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Realization of Positive Linear Systems

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ABSTRACT

Positive linear systems are frequently used as mathematical models in research areas like biology and economics. The problem of classifying all minimal realizations of these systems is treated in this paper. Extensive use is made of the theory of polyhedral cones. Sufficient and necessary conditions for the existence of a positive realization are given, but the problem of minimality leads to the factorization problem for positive matrices. Ideas and results are given to come towards a solution of this factorization problem. © Elsevier Science Inc., 1997

1. INTRODUCTION

The purpose of this paper is to present results on the realization of positive linear systems.

The motivation for the study of positive realizations is the use of positive linear systems in biomathematics, economics, chemometrics, and other research areas. A *finite-dimensional positive linear system* is a linear dynamical system in which the input, state, and output space are spaces over the positive real numbers. Systems in this class are useful models in biomathematics, where they are called *linear compartmental systems*; see [9]. The identification problem for this class of systems is unsolved. There are no

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conditions known for global structural identifiability of such systems [8]. These conditions may be based on realization theory for positive linear systems. An early reference on structural identifiability is [1].

In this paper we will deal with time-invariant finite-dimensional positive linear systems in discrete time; for short we will call them positive linear systems. A positive realization of a given positive impulse response function is a positive linear system whose impulse response function equals the given one. A positive realization of an impulse response function is said to be minimal if the state space as a vector space over the positive real numbers is of minimal dimension. The positive realization problem is to show the existence of a positive realization of a positive impulse response function and to classify all minimal positive realizations. To solve the problem we will use techniques of the theory for polyhedral cones. Kodama and coworkers have worked on this problem [10, 11, 13]. Only in the last paper have they used polyhedral cones explicitly. Another reference on positive linear systems is [2], and other references on positive realizations are [12, 19].

The realization problem for positive linear systems is closely related to the stochastic realization problem for finite-valued processes, which has been studied by G. Picci [14, 16].

Up to now the positive realization problem is unsolved. In this paper we will give a necessary and sufficient condition for the existence of a positive realization. Unsolved questions are the characterization of minimality of the state space, which leads to a factorization problem for positive matrices, and the classification of all minimal positive realizations.

The outline of this paper is as follows. The positive realization problem is formulated in Section 2. In Section 3 some theory of polyhedral cones is presented. The existence of a positive realization is proven in Section 4. In Section 5 some results on the characterization of minimality are given. In Section 6 we will say something about the relation between minimal positive realizations.

2. PROBLEM FORMULATION

In this section some notation is introduced and the problem is posed.

The set $R_+ = [\,0, +\infty)$ is the set of the *positive real numbers*. Let $Z_+ = \{1, 2, \dots\}$ denote the set of positive integers, $Z_n = \{1, \dots, n\}$, and $\mathbb{N} = \{0, 1, 2, \dots\}$. Denote by R_+^n the set of n-tuples of the positive real numbers. The set $R_+^{n \times m}$ is the set of *positive matrices* of size n by m. Note that R_+^n is not a vector space, because it does not admit an inverse with respect to addition. For two sets A and B, the notation $A \subsetneq B$ means that A is a strict subset of B.

DEFINITION 2.1. A *positive linear system* (in place of the more formal term "time-invariant finite-dimensional positive linear system in discrete time") is a linear dynamical system in which the input, state, and output space are respectively $U = R_+^m$, $X = R_+^n$, and $Y = R_+^k$, and the time index set is $T = \mathbb{N}$. The system will be represented by

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t), \qquad x(0) = x_0, \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1}$$

in which $A \in R_+^{n\times n}$, $B \in R_+^{n\times m}$, $C \in R_+^{k\times n}$, and $D \in R_+^{k\times m}$.

A positive matrix may be regarded not only as a matrix, but also as a polyhedral cone, as we shall see. The relationship between the matrix and the geometric approach is very useful. Because R_+^n is not a vector space, we cannot use the usual linear algebra. Therefore we treat the positive-realization problem through convex-cone analysis.

PROBLEM 2.2 (The *positive-realization problem* for a positive impulse response function).

a. Formulate necessary and sufficient conditions for the existence of a positive linear system whose impulse response function equals the given impulse response function. If such a system exists, it is called a *positive realization* of the given impulse response function.

b. Determine the minimal dimension of the state space of a positive realization.

c. Classify all positive realizations of the given impulse response function.

d. If two positive realizations of the same impulse response function are minimal, then indicate the relation between them.

3. POLYHEDRAL CONES

This section contains definitions and results on polyhedral cones which will be required in this paper. References on this subject are [2, 3, 6, 7].

3.1. *Convex Cones*

For $k \in Z_+ \cup \{+\infty\}$, we shall first consider the vector space R^k.

DEFINITION 3.1. A subset C of R^k is called a *convex cone* if the following two conditions hold:

1. $v_1, v_2 \in C$ implies $v_1 + v_2 \in C$,
2. $v \in C$ and $\lambda \in R_+$ imply $\lambda v \in C$.

If we have a set $S \subseteq R^k$, we denote by $C(S)$ the set consisting of all finite nonnegative linear combinations of elements of S. We say that $C(S)$ is generated or *spanned* by S. It is easy to see that $C(S)$ is a convex cone. Therefore we will write cone(S) instead of $C(S)$.

DEFINITION 3.2. A convex cone C is said to be a *polyhedral cone* if it is spanned by a finite number of vectors $v_1, v_2, \dots, v_n \in R^k$.

Thus C is a polyhedral cone if and only if there exists a finite set $S \subset R^k$ such that $C = \text{cone}(S)$. We call S the *set of spanning vectors* of C.

DEFINITION 3.3. C is said to be a *subcone* of C_1 if C and C_1 are convex cones and $C \subseteq C_1$.

A subcone of a polyhedral cone need not be polyhedral.

DEFINITION 3.4. Let $D\{C\}$ denote the intersection of all subspaces containing a convex cone C. Let $d(C)$ denote the dimension of $D\{C\}$. Then $d(C)$ is said to be the *dimension* of C.

The *relative boundary* of C, denoted by $\partial(C)$, is the boundary of C when C is considered embedded in $D\{C\}$.

From now on we only consider convex cones which are contained in R^k_+, for $k \in Z_+ \cup \{+\infty\}$. We have $S \subseteq R^k_+$ if and only if $\text{cone}(S) \subseteq R^k_+$. To see the relationship between polyhedral cones and positive matrices, we state the following result.

PROPOSITION 3.5. *$C \in R^k_+$ is a polyhedral cone if and only if there exist an $n \in Z_+$ and an $A \in R^{k \times n}_+$ such that $C = \{Ax \in R^k_+ \mid x \in R^n_+\}$.*

Proof. Letting the columns of A be the vectors $v_1, v_2, \dots, v_n \in R^k_+$ of Definition 3.2, we immediately see the equivalence. ∎

Because of this proposition, we will also use the notation cone(A) for the cone generated by the matrix A. Another important tool we need is given in the following definition.

DEFINITION 3.6. A finite set of vectors in R_+^k, say $\{v_1, \dots, v_m\} \subset R_+^k$, is said to be *positively dependent* if there exists an $i \in Z_m$ such that v_i can be written as a nonnegative linear combination of $\{v_j,\ j \in Z_m,\ j \neq i\}$, or, equivalently,

$$v_i = \sum_{j=1, j \neq i}^{m} \lambda_j v_j, \qquad \text{in which} \quad \lambda_j \in R_+.$$

It is said to be *positively independent* otherwise.

DEFINITION 3.7. A finite set of vectors (nonempty, not all zero) $\{v_1, \dots, v_m\} \subset R_+^k$ is said to be a *frame* of the polyhedral cone $C \subseteq R_+^k$ if (1) the set $\{v_1, \dots, v_m\}$ is positively independent and (2) the set spans the cone C. The integer m is said to be the *size* of the frame.

Let $k, m \in Z_+$, $m \leqslant k$. Denote the set of polyhedral cones with a frame of size m as

$$C_{k,m} = \left\{ C \subseteq R_+^k \,\middle|\, C \text{ is a polyhedral cone with a frame of } m \text{ vectors} \right\}.$$

The following definition and propositions come from [7]. We need them for Definition 3.11.

DEFINITION 3.8. Let C be a polyhedral cone in R_+^k of dimension m. Then C has one m-facet, itself, and no r-facets for $r > m$. If $r < m$, then F is an r-facet of C if

1. F is a subcone of an $(r+1)$-facet G;
2. $F \subseteq \partial(G)$;
3. no subcone of G contained in $\partial(G)$ properly contains F;
4. $F \neq \emptyset$.

Denote by $\mathcal{F}_r(C)$ the set of r-facets of C.

PROPOSITION 3.9. *Let C be a polyhedral cone and F an r-facet of C. Then*

1. $d(F) = r$;
2. *F is a polyhedral cone*;
3. *any frame of C contains a subset spanning F.*

PROPOSITION 3.10. *Let C be a polyhedral cone and F a subcone of C. Then F is a facet of C if and only if $a, b \in C$ and $a + b \in F$ imply $a, b \in F$.*

Define on $C_{k,m}$ an order relation by inclusion of cones. New in this context is the notion of extremal cone, which is defined below.

DEFINITION 3.11. Let $k, m \in Z_+$, $m \leqslant k$. A cone in $C_{k,m}$ is said to be an *extremal cone* if it is a maximal element in $C_{k,m} \setminus \mathscr{F}_m(R_+^k)$ with respect to the order relation; denote

$$CE_{k,m} = \left\{C \in C_{k,m} \setminus \mathscr{F}_m\left(R_+^k\right) \middle| C \text{ an extremal cone}\right\}.$$

Note that R_+^k is the only maximal element in $C_{k,k}$, and $\mathscr{F}_k(R_+^k) = \{R_+^k\}$. In Section 5 we will characterize the matrices that correspond to extremal cones. For that purpose we need the concept of primes, which is the topic of the next subsection.

3.2. *Primes*

The following definitions are stated in [17]. For more results, see [15].

DEFINITION 3.12. A positive matrix $M \in R_+^{n\times n}$ is said to be a *monomial matrix* if every row and every column contains exactly one strictly positive element.

DEFINITION 3.13. A *prime* in the positive matrices is a matrix $A \in R_+^{n\times n}$ such that

1. A is not a monomial matrix;
2. if $A = BC$ with $B, C \in R_+^{n\times n}$, then either B or C is a monomial matrix.

EXAMPLES. In $R_+^{2\times 2}$ are no primes. In $R_+^{3\times 3}$ all primes can be written as

$$A = M_1 \begin{pmatrix} 1-s & s & 0 \\ s & 0 & 1-s \\ 0 & 1-s & s \end{pmatrix} M_2$$

for M_1, M_2 monomials and $s \in (0, 1)$. In $R_+^{n\times n}$ examples of primes are given by

$$A = M_1 \begin{pmatrix} s & 0 & \cdots & 0 & 1-s \\ 1-s & s & & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & & s & 0 \\ 0 & 0 & \cdots & 1-s & s \end{pmatrix} M_2$$

for M_1, M_2 monomials and $s \in (0, 1)$. More examples can be found in [15, 17].

A new definition in this context is the following:

DEFINITION 3.14. Let $k, m \in Z_+$, $m \leqslant k$. A positive matrix $A \in R_+^{k \times m}$ is said to be *part of a monomial* in $R_+^{k \times k}$ if there exists a $B \in R_+^{k \times (k-m)}$ such that

$$(A \quad B)$$

is a monomial in $R_+^{k \times k}$.

It follows that A is part of a monomial if and only if A has exactly one strictly positive element in every column and at most one strictly positive element in every row. At this point we can give a relation between facets of R_+^k and matrices that are parts of a monomial.

PROPOSITION 3.15. *Let* $k, m \in Z_+$, $m \leqslant k$, *and* $A \in R_+^{k \times m}$. *Then* cone(A) *is an m-facet of* R_+^k *if and only if A is part of a monomial.*

Proof. $\Rightarrow$: Let cone(A) be an m-facet. With Proposition 3.9, $d(\text{cone}(A)) = m$. Assume A is not part of a monomial. Then we can consider three cases:

1. A has a column that has no strictly positive elements;
2. A has a column that has more than one strictly positive element;
3. A has a row that has more than one strictly positive element.

In case 1 we immediately see that $d(\text{cone}(A)) \leqslant m - 1$, which is a contradiction.

In case 2, let a be the column with $a_{i_1}, \dots, a_{i_r} > 0$ for $r > 1$, and the other elements zero. Because $a = a_{i_1}e_{i_1} + \cdots + a_{i_r}e_{i_r}$, with e_j the jth unit vector in R^k, it follows from Proposition 3.10 that $a_{i_1}e_{i_1}, \dots, a_{i_r}e_{i_r} \in \text{cone}(A)$. But then $a_{i_1}e_{i_1}, \dots, a_{i_r}e_{i_r}$ are spanning vectors of cone(A), so they have to be columns of A. Since a is linearly dependent of $a_{i_1}e_{i_1}, \dots, a_{i_r}e_{i_r}$, $d(\text{cone}(A)) \leqslant m - 1$. This contradicts the fact that $d(\text{cone}(A)) = m$.

Case 3 is not possible, because in cases 1 and 2 we have already seen that A has exactly one positive element in every column, and the columns have to be linearly independent. So every row can contain at most one strictly positive element.

It follows that A is part of a monomial.

$\Leftarrow$: Let A be part of a monomial. Without loss of generality we may assume that $a_1, \dots, a_m$ are unit vectors in R^k_+. Consider the case $m = k - 1$. Let $a_{m+1} = a_k = e_j$, such that $(A \;\; e_j)$ is a monomial. Then cone(A) satisfies the conditions of Definition 3.8 with $G = R^k_+$, so cone(A) is a $(k - 1)$-facet. By embedding an $(m + 1)$-facet in R^{m+1}_+, the result follows by induction. ■

4. EXISTENCE OF A POSITIVE REALIZATION

The main theorems are given below. They give necessary and sufficient conditions for the existence of a realization of a positive impulse response function as a positive linear system. First we will consider the case without input, and then the general input-output case.

Let $T = \mathbb{N}$, $Y = R^k_+$. Let σ denote the *backward shift operator*

$$(\sigma y)(t) = y(t + 1) \qquad \text{for} \quad y : T \to Y.$$

A cone $C_1 \subseteq R^\infty_+$ is said to be *backward-shift-invariant* if $y_1 \in C_1$ implies $\sigma y_1 \in C_1$.

4.2. *Case of a System without Input*

THEOREM 4.1. *Let $T = \mathbb{N}$, $Y = R^k_+$. Consider a set V of trajectories, each of which is a function $y : T \to Y$. Let*

$$\text{cone}(V) = \text{cone}\left\{\begin{pmatrix} y(0)^T & y(1)^T & y(2)^T & \cdots \end{pmatrix}^T \in R^\infty_+ \,\middle|\, y \in V\right\} \subseteq R^\infty_+,$$

where y^T denotes the transpose of y. There exists a positive linear system

$$x(t + 1) = Ax(t), \qquad x(0) = x_0,$$
$$y(t) = Cx(t),$$

such that any element of V is represented by the output of this system for some $x_0 \in R_+^n$ *if and only if there exists a set* $C_1 \subseteq R_+^\infty$ *such that*

1. C_1 *is a polyhedral cone;*
2. $\text{cone}(V) \subseteq C_1$;
3. C_1 *is backward-shift-invariant.*

Proof. $\Rightarrow$: Assume the set V is represented by the positive linear system

$$x(t+1) = Ax(t), \qquad x(0) = x_0,$$

$$y(t) = Cx(t),$$

with $X = R_+^n$, $A \in R_+^{n\times n}$, $C \in R_+^{k\times n}$, for $n \in Z_+$. Let

$$S = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \end{pmatrix} \in R_+^{\infty \times n},$$

$$C_2 = \text{cone}(S) = SR_+^n \subseteq R_+^\infty.$$

Since C_2 has n spanning vectors, C_2 is a polyhedral cone. The set of trajectories generated by this system equals the cone C_2. Because the system is a realization,

$$\text{cone}(V) \subseteq C_2.$$

Let $y \in C_2$. From the definition of C_2 it follows that there exists an $x_0 \in R_+^n$ such that $y(t) = CA^t x_0$ for all $t \in T$. Then

$$(\sigma y)(t) = y(t+1) = CA^{t+1}x_0 = CA^t(Ax_0).$$

Because $Ax_0 \in R_+^n$, $\sigma y \in C_2 = \text{cone}(S)$. Therefore C_2 is backward-shift-invariant.

$\Leftarrow$: (a) Let $C_1 \subseteq R_+^\infty$ be a set satisfying conditions 1, 2, and 3. Because C_1 is a polyhedral cone, there exist an $n \in Z_+$ and an $S \in R_+^{\infty\times n}$ such that

$$C_1 = \text{cone}(S).$$

(b) Define $C \in R_+^{k \times n}$ by

$$C_{ij} = S_{ij}, \qquad i = 1, \dots, k, \quad j = 1, \dots, n.$$

(c) Let $e_i \in R_+^n$ be the ith unit vector. Define, for $i \in Z_n$, $y_i = Se_i \in R_+^{\infty}$. Then $y_i \in C_1$. Because C_1 is backward-shift-invariant, $\sigma y_i \in C_1$. Then there exists an $x_i \in R_+^n$ such that

$$\sigma y_i = Sx_i, \qquad i = 1, \dots, n.$$

Define $A \in R_+^{n \times n}$ by

$$A = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}.$$

Then $\sigma y_i = Sx_i = SAe_i$, $i = 1, \dots, n$.

(d) Our task is now to show that if $y = Sx$ for some $y \in \text{cone}(S) = C_1$ and $x \in R_+^n$, then $\sigma y = SAx$. To this end, suppose $y \in \text{cone}(S) = C_1$. Then there exists an $x \in R_+^n$ such that

$$y = Sx = S \sum_{i=1}^{n} \alpha_i e_i.$$

Then

$$\begin{aligned} \sigma y &= \sigma Sx = \sigma S \sum_{i=1}^{n} \alpha_i e_i = \sum_{i=1}^{n} \alpha_i \sigma Se_i \\ &= \sum_{i=1}^{n} \alpha_i \sigma y_i = \sum_{i=1}^{n} \alpha_i SAe_i = SA \sum_{i=1}^{n} \alpha_i e_i = SAx. \end{aligned}$$

(e) Write S as follows:

$$S = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ \vdots \end{pmatrix} \qquad \text{with} \quad S_t \in R_+^{k \times n} \quad \forall t \in T.$$

We shall prove that $S_t = CA^t$ $\forall t \in T$. By definition of C, we have $S_0 = C$. Suppose that $S_\tau = CA^\tau$, $\tau = 0, 1, \dots, t$. It will be shown that $S_{t+1} = CA^{t+1}$.

Let $y \in C_1$. Hence there exists an $x_0 \in R_+^n$ such that $y = Sx_0$. By step (d) we have $\sigma y = SAx_0$. Then

$$S_{t+1}x_0 = y(t+1) = (\sigma y)(t) = S_t Ax_0 = (CA^t) Ax_0 = CA^{t+1}x_0.$$

This holds for all $y \in C_1$, in particular for $y_i = Se_i$ as defined in (c). Hence $S_{t+1} = CA^{t+1}$. It follows that

$$C_1 = \operatorname{cone}(S) = \operatorname{cone}\begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \end{pmatrix}.$$

(f) Condition 2, $\operatorname{cone}(V) \subseteq C_1$, implies that for any trajectory y in $V \subseteq \operatorname{cone}(V)$, hence in C_1, there exists an $x_0 \in R_+^n$ such that $y = Sx_0$ or $y(t) = CA^t x_0$ for all $t \in T$. So y is the output of a positive linear system due to $x_0 \in R_+^n$. ■

REMARK 4.2. The realization problem for positive linear systems is closely related to the stochastic realization problem for finite-valued processes. The above theorem is inspired by the work of G. Picci [14].

REMARK 4.3. This result is entirely different from the work of Maeda, Kodama, et al. [10, 11, 13]. They start with a minimal realization in the usual sense. From this system they derive a positive linear system. They do not use the notion of backward-shift invariance.

4.2. *General Case*

THEOREM 4.4. *Let $T = \mathbb{N}$, $Y = R_+^k$, $U = R_+^m$. Consider a positive impulse response function $W : T \to R_+^{k\times m}$. Define*

$$H = \begin{pmatrix} W(1)^T & W(2)^T & W(3)^T & \cdots \end{pmatrix}^T,$$

and consider $\operatorname{cone}(H)$. *There exists a positive linear system*

$$x(t+1) = Ax(t) + Bu(t), \qquad x(0) = x_0,$$

$$y(t) = Cx(t) + Du(t),$$

whose impulse response function equals W *if and only if there exists a set* $C_1 \subseteq R_+^\infty$ *such that*

1. C_1 *is a polyhedral cone*;
2. $\operatorname{cone}(H) \subseteq C_1$;
3. C_1 *is backward-shift-invariant.*

Proof. $\Rightarrow$: Assume W is the impulse response function of the positive linear system

$$x(t+1) = Ax(t) + Bu(t), \qquad x(0) = x_0,$$
$$y(t) = Cx(t) + Du(t),$$

with $X = R_+^n$, $A \in R_+^{n \times n}$, $B \in R_+^{n \times m}$, $C \in R_+^{k \times n}$, $D \in R_+^{k \times m}$, for $n \in Z_+$. Let

$$S = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \end{pmatrix} \in R_+^{\infty \times n},$$

$$C_2 = \operatorname{cone}(S) = SR_+^n \subseteq R_+^\infty.$$

As in the proof of Theorem 4.1, C_2 is a polyhedral cone and backward-shift-invariant. Finally, we have

$$\operatorname{cone}(H) = \operatorname{cone}(S.B) \subseteq \operatorname{cone}(S) = C_2.$$

$\Leftarrow$: Steps (a) to (e) are as in the proof of Theorem 4.1. Step (f) is slightly different:

(f) Write H as follows:

$$H = \begin{pmatrix} H_1 & H_2 & \cdots & H_m \end{pmatrix} \quad \text{with} \quad H_i \in R_+^\infty, \quad i = 1, \dots, m.$$

Condition 2, $\operatorname{cone}(H) \subseteq C_1$, implies that for any H_i $(i = 1, \dots, m)$, $H_i \in \operatorname{cone}(H)$; hence in C_1, there exists a $B_i \in R_+^n$ such that $H_i = SB_i$. Define $B = (B_1 \ B_2 \ \cdots \ B_m)$. It follows that

$$SB = S\begin{pmatrix} B_1 & B_2 & \cdots & B_m \end{pmatrix} = \begin{pmatrix} SB_1 & SB_2 & \cdots & SB_m \end{pmatrix}$$
$$= \begin{pmatrix} H_1 & H_2 & \cdots & H_m \end{pmatrix} = H,$$

or $CA^tB = W(t+1)$ for all $t \in T$. Define $D = W(0)$. ∎

5. CHARACTERIZATION OF MINIMALITY

In this section we will give some results on the characterization of minimality.

Realization theory for linear dynamical systems over R^n gives necessary and sufficient conditions for minimality. See [18]. For positive linear systems these conditions are sufficient, but not necessary. If a positive linear system is such that (A, B) is a reachable pair and (A, C) is an observable pair, then the system is a minimal positive realization of its impulse response. The converse is not true in general. At the end of this section, we will give an example of a system that is a minimal positive linear system, but that is not minimal as a linear system over R^n. The problem is to find necessary and sufficient conditions for a positive linear system to be a minimal positive realization of the impulse response function.

5.1. *Positive Rank*

The complete characterization of minimality is currently unsolved. It appears to be a difficult problem. Attention is therefore restricted to a problem of positive linear algebra of which the solution is expected to contribute to the characterization of minimality. The problem requires a definition.

DEFINITION 5.1. Let $A \in R_+^{k \times m}$ for $k, m \in Z_+$. The *positive rank* of the matrix A is defined as the least integer $n \in Z_+$ for which there exists a factorization

$$A = BC \tag{2}$$

with $B \in R_+^{k \times n}$ and $C \in R_+^{n \times m}$. Let pos-rank(A) denote this integer.

A *positive matrix factorization* of A is any factorization of A of the form (2) for arbitrary $n \in Z_+$. A *minimal positive matrix factorization* of A is any positive matrix factorization of A in which $n =$ pos-rank(A). The matrix A is called *strictly factorizable* if there exists a positive matrix factorization of the form (2), with $n \leqslant \min\{k, m\}$, in which none of the matrices $\{B, B^T, C, C^T\}$ is part of a monomial.

The concept of positive rank is used in [4], where it is called *nonnegative rank*.

PROBLEM 5.2. Let $A \in R_+^{k \times m}$. Determine the positive rank of A and all minimal positive matrix factorizations of this matrix.

A related problem is the following.

PROBLEM 5.3. Let $H \in R_+^{k \times m}$, $A \in R_+^{k \times n}$, $B \in R_+^{n \times m}$ such that $H = AB$. Give necessary and sufficient conditions on A and B for the factorization $H = AB$ to be minimal.

It is known that for positive matrices in $R_+^{n \times n}$ with $n \leqslant 3$ the positive rank equals the ordinary rank over R, but in $R_+^{4 \times 4}$ there exists a positive matrix for which the positive rank is strictly larger than the ordinary rank; see the example at the end of Section 5.3. In general we have

$$\text{pos-rank}(AB) \leqslant \min\{\text{pos-rank}(A), \text{pos-rank}(B)\}.$$

For more details and results see [4].

The positive matrix factorization (2) can be interpreted in geometric terms as $\text{cone}(A) \subseteq \text{cone}(B)$, from which an interpretation of the positive rank follows.

The problem defined above is currently unsolved. The concept of an extremal cone may be useful for the determination of the positive rank.

5.2. *Extremal Cones*

In this subsection an approach to deal with the problems defined in the last subsection is sketched. We will start with the square case.

PROPOSITION 5.4. *Let $k \in Z_+$ and $A \in R_+^{k \times k}$. Then* $\text{cone}(A) \in CE_{k,k}$ *if and only if A is a prime in the positive matrices.*

Proof. $\Rightarrow$: If A is not a prime, then either it is a monomial or there exist B and F such that $A = BF$ and neither B nor F is a monomial. If A is a monomial, then $\text{cone}(A) = R_+^k$, contradicting the assumption. Suppose A admits the indicated factorization. Because B is not a monomial, $\text{cone}(B) \neq R_+^k$. Now $\text{cone}(A) \subsetneq \text{cone}(B)$, since F is not a monomial. It follows that $\text{cone}(A)$ is not an extremal cone, contradicting the assumption.

$\Leftarrow$: Assume $\text{cone}(A) \notin CE_{k,k}$. Then either it is identical to R_+^k, or $\text{cone}(A)$ is not maximal. If $\text{cone}(A) = R_+^k$, then A is a monomial, hence not a prime. Suppose $\text{cone}(A) \in C_{k,k} \setminus \{R_+^k\}$ is not a maximal element. Then there exists a $C_1 \in C_{k,k} \setminus \{R_+^k\}$ such that cone$(A) \subsetneq C_1$. Let $C_1 = \text{cone}(A_1)$ for a matrix A_1 which is not a monomial. Then cone$(A) \subsetneq$ cone(A_1) or $A = MA_1Q$, for $Q \in R_+^{k\times k}$ not a monomial, and for a monomial $M \in R_+^{k\times k}$. Because MA_1 and Q are not monomials, A is not a prime. ∎

PROPOSITION 5.5. *Let $k \in Z_+$ and $A \in R_+^{k\times k}$. If A is prime in the positive matrices, then* pos-rank$(A) = k$.

Proof. Suppose $n =$ pos-rank$(A) < k$. Then there exists a factorization

$$A = BC$$

with $B \in R_+^{k\times n}$ and $C \in R_+^{n\times k}$, neither of which is a monomial. This contradicts that A is a prime in the positive matrices. ∎

A necessary condition for minimality of the factorization mentioned in Problem 5.3 is pos-rank$(A) =$ pos-rank$(B) = n$. This is not a sufficient condition. Indeed, consider the following two positive matrices:

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Because A and B are prime matrices (see [17]), we have with Proposition 5.5 that pos-rank$(A) =$ pos-rank$(B) = 4$. But

$$AB = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 0 \\ 2 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \frac{2}{5} & 0 \\ 0 & 1 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{5} & 1 \end{pmatrix},$$

so pos-rank$(AB) \leqslant 3$, and because $3 =$ rank$(AB) \leqslant$ pos-rank(AB), we have pos-rank$(AB) = 3$. For the nonsquare case we can state the following proposition:

PROPOSITION 5.6. *Let $k, m \in Z_+$, $m < k$, and $A \in R_+^{k\times m}$. Then* cone$(A) \in CE_{k,m}$ *if and only if A is not strictly factorizable and not part of a monomial.*

Proof. $\Leftarrow$: Assume cone$(A) \notin CE_{k,m}$. Then we can consider the following two cases:

1. cone(A) is an m-facet of R_+^k;
2. cone(A) is not maximal.

In case 1 we have, with Proposition 3.15, that A is part of a monomial: contradiction. For case 2, suppose cone$(A) \in C_{k,m} \setminus \mathscr{F}_m(R_+^k)$ is not a maximal element; hence there exists a $C_1 \in C_{k,m} \setminus \mathscr{F}_m(R_+^k)$ such that cone $(A) \subsetneq C_1$. Let $C_1 = \text{cone}(A_1)$ for a matrix A_1 which is not part of a monomial. Then cone$(A) \subsetneq$ cone(A_1) or $A = MA_1Q$, for $Q \in R_+^{m\times m}$ not a monomial, and for a monomial $M \in R_+^{k\times k}$. Because MA_1 is not part of a monomial and Q is not a monomial, A is strictly factorizable: contradiction. Hence if A is not strictly factorizable and A is not part of a monomial, then cone$(A) \in CE_{k,m}$.

$\Rightarrow$: Because cone$(A) \notin \mathscr{F}_m(R_+^k)$, it follows from Proposition 3.15 that A is not part of a monomial. Assume there exists a factorization $A = DF$, in which $D \in R_+^{k\times m}$ is not part of a monomial and $F \in R_+^{m\times m}$ is not a monomial. Because D is not part of a monomial, cone$(D) \notin \mathscr{F}_m(R_+^k)$. Since F is not a monomial, cone$(A) \subsetneq$ cone(D). It follows that cone(A) is not an extremal cone, contradicting the assumption. Hence if $A = DF$, then either D is part of a monomial or F is a monomial. So A is not strictly factorizable. ■

Proposition 5.7. *Let* $k, m \in Z_+$, $m \leqslant k$, *and* $A \in R_+^{k\times m}$. *If* cone(A) $\in CE_{k,m} \cup \mathscr{F}_m(R_+^k)$, *then* pos-rank$(A) = m$.

Proof. For cone$(A) \in \mathscr{F}_m(R_+^k)$, A is part of a monomial, so $m =$ rank$(A) \leqslant$ pos-rank(A). Hence pos-rank$(A) = m$. For cone$(A) \in CE_{k,m}$, suppose $n =$ pos-rank$(A) < m$. Then there exists a factorization $A = BC$ with $B \in R_+^{k\times n}$ and $C \in R_+^{n\times m}$. Then

$$A = \begin{pmatrix} B & 0 \end{pmatrix} \begin{pmatrix} C \\ 0 \end{pmatrix}$$

is also a factorization, in which $(B\ \ 0) \in R_+^{k\times m}$ is not part of a monomial and

$$\begin{pmatrix} C \\ 0 \end{pmatrix} \in R_+^{m\times m}$$

is not a monomial. So A is strictly factorizable, contradicting the fact that $A \in CE_{k,m}$. ■

Now we can reduce the problem of the determination of the positive rank of a matrix to the problem of finding extremal cones as follows:

PROCEDURE 5.8 (Determination of the positive rank). Let $A \in R_+^{k \times m}$, $m \leqslant k$. For $n = m, m-1, \dots, \operatorname{rank}(A)$, determine $C \in CE_{k,n} \cup \mathcal{F}_n(R_+^k)$ such that $\operatorname{cone}(A) \subseteq C$. The lowest n for which we can find such a C is the positive rank of A.

5.3. *Consequences for Positive Linear Systems*

What can be said about the positive rank in relation with a positive linear system? Consider a positive linear system (A, B, C) with $A \in R_+^{n \times n}$, $B \in R_+^{n \times m}$, $C \in R_+^{k \times n}$. Define $H(p, q)$ to be the Hankel matrix

$$\begin{pmatrix} CB & CAB & \cdots & CA^{q-1}B \\ CAB & CA^2B & & \vdots \\ \vdots & & \ddots & \vdots \\ CA^{p-1}B & \cdots & \cdots & CA^{p+q-2}B \end{pmatrix}$$

for $p, q \geqslant 1$.

PROPOSITION 5.9. *Let the positive linear system (A, B, C) be given as above. Then for every $p, q \geqslant 1$,* $\operatorname{pos-rank}(H(p,q)) \leqslant n$.

Proof. We can factorize $H(p, q)$ as follows:

$$H(p,q) = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{p-1} \end{pmatrix} \begin{pmatrix} B & AB & \cdots & A^{q-1}B \end{pmatrix}.$$

It follows that

$$\operatorname{pos-rank}(H(p,q)) \leqslant \min\left\{\operatorname{pos-rank}\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{p-1} \end{pmatrix},\right.$$

$$\left.\operatorname{pos-rank}\begin{pmatrix} B & AB & \cdots & A^{q-1}B \end{pmatrix}\right\}$$

$$\leqslant \min\{pk, n, qm, n\} \leqslant n.$$

■

A sufficient condition for minimality will now be given. This condition is stronger than the reachability/observability condition.

PROPOSITION 5.10. *Let the positive linear system* (A, B, C) *be given as above. If there exist* $p, q \geqslant 1$ *such that* $\operatorname{pos-rank}(H(p,q)) = n$, *then* (A, B, C) *is a minimal positive linear system.*

In general we will have that p and q are such that $kp \geqslant n$ and $mq \geqslant n$. For the converse of Proposition 5.10 we may need something more.

Proof. Assume (A, B, C) is not minimal. Then there exists a triple (A_1, B_1, C_1), $A_1 \in R_+^{n_1 \times n_1}$, $B_1 \in R_+^{n_1 \times m}$, $C_1 \in R_+^{k \times n_1}$, with $n_1 < n$, that has the same impulse response function as (A, B, C), i.e., $CA^rB = C_1 A_1^r B_1$ for all $r \geqslant 0$. Then

$$H(p,q) = \begin{pmatrix} CB & \cdots & CA^{q-1}B \\ \vdots & & \vdots \\ CA^{p-1}B & \cdots & CA^{p+q-2}B \end{pmatrix}$$

$$= \begin{pmatrix} C_1 B_1 & \cdots & C_1 A_1^{q-1} B_1 \\ \vdots & & \vdots \\ C_1 A_1^{p-1} B_1 & \cdots & C_1 A_1^{p+q-2} B_1 \end{pmatrix}$$

$$= \begin{pmatrix} C_1 \\ C_1 A_1 \\ \vdots \\ C_1 A_1^{p-1} \end{pmatrix} \begin{pmatrix} B_1 & A_1 B_1 & \cdots & A_1^{q-1} B_1 \end{pmatrix},$$

and the matrices after the last equality have sizes $kp \times n_1$ and $n_1 \times qm$ respectively. It follows that pos-rank$(H(p,q)) \leqslant n_1 < n$. This is a contradiction. ■

EXAMPLE 5.11. Consider the positive linear system (A, B, C) with

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix}.$$

Then

$$H(4,4) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix},$$

and this is a prime matrix; hence pos-rank$(H(4,4)) = 4$. With Proposition 5.10 it follows that the system is minimal as a positive linear system. On the other hand, note that the system is *not* minimal as a linear system over R^4, since (A, C) is not an observable pair.

6. CLASSIFICATION AND EQUIVALENCE OF POSITIVE REALIZATIONS

For a relation between minimal positive realizations of a positive impulse response function, we can state the following.

PROPOSITION 6.1. *Let the positive linear system of the form* (1), *with matrices* (A, B, C, D), *be a minimal positive realization of a positive impulse response function. For any monomial matrix* $M \in R_+^{n \times n}$, *the positive linear system of the form* (1), *with matrices* $(MAM^{-1}, MB, CM^{-1}, D)$, *is also a minimal positive realization of the same impulse response function.*

Proof. This immediately follows from the fact that for a monomial matrix $M \in R_+^{n \times n}$, also $M^{-1} \in R_+^{n \times n}$. ■

The converse of this proposition does not hold. That is, let the positive linear system of the form (1), with matrices (A, B, C, D), be a minimal positive realization of a positive impulse response function. The set S of matrices given by

$$S = \{T \in R^{n \times n} \; T \text{ nonsingular}, \quad TAT^{-1} \in R_+^{n \times n}, TB \in R_+^{n \times m}, CT^{-1} \in R_+^{k \times n}\}$$

contains the set of monomial matrices $M \in R_+^{n \times n}$, but it is possible that S is not equal to the set of monomial matrices. Indeed, consider a positive linear system of the form (1) with matrices

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \qquad D = 0.$$

This system is minimal. Take

$$T = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}.$$

Then $(TAT^{-1}, TB, CT^{-1}, D)$, with

$$TAT^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}, \qquad TB = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad CT^{-1} = \begin{pmatrix} 1 & 2 \end{pmatrix},$$

is also a minimal positive realization with the same impulse response function, but T is not a monomial matrix.

So, in general, two minimal positive realizations, (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2), of the same impulse response functions are related by a matrix T that can have negative elements.

7. CONCLUSIONS

The technique of the theory for polyhedral cones seems to be a useful way to deal with the realization problem for positive linear systems. We have given necessary and sufficient conditions for the existence of a positive realization. For the characterization of minimality we have found a sufficient condition that is stronger than the reachability/observability condition. Further research has to be done on the positive rank of matrices and the backward shift invariance, to find a sufficient and necessary condition. If we have a minimal positive realization, we can give a class of minimal positive realizations that are equivalent to it, but this is not the complete class. So the open problem here is to find a class of matrices T such that all positive realizations $(TAT^{-1}, TB, CT^{-1}, D)$ equivalent to the positive realization (A, B, C, D) can be given.

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