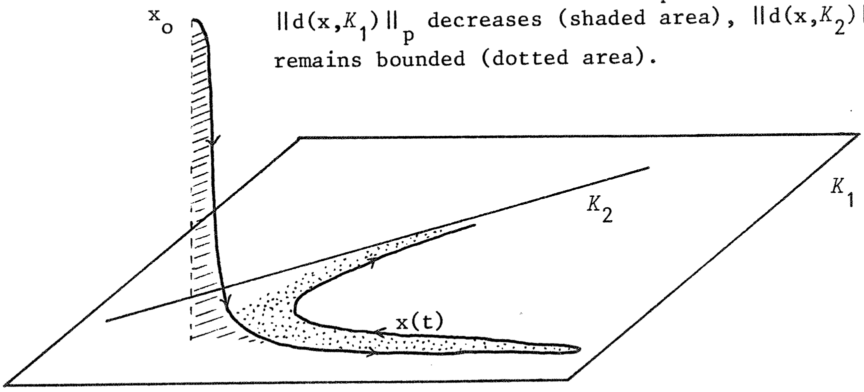


fig. 4.2. A typical point in $V_p(K_1,K_2)$: while $\|d(x,K_1)\|_p$ decreases (shaded area), $\|d(x,K_2)\|_1$ remains bounded (dotted area).

Note that it is immediate from the definition that the subspace $V_g(K_1,K_2)$ (see DEF. 4.4) is contained in $V_p(K_1,K_2)$ (with $\mathbb{C}_g = \mathbb{C}^-$). Next, in addition to DEF 4.11, we will introduce the following subspace of $V_b^*(K_1)$ (see also COR. 3.33) :

DEFINITION 4.12. Given a pair of subspaces $K_2 \subset K_1$, define:

$$\begin{aligned} H(K_1,K_2) := \{x_o \in X \mid \exists\, C \in \mathbb{R} \text{ and } r \in \mathbb{N} \text{ such that for all } \\ \varepsilon > 0\ \exists\, (\xi,\omega)\text{-representation for } x_o \text{ with } \\ \xi(s) \in X_+(s),\ \omega(s) \in U_+(s),\ \deg \xi \leq r, \\ \|d(\xi,K_1)\|_\infty \leq \varepsilon \text{ and } \|d(\xi,K_2)\|_\infty \leq C\}. \end{aligned}$$

By definition, for $p \in \{1,\infty\}$, if $(\text{ADDPBP})_p$ is solvable then $\operatorname{im} G \subset V_1(K_1,K_2)$ and if $(\text{ADDPBP})_2$ is solvable, then $\operatorname{im} G \subset H(K_1,K_2)$. Of course, this observation is of little use unless we can obtain convenient, in principle computable, expressions for the subspaces that we have defined. In this chapter, it will turn out that these can indeed be obtained. In order to proceed, it is convenient to introduce the following subspace in terms of Bohl distributional trajectories. In the sequel, if x is an X-valued distribution and K a subspace of X, then

x/K will denote the X/K-valued distribution defined by $\langle x/K,\varphi\rangle := [\langle x,\varphi\rangle]$, $\varphi \in D(\mathbb{R})$, where [ ] denotes the equivalence class modulo K. Also, if $x \in D_B'^n$ (see DEF. 2.4) then we define its spectrum to be the spectrum of its regular part : $\sigma(x) := \sigma(x_{reg})$ (SECTION 2.7). The order of x is defined as the order of its impulsive part: ord $x := \text{ord } x_{imp}$ (SECTION 2.2). Define

$$V_b(K_1,K_2) := \{x_o \in X \mid \exists\, u \in D_B'^m \text{ such that } x^+(x_o,u) \text{ lies in } K_1,\ \sigma(x^+(x_o,u)/K_2) \subset \mathbb{C}^- \text{ and } \text{ord } x^+(x_o,u)/K_2 \leq 0\}.$$

A point x_o lies in $V_b(K_1,K_2)$ if it can serve as initial condition for a Bohl distributional trajectory that lies in K_1, while simultaneously its components modulo K_2 have their spectrum contained in $\mathbb{C}^-$ and have impulsive parts of at most distributional order zero (i.e. the impulsive part of x^+/K_2 is either zero or a Dirac delta). Of course, we could also introduce $V_b(K_1,K_2)$ in terms of (ξ,ω)-representations. Let H_2 be such that ker $H_2 = K_2$. Then $V_b(K_1,K_2)$ is exactly the subspace of points in X that have a (ξ,ω)-representation with $\xi(s) \in K_1(s)$, $\omega(s) \in U(s)$ and $H_2\xi(s)$ *proper* and *asymptotically stable*. It is immediate from the definition that the subspace $V_g(K_1,K_2)$ (DEF. 4.4) is contained in $V_b(K_1,K_2)$ (with $\mathbb{C}_g = \mathbb{C}^-$).

It will now be shown that both $V_p(K_1,K_2)$ and $H(K_1,K_2)$ are contained in $V_b(K_1,K_2)$. After that, we will show that $V_b(K_1,K_2)$ has an expression in terms of supremal controlled invariant subspaces and supremal L_p-almost controllability subspaces. We will need the following result on the limiting behaviour of sequences of rational functions:

LEMMA 4.13. *Let* $\{f_n\}_{n \in \mathbb{N}}$ *be a sequence of strictly proper rational functions. Let* $r \in \mathbb{N}$ *be such that* deg $f_n \leq r$, $\forall n$. *Assume that* $\lim_{n\to\infty} f_n(s)$ *exists for infinitely many* $s \in \mathbb{C}$. *Then there exists a rational function* f *(not necessarily proper) such that* $\lim_{n\to\infty} f_n(s) = f(s)$ *for all but finitely many* $s \in \mathbb{C}$.

PROOF: A proof of this can be found in HAZEWINKEL (1980, p. 169).

□

The above result is not valid without the assumption on the uniform bound of the McMillan degrees. The limiting function f will of course in general not be strictly proper or proper. Using the above result we may prove that every sequence of strictly proper asymptotically stable rational vectors with an upper bound to their McMillan degree, uniformly bounded in the closed right half plane, has a subsequence that converges to a proper asymptotically stable rational vector:

LEMMA 4.14. *Let* Z *be a finite dimensional linear space and let* $\{\zeta_n\}_{n \in \mathbb{N}}$ *be a sequence in* $Z_+(s)$ *such that* $\sigma(\zeta_n) \subset ¢^-$, $\forall n$. *Assume that there is* $r \in \mathbb{N}$ *such that* $\deg \xi_n \leq r$, $\forall n$. *Also, suppose that there exists a* $C \in \mathbb{R}$ *such that* $\|\zeta_n(s)\| \leq C$ *for all* $s \in ¢^+$. *Then there exists* $\zeta(s) \in Z(s)$ *, with* $\zeta(s)$ *proper and* $\sigma(\zeta) \subset ¢^-$, *and a subsequence* $\{\zeta_{n_m}\}$ *such that* $\lim_{m\to\infty} \zeta_{n_m}(s) = \zeta(s)$, *pointwise for all but finitely many* $s \in ¢$.

PROOF: The proof of this lemma uses Arzela-Ascoli's theorem (AHLFORS (1966 , TH. 12, p. 216)), which states that if $\{f_n\}$ is a sequence of complex functions, analytic in an (open) region $\Omega \subset ¢$ and uniformly bounded on every compact set in Ω, then it has a subsequence which converges to a function f, uniformly on every compact set in Ω. This function f is again analytic in Ω (AHLFORS (1966 , TH.1, p. 174). Now, by applying this result with $\Omega = ¢_o^+$ to every component of $\{\zeta_n\}$, we may conclude that there is a subsequence $\{\zeta_{n_m}\}$ converging to a vector $\zeta(s)$, analytic in $¢_o^+$. Moreover, this convergence holds, a fortiori for infinitely many $s \in ¢_o^+$. Thus, by LEMMA 4.1 , we may assume that it holds for all but finitely many $s \in ¢$ and that $\zeta(s) \in Z(s)$. It then remains to prove that $\zeta(s)$ has no poles on the imaginary axis and that it is proper. Define:

$$J := \{s \in ¢ \mid \mathrm{Re}\, s = 0,\ s \notin \sigma(\zeta) \text{ and } \lim_{m\to\infty} \zeta_{n_m}(s) = \zeta(s)\}.$$

Denote $I := \{s \in ¢ \mid \mathrm{Re}\ s = 0\}$. Obviously, $I \setminus J$ is a finite set. Suppose that $s_o \in J$. For m sufficiently large we have $\|\zeta_{n_m}(s_o) - \zeta(s_o)\| \leq 1$ and hence

$$\|\zeta(s_o)\| \leq \|\zeta(s_o) - \zeta_{n_m}(s_o)\| + \|\zeta_{n_m}(s_o)\| \leq 1 + C. \tag{4.11}$$

It follows that $\zeta(s)$ is bounded on J and hence, since it is rational, on the entire imaginary axis I. We conclude that $\zeta(s)$ has no poles in I and, by letting $s_o \to \infty$ in (4.11), that $\zeta(s)$ is proper. Thus, we have proven the lemma.

□

As a consequence of this convergence result, it is now possible to show that the subspaces V_p and H are contained in V_b. This will be done by analyzing in a straightforward way the asymptotic behaviour of the sequences of Bohl trajectories and (ξ,ω)-representations in terms of which V_p and H are defined.

<u>LEMMA 4.15</u>. *Let* $1 \leq p < \infty$ *and let* $K_2 \subset K_1$ *be a pair of subspaces of* X. *Then we have* $V_p(K_1,K_2) \subset V_b(K_1,K_2)$ *and* $H(K_1,K_2) \subset V_b(K_1,K_2)$.

<u>PROOF</u>: Let H_i be a mapping such that $K_i = \ker H_i$. Suppose that $x_o \in V_p$. By DEF. 4.11, there is $C \in \mathbb{R}$ and $r \in \mathbb{N}$ and a sequence of regular Bohl inputs $\{u_n\}$ such that the resulting trajectories $\{x_n\}$ through x_o satisfy $\deg x_n \leq r$, $\|H_1x_n\|_p \to 0$ $(n \to \infty)$ and $\|H_2x_n\|_1 \leq C$, $\forall n$. Denote $z_{i,n} := H_ix_n$. Let $\zeta_{i,n}(s)$, $\xi_n(s)$ and $\omega_n(s)$ be the Laplace transforms of $z_{i,n}$, x_n and u_n respectively. For all $s \in \mathbb{C}^+$ we have

$$\|\zeta_{2,n}(s)\| \leq \int_o^\infty e^{-(\mathrm{Re}\,s)t}\|z_{2,n}(t)\|dt \leq \|z_{2,n}\|_1 \leq C, \forall n .$$

Hence, by the foregoing lemma, there is an asymptotically stable proper $\zeta(s) \in Z(s)$ and a subsequence of $\{\zeta_{2,n}\}$, that we will again denote $\{\zeta_{2,n}\}$, such that $\zeta_{2,n}(s) \to \zeta(s)$ for all but finitely many $s \in \mathbb{C}$. Also, for all $s \in \mathbb{C}_o^+$ we have

$$\|\zeta_{1,n}(s)\| \leq \int_o^\infty e^{-(\mathrm{Re}\,s)t}\|z_{1,n}(t)\|dt \leq K\|z_{1,n}\|_p ,$$

for some $K \in \mathbb{R}$ and hence $\zeta_{1,n}(s) \to 0$ for all but finitely many $s \in \mathbb{C}$. Now, let $F \in \underline{F}(V^*(K_2))$, let $P\colon X \to X/V^*(K_2)$ be the canonical projection and let $\bar{A}_F$ denote the quotient mapping of A_F modulo $V^*(K_2)$. Let $\bar{B} := PB$ and let $\bar{H}_1$ and $\bar{H}_2$ be mappings such that $\bar{H}_1P = H_1$ and $\bar{H}_2P = H_2$. Decompose $U = U_1 \oplus U_2$ with $U_1 := \ker \bar{B}$ and U_2 an arbitrary complement. Accordingly, partition $\bar{B} = (0\ \ \bar{B}_2)$. Then $\bar{B}_2$ is injective. Denote $\bar{G}(s) := \bar{H}_2(Is - \bar{A}_F)^{-1}\bar{B}_2$. Let $\bar{R}^*(\bar{K}_2)$ denote the supremal controllability subspace in $\bar{K}_2$ with respect to $(\bar{A}_F, \bar{B})$. By WONHAM (1979, ex. 5.8), $\bar{R}^*(\bar{K}_2) = \{0\}$. Hence, by TH. 3.16, $\bar{G}(s)$ is $\mathbb{R}(s)$-injective and has a left-inverse $\bar{G}^+(s)$. Let $\tilde{\omega}_n(s) := \omega_n(s) + F\xi_n(s)$. Then $x_o = (Is-A_F)\xi_n(s) + B\tilde{\omega}_n(s)$. In the decomposition $U = U_1 \oplus U_2$, let $\tilde{\omega}_n(s) = (\tilde{\omega}_{1,n}(s)^T, \tilde{\omega}_{2,n}(s)^T)^T$. Then $Px_o = (Is - \bar{A}_F)\, P\xi_n(s) + \bar{B}_2\tilde{\omega}_{2,n}(s)$ and we obtain

$$\widetilde{\omega}_{2,n}(s) = \bar{G}(s)^{+}[\bar{H}_2(Is-\bar{A}_F)^{-1}\ Px_o - \zeta_{2,n}(s)],$$

and

$$\dot{P}\xi_n(s) = (Is-\bar{A}_F)^{-1}\ (Px_o - \bar{B}_2\widetilde{\omega}_{2,n}(s)).$$

Since $\zeta_{2,n}(s) \to \zeta(s)$, there are rational vectors $\widetilde{\omega}_2(s)$ and $\bar{\xi}(s)$ such that $\widetilde{\omega}_{2,n}(s) \to \widetilde{\omega}_2(s)$ and $P\xi_n(s) \to \bar{\xi}(s)$ for all but finitely many $s \in \mathbb{C}$. Define now $\widetilde{\omega}(s) := (0^T, \widetilde{\omega}_2(s)^T)^T$. Then we have $Px_o = (Is-\bar{A}_F)\bar{\xi}(s) + \bar{B}\widetilde{\omega}(s)$. Moreover, since $\bar{H}_1 P\xi_n(s) = \zeta_{1,n}(s) \to 0$, we have $\bar{H}_1\bar{\xi}(s) = 0$. Also since $\bar{H}_2 P\xi_n(s) = \zeta_{2,n}(s) \to \zeta(s)$, $\bar{H}_2\xi(s)$ is proper and asymptotically stable. To conclude the proof, let $\xi(s)$ be a rational vector such that $P\xi(s) = \bar{\xi}(s)$. Then we have $H_1\xi(s) = 0$ and $H_2\xi(s)$ is proper and asymptotically stable. Moreover, since $Px_o = (Is-\bar{A}_F)\bar{\xi}(s) + \bar{B}\widetilde{\omega}(s)$, there is a vector $x_1 \in V^*(K_2)$ such that $x_o = (Is-A_F)\xi(s) + B\widetilde{\omega}(s) - x_1$. It follows that $x_o - x_1 \in V_b(K_1,K_2)$. Since obviously $V^*(K_2) \subset V_b(K,K_2)$, it follows that $x_o \in V_b(K_1,K_2)$ and hence we have proven that $V_p(K_1,K_2) \subset V_b(K_1,K_2)$.

The inclusion $H(K_1,K_2) \subset V_b(K_1,K_2)$ may be proven in a similar way. This completes the proof of the lemma. □

In the sequel, it will turn out that under certain additional assumptions on the system (A,B,H_1,H_2) the inclusions that we have established above are, in fact, equalities. To be able to prove this we need more information on the subspace $V_b(K_1,K_2)$. The following result is very useful since it tells us that $V_b(K_1,K_2)$ has a representation in terms of subspaces that we have already studied:

<u>THEOREM 4.16.</u> *Let* $V_g(K_1,K_2)$ *be taken with respect to the stability set* $\mathbb{C}_g = \mathbb{C}^-$. *Then we have*

$$V_b(K_1,K_2) = V_g(K_1,K_2) + B + A[R_b^*(K_2) \cap K_1].$$

<u>PROOF:</u> In this proof, again let H_2 be a mapping such that $K_2 = \ker H_2$. Assume that $x_o \in V_b$. Then $x_o = (Is-A)\xi(s) + B\omega(s)$, with $\xi(s) \in K_1(s)$, $\omega(s) \in U(s)$ and $H_2\xi(s)$ proper and asymptotically stable. Decompose $\xi(s) = \xi_+(s) + \xi_-(s)$ and $\omega(s) = \omega_+(s) + \omega_-(s)$ into their strictly proper and polynomial parts. Then $\xi_-(s) \in K_1[s]$, $\xi_+(s) \in K_{1,+}(s)$. Also, $H_2\xi_-(s)$ is a constant and $H_2\xi_+(s)$ is asymptotically stable. Moreover,

$$x_o - (Is-A)\xi_+(s) - B\omega_+(s) = (Is-A)\xi_-(s) + B\omega_-(s).$$

In this equation, the left hand side is proper and the right hand side a polynomial. Therefore, both sides must in fact be equal to the same constant vector $x_1 \in X$ and we find

$$(4.12) \qquad x_1 = (Is - A)\xi_-(s) + B\omega_-(s),$$

$$(4.13) \qquad x_o - x_1 = (Is - A)\xi_+(s) + B\omega_+(s).$$

It follows from REMARK 4.6 that $x_o - x_1 \in V_g(K_1,K_2)$ (with $\mathbb{C}_g = \mathbb{C}^-$). It remains to prove that $x_1 \in B + A[R_b^*(K_2) \cap K_1]$. Let $\xi_-(s) = \sum_{i=o}^{N} \xi_i s^i$ and $\omega_-(s) = \sum_{i=o}^{N+1} \omega_i s^i$. There is $\xi_1(s) \in K_2[s]$ and $\omega_1(s) \in U[s]$ such that $\xi_-(s) = \xi_o + s\xi_1(s)$ and such that $\omega_-(s) = \omega_o + s\omega_1(s)$. Hence, by (4.12),

$$x_1 = B\omega_o - A\xi_o + s\xi_o + s^2\xi_1(s) - As\xi_1(s) + Bs\omega_1(s).$$

Thus, by equating powers it follows that $x_1 = -A\xi_o + B\omega_o$ and

$$-\xi_o = (Is - A)\xi_1(s) + B\omega_1(s).$$

Therefore, by TH. 3.9, $\xi_o \in R_b^*(K_2)$. Since also $\xi_o \in K_1$, we obtain $x_1 \in B + A[R_b^*(K_2) \cap K_1]$.
To prove the converse inclusion, note that $V_g \subset V_b$ follows immediately from the definitions. The proof may then be completed using a similar manipulation as above.

□

We conclude this section by noting that the above results immediately lead to the subspace inclusion

$$(4.14) \qquad \text{im } G \subset V_g(K_1,K_2) + B + A[R_b^*(K_2) \cap K_1]$$

(with $V_g(K_1,K_2)$ defined with respect to $\mathbb{C}_g = \mathbb{C}^-$) as a necessary condition for solvability of $(\text{ADDPBP})_p$ for $p \in \{1,2,\infty\}$. In the next section it will be shown that under certain assumptions (4.14) also provides a sufficient condition for solvability of $(\text{ADDPBP})_p$ for all $1 \leq p \leq \infty$.

4.4 SUFFICIENT CONDITIONS FOR SOLVABILITY OF $(\text{ADDPBP})_p$

Whereas in the previous section we have been discussing the L_p-almost disturbance decoupling with bounded peaking from the point of view of finding a necessary condition for its solvability, in the present section we will be mainly concerned with establishing sufficient conditions. Our main goal will be to show that under certain assumptions the subspace inclusion (4.14), that was shown to be a necessary condition, in fact also provides a sufficient condition for the solvability of $(\text{ADDPBP})_p$.

In this section we will assume throughout that the stability set $\mathbb{C}_g$ is equal to the open left half plane $\mathbb{C}^-$. Accordingly, the subspaces $V_g(K_1,K_2)$ and $V_g^*(K)$ are always understood to be taken with respect to this stability set.

The development in this section will go along the following lines. First, we will show that for a large class of systems the subspace $V_b(K_1,K_2)$ admits a decomposition into the direct sum of $V_g(K_1,K_2)$ together with a number of singly generated almost controllability subspaces of a particular form. Indeed, it will turn out that these singly generated almost controllability subspaces can be chosen to be spanned by vectors $b, A_Fb,\dots, A_F^kb$ with the property that $b, A_Fb,\dots, A_F^{k-1}b$ all lie in K_1, while the vectors $b, A_Fb,\dots, A_F^{k-2}b$ lie in K_2. Next, for the class of singly generated almost controllability subspaces that have the latter property, we will prove a result concerning the canonical approximation by controlled invariant subspaces, in the spirit of TH.2.35 and TH. 3.27. As a final ingredient, we will establish the existence of a stable controlled invariant complement of $V_b(K_1,K_2)$ in the subspace $V_g(K_1,K_2)$ + $\langle A|B\rangle$ (in analogy to LEMMA 3.30). Combining all these partial results we will then show that under mild assumptions on the systems under consideration, starting in the subspace $V_b(K_1,K_2)$, one may travel along trajectories that are generated by state feedback such that the L_p-norms of the distance of these trajectories to K_1 are arbitrary small, while the L_1-norms of the distances to K_2 remain bounded. For the class of systems under consideration this will immediately imply that (4.14) is a sufficient condition for the solvability of $(\text{ADDPBP})_p$ for all $1 \leq p \leq \infty$.

To start with, denote

$$W(K_1,K_2) := B + A[R_b^*(K_2) \cap K_1].$$

It was shown in TRENTELMAN (1984, TH. 5.3) that $W(K_1,K_2)$ is exactly the subspace consisting of all points in X that have a (ξ,ω)-representation with $\xi(s) \in K_1[s]$, $\omega(s) \in U[s]$ and $H_2\xi(s)$ constant (i.e., if $\xi(s) = \sum_{i=o}^{N} \xi_i s^i$, then $H_2\xi_i = 0$ for $i \geq 1$). As a consequence of this, we can immediately conclude that $R_b^*(K_2) \subset W(K_1,K_2) \subset R_b^*(K_1)$. Let $R^\mu(K_2)$ be the sequence of subspaces generated by (ACSA)' (SECTION 3.2). Define now:

$$F^\mu(K_1,K_2) := R^\mu(K_2) \cap K_1.$$

The sequence F^μ inherits its properties from the algorithm (ACSA)'. In particular, it is immediate (TH. 3.7) that F^μ is monotonically non-decreasing. Moreover, if we define $k := 1 + \dim K_2$, then

$$W(K_1,K_2) = B + AF^k(K_1,K_2). \tag{4.15}$$

We will need the following property of the sequence F^μ :

LEMMA 4.17. *For each* $\mu \in \mathbb{N}$ *there is a chain* $\{B_i\}_{i=1}^{\mu}$ *in* B *and a mapping* $F : X \to U$ *such that*

$$F^\mu(K_1,K_2) = B_1 \oplus A_F B_2 \oplus \ldots \oplus A_F^{\mu-1} B_\mu, \tag{4.16}$$

$$B_2 \oplus A_F B_3 \oplus \ldots \oplus A_F^{\mu-2} B_\mu \subset K_2, \tag{4.17}$$

$$\dim B_i = \dim A_F^{i-1} B_i = \dim F^i - \dim F^{i-1}, \quad i \in \underline{\mu}. \tag{4.18}$$

PROOF: A proof of this may be given along the lines of the proof of TH. 1.10 and will be omitted here. For a detailed proof, we refer to TRENTELMAN (1984, LEMMA B.1).

□

Note that if we take $K_1 = K_2 = K$, then $F^\mu = S^\mu(K)$, as generated by the almost controllability subspace algorithm ACSA (see SECTION 1.3). Hence, statement (ii) of TH. 1.10 can be recovered as a special case of the previous lemma.

Using the above result, the following may now be proven:

LEMMA 4.18. *Assume that* $R^*(K_1) = \{0\}$. *Then there is a chain* $\{B_i\}_{i=1}^{k}$ *in* B *and a mapping* $F : X \to U$ *such that*

$$W(K_1,K_2) = B \oplus A_F B_1 \oplus \ldots \oplus A_F^k B_k, \tag{4.19}$$

$$(4.20)\qquad \mathcal{B}_1 \oplus A_F\mathcal{B}_2 \oplus \ldots \oplus A_F^{k-1}\mathcal{B}_k \subset \mathcal{K}_1,$$

$$(4.21)\qquad \mathcal{B}_2 \oplus A_F\mathcal{B}_3 \oplus \ldots \oplus A_F^{k-2}\mathcal{B}_k \subset \mathcal{K}_2,$$

$$(4.22)\qquad \dim \mathcal{B}_i = \dim A_F^i\mathcal{B}_i = \dim \mathcal{F}^i - \dim \mathcal{F}^{i-1},\ i \in \underline{k}\ .$$

PROOF: Apply LEMMA 4.17 to obtain a representation $\mathcal{F}^k = \mathcal{B}_1 \oplus A_F\mathcal{B}_2 \oplus \ldots \oplus A_F^{k-1}\mathcal{B}_k$. From (4.17) and the fact that, by definition, $\mathcal{F}^k \subset \mathcal{K}_1$, we obtain (4.20) and (4.21) above. Now, we claim that $\ker A_F \cap \mathcal{F}^k = \{0\}$. For assume this is not true. Then there is a vector $0 \neq v \in \mathcal{K}_1$ such that the one-dimensional subspace span $\{v\}$ is controlled invariant. Thus, $v \in \mathcal{V}^*(\mathcal{K}_1)$. Also, $v \in \mathcal{F}^k \subset \mathcal{R}^k(\mathcal{K}_1) \cap \mathcal{K}_1 \subset \mathcal{R}_a^*(\mathcal{K}_1)$ and hence $v \in \mathcal{V}^*(\mathcal{K}_1) \cap \mathcal{R}_a^*(\mathcal{K}_1) = \mathcal{R}^*(\mathcal{K}_1)$ (see TH. 1.32). Since, by assumption, $\mathcal{R}^*(\mathcal{K}_1) = \{0\}$, this yields a contradiction. This proves our claim. We may then immediately conclude that $\dim A_F^i\mathcal{B}_i = \dim A_F^{i-1}\mathcal{B}_i$ for all $i \in \underline{k}$, which proves (4.22). Finally, to prove (4.19), note that

$$(4.23)\qquad \mathcal{W}(\mathcal{K}_1,\mathcal{K}_2) = \mathcal{B} + A_F\mathcal{F}^k = \mathcal{B} + \sum_{i=1}^{k} A_F^i\mathcal{B}_i.$$

Again by the assumption that $\mathcal{R}^*(\mathcal{K}_1) = \{0\}$ it may be verified that $\mathcal{B} \cap A_F\mathcal{F}^k = \{0\}$. Therefore, the dimension of $\mathcal{W}(\mathcal{K}_1,\mathcal{K}_2)$ must be equal to $\dim \mathcal{B} + \dim \mathcal{F}^k = \dim \mathcal{B} + \sum_{i=1}^{k}\dim \mathcal{B}_i$. This implies that all sums appearing in (4.23) must, in fact, be direct sums.

□

In the following, again let $\mathcal{L}(u,F,k)$ denote the k-dimensional singly generated almost controllability subspace spanned by the vectors $Bu, A_FBu, \ldots, A_F^{k-1}Bu$ (see SECTION 2.4). We will now show that if $\mathcal{R}^*(\mathcal{K}_1) = \{0\}$, then $\mathcal{V}_b(\mathcal{K}_1,\mathcal{K}_2)$ can be decomposed into the direct sum of $\mathcal{V}_g(\mathcal{K}_1,\mathcal{K}_2)$ and a number of singly generated almost controllability subspaces $\mathcal{L}(u_i,F,r_i)$ with the property that the first r_i-1 vectors $Bu_i, A_FBu_i, \ldots, A_F^{r_i-2}Bu_i$ are contained in $\mathcal{K}_1$, while the first r_i-2 vectors $Bu_i, A_FBu_i, \ldots, A_F^{r_i-3}Bu_i$ are contained in $\mathcal{K}_2$:

LEMMA 4.19. *Let $\mathcal{K}_2 \subset \mathcal{K}_1$ be a given pair of subspaces and assume that $\mathcal{R}^*(\mathcal{K}_1) = \{0\}$. Then there exists an $r \in \mathbb{N}$ and, for $i \in \underline{r}$, integers $r_i \in \mathbb{N}$, vectors $u_i \in \mathcal{U}$ and a mapping $F : \mathcal{X} \to \mathcal{U}$ such that*

$$(4.24)\qquad V_b(K_1,K_2) = V_g(K_1,K_2) \oplus \bigoplus_{i=1}^{r} \mathcal{L}(u_i,\ F,\ r_i),$$

with $\mathcal{L}(u_i,\ F,\ r_i-1) \subset K_1$ *and* $\mathcal{L}(u_i,\ F,\ r_i-2) \subset K_2$. *Here, we define* $\mathcal{L}(u_i,\ F,\ -1) = \mathcal{L}(u_i,\ F,\ 0) := \{0\}$.

PROOF: By TH. 4.16, $V_b(K_1,K_2) = V_g(K_1,K_2) + W(K_1,K_2)$. We contend that this sum is a direct one. To see this, note that $V_g(K_1,K_2) \subset V^*(K_1)$ and that $W(K_1,K_2) \subset R_b^*(K_1)$. Consequently, it follows from COR. 3.12 that $V_g \cap W \subset R^*(K_1) = \{0\}$. To complete the proof, it suffices to show that $W(K_1,K_2)$ can be decomposed into a direct sum of singly generated almost controllability subspaces with the required properties. This may be proven using LEMMA 4.18: let $\{B_i\}_{i=1}^{k}$ be a chain in B and let F be a mapping such that (4.19) to (4.22) hold. Choose a basis for $W(K_1,K_2)$ as follows: first choose a basis for B_k, extend this to a basis for B_{k-1} and proceed in this way to obtain a basis for B. Using (4.22), this yields a basis for $W(K_1,K_2)$ that can be rearranged into singly generated almost controllability subspaces with the required properties with respect to K_2 and K_1 (see also REMARK 2.28).

□

The main point of all this is, that we want to be able to decompose $V_b(K_1,K_2)$ according to (4.24), while the singly generated almost controllability subspaces have the above particular position with respect to K_1 and K_2. We proved that this can be done if $R^*(K_1) = \{0\}$. There is yet another important case in which such decomposition can be established. In the sequel, let $\tilde{B} \subset B$ be such that $\tilde{B} \oplus (B \cap V^*(K_1)) = B$ (see also SECTION 2.3). Let V be a mapping such that $\tilde{B} = \text{im } BV$. Let $\tilde{R}_b^*(K_1)$ denote the supremal L_p-almost controllability subspace of K_1, with respect to the system (A,BV). It was shown in the proof of LEMMA 3.29 that

$$V_b^*(K_1) = V^*(K_1) \oplus \tilde{R}_b^*(K_1).$$

Now, define

$$(4.25)\qquad \tilde{W}(K_1) := \tilde{B} + A(\tilde{B} \cap K_1).$$

We will show that if $K_2 = \{0\}$, then $V_b(K_1,K_2)$ has a decomposition into the direct sum of $V_g(K_1,K_2)$ (which, in that case, is equal to $V_g^*(K_1)$) and the subspace $\widetilde{W}(K_1)$:

<u>LEMMA 4.20</u>. *Let K_1 be a subspace of X. Then*

$$V_b(K_1, \{0\}) = V_g(K_1, \{0\}) \oplus \widetilde{W}(K_1). \tag{4.26}$$

<u>PROOF</u>: In this proof, denote $V_g := V_g(K_1, \{0\})$. Also, let $B_1 := B \cap V^*(K_1)$ It follows from TH. 4.16 that

$$\begin{aligned} V_b(K_1, \{0\}) &= V_g + B + A[B \cap K_1] \\ &= V_g + B + A[(B_1 \oplus \widetilde{B}) \cap K_1] \\ &= V_g + B + A[\ B_1 + (\widetilde{B} \cap K_1)]. \end{aligned}$$

Now, note that $B_1 \subset R^*(K_1)$ (see WONHAM (1979 , TH.5.5)). Consequently, $AB_1 \subset R^*(K_1) + B \subset V_g + B$. Hence, we find

$$\begin{aligned} V_b(K_1, \{0\}) &= V_g + B + A(\widetilde{B} \cap K_1). \\ &= V_g + B_1 + \widetilde{B} + A(\widetilde{B} \cap K_1). \end{aligned}$$

Again by the fact that $B_1 \subset R^*(K_1) \subset V_g$, we have

$$V_b(K_1, \{0\}) = V_g + \widetilde{B} + A(\widetilde{B} \cap K_1) = V_g + \widetilde{W}(K_1)$$

Finally, since $V_g \subset V^*(K_1)$ and $W(K_1) \subset \widetilde{R}_b^*(K_1)$, it follows that the sum in (4.26) is direct.

□

This indeed yields the desired decomposition of $V_b(K_1, \{0\})$ into the direct sum of $V_g(K_1, \{0\})$ and a number of singly generated almost controllability subspaces with the desired properties. In this special case, the singly generated almost controllability subspaces are either one-dimensional or two-dimensional. The mapping F can be taken to be zero:

<u>LEMMA 4.21.</u> *Let K_1 be a subspace of X. There is* $r \in \mathbb{N}$ *and, for* $i \in \underline{r}$ *, there are integers* $r_i \in \{1, 2\}$ *and vectors* $u_i \in U$ *such that*

(4.27) $\quad V_b(K_1, \{0\}) = V_g(K_1, \{0\}) \oplus \bigoplus_{i=1}^{r} \mathcal{L}(u_i, 0, r_i),$

with $\mathcal{L}(u_i, 0, r_i-1) \subset K_1$.

PROOF: It is easy to see that $\dim \widetilde{B} \cap K_1 = \dim A(\widetilde{B} \cap K_1)$. Choose a basis for $\widetilde{W}(K_1)$ as follows. First choose a basis $b_1,\ldots, b_\ell$ for $\widetilde{B} \cap K_1$. Extend this to a basis for $\widetilde{B}$ by adding the vectors $b_{\ell+1},\ldots, b_r$. By the dimensional equality above, the system $b_1, b_2,\ldots, b_\ell, Ab_1,\ldots, Ab_\ell, b_{\ell+1},\ldots, b_r$ forms a basis for $\widetilde{W}(K_1)$. This basis can be rearranged into one and two-dimensional singly generated almost controllability subspaces with the required properties.

□

REMARK 4.22. In LEMMA 4.21, we have omitted the statement $\mathcal{L}(u_i, 0, r_i-2) \subset K_2$, since this will be automatically satisfied if the integers r_i are either 1 or 2 and if $K_2 = \{0\}$ (of course, again with the convention that $\mathcal{L}(u_i, F, 0) = \mathcal{L}(u_i, F, -1):= \{0\}$).

As the next step in the development outlined in the introduction to this section we will prove a result analogous to TH. 2.35 and TH. 3.27, concerning the canonical sequence $\{\mathcal{L}_n\}$ of controlled invariant subspaces that converges to the singly generated almost controllability subspace $\mathcal{L}(u, F, k)$. Recall that we defined $\mathcal{L}(u, F, k):= \operatorname{span}\{x_1(n,u),\ldots, x_k(n,u)\}$, with the vectors $x_i(n,u)$ defined by (2.6). Given $\mathcal{L}(u, F, k)$, we denote $\mathcal{L}':= \mathcal{L}(u, F, k-1)$ and $\mathcal{L}'':= \mathcal{L}(u, F, k-2)$ (with the convention that $\mathcal{L}(u, F, 0) = \mathcal{L}(u, F, -1) = \{0\}$). We have the following result:

THEOREM 4.23. *Let* $F_n : \mathcal{L}_n(u, F, k) \to U$ *be defined by* $F_n x_i(n,u) = -n^i u$, $(i \in \underline{k})$. *Fix* $1 \le p_o \le \infty$. *Then there exists a constant* $C \in \mathbb{R}$ *and for all* $\varepsilon > 0$ *there exists a* $K \in \mathbb{N}$ *such that for all* $i \in \underline{k}$ *and* $p \in [1,p_o]$

$$\| d(\mathcal{L}', e^{(A_F+BF_n)t} x_i(n,n))\|_p \le \varepsilon \;, \quad \forall n \ge K$$

and

$$\| d(\mathcal{L}'', e^{(A_F+BF_n)t} x_i(n,u))\|_1 \le C \;, \quad \forall n \in \mathbb{N}.$$

PROOF: The proof of this is an extension of the proof of TH. 3.27. Again, let $x_i(n,u)$ be expanded as in (2.10). In this expansion all terms but the last two composite sums between brackets are contained in $\mathcal{L}'$, whereas all terms but the last *three* composite sums are contained in $\mathcal{L}''$. Denote the last composite sum between brackets by v(n) and let

$$a_1 := \sum_{\ell_1=1}^{i} \sum_{\ell_2=1}^{\ell_1} \cdots \sum_{\ell_{k-i}=1}^{\ell_{k-i-1}} A_F^{k-1}Bu \quad ,$$

$$a_2 := \sum_{\ell_1=1}^{i} \sum_{\ell_2=1}^{\ell_1} \cdots \sum_{\ell_{k-i-1}=1}^{\ell_{k-i-2}} A_F^{k-2}Bu \ .$$

By LEMMA 3.28, we have $\lim_{n\to\infty} n^{k-i}\, d(\mathcal{L}', x_i(n,u)) < \infty$. However, by (2.10) we also have

$$d(\mathcal{L}'',x_i(n,u)) = d(\mathcal{L}'', \frac{(-1)^{k-i-1}}{n^{k-i-1}} a_2 + \frac{(-1)^{k-i}}{n^{k-i}} a_1 + \frac{(-1)^{k-i+1}}{n^{k-i+1}} v(n)).$$

Using the triangular inequality, this implies that also

$$\lim_{n\to\infty} n^{k-i-1}\, d(\mathcal{L}'', x_i(n,u)) < \infty.$$

Having established the existence of the above two limits, the proof of the theorem may then be completed along the lines of the proof of TH. 3.27. Again, a detailed proof can be found in TRENTELMAN (1984).

□

Finally, we need to prove an extension of LEMMA 3.30. It is here that we will use the results on exact disturbance decoupling with output stability from SECTION 4.2. Let us however first recall what we have established so far. First, we have shown that if either $R^*(K_1) = \{0\}$ or $K_2 = \{0\}$, then $V_b(K_1,K_2)$ can be decomposed into the direct sum of $V_g(K_1,K_2)$ and a number of singly generated almost controllability subspaces $\mathcal{L}(u, F, k)$ such that $\mathcal{L}(u, F, k-1) \subset K_1$ and $\mathcal{L}(u, F, k-2) \subset K_2$. Moreover, we have proven a theorem on the canonical approximation of $\mathcal{L}(u, F, k)$ by controlled invariant subspaces.

In the following, denote $V_b := V_b(K_1,K_2)$, etc. . Recall that in this section $\mathbb{C}_g = \mathbb{C}^-$. The reason for this is that in the definition of V_b we have given, this choice of stability set assures that $V_p \subset V_b$ and $H \subset V_b$ (see SECTION 4.3). Accordingly, LEMMA 4.19 and LEMMA 4.21 are only valid if the subspace V_g appearing in the statements of these lemmas is taken with with respect to the stability set $\mathbb{C}^-$. The following lemma is however also valid if we replace $\mathbb{C}^-$ by an arbitrary symmetric subset of $\mathbb{C}$.

LEMMA 4.24. *Consider the system* (A,B). *Let* Λ *be a symmetric set of* $\dim[\langle A|B\rangle + V_g] - \dim V_b$ *complex numbers. Then there exists a subspace* S *and, for every mapping* $F_o \in \underline{F}(V^*(K_2))$, *a mapping* $F_1 : X \to U$ *such that*

$$(4.28) \qquad F_1|V^*(K_2) = F_o|V^*(K_2),$$

$$(4.29) \qquad (A+BF_1)V_g \subset V_g,$$

$$(4.30) \qquad \sigma(A+BF_1|V_g/V^*(K_2)) \subset \mathbb{C}^-,$$

$$(4.31) \qquad V_b \oplus S = V_g + \langle A|B\rangle,$$

$$(4.32) \qquad (A+BF_1)(V_g \oplus S) \subset V_g \oplus S,$$

$$(4.33) \qquad \sigma(A+BF_1|(V_g \oplus S)/V_g) = \Lambda.$$

PROOF: Let $F_o \in \underline{F}(V^*(K_2))$. By TH. 4.7, there is a $F \in \underline{F}(V_g)$ such that $\sigma(A_F|V_g/V^*(K_2)) \subset \mathbb{C}^-$. The rest of the proof follows closely the lines of the proof of TH. 2.48. Let $P : X \to X/V_g$ denote the canonical projection. The idea is to apply TH. 2.39 to the system with system mapping $\hat{A} := A_F|X/V_g$ and input mapping $\hat{B} := PB$ to find a complement $\bar{S}$ of PV_b in the reachable space $\langle\hat{A}|\operatorname{im}\hat{B}\rangle$. This may indeed be done since $W(K_1,K_2) \in \underline{R}_a$ (see LEMMA 4.17) and, consequently, $PV_b = PW \in \underline{R}_a(\hat{A},\hat{B})$.
□

Combining our previous results we now arrive at the main result of this section. It turns out that, if $R^*(K_1) = \{0\}$ or if $K_2 = \{0\}$, then starting in the subspace $V_b(K_1,K_2)$, one can make the L_p-norm of the pointwise distance to K_1 smaller than any positive real number ε, while the L_1-norm of the distance to K_2 is dominated by a constant, independent of ε. Moreover, all this can be achieved moving along trajectories that are generated by state feedback:

THEOREM 4.25. *Let* $K_2 \subset K_1$ *be a pair of subspaces. Assume that* $R^*(K_1) = \{0\}$ *or that* $K_2 = \{0\}$. *Let* $1 < p_o < \infty$. *Then there exists a constant* $C \in \mathbb{R}$ *and for all* $\varepsilon > 0$ *a mapping* $F: X \to U$ *such that*

$$\|d(e^{A_F t}x_o, K_1)\|_p \leq \varepsilon$$

and

$$\| d(e^{A_F t} x_o, K_2) \|_1 \leq C$$

for all $x_o \in V_b(K_1,K_2)$ *with* $\|x_o\| \leq 1$ *and* $p \in [1,p_o]$.

PROOF: For both cases, $R^*(K_1) = \{0\}$ or $K_2 = \{0\}$, the proof of this result may be given by adapting in a straightforward way the proof of TH. 2.47 or that of TH.3.25, using the results obtained in this section. In the proof, a sequence of feedback mappings $\{F_n\}$ is constructed. In the case that $R^*(K_1) = \{0\}$, the mappings $A + BF_n$ have the following invariant subspaces: $V^*(K_2)$, $V_g(K_1,K_2)$, $V_g(K_1,K_2) \oplus V(n)$ (with the latter converging to $V_b(K_1,K_2)$) and finally, $V_g(K_1,K_2) + \langle A|B\rangle$. In this case, the situation with the spectrum of $A+BF_n$ is depicted in the following diagram:

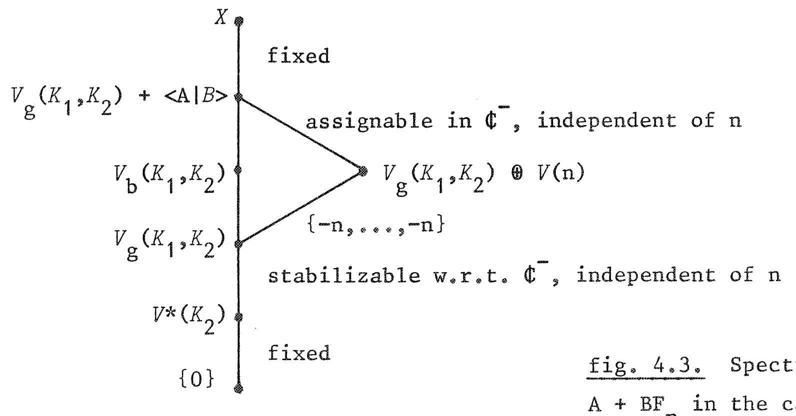


fig. 4.3. Spectrum of $A + BF_n$ in the case that $R^*(K_1) = \{0\}$.

In the case that $K_2 = \{0\}$, the subspace $V_g(K_1,K_2)$ is equal to $V_g^*(K_1)$. In this case, the mappings $A + BF_n$ can be constructed to have the following invariant subspaces: $R^*(K_1)$, $V_g^*(K_1)$, $V_g^*(K_1) \oplus V(n)$ (with the latter converging to $V_b(K_1, \{0\}) = V_g^*(K_1) + B + A(B \cap K_1)$) and, finally, $V_g^*(K_1) + \langle A|B\rangle$. The situation with the spectrum of $A + BF_n$ is depicted in the following lattice diagram:

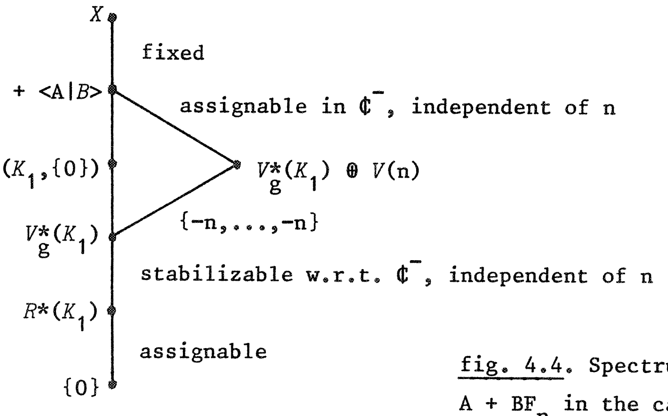


fig. 4.4. Spectrum of $A + BF_n$ in the case that $K_2 = \{0\}$.

Note that in both cases $\sigma(A+BF_n|(V_g \oplus V(n))/V_g) = \{-n,\ldots, -n\}$: these eigenvalues tend to 'minus infinity' as $n \to \infty$. □

As an immediate consequence of the previous result, we obtain the following conditions for the solvability of the L_p-almost disturbance decoupling problem with bounded peaking. We will first consider the case that $R^*(K_1) = \{0\}$. Consider the system (4.1) with the output equations (4.2). Denote $K_1 := \ker H_1$ and $K_2 := \ker H_2$. Then we have:

COROLLARY 4.26. *Let* $p \in \{1,2,\infty\}$. *Assume that the system* (A,B,H_1) *is left-invertible. Then* $(\text{ADDPBP})_p$ *is solvable if and only if*

$$(4.34) \qquad \operatorname{im} G \subset V_g(K_1,K_2) + B + A[R^*_b(K_2) \cap K_1].$$

PROOF: The necessity part of this result was already established in SECTION 4.3 (in fact, (4.34) is a necessary condition even without the premisse that (A,B,H_1) is left-invertible). To prove that (4.34) is a sufficient condition if (A,B,H_1) is left-invertible, first note from TH. 3.16 that (A,B,H_1) is left-invertible if and only if $R^*(K_1) = \{0\}$. The result then follows from TH. 4.25 upon noting that, for all $1 \leq p \leq \infty$, then L_p-L_p induced norm of a convolution operator is dominated by the

L_1-norm of its kernel (this shows that the subspace inclusion (4.34) is, in fact, a sufficient condition for solvability of $(\mathrm{ADDPBP})_p$ for all $1 \leq p \leq \infty$, provided that $R^*(K_1) = \{0\}$).

□

Next, we will consider the case that $K_2 = \{0\}$. Obviously, this holds in the important special case that $H_2 = I$, which corresponds to the L_p-almost disturbance decoupling problem with bounded peaking of the entire state vector. It turns out that in this case we do not need the condition that requires left-invertibility of the system (A,B,H_1):

<u>COROLLARY 4.27.</u> *Let* $p \in \{1, 2, \infty\}$. *Assume that* $K_2 = \{0\}$. *Then* $(\mathrm{ADDPBP})_p$ *is solvable if and only if*

$$\text{(4.35)} \qquad \operatorname{im} G \subset V^*_g(K_1) + B + A(B \cap K_1).$$

<u>PROOF</u>: If $K_2 = \{0\}$, then $V_b(K_1,K_2)$ is equal to the subspace on the right in (4.35). The fact that (4.35) is a necessary condition was established in SECTION 4.3. Sufficiency follows immediately from TH. 4.2 .

□

<u>REMARK 4.28</u>. In the above we have established for two cases the fact that the subspace inclusion $\operatorname{im} G \subset V_b(K_1,K_2)$ provides a necessary and sufficient condition for the solvability of $(\mathrm{ADDPBP})_p$. The first case we considered was the case that $R^*(K_1) = \{0\}$, the second case that $K_2 = \{0\}$. At this point however we make the important observation that the only reason for making these assumptions was that we wanted to obtain a direct sum decomposition of the subspace $V_b(K_1,K_2)$ into the direct sum of $V_g(K_1,K_2)$ and a number of singly generated almost controllability subspaces $\mathcal{L}(u_i, F, r_i)$ with the properties $\mathcal{L}(u_i, F, r_i-1) \subset K_1$ and $\mathcal{L}(u_i, F, r_i-2) \subset K_2$. The conditions $R^*(K_1) = \{0\}$ and $K_2 = \{0\}$ provide only two possible assumptions under which such a decomposition can indeed be established. In fact, it is easy to obtain other conditions that make it possible to decompose $V_b(K_1,K_2)$ in the above way.

We note that the subspace $V_b(K_1,K_2)$ can in principle be computed using recursive algorithms. The subspace $V_g(K_1,K_2) = V^*(K_2) + V^*_g(K_1)$ may be computed using the invariant subspace algorithm ISA (SECTION 1.6) and a construction as in WONHAM (1979, p. 114). The subspace $R^*_b(K_2)$ can be calculated via the algorithm (ACSA)'(SECTION 3.2). Also the sequences of feedback mappings $\{F_n\}$ required may be obtained constructively using the method of approximating $V_b(K_1,K_2)$ by a sequence of controlled invariant subspaces. We will present an illustrative worked example in SECTION 4.6.

It is noted that the condition for solvability of the 'ordinary' L_p-almost disturbance decoupling problem $(\mathrm{ADDP})_p$ is recovered from COR. 4.26 by taking $K_1 = K_2 = K$. Indeed, since $V_g(K,K) = V^*(K)$ and since $B + A[R_b^*(K) \cap K] = B + AR_a^*(K) = R_b^*(K)$, we may conclude that, under the assumption $R^*(K) = \{0\}$, the inclusion $\mathrm{im}\ G \subset V_b^*(K)$ is a necessary and sufficient condition for solvability of $(\mathrm{ADDP})_p$ for $p \in \{1, 2, \infty\}$ (of course, this fact was already established in SECTION 3.3 without the condition $R^*(K) = \{0\}$).

Another implication of COR. 4.26. and COR. 4.27 is that for any given pair of subspaces $K_2 \subset K_1$ such that either $R^*(K_1) = \{0\}$ or $K_2 = \{0\}$, the subspaces $V_p(K_1,K_2)$ (DEF. 4.11) are the same for all $p \in [1,\infty)$. Moreover, this one subspace also admits a H^∞ approximate frequency domain characterization (DEF. 4.12) and a characterization in terms of Bohl distributional inputs. Finally, it may be expressed in a simple way in terms of subspaces that are computable using recursive algorithms:

COROLLARY 4.29.

(i) *Let $K_2 \subset K_1$ be a pair of subspaces of X. Assume that $R^*(K_1) = \{0\}$. Then for all $p \in [1,\infty)$ we have*

$$V_p(K_1,K_2) = H(K_1,K_2) = V_b(K_1,K_2) = V_g(K_1,K_2) + B + A[R_b^*(K_2) \cap K_1] .$$

(ii) *Let K be a subspace of X. Then for all $p \in [1,\infty)$ we have*

$$V_p(K, \{0\}) = H(K, \{0\}) = V_b(K, \{0\}) = V_g^*(K) + B + A(B \cap K).$$

PROOF: (i) The inclusions $V_p \subset V_b$ and $V_p \subset H$ were proven in LEMMA 4.15. The inclusion $V_b \subset V_p$ follows from TH. 4.25 and the inclusion $V_b \subset H$ may be proven using an argument similar to that in the proof of COR.3.33. Finally, the fourth equality was established in TH. 4.16, (ii) This may be proven along the same lines.

□

We want to conclude this section with some final remarks. First, although we only discussed the space $V_p(K_1,K_2)$ for the case that $p \in [1,\infty)$, it is also possible to obtain characterizations of $V_\infty(K_1,K_2)$. The main distinction of this space is, that it is contained in K_1, which is of course not the case if $p \in [1,\infty)$. It can be shown that if $R^*(K_1) = \{0\}$ or $K_2 = \{0\}$ then $V_\infty(K_1,K_2)$ is equal to the space of all points in K_1 that have a (ξ,ω)-representation with $\xi(s) \in K_1(s)$, $\omega(s) \in U(s)$ and $H_2\xi(s)$ proper and asymptotically stable. Since $V_b(K_1,K_2)$ is defined as the subspace of all such points in X (instead of K_1) we find that if $R^*(K_1) = \{0\}$ or $K_2 = \{0\}$, then $V_\infty(K_1,K_2) = V_b(K_1,K_2) \cap K_1$.

Next, we would like to point out that, parallel to the L_p-almost disturbance decoupling with bounded peaking as treated in this chapter, we could also have considered a version of this problem which extends the L_p/L_q-almost disturbance decoupling problem as treated in SECTION 2.6 of this tract. This would lead to the L_p/L_q *almost disturbance decoupling problem with bounded peaking*, (ADDPBP)'. This problem will be said to be solvable if there is a constant $C \in \mathbb{R}$ and for all $\varepsilon > 0$ a mapping $F : X \to U$ such that in the closed loop system with $x(0) = 0$,

$$\|z_1\|_p \leq \varepsilon \|d\|_q$$

and

$$\|z_2\|_p \leq C \|d\|_q \quad ,$$

for all $d \in L_q(\mathbb{R}^+,D)$ and *for all* $1 \leq p \leq q \leq \infty$. It may be shown that the solvability of (ADDPBP)' is equivalent to the existence of a constant C and a sequence $\{F_n\}$ such that $\|W_{1,F_n}\|_p \to 0$ $(n\to\infty)$ and $\|W_{2,F_n}\|_p \leq C$, $\forall n$, for $p = 1$ ánd $p = \infty$. The solvability of the latter problem involves the subspace of all points in K_1 that have a (ξ,ω)-representation with $\xi(s) \in K_1(s)$, $\omega(s) \in U(s)$ and $H_2\xi(s)$ *strictly proper* and asymptotically stable:

$$V_a(K_1,K_2) := \{x_0 \in K_1 \mid \exists\, u \in D_B'^m \text{ such that } x^+(x_0,u) \text{ lies in } K_1,\ \sigma(x^+(x_0,u)/K_2) \subset \mathbb{C}^- \text{ and } \operatorname{ord} x^+(x_0,u)/K_2 = -1\}.$$

A theory along the lines of the one sketched in this chapter may be developed around this subspace. It turns out that

$$V_a(K_1,K_2) = V_g(K_1,K_2) + [R_b^*(K_2) \cap K_1] ,$$

and that if $R^*(K_1) = \{0\}$ or if $K_2 = \{0\}$, then (ADDPBP)' is solvable if and only if im $G \subset V_a(K_1,K_2)$. Again, the solvability conditions for (ADDP)' (see SECTION 2.6) are recovered from this result by taking $K_1 = K_2 = K$. The subspace $V_a(K_1,K_2)$ will also appear in section 4.7 in the context of the problem of perfect regulation.

We conclude this section with a lattice diagram in which the interrelations between the various 'V'-spaces discussed is depicted. The spaces $V_g^*(K_2)$, $V_g^*(K_1)$ and $V_g(K_1,K_2)$ are taken relative to the stability set $\mathbb{C}_g = \mathbb{C}^-$.

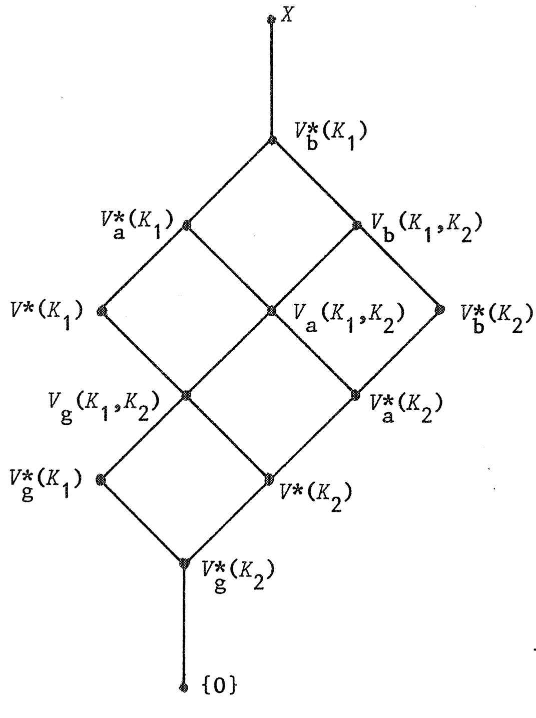


fig. 4.5. Interrelations of 'V' -spaces.

4.5 BOUNDED PEAKING AND SPECTRAL ASSIGNABILITY

In this section we will briefly discuss how the results of the previous sections lead to necessary and sufficient conditions for the solvability of the L_p-almost disturbance decoupling problem with pole placement as defined in DEF. 4.2. In order to prevent confusion between the stability sets that will appear in the formulations in this section, we will denote the subspace $V_g(K_1,K_2)$ taken with respect to the stability set $\mathbb{C}^-$ by $V_-(K_1,K_2)$. All other stability sets in this section will be denoted by $\mathbb{C}_g$ and will always be assumed to satisfy (2.27) and (2.28), i.e. to be symmetric with respect to the real axis and to contain an interval $(-\infty, c]$ for some $c \in \mathbb{R}$.

As before, the solvability of $(\text{ADDPBPPP})_p$ may be expressed in terms of the closed loop impulse response matrices:

<u>LEMMA 4.30</u>. *Let* $p \in \{1,\infty\}$. *Then* $(\text{ADDPBPPP})_p$ *is solvable if and only if there exists a constant* $C \in \mathbb{R}$ *and for all* $\varepsilon > 0$ *and all* $\mathbb{C}_g$ *a mapping* $F : X \to U$ *such that* $\|W_{1,F}\|_1 \leq \varepsilon$, $\|W_{2,F}\|_1 \leq C$ *and* $\sigma(A_F) \subset \mathbb{C}_g$. $(\text{ADDPBPPP})_2$ *is solvable if and only if there exists a constant* $C \in \mathbb{R}$ *and for all* $\varepsilon > 0$ *and all* $\mathbb{C}_g$ *a mapping* $F : X \to U$ *such that* $\|\hat{W}_{1,F}\|_\infty \leq \varepsilon$, $\|\hat{W}_{2,F}\|_\infty \leq C$ *and* $\sigma(A_F) \subset \mathbb{C}_g$. □

Our discussion on the conditions for solvability of the above problems should therefore obviously involve the following spaces. For $1 \leq p < \infty$ we define:

$$R_p(K_1,K_2) := \{x_o \in X \mid \exists\ C \in \mathbb{R} \text{ and } r \in \mathbb{N} \text{ such that } \forall\, \varepsilon > 0 \text{ and } \forall\, \mathbb{C}_g\ \exists\, x \in \Sigma_g^{B(r)}(A,B) \text{ with } x(0) = x_o,\ \|d(x,K_1)\|_p \leq \varepsilon \text{ and } \|d(x,K_2)\|_1 \leq C\}.$$

Furthermore, we define

$$H(K_1,K_2) := \{x_o \in X \mid \exists\ C \in \mathbb{R} \text{ and } r \in \mathbb{N} \text{ such that } \forall\, \varepsilon > 0 \text{ and } \forall\, \mathbb{C}_g\ \exists\ (\xi,\omega)\text{-representation with } \xi(s) \in X_+(s),\ \omega(s) \in U_+(s),\ \deg \xi \leq r,\ \sigma(\xi) \subset \mathbb{C}_g,\ \|d(\xi,K_1)\|_\infty \leq \varepsilon \text{ and } \|d(\xi,K_2)\|_\infty \leq C\}.$$

A point x_o lies in $R_p(K_1,K_2)$ if, starting in x_o, we can make the L_p-norm of the distance to K_1 smaller than any positive real number ε, while moving along Bohl trajectories with an upper bound to their McMillan degree and characteristic values located arbitrarily in $\mathbb{C}$. The L_1-norms of the distances to K_2 are dominated by a constant, independent of ε. From the results in SECTION 3.4 we see that $R^*_b(K_2) \subset R_p(K_1,K_2) \subset R^*_b(K_1)$ for all $p \in [1,\infty)$. It is also immediate that $R^*(K_1) \subset R_p(K_1,K_2)$. The same inclusions hold for $H(K_1,K_2)$. Of course, both R_p and S are contained in $V_b(K_1,K_2)$.

By definition, if $p \in \{1,\infty\}$ then a necessary condition for solvability of $(\text{ADDPBPPP})_p$ is that im $G \subset R_1(K_1,K_2)$ (i.e. R_p with $p = 1$). Also, a necessary condition for solvability of $(\text{ADDPBPPP})_2$ is that im $G \subset H(K_1,K_2)$. Of course, a necessary condition in all cases is that (A,B) is controllable. In the sequel it will be shown that, provided that one of the two assumptions $R^*(K_1) = \{0\}$ or $K_2 = \{0\}$ holds, the subspace $H(K_1,K_2)$ and all subspaces $R_p(K_1,K_2)$ for $p \in [1,\infty)$ are equal to one and the same subspace $R_b(K_1,K_2)$, where

$$(4.37) \qquad R_b(K_1,K_2) := R^*(K_1) + B + A[R^*_b(K_2) \cap K_1].$$

It will also turn out that, again if $R^*(K_1) = \{0\}$ or if $K_2 = \{0\}$, then controllability of (A,B) together with the inclusion im $G \subset R_b(K_1,K_2)$ provide sufficient conditions for the solvability of $(\text{ADDPBPPP})_p$. The following result is obtained in a fairly simple way:

<u>LEMMA 4.31</u>. *Let* $p \in [1,\infty)$. *Then* $R_p(K_1,K_2) \subset R_b(K_1,K_2)$ *and* $H(K_1,K_2) \subset R_b(K_1,K_2)$.

<u>PROOF:</u> Denote $R_p := R_p(K_1,K_2)$, $V_- := V_-(K_1,K_2)$, $W := W(K_1,K_2)$, etc. We have $R_p \subset V_b \cap R^*_b(K_1) = (V_- + W) \cap R^*_b(K_1)$. Since $W \subset R^*_b(K_1)$. this equals $(V_- \cap R^*_b(K_1)) + W$. We contend that $V_- \cap R^*_b(K_1) = R^*(K_1)$. To show this, let $\tilde{R}^*_b(K_1)$ be as in SECTION 4.4. Then we have (see also the proof of LEMMA 3.29)

$$\begin{aligned} V_- \cap R^*_b(K_2) &= V_- \cap [R^*(K_1) \oplus \tilde{R}^*_b(K_1)] \\ &= R^*(K_1) + (V_- \cap \tilde{R}^*_b(K_1)). \end{aligned}$$

Since $V_- \subset V^*(K_1)$, and since $\tilde{R}^*_b(K_1) \cap V^*(K_1) = \{0\}$, we have $V_- \cap \tilde{R}^*_b(K_1) = \{0\}$. We conlude that $R_p \subset R^*(K_1) + W = R_b$. The inclusion $H \subset R_b$ follows in the same way.

□

From the above, the subspace inclusion $\operatorname{im} G \subset R_b(K_1,K_2)$ is a necessary condition for solvability of $(\text{ADDPBPPP})_p$ for the cases $p = 1$, $p = 2$ and $p = \infty$. Our next result will lead to sufficient conditions:

<u>THEOREM 4.32.</u> *Let* $K_2 \subset K_1$ *be a pair of subspaces. Assume that* $R^*(K_1) = \{0\}$ *or that* $K_2 = \{0\}$. *Let* $1 \leq p_o < \infty$. *Then there exists a constant* C *and for all* $\varepsilon > 0$ *and for every* $\mathbb{C}_g$ *a mapping* $F : X \to U$ *such that*

$$\|d(e^{A_F t}x_o, K_1)\|_p \leq \varepsilon$$

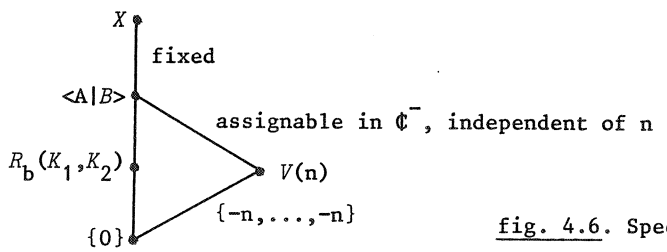
and

$$\|d(e^{A_F t}x_o, K_2)\|_1 \leq C \text{ for all } x_o \in R_b^*(K_1,K_2) \text{ with } \|x_o\| \leq 1 \text{ and } p \in [1,p_o],$$

while

$$\sigma(A + BF\,|\,\langle A|B\rangle) \subset \mathbb{C}_g.$$

<u>PROOF:</u> If $R^*(K_1) = \{0\}$, then $R_b(K_1,K_2) = W(K_1,K_2)$ as defined in the previous section. It was shown that, under the assumption that $R^*(K_1) = \{0\}$, $W(K_1,K_2)$ has a direct sum decomposition into a number of singly generated almost controllability subspaces $\mathcal{L}(u_i, F, r_i)$ with $\mathcal{L}(u_i, F, r_i-1) \subset K_1$ and $\mathcal{L}(u_i, F, r_i-2) \subset K_2$. As in the proof of TH. 4.25 the proof can be given by approximating these singly generated almost controllability subspaces by controlled invariant subspaces. In the proof, a sequence of feedback mappings $\{F_n\}$ is contructed such that $A + BF_n$ has as invariant subspaces $V(n)$ (which converges to $R_b(K_1,K_2)$) and $\langle A|B\rangle$. This yields the following lattice diagram:



<u>fig. 4.6.</u> Spectrum of $A + BF_n$ in the case that $R^*(K_1) = \{0\}$.

If $K_2 = \{0\}$ then it can be shown as in LEMMA 4.20 that $R_b(K_1,K_2) = R^*(K_1) \oplus \tilde{B} \oplus A(\tilde{B} \cap K_1)$, with $\tilde{B} \subset B$ such that $\tilde{B} \oplus (B \cap V^*(K_1)) = B$. This leads to a direct sum of $R^*(K_1)$ together with a number of one and two-dimensional singly generated almost controllability subspaces. Approximating the latter in the canonical way by controlled invariant subspaces leads to a sequence of feedback mappings $\{F_n\}$ such that $A + BF_n$ has the following invariant subspaces: $R^*(K_1)$, $R^*(K_1) \oplus V(n)$ (with $R^*(K_1) \oplus V(n)$ comverging to $R^*(K_1) \oplus \tilde{B} \oplus A(\tilde{B} \cap K_1)$) and $\langle A|B\rangle$. The situation with the spectrum of $A + BF_n$ is as follows:

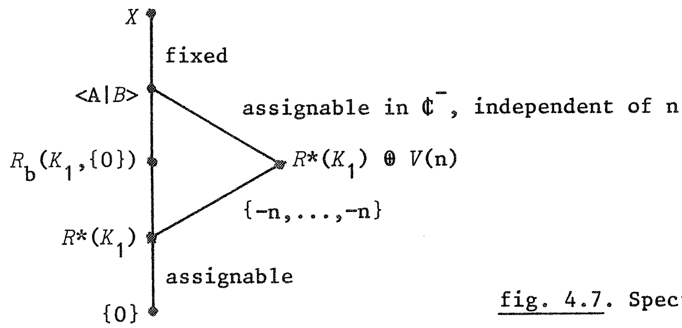


fig. 4.7. Spectrum of $A + BF_n$ in the case that $K_2 = \{0\}$.

□

The following result now provides necessary and sufficient conditions for solvability of $(\text{ADDPBPPP})_p$ for the case that the system (A,B,H_1) is left-invertible. Again, denote $K_1 := \ker H_1$ and $K_2 := \ker H_2$:

COROLLARY 4.33. *Let* $p \in \{1,2,\infty\}$ *and assume that* (A,B,H_1) *is left-invertible. Then* $(\text{ADDPBPPP})_p$ *is solvable of and only if*

$$\operatorname{im} G \subset B + A[R_b^*(K_2) \cap K_1]$$

and (A,B) *is controllable.*

PROOF: This follows immediately from the foregoing, using the fact that left-invertibility of (A,B,H_1) is equivalent to $R^*(K_1) = \{0\}$.

□

The next corollary deals with the case that $K_2 = \{0\}$. Again, in this case no invertibility assumptions are required (compare with COR. 4.27):

COROLLARY 4.34. *Let* $p \in \{1, 2, \infty\}$ *and assume that* $K_2 = \{0\}$. *Then* $(\text{ADDPBPPP})_p$ *is solvable if and only if*

$$\operatorname{im} G \subset R^*(K_1) + B + A(B \cap K_1).$$

□

Our final result establishes the equalities between the various subspaces introduced in the beginning of this section:

COROLLARY 4.35.

(i) *Let* $K_2 \subset K_1$ *be a pair of subspaces of* X *and assume that* $R^*(K_1) = \{0\}$. *Then for all* $p \in [1,\infty)$ *we have*

$$R_p(K_1,K_2) = H(K_1,K_2) = B + A[R^*_b(K_2) \cap K_1].$$

(ii) *Let* K *be a subspace of* X. *Then for all* $p \in [1,\infty)$ *we have*

$$R_p(K, \{0\}) = H(K, \{0\}) = R^*(K) + B + A(B \cap K).$$

□

To conclude this section, we want to make some remarks. First, we note that the subspaces $R_b(K_1,K_2)$, defined by (4.37) can also be characterized in terms of Bohl distributions of (ξ,ω)-representations. We will confine ourselves to simply stating the result. Indeed, $R_b(K_1,K_2)$ consists exactly of those points in X that for each symmetric subset $\mathbb{C}_g$ of $\mathbb{C}$ with the property that $\mathbb{C}_g \cap \mathbb{C}^- = \emptyset$ have a (ξ,ω)-representation with $\xi(s) \in K_1(s)$, $\omega(s) \in U(s)$, $\sigma(\xi) \subset \mathbb{C}_g$ and $H_2\xi(s)$ proper and asymptotically stable (i.e. $\sigma(\xi(s)/K_2) \subset \mathbb{C}^-$). It is also possible to characterize $R_b(K_1,K_2)$ in terms of its finite time controllability properties: starting in a point $x_0 \in R_b(K_1,K_2)$ one may travel to the origin in a given finite time, moving along regular trajectories in such a way that the L_p-norms of the distances to K_1 are arbitrarily small, while the L_1-norms of the distances to K_2 remain bounded. Of course in order to achieve this behaviour, one has to leave the realm of regular *Bohl* trajectories. In order to obtain a rigorous proof of this characterization, one has to apply methods different from those used in this chapter (the methods used in this chapter are based heavily on the fact that everything is rational, see LEMMA 4.13 and the proof of LEMMA 4.14). It is however possible to give a proof of the above-mentioned property of $\mathcal{R}_b(\mathcal{K}_1,\mathcal{K}_2)$ based on the properties of the factor system modulo $\mathcal{S}^\infty$ (see SECTION 1.4).

Again, conditions for solvability of the L_p-almost disturbance decoupling problem with pole placement, $(\text{ADDPPP})_p$ (see SECTION 3.4) may be recovered from COR. 4.33 by taking $\mathcal{K}_1 = \mathcal{K}_2 = \mathcal{K}$.

Finally, it is possible to define the L_p/L_q version of the problem discussed in this section. This problem, denoted by (ADDPBPPP)', is obtained by adding the requirement of spectral assignability to the problem formulation of (ADDPBP)' (see the concluding remarks at the end of the previous section). It may be shown that if (A,B,H_1) is left-invertible then this problem is solvable if and only if $\text{im } G \subset \mathcal{R}_b^*(\mathcal{K}_2) \cap \mathcal{K}_1$ and (A,B) is controllable. If $\mathcal{K}_2 = \{0\}$, then it is solvable if and only if $\text{im } G \subset \mathcal{R}^*(\mathcal{K}_1) + \mathcal{B} \cap \mathcal{K}_1$. In fact, both subspaces appearing here are special cases of the subspace $\mathcal{R}_a(\mathcal{K}_1,\mathcal{K}_2)$ defined by

$$\mathcal{R}_a(\mathcal{K}_1,\mathcal{K}_2) := \mathcal{R}^*(\mathcal{K}_1) + [\mathcal{R}_b^*(\mathcal{K}_2) \cap \mathcal{K}_1] \ .$$

This subspace is the 'controllability' analogue of the subspace $\mathcal{V}_a(\mathcal{K}_1,\mathcal{K}_2)$ defined by (4.36). Using methods similar to those in SECTION 4.3 it can be proven that the inclusion $\text{im } G \subset \mathcal{R}_a(\mathcal{K}_1,\mathcal{K}_2)$ *is always a necessary condition for solvability of* (ADDPBPPP)'. We conclude this section with two lattice diagrams that display the interrelations between the various subspaces:

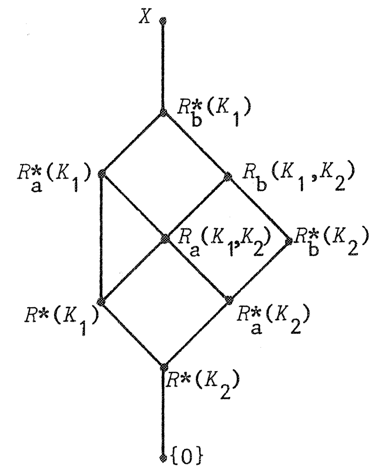


fig. 4.8. Interrelations of '$\mathcal{R}$'-spaces.

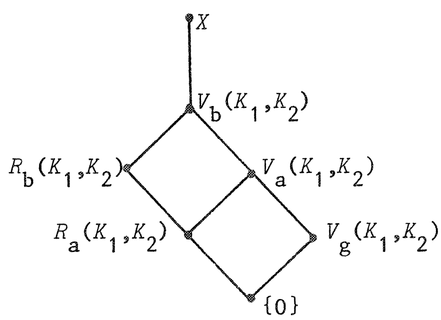


fig. 4.9. Interrelations of two-output constrained subspaces.

4.6 A WORKED EXAMPLE

To illustrate the theory developed in this chapter and to demonstrate its computational feasability, in this section we will present a worked example. We will consider a linear system with two outputs, and check whether $(\mathrm{ADDPBP})_p$ is solvable for this system. Next, we will actually compute a sequence of feedback mappings that achieves our design purpose. The system that will be considered is given by $\dot{x}(t) = Ax(t) + Bu(t) + Gd(t)$, $z_1(t) = H_1x(t)$, $z_2(t) = H_2x(t)$, with

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad H_1 = (0\ 0\ 0\ 1\ 0), \quad H_2 = I_{5\times 5}$$

and

$$G = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} .$$

Thus, $X = \mathbb{R}^5$ and $U = \mathbb{R}^2$. Denote $K_i = \ker H_i$. The route that we will take is as follows. First, we will check whether the subspace inclusion (4.35) holds, to see if $(\mathrm{ADDPBP})_p$ is solvable. It will turn out that this is indeed true. After this, we will follow closely the the lines of the development in SECTION 4.4 and construct a required sequence $\{F_n\}$. As before, $\mathbb{C}_g = \mathbb{C}^-$ and the subspaces $V_g^*(K_1)$ and $V_g(K_1,K_2)$ are taken with respect to this stability set. Let the standard basis vectors in $\mathbb{R}^5$ be denoted by e_i and those in $\mathbb{R}^2$ by $\tilde{e}_i$.

Using the algorithm ISA (1.46) and a construction as in WONHAM (1979, p. 114), we calculate that $V_g(K_1,K_2) = V_g^*(K_1) = \mathrm{span}\,\{e_1,e_2\}$ (since $V^*(K_2) = \{0\}$). Thus, by TH. 4.8, DDPOS is not solvable for the above system. Since $K_2 = \{0\}$, by COR. 4.27 we should check if the subspace inclusion $\mathrm{im}\,G \subset V_g^*(K_1) + B + A(B \cap K_1)$ holds. It may be calculated that $V_g^*(K_1) + B + A(B \cap K_1) = \mathrm{span}\,\{e_1, e_2, e_4, e_5\}$. Since im G is indeed contained in this subspace , $(\mathrm{ADDPBP})_p$ is solvable for all $1 \leq p \leq \infty$. Unfortunately, $(\mathrm{ADDPBPPP})_p$ is *not* solvable because (A,B) is an uncontrollable pair. We will now construct a required sequence of feedback mappings:

step 1: decomposition. We decompose $V_b = V_g \oplus \tilde{W}$, with $\tilde{W} := \tilde{B} + A(\tilde{B} \cap K_1)$ and $\tilde{B}$ such that $\tilde{B} \oplus (B + V^*(K_1)) = B$. Then $\tilde{W} = \text{span}\{e_4, e_5\}$. Since $e_5 \in B$ and $e_4 = Ae_5$, $\tilde{W}$ is equal to the 2-dimensional singly generated almost controllability subspace $\mathcal{L}(\tilde{e}_2, 0, 2)$ (= span $\{B\tilde{e}_2, AB\tilde{e}_2\}$). Note that indeed $\mathcal{L}(\tilde{e}_2, 0, 1) = \text{span}\{e_5\} \subset K_1$ and that, trivially, $\mathcal{L}(e_2, 0, 0) = \{0\} \subset K_2$.

step 2: approximation. Next, we look for the canonical sequence $\mathcal{L}_n := \mathcal{L}_n(\tilde{e}_2, 0, 2) = \text{span}\{x_1(n, \tilde{e}_2), x_2(n, \tilde{e}_2)\}$ of controlled invariant subspaces converging to $\mathcal{L}(\tilde{e}_2, 0, 2)$. We calculate $x_1(n, \tilde{e}_2) = (I + n^{-1}A)^{-1}B\tilde{e}_2 = (0, 0, -n^{-2}, -n^{-1}, 1)^T$ and $x_2(n, \tilde{e}_2) = (I + n^{-1}A)^{-1}x_1(n, \tilde{e}_2) = (0, 0, 0, 1, 0)^T$. For each n, we define a mapping $\tilde{F}_n$ on $\mathcal{L}_n$ by $\tilde{F}_n x_1(n, \tilde{e}_2) := -n^{-1}\tilde{e}_2$ and $\tilde{F}_n x_2(n, \tilde{e}_2) := -n^{-2}\tilde{e}_2$.

step 3: a feedback mapping outside $\mathcal{L}_n$: In this step we look, given a symmetric set $\Lambda \subset \mathbb{C}^-$ of $\dim[\langle A|B\rangle + V_g] - \dim V_b$ complex numbers, for a mapping F and a subspace S such that (4.28) to (4.33) are satisfied. Since $\langle A|B\rangle + V_g = \mathbb{R}^5$, Λ will consist of one element. It may be verified that with $\Lambda = \{-3\}$, $S = \text{span}\{(0, 0, 1, -3, 9)\}$ and F given by

$$F = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix},$$

we have $A_F V_g \subset V_g$, $V_b \oplus S = \mathbb{R}^5$, $A_F(V_g \oplus S) \subset V_g \oplus S$ and $\sigma(A_F|(V_g \oplus S)/S) = \{-3\}$. We calculate that $\sigma(A_F\, V_g) = \{-1, -1\}$.

step 4: definition of the required sequence $\{F_n\}$: We have now decomposed $\mathbb{R}^5$ into the direct sum $V_g \oplus \mathcal{L}_n \oplus S$. Next, define for each n a mapping $F_n : \mathbb{R}^5 \to \mathbb{R}^2$ by $F_n|V_g \oplus S := F|V_g \oplus S$ (as defined in step 4) and $F_n|\mathcal{L}_n := \tilde{F}_n|\mathcal{L}_n$. The matrix of F_n in the standard bases of $\mathbb{R}^5$ and $\mathbb{R}^2$ is calculated to be

$$F_n = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & f_{23}(n) & -n^2 & f_{25}(n) \end{pmatrix},$$

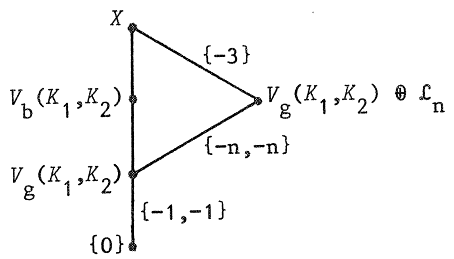
with $f_{23}(n) = \dfrac{-27n^2 + 18n^3 - 3n^4}{n^2 + 9}$ and $f_{25}(n) = \dfrac{27 - 3n^2 - 2n^3}{n^2 + 9}$

Evaluating $A + BF_n$ in the basis suggested by the decomposition $\mathbb{R}^5 = V_g \oplus \mathcal{L}_n \oplus S$, we may calculate the closed loop impulse response matrices from d to z_1 and z_2 respectively:

$$W_{1,n}(t) := H_1 e^{(A+BF_n)t}\, G = (nt+1)e^{-nt} \;,$$

$$W_{2,n}(t) := H_2 e^{(A+BF_n)t}\, G = \begin{pmatrix} 0 \\ 0 \\ t \\ nt+1 \\ -n^2 t \end{pmatrix} e^{-nt} \;.$$

Straightforward calculation shows that $\|W_{1,n}\|_1 = \frac{2}{n}$ and that $\|W_{2,n}\|_1 \leq 1 + \frac{2}{n} + \frac{1}{n^2}$ (in the latter,we have integrated the standard Euclidean norm of $W_{2,n}(t)$). From this, it is seen that that indeed for every $1 \leq p \leq \infty$ the $L_p - L_p$ induced norm of the closed loop operator from d to z_1 tends to zero as $n \to \infty$ and that the induced norm of the operator from d to z_2 is bounded with respect to n. Note that $\|F_n\| \to \infty$ as $n \to \infty$. Finally, the distribution of the closed loop eigenvalues over the various invariant subspaces is depicted in the following diagram:



4.7 PERFECT REGULATION WITH L_2-BOUNDED PEAKING

To conclude this chapter, we want to outline how the methods developed here can be adapted and extended to tackle the problem of *perfect regulation*. The origins of this problem lie in the realm of the nearly singular linear quadratic or *cheap control problem*. In the latter context the perfect regulation problem may be formulated as follows. Given a system $\dot{x}(t) = Ax(t) + Bu(t)$, $x(0) = x_o$, $z(t) = Hx(t)$ and a family of quadratic performance criteria $J_\varepsilon(x_o) = \|z\|_2^2 + \varepsilon^2\|u\|_2^2$, with $\varepsilon > 0$, let the optimal state feedback control law (associated with a particular ε)

be given by F_ε. Denote the corresponding optimal cost by $J^*_\varepsilon(x_o)$. Then perfect regulation is said to be achieved for this system if for every $x_o \in X$ the optimal cost $J^*_\varepsilon(x_o)$ converges to zero as ε tends to zero. The motivation behind this problem formulation is the following. In the above, an initial condition $x_o \neq 0$ should be interpreted as a *disturbance* and it should be understood that it is desired to keep the output variable $z(t)$ at a *nominal value zero*. Thus, in the presence of a disturbance $x(0) = x_o$, it is required to find a state feedback control law such that in the closed loop system with $x(0) = x_o$, $z(t) \to 0$ as $t \to \infty$. Solving the above linear quadratic problem for a fixed value of $\varepsilon > 0$ provides a method to find such state feedback control law. Indeed, in the problem formulation it is implicitly assumed that the closed loop output $z(t)$ will lie in $L_2(\mathbb{R}^+, Z)$, which automatically yields the desired convergence to zero as t tends to infinity†. Now, we still have a degree of freedom brought into the problem by introducing the parameter ε. Since in a sense ε weights the amount of control energy in the criterion, it is expected that in certain cases the convergence properties of $z(t)$ can be improved by decreasing the parameter ε. Let $z_\varepsilon(x_o)$ denote the output of the closed loop system when the optimal control law $u = F_\varepsilon x$ is applied to the system with initial condition $x(0) = x_o$. Then we have the following inequality:

$$\|z_\varepsilon(x_o)\|_2^2 \leq J^*_\varepsilon(x_o).$$

We conclude that if perfect regulation is achieved (in the sense of the definition above) then by decreasing ε, the output $z_\varepsilon(x_o)$ can be forced to return back to its nominal value zero asymptotically *with any desired 'speed'* (and, in fact, in the limit for $\varepsilon \to 0$, $z_\varepsilon(x_o)$ will become a jump from Hx_o to 0). In this form, the problem of perfect regulation was studied e.g. in KWAKERNAAK & SIVAN (1972), JAMESON & O'MALLEY (1975) and FRANCIS (1979) (see also FUJI (1982) and WILLEMS & WILLEMS (1983, APP. B)).

In the above, due to the decrease in the weighting of the control energy, the optimal feedback mappings F_ε will in general be unbounded, i.e. we will have $\|F_\varepsilon\| \to \infty$ as $\varepsilon \to 0$. Again, this will result in the phenomenon that we already discussed in the introduction to this chapter: certain state variables will become 'too large' as ε becomes small. This observation leads to the formulation of problems of perfect regulation with bounded peaking as studied in FRANCIS & GLOVER (1978).

† since the optimal closed loop output z(.) is generated by state feedback and is therefore Bohl.

Of course, the use of linear quadratic techniques in the above context is only one way to handle the problem of 'speeding up output response'. One might even argue that using these techniques is somewhat artificial and that one should consider the problem from a more 'structural' point of view. In fact, this was recognized in KIMURA (1981). There the perfect regulation problem was posed purely in terms of finding state feedback mappings such that the closed loop output has arbitrarily small L_2-norm (with, in addition, an internal stability constraint). Here, we will follow the latter line of thought and discuss briefly a few problems of perfect regulation that fit nicely in the framework developed in this tract. Due to space limitations, most of the details will be omitted. We consider the system $\dot{x}(t) = Ax(t) + Bu(t)$, $z(t) = Hx(t)$. If F is a mapping from X to U we denote $W_F(t) := He^{A_F t}$. The first problem we consider is a plain regulation problem without any stability requirements: PPR, *the problem of perfect regulation*, will be said to be solvable if for all $\varepsilon > 0$ there exists $F : X \to U$ such that $\|W_F\|_2 \leq \varepsilon$. The following is a version of this problem with spectral assignability: PPRPP, *the problem of perfect regulation with pole placement*, will be said to be solvable if for all $\varepsilon > 0$ and for all $\mathbb{C}_g$ (satisfying (2.27) and (2.28) there is $F : X \to U$ such that $\|W_F\|_2 \leq \varepsilon$ and $\sigma(A_F) \subset \mathbb{C}_g$. Conditions for solvability of these problems are easily obtained by combining the results of TH. 3.6, TH. 3.25. COR. 3.37 and TH. 3.36 : PPR is solvable if and only if $X = V_b^*(\ker H)$ (or, equivalently, if and only if (A,B,H_1) is right-invertible, see TH. 3.15). Moreover , PPRPP is solvable if and only if $X = R_b^*(\ker H)$ (or, equivalently, if and only if (A,B,H_1) has a polynomial right inverse). Note that if $X = R_b^*(\ker H)$, then the pair (A,B) is automatically controllable.

In KIMURA (1981) (see also KIMURA (1982)), a version of the above is considered with a constraint of internal asymptotic stability (rather than pole placement). The conditions found there involve right-invertibility, together with a minimum phase condition. Also in that paper, the problem of perfect regulation with L_2-bounded peaking is discussed. Consider the two-output system $\dot{x}(t) = Ax(t) + Bu(t)$, $z_1(t) = H_1x(t)$, $z_2(t) = H_2x(t)$. As before (see SECTION 4.1) we assume that z_2 is an enlargment of z_2, i.e., that $K_2 := \ker H_2 \subset \ker H_1 =: K_1$. In the sequel, denote $W_{i,F}(t) := H_ie^{A_F t}$. We will say that PPRBP, the *problem of perfect regulation with bounded peaking*, is solvable if there exists a constant $C \in \mathbb{R}$ and for all $\varepsilon > 0$ a mapping $F : X \to U$ such that $\|W_{1,F}\|_2 \leq \varepsilon$ and

$\|W_{2,F}\|_2 \leq C$. Moreover, we will say that PPRBPPP, the *problem of perfect regulation with bounded peaking and pole placement*, is solvable if there exists a constant $C \in \mathbb{R}$ and for all $\varepsilon > 0$ and for all $\mathbb{C}_g$ (satisfying (2.27) and (2.28)) a mapping $F : X \to U$ such that $\|W_{1,F}\|_2 \leq \varepsilon$, $\|W_{2,F}\|_2 \leq C$ and $\sigma(A + BF) \subset \mathbb{C}_g$. We will briefly outline how conditions for the solvability of these problems may be obtained. If PPRBP is solvable, then for every $x_o \in X$, one can find a constant $C \in \mathbb{R}$ and a sequence of Bohl trajectories $x_n(\cdot)$ such that $x_n(0) = x_o$, $\|H_1 x_n\|_2 \to 0$ and $\|H_2 x_n\|_2 \leq C$, $\forall n$. Denote $z_{2,n} := H_2 x_n$. Since the unit ball in L_2 is compact in the weak topology, $z_{2,n}$ has a weakly convergent subsequence $z_{2,n_m} \to z_2$ $(m \to \infty)$ with $z_2 \in L_2(\mathbb{R}^+, Z_2)$. In particular, this implies that $\langle z_{2,n_m}, e^{-st}\rangle \xrightarrow{m\to\infty} \langle z_2, e^{-st}\rangle$ for every $s \in \mathbb{C}_o^+ (=\{s \mid \mathrm{Re}\ s > 0\})$. Hence, we have pointwise convergence $\zeta_{2,n_m}(s) \to \zeta_2(s)$ of the Laplace tranforms. By LEMMA 4.12, $\zeta_2(s)$ is rational. However, being the Laplace transform of $z_2 \in L_2(\mathbb{R}^+, Z_2)$ it is also *strictly proper* and *asymptotically stable*. It may thus be proven along the lines of LEMMA 4.14 that *if* PPRBP *is solvable then* $X = T_b(K_1, K_2)$, *with*

$$T_b(K_1,K_2) := \{x_o \in X \mid \exists\ n \in D_B'^m \text{ such that } x^+(x_o,u) \text{ lies in } K_1,\ \sigma(x^+(x_o,u)/K_2) \subset \mathbb{C}^- \text{ and } \mathrm{ord}\ x^+(x_o,u)/K_2 = -1\}.$$

(Recall: the order of a distribution with support in $\mathbb{R}^+$ is defined to be -1 if it is equal to the zero-distribution). In the frequence domain, T_b consists of those points in X that have a (ξ,ω)-representation with $\xi(s) \in K_1(s)$, $\omega(s) \in U(s)$ and $H_2\xi(s)$ asymptotically stable and strictly proper. It may be shown that $T_b(K_1,K_2) = V_g(K_1,K_2) + R_b^*(K_2)$. To obtain necessary conditions for the solvability of PPRBP, we may use the ideas of SECTION 4.4. It is found that *if the system* (A, B, H_1) *is left-invertible, then* PPRBP *is solvable if and only if* $X = V_g(K_1,K_2) + R_b^*(K_2)$ (note that under the assumption that (A,B,H_1) is left-invertible, or equivalently that $R^*(K_1) = \{0\}$, this sum is a direct sum). If we assume that $K_2 = \{0\}$ (which occurs in the important case that $H_2 = I$), then the left-invertibility assumption may be omitted: *if* $K_2 = \{0\}$, *then* PPRBP *is solvable if and only if* $X = V_g^*(K_1) + B$.

It can be shown that a necessary condition for the solvability of PPRBPPP is that $X = R^*(K_1) + R_b^*(K_2)$. Moreover, *if* (A,B,H_1) *is left-invertible then* PPRBPPP *is solvable if and only if* $X = R_b^*(K_2)$. *If* $K_2 = \{0\}$, *then* PPRBPPP *is solvable if and only if* $X = R^*(K_1) + B$.

To conclude this section, note that

$$T_b(K_1,K_2) \cap K_1 = V_a(K_1,K_2),$$

as defined on page 164 of this tract and

$$[R^*(K_1) + R_b^*(K_2)] \cap K_1 = R_a(K_1,K_2) \quad ,$$

as defined in SECTION 4.5, page 171.

CHAPTER 5

ALMOST CONDITIONALLY INVARIANT SUBSPACES

The two main purposes of this chapter are, first, to introduce the dual notions of the various almost controlled invariant subspaces we considered in previous chapters and to apply the new subspaces to problems involving the design of reduced and minimal order PID-observers and, secondly, to discuss in detail the problem of almost disturbance decoupling by measurement feedback. The chapter is split up into seven sections.

In section 1, we will give definitions of the concepts of almost conditionally invariant subspace, almost observability subspace and almost detectability subspace. The definitions that we will give are in terms of the approximate invariance properties of these subspaces under output injection mappings. We will give interpretations of the new subspaces in terms of the existence of approximate observers and PID-observers. Also in this section, we will introduce the notions of infimal L_p-almost conditionally invariant and infimal L_p-almost observability subspace. The sections 2, 3 and 4 of this chapter are devoted to the problem of almost disturbance decoupling by measurement feedback. In section 2, we will introduce this problem and derive necessary and sufficient conditions for its solvability.
In contrast to the previous chapters, the methods we use here are frequency domain oriented and leave the framework of the geometric approach. In section 3, we define a version of the almost disturbance decoupling problem by measurement feedback, with a constraint on the high gain behaviour of the closed loop transfer matrices from the disturbance to the control input. This constraint is called the constraint of guaranteed roll-off. Also, section 3 contains some preliminary results concerning this new synthesis problem. In particular, we consider versions of the exact disturbance decoupling problem by state feedback and the L_p-almost disturbance decoupling problem by state feedback under the constraint of guaranteed roll-off. In section 4, we establish necessary and sufficient conditions for the solvability of the problem we posed in section 3.

Sections 5, 6 and 7 contain a discussion on the design of low order PID-observers. In section 5, we show that the classical result by Luenberger on the existence of reduced order state observers is a special case of a more general result. In fact, it is shown that the dynamic order of a state observer can be reduced even more, by allowing the observer to be a PID-observer, i.e., by allowing the observer to contain differentiators. In section 6, we generalize a result by Wonham and Morse in the context of the minimal dimension cover problem. Finally, in section 7, this result is dualized to establish the existence of minimal order PID-observers for scalar valued linear functions of the state.

5.1 DUALITY: ALMOST CONDITIONALLY INVARIANT SUBSPACES

In this section, we will dualize the basic concepts discussed in this tract and obtain the notions of almost conditionally invariant subspace, almost observability subspace and almost detectability subspace. These notions will be the duals of almost controlled invariant subspace, almost controllability subspace and almost stabilizability subspace respectively, and could, as such, be introduced by formal dualization. This procedure was for example used in WONHAM (1979, ex. 5.17) to dualize the concept of controlled invariance (see also BHATTACHARYYA (1978)). Another point of view is to define the dual concepts more intrinsically in terms of the existence of dynamic observers. In fact, in this way the concept of conditionally invariant subspace was introduced in WILLEMS & COMMAULT (1981).

Here, we prefer to take a different starting point. We will define the dual concepts rather in terms of their (approximate) invariance properties under output injection mappings. In fact, we already gave definitions of conditionally invariant subspace and detectability subspace based on these properties (see DEF. 3.48). Starting from the definitions that we will give, we will explain in what sense the subspaces introduced are related to the existence of dynamic observers. Thus, the observer interpretations will appear as consequences of our definitions, rather than as definitions themselves.

Also, in this section we will define the duals of the notions of supremal L_p-almost controlled invariant and supremal L_p-almost controllability subspace associated with a given subspace of the state space. The dual concepts will be called infimal L_p-almost conditionally invariant and L_p-almost observability subspaces. Our starting point with these subspaces will be to define them in terms of the 'normal' infimal (L_∞-) almost conditionally and almost observability subspaces containing a given subspace of the state space. Using the obvious duality arguments, we will then characterize these subspaces in terms of (L_p-) approximate invariance properties under output injection and in terms of the existence of observers.

The prevailing tenor in the present section will be to make use of the underlying duality as much as possible when we want to obtain results on the subspaces that will be introduced. Only some of the most important results will be stated separately. The reader should however keep in mind that all results obtained in this tract so far can be dualized to obtain results in the context of observer design.

Consider the observed linear time invariant flow

$$(5.1) \qquad \dot{x}(t) = Ax(t) \ , \quad y(t) = Cx(t) \ .$$

Here, as usual, $x(t)$ takes its values in the n-dimensional real linear space $\mathcal{X}$. The vector $y(t)$ takes its values in the p-dimensional real linear space $\mathcal{Y}$. A and C are linear mappings and C is assumed to be surjective. We will refer to (5.1) as the system (C,A). Recall (DEF. 3.48) that a subspace $\mathcal{S} \subset \mathcal{X}$ is called conditionally invariant (or (C,A)-invariant) if there exists a mapping $G: \mathcal{Y} \to \mathcal{X}$ such that $(A+GC)\mathcal{S} \subset \mathcal{S}$. Obviously, this inclusion is equivalent to the statement

$$d(e^{(A+GC)t}x_0, \mathcal{S}) = 0 \quad \text{for all } t \in \mathbb{R}^+ \text{ and } x_0 \in \mathcal{S} \ .$$

Allowing this distance function to be arbitrarily small by properly choosing the output injection yields:

<u>DEFINITION 5.1.</u> A subspace $\mathcal{S}_a \subset X$ will be called *almost conditionally invariant* if for all $\varepsilon > 0$ there exists a mapping $G: \mathcal{Y} \to \mathcal{X}$ such that $d(e^{(A+GC)t}x_0, \mathcal{S}_a) \leq \varepsilon$ for all $t \in \mathbb{R}^+$ and $x_0 \in \mathcal{S}_a$ with $\|x_0\| \leq 1$.

Before we continue, we will explain how almost conditionally invariant subspaces are related to observers. In the first part of this section, an observer will be a system

$$(5.2) \qquad \begin{aligned} \dot{w}(t) &= Nw(t) + My(t) \ , \\ v(t) &= Lw(t) \ , \end{aligned}$$

without a direct feedthrough term, that has the observed output of the system (5.1) as its input. The vector $v(t)$ is called the estimate. We will show that if S_a is almost conditionally invariant then for all $\varepsilon > 0$ there exists an observer with state space X and output space X/S_a such that if $w(0) = 0$ and if the initial condition x_0 in S_a satisfies $\|x_0\| \leq 1$ then the difference $\|v(t) - x(t)/S_a\|$ between the output of the observer and the state trajectory modulo S_a is less than ε for all $t \in \mathbb{R}^+$. Thus, an almost conditionally invariant subspace S_a has the property that if $x(0) \in S_a$, then $x(t)/S_a$ can be estimated arbitrarily accurately using the observed output $y(t) = Cx(t)$. To see this, let S_a be almost conditionally invariant and let $\varepsilon > 0$. Let G be as in DEF. 5.1 and define $N := A + GC$, $M := -G$. Let L be equal to the canonical projection $X \to X/S_a$. Assume $x(0) \in S_a$, $\|x_0\| \leq 1$ and $w(0) = 0$. It may be seen that the vector $x(t) - w(t)$ satisfies $\frac{d}{dt}(x-w) = (A+GC)(x-w)$, $(x-w)(0) = x(0)$. Hence we find $d(x(t)-w(t),S_a) \leq \varepsilon$ for all $t \in \mathbb{R}^+$ and consequently

$$\|x(t)/S_a - v(t)\| = \|Px(t) - Pw(t)\| = \|(x(t) - w(t))/S_a\| \leq \varepsilon$$

for all $t \in \mathbb{R}^+$. Note from the above that the distance $d(e^{(A+GC)t}x_0, S_a)$ appearing in DEF. 5.1 is a measure of the estimation error between the estimate $v(t)$ and the state trajectory modulo S_a. If S_a is conditionally invariant, then this estimation error can be kept zero for all $t \in \mathbb{R}^+$ (since in that case G may be chosen such that $(A+GC)S \subset S$).

Recall also the definition of observability subspace (see WILLEMS & COMMAULT (1981), where the slightly different terminology complementary observability subspace was used): a subspace N of X is called an observability subspace if for every symmetric subset $\mathbb{C}_g$ of $\mathbb{C}$ (see SECTION 2.5) there exists a mapping $G: Y \to X$ such that $(A+GC)N \subset N$ and $\sigma(A+GC \mid X/N) \subset \mathbb{C}_g$. In the sequel, let $\langle\ker C|A\rangle$ denote the unobservable subspace of (C,A) (i.e. the supremal A-invariant subspace in ker C: $\bigcap_{i=1}^{n} A^{-i}\ker C$). It is known that if N is an observability subspace, then we always have $\langle\ker C|A\rangle \subset N$. Moreover, a subspace N is an observability subspace if and only if for every symmetric $\mathbb{C}_g$ there exists a mapping $G: Y \to X$ such that $(A+GC)N \subset N$ and $\sigma(A+GC \mid X/\langle\ker C|A\rangle) \subset \mathbb{C}_g$. Thus, under the constraint of making N invariant by output injection we do not only have freedom of spectral assignability on X/N, but even on the larger space $X/\langle\ker C|A\rangle$. (This may be seen by dualizing SCHUMACHER (1981, TH. 1.9).) It is the latter property that will serve as the basis of our definition of almost observability subspace:

<u>DEFINITION 5.2</u>. A subspace $N_a \subset X$ will be called an almost observability subspace if for every $\mathbb{C}_g$ (satisfying (2.27) and (2.28)) and for all $\varepsilon > 0$ there exists a mapping $G: Y \to X$ such that $d(e^{(A+GC)t}x_0, N_a) \le \varepsilon$ for all $t \in \mathbb{R}^+$ and $x_0 \in N_a$ with $\|x_0\| \le 1$ and $\sigma(A+GC|X/\langle\ker C|A\rangle) \subset \mathbb{C}_g$.

Let us see what this definition means in terms of estimating the state trajectory modulo N_a. Let $\varepsilon > 0$ and let $\mathbb{C}_g$ be a subset of $\mathbb{C}$ satisfying (2.27) and (2.28) (these assumptions will be needed later on in this section when we establish the duality between almost observability subspaces and almost controllability subspaces). Let $G: Y \to X$ as in DEF. 5.2 and again define an observer by $N := A + GC$, $M := -G$ and L the canonical projection of X onto X/N_a. As before, if $x(0) \in N_a$, $\|x_0\| \le 1$ and $w(0) = 0$, then the estimation error $e(t) := x(t)/N_a - v(t)$ satisfies $\|e(t)\| \le \varepsilon$, $\forall\, t \in \mathbb{R}^+$. However, in addition we now have that *for all* initial condition pairs $(x(0),w(0))$ the spectrum of $(x-w)/\langle\ker C|A\rangle$ is contained in $\mathbb{C}_g$. In the sequel, it will be proven that every almost observability subspace N_a contains in fact the subspace $\langle\ker C|A\rangle$. Thus, in particular, the spectrum of the estimation error $e(t)$ (being equal to $(x-w)/N_a$) is contained in $\mathbb{C}_g$. We conclude that if N_a is an almost observability subspace, then for every $\varepsilon > 0$ and for every $\mathbb{C}_g$ there exists an observer (5.2) such that (i) if $x(0) \in N_a$, $\|x(0)\| \le 1$, and $w(0) = 0$ then the error $\|x(t)/N_a - v(t)\| \le \varepsilon$, $\forall\, t \in \mathbb{R}^+$, and (ii) for *every* pair of initial condition $(x(0),w(0))$ the spectrum of the estimation error $x/N_a - v$ is contained in $\mathbb{C}_g$.

If N is an observability subspace, then the estimation error can be made exactly equal to zero if $x(0) \in N$ and $w(0) = 0$, while for arbitrary initial conditions the spectrum of the error can be located arbitrarily in $\mathbb{C}$.

Finally, recall the definition of detectability subspace (see DEF. 3.48 or SCHUMACHER (1981), where the terminology outer detectability subspace was used): given a fixed symmetric subset $\mathbb{C}_g$, a subspace S_g of X is called a detectability subspace if there exists a mapping $G: Y \to X$ such that $(A+CG)S_g \subset S_g$ and $\sigma(A+GC|X/S_g) \subset \mathbb{C}_g$. Let X_{det} denote the undetectable subspace of (C,A) (i.e. the smallest detectability subspace contained in X, see also SCHUMACHER (1981, p. 26)). It can be shown that S_g is a detectability subspace if and only if there exists a mapping $G: Y \to X$ such that $(A+GC)S_g \subset S_g$ and $\sigma(A+GC|X/X_{det}) \subset \mathbb{C}_g$. Thus, under the constraint of making S_g invariant using output injection, not only the spectrum on X/S_g but also the spectrum on the (larger) space X/X_{det} can be located in $\mathbb{C}_g$. This property serves as the basis of the following:

<u>DEFINITION 5.3</u>. Let $\mathbb{C}_g$ be a subset of $\mathbb{C}$ that satisfies (2.27) and (2.28). Then a subspace $S_{a,g} \subset X$ will be called an *almost detectability subspace* if there exists a closed subset $D \subset \mathbb{C}_g$ and for all $\varepsilon > 0$ a mapping $G: Y \to X$ such that $d(e^{(A+GC)t} x_0, S_{a,g}) \leq \varepsilon$ for all $t \in \mathbb{R}^+$ and $x_0 \in S_{a,g}$ with $\|x_0\| \leq 1$ and $\sigma(A+GC|X/X_{det}) \subset D$.

The subset D enters into our considerations in order to make sure that this definition indeed yields a concept that is dual to the concept of almost stabilizability subspace (see SECTION 2.8). In terms of estimating the state trajectory modulo $S_{a,g}$, this definition leads to the existence of observers with the property that for initial conditions $w(0) = 0$ and $x(0) \in S_{a,g}$ the estimation error can be made arbitrarily small, while for arbitrary initial condition pairs $(x(0),w(0))$ the spectrum of the error is contained in $D \subset \mathbb{C}_g$ (here, the fact is used that for every almost detectability subspace $S_{a,g}$ we have $X_{det} \subset S_{a,g}$; this will be shown in the sequel). The families of all almost conditionally invariant, almost observability and almost detectability subspaces associated with (C,A) will be denoted respectively by $\underline{S}_a(C,A)$, $\underline{N}_a(C,A)$ and $\underline{S}_{a,g}(C,A)$. In the following we will establish the desired duality:

<u>THEOREM 5.4</u>. *Consider the system* (C,A). *Then we have*

(i) $$S_a \in \underline{S}_a(C,A) \Leftrightarrow S_a^{\perp} \in \underline{V}_a(A^T,C^T) ,$$

(ii) $$N_a \in \underline{N}_a(C,A) \Leftrightarrow N_a^{\perp} \in \underline{R}_a(A^T,C^T) ,$$

(iii) $$S_{a,g} \in \underline{S}_{a,g}(C,A) \Leftrightarrow S_{a,g}^{\perp} \in \underline{V}_{a,g}(A^T,C^T) .$$

<u>PROOF</u>: In this proof, for a given subspace V of X its *canonical injection* will be defined as the mapping $Q: V \to X$ defined by $Q := I|V$, the identity mapping restricted to V. We will only prove the equivalence (ii). The other statements follow in the same way by applying COR. 2.52 and TH. 2.80. Let $N_a \in \underline{N}_a(C,A)$. Let Q be the canonical injection $N_a \to X$ and let P be the canonical projection $X \to X/N_a$. For all $\varepsilon > 0$ and all $\mathbb{C}_g$ there is a mapping G such that $\|Pe^{(A+GC)t} Q\| \leq \varepsilon$, $\forall t \in \mathbb{R}^+$ and $\sigma(A+GC|X/\langle \ker C|A\rangle) \subset \mathbb{C}_g$. By transposition we obtain

$$\|Q^T e^{(A^T+C^TG^T)t} P^T\| \leq \varepsilon \quad \text{and} \quad \sigma(A^T+C^TG^T \mid \langle A^T|\operatorname{im} C^T\rangle) \subset \mathbb{C}_g .$$

Since $\ker Q^T = \mathrm{im}\, P^T = N_a^{\perp}$, we infer from COR. 2.61 that $N_a^{\perp} \in \underline{R}_a(A^T,C^T)$. The fact that if $R_a \in \underline{R}_a(A,B)$ then $R_a^{\perp} \in \underline{N}_a(B^T,A^T)$ may be provem in an analogous way.

□

Note that the assumptions (2.27) and (2.28) (i.e. the stability sets $\mathbb{C}_g$ are symmetric with respect to the real axis and contain a negative semi-infinite interval) are essential here: only under these assumptions the results of COR. 2.52, COR. 2.61 and TH. 2.80 are valid. In the observer context these conditions appear because decreasing the estimation error forces part of the error spectrum to run off to minus infinity.

Of course, TH. 5.4 enables us to obtain several results on the families of subspaces we introduced in this section by straightforward dualization of previous results in this tract. We will only state some of the main results here. For an overview of results on almost conditionally invariant subspaces, we refer to WILLEMS (1982a). First, it follows immediately that the families $\underline{S}_a$, $\underline{N}_a$ and $\underline{S}_{a,g}$ are closed under subspace intersection. Hence, for any given subspace G of X, there exists an infimal almost conditionally invariant, an infimal almost observability and an infimal almost detectability subspace containing G. These subspaces will be denoted by $S_a^*(G)$, $N_a^*(G)$ and $S_{a,g}^*(G)$ respectively. In the sequel, a family of subspaces $\{K_i\}_{i=1}^r$ will be called *a chain around* ker C if

$$\ker C \subset K_1 \subset K_2 \subset \ldots \subset K_r \ .$$

We will denote $A_G := A + GC$. The following statements may be proven by dualizing results from CH. 1 and SECTION 2.8:

PROPOSITION 5.5.

(i) *A subspace N_a is an almost observability subspace if and only if there exists a chain $\{K_i\}_{i=1}^r$ around* ker C *such that*

$$N_a = K_1 \cap (A_G)^{-1} K_2 \cap (A_G)^{-2} K_3 \cap \ldots \cap (A_G)^{-r+1} K_r \ .$$

(ii) *A subspace S_a is an almost conditionally invariant subspace if and only if $S_a = S \cap N_a$, where S is conditionally invariant and N_a is an almost observability subspace.*

(iii) *A subspace* $S_{a,g}$ *is an almost detectability subspace if and only if* $S_{a,g} = S_g \cap N_a$, *where* S_g *is a detectability subspace and* N_a *an almost observability subspace.*

□

Whereas almost controlled invariant subspaces are connected with distributional inputs, almost conditionally invariant subspaces are connected with PID-*observers*. This fact was established in WILLEMS (1982a). A PID-observer will be a possibly noncausal (in the sense that it contains differentiators) system

$$\text{(5.3)} \qquad \begin{aligned} \dot{w}(t) &= Nw(t) + My(t) \;, \\ v(t) &= Lw(t) + \left(K_0 + K_1 \frac{d}{dt} + \ldots + K_\ell \frac{d^\ell}{dt^\ell}\right) y(t) \;, \end{aligned}$$

that has the observed output of the system (5.1) as its input. Its state space W and its output space V are finite dimensional real linear spaces. In 5.3, $K_0, K_1, \ldots, K_\ell$ are mappings from Y to V. Moreover, L, M and N are mappings respectively from W to V, from Y to W and from W to W.

In interpreting the action of (5.3) we have to be a little bit precise. We will take the following point of view. The (possibly nonproper) transfer matrix of (5.3) is given by $R(s) = L(Is-N)^{-1} M + K(s)$, where $K(s) = K_0 + K_1 s + \ldots + K_\ell s^\ell$. Thus, since the Laplace transform of the observed output is given by $C(Is-A)^{-1} x(0)$, an initial condition $w(0)$ leads to an observer output v with Laplace transform

$$\text{(5.4)} \qquad \hat{v}(s) = R(s)C(Is-A)^{-1} x(0) + L(Is-N)^{-1} w(0) \;.$$

Clearly, $\hat{v}(s)$ need not be strictly proper. However, since it is rational, $\hat{v}(s)$ may be written uniquely as the sum $\hat{v}_+(s) + \hat{v}_-(s)$ of its strictly proper part and its polynomial part. The strictly proper part corresponds to a part of the signal v that is a regular Bohl function and will be denoted by v_{reg}. The polynomial part corresponds to a part of the signal v that is an impulsive Bohl distribution and will be denoted by v_{imp}. We will interpret the action of (5.3) in the above way: an initial condition $w(0)$ and an observed output $y(t) = Ce^{At} x(0)$ yield an observer output $v = v_{reg} + v_{imp}$. This output will be called the *estimate*.

If in (5.3) the mappings $K_0, K_1, \ldots, K_\ell$ are zero, then we will say that it is a I-*observer* (integral). If the mappings $K_1, K_2, \ldots, K_\ell$ are zero, it will be called a PI-*observer* (proportional/integral). Note that I-observers as well as PI-observers always give estimates v that are regular Bohl functions. If in (5.3) the mappings N, M and L are zero, i.e. if the observer has a polynomial transfer matrix, then it will be called a PD-*observer* (proportional/derivative). The relation of the above concepts with almost conditionally invariant subspaces and almost observability subspaces is established in the following:

THEOREM 5.6.

(i) *A subspace* S_a *is almost conditionally invariant if and only if there exists a* PID-*observer with the properties that if* $w(0) = 0$ *and* $x(0) \in S_a$, *then the estimate* v *is a regular Bohl function (i.e.* $v_{imp} = 0$*) and* $v(t) = x(t)/S_a$, $\forall t \in \mathbb{R}^+$.

(ii) *A subspace* N_a *is an almost observability subspace if and only if there exists a* PD-*observer with the properties that if* $x(0) \in N_a$, *then the estimate* v *is a regular Bohl function and* $v(t) = x(t)/N_a$, $\forall t \in \mathbb{R}^+$.

In order to prove this, it is convenient to proceed via the following characterizations of almost controlled invariance and almost controllability subspaces:

LEMMA 5.7. *Let* V *be a subspace of* X. *Let* Q *be the canonical injection of* V *and let* P *be the canonical projection* $X \to X/V$. *Then we have*

(i) V *is almost controlled invariant if and only if there is a rational matrix* W(s) *such that* $P(Is-A)^{-1} BW(s) = P(Is-A)^{-1} Q$.

(ii) V *is an almost controllability subspace if and only if there is a polynomial matrix* W(s) *such that* $P(Is-A)^{-1} W(s) = P(Is-A)^{-1} Q$.

PROOF: (i) ($\Leftarrow$) Suppose such W(s) exists. Let $x_0 = Qx_0 \in V \subset X$. Define $\omega(s) := W(s)x_0$ and $\xi(s) := (Is-A)^{-1}(Qx_0 - BW(s)x_0)$. Then

$$x_0 = (Is-A)\xi(s) + B\omega(s) \quad \text{and} \quad P\xi(s) = 0 \ .$$

Thus, every $x_0 \in V$ has a (ξ,ω)-representation with $\xi(s) \in V(s)$ and $\omega(s) \in U(s)$. It follows from TH. 2.15 that $V \in \underline{V}_a(A,B)$. ($\Rightarrow$) This inclusion may be proven in a similar way. Part (ii) of the lemma may be proven analogously, using the result of TH. 2.16.

□

<u>PROOF OF TH. 5.6</u>. If $S_a \in \underline{S}_a(C,A)$, then $S_a^{\perp} \in \underline{V}_a(A^T,C^T)$. Let P and Q denote the canonical projection and injection of $\underline{S}_a^{\perp}$. By the previous lemma, there is a rational matrix W(s) such that

$$P(Is-A^T)^{-1}\ C^T\ W(s) = P(Is-A^T)^{-1}\ Q\ .$$

Transposition and putting $R(s) := W(s)^T$ yields

$$R(s)C(Is-A)^{-1}\ P^T = Q^T(Is-A)^{-1}\ P^T\ . \tag{5.5}$$

Note that im P^T = ker $Q^T = S_a$. Thus, (5.5) states that if $x(0) \in S_a$, then the PID-observer with transfer matrix R(s) yields an estimate v with Laplace transform

$$\hat{v}(s) = R(s)C(Is-A)^{-1}\ x(0) = Q^T(Is-A)^{-1}\ x(0)\ ,$$

or, equivalently, the estimation v is a regular Bohl function and is equal to $x(t)/S_a$, $\forall t \in \mathbb{R}^+$.

To prove the converse implication, note that if there exists a PID-observer such that for every $x(0) \in S_a$ the estimation is equal to $x(t)/S_a$, then the equation (5.5) has a rational solution. By transposing (5.5) and again applying LEMMA 5.7, this implies that $S_a^{\perp}$ is in $\underline{V}_a(A^T,C^T)$ and hence that S_a is almost conditionally invariant.

Part (ii) of TH. 5.6 may be proven analogously, using LEMMA 5.7 (ii). □

<u>REMARK 5.8</u>. Similar characterizations may of course be given for conditionally invariant and observability subspaces. It turns out that a subspace S is conditionally invariant if and only if there is a I-observer that yields estimates x(t) modulo S and that a subspace N is an observability subspace if and only if both a I-observer and a PD-observer exist whose estimates are equal to the state trajectory modulo N. In the same spirit, characterizations for detectability subspaces and almost detectability subspaces can be obtained.

To conclude this section, we will introduce the concepts of infimal L_p-almost conditionally invariant subspace and infimal L_p-almost observability subspace. We will define these as follows:

<u>DEFINITION 5.9</u>. Consider the system (C,A) and let G be a subspace of X. Then the *infimal* L_p*-almost observability subspace of* G is defined as

$$N_b^*(G) := \ker C \cap A^{-1} N_a^*(G) .$$

The *infimal* L_p*-almost conditionally invariant subspace of* G is defined as

$$S_b^*(G) := S^*(G) \cap N_b^*(G) .$$

Here, $S^*(G)$ denotes the infimal conditionally invariant subspace containing G.

By dualization of results on supremal L_p-almost controlled invariant subspaces and supremal L_p-almost controllability subspaces established in CH. 3, we may obtain the following output injection characterizations of these subspaces:

THEOREM 5.10. *Let* G *be a subspace of* X. *Then we have:*

(i) *If* $1 \leq p < \infty$, *then* $x_0 \in S_b^*(G)$ *if and only if for all* $\varepsilon > 0$ *there is a mapping* $G: Y \to X$ *such that*

$$\|d(e^{A_G t} x_0, G)\|_p \leq \varepsilon . \tag{5.6}$$

Moreover, on bounded intervals (5.6) *may be achieved uniformly in* p*:*

(ii) *If* $1 \leq p_0 < \infty$, *then for all* $\varepsilon > 0$ *there exists a mapping* $G: Y \to X$ *such that the inequality* (5.6) *holds for all* $p \in [1,p_0]$ *and for all* $x_0 \in S_b^*(G)$ *with* $\|x_0\| \leq 1$.

(iii) *If* $1 \leq p < \infty$, *then* $x_0 \in N_b^*(G)$ *if and only if for all* $\varepsilon > 0$ *and for all* $\mathbb{C}_g$ *(satisfying* (2.27) *and* (2.28)*) there exists a mapping* $G: Y \to X$ *such that* (5.6) *holds and, simultaneously,*

$$\sigma(A_G \mid X/\langle \ker C | A \rangle) \subset \mathbb{C}_g . \tag{5.7}$$

Moreover, the inequality can again be achieved uniformly in p*:*

(iv) *If* $1 \leq p_0 < \infty$, *then for all* $\varepsilon > 0$ *and for all* $\mathbb{C}_g$ *(satisfying* (2.27) *and* (2.28)*) there exists a mapping* $G: Y \to X$ *such that simultaneously* (5.6) *holds for all* $p \in [1,p_0]$ *and all* $x_0 \in N_b^*(G)$ *with* $\|x_0\| \leq 1$ *and* (5.7) *holds.*

PROOF: (i) and (ii) follows from the dual versions of TH. 3.6 and TH. 3.25, (iii) and (iv) follow from the dual versions of LEMMA 3.35 and TH. 3.36.

□

We note that, in general, the subspace G will not be contained in $S_b^*(G)$ or $N_b^*(G)$. We will now briefly discuss the interpretation of the above in the observer context. By again putting $N := A + GC$, $M := -G$ and by letting $L := P$, the canonical projection $X \to X/G$, it may be seen that, for fixed p_0, for all $\varepsilon > 0$ there is a (I-) observer (5.2) with state space X and output space X/G such that if $w(0) = 0$ and if the initial condition $x(0)$ lies in $S_b^*(G)$ and satisfies $\|x(0)\| \leq 1$, then the difference $\|x(t)/G - v(t)\|$ between the estimate and the state trajectory modulo G has an L_p-norm smaller than ε for all $p \in [1,p_0]$. Thus, if a trajectory of the observed flow starts in $S_b^*(G)$, then its components modulo G can be estimated arbitrarily accurately in L_p-norm for all p in a bounded interval of $[1,\infty)$. For trajectories starting in $N_b^*(G)$ this may be done while simultaneously for all initial condition pairs $(x(0),w(0))$ the spectrum of $(x-w)/\langle \ker C|A\rangle$ is located arbitrarily in $\mathbb{C}$. Thus, in particular, if (C,A) is observable, then x/G may be estimated *arbitrarily fast* (in the sense that $\|x/G - v\|_p$ can be made arbitrarily small), while the observer state w will ultimately be a good estimate of the entire state trajectory (take for example $\mathbb{C}_g = \mathbb{C}^-$).

Finally, we note that the above concepts play a role in obtaining conditions for solvability of versions of the almost disturbance decoupled estimation problem. These problems are the duals of the almost disturbance decoupling problems discussed in this tract. For details we refer to WILLEMS (1982a). For all subspaces introduced in this section, recursive algorithms may be set up by straightforward dualization. This leads in particular to some striking equalities between subspaces. It turns out that, for example, $V^*(\ker C) = N_b^*(\mathrm{im}\, B)$ and that $S_b^*(\mathrm{im}\, B) = R^*(\ker C)$. For a glossary of algorithms and subspace equalities, we refer to MALABRE (1982).

5.2 ALMOST DISTURBANCE DECOUPLING BY MEASUREMENT FEEDBACK

In all problems of approximate disturbance decoupling that we have discussed so far, it was assumed that the entire state vector of the plant under consideration was available for feedback control. We will now impose the more restrictive assumption that only a part of the plant state can be used for control purposes and consider problems of approximate disturbance decoupling by *measurement feedback*. Again, we will look at the linear system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Gd(t) \ , \\ z(t) &= Hx(t) \ , \end{aligned} \tag{5.8}$$

where d should be interpreted as an unknown disturbance and z as a to-be-controlled output (for the dimensions of the linear spaces in which the above variables take their values, see SECTION 2.6). We already studied the problem of designing state feedback control laws $u(t) = Fx(t)$ such that in the closed loop system the influence of d on z was made arbitrarily small. Suppose now however that, instead of the entire state vector, only a linear function of the state vector is available for feedback. More precisely, suppose that we have only access to the vector

$$y(t) = Cx(t) \ , \tag{5.9}$$

where $y(t)$ takes its values in the p-dimensional linear space $\mathcal{Y}$ and where C is a mapping from $\mathcal{X}$ to $\mathcal{Y}$. The output y will thus be interpreted as a measurement. It will be assumed that this measurement may be used as input for the finite dimensional time invariant linear system

$$\begin{aligned} \dot{w}(t) &= Nw(t) + My(t) \ , \\ u(t) &= Lw(t) + Ky(t) \ , \end{aligned} \tag{5.10}$$

referred to as the feedback processor (K,L,M,N). If the plant (5.8) is connected with the feedback processor (5.10) via the measurement vector (5.9), then we obtain a closed loop system that is given by the equations

$$\begin{aligned} \begin{pmatrix} \dot{x}(t) \\ \dot{w}(t) \end{pmatrix} &= \begin{pmatrix} A+BKC & BL \\ N & MC \end{pmatrix} \begin{pmatrix} x(t) \\ w(t) \end{pmatrix} + \begin{pmatrix} G \\ 0 \end{pmatrix} d(t) \ , \\ z(t) &= (H \quad 0) \begin{pmatrix} x(t) \\ w(t) \end{pmatrix} \ . \end{aligned} \tag{5.11}$$

The problem of disturbance decoupling by measurement feedback, DDPM, is to design for the system (5.8), (5.9) a feedback processor (K,L,M,N) such that in the closed loop system (5.11) the transfer matrix from d to z is zero. It is well known that DDPM is solvable if and only if $\mathcal{S}^*(\text{im } G) \subset \mathcal{V}^*(\ker H)$ (see SCHUMACHER (1980) or IMAI & AKASHI (1979) and also BASILE & MARRO (1969a)). The extension of this problem to the situation that, in addition, we require internal stability or pole placement was studied in WILLEMS & COMMAULT (1981) and in IMAI & AKASHI (1981).

Here, we will consider the following question: if DDPM is not solvable, is it then possible to design feedback processors such that in the closed loop system the influence of the disturbances on the to-be-controlled outputs is arbitrarily small? Generalizing the formal definition of the L_p-almost disturbance decoupling problem with state feedback (DEF. 3.18), we are thus led to the following:

DEFINITION 5.11. Let $1 \leq p \leq \infty$. $(\mathrm{ADDPM})_p$, *the* L_p*-almost disturbance decoupling problem with measurement feedback*, is said to be solvable if for all $\varepsilon > 0$ there exists a feedback processor (K,L,M,N) such that in the closed loop system with $(x(0),w(0)) = 0$, $\|z\|_p \leq \varepsilon \|d\|_p$ for all $d \in L_p(\mathbb{R}^+, \mathcal{D})$.

This problem was the main subject of WILLEMS (1982a) and, indeed, necessary and sufficient conditions in the form of subspace inclusions involving almost controlled invariant and almost conditionally invariant were obtained. In this section, we will give a brief review, together with some extensions, of the material on the above problem as presented in WILLEMS (1982a). In the following sections, extensions of the problem to include 'roll-off' constraints will be discussed. The methods that will be used will have a tendency towards the frequency domain approach (in contrast to the state-space approach of the previous chapters). In the sequel, denote

$$\begin{aligned} G_{11}(s) &:= H(Is-A)^{-1}\,G\ , \quad G_{12}(s) := H(Is-A)^{-1}\,B\ , \\ G_{21}(s) &:= C(Is-A)^{-1}\,G\ , \quad G_{22}(s) := C(Is-A)^{-1}\,B\ , \end{aligned} \tag{5.12}$$

for the open loop transfer matrices from d to z, u to z, d to y and u to y, respectively. The transfer matrix of the feedback processor (5.10) will be denoted by

$$F(s) := L(Is-N)^{-1}\,M + K\ . \tag{5.13}$$

If we connect the plant (5.8), (5.9) with the processor (5.10), then in the closed loop system (5.11) the transfer matrix from d to z is given by

$$G_{cl}(s) := G_{12}(s)[I - F(s)G_{22}(s)]^{-1}\,F(s)G_{21}(s) + G_{11}(s)\ . \tag{5.14}$$

Note that the inverse in the above expression always exists. Indeed, the rational matrix $I - F(s)G_{22}(s)$ is invertible because its determinant is not identically equal to zero. This may be seen by noting that

$$\lim_{|s|\to\infty} \det(I - F(s)G_{22}(s)) = 1\ .$$

In the following, if $G(s)$ is an asymptotically stable, strictly proper rational matrix, then for $1 \leq p < \infty$, $\|G\|_p$ will denote the L_p-norm of the inverse Laplace transform $L^{-1}G$. $\|G\|_\infty$ will denote the H^∞-norm of $G(s)$ (see also SECTION 3.3). We will denote by $\mathbb{R}_0^{m\times p}(s)$ the set of all proper rational $m \times p$ matrices with real coefficients. As before, solvability of $(\text{ADDPM})_p$ can be formulated in terms of the L_1-norm and H^∞-norm of the closed loop transfer matrix (see also LEMMA 3.19 and LEMMA 3.21):

<u>LEMMA 5.12</u>. *Let* $p \in \{1,\infty\}$. *Then* $(\text{ADDPM})_p$ *is solvable if and only if for all* $\varepsilon > 0$ *there is* $F(s) \in \mathbb{R}_0^{m\times p}(s)$ *such that* $\|G_{cl}\|_1 \leq \varepsilon$. $(\text{ADDPM})_2$ *is solvable if and only if for all* $\varepsilon > 0$ *there is* $F(s) \in \mathbb{R}_0^{m\times p}(s)$ *such that* $\|G_{cl}\|_\infty \leq \varepsilon$.

□

Thus, solvability of $(\text{ADDPM})_p$ amounts to finding proper rational matrices $F(s)$ such that the expression (5.14) is small in a suitable norm. Since $F(s)$ enters this expression in a highly nonlinear way, this promises to be a hard problem. It turns out however, that by a suitable transformation the problem may be turned into a linear one. Consider the following linear rational matrix equation

$$\text{(RME)} \qquad G_{12}(s)X(s)G_{21}(s) + G_{11}(s) = 0 \ .$$

<u>DEFINITION 5.13</u>. Let $1 \leq p < \infty$. We will say that RME is L_p-*almost solvable over* $\mathbb{R}_0^{m\times p}(s)$ if for all $\varepsilon > 0$ there is an $X(s) \in \mathbb{R}_0^{m\times p}(s)$ such that $\|G_{12}XG_{21} + G_{11}\|_p \leq \varepsilon$. In the same way, RME will be called H^∞-*almost solvable over* $\mathbb{R}_0^{m\times p}(s)$ if such $X(s)$ exists such that the H^∞-norm $\|G_{12}XG_{21} + G_{11}\|_\infty \leq \varepsilon$.

The next observation then reduces our (nonlinear) almost disturbance decoupling problem to the basically linear problem of almost solvability of RME:

<u>LEMMA 5.14</u>. *Let* $p \in \{1,\infty\}$. *Then* $(\text{ADDPM})_p$ *is solvable if and only if* RME *is* L_1-*almost solvable over* $\mathbb{R}_0^{m\times p}(s)$. $(\text{ADDPM})_2$ *is solvable if and only if* RME *is* H^∞-*almost solvable over* $\mathbb{R}_0^{m\times p}(s)$.

<u>PROOF</u>: This follows in an easy way from the fact that each 'almost' solution of RME yields an 'almost disturbance decoupling' $F(s)$ by putting $F(s) := X(s)[I + G_{22}(s)X(s)]^{-1}$. Conversely, every such $F(s)$ yields of course an 'almost' solution $X(s)$ (see also WILLEMS (1982a, LEMMA 2).

□

In order to obtain conditions for the almost solvability of RME, we will, in addition to RME, consider the following linear rational matrix equation:

(RME)' $\quad G_{12}(s)X(s) + G_{11}(s) = 0$.

Here, $G_{12}(s)$ and $G_{11}(s)$ are given by (5.12). Recall from SECTION 2.6 that $G_{11}(s)$ is an $\ell \times r$ matrix and that $G_{12}(s)$ is an $\ell \times m$ matrix. We will say that (RME)' is L_p*-almost solvable over* $\mathbb{R}_0^{m\times r}(s)$ if for all $\varepsilon > 0$ there is $X(s) \in \mathbb{R}_0^{m\times r}(s)$ such that $\|G_{12}X + G_{11}\|_p \leq \varepsilon$. In a similar way we define H^∞-almost solvability. Solvability and almost solvability of (RME)' has already been studied implicitly in CH. 3:

THEOREM 5.15. *Let* $1 \leq p < \infty$. *The following statements are equivalent:*

(i) (RME)' *is* L_p*-almost solvable over* $\mathbb{R}_0^{m\times r}(s)$,

(ii) (RME)' *is* H^∞*-almost solvable over* $\mathbb{R}_0^{m\times r}(s)$,

(iii) (RME)' *is solvable over* $\mathbb{R}^{m\times r}(s)$,

(iv) $\operatorname{im} G \subset V_b^*(\ker H)$.

PROOF: A proof of this may be given by applying the results of CH. 3 and using the ideas in WILLEMS (1982a, APP. A) (see also the proof of LEMMA 5.7). □

As a very important and direct consequence of the above result, we find that if M(s) and N(s) are two arbitrary strictly proper rational matrices with real coefficients, then the solvability of the equation $M(s)X(s) + N(s) = 0$ over the set of *all* rational matrices of dimensions compatible with M(s) and N(s) is equivalent to both the L_1-almost solvability and the H^∞-almost solvability of this equation over the set of all *proper* rational matrices. This may be seen as follows. Given M(s) and N(s), we can realize the composite matrix $(M(s) \mid N(s))$ in state space form as

$$(M(s) \mid N(s)) = H_0(Is-A_0)^{-1}(B_0 \mid G_0) .$$

Obviously, this is always possible for suitable real matrices H_0, B_0, G_0 and A_0, Using this, the equation $M(s)X(s) = N(s)$ takes the form

$$H_0(Is-A_0)^{-1} B_0 X(s) = H_0(Is-A_0)^{-1} G_0$$

and the claim follows immediately from TH. 5.15.

Now, the point is that this important observation of course also applies to the linear equation RME. Indeed, for suitable strictly proper rational matrices $M(s)$ and $N(s)$, RME may be written in the form $M(s)X(s) + N(s) = 0$. Thus we find:

<u>COROLLARY 5.16</u>. *Let* $1 \leq p < \infty$. *The following statements are equivalent:*

(i) RME *is* L_p*-almost solvable over* $\mathbb{R}_0^{m\times p}(s)$,

(ii) RME *is* H^∞*-almost solvable over* $\mathbb{R}_0^{m\times p}(s)$,

(iii) RME *is solvable over* $\mathbb{R}^{m\times p}(s)$.

□

If we now combine LEMMA 5.14 and COR. 5.16, we see that for $p \in \{1,2,\infty\}$ the solvability of $(\mathrm{ADDPM})_p$ is equivalent to the (exact) solvability of RME. Solvability of RME is a purely algebraic problem and, as will turn out, a very tractable one. In the following, let $\mathbb{F}$ be an arbitrary field and let $\mathbb{F}^{\ell\times r}$ denote the set of all $\ell \times r$ matrices over $\mathbb{F}$. Let $M \in \mathbb{F}^{\ell\times m}$, $S \in \mathbb{F}^{p\times r}$ and $N \in \mathbb{F}^{\ell\times r}$ be given. Consider the linear equations L_1: $MX_1 = N$, L_2: $X_2S = N$ and L_3: $MX_3S = N$ in the unknown matrices $X_1 \in \mathbb{F}^{m\times r}$, $X_2 \in \mathbb{F}^{\ell\times p}$ and $X_3 \in \mathbb{F}^{m\times p}$. It was shown in WILLEMS (1982a, APP. B) that in this general set-up, *the equation* L_3 *is solvable if and only if both* L_1 *and* L_2 *are solvable*. Since RME is a special case of L_3 with $\mathbb{F} = \mathbb{R}(s)$, we find that RME is solvable if and only if (RME)' is solvable and (RME)" is solvable, with

$$\text{(RME)"} \qquad X(s)G_{21}(s) + G_{11}(s) = 0 \; .$$

By dualization of the equivalence between (iii) and (iv) in TH. 5.15, we find that (RME)" is solvable if and only if the subspace inclusion $S_b^*(\mathrm{im}\ G) \subset \ker H$ holds (see SECTION 5.1). Consequently, we find the following necessary and sufficient conditions for the solvability of $(\mathrm{ADDPM})_p$ for the cases that $p = 1$, $p = 2$ or $p = \infty$:

<u>THEOREM 5.17</u>. *Let* $p \in \{1,2,\infty\}$. *Then* $(\mathrm{ADDPM})_p$ *is solvable if and only if* $\mathrm{im}\ G \subset V_b^*(\ker H)$ *and* $S_b^*(\mathrm{im}\ G) \subset \ker H$.

□

In WILLEMS (1982a), the condition $S_b^*(\mathrm{im}\ G) \subset \ker H$ was shown to be a necessary and sufficient condition for the solvability of the L_p-almost disturbance decoupled estimation problem $(\mathrm{ADDEP})_p$. Thus, $(\mathrm{ADDPM})_p$ is solvable if and only if $(\mathrm{ADDP})_p$ ánd $(\mathrm{ADDEP})_p$ are solvable.

5.3 GUARANTEED ROLL-OFF: PROBLEM FORMULATION AND PRELIMINARIES

In the previous section, we have discussed the problem of almost disturbance decoupling by measurement feedback. It was shown that, depending on p, solvability of this problem is equivalent to the existence for each $\varepsilon > 0$ of a proper rational matrix $F_\varepsilon(s)$ such that the L_1-norm of the closed loop impulsive response or the H^∞-norm of the closed loop transfer matrix from d to z is smaller than ε. As was the case in the problem with state feedback (where the to-be-designed state feedback control laws turned out to be unbounded functions of the decoupling accuracy ε), the coefficients in the transfer matrices $F_\varepsilon(s)$ (or, equivalently, the mappings K_ε, L_ε, M_ε and N_ε in the to-be-designed processor) will run off to infinity as $\varepsilon \downarrow 0$. In this section we will study the behaviour of the closed loop transfer matrices from the disturbance d to the control u as ε tends to zero. Assume that the system is controlled by the feedback processor $F_\varepsilon(s)$ in such a way that our design purpose is achieved, i.e. in such a way that $\|G_{cl}\|_1 \leq \varepsilon$ or $\|G_{cl}\|_\infty \leq \varepsilon$ (cf. (5.14)). In the closed loop system, the transfer matrix from d to the control u is given by

$$(5.15) \qquad T_\varepsilon(s) := (I - F_\varepsilon(s)G_{22}(s))^{-1} F_\varepsilon(s)G_{21}(s) \; .$$

Obviously, since $F_\varepsilon(s)$ is proper, $T_\varepsilon(s)$ will be strictly proper. Therefore, if the initial condition in the closed loop system is zero, then the control u resulting from a regular disturbance d, will again be regular. However, in general the sequence $T_\varepsilon(s)$ will, if it converges at all, converge to a rational matrix $T_0(s)$ that will not be proper itself. Thus, since the action of $T_0(s)$ interpreted in the time domain will consequently involve differentiation, some regular disturbances d will in the limit for $\varepsilon \downarrow 0$ give rise to distributional controls. To illustrate this, consider the situation that $T_\varepsilon(s)$ is given by $T_\varepsilon(s) = s(1+\varepsilon s^2)^{-1}$. The disturbance $d(t) = e^{-t}$ will result in an on-line control input u_ε with Laplace transform $\hat{u}_\varepsilon(s) = s(1+\varepsilon s^2)^{-1}(s+1)^{-1}$. Now, as we increase the required accuracy of decoupling, i.e. as we let $\varepsilon \downarrow 0$, we will have $T_\varepsilon(s) \to s$ and $\hat{u}_\varepsilon(s) \to s(s+1)^{-1} =: \hat{u}(s)$. The regular control inputs u_ε will converge to the distribution $u = \delta - e^{-t}$. In certain applications this may be unacceptable. Physical limitations might impose the a priori constraint that the on-line control action in the closed loop system should not be 'too large'. Thus, a designer might want to choose the processor transfer matrices $F_\varepsilon(s)$ in such a way that the limiting transfer matrix $T_0(s)$ from d to u is proper. In the same spirit, it is possible that the designer wants to impose an a priori upper bound to the degree of the polynomial part of $T_0(s)$. Even if the limiting transfer matrix $T_0(s)$ does have the property that it is proper, it might still be desired to make it 'as proper as possible', in the sense that if the Laurent expansion of $T_0(s)$ is given by $T_0(s) = T_0 + T_1 s^{-1} + T_2 s^{-2} + \ldots$, then the coefficients T_i from $i = 0$ up to some preferably as high as possible index k are zero. In a single input single output context, this would be equivalent to a preference of an as large as possible excess of poles over zeros. A large pole-zero excess increases the capability to attenuate possibly unmodelled signals of high frequency.

Summarizing, it is of interest to have a design procedure in which a designer can choose sequences of almost disturbance decoupling transfer matrices $F_\varepsilon(s)$, while simultaneously the limiting transfer matrix $T_0(s)$ has a certain prespecified maximum power of s in its Laurent expansion around infinity. This maximum power of s will be called the high frequency roll-off of the rational matrix $T_0(s)$:

DEFINITION 5.18. Let $T(s)$ be a rational matrix. Then its *high frequency roll-off* $r(T)$ is defined as

$$r(T) = \max \{k \in \mathbb{Z} \mid \lim_{|s|\to\infty} s^k\, T(s) < \infty\} .$$

If $T = 0$, we define $r(T) := \infty$. If this is not the case, then $r(T) \in \mathbb{Z}$. In fact, if $\rho \in \mathbb{Z}$ and if $T(s)$ is a rational matrix, then $r(T) \leq \rho$ if and only if in the Laurent expansion of $T(s)$ the terms corresponding to the powers $s^{-\rho+k}$, $k \in \mathbb{N}$, vanish identically. Thus $r(T) \geq 0$ if and only if $T(s)$ is proper and $r(T) \geq 1$ if and only if $T(s)$ is strictly proper.

In the coming two sections, we will consider the following extension of the almost disturbance decoupling problem with measurement feedback. Given an integer $\rho \in \mathbb{Z}$, we ask whether it is possible to find, for each $\varepsilon > 0$, a proper rational matrix $F_\varepsilon(s)$ (a feedback processor), such that the transfer matrices $T_\varepsilon(s)$ from d to u converge to a rational matrix $T_0(s)$ with $r(T_0) \geq \rho$. A few words on the type of convergence of the sequence $T_\varepsilon(s)$ that will be considered here are at order. Let $D_+'^{m\times r}$ be the space of all $m \times r$ matrices with entries in D_+'. Let $S_+'^{m\times r} \subset D_+'^{m\times r}$ be the subspace of matrices with entries in the space $S_+' \subset D_+'$ of tempered distributions with support in $\mathbb{R}^+$

(see APP.). If $x \in D'_+$ and $\sigma \in \mathbb{R}$, then x^σ will denote the distribution in D'_+ defined by $\langle x^\sigma,\varphi\rangle := \langle x,e^{-\sigma t}\varphi\rangle$. If $\sigma_0 \in \mathbb{R}$, then $S'_+(\sigma_0)$ will denote the space of all distributions $x \in D'_+$ with the property that $x^\sigma \in S'_+$ for all $\sigma > \sigma_0$. We define a topology on $S'_+(\sigma_0)$ as follows. A sequence $\{x_n\}$ in $S'_+(\sigma_0)$ will be said to converge to x if for each $\sigma > \sigma_0$ the sequence $\{x_n^\sigma\}$ converges to x^σ in the topology of S'_+ (see APP.). We will denote by $(S'_+(\sigma_0))^{m\times r}$ the subspace of $D'^{m\times r}_+$ of all matrices with entries in $S'_+(\sigma_0)$. A sequence T_n in this space will be said to converge to T if the respective entries converge in $S'_+(\sigma_0)$. It is the latter kind of convergence that we will consider for the transfer matrices $T_\varepsilon(s)$ between d and u. In the following, let L denote Laplace transform. We define:

DEFINITION 5.19. For $n \in \mathbb{N}$ let $T_n(s),T(s) \in \mathbb{R}^{m\times r}(s)$. We will say that $T_n(s)$ converges to $T(s)$ in $(S'_+(\sigma_0))^{m\times r}$ if $L^{-1}T_n$ converges to $L^{-1}T$ in the topology of $(S'_+(\sigma_0))^{m\times r}$ as defined above.

We will extensively use the following result, that relates the convergence of rational matrices in the sense of the latter definition to the convergence of these matrices as functions of the complex variable s:

LEMMA 5.20. *For* $N \in \mathbb{N}$, *let* $T_n(s),T(s) \in \mathbb{R}^{m\times r}(s)$ *and let* $\sigma_0 \in \mathbb{R}$. *Suppose that* $T_n(s) \to T(s)$ $(n \to \infty)$, *uniformly on compact subsets of* $\{s \in \mathbb{C} \mid \mathrm{Re}\ s > \sigma_0\}$. *Moreover, assume that there exists a polynomial* $p(s)$ *(independent of* n*) such that the Euclidean norms* $\|T_n(s) - T(s)\|$ *satisfy* $\|T_n(s) - T(s)\| \leq p(|s|)$ *for all* $s \in \{s \in \mathbb{C} \mid \mathrm{Re}\ s > \sigma_0\}$. *Then* $T_n(s)$ *converges to* $T(s)$ *in* $(S'_+(\sigma_0))^{m\times r}$ *as* $n \to \infty$.

PROOF: For a proof, we refer to SCHWARTZ (1966, REMARK 1, p. 307).

□

We are now in a position to formulate the main problem we will discuss in the sequel. In the following, if (K,L,M,N) defines the feedback processor (5.10), let F(s) denote its transfer matrix (cf. (5.13)) and let T(s) denote the closed loop transfer matrix from d to u (cf. (5.15)).

DEFINITION 5.21. Let $\rho \in \mathbb{Z}$ and let $1 \leq p \leq \infty$. We will say that $(\mathrm{ADDPM})^\rho_p$, *the* L_p*-almost disturbance decoupling problem with measurement feedback and guaranteed roll-off* ρ, is solvable if there are $\sigma_0 \in \mathbb{R}$ and $T_0(s) \in \mathbb{R}^{m\times r}(s)$

with $r(T_0) \geq \rho$ such that the following holds: for all $\varepsilon > 0$ there is a feedback processor $(K_\varepsilon, L_\varepsilon, M_\varepsilon, N_\varepsilon)$ such that in the closed loop system with $(x(0), w(0)) = 0$ simultaneously

$$\|z\|_p \leq \varepsilon \|d\|_p \quad \text{for all } d \in L_p(\mathbb{R}^+, D)$$

and

$$T_\varepsilon(s) \to T_0(s) \quad (\varepsilon \downarrow 0) \quad \text{in} \quad (S'_+(\sigma_0))^{m \times r} .$$

Necessary and sufficient conditions for solvability of the above problem will be established in section 5.4. Before we can do so, we will first prove some preliminary results, respectively on a problem of exact disturbance decoupling by state feedback with roll-off constraint, and on a problem of almost disturbance decoupling by state feedback with roll-off constraint. This will be the subject of the remainder of the present section. Consider the system $\dot{x}(t) = Ax(t) + Bu(t) + Gd(t)$, $z(t) = Hx(t)$. If F is a mapping from X to U, denote the closed loop transfer matrix from d to u by $T_F(s) := F(Is - A - BF)^{-1} G$. We will look at the following extension of DDP, the 'ordinary' disturbance decoupling problem with state feedback (WONHAM (1979)).

DEFINITION 5.22. Let $\rho \in \mathbb{N}$. We will say that $(DDP)_\rho$, *the disturbance decoupling problem with guaranteed roll-off* ρ, is solvable if there exists a mapping $F: X \to U$ such that $He^{A_F t} G = 0$ for all $t \in \mathbb{R}$ and $r(T_F) \geq \rho$.

Note that DDP is recovered from this definition by taking $\rho = 1$, i.e. by requiring $T_F(s)$ to be strictly proper (since this will be the case for all F, the constraint $r(T_F) \geq 1$ is an empty one). In the following, if L is a subspace of X and $\rho \in \mathbb{N}$, define

$$N^{\rho-1}(L) := \bigcap_{i=1}^{\rho} A^{-i+1} L .$$

Here, $A^{-i+1} L := \{x \in X \mid A^{i-1} x \in L\}$. The subspaces $N^i(L)$ are nested according to $L = N^0(L) \supset N^1(L) \supset \ldots$. Let $V^*(\ker H)$ be the supremal controlled invariant subspace contained in ker H. It turns out that a necessary and sufficient condition for $(DDP)_\rho$ to be solvable is that the disturbances enter $V^*(\ker H)$ sufficiently 'deeply':

THEOREM 5.23. *Let* $\rho \in \mathbb{N}$. *Then* $(\mathrm{DDP})_\rho$ *is solvable if and only if*

$$\operatorname{im} G \subset N^{\rho-1}(V^*(\ker H)) .$$

PROOF: In the proof, denote $V^* := V^*(\ker H)$. ($\Leftarrow$) Let $x_1,\dots,x_k$ be a basis of $N^{\rho-1}(V^*)$. For $j \in \underline{k}$ and $i = 0,1,\dots,\rho-1$ we have

$$A^i x_j \in N^{\rho-i-1}(V^*) \subset V^* .$$

Define

$$V := \operatorname{span}\{A^i x_j \mid j \in \underline{k},\ i = 0,1,\dots,\rho-2\} .$$

Let W be a subspace such that $V \oplus W = V^*$. Choose $F_0 \in \underline{F}(V^*)$. Define $F\colon X \to U$ by $F|V := 0$, $F|W := F_0|W$ and extend F arbitrarily to a mapping on X. Then we have $F \in \underline{F}(V^*)$ and also, since $F|V = 0$ and $A^i N^{\rho-1}(V^*) \subset V$, $FA^i G = 0$ for $i = 0,1,\dots,\rho-2$. ($\Rightarrow$) If $(\mathrm{DDP})_\rho$ is solvable, then there is F such that $\langle A_F|\operatorname{im} G\rangle \subset \ker H$ and $FA_F^i G = 0$ for $i = 0,1,\dots,\rho-2$. Thus, for $i = 0,1,\dots,\rho-1$ we find that

$$\langle A_F|A^i \operatorname{im} G\rangle = \langle A_F|A_F^i \operatorname{im} G\rangle = \langle A_F|\operatorname{im} G\rangle \subset \ker H .$$

It follows that for $i = 0,1,\dots,\rho-1$, $A^i \operatorname{im} G \subset V^*$ and consequently that $\operatorname{im} G \subset N^{\rho-1}(V^*)$.

□

For our purpose, it is convenient to state the above result in terms of the solvability of the linear rational matrix equation (RME)' (see SECTION 5.2). In the sequel, if $\rho \in \mathbb{Z}$, let $\mathbb{R}_\rho^{m\times r}(s) \subset \mathbb{R}^{m\times r}(s)$ be the subspace of rational matrices T with the property that $r(T) \geq \rho$. It follows immediately from the foregoing that solvability of $(\mathrm{DDP})_\rho$ is equivalent to the solvability of (RME)' over the space $\mathbb{R}_\rho^{m\times r}(s)$:

COROLLARY 5.24. *Let* $\rho \in \mathbb{N}$. *Then* (RME)' *is solvable over* $\mathbb{R}_\rho^{m\times r}(s)$ *if and only if* $\operatorname{im} G \subset N^{\rho-1}(V^*(\ker H))$.

□

Next, we will discuss a generalization of the L_p-almost disturbance decoupling problem by state feedback (see SECTION 3.3). In addition to approximate decoupling we require the degree of the polynomial part of the rational matrix $T_0(s)$, obtained as the limit of the sequence

$T_\varepsilon(s) := F_\varepsilon(Is - A - BF_\varepsilon)^{-1}G$ of disturbance-to-control transfer matrices, to have a certain a priori given upper bound. In our terminology, for $\rho \in \mathbb{Z}$, $\rho \leq 0$, an upper bound $-\rho$ to this degree is equivalent to the requirement $r(T_0) \geq \rho$. Consider the following definition:

DEFINITION 5.25. Let $\rho \in \mathbb{Z}$, $\rho \leq 0$, and let $1 \leq p \leq \infty$. We will say that $(\mathrm{ADDP})_p^\rho$, *the* L_p*-almost disturbance decoupling problem with guaranteed roll-off* ρ, is solvable if there exists a $T_0(s) \in \mathbb{R}_\rho^{m\times r}(s)$ and for all $\varepsilon > 0$ a mapping $F : X \to U$ such that in the closed loop system with $x(0) = 0$ simultaneously $\|z\|_p \leq \varepsilon\|d\|_p$ for all $d \in L_p(\mathbb{R}^+, D)$ and $T_\varepsilon(s) \to T_0(s)$ $(\varepsilon \downarrow 0)$ in $(S'_+(0))^{r\times m}$.

If F_ε is a mapping from X to U, let $W_\varepsilon(t) := He^{(A+BF_\varepsilon)t}G$ and $\hat{W}_\varepsilon(s) := H(Is - A - BF_\varepsilon)^{-1}G$. The following analogue of LEMMA 3.19 and LEMMA 3.21 is immediate:

LEMMA 5.26. *Let* $\rho \in \mathbb{Z}$ *and* $\rho \leq 0$. *Let* $p \in \{1,\infty\}$. *Then* $(\mathrm{ADDP})_p^\rho$ *is solvable if and only if there is a* $T_0(s) \in \mathbb{R}_\rho^{m\times r}(s)$ *and for all* $\varepsilon > 0$ *a mapping* $F_\varepsilon : X \to U$ *such that* $\|W_\varepsilon\|_1 \leq \varepsilon$ *and* $T_\varepsilon(s) \to T_0(s)$ $(\varepsilon \downarrow 0)$ *in* $(S'_+(0))^{m\times r}$. $(\mathrm{ADDP})_2^\rho$ *is solvable if and only if there is a* $T_0(s) \in \mathbb{R}_\rho^{m\times r}(s)$ *and for all* $\varepsilon > 0$ *a mapping* $F_\varepsilon : X \to U$ *such that* $\|\hat{W}_\varepsilon\|_\infty \leq \varepsilon$ *and* $T_\varepsilon(s) \to T_0(s)$ $(\varepsilon \downarrow 0)$ *in* $(S'_+(0))^{m\times r}$.

□

It turns out that in order to obtain conditions for solvability of the above problem, we should consider the sequence of subspaces $R^\mu(\ker H)$, generated recursively by the algorithm (ACSA)' (cf. (3.6)). Recall from TH. 3.8 that if $k := \dim\ker H + 1$, then $V_b^*(\ker H) = V^*(\ker H) + R^k(\ker H)$. Also, $R^\mu(\ker H)$ is monotonically nondecreasing. Therefore, the following result says that for $p \in \{1,2,\infty\}$, $(\mathrm{ADDP})_p^\rho$ is solvable if and only if the disturbances enter V_b^* sufficiently 'deeply'.

THEOREM 5.27. *Let* $\rho \in \mathbb{Z}$ *and* $\rho \leq 0$. *Let* $p \in \{1,2,\infty\}$. *Then* $(\mathrm{ADDP})_p^\rho$ *is solvable if and only if* $\mathrm{im}\, G \subset V^*(\ker H) + R^{-\rho+1}(\ker H)$.

PROOF: (outline) ($\Rightarrow$) If $p \in \{1,2,\infty\}$, then, depending on p, there are mappings F_ε such that $\|W_\varepsilon\|_1 \to 0$ or $\|\hat{W}_\varepsilon\|_\infty \to 0$ $(\varepsilon \downarrow 0)$. Obviously, in both cases we have $\hat{W}_\varepsilon(s) \to 0$ pointwise for $s \in \mathbb{C}^+$. Also, there is $T_0(s) \in \mathbb{R}_\rho^{m\times r}(s)$ such that $T_\varepsilon(s) \to T_0(s)$ in $(S'_+(0))^{m\times r}$. Using the definition of convergence of tempered distributions, it may be shown that $T_\varepsilon(s) \to T_0(s)$ pointwise for $s \in \mathbb{C}_o^+$. Now, for all ε we have $\hat{W}_\varepsilon(s) = G_{11}(s) + G_{12}(s)T_\varepsilon(s)$ (see (5.12)). Letting $\varepsilon \downarrow 0$ we therefore obtain $G_{11}(s) + G_{12}(s)T_0(s) = 0$, i.e. $T_0(s)$ is a solution of (RME)'. Using TH. 3.9 (i) and COR. 2.12 (ii) it is then easy to verify that $\operatorname{im} G \subset V^*(\ker H) + R^{-\rho+1}(\ker H)$. ($\Leftarrow$) A proof of this can be given along the lines of the proof of TH. 3.25. The idea is to decompose $V^*(\ker H) + R^{-\rho+1}(\ker H)$ into the direct sum of $V^*(\ker H)$ and a number of singly generated almost controllability subspaces $\mathcal{L}(u_i,F,r_i)$. Using TH. 1.10, it can be shown that such a decomposition exists with $\max r_i \leq -\rho+1$. Following the line of the proof of TH. 3.25, we obtain a sequence F_n such that $T_n(s) := F_n(Is - A - BF_n)^{-1} G$ converges to a $T_0(s) \in \mathbb{R}_\rho^{m\times r}(s)$ $(n \to \infty)$ uniformly on compact subsets in $\mathbb{C}_o^+$, while the Euclidean norms $\|T_n(s) - T_0(s)\|$ are dominated by a polynomial in $|s|$, independent of n. It follows from LEMMA 5.20 that $T_n(s) \to T_0(s)$ in $(S'_+(0))^{m\times r}$ (see also TRENTELMAN & WILLEMS (1983)). □

We will now formulate the above result again in terms of solvability of the rational matrix equation (RME)'. Note that $\mathbb{R}_1^{m\times r}(s)$ (i.e. the space $\mathbb{R}_\rho^{m\times r}(s)$ with $\rho = 1$) is the space of all strictly proper rational $m \times r$ matrices with real coefficients. Let $1 \leq p < \infty$. We will say that (RME)' is L_p-*almost* ρ-*solvable over* $\mathbb{R}_1^{m\times r}(s)$, if there exists $X(s) \in \mathbb{R}_\rho^{m\times r}(s)$ and for all $\varepsilon > 0$ a $X_\varepsilon(s) \in \mathbb{R}_1^{m\times r}(s)$ such that $\|G_{12}(s)X_\varepsilon(s) + G_{11}(s)\|_p \leq \varepsilon$ and $X_\varepsilon(s) \to X(s)$ $(\varepsilon \downarrow 0)$ in $(S'_+(0))^{m\times r}$. Similarly, we define (RME)' to be H^∞-*almost* ρ-*solvable* if the above holds with the L_p-norm replaced by the H^∞-norm. The following is now easily verified:

THEOREM 5.28. *Let* $1 \leq p < \infty$ *and let* $\rho \in \mathbb{Z}$, $\rho \leq 0$. *The following statements are equivalent:*

(i) (RME)' *is* L_p-*almost* ρ-*solvable over* $\mathbb{R}_1^{m\times r}(s)$,

(ii) (RME)' *is* H^∞-*almost* ρ-*solvable over* $\mathbb{R}_1^{m\times r}(s)$,

(iii) (RME)' *is solvable over* $\mathbb{R}_\rho^{m\times r}(s)$,

(iv) $\operatorname{im} G \subset V^*(\ker H) + R^{-\rho+1}(\ker H)$.

□

5.4 GUARANTEED ROLL-OFF: MAIN RESULTS

We will now consider our main problem, the L_p-almost disturbance decoupling problem with measurement feedback and guaranteed roll-off, as defined in DEF. 5.21. It turns out that for integers $\rho > 0$ necessary and suffient conditions for the solvability of $(\text{ADDPM})_p^\rho$ are obtained by requiring the solvability of both $(\text{DDP})_\rho$ and the L_p-almost disturbance decoupled estimation problem $(\text{ADDEP})_p$ (see WILLEMS (1982a)). Moreover, for integers $\rho \leq 0$ necessary and sufficient conditions are obtained by requiring the solvability of both $(\text{ADDP})_p^\rho$ and $(\text{ADDEP})_p$. The L_p-almost disturbance decoupled estimation problem is the dual of $(\text{ADDP})_p$. It requires for the system $\dot{x}(t) = Ax(t) + Gd(t)$, $y(t) = Cx(t)$, $z(t) = Hx(t)$ the existence of I-observers having the measurement y as their input and an estimate $\hat{z}$ of z as their output, such that the L_p-L_p induced norm of the operator from d to the estimation error $e := z - \hat{z}$ is arbitrarily small. For $p \in \{1,2,\infty\}$, a necessary and sufficient condition for the solvability of $(\text{ADDEP})_p$ is $S_b^*(\text{im } G) \subset \ker H$ (see also SECTION 5.1). We will now state the main result of this section:

THEOREM 5.9.

(i) *(positive guaranteed roll-off). Let* $p \in \{1,2,\infty\}$ *and* $\rho \in \mathbb{N}$. *Then* $(\text{ADDPM})_p^\rho$ *is solvable if and only if*

$$\text{im } G \subset N^{\rho-1}(V^*(\ker H)) \quad \textit{and} \quad S_b^*(\text{im } G) \subset \ker H \ .$$

(ii) *(nonpositive guaranteed roll-off). Let* $p \in \{1,2,\infty\}$ *and* $\rho \in \mathbb{Z}$, $\rho \leq 0$. *Then* $(\text{ADDPM})_p^\rho$ *is solvable if and only if*

$$\text{im } G \subset V^*(\ker H) + R^{-\rho+1}(\ker H) \quad \textit{and} \quad S_b^*(\text{im } G) \subset \ker H \ .$$

The proof of this theorem will be given through a series of lemmas involving the solvability and almost solvability of the linear rational matrix equation RME (see SECTION 5.2). Recall from SECTION 5.3 that with a feedback processor $F_\varepsilon(s)$, in the closed loop system the transfer matrix from d to u is given by $T_\varepsilon(s) = (I - F_\varepsilon(s)G_{22}(s))^{-1} F_\varepsilon(s)G_{21}(s)$. Make the following observation:

LEMMA 5.30. *Let* $\rho \in \mathbb{Z}$. *If* $p \in \{1,\infty\}$, *then* $(\mathrm{ADDPM})_p^\rho$ *is solvable if and only if there is a* $T_0(s) \in \mathbb{R}_\rho^{m\times r}(s)$, $\sigma_0 \in \mathbb{R}$ *and for all* $\varepsilon > 0$ *a* $F_\varepsilon(s) \in \mathbb{R}_0^{m\times p}(s)$ *such that* $\|G_{12}(s)T_\varepsilon(s) + G_{11}(s)\|_1 \leq \varepsilon$ *and* $T_\varepsilon(s) \to T_0(s)$ $(\varepsilon \downarrow 0)$ *in* $(S_+'(\sigma_0))^{m\times r}$. $(\mathrm{ADDPM})_2^\rho$ *is solvable if and only if the above holds with* L_1-*norm replaced by* H^∞-*norm.*

□

Consider the equation RME. For $\rho \in \mathbb{Z}$, we will say that RME is ρ-*solvable over* $\mathbb{R}^{m\times p}(s)$ if there is $X(s) \in \mathbb{R}^{m\times p}(s)$ such that $X(s)G_{21}(s) \in \mathbb{R}_\rho^{m\times r}(s)$. Let $1 \leq p < \infty$. We will say that RME is L_p-*almost* ρ-*solvable over* $\mathbb{R}_0^{m\times p}(s)$ if there is $T_0(s) \in \mathbb{R}_\rho^{m\times r}(s)$, $\sigma_0 \in \mathbb{R}$ and for all $\varepsilon > 0$ a matrix $X_\varepsilon(s) \in \mathbb{R}_0^{m\times r}(s)$ such that $\|G_{12}(s)X_\varepsilon(s)G_{21}(s) + G_{11}(s)\|_p \leq \varepsilon$ and $X_\varepsilon(s)G_{21}(s) \to T_0(s)$ $(\varepsilon \downarrow 0)$ in $(S_+'(\sigma_0))^{m\times r}$. Similarly, RME is said to be H^∞-*almost* ρ-*solvable* if this holds with L_p-norm replaced by H^∞-norm.

LEMMA 5.31. *Let* $\rho \in \mathbb{Z}$. *If* $p \in \{1,\infty\}$, *then* $(\mathrm{ADDPM})_p^\rho$ *is solvable if and only if* RME *is* L_1-*almost* ρ-*solvable over* $\mathbb{R}_0^{m\times p}(s)$. $(\mathrm{ADDPM})_2^\rho$ *is solvable if and only if* RME *is* H^∞-*almost* ρ-*solvable over* $\mathbb{R}_0^{m\times p}(s)$.

PROOF: Let $p \in \{1,\infty\}$ and assume $(\mathrm{ADDPM})_p^\rho$ solvable. Then, by LEMMA 5.30, there is $T_0(s) \in \mathbb{R}_\rho^{m\times q}(s)$, $\sigma_0 \in \mathbb{R}$, and there are proper rational matrices $F_\varepsilon(s)$ such that $T_\varepsilon(s) \to T_0(s)$ $(\varepsilon \downarrow 0)$ and $\|G_{12}(s)T_\varepsilon(s) + G_{11}(s)\|_1 \leq \varepsilon$. By taking $X_\varepsilon(s) := (I - F_\varepsilon(s)G_{22}(s))^{-1} F_\varepsilon(s)$, we find that RME is L_1-almost ρ-solvable over $\mathbb{R}_0^{m\times p}(s)$. Conversely, if RME is L_1-almost ρ-solvable over $\mathbb{R}_0^{m\times p}(s)$, then there is $T_0(s) \in \mathbb{R}_\rho^{m\times r}(s)$, $\sigma_0 \in \mathbb{R}$, and there are matrices $X_\varepsilon(s) \in \mathbb{R}_0^{m\times p}(s)$ such that $\|G_{12}(s)X_\varepsilon(s)G_{21}(s) + G_{11}(s)\|_1 \leq \varepsilon$ and $X_\varepsilon(s)G_{21}(s) \to T_0(s)$. Now, define

$$F_\varepsilon(s) := X_\varepsilon(s)(I + G_{22}(s)X_\varepsilon(s))^{-1}\ . \tag{5.16}$$

Then $F_\varepsilon(s) \in \mathbb{R}_0^{m\times p}(s)$ and $X_\varepsilon(s) = (I - F_\varepsilon(s)G_{22}(s))^{-1} F_\varepsilon(s)$. Since therefore $T_\varepsilon(s) = X_\varepsilon(s)G_{21}(s)$, LEMMA 5.30 implies that $(\mathrm{ADDPM})_p^\rho$ is solvable for $p \in \{1,\infty\}$. The case that $p = 2$ admits the same proof.

□

Thus, we have transformed our problem into the problems of L_1-almost and H^∞-almost ρ-solvability of RME. Our next result states that for both cases, almost ρ-solvability over the space of proper rational matrices is equivalent to (exact) ρ-solvability over the space of all rational matrices (compare this with COR. 5.16). This nice result will enable us to continu the proof of TH. 5.29 in a purely algebraic way.

LEMMA 5.32. *Let $\rho \in \mathbb{Z}$ and $1 \leq p < \infty$. Then the following statements are equivalent:*

(i) RME *is* L_p*-almost* ρ*-solvable over* $\mathbb{R}_0^{m\times p}(s)$,

(ii) RME *is* H^∞*-almost* ρ*-solvable over* $\mathbb{R}_0^{m\times p}(s)$,

(iii) RME *is* ρ*-solvable over* $\mathbb{R}^{m\times p}(s)$.

PROOF: (iii) $\Rightarrow$ (i). As was already noted in SECTION 5.2, the equation RME can be written as $M(s)X(s) + N(s) = 0$ for suitable rational matrices $M(s) \in \mathbb{R}^{r\ell\times pm}(s)$, $N(s) \in \mathbb{R}^{r\ell}(s)$ in the unknown $X(s) \in \mathbb{R}^{mr}(s)$. Moreover, the constraint $X(s)G_{21}(s) \in \mathbb{R}_\rho^{m\times r}(s)$ can be written as $R(s)X(s) \in \mathbb{R}_\rho^{mr}(s)$ for some $R(s) \in \mathbb{R}^{mr\times pm}$. Now, assume that RME is ρ-solvable over $\mathbb{R}^{m\times p}(s)$. Then there is a solution $\bar{X}(s) \in \mathbb{R}^{pm}(s)$ such that $R(s)\bar{X}(s) \in \mathbb{R}_\rho^{mr}(s)$. Also, since the equation $M(s)X(s) + N(s) = 0$ is a special case of (RME)' (see SECTION 5.2), it follows from TH. 5.15 and the proof of TH. 5.27 that there is $X_0(s) \in \mathbb{R}^{pm}(s)$ and that there are $X_\varepsilon(s) \in \mathbb{R}_1^{pm}(s)$ such that

(i) $\|M(s)X_\varepsilon(s) + N(s)\|_p \to 0 \quad (\varepsilon \downarrow 0)$,

(ii) $X_\varepsilon(s) \to X_0(s)$ $(\varepsilon \downarrow 0)$ uniformly on compact sets in $\mathbb{C}_0^+$,

(iii) the Euclidean norms $\|X_\varepsilon(s) - X(s)\|$ are bounded from above by a polynomial in $|s|$ independent of ε.

Now consider $\mathbb{R}^{pm}(s)$ as a linear space over the field $\mathbb{R}(s)$. Obviously, the $M(s)$ above defines an $\mathbb{R}(s)$-linear mapping from $\mathbb{R}^{pm}(s)$ to $\mathbb{R}^{r\ell}(s)$. Let $M := \ker M(s)$ (M is a subspace of $\mathbb{R}^{pm}(s)$). Define $X_1 := M$ and let X_2 be an arbitrary subspace of $\mathbb{R}^{pm}(s)$ such that $X_1 \oplus X_2 = \mathbb{R}^{pm}(s)$. In this decomposition, $M(s)$ has a matrix of the form $(0 \mid M_2(s))$, where $M_2(s)$ is $\mathbb{R}(s)$-injective. Accordingly, decompose

$$\bar{X}(s) = (\bar{X}_1(s)^T, \bar{X}_2(s)^T)^T ,$$

$$X_\varepsilon(s) = (X_{\varepsilon,1}(s)^T, X_{\varepsilon,2}(s)^T)^T$$

and

$$X_0(s) = (X_{0,1}(s)^T, X_{0,2}(s)^T)^T .$$

From (i) above, it follows that

$$M_2(s)X_{\varepsilon,2}(s) + N(s) = M(s)X_\varepsilon(s) + N(s) \to 0 \qquad (\varepsilon \downarrow 0) ,$$

pointwise in C_o^+. On the other hand, from (ii) above it follows that

$$M_2(s)X_{\varepsilon,2}(s) + N(s) \to M_2(s)X_{0,2}(s) + N(s) \qquad (\varepsilon \downarrow 0) ,$$

pointwise in $\mathbb{C}_o^+$. From the fact that $M_2(s)$ is injective, we therefore obtain that $X_{0,2}(s) = \bar{X}_2(s)$. Next, decompose $\bar{X}_1(s) = \bar{X}_{1,+}(s) + \bar{X}_{1,-}(s)$ into its strictly proper respectively polynomial part. Let $k \in \mathbb{N}$ be sufficiently large to guarantee that

$$(\varepsilon s+1)^{-k}\, \bar{X}_{1,-}(s)$$

Is strictly proper. Define

$$X^*_{\varepsilon,1}(s) := \bar{X}_{1,+}(s) + (\varepsilon s+1)^{-k}\, \bar{X}_{1,-}(s) .$$

Clearly, $X^*_{\varepsilon,1}(s)$ is strictly proper. Finally, define

$$X^*_\varepsilon(s) := (X^*_{\varepsilon,1}(s)^T, X_{\varepsilon,2}(s)^T)^T .$$

It may then be verified that

(i) $\|M(s)X^*_\varepsilon(s) + N(s)\|_p \to 0 \quad (\varepsilon \downarrow 0)$,

(ii) $X^*_\varepsilon(s) \to \bar{X}(s)$ $(\varepsilon \downarrow 0)$ uniformly on compact subsets in $\mathbb{C}_o^+$,

(iii) the Euclidean norms $\|X^*_\varepsilon(s) - \bar{X}(s)\|$ are bounded from above by a polynomial in $|s|$, independent of ε.

From (ii) and (iii) it follows that $R(s)X^*_\varepsilon(s) \to R(s)\bar{X}(s)$ in $(S'_+(\sigma_0))^{m\times r}$ for some $\sigma_0 \in \mathbb{R}$ depending on the rational matrix $R(s)$. Since $R(s)\bar{X}(s) \in \mathbb{R}^{mr}_\rho(s)$, we conclude that RME is L_p-almost ρ-solvable over $\mathbb{R}^{m\times p}_0(s)$ (in fact, even over $\mathbb{R}^{m\times p}_1(s)$).

(i) $\Rightarrow$ (iii). If RME is L_p-almost ρ-solvable over $\mathbb{R}^{m\times p}_0(s)$, then there is a vector $T_0(s) \in \mathbb{R}^{mr}_\rho(s)$, a $\sigma_0 \in \mathbb{R}$ and there are $X_\varepsilon(s) \in \mathbb{R}^{pm}_0(s)$ such that

$$\begin{pmatrix} M(s) \\ R(s) \end{pmatrix} X_\varepsilon(s) + \begin{pmatrix} N(s) \\ T_0(s) \end{pmatrix} \to 0 \qquad (\varepsilon \downarrow 0) ,$$

pointwise for $s \in \{s \in \mathbb{C} \mid \mathrm{Re}\ s > \max\{0,\sigma_0\}\}$. It may then be shown that a vector $\bar{X}(s) \in \mathbb{R}^{pm}(s)$ exists such that $M(s)\bar{X}(s) + N(s) = 0$ and $R(s)\bar{X}(s) + T_0(s) = 0$. Thus, RME is ρ-solvable over $\mathbb{R}^{m\times p}(s)$.

The equivalence between statements (i) and (ii) of the lemma follows in a completely similar way.

□

Now, if we combine the previous result with LEMMA 5.31, we find that for $p \in \{1,2,\infty\}$ and $\rho \in \mathbb{Z}$, the solvability of $(\text{ADDPM})_p^\rho$ is equivalent to the single requirement of (exact) ρ-solvability of the linear rational matrix equation RME over $\mathbb{R}^{m\times p}(s)$. At this point, note the similarity with the development of the theory around the problem $(\text{ADDPM})_p$ (see SECTION 5.2). Indeed, it was shown that for $p \in \{1,2,\infty\}$, the solvability of $(\text{ADDPM})_p$ is equivalent to the solvability of RME over $\mathbb{R}^{m\times p}(s)$.

As the last major step in our proof of TH. 5.29, we will show that the ρ-solvability of RME over $\mathbb{R}^{m\times p}(s)$ is equivalent to simultaneously the solvability of (RME)' over $\mathbb{R}_\rho^{m\times r}(s)$ and the solvability of (RME)" (see SECTION 5.2) over $\mathbb{R}^{\ell\times p}(s)$. For this, we need a rather special result on the existence of a canonical form for rational matrices. In the sequel, a *permutation matrix* will be a $r \times r$ square matrix P, obtained by interchanging the columns of the $r \times r$ identity matrix I_r arbitrarily.

<u>LEMMA 5.33</u>. *Let* $M(s)$ *be a rational* $r \times q$ *matrix* $(r \geq q)$ *with real coefficients. Suppose that* $M(s)$ *has full rank* q. *Then there exists a* $r \times r$ *permutation matrix* P *and a bijective rational* $q \times q$ *matrix* $\Lambda(s)$ *such that*

$$M(s) = PQ(s)\Lambda(s) \ , \tag{5.17}$$

where $Q(s)$ *is a rational* $r \times q$ *matrix with real coefficients, of the form*

$$Q(s) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ * & 1 & & \vdots \\ \vdots & * & & \\ \vdots & & & 0 \\ \vdots & & & 1 \\ \vdots & & & * \\ \vdots & & & \vdots \\ * & \cdots & & * \end{pmatrix} , \tag{5.18}$$

with zeros above the diagonal, ones on the diagonal and proper rational functions below the diagonal.

PROOF: Let $M(s) = (m_{ij}(s))$. Without loss of generality, assume that $m_{11}(s)$ has the property that $r(m_{11}) \leq r(m_{i1})$ $(i = 1,\ldots,q)$ and that $m_{11} \neq 0$ (if this does not hold, we may always apply a premultiplication with a $r \times r$ permutation matrix to shift the nonzero element in the first column with smallest roll-off to the (1,1)-position). Apply a postmultiplication with the $r \times r$ diagonal matrix $\mathrm{diag}(m_{11}(s)^{-1},1,\ldots,1)$. The result is a matrix $M'(s)$ that has ab element 1 in its (1,1)-position and proper rational functions in the entire first column. Next, apply to $M'(s)$ a postmultiplication with a bijective rational $q \times q$ matrix such that the resulting matrix $M''(s)$ has zeros in its $(1,i)$-positions for $i \geq 2$. Now, repeat the above procedure for the lower $(r-1) \times (q-1)$ block of $M''(s)$. Doing this, we may achieve by premultiplication using a permutation matrix and by postmultiplication using a bijective rational matrix that we obtain an element 1 in the (2,2)-position, proper rational functions in the second column below the (2,2)-position and zeros in the $(2,i)$-position for $i \geq 3$. Carrying on in this way, we find a canonical decomposition (5.17), with $Q(s)$ given by (5.18). This completes the proof of the lemma.

$\square$

For a linear space X, decomposed as $X = X_1 \oplus X_2$, we recall that the projection of X onto X_1 along X_2 is defined as the mapping $\pi: X \to X$ given by $\pi(x_1 \oplus x_2) = x_1$. In the following, let E_0 denote the $r \times (r-q)$ matrix defined by

$$(5.19) \qquad E_0 := \begin{pmatrix} 0 \\ I_{r-q} \end{pmatrix} .$$

Note that if $Q(s)$ has the canonical form (5.18), then the composite $q \times q$ rational matrix $(Q(s) \vdots E_0)$ is *bicausal*, i.e. it is proper and has a proper inverse. This property will be crucial in the proof of the following:

LEMMA 5.34. *Consider the linear space* $\mathbb{R}^r(s)$. *Let* V *be a subspace of* $\mathbb{R}^r(s)$. *Then there exists a subspace* $E \subset \mathbb{R}^r(s)$ *such that* $V \oplus E = \mathbb{R}^r(s)$ *and such that the projection* π *of* $\mathbb{R}^r(s)$ *onto* E *along* V *has the property that*

$$\pi \mathbb{R}_0^r(s) \subset \mathbb{R}_0^r(s) .$$

REMARK 5.35. The above result says that for a given subspace of $\mathbb{R}^r(s)$, we can always find a complementary subspace such that the subset of all proper vectors in $\mathbb{R}^r(s)$ is invariant under the projection along the first subspace onto the second.

PROOF OF LEMMA 5.34: Suppose that dim V = q and let M(s) be a full-rank rational r × q matrix such that $V = \text{im } M(s)$. Factorize

$$M(s) = PQ(s)\Lambda(s) ,$$

where Q(s) has the canonical form (5.18). Define a subspace E_0 of $\mathbb{R}^r(s)$ by $E_0 := \text{im } E_0$ (with E_0 given by (5.19)) and let $E := PE_0$. Note that $V = P \text{ im } Q(s)$ and that $\mathbb{R}^r(s) = \text{Im } Q(s) \oplus E_0$. It follows that $\mathbb{R}^r(s) = V \oplus E$. Moreover, it may be seen that the matrix of the projection π of $\mathbb{R}^r(s)$ onto E along V is given by

$$\Pi(s) := (0_{r\times q} \mid PE_0)(PQ(s) \mid PE_0)^{-1} .$$

Since $(PQ(s) \mid PE_0)^{-1} = (Q(s) \mid E_0)^{-1} P^{-1}$ and since $(Q(s) \mid E_0)$ is bicausal, we find that $\Pi(s)$ is proper or, equivalently, that the subset $\mathbb{R}_0^r(s)$ is invariant under π.

□

We are now in the position to prove the following important lemma:

LEMMA 5.35. *Let* $\rho \in \mathbb{Z}$. *Then* RME *is* ρ*-solvable over* $\mathbb{R}^{m\times p}(s)$ *if and only if* (RME)' *is solvable over* $\mathbb{R}_\rho^{m\times r}(s)$ *and* (RME)" *is solvable over* $\mathbb{R}^{\ell\times p}(s)$.

PROOF: (⇒) This implication is trivial. (⇐) Let $X_1(s) \in \mathbb{R}_\rho^{m\times r}(s)$ and $X_2(s) \in \mathbb{R}^{\ell\times p}(s)$ be solutions to (RME)' and (RME)" respectively. Apply the foregoing lemma to the space $\mathbb{R}^r(s)$ with $V := \ker G_{21}(s)$: let E be a subspace such that $V \oplus E = \mathbb{R}^r(s)$ and let $\Pi(s)$ be the proper rational r × r matrix representing the projection onto E along V. Define $X_1'(s) := X_1(s)\Pi(s)$. We contend that $X_1'(s)$ is a solution of (RME)'. To prove this, first note that from the fact that $X_2(s)G_{21}(s) + G_{11}(s) = 0$, we have $\ker G_{21}(s) \subset \ker G_{11}(s)$. Therefore,

$$G_{12}(s)X_1'(s) \mid \ker G_{21}(s) = 0 = - G_{11}(s) \mid \ker G_{21}(s) ,$$

$$G_{12}(s)X_1'(s) \mid E = G_{12}(s)X_1(s) \mid E = - G_{11}(s) \mid E ,$$

which proves that $G_{12}(s)X_1'(s) + G_{11}(s) = 0$. Next, we claim that $X_1'(s) \in \mathbb{R}_\rho^{m\times r}(s)$. Indeed, this is trivial since $X_1(s) \in \mathbb{R}_\rho^{m\times r}(s)$ and $\Pi(s)$ is proper. Finally, since $\ker G_{21}(s) \subset \ker X_1'(s)$, $X_1'(s)$ may be written as $X_1'(s) = X(s)G_{21}(s)$, for some $X(s) \in \mathbb{R}^{m\times p}(s)$. Summarizing, we have that $X(s)$ is a solution of RME and that $X(s)G_{21}(s) = X_1'(s) \in \mathbb{R}_\rho^{m\times r}(s)$. We conclude that RME is ρ-solvable over $\mathbb{R}^{m\times p}(s)$.

□

The proof of our main result, TH. 5.29, can now be given as follows:

PROOF OF THEOREM 5.29. By LEMMA 5.31 and LEMMA 5.32, for $\rho \in \mathbb{Z}$ and $p \in \{1,2,\infty\}$, solvability of $(\text{ADDPM})_p^\rho$ is equivalent to the ρ-solvability of RME over $\mathbb{R}^{m\times p}(s)$. By LEMMA 5.35, the latter is equivalent to solvability of (RME)' over $\mathbb{R}_\rho^{m\times r}(s)$ ánd solvability of (RME)" over $\mathbb{R}^{\ell\times p}(s)$. Now, for $\rho \in \mathbb{N}$, solvability of (RME)' over $\mathbb{R}_\rho^{m\times r}(s)$ is equivalent to $\text{im } G \subset \mathcal{N}^{\rho-1}(\mathcal{V}^*(\ker H))$ (COR. 5.24). For $\rho \in \mathbb{Z}$, $\rho \leq 0$, solvability of (RME)' over $\mathbb{R}_\rho^{m\times r}(s)$ is equivalent to $\text{im } G \subset \mathcal{V}^*(\ker H) + \mathcal{R}^{-\rho+1}(\ker H)$ (TH. 5.28). Finally, it follows from the dual version of the equivalence between (iii) and (iv) in COR. 5.15 that solvability of (RME)" over $\mathbb{R}^{\ell\times p}(s)$ is equivalent to $\mathcal{S}_b^*(\text{im } G) \subset \ker H$. This completes the proof of the theorem.

□

Note that the proof of TH. 5.29 required some more analysis than the analogous result in TH. 5.17 on the solvability of the problem $(\text{ADDPM})_p$ without roll-off constraint. The solvability of the latter problem could be formulated purely in terms of the solvability of a linear equation over an arbitrary field (cf. the remarks preceding TH. 5.17).

There are two important special cases of TH. 5.29 that we want to state in a separate corollary:

COROLLARY 5.36. *Let* $p \in \{1,2,\infty\}$.

(i) $(\text{ADDPM})_p^1$ *is solvable if and only if* $\text{im } G \subset \mathcal{V}^*(\ker H)$ *and* $\mathcal{S}_b^*(\text{im } G) \subset \ker H$.

(ii) $(\text{ADDPM})_p^0$ *is solvable if and only if* $\text{im } G \subset \mathcal{V}^*(\ker H) + \mathcal{B}$ *and* $\mathcal{S}_b^*(\text{im } G) \subset \ker H$.

□

The first result says that the conditions for solvability of the L_p-almost disturbance decoupling problem by measurement feedback with disturbance-to-control transfer matrices having a strictly proper limit, are that both the disturbance decoupling problem by state feedback, DDP, ánd the

L_p-almost disturbance decoupled estimation problem $(ADDEP)_p$ (see WILLEMS (1982a)) are solvable. If in the latter statement we change strictly proper by proper, then the conditions become equivalent to the solvability of both $(ADDP)_p^0$ and $(ADDEP)_p$.

REMARK 5.37. The results of this section and the previous one can be dualized to obtain conditions for solvability of a class of (almost) disturbance decoupled estimation problems. For this, we refer to TRENTELMAN & WILLEMS (1983). We also note that the theory around the problem $(ADDP)_p^\rho$ has a nice interpretation in the context of disturbance decoupling by feedforward (see WILLEMS (1982b)). In fact, the condition $\text{im } G \subset \mathcal{V}^*(\ker H) + \mathcal{R}^{\rho-1}(\ker H)$ can be shown to be equivalent to the existence of a disturbance decoupling feedforward control law that involves a feedback component together with differentiating elements d^i/dt^i from $i = 0$ up to the order $i = \rho-2$.

REMARK 5.38. A pressing and as yet unsolved problem is the extension of the results of this section to the situation that, in addition, we have a constraint of internal stability or pole placement in the closed loop system. Even for the case without roll-off constraint, i.e. for the 'plain' problem $(ADDPM)_p$, this extension is still an open problem. Some preliminary results (see e.g. WILLEMS & IKEDA (1984) or SABERI (1984)) seem to indicate however that, for example, conditions for solvability of $(ADDPM)_p$ with internal stability involves the 'stabilizability' and 'detectability' versions $\mathcal{V}_g^*(\ker H) + \mathcal{R}_b^*(\ker H)$ and $\mathcal{S}_g^*(\text{im } G) \cap \mathcal{N}_b^*(\text{im } G)$ of $\mathcal{V}_b^*(\ker H)$ and $\mathcal{S}_b^*(\text{im } G)$, respectively.

5.5 PID-OBSERVERS AND REDUCTION OF OBSERVER ORDER

In the next three sections, we will discuss some applications of the concept of almost conditionally invariant subspace to the design of reduced and minimal order dynamic observers for the observed linear flow (5.1). We will start off in this section by establishing a generalization of a well-known result by LUENBERGER (1964) on the existence of reduced order state observers. Consider the finite dimensional linear time invariant flow $x(t) = Ax(t)$, $y(t) = Cx(t)$. As before, let $\mathring{x}(t) \in \mathcal{X} \simeq \mathbb{R}^n$ and $y(t) \in \mathcal{Y} \simeq \mathbb{R}^p$. Also, assume that C is a surjective mapping. It is well-known (see for example KWAKERNAAK & SIVAN (1972)) that if (C,A) is observable, then a 'full order' dynamic observer for the state of the flow (C,A) can be found as follows. Let Λ be a symmetric set of n complex numbers. Let $G: Y \to X$ be a mapping such that $\sigma(A+GC) = \Lambda$ and define a I-observer (see SECTION 5.1) $\dot{w}(t) = Nw(t) + My(t)$, $\hat{x}(t) = Lw(t)$ by $N := A + GC$, $M := -G$ and $L := I_X$ (the identity mapping of X). Obviously, the estimation error $e(t) := \hat{x}(t) - x(t)$ satisfies $\dot{e}(t) = Ne(t)$. Consequently, if we take $\Lambda \subset \mathbb{C}^-$, then $e(t) \to 0$ $(t \to \infty)$ for all initial conditions $(x(0),w(0))$, i.e., $\hat{x}(t)$ ultimately identifies $x(t)$.

It was pointed out in LUENBERGER (1964), that the dynamic order of the above observer (being equal to $n = \dim X$) is unnecessarily large, since from the observation $y(t)$ it is possible at once to recover the part of the state vector modulo $\ker C$. In this way, it is possible to reduce the order of the observer to $n-p$. In WONHAM (1970), the existence of this 'reduced order' observer was established using the dual version of the following

<u>PROPOSITION 5.39</u>. *Assume that* (C,A) *is observable. Let* Λ *be a symmetric set of* $n-p$ *complex numbers. Then there exists a conditionally invariant subspace* S *of* X *and a mapping* $G: Y \to X$ *such that*

(5.20) $\quad \ker C \oplus S = X$,

(5.21) $\quad (A+GC)S \subset S$,

(5.22) $\quad \sigma(A+GC\,|\,X/S) = \Lambda$.

□

We will briefly recall the construction leading to the 'reduced order' observer. The idea is, that the state modulo S may be estimated by a I-observer (using the fact that S is conditionally invariant), while the state modulo $\ker C$ may be estimated by a P-observer (a PD-observer without differentiators, see SECTION 5.1). Suitably combining these two actions will yield a single PI-observer for the entire state x. Let $P: X \to X/S$ be the canonical projection. Indeed, from (5.20) we have $\ker \binom{C}{P} = \{0\}$. Hence, there exist mappings $K: Y \to X$ and $L: X/S \to X$ such that $KC + LP = I_X$. Let G be as in PROP. 5.39. Define $N := (A+GC)|X/S$ and $M := -PG$. Consider the PI-observer $\dot{w}(t) = Nw(t) + My(t)$, $\hat{x}(t) = Lw(t) + Ky(t)$. Note that the dynamic order of this observer is $n-p$. We contend that, if $\Lambda \subset \mathbb{C}^-$, then $\hat{x}(t) - x(t) \to 0$ $(t \to \infty)$, for every initial condition pair $(x(0),w(0))$. Indeed, $w(t) - x(t)/S$ satisfies the differential equation $\dot{z} = Nz$ and therefore $w(t) \to x(t)/S$ $(t \to \infty)$. Now, by (5.20), $x(t) = x_1(t) \oplus x_2(t)$, with $x_1(t) \in \ker C$ and

$x_2(t) \in S$. Since $KC + LP = I_X$, we have $x_1(t) = LPx(t) = L(x(t)/S)$. Thus, $Lw(t) \to x_1(t)$ $(t \to \infty)$, i.e. $Lw(t)$ *ultimately* identifies $x_1(t)$. Moreover, $x_2(t) = KCx(t) = Ky(t)$, i.e. $Ky(t)$ *instantaneously* identifies $x_2(t)$. Consequently, $\hat{x}(t) = Lw(t) + Ky(t)$ ultimately identifies $x_1(t) \oplus x_2(t) = x(t)$. It is vital to note here that the above instantaneous identification property is provided by the presence of the direct feedthrough term $Ky(t)$ in the 'reduced order' observer. Summarizing, the order reduction from n to $n-p$ stems from a change in type of observer: the full order observer is a I-observer, the reduced order observer is a PI-observer. In the present section we will show that it is possible to reduce the dynamic order of the observer even more by allowing direct feedthrough of derivatives of the observation $y(t)$, i.e. by allowing the observer to be a PID-*observer* (see SECTION 5.1).

Instead of assuming the entire observation $y(t)$, together with all its derivatives $y^{(1)}(t), y^{(2)}(t), \ldots$ to be available for estimation of the state $x(t)$, the following option will be taken. Let $y(t) = (y_1(t), \ldots, y_p(t))^T$. For each component $y_i(t)$, specify an integer κ_i such that $-1 \leq \kappa_i \leq n-1$. Here, a value $\kappa_i \geq 0$ will mean that $y_i(t), y_i^{(1)}(t), \ldots, y_i^{(\kappa_i)}(t)$ may be used for direct feedthrough. A value $\kappa_i = -1$ will mean that nor $y_i(t)$ nor any of its derivatives may be used for direct feedthrough. In a suitable basis for Y we can arrange that $\kappa_1 \geq \kappa_2 \geq \ldots \geq \kappa_p$. Denote $\kappa := \kappa_1$ and define integers ν_i $(i = 0, 1, \ldots, \kappa)$ by

$$\nu_i := \text{the number of integers (counting multiplicity) in the set } \{\kappa_1, \ldots, \kappa_p\} \text{ that are } \geq i.$$

Clearly, $\nu_0 \geq \nu_1 \geq \ldots \geq \nu_\kappa$. Moreover, the above is equivalent to saying that the subvector $(y_1(t), \ldots, y_{\nu_i}(t))^T \in Y_i := \mathbb{R}^{\nu_i}$ may be used for direct feedthrough, together with all its derivatives up to the order i. More concretely, let $L_i \colon Y \to Y_i$ be mappings such that $(y_1, \ldots, y_{\nu_i})^T = L_i y$. Then the output equation of our observer should be of the form

$$\hat{x}(t) = F\left(w(t),\ L_0 y(t),\ \frac{d}{dt} L_1 y(t), \ldots, \frac{d^\kappa}{dt^\kappa} L_\kappa y(t)\right), \tag{5.23}$$

for some (linear) mapping $F\colon W \oplus Y_0 \oplus Y_1 \oplus \ldots \oplus Y_\kappa \to X$. Here, $w \in W$ is the state of the observer, which will be assumed to be driven only by $y(t)$. That is, the dynamic part of our observer will be assumed to be of the form

$$\dot{w}(t) = G(w(t), y(t)),$$

for some (linear) mapping G: $W \oplus Y \to W$. Now, with mappings L_i as defined above, we obviously have $\ker L_0 \subset \ker L_1 \subset \ldots \subset \ker L_\kappa$. Define a mapping R: $X \to Y_0 \oplus Y_1 \oplus \ldots \oplus Y_{\kappa_1}$ by

$$(5.24) \qquad R_\kappa := \begin{pmatrix} L_0 C \\ L_1 CA \\ \vdots \\ L_\kappa CA^\kappa \end{pmatrix} x \ .$$

Denote $K_i := \ker L_i C$. We then have the following inclusion:

$$\ker C \subset K_0 \subset K_1 \subset \ldots \subset K_\kappa \ .$$

This says that $\{K_i\}_{i=1}^{\kappa}$ is *a chain around* ker C (SECTION 5.1). Moreover,

$$(5.25) \qquad \ker R = \bigcap_{i=0}^{\kappa} A^{-i} K_i \ ,$$

and from PROP. 5.5 we have that ker R is *an almost observability subspace.* The following theorem now yields a direct generalization of PROP 5.39 of this section:

<u>THEOREM 5.40</u>. *Assume that* (C,A) *is observable and suppose that* $N_a \subset X$ *is an almost observability subspace. Let* Λ *be a symmetric set of* dim N_a *complex numbers. Then there exists a conditionally invariant subspace* S *of* X *and a mapping* G: $Y \to X$ *such that*

$$(5.26) \qquad N_a \oplus S = X \ ,$$

$$(5.27) \qquad (A+GC)S \subset S \ ,$$

$$(5.28) \qquad \sigma(A+GC \,|\, X/S) = \Lambda \ .$$

PROOF: The proof follows immediately upon noting that the statement of the theorem is the dual of the version of TH. 2.39 (see SECTION 2.5) in which the system is assumed to be controllable.

□

It will now be shown how TH. 5.40 can be applied to construct a reduced order PID-observer for the state of the flow (C,A), that uses derivatives of the observation y(t) for direct feedthrough according to the prespecified integers κ_i. In the above theorem, take $\mathcal{N}_a$ = ker R. Let Λ be a symmetric set of $n_a := \dim \mathcal{N}_a$ complex numbers. Let $\mathcal{S}$ be a conditionally invariant subspace and let G: $\mathcal{Y} \to \mathcal{X}$ be a mapping such that (5.26) to (5.28) hold. Now, the idea is that $x/\mathcal{S}$ may be estimated by a I-*observer* as before. On the other hand, since $\mathcal{N}_a$ is an almost observability subspace, the component $x/\mathcal{N}_a$ may be estimated by a PD-*observer* (see TH. 5.6 (ii)). We will show that these actions can be combined *into a single* PID-*observer*. Again, let P: $\mathcal{X} \to \mathcal{X}/\mathcal{S}$ be the canonical projection. From (5.26), ker $\binom{R}{P} = \{0\}$. Consequently, there are mappings K_e: $\mathcal{Y}_0 \oplus \mathcal{Y}_1 \oplus \ldots \oplus \mathcal{Y}_\kappa \to \mathcal{X}$ and L: $\mathcal{X}/\mathcal{S} \to \mathcal{X}$ such that $K_eR + LP =$ $= I_X$. Define $N := (A+GC)|\mathcal{X}/\mathcal{S}$ and $M := -PG$. Partition $K_e = (K_0, K_1, \ldots, K_\kappa)$ compatible with $\mathcal{Y}_0 \oplus \mathcal{Y}_1 \oplus \ldots \oplus \mathcal{Y}_\kappa$. Consider the PID-observer

$$\dot{w}(t) = Nw(t) + My(t) \ ,$$

(5.29)
$$\hat{x}(t) = Lw(t) + K_0 L_0 y(t) + K_1 \frac{d}{dt} L_1 y(t) + \ldots + K_\kappa \frac{d^\kappa}{dt^\kappa} L_\kappa y(t) \ .$$

Note that the output equation of this observer is indeed of the required form (5.23). Moreover, $w(t) - x(t)/\mathcal{S}$ satisfies $\dot{z} = Nz$. Thus, taking $\Lambda \subset \mathbb{C}^-$ yields $w(t) \to x(t)/\mathcal{S}$ $(t \to \infty)$. Again, by (5.26), $x(t) = x_1(t) \oplus x_2(t)$, with $x_1(t) \in \mathcal{N}_a$ and $x_2(t) \in \mathcal{S}$. Since $K_eR + LP = I_X$, we have $x_1(t) = LPx(t) =$ $= L(x(t)/\mathcal{S})$ and therefore $Lw(t) \to x_1(t)$ $(t \to \infty)$, i.e. Lw(t) ultimately identifies $x_1(t)$. On the other hand,

$$x_2(t) = (K_eR + LP)x_2(t) = K_e R x_2(t) = K_e R x(t) =$$

$$= (K_0 L_0 C + K_1 L_1 CA + \ldots + K_\kappa L_\kappa CA^\kappa)x(t) \ .$$

This is however equal to

$$K_0 L_0 y(t) + K_1 \frac{d}{dt} L_1 y(t) + \ldots + K_\kappa \frac{d^\kappa}{dt^\kappa} L_\kappa y(t) \ .$$

Thus, the state component $x_2(t)$ is identified instantaneously by the PD-part of the observer. The dynamic order of the observer (5.29) is equal to n_a = dim ker R and consequently depends on the prespecified integers $\kappa_1, \ldots, \kappa_p$. From (5.24) it is clear that

(5.30)
$$n_a \geq \max \left\{0 \, , \, n - p - \sum_{i=1}^{p} \kappa_i\right\} ,$$

with equality if and only if R has full rank. Therefore, (5.30) only provides a lower bound for the order of the observer (5.29) and this lower bound is achieved if and only if the mapping R has full rank. One possible way to make sure that R has full rank is to choose the integers κ_i in the following way. Let $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_p \geq 1$ be the observability indices of (C,A). Let $c_1, c_2, \dots, c_p$ be the rows of C. Dualizing a result in WONHAM (1957, § 5.7), it can be shown that, possibly after relabeling the basis vectors of Y, the following row vectors are linearly independent:

$$\begin{array}{llllllll}
c_1 & , c_1A & , \dots & \dots & \dots & \dots & c_1A^{\lambda_1-1} & , \\
c_2 & , c_2A & , \dots & \dots & \dots & c_2A^{\lambda_2-1} & , & \\
\vdots & & & & & & & \\
c_{p-1} & , c_{p-1}A & , \dots & \dots & , c_{p-1}A^{\lambda_{p-1}-1} & , & & \\
c_p & , c_pA & , \dots & , c_pA^{\lambda_p-1} & & & &
\end{array}$$

Hence, if we take $\kappa_1 \geq \kappa_2 \geq \dots \geq \kappa_p$ such that $-1 \leq \kappa_i \leq \lambda_i - 1$, then R as defined by (5.24) is surjective and the dynamic order of the observer (5.29) becomes $n_a = n - p - \Sigma_{i=1}^{p} \kappa_i$.

REMARK 5.41. We note the the 'full order' dynamic observer may be recovered from this result by taking $\kappa_i = -1$ for all i, i.e. by not allowing direct feedthrough of any of the components of y(t) (or, equivalently, R = 0). Also note that the 'reduced order' observer originating from PROP 5.39 may be recovered from the above result by taking $\kappa_i = 0$ for all i, i.e. by allowing direct feedthrough of y(t) only and not of its derivatives (equivalently: R = C). If we take $\kappa_i \geq \lambda_i - 1$, then we obtain $n_a = 0$. In this case the observer (5.29) degenerates into a PD-observer.

REMARK 5.42. Finally, we note that a theory similar to the one above may be developed if the system (C,A), instead of observable, is only *detectable*. In this case, TH. 5.40 should be replaced by the dual version of COR. 2.42: if (C,A) is detectable and N_a is an almost observability subspace, then there exists a detectability subspace S_g such that $N_a \oplus S_g = X$. Note that this result generalizes SCHUMACHER (1981, LEMMA 2.9). His result may be recovered from ours by taking $N_a = \ker C$.

5.6 ALMOST CONTROLLABILITY SUBSPACE COVERS

The purpose of this section is to present some results on the role of almost controllability subspaces in the context of the minimal dimension cover problem. We will extend some well-known results by WONHAM & MORSE (1972) on the existence of minimal dimension covers for one dimensional subspaces of the state space. In the next section these results will be dualized to obtain a result on the existence of minimal order observers for a single linear functional of the state. The proofs in this section will be omitted. For the details of these proofs, we refer to TRENTELMAN (1984).

Consider the controllable system $\dot{x}(t) = Ax(t) + Bu(t)$, with $x(t) \in X \simeq \mathbb{R}^n$, $u(t) \in U \simeq \mathbb{R}^m$ and B an injective mapping. Let L be a subspace of X. Recall from WONHAM & MORSE (1972) that a controlled invariant subspace V is called a *cover for* L if $L \subset B+V$. Now, let R_a be an almost controllability subspace. Generalizing the previous definition, an (A,B)-invariant subspace V will be called an R_a*-cover for* L if $L \subset R_a+V$. Thus, a cover in the sense of WONHAM & MORSE (1972) would, in our terminology, be called a B*-cover*. Recall from SECTION 2.5 that a subset Λ of $\mathbb{C}$ is called symmetric if $\Lambda \cap \mathbb{R} \neq \emptyset$ and if $\Lambda = \bar{\Lambda}$, where $\bar{\Lambda}$ denotes the set obtained by taking the complex conjugate of each element in Λ. In the sequel, we will also consider the empty set $\emptyset$ to be symmetric. With this convention, we define the R_a*-cover index of* L to be the smallest integer $\nu \geq 0$ such that the following holds: for every symmetric set Λ of ν complex numbers (counting multiplicity), there exists an R_a-cover V for L and a mapping $F \in \underline{F}(V)$ such that $\dim V = \nu$ and $\sigma(A_F|V) = \Lambda$. From TH. 2.39, we immediately obtain that every subspace $L \subset X$ has an R_a-cover of dimension $n - \dim R_a$. Consequently, the R_a-cover index ν of L is well-defined and satisfies $0 \leq \nu \leq n - \dim R_a$.

The problem of computing the B-cover index and the corresponding B-covers for an arbitrary subspace L is as yet an unsolved problem. However, for the case that $\dim L = 1$, a complete solution was described in WONHAM & MORSE (1972). In the following, we will extend this result to the problem of computing the R_a-cover index for one-dimensional subspaces L, for the case that the underlying almost controllability subspace R_a is equal to H_k, where

$$H_k := \begin{cases} B + AB + \dots + A^{k-1}B & \text{if } k > 0 \ , \\ \{0\} & \text{if } k = 0 \ . \end{cases}$$

It is well-known (see e.g. WONHAM & MORSE (1972), or WONHAM (1979, § 5.7)) that if (A,B) is controllable, then there are controllability subspaces R_i with $\dim R_i = \mu_i$, $R_i \cap B =: b_i$ one-dimensional and $X = R_1 \oplus R_2 \oplus \oplus \ldots \oplus R_m$, together with a mapping $F: X \to U$ such that $A_F R_i \subset R_i$ and $R_i = b_i \oplus A_F b_i \oplus \ldots \oplus A_F^{\mu_i - 1} b_i$. The integers μ_i are the *controllability indices* of (A,B). We will assume that $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_m$. Define now $R_{m+1} := \{0\}$ and $\mu_{m+1} := \{0\}$. For every integer $k \geq 0$ and subspace $L \subset X$, define

$$\ell := \max \left\{ i \mid 1 \leq i \leq m+1 \text{ and } L \subset H_k + \sum_{j=i}^{m+1} R_j \right\} .$$

Note that ℓ depends on L and k. Then we have the following lemma, which generalizes WONHAM & MORSE (1972, LEMMA 3.2):

LEMMA 5.43. *Assume that* $\dim L = 1$. *Suppose that* Λ *is a symmetric set of* $\max\{0, \mu_\ell - k\}$ *complex numbers. Then there is an* H_k*-cover* V *for* L *and a mapping* $F \in \underline{F}(V)$ *such that* $\dim V = \max\{0, \mu_\ell - k\}$ *and* $\sigma(A_F | V) = \Lambda$.

□

A proof of this lemma uses TH. 2.39 and can be found in TRENTELMAN (1984). Using this result, it turns out that for one-dimensional subspaces L it is indeed possible to computer their H_k-cover index for all integer $k \geq 0$:

THEOREM 5.44. *Let* $k \in \mathbb{N} \cup \{0\}$ *and assume that* L *is a one-dimensional subspace of* X. *Then the* H_k*-cover index* $\nu(k)$ *of* L *is equal to* $\nu(k) = \max\{0, \mu_\ell - k\}$.

□

The fact that $\nu(k) \leq \max\{0, \mu_\ell - k\}$ is immediate from LEMMA 5.43. The proof of the reverse equality may be given by adapting the proof of the corresponding result in WONHAM & MORSE (1972, p. 99). Note that their result can be recovered from TH. 5.44 by taking $k = 1$.

5.7 MINIMAL ORDER PID-OBSERVERS

In the present section we will introduce formal definitions of the concepts of PID-observer and minimality of observer order. With these definitions as a starting point, we will explain in which sense the 'full order' observer, the 'reduced order' observer and the PID-observer (5.29) (see SECTION 5.5) are minimal. Finally, we will dualize the results from SECTION

5.6 to obtain the existence of minimal order PID-observers for a single linear functional of the state. This result will generalize the existence results on minimal order PI-observers in WONHAM & MORSE (1972) (see also WONHAM (1979, p. 77) and LUENBERGER (1966)).

Consider the observed linear flow $\dot{x}(t) = Ax(t)$, $y(t) = Cx(t)$ as in SECTION 5.5. Assume that a second output equation

(5.31) $\quad z(t) = Dx(t)$

is given. Here, Z is assumed to be a finite dimensional linear space and D a mapping from X to Z. The variable z(t) should be interpreted as a variable to be estimated by an observer. In the following, let N_a be an almost observability subspace (with respect to (C,A)). A (C,A)-invariant subspace will be called a N_a-PID-*observer for* Dx if $S \cap N_a \subset \ker D$. The N_a-*observer index of* Dx is defined as the smallest integer $\nu \geq 0$ such that the following holds: for every symmetric set Λ of ν complex numbers there exists a N_a-PID-observer S for Dx and a mapping G: $Y \to X$ such that $\dim X/S = \nu$, $(A+GC)S \subset S$ and $\sigma(A+GC|X/S) = \Lambda$. A N_a-PID-observer S for Dx will be said to have minimal order if $\dim X/S$ is equal to the N_a-observer index of Dx. The above definitions generalize definitions in WONHAM & MORSE (1972). In fact, their definitions of observer and observer index are recovered from ours by taking $N_a = \ker C$. Note the duality between the notions of N_a-PID-observer and R_a-cover. A little thought reveals that a conditionally invariant subspace S is a N_a-PID-observer if and only if $S^\perp$ is a $N_a^\perp$-cover for im D^T (with respect to the system (A^T,C^T). Moreover, the N_a-observer index of Dx is equal to the $N_a^\perp$-cover index of im D^T. We will now explain how the above formal definition of N_a-PID-observer yields a 'real' PID-observer for z = Dx. The idea is, that x/S may be estimated by a I-observer (since S is conditionally invariant). On the other hand, x/N_a may be estimated by a PD-observer (TH. 5.6). Thus, it will be possible to estimate $x/(S \cap N_a)$ using a PID-observer. Since $S \cap N_a \subset \ker D$, the same will be true for x/ker D. More concretely, specify integers $\kappa_1 \geq \kappa_2 \geq \ldots \geq \kappa_p$ as in SECTION 5.5, and take $N_a = \ker R$, where R is given by (5.24). Now, let S be a N_a-PID-observer for Dx and suppose S has minimal order. Let $\nu = \dim X/S$ be the N_a-observer index of Dx. Let P: $X \to X/S$ be the canonical projection. Let Λ be a symmetric set of ν complex numbers. Clearly, since $S \cap N_a \subset \ker D$, we have $\ker \binom{R}{P} \subset \ker D$. Therefore, mappings $K : Y_0 \oplus Y_1 \oplus \ldots \oplus Y_\kappa \to Z$ ($\kappa := \kappa_1$, see SECTION 5.5) and L: $X/S \to Z$ exist such that $K_eR + LP = D$. Define $N := (A+GC)|X/S$ and $M := -PG$. Decompose the mapping $K_e = (K_0,K_1,\ldots,K_\kappa)$ and consider the PID-observer

$$\dot{w}(t) = Nw(t) + My(t) \; ,$$

(5.32)
$$\hat{z}(t) = Lw(t) + K_0 L_0\, y(t) + K_1 \frac{d}{dt} L_1 y(t) + \ldots + K_\kappa \frac{d^\kappa}{dt^\kappa} L_\kappa y(t) \; .$$

Then the vector $w(t) - x(t)/S$ satisfies $\dot{r} = Nr$ and we have $\hat{z} - z = Lr$. Hence, taking $\Lambda \subset \mathbb{C}^-$ yields $\hat{z}(t) \to z(t)$ $(t \to \infty)$. We see that (5.32) indeed defines a system that estimates $z(t)$. The PID-observer (5.32) has minimal order in the sense that its dynamic order is equal to the N_a-observer index of Dx (with $N_a = \ker R$). Again, in general the problem of computing the N_a-observer index of Dx, being dual to a cover problem, is very difficult. We do have a result which treats the case that $D = I$:

<u>THEOREM 5.45</u>. *Let* N_a *be an almost observability subspace. Then the* N_a-*observer index of* x *is equal to* $\dim N_a$.

<u>PROOF</u>: From TH. 5.40, the N_a-observer index ν of x satisfies $\nu \leq \dim N_a$. On the other hand, every N_a-PID-observer S for x must satisfy $N_a \cap S = \{0\}$. Thus, we also have $\nu \geq \dim N_a$.

□

From the previous theorem, note that the PID-observer (5.29) has minimal order, in the sense that its dynamic order n_a $(= \dim N_a)$ is equal to the N_a-observer index of x. As a special case of this, we find that the 'full order' observer has minimal order (its dynamic order $n_a = n$ is equal to the N_a-observer index of x with $N_a = X$). As another special case, we obtain that the 'reduced order' observer has minimal order (its dynamic order $n_a = n - p$ is equal to the N_a-observer index of x with $N_a = \ker C$).

Finally, we will dualize the results from SECTION 5.6 to establish the existence of minimal order PID-observers for a single linear functional of the state. In (5.31), assume that $Z = \mathbb{R}$ and to stress this, write $D = d$, where d is a linear functional on X. We will assume that the entire observation $y(t)$, together with all its derivatives up to the order k-1 may be used for direct feedthrough. This corresponds to taking $\kappa_1 = \kappa_2 = \ldots = \kappa_p = k - 1$. Here, $k = 0$ means that no direct feedthrough is allowed at all. The mapping R (see (5.24)) corresponding to this choice is $R = R_k$, where

$$R_k := \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{pmatrix} \; ; \quad R_0 := 0 \; .$$

Note that for all i, $L_i = I$ and $Y_i = Y$. Denote $N_k = \ker R_k$. As noted before, the N_k-observer index of dx is equal to the $N_k^{\perp}$-cover index of im d^T (with respect to (A^T, C^T)). The latter integer $\nu(k)$ can be found using TH. 5.44. It follows immediately that for each symmetric set Λ of $\nu(k)$ complex numbers, a minimal order N_k-PID-observer S for dx exists, and a mapping $G: Y \to X$ such that $(A+GC)S \subset S$ and $\sigma(A+GC|X/S) = \Lambda$. This leads to a PID-observer for $z(t) = dx(t)$:

$$\dot{w}(t) = Nw(t) + My(t) \; ,$$

$$\hat{z}(t) = gw(t) + f_0 y(t) + f_1 \frac{d}{dt} y(t) + \dots + f_{k-1} \frac{d^{k-1}}{dt^{k-1}} y(t) \; ,$$

with $\sigma(N) = \Lambda$, of dynamic order $\nu(k)$. Here g and f_i are linear functionals on W and Y respectively. The observer (5.33) has minimal order in the sense that its dynamic order is equal to the N_k-observer index of dx. Note that the original result in WONHAM & MORSE (1972) is recovered from the above by taking $k = 1$.

In particular, for a given $d: X \to \mathbb{R}$, *it is possible to find a* PID-*observer for* $z(t) = dx(t)$, *using* $y(t), y^{(1)}(t), \dots, y^{(k-1)}(t)$ *for direct feedthrough of dynamic order* $\max\{0, \lambda_1 - k\}$. *Here,* λ_1 *is the largest observability index of the pair* (C,A). It is always possible to find a I-observer for $z(t) = dx(t)$ (no direct feedthrough at all) of dynamic order λ_1.

CONCLUDING REMARKS

In chapter 1 we have introduced the basic concepts of almost controlled invariant subspace and almost controllability subspace. These are defined in terms of their approximate holdability properties with respect to the state trajectories of the underlying linear system. We show that these subspaces can be characterized geometrically in terms of the linear maps defining our system. Rigorous proofs of these geometric characterizations turn out to be quite involved. In order to streamline these proofs we introduce the notion of 'factor system modulo S^{∞}'.

In chapter 2 we first show that almost controlled invariant subspaces and almost controllability subspaces can be regarded as 'ordinary' controlled invariant and controllability subspaces if we allow distributions as inputs. Subsequently, we establish frequency domain descriptions of the subspaces under consideration by characterizing them in terms of (ξ,ω)-representations (see also HAUTUS (1980) and SCHUMACHER (1983a)).

The purpose of the second part of chapter 2 is to make the concepts of almost controlled invariant subspace and almost controllability subspace applicable to feedback synthesis problems. As a first application, in section 2.6 we treat the L_p/L_q-almost disturbance decoupling problem. The latter problem was introduced originally in WILLEMS (1980) (see also WILLEMS (1981)). Here, we give a detailed and rigorous treatment of this problem, based on the approximation of almost controlled invariant subspaces by 'ordinary' controlled invariant subspaces (see section 2.4) and on the construction of controlled invariant complements of almost controllability subspaces (see section 2.5). Section 2.7 deals with almost controllability subspaces and their application to the L_p/L_q-almost disturbance decoupling problem with pole placement. The chapter closes down with a section on almost stabilizability subspaces. These were introduced orginally in SCHUMACHER (1984). Here, we establish a geometric characterization of these subspaces for arbitrary stability sets.

Chapter 3 deals with L_p-almost controlled invariant subspaces and L_p-almost controllability subspaces. The basic set-up on approximation of almost controlled invariant subspaces as treated in chapter 2 is used here to establish a rigorous treatment of the L_p-almost disturbance decoupling problem. We also discuss the version of this problem with pole placement.

An open problem still remains to find necessary and sufficient conditions for solvability of the L_p-almost disturbance decoupling problem with internal asymptotic stability (ADDPS).

In section 3.5 we use the concepts introduced to obtain some new results on the classical problem of finding low order stabilizing dynamic compensators. It is shown that invertible minimum phase system can always be stabilized using dynamic output feedback with dynamic order equal to the system's pole/zero excess minus the number of inputs. The latter result provides an interesting addition to a series of earlier results as for example described in LUENBERGER (1964), BRASCH & PEARSON (1970), KIMURA (1975), WANG & DAVISON (1975), KIMURA (1977) and HERMANN & MARTIN (1977).

Chapter 4 is devoted to problems of constrained L_p-almost disturbance decoupling: apart from approximate decoupling up to any desired degree of accuracy from the disturbances to a first to-be-controlled output, it is required that the operator from the disturbances to a second to-be-controlled output is a bounded function of the decoupling accuracy. In our treatment of the above problem it turns out that we first need to resolve a synthesis problem of combined exact disturbance decoupling and output stabilization. This is done in section 4.2. The results of the latter section can be specialized to obtain results in HAUTUS (1981), BHATTACHARRYA, PEARSON & WONHAM (1972) and WONHAM (1979) as special cases. We remark that recently the results of section 4.2 have been extended to the situation that, instead of allowing state feedback, we restrict ourselves to output feedback (see VAN DER WOUDE (1986)). As an area of future research we mention a possible application of the results of chapter 4 to problems of exact and almost non-interacting control.

In the first section of chapter 5 the dual concepts of almost conditionally invariant subspace, almost observability subspace and almost detectability subspace are discussed. Different from the definitions in WILLEMS (1982a) these subspaces are introduced by dualizing the feedback characterizations of the relevant almost controlled invariant subspaces as established in earlier chapters. Subsequently, we treat the almost disturbance decoupling problem by measurement feedback. Our treatment is based entirely on WILLEMS (1982a). In sections 5.3 and 5.4 we discuss a version of the latter problem with 'roll-off' constraints. Finally, in the last three sections of this tract we show the relevance of almost observability subspaces and almost controllability subspaces in the context of design of reduced and minimal order PID-observers. The results described here are extensions of earlier results on observer design in LUENBERGER (1964), WONHAM (1970) and WONHAM & MORSE (1972).

A striking fact in our treatment of the almost disturbance decoupling problem by measurement feedback and, subsequently, its constrained version with 'roll-off' constraints, is that the methods we use are strongly frequency domain oriented. An open question remains how these problems can be treated in a more geometric style, perhaps comparable with treatments of the problem of exact disturbance decoupling by measurement feedback as in SCHUMACHER (1980) or WILLEMS & COMMAULT (1981). Undoubtedly, such a geometric treatment of ADDPM would also yield to resolving the as yet unsolved problems of almost disturbance decoupling with measurement feedback and internal stability, ADDPMS, and the version of the same problem with pole placement, ADDPMPP.

APPENDIX

SOME BASIC FACTS ON DISTRIBUTIONS

In this appendix we have collected some basic material on distributions. For more details, the reader is referred to, for example, RUDIN (1973) or SCHWARTZ (1951).

Let $C^\infty(\mathbb{R}, \mathbb{R})$ denote the space of all functions $\varphi : \mathbb{R} \to \mathbb{R}$ that have derivatives of arbitrary order. For any function $\varphi : \mathbb{R} \to \mathbb{R}$, its support is defined as the closure of the set $\{t \in \mathbb{R} | \varphi(t) \neq 0\}$. The support of φ is denoted by supp φ. In CH. 2 of this tract, the following space of test-functions is used:

$$D(\mathbb{R}) : = \{\varphi \in C^\infty(\mathbb{R}, \mathbb{R}) \mid \varphi \text{ has compact support}\}.$$

There is a standard way to endow the space $D(\mathbb{R})$ with a topology. With respect to this topology, we define the space of distributions D' as the dual of the space $D(\mathbb{R})$:

$$D' : = \{x : D(\mathbb{R}) \to \mathbb{R} \mid x \text{ is linear and continuous}\}.$$

The value of $x \in D'$ at $\varphi \in D(\mathbb{R})$ will be denoted by $\langle x,\varphi\rangle$, If $\Omega \subset \mathbb{R}$ is open and $x \in D'$, then we will say that $x = 0$ on Ω if $\langle x,\varphi\rangle = 0$ for all $\varphi \in D(\mathbb{R})$ with supp $\varphi \subset \Omega$. Given $x \in D'$, let $A_x : = \{\Omega \subset \mathbb{R} | \Omega \text{ is open and } x = 0 \text{ on } \Omega\}$. Then the support of x is defined by

$$\text{supp } x =: \mathbb{R} \setminus \bigcup_{\Omega \in A_x} \Omega.$$

In this tract, D'_+ denotes the subspace of all $x \in D'$ with supp $x \subset \mathbb{R}^+$. Given $T \geq 0$, $D'_{[0,T]}$ denotes the subspace of all $x \in D'$ with supp $x \subset [0,T]$.

Let $L_{1,loc}(\mathbb{R}, \mathbb{R})$ denote the space of all locally integrable functions, i.e. the space of all measurable functions $\psi : \mathbb{R} \to \mathbb{R}$ such that $\int_K |\psi| < \infty$, for every compact set $K \subset \mathbb{R}$. This space can be identified with a subspace of D' by the identification

$$\langle \psi,\varphi \rangle : = \int_{\mathbb{R}} \psi(t)\varphi(t)dt \ , \quad \text{for } \varphi \in D(\mathbb{R}).$$

We will now define what we mean by the convolution of two distributions in D'_+. One way to do this, is to enlarge the underlying space of testfunctions. Define a space of testfunctions by

$$D_\infty(\mathbb{R}) := \{\varphi \in C^\infty(\mathbb{R}, \mathbb{R}) \mid \exists\, \lambda \in \mathbb{R} \text{ with supp } \varphi \subset (-\infty, \lambda]\}.$$

Again, there is a standard way to put a topology on this space (see e.g. SCHWARTZ (1951)). Taking the dual space of $D_\infty(\mathbb{R})$ yields a space of distributions that will be denoted by D'_∞. Since $D(\mathbb{R}) \subset D_\infty(\mathbb{R})$, D'_∞ may be considered as a subspace of D'. Given a function $\varphi : \mathbb{R} \to \mathbb{R}$ and a $\tau \in \mathbb{R}$, we define $(\sigma^\tau \varphi)(t) := \varphi(t+\tau)$ and $\tilde{\varphi}(t) := \varphi(-t)$. In addition to $D_\infty(\mathbb{R})$, we consider the space

$$D_{-\infty}(\mathbb{R}) := \{\psi \in C^\infty(\mathbb{R}, \mathbb{R}) \mid \exists\, \lambda \in \mathbb{R} \text{ with supp } \psi \subset [\lambda, \infty)\}.$$

$D_{-\infty}(\mathbb{R})$ can be considered as a subspace of D'_∞ by the obvious identification. Now, if $x \in D'$ and $\psi \in D_{-\infty}(\mathbb{R})$, then we define their convolution as the function

$$(x * \psi)(t) := \langle x, \sigma^{-t} \tilde{\psi} \rangle, \quad t \in \mathbb{R}.$$

The function $x * \psi$ is in $D_{-\infty}(\mathbb{R})$. Next, if x and y are two distributions in D'_∞, then we define their convolution as the distribution

$$\langle x * y, \varphi \rangle := \langle x, (y * \tilde{\varphi})^\sim \rangle \quad , \quad \varphi \in D_\infty(\mathbb{R}).$$

The distribution $x * y$ is in D'_∞. Since D'_+ can be identified with a subspace of D'_∞ and since $D(\mathbb{R}) \subset D_\infty(\mathbb{R})$, this also defines a convolution operation for distributions in D'_+. Moreover, if $x, y \in D'_+$, then $x * y \in D'_+$. It is in this sense that the convolution appearing in CH. 2, DEF. 2.2 should be interpreted.

Finally, we will spend a few words on tempered distributions. For this, we will consider yet another space of testfunctions. A function $\varphi \in C^\infty(\mathbb{R}, \mathbb{R})$ will be called rapidly decreasing if

$$\sup_{t \in \mathbb{R}} \left| t^\alpha \frac{d^\beta \varphi}{dt^\beta} \right| < \infty,$$

for all nonnegative integers α and β. Denote the space of smooth, rapidly decreasing functions by

$$S(\mathbb{R}) := \{\varphi \in C^{\infty}(\mathbb{R}, \mathbb{R}) \mid \varphi \text{ is rapidly decreasing}\}.$$

With a standard topology on $S(\mathbb{R})$, the dual S' of $S(\mathbb{R})$ is called the space of tempered distributions. Obviously $D(\mathbb{R}) \subset S(\mathbb{R})$ and S' can be considered as a subspace of D'. We will denote by S'_+ the subspace of tempered distributions with support in $\mathbb{R}^+$. A sequence x_n in S' is said to converge to x if $\langle x_n, \varphi\rangle \to \langle x, \varphi\rangle$ $(n \to \infty)$ for every $\varphi \in S(\mathbb{R})$.

We conclude this appendix by noting that the vector valued versions of the above concepts are defined in the obvious way: D'^n, D'^n_+, S'^n_+, etc. are the spaces of n-vectors with components in D', D'_+, S'_+. If $K \in D'^{p\times m}_+$ and $n \in D'^m_+$, then the convolution $K * u$ is defined componentwise.

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SUBJECT INDEX

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SYMBOL INDEX

Symbols are followed by a brief explanation of their meaning and the number of the page on which they are defined or used in a typical way.

A	system mapping	2
(A,B)	system with system mapping A and input mapping B	5
$\langle A \mid \mathcal{B} \rangle$	infimal A-invariant subspace containing $\mathcal{B}$ (reachable subspace)	7
$(\bar{A},\bar{B})$	factor system modulo $\mathcal{S}^{\infty}$ (unless otherwise stated)	21, 26
(A,B,C)	system with system mapping A, input mapping B and output mapping C	106
A_F	closed loop system mapping A + BF	7
B	control input mapping	3
$\mathcal{B}$	the image of the mapping B in $\mathcal{X}$	7
C	observation output mapping	106
$\mathbb{C}$	field of complex numbers	64
$\mathbb{C}^+$	closed right half complex plane	148
$\mathbb{C}_o^+$	open right half complex plane	110
$\mathbb{C}^-$	open left half complex plane	73
(C,A)	system with system mapping A and output mapping C	125
$C^{\infty}(\mathbb{R},\mathcal{X})$	smooth functions $\mathbb{R} \to \mathcal{X}$	20
$\mathcal{D}$	linear space in which disturbances take their values	69
D'	space of distributions	225
D'_+	space of distributions with support in $\mathbb{R}^+$	225
D'_B	space of Bohl distributions	43
D'^n	n-vectors with components in D'	227
D'^n_+	n-vectors with components in D'_+	227
D'^n_B	n-vectors with components in D'_B	43

$D_+^{\prime m\times r}$	$m \times r$ matrices with components in $D_+^{\prime}$	197
deg f	McMillan degree of f	76
$D(\mathbb{R})$	space of test functions	225
F	feedback mapping (from X to U)	3, 69
$\underline{F}(V)$	set of all F such that V is (A+BF)-invariant	7
G	disturbance input mapping	69
G	output injection mapping (in CH. 5)	128
$G_{cl}(s)$	closed loop transfer matrix	192
$\underline{G}(q,X)$	space of q-dimensional subspaces of X (Grassmannian manifold)	58
H, H_1, H_2	output mappings (to-be-controlled outputs)	69
H^∞	Hardy space with respect to $\mathbb{C}_o^+$	110
I_X	identity mapping of the space X	65
K, L, M, N	mappings defining a compensator, a feedback processor or an observer	125,181,191
$K(s)$	all vectors in $X(s)$ that lie in K for all s	45
$K[s]$	all vectors in $X[s]$ that lie in K for all s	45
$K_+(s)$	all vectors in $X_+(s)$ that lie in K for all s	45
$\langle \ker C \mid A \rangle$	supremal A-invariant subspace contained in ker C (unobservable subspace)	182
L	one-sided Laplace transform	45
ℓ	dimension of the output space Z	69
$L_p(\mathbb{R}^+,X)$	for $p \in [1,\infty)$: space of all measurable functions $x\colon \mathbb{R}^+ \to X$ such that $\int_{\mathbb{R}^+} \lVert x \rVert^p < \infty$	70
$L_\infty(\mathbb{R}^+,X)$	space of all measurable functions $x\colon \mathbb{R}^+ \to X$ such that $\operatorname{ess\,sup}_{t\in\mathbb{R}^+} \lVert x(t) \rVert < \infty$	70
$\mathcal{L}(u,F,k)$	k-dimensional singly generated almost controllability subspace	59
$\mathcal{L}_n(u,F,k)$	canonical approximating sequence for $\mathcal{L}(u,F,k)$	60 ,61

m	dimension of the input space $\mathcal{U}$	6
n	dimension of the state space $\mathcal{X}$	5
$\mathbb{N}$	set of natural numbers $\{1,2,\ldots\}$	14
$\underline{\mathcal{N}}_a(C,A)$	set of almost observability subspaces	185
$\mathcal{N}_a^*(\mathcal{G})$	infimal almost observability subspace containing $\mathcal{G}$	185
$\mathcal{N}_b^*(\mathcal{G})$	infimal L_p-almost observability subspace of $\mathcal{G}$	188
ord x	distributional order of impulsive distribution x	47
P	(often) canonical projection	6
p	dimension of the output space $\mathcal{Y}$	106
r	dimension of the disturbance space $\mathcal{D}$	69
$\mathbb{R}$	field of real numbers	6
$\mathbb{R}^+$	nonnegative real numbers	6
$\mathbb{R}^-$	negative real numbers	6
$\underline{\mathcal{R}}(A,B)$	set of all controllability subspaces	7
$\mathcal{R}_a^*(\mathcal{K})$	supremal almost controllability subspace in $\mathcal{K}$	10
$\underline{\mathcal{R}}_a(A,B)$	set of all almost controllability subspaces	9
$\mathcal{R}^*(\mathcal{K})$	supremal controllability subspace in $\mathcal{K}$	8
$\mathcal{R}_b^*(\mathcal{K})$	supremal L_p-almost controllability subspace of $\mathcal{K}$	99
$\mathbb{R}(s)$	field of rational functions with coefficients in $\mathbb{R}$	106
$\mathcal{R}_b(\mathcal{K}_1,\mathcal{K}_2)$	two-output constrained almost controllability subspace	167
r(T)	roll-off of rational matrix T	197
$\mathbb{R}^{m\times p}(s)$	space of all real rational $m \times p$ matrices	195
$\mathbb{R}_\rho^{m\times p}(s)$	space of all elements T in $\mathbb{R}^{m\times p}(s)$ with $r(T) \geq \rho$	200
$\mathbb{R}_0^{m\times p}(s)$	space of all proper real rational $m \times p$ matrices	193
$\mathcal{S}^\infty$, $\mathcal{S}^\infty(\mathcal{K})$	limiting subspace of ACSA	18, 19
$\mathcal{S}_a^*(\mathcal{G})$	infimal almost conditionally invariant subspace containing $\mathcal{G}$	185
$\mathcal{S}_b^*(\mathcal{G})$	infimal L_p-almost conditionally invariant subspace of $\mathcal{G}$	189

S'_+	space of tempered distributions with support in $\mathbb{R}^+$	227
$S'^{m\times r}_+$	space of m × r matrices with components in S'_+	197
$S'_+(\sigma_0)$	subspace of D'_+	198
$[S'_+(\sigma_0)]^{m\times r}$	space of m × r matrices with components in $S'_+(\sigma_0)$	198
$\underline{S}_a(C,A)$	set of all almost conditionally invariant subspaces	185
$S^*_{a,g}(G)$	infimal almost detectability subspace containing G with respect to the stability set $\mathbb{C}_g$	185
$\underline{S}_{a,g}(C,A)$	set of all almost detectability subspaces w.r.t. $\mathbb{C}_g$	185
U	input space (domain of B)	3
U_D	space of admissible distributional inputs	41
$\underline{V}(A,B)$	set of all controlled invariant subspaces	7
$\underline{V}_a(A,B)$	set of all almost controlled invariant subspaces	9
$\underline{V}_{a,g}(A,B)$	set of all almost stabilizability subspaces with respect to $\mathbb{C}_g$	84
$\underline{V}_g(A,B)$	set of all stabilizability subspaces with respect to $\mathbb{C}_g$	84
$V^*_a(K)$	supremal almost controlled invariant subspace in K	10
$V^*_{a,g}(K)$	supremal almost stabilizability subspace with respect to $\mathbb{C}_g$ in K	84
$V^*_b(K)$	supremal L_p-almost controlled invariant subspace of K	99
$V^*_g(K)$	supremal stabilizability subspace with respect to $\mathbb{C}_g$ in K	84
$V^*(K)$	supremal controlled invariant subspace in K	8
$V_b(K_1,K_2)$	two-output constrained almost controlled invariant subspace	147
$V_g(K_1,K_2)$	two-output constrained controlled invariant subspace	140
X	state space	5
$x(x_0,u)$	state trajectory with initial condition x_0 and output $u \in U_D$	42
$x^+(x_0,u)$	restriction of $x(x_0,u)$ to $\mathbb{R}^+$	42
$x_i(n,u)$	sequence of vectors converging to $A_F^{i-1}\,Bu$	61

X_{det}	undetectable subspace of (C,A) (with respect to $\mathbb{C}_g$)	183
X_{stab}	stabilizable subspace of (A,B) (with respect to $\mathbb{C}_g$)	68
$X(s)$	space of n-vectors with components in $\mathbb{R}(s)$	45
$X_+(s)$	space of n-vectors whose components are strictly proper elements of $\mathbb{R}(s)$	45
$X[s]$	space of n-vectors whose components are polynomials with real coefficients	45
Y	output space (measurements)	106
Z, Z_1, Z_2	output spaces (to-be-controlled outputs)	69, 138
$\mathbb{Z}$	set of integers $\{\ldots,-1,0,1,\ldots\}$	197
δ, $\delta^{(0)}$	Dirac distribution	11
$\delta^{(i)}$	i th derivative of Dirac distribution	11
λ_i	observability indices	216
μ_i	controllability indices	218
σ^*	set of invariant zeros	126
$\sigma(f)$	spectrum of Bohl function f	76
$\sigma(A)$	spectrum of mapping A	65
$\Sigma(A,B)$	absolutely continuous state trajectories	6
$\tilde{\Sigma}(A,B)$	smooth state trajectories	20
$\Sigma^B(A,B)$	state trajectories that are regular Bohl functions	76
$\Sigma^{B(r)}(A,B)$	elements x in $\Sigma^B(A,B)$ with deg $x \leqq r$	76
$\Sigma_g^{B(r)}(A,B)$	elements x in $\Sigma^{B(r)}(A,B)$ with $\sigma(x) \subset \mathbb{C}_g$	76
$\Sigma_D(A,B)$	state trajectories generated by inputs in U_D	42
$\dot{\cup}$	disjoint union of spectra	81
$\underline{n}$	set $\{1,2,3,\ldots,n\}$	6
$\perp$	orthogonal complement	133

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