Discrete Mathematics 39 (1982) 263–281
North-Holland Publishing Company

MORE MUTUALLY ORTHOGONAL LATIN SQUARES

A.E. BROUWER and G.H.J. van REES
Dept. of Pure Mathematics, Stichting Mathematisch Centrum, Kruislaan 413, 1098 SJ Amsterdam, The Netherlands

Received 29 January 1981

Wilson's construction for mutually orthogonal Latin squares is generalized. This generalized construction is used to improve known bounds on the function n_r (the largest order for which there do not exist r MOLS). In particular we find

$$n_7 \leqslant 780, \quad n_8 \leqslant 4738, \quad n_9 \leqslant 5842, \quad n_{10} \leqslant 7222,$$
$$n_{11} \leqslant 7478, \quad n_{12} \leqslant 9286, \quad n_{13} \leqslant 9476, \quad n_{15} \leqslant 10632.$$

0. Introduction

For the definition of a Latin square and a set of mutually orthogonal Latin squares, (MOLS), see Dénes and Keedwell [8]. Let $N(v)$ denote the maximum number of MOLS of order v. (For $v > 1$ we have $N(v) \leqslant v-1$; it is convenient to put $N(0) = N(1) = +\infty$.) Chowla, Erdös and Straus [7] showed that $\lim_{v\to\infty} N(v) = +\infty$. Consequently we may define

$$n_r := \max\{v \mid N(v) < r\} \quad (\text{for } r \geqslant 2).$$

Wilson [23] proved that $n_r < r^{17}$ when r is sufficiently large. For small values of r explicit upper bounds for n_r have been obtained. The current state of affairs is:

$n_2 = 6$ (Bose, Shrikhande and Parker [2]),
$n_3 \leqslant 14$ (Wang and Wilson [22]),
$n_4 \leqslant 52$ (Guérin [9]),
$n_5 \leqslant 62$ (Hanani [10]),
$n_6 \leqslant 76$ (Wojtas [26]),
$n_7 \leqslant 780$, $n_8 \leqslant 4738$, $n_9 \leqslant 5842$, $n_{10} \leqslant 7222$,
$n_{11} \leqslant 7478$, $n_{12} \leqslant 9286$, $n_{13} \leqslant 9476$, $n_{14} \leqslant n_{15} \leqslant 10632$ (this paper),
$n_{30} \leqslant 65278$ (Brouwer [3]).

(The very good bounds on n_r for $r \leqslant 6$ are obtained using the fact that 7, 8, 9 are consecutive prime powers. The bounds on n_{15} and n_{30} come from 16, 17 and 31, 32 respectively.) For a list of lower bounds for $N(v)$, $v < 10000$, see Brouwer [3].

As is well known, the existence of r mutually orthogonal Latin squares of order

v is equivalent to the existence of a transversal design $TD[r+2; v]$ (with blocks of size $r+2$ and $r+2$ groups of size v) (see, e.g. Wilson [23]). We shall use the language of transversal designs in the sequel.

In [23] Wilson describes a recursive construction for transversal designs. This construction was generalized by Wojtas [27, 28] and Stinson [18]. This construction is now further generalized to subsume the other constructions. (Both authors arrived independently at essentially the same theorem – the logical conclusion of the work of Wojtas and Stinson. A much more general construction for group divisible designs, generalizing almost every known recursive construction, has just been found by Stinson (oral communication) but it seems that the specialization of this very general result to the case of transversal designs is almost equivalent to our result.)

1. The construction

As auxiliary structures in the construction we need 'transversal designs with holes', things that look like a transversal design from which one or more (disjoint) subdesigns have been removed. (This concept – in the case of one hole – occurs in Horton [11] under the name 'incomplete array'.) Specifically, we write $TD[k; v] - \sum_{i=1}^{r} TD[k; u_i]$ for a structure $(X, \mathcal{G}, \mathcal{A}, (Y_i)_{i\leq r})$ where X is a set of kv points, $\mathcal{G} = \{G_1, G_2, \ldots, G_k\}$ is a partition of X into k *groups* of v elements each, each Y_i $(1 \leq i \leq r)$ is a set of ku_i points such that $|Y_i \cap G_j| = u_i$ for $1 \leq j \leq k$, and $\mathcal{A}$ is a set of subsets of X called *blocks*, each containing exactly one element from each group, such that each pair $\{x, y\}$ of elements from different groups is either contained in Y_i for some i or occurs in a unique block of $\mathcal{A}$ (but not both).

Thus it follows that each block contains k elements and there are $v^2 - \sum_{i=1}^{r} u_i^2$ blocks. Notice that for $r=0$ the concept 'transversal design with zero holes' coincides with the usual concept of transversal design. Also, that if a transversal design contains r disjoint subdesigns we obtain a 'transversal design with r holes' by removing the blocks of these subdesigns. Note however that a transversal design with holes might exist where the full design does not exist. For example, Horton constructed $TD[4; 6] - TD[4; 2]$.

The following is our main theorem.

Theorem 1.1. *Let* $(X, \mathcal{G}, \mathcal{A})$ *be a* $TD[k+l; t]$ *where* $\mathcal{G} = \{G_1, \ldots, G_k, H_1, \ldots, H_l\}$. *For* $1 \leq i \leq l$ *let* $H_i = \sum_{j=1}^{p_i} H_{ij}$ *be a partition of* H_i. *Let nonnegative numbers* m, m_{ij} *be given such that the following two conditions are satisfied.*

(i) *For* $1 \leq i \leq l$ *there exists a transversal design*

$$TD\left[k; \sum_{j=1}^{t} m_{ij} h_{ij}\right] \quad \textit{where } h_{ij} := |H_{ij}|.$$

(ii) *For any block* $A \in \mathscr{A}$ *intersecting* $H_{ij(i)}$ $(1 \leq i \leq l)$ *there exists an incomplete transversal design (transversal design with l holes)*

$$\mathrm{TD}\left[k; m + \sum_{i=1}^{l} m_{ij(i)}\right] - \sum_{i=1}^{l} \mathrm{TD}[k; m_{ij(i)}].$$

Then a $\mathrm{TD}[k; mt + \sum_{i=1}^{l} \sum_{j=1}^{p_i} m_{ij} h_{ij}]$ *exists.*

Proof. Let $I_k = \{1, 2, \ldots, k\}$ be some set of cardinality k. Let M, M_{ij} be sets of cardinality m, m_{ij}, respectively. Let $X_0 = G_1 \cup \cdots \cup G_k$. For each block $A \in \mathscr{A}$, put $A_0 = A \cap X_0$ and $A_i = A \cap H_i$ $(1 \leq i \leq l)$. The design we construct will have pointset

$$X^* = (X_0 \times M) \cup \bigcup_{i,j} (I_k \times M_{ij} \times H_{ij})$$

and collection of groups $\mathscr{G}^* = \{G_1^*, \ldots, G_k^*\}$, where

$$G_i^* = (G_i \times M) \cup \bigcup_{i,j} (\{i\} \times M_{ij} \times H_{ij}) \quad (1 \leq i \leq k).$$

It remains to describe the blocks.

For each block $A \in \mathscr{A}$ construct a $\mathrm{TD}[k; m + \sum_i m_{ij(i)}] - \sum_i \mathrm{TD}[k; m_{ij(i)}]$ on the set

$$A^* = (A_0 \times M) \cup \bigcup_i (I_k \times M_{ij(i)} \times A_i)$$

(where $j(i)$ is defined by $A_i \subset H_{ij(i)}$ $(1 \leq i \leq l)$) with groups $A^* \cap G_i^*$ $(1 \leq i \leq k)$ and holes $I_k \times M_{ij(i)} \times A_i$ $(1 \leq i \leq l)$. Let its family of blocks be $\mathscr{B}_A$.

Next, for $1 \leq i \leq l$, let $\mathscr{C}_i$ be the family of blocks of a transversal design $\mathrm{TD}[k; \sum_j m_{ij} h_{ij}]$ with pointset $H_i^* = \bigcup_j I_k \times M_{ij} \times H_{ij}$ and groups $H_i^* \cup G_g^*$ $(1 \leq g \leq k)$. Put $\mathscr{A}^* = \bigcup_A \mathscr{B}_A \cup \bigcup_i \mathscr{C}_i$. Then $(X^*, \mathscr{G}^*, \mathscr{B}^*)$ is the required design, as one readily checks. □

Sometimes one needs another distribution of the holes. A still more general theorem tells us where we may avoid holes.

Theorem 1.2. *Let* $(X, \mathscr{G}, \mathscr{A})$ *be a* $\mathrm{TD}[k + l; t]$ *where* $\mathscr{G} = \{G_1, \ldots, G_k, H_1, \ldots, H_l)$. *Let* $H = \bigcup_{i=1}^{l} H_i$. *Choose a nonnegative integer m and maps* $w : H \to \mathbb{N}_0$, $g : H \to \mathscr{G} \cup \mathscr{A}$ *such that* $x \in g(x)$ *for each* $x \in H$. *If*

(i) *there exist* $\mathrm{TD}\left[k; \sum_{x \in H_i} w(x)\right] - \sum_{\substack{x \in H_i \\ g(x) \neq H_i}} \mathrm{TD}[k; w(x)]$ $(1 \leq i \leq l)$,

and

(ii) *there exist* $\mathrm{TD}\left[k; m + \sum_{x \in A \cap H} w(x)\right] - \sum_{\substack{x \in A \cap H \\ g(x) \neq A}} \mathrm{TD}[k; w(x)]$ $(\forall A \in \mathscr{A})$

then there exists a $\mathrm{TD}[k; mt + \sum_{x \in H} w(x)]$.

The proof is similar to that of Theorem 1.1. We shall call the members m_{ij} from Theorem 1.1 and $w(x)$ from Theorem 1.2 *weights*. The most useful applications are those where all nonzero weights occur on one or two groups or on one block. Let us formulate these explicitly.

Corollary 1.3 (Brouwer [3(b)]). *If* $\mathrm{TD}[k+1;t]$ *and* $\mathrm{TD}[k;\sum_{j=1}^{p} m_j h_j]$ *and (for* $j=1,\ldots,p$*)* $\mathrm{TD}[k;m+m_j]-\mathrm{TD}[k;m_j]$ *all exist (where* $t=\sum_{j=1}^{p} h_j$*), then also* $\mathrm{TD}[k;mt+\sum_{j=1}^{p} m_j k_j]$ *exists.*

Proof. This is the case $l=1$ of Theorem 1.1 □

Corollary 1.4 (Brouwer [5]). *If* $\mathrm{TD}[k+l;t], \mathrm{TD}[k;m], \mathrm{TD}[k;m+w]$ *and (for* $i=1,\ldots,l$*)* $\mathrm{TD}[k;m+w_i]-\mathrm{TD}[k;w_i]$ *all exist (where* $w=\sum_{i=1}^{l} w_i$*), then also* $\mathrm{TD}[k;mt+w]$ *exists.*

Proof. This is the case $w(x)=0$ for $x\notin A$ and $g(x)=A$ for $x\in A$ (where A is some fixed block) of Theorem 1.2. Note that we do not need $\mathrm{TD}[k;m]$ in case $k+l=t+1$. □

Remarks. Theorem 1.2 generalizes most known variants of Wilson's theorem. One obtains Theorem 1.1 by taking $g(x)=H_i$ for $x\in H_i$. Wilson's construction [23] is Theorem 1.1 with all weights either zero or one. Stinson's construction [18] is the case of Theorem 1.1 with weights $\in\{0,n\}$. Wojtas's construction [27] is Corollary 1.3 with weights $\in\{0,1,m_1\}$ and $m=m_1m_2$. Corollary 1.4 is a generalization of Wojtas's lemma 2.1 [28].

Of course in this kind of situation the merit lies not so much in finding new generalizations, as well in finding new specializations of the parameters in one of these very general theorems so as to produce working corollaries. For example, not until four years after Wilson's theorem was published did Wojtas (in [26]) show that $N(90)\geq 6$ was a corollary.

So let us justify these beautiful theorems by improving the known results on n_r $(7\leq r\leq 15)$. [This is a nice test case. Previous results are (approximately in chronological order):

$n_7\leq 5036$ (Bussemaker and Kamps, 1974 [12]),
$n_7\leq 4922$ (Wojtas, 1977 [25]),
$n_7\leq 4146$ and $n_8\leq 9402$ (Mullin et al., 1978 [16]),
$n_7\leq 4298$ (Wojtas, 1978 [27]),
$n_7\leq 2862$ and $n_8\leq 7768$ (Brouwer, 1978 [3]),
$n_7\leq 2862$ (Stinson, 1978 [17]),
$n_7\leq 1750$ (Wojtas, 1979 [28]),
$n_7\leq 1726$ and $n_8\leq 7464$ (Brouwer, 1979 [4]),
$n_8\leq 7474$ (Stinson, 1979 [18]).

Here we show $n_7\leq 780$ and $n_8\leq 4738$, a great leap forward.]

2. Holes of size one

A TD$[k; v]-$TD$[k; 0]$ exists if and only if TD$[k; v]$ exists; they are the same object. Also for holes of size one we have easy criteria.

Lemma 2.1. (a) *Suppose a* TD$[k; v]-$TD$[k; u]$ *exists. Then* $v = u$ *or* $v \geq (k-1)u$. *A* TD$[k; v]-$TD$[k; u]-$TD$[k; 1]$ *exists iff* $v > (k-1)u$.

(b) *Suppose a* TD$[k; v]-\sum_{i=1}^{r}$TD$[k; u_i]$ *exists, where* $r \geq 2$ *and* $u_1 \geq u_2 \geq \cdots \geq u_r \geq 0$. *Then* $v \geq (k-1) \cdot u_1 + u_2$. *If* $v > (k-1)\sum_{i=1}^{r} u_i$, *then a* TD$[k; v]-\sum_{i=1}^{r}TD[k; u_i]-TD[k; 1]$ *exists.*

Proof. In order to obtain a hole of size one, remove a block disjoint from the given holes. □

Lemma 2.2. (a) *Suppose a* TD$[k+1; v]$ *exists. Then a* TD$[k; v]-\sum_{i=1}^{v}$TD$[k; 1]$ *exists.*

(b) *Suppose a* TD$[k+1; v]-\sum_{i=1}^{r}$TD$[k+1; u_i]$ *exists, where* $f := v - \sum_{i=1}^{r} u_i > 0$. *Then a* TD$[k; v]-\sum_{i=1}^{r}TD[k; u_i]-\sum_{i=1}^{f}TD[k; 1]$ *exists.*

Proof. Obvious. □

The conclusion of Lemma 2.1(a) can be strengthened slightly:

Lemma 2.3. *Suppose that* $k \geq 3$, $v > (k-1)u$ *and that a* TD$[k; v]-$TD$[k; u]$ *exists. Then a* TD$[k; v]-$TD$[k; u]-2$TD$[k; 1]$ *exists.*

Proof. Consider the graph with the blocks of TD$[k; v]-$TD$[k; u]$ which are disjoint from the hole as vertices, two blocks being adjacent if they have nonempty intersection. By Lemma 2.1(a) the set of vertices V is nonempty. In fact $|V| = v^2 - u^2 - ku(v-u)$, and the graph is regular of degree $d := k(v-1-(k-1)u)$. Since $v > (k-1)u$ and $k \geq 3$ it follows that $|V| - 1 > d$ ($|V| - d - 1 = (v-(k-1)u)(v-u-k) + k - 1 > 0$), i.e., the graph is not complete so that there exist two nonadjacent vertices. □

Corollary 2.4. *Suppose that* $v \geq k \geq 3$ *and that a* TD$[k; v]$ *exists. Then a* TD$[k; v]-3$TD$[k; 1]$ *exists.*

3. Input designs

In order to apply our theorems we need some constructions for transversal designs with holes. First remark that if we have a TD$[k; v]$ with subdesign

TD$[k;u]$ then by removing the blocks of the subdesign we get TD$[k;v]-$TD$[k;u]$. Usually we shall construct transversal designs with holes in this way. However, some of the following propositions yield transversal designs with holes that perhaps cannot be filled.

Proposition 3.1. *Let $(X, \mathcal{G}, \mathcal{A})$ be a group divisible design such that for each $A \in \mathcal{A}$ a* TD$[k+1; |A|]$ *exists. Then a* TD$[k; |X|]-\sum_{G\in\mathcal{G}}TD[k; |G|]$ *exists.*

Proof. This is the well-known 'pairwise balanced design' – construction. (It is of course sufficient to require the existence of TD$[k;a]-\sum_{i=1}^{a}$TD$[k;1]$ for $a=|A|, A\in\mathcal{A}$.) □

Proposition 3.2 (MacNeish [13], Bush [6]). *If there exists a* TD$[k;m]$ *and a* TD$[k;n]$ *then there exists a* TD$[k;mn]$ *which contains a sub-*TD$[k;n]$.

More generally we have

Proposition 3.3. *If there exists a* TD$[k;n]$ *and a* TD$[k;v]-\sum_i$TD$[k;u_i]$, *then there exists a* TD$[k;nv]-\sum_i$TD$[k;nu_i]$.

Proof. Obvious. □

The design that we constructed in the conclusion of Theorem 1.1 is full of subdesigns. And even if some of the ingredients are missing we at least get a design with holes. More precisely:

(A) *Under the assumptions of Theorem* 1.1 *except for those under* (i) *we find that*

$$\mathrm{TD}\left[k; mt+\sum_{i=1}^{l}\sum_{j=1}^{p_i} m_{ij}h_{ij}\right]-\sum_{i=1}^{l}\mathrm{TD}\left[k;\sum_{j=1}^{p_i} m_{ij}h_{ij}\right]$$

exists.

(B) *Under the assumptions of Theorem* 1.1, *if* (ii) *is replaced by the slightly stronger condition* (ii)′: *for any block A there exists a*

$$\mathrm{TD}\left[k; m+\sum_{i=1}^{l} m_{ij(i)}\right]-\sum_{i=1}^{l}\mathrm{TD}[k; m_{ij(i)}]-\mathrm{TD}[k;1],$$

then we may construct the design in the conclusion in such a way that it contains a subdesign $T[k;t]$.

Proof. Construct this subdesign on the set $X_0\times\{0\}$ (where 0 is some fixed element of M). (Clearly, by strengthening (ii)′ further, we may obtain more disjoint subdesigns $T[k;t]$.) □

(C) *Under the assumptions of Theorem* 1.1, *if* (i) *is strengthened by requiring that each* $\mathrm{TD}[k; \sum_{j=1}^{p_i} m_{ij}h_{ij}]$ *contains a* sub-$\mathrm{TD}[k; m_{ij(i)}]$ $(1 \leq i \leq l)$, *then we may construct the design in the conclusion in such a way that it contains a subdesign* $\mathrm{TD}[k; m + \sum_{i=1}^{l} m_{ij(i)}]$.

(C1) *In fact, disjoint blocks A give rise to disjoint such subdesigns.*

[Hundreds of variants can be written down – e.g. if under (B) the '1' in condition (ii)′ is replaced by an 'a', then we may conclude to a subdesign $T[k; at]$ – but these seem useless if one's only purpose is to obtain good bounds on n_r.]

Similarly the design constructed in Theorem 1.2 is full of subdesigns; we refrain from any explicit formulation.

Specializing parameters we may again convert these general remarks into useful propositions.

Propositions 3.4. *Let* $m > 1$ *and suppose that a* $\mathrm{TD}[k+1; t]$, *a* $\mathrm{TD}[k; m]$ *and a* $\mathrm{TD}[k; m+1]$ *exist; and that* $0 \leq s \leq t$. *Then a* $\mathrm{TD}[k; mt+s] - \mathrm{TD}[k; s]$ *exists. If, moreover* $\mathrm{TD}[k; s]$ *exists, then a* $\mathrm{TD}[k; mt+s]$ *exists which contains a* sub-$\mathrm{TD}[k; t]$, *a* sub-$\mathrm{TD}[k; m]$ *if* $s \neq t$, *a* sub-$\mathrm{TD}[k; m+1]$ *if* $s \neq 0$, *and a* sub-$\mathrm{TD}[k; s]$.

Proof. In Theorem 1.1 put $l = 1, p_1 = 2, m_{11} = 1, m_{12} = 0$. By Remark (A) $\mathrm{TD}[k; mt+s] - \mathrm{TD}[k; s]$ exists. The sub-$\mathrm{TD}[k; t]$ is found using Remark (B) – note that the requirement is that $\mathrm{TD}[k; m] - \mathrm{TD}[k; 1]$ exists (i.e. $m \neq 0$) and that $\mathrm{TD}[k; m+1] - 2\,\mathrm{TD}[k; 1]$ exists (i.e. $k \leq m + 1$, which follows from the existence of $\mathrm{TD}[k; m]$). The sub-$\mathrm{TD}[k; m+i]$ $(i = 0, 1)$ are guaranteed by Remark (C). □

Proposition 3.5. *Let* $m > 1$ *and suppose that a* $\mathrm{TD}[k+w; t]$, *a* $\mathrm{TD}[k; m]$ *and a* $\mathrm{TD}[k; m+1]$ *exist. Then a* $\mathrm{TD}[k; mt+w] - \mathrm{TD}[k; m+w]$ *exists. If, moreover,* $\mathrm{TD}[k; m+w]$ *exists, then there exists a* $\mathrm{TD}[k; mt+w]$ *which contains a* sub-$\mathrm{TD}[k; t]$, *a* sub-$\mathrm{TD}[k; m]$, *a* sub-$\mathrm{TD}[k; m+1]$ *if* $w > 0$, *and a* sub-$\mathrm{TD}[k; m+w]$.

Proof. In Corollary 1.4 put $l = w$ and $w_1 = \cdots = w_l = 1$ (thus we obtain a theorem of Wojtas [25, 28]). The claims again follow from (A)–(C) or their analogues for Theorem 1.2. □

3.1. *Separable designs*

Bose, Shrikhande and Parker [2, Theorem 4] proved a theorem the most important special case of which was reproved in Van Lint [12, Theorem 13.2.2]:

If there is a symmetric BIBD$(v, k, 1)$ *then* $N(k^2+1) \geq \min\{N(k), N(k+1) - 1\}$.

But the design constructed contains a subdesign of order k – in fact Van Lint proves

Proposition 3.6. *If there is a symmetric* $B[k; v]$ *and a* $\mathrm{TD}[c+1; k+1]$, *then there is a* $\mathrm{TD}[c; v+k]-\mathrm{TD}[c; k]$.

A *separable* pairwise balanced design in the sense of Bose, Shrikhande and Parker is a PBD $(X, \mathscr{B})$ with $\lambda = 1$ where the collection of blocks can be partitioned into classes $\mathscr{B}_i$ such that each $(X, \mathscr{B}_i)$ is a 1-design with $r_i = k_i$ (type I) or $r_i = 1$ (type II). Let $v := |X|$. By 'partially completing' this design by adding 'points at infinity' to the blocks of some of the classes $\mathscr{B}_i$ (say, those with $i \in I$, where I is some index set) and then performing the PBD construction for transversal designs one obtains a transversal design on $v+x$ points, where $x = \sum_{i \in I} r_i$. If only classes of type II are present this corresponds to ordinary completion followed by an application of Proposition 3.1; in the presence of type I classes there is no intermediate pairwise balanced design but Bose, Shrikhande and Parker showed how to proceed in this case.

A direct generalization of a slight improvement of their theorem is

Theorem 3.7. *Let* $(X, \mathscr{B})$ *be a separable* PBD *on* v *points with* $\lambda = 1$ *and with separation* $\mathscr{B} = \sum_{i \in J} \mathscr{B}_i$, *where each* $\mathscr{B}_i$ *is a* $1-(v, k_i, r_i)$ *design with* $r_i = k_i$ *or* $r_i = 1$. *Let* $I \subset J$ *and let* $x = \sum_{i \in I} r_i$.

Suppose that there exist $\mathrm{TD}[c+\varepsilon_i; k_i+1]$ *for* $i \in I$ *and* $\mathrm{TD}[c+\varepsilon_i; k_i]$ *for* $i \in J \backslash I$.

(i) *If* $\varepsilon_i \geq 1$ *for all* $i \in J$, *or if there is an index* i_0 *such that* $\mathscr{B}_{i_0}$ *is of type* II (*i.e.* $r_{i_0} = 1$) *and* $\varepsilon_{i_0} \geq 0$ *and* $\varepsilon_i \geq 1$ *for all* $i \in J \backslash \{i_0\}$, *then there exists a* $\mathrm{TD}[c; v+x]-\mathrm{TD}[c; x]$.

(ii) *If there is an index* i_0 *such that* $i_0 \notin I$ *and* $\mathscr{B}_{i_0}$ *is of type* II *and* $\varepsilon_i \geq 1$ *for* $i \in J \backslash \{i_0\}$ (*and* ε_{i_0} *is arbitrary*), *then there exists a* $\mathrm{TD}[c; v+x]-\mathrm{TD}[c; x]-\sum_{i=1}^{s} \mathrm{TD}[c; k]$ *where* $s = v/k$ *and* $k = k_{i_0}$.

We omit the proof. As usual, everywhere where $\mathrm{TD}[c+1; u]$ was required, $\mathrm{TD}[c; u]-\sum_{i=1}^{u} \mathrm{TD}[c; 1]$ suffices. Also, if e.g. in case (ia) a $\mathrm{TD}[c; x]$ exists, then a $\mathrm{TD}[c; v+x]$ exists with subdesigns $\mathrm{TD}[c; k_i]$ for $i \in J \backslash I$ and $\mathrm{TD}[c; k_i+1]$ for $i \in I$ and $\mathscr{B}_i$ of type II.

Apart from some sporadic examples containing small blocks (say of size less than six) all separable designs we know are either resolvable or come from the next theorem.

Theorem 3.8 (Brouwer [4]). *Let* q *be the power of a prime, and* $0 < t \leq q^2-q+1$. *Then there exists a pairwise balanced design* $B[\{t\ q+t\}, t(q^2+q+1)]$ *such that it is the union of a symmetric* $1\text{-}(v, q+t, q+t)$ *design and* $(q^2-q+1-t)$ $1\text{-}(v, t, 1)$ *designs.*

3.2. *A difference method*

Wilson [24] has given a direct construction for incomplete transversal designs.

Proposition 3.9. *Let $q = mt + 1$ be a prime power. Let $k = m + 2$. If there may be found a matrix-minus-diagonal of field elements $a_{ij} \in \mathbb{F}_q$ $(1 \le i, j \le k; i \ne j)$ such that for each j_1, j_2 $(1 \le j_1 < j_2 \le k)$ the m differences $a_{ij_2} - a_{ij_1}$ $(1 \le i \le k; i \ne j_1, j_2)$ form a system of representatives for the cyclotomic classes of index m in $\mathbb{F}_q$, then $T[k; q+t] - T[k; t]$ exists.*

Mullin et al. [16] introduced the notation $V(m, t)$ for a vector of length $m+1$ such that the circulant matrix with empty diagonal and $V(m, t)$ as first row has the properties required in Proposition 3.9. They constructed $V(8, 9)$ and $V(8, 11)$.

In Appendix A we construct $V(m, t)$ for $4 \le m \le 8$ and $q = mt + 1 < 2000$ for all relevant primes (but not prime powers) q.

(It is remarkable that the time required to find such a vector for given m at first increases strongly with t while it decreases again for large t: if the cyclotomic classes are large enough, then there are many solutions. On the other hand, increasing m by one makes the problem an order of magnitude more difficult. I could not find any $V(m, t)$ with $m > 8$.)

4. An example

Several authors paid attention to $o_r := \max\{v \mid v \text{ odd and } N(v) < r\}$, mainly because usually one can obtain much better upper bounds for o_r than for n_r. (The reason must be that prime powers are usually odd. One exception was $r = 29$ where Hanani found $e_{29} \le 2733666$, $n_{29} \le 34115553$ [10] – in his case just the even numbers were simpler to deal with – but recently Brouwer [5] showed $(n_{29} \le)$ $n_{30} \le 65278$ and the only possible exceptions above 60000 are even so that $o_{30} < 60000$.)

Some results are:

$o_7 \le 469$ and $o_{15} \le 54047$ (Szajowski, 1976 [20]),
$o_7 \le 335$ (Wojtas, 1977 [25]),
$o_8 \le 2343$ (Stinson, 1978 [17]).

For small r one finds from existing tables: $o_3 = 3$, $o_4 \le 33$, $o_5 \le 51$, $o_6 \le 75$. A computer program produced the bounds $o_9 \le 2607$, $o_{10} \le 2863$, $o_{11} \le 3471$, $o_{12} \le 3565$, $o_{15} \le 5467$. But in fact 5467 was the only possible exception above 3603, so that $o_{15} \le 3603$ as soon as we show that $N(5467) \ge 15$. This motivates us to prove the following lemma. (The proof is a nice illustration of how Theorem 1.2 may be used.)

Lemma 4.1. $N(5467) \ge 15$.

Proof. $5467 = 19 \cdot 271 + 289 + 29$, $289 = 17 \cdot 17$, $29 = 1 \cdot 17 + 12 \cdot 1$.

Apply Theorem 1.2 with $k = 17$, $t = 19$, $m = 271$, $l = 2$. Give in H_1 two points weight 0 and seventeen points weight 17. Give in H_2 one point (x_0) weight 17, twelve points weight 1, and six points weight 0. Let for $x \in H_1$, $g(x)$ be the block through x and x_0, and let $g(x) = H_2$ for $x \in H_2$. We need the following ingredients:

(1) TD[17; 289] − 17 TD[17; 17]. This is found using Proposition 3.3 with $k = n = v = 17$, $u_i = 1$ $(1 \leq i \leq 17)$ and Lemma 2.2(a).

(2) TD[17, 29], which exists since 29 is prime.

(3) TD[17; 271], which exists since 271 is prime.

(4) TD[17; 272] − TD[17; 1], which exists since 272 = 16.17.

(5) TD[17; 288] − TD[17; 17]. This is found using Proposition 3.4 with $m = 16$, $k = t = 17$, $s = 16$.

(6) TD[17; 305] − TD[17; 17]. This is found using Proposition 3.4 with $m = 16$, $k = 17$, $t = 19$, $s = 1$. Since all necessary ingredients exist, Theorem 1.2 gives us a TD[17; 5467]. □

5. Seven squares

Let us show how to use our theorems to obtain $n_7 \leq 780$. Wojtas [28] showed $n_7 \leq 1750$ and Brouwer [3] gives a list of orders for which there may not exist seven mutually orthogonal Latin squares. For each such order >780 we indicate a construction. Let us give an example,

$$876 = 11 \cdot 72 + (7 \times 8 + 1 \times 1 + 3 \times 0) + (3 \times 9 + 8 \times 0)$$

means (apart from arithmetic equality) that $N(876) \geq 7$ follows from an application of Theorem 1.1 with $(k = 9)$, $t = 11$, $m = 72$, $l = 2$,

$$(h_{ij}) = \begin{pmatrix} 7 & 1 & 3 \\ 3 & 8 & \end{pmatrix}, \qquad (m_{ij}) = \begin{pmatrix} 8 & 1 & 0 \\ 9 & 0 & \end{pmatrix}.$$

In this particular case we may check the availability of the ingredients as follows: $N(57) \geq 7$ follows from $57 = 7^2 + 7 + 1$ and the existence of PG(2, 7), $N(27) \geq 7$ since 27 is a prime power, $N(72) \geq 7$ since $72 = 8 \cdot 9$, $N(73) \geq 7$ since 73 is prime, the existence of TD[9; 80] − TD[9; 8] follows from Proposition 3.4 and $80 = 9 \cdot 8 + 8$, that of TD[9; 81] − TD[9; 9] from Proposition 3.2, that of TD[9; 82] − TD[9; 9] − TD[9; 1] from the existence of $V(8, 9)$, and finally that of TD[9; 89] − TD[9; 9] − TD[9; 8] from Proposition 3.4 and the preceding Remark (C_1) and $89 = 11 \cdot 8 + 1$.

For the designs below it is easy to verify that the required ingredients exist. For shortness we drop terms $h \times 0$ and write h instead of $h \times 1$ so that the above line becomes '$876 = 11 \cdot 72 + (7 \times 8 + 1) + 3 \times 9$'. (Concerning the last line of Table 1, that for $v = 796$, note that by a remark due to Wojtas [26] we may choose sets H_{ij} with $|H_{11}| = 8$, $|H_{21}| = 9$, $|H_{31}| = 9$, ($m_{i1} = 1$, $m_{i2} = 0$) in such a way that each block A intersects at least one of the H_{i1} so that we do not need the ingredient TD[9; 70].)

Table 1
Existence of TD[9; v]

$1750=23\cdot 72+9\times 9+13$	$1006=13\cdot 71+8\times 9+11$
$1740=23\cdot 71+(11\times 9+8)$	$994=13\cdot 71+(7\times 10+1)$
$1734=11\cdot 151+(8\times 9+1)$	$982=13\cdot 71+(6\times 9+5)$
$1726=23\cdot 71+(9\times 9+1)+11$	$966=13\cdot 71+(4\times 9+7)$
$1722=23\cdot 71+8\times 9+17$	$914=13\cdot 64+(10\times 8+2)$
$1718=23\cdot 71+8\times 9+13$	$876=11\cdot 72+(7\times 8+1)+3\times 9$
$1260=16\cdot 72+11\times 9+9$	$868=11\cdot 72+(6\times 8+1)+3\times 9$
$1258=17\cdot 71+(4\times 9+7)+8$	$866=13\cdot 56+(10\times 8+2)+7\times 8$
$1230=16\cdot 71+9\times 9+13$	$844=11\cdot 72+(3\times 8+1)+3\times 9$
$1206=11\cdot 103+(8\times 9+1)$	$836=11\cdot 71+(5\times 9+2)+8$
$1202=11\cdot 99+8\times 13+9$	$828=11\cdot 72+3\times 9+9$
$1198=11\cdot 103+(7\times 9+2)$	$826=11\cdot 71+(4\times 9+1)+8$
$1190=16\cdot 72+3\times 9+11$	$822=11\cdot 71+(4\times 9+5)$
$1182=11\cdot 100+(9\times 9+1)$	$820=11\cdot 72+3\times 9+1$
$1180=16\cdot 72+3\times 9+1$	$818=11\cdot 71+(4\times 9+1)$
$1126=11\cdot 99+(4\times 8+5)$	$814=11\cdot 71+(2\times 9+7)+8$
$1026=13\cdot 72+9\times 9+9$	$806=11\cdot 71+(2\times 9+7)$
$1022=13\cdot 71+11\times 9$	$804=11\cdot 71+(2\times 9+5)$
$1020=13\cdot 71+(7\times 10+3\times 9)$	$802=11\cdot 72+1\times 9+1$
$1012=13\cdot 71+9\times 9+8$	$(796=11\cdot 70+8+9+9)$

[*Note.* $N(56)\geqslant 7$ is proved in Mills [14], $N(57)\geqslant 7$ in Bose and Shrikhande [1], $N(65)\geqslant 7$ follows from Proposition 3.6, the existence of TD[9; 81]−TD[9; 10], TD[9; 82]−TD[9; 9] and of TD[9; 100]−TD[9; 11] follows from the existence of $V(7, 10)$, $V(8, 9)$ and $V(8, 11)$, respectively.]

Thus we proved:

Theorem 5.1. $n_7 \leqslant 780$.

6. Fifteen squares

First we ran a program with some knowledge about Latin squares to find an upper bound on n_{15}. It proved $n_{15}\leqslant 59942$. (As follows: as a corollary to Wilson's theorem we have

(*) If $N(t)\geqslant 16$ *and* $0\leqslant h\leqslant t$ *and* $N(h)\geqslant 15$, *then* $N(16+h)\geqslant 15$. Given n, if we know enough numbers h in the residue class of $n \pmod{16}$ such that $N(h)\geqslant 15$, then among the numbers t we get when writing $n=16t+h$ at least one is coprime to $2\cdot 3\cdot 5\cdot 7\cdot 11\cdot 13$ so that for this t we have $N(t)\geqslant 16$. By (*) it follows that $N(n)\geqslant 15$ provided that $t\geqslant h$. Hence one finds that this works for $n\geqslant 17h_{\max}$, $h_{\max}$ being the largest element in some fixed good collection of numbers h. As an explicit example, consider the residue class $1 \pmod{16}$. The program proved $N(h)\geqslant 15$ for

$$h\in\{1, 17, 49, 81, 97, 113, 193, 241, 257, 273, 289, 305, 321, 337, 353, 369, 385, 401, 417, 433, 449, 465, 481, 497, 513\}.$$

(And indeed, $N(1)=+\infty$ and all other numbers are prime powers or of the form $16q+1$ or $16q+17$ where q is a primepower ≥ 17.)

Now if we write $n=16t_0+1$, then we have $n=16t+h$ with h in the above set and

$$t \in t_0=\{0,1,3,5,6,7,12,15,16,17,18,19,20,21,22,23,\\ 24,25,26,27,28,29,30,31,32\}.$$

We claim that at least one of these t has no factors 2, 3, 5, 7, 11 or 13. Consider six cases according to the residue class of $t_0 \pmod 6$.

(α) $t_0 \equiv 1 \pmod 6$. Choose $t \in t_0-\{0,6,12,18,20,24,26,30,32\}$. At most three of these numbers are divisible by 5, at most two by 7, at most one by 11 and at most two by 13. But we have nine choices and $9-3-2-1-2>0$, so we may pick t in such a way that $(t, 2\cdot 3\cdot 5\cdot 7\cdot 11\cdot 13)=1$.

(β) $t_0 \equiv 2 \pmod 6$. Choose $t \in t_0-\{1,7,15,19,21,25,27,31\}$. At most three of these numbers have a factor 5, at most two a factor of 7, at most one a factor of 11 and at most two a factor 13. But unfortunately $8-3-2-1-2=0$. Looking somewhat closer we see that three five's occur only when $t_0 \equiv 1 \pmod 5$. Now choose $t \in t_0-\{7,15,19,25,27\}$. There is at most one 7 or 11 or 13 so that two choices are left.

The other cases are similar.

This proves that $N(n)\geq 15$ for $n \equiv 1 \pmod{16}$, $n \geq 17\cdot 513=8721$. (By hand one finds $N(n)\geq 15$ for $n\equiv 1 \pmod{16}$ and $n>3505$ – all n admit a decomposition $n=16t+h$ such that (*) applies, or with t prime, $0\leq h\leq t-15$, $N(h+16)\geq 15$ where Proposition 3.5 applies, except for $n=4833=27\cdot 179$, $3537=27\cdot 131$, $3521=31\cdot 113+18$.)

In a similar way one finds $N(n)\geq 15$ for $n\geq 17\cdot h_{\max}$ for the other residue classes mod 16:

$n \pmod{16}$	0	1	2	3	4	5	6	7
$h_{\max}$	720	513	3154	643	3172	869	3526	615

$n \pmod{16}$	8	9	10	11	12	13	14	15
$h_{\max}$	2840	841	2570	875	3212	797	2590	847

It follows that $n_{15}<17\cdot 3526=59942$ and $o_{15}<17\cdot 875=14875$.)

Next with a short run it turned out that in fact the above method ($n=16t+h$) also works in the interval $31000\leq n\leq 60000$. Covering the interval $10000\leq n\leq 31000$ with a somewhat smarter program, and $0\leq n\leq 10699$ with the full strength of the program that knows all recursive constructions described in [3], we get the results mentioned in the introduction.

[*Note.* Recently I learned that Stinson [19] used a similar method to obtain a bound for n_{30}. Given his result the above work may be replaced by a search through the interval $10000 \leq n \leq 121605$.]

References

[1] R.C. Bose and S.S. Shrikhande, On the construction of sets of mutually orthogonal Latin squares and the falsity of a conjecture of Euler, Trans. Amer. Math. Soc. 95 (1960) 191–209.
[2] R.C. Bose, S.S. Shrikhande and E.T. Parker, Further results on the construction of mutually orthogonal Latin squares and the falsity of Euler's conjecture, Canad. J. Math. 12 (1960) 189–203.
[3] A.E. Brouwer, (a) Mutually orthogonal Latin squares, Math. Centr. report ZN 81, August 1978. (b) The number of mutually orthogonal Latin squares, Math. Centr. report ZW 123, June 1979. [The latter is an improved version of the former. A third edition is in preparation.]
[4] A.E. Brouwer, A series of separable designs with application to pairwise orthogonal Latin squares, Math. Centr. report ZW 77, August 1979, European J. Combinatories 1 (1980) 39–41.
[5] A.E. Brouwer, On the existence of 30 mutually orthogonal Latin squares, Math. Centr. report ZW 136/80, Jan. 1980.
[6] K.A. Bush, A generalization of a theorem due to MacNeish, Ann. Math. Stat. 23 (1952) 293–295.
[7] S. Chowla, P. Erdös and E.G. Straus, On the maximal number of pairwise orthogonal Latin squares of a given order, Canad. J. Math. 12 (1960) 204–208.
[8] J. Dénes and A.D. Keedwell, Latin Squares and Their Applications (Academic Press, New York, 1974).
[9] R. Guérin, Existence et propriétés des carrés latins orthogonaux II, Publ. Inst. Statist. Univ. Paris 15 (1966) 215–293. MR 35 (1968) #4118.
[10] H. Hanani, On the number of orthogonal Latin squares, J. Combin. Theory 8 (1970) 247–271.
[11] J.D. Horton, Sub-Latin squares and incomplete orthogonal arrays, J. Combin. Theory (A) 16 (1974) 23–33.
[12] J.H. van Lint, Combinatorial Theory Seminar, Lecture Notes in Math., Vol. 382, (Springer-Verlag, Berlin, 1974).
[13] H.F. MacNeish, Euler squares, Ann. Math. 23 (1922) 221–227.
[14] W.H. Mills, Some mutually orthogonal Latin squares, Proc. 8th S-E Conf. on Combinatorics, Graph Theory and Computing (1977) 473–487.
[15] R.C. Mullin, P.J. Schellenberg, D.R. Stinson and S.A. Vanstone, Some results on the existence of squares, Annals Discrete Math. 6 (1980) 257–274.
[16] R.C. Mullin, P.J. Schellenberg, D.R. Stinson and S.A. Vanstone, On the existence of 7 and 8 mutually orthogonal Latin squares, Dept. of Combinatorics and Optim. Research Rept. CORR 78-14 (1978), Univ. of Waterloo.
[17] D.R. Stinson, A note on the existence of 7 and 8 mutually orthogonal Latin squares, Ars Combinatoria 6 (1978) 113–115.
[18] D.R. Stinson, A generalization of Wilson's construction for mutually orthogonal Latin squares, Ars. Combinatoria 8 (1979) 95–105.
[19] D.R. Stinson, On the existence of 30 mutually orthogonal Latin squares, Ars Combinatoria 7 (1979) 153–170.
[20] K. Szajowski, The number of orthogonal Latin squares, Applicationes Mathematicae 15 (1976) 85–102.
[21] G.H.J. van Rees, A corollary to a theorem of Wilson, Dept. of Combinatorics and Optim. Research Rept. CORR 78-15 (1978), Univ. of Waterloo.
[22] S.M.P. Wang and R.M. Wilson, A few more squares II, Proc. 9th S-E Conf. on Combinatorics, Graph Theory and Computing (1978) 688 (Abstract).
[23] R.M. Wilson, Concerning the number of mutually orthogonal Latin squares, Discrete Math. 9 (1974) 181–198.
[24] R.M. Wilson, A few more squares, Proc. 5th S-E Conf. on Combinatorics, Graph Theory and Computing (1974) 675–680.

[25] M. Wojtas, On seven mutually orthogonal Latin squares, Discrete Math. 20 (1977) 193–201.
[26] M. Wojtas, A note on mutually orthogonal Latin squares, preprint October 1978, Komunikat nr. 236, Instytut Matematyki Politechniki Wrocławskiej.
[27] M. Wojtas, The construction of mutually orthogonal Latin squares, preprint May 1978, Komunikat nr.172, Instytut Mathematyki Politechniki Wrocławskiej.
[28] M. Wojtas, New Wilson-type constructions of mutually orthogonal Latin squares II, Seria Preprinty nr.4, Instytut Matematyki Politechniki Wrocławskeij, March 1979.

Appendix A

Below we list an example of a vector $V(m,t)$ (Cf. Section 3.2 and references [24], [16], [3b]) for $4 \leq m \leq 8$ and for all $t \geq t_0$ such that $q = mt+1$ is prime, and m and t are not both even, up to values of q around 2000.

m	4	5	6	7	8
t_0	3	6	5	6	9

**** m = 4 ****

q = 13,	q+t = 16,	V(4,3):	0	1	3	7	2
q = 29,	q+t = 36,	V(4,7):	0	1	3	7	19
q = 37,	q+t = 46,	V(4,9):	0	1	3	2	8
q = 53,	q+t = 66,	V(4,13):	0	1	3	7	19
q = 61,	q+t = 76,	V(4,15):	0	1	3	7	5
q = 101,	q+t = 126,	V(4,25):	0	1	3	2	31
q = 109,	q+t = 136,	V(4,27):	0	1	3	11	2
q = 149,	q+t = 186,	V(4,37):	0	1	3	2	5
q = 157,	q+t = 196,	V(4,39):	0	1	3	2	65
q = 173,	q+t = 216,	V(4,43):	0	1	3	2	7
q = 181,	q+t = 226,	V(4,45):	0	1	3	7	38
q = 197,	q+t = 246,	V(4,49):	0	1	3	2	23
q = 229,	q+t = 286,	V(4,57):	0	1	3	7	59
q = 269,	q+t = 336,	V(4,67):	0	1	3	2	21
q = 277,	q+t = 346,	V(4,69):	0	1	3	7	125
q = 293,	q+t = 366,	V(4,73):	0	1	3	2	22
q = 317,	q+t = 396,	V(4,79):	0	1	3	2	29
q = 349,	q+t = 436,	V(4,87):	0	1	3	2	10
q = 373,	q+t = 466,	V(4,93):	0	1	3	2	56
q = 389,	q+t = 486,	V(4,97):	0	1	3	2	5
q = 397,	q+t = 496,	V(4,99):	0	1	3	2	74
q = 421,	q+t = 526,	V(4,105):	0	1	3	7	66
q = 461,	q+t = 576,	V(4,115):	0	1	3	2	5
q = 509,	q+t = 636,	V(4,127):	0	1	3	2	21
q = 541,	q+t = 676,	V(4,135):	0	1	3	7	45
q = 557,	q+t = 696,	V(4,139):	0	1	3	2	16
q = 613,	q+t = 766,	V(4,153):	0	1	3	2	8
q = 653,	q+t = 816,	V(4,163):	0	1	3	2	67
q = 661,	q+t = 826,	V(4,165):	0	1	3	2	139
q = 677,	q+t = 846,	V(4,169):	0	1	3	2	85
q = 701,	q+t = 876,	V(4,175):	0	1	3	2	79
q = 709,	q+t = 886,	V(4,177):	0	1	3	7	38
q = 733,	q+t = 916,	V(4,183):	0	1	3	7	31
q = 757,	q+t = 946,	V(4,189):	0	1	3	7	48
q = 773,	q+t = 966,	V(4,193):	0	1	3	2	7
q = 797,	q+t = 996,	V(4,199):	0	1	3	2	7

q = 821, q+t = 1026, V(4,205):	0	1	3	2	20
q = 829, q+t = 1036, V(4,207):	0	1	3	7	272
q = 853, q+t = 1066, V(4,213):	0	1	3	2	208
q = 877, q+t = 1096, V(4,219):	0	1	3	2	17
q = 941, q+t = 1176, V(4,235):	0	1	3	2	5
q = 997, q+t = 1246, V(4,249):	0	1	3	2	166
q = 1013, q+t = 1266, V(4,253):	0	1	3	2	7
q = 1021, q+t = 1276, V(4,255):	0	1	3	2	105
q = 1061, q+t = 1326, V(4,265):	0	1	3	2	5
q = 1069, q+t = 1336, V(4,267):	0	1	3	2	10
q = 1093, q+t = 1366, V(4,273):	0	1	3	7	398
q = 1109, q+t = 1386, V(4,277):	0	1	3	2	373
q = 1117, q+t = 1396, V(4,279):	0	1	3	7	15
q = 1181, q+t = 1476, V(4,295):	0	1	3	2	217
q = 1213, q+t = 1516, V(4,303):	0	1	3	7	15
q = 1229, q+t = 1536, V(4,307):	0	1	3	2	10
q = 1237, q+t = 1546, V(4,309):	0	1	3	7	2
q = 1277, q+t = 1596, V(4,319):	0	1	3	2	55
q = 1301, q+t = 1626, V(4,325):	0	1	3	2	115
q = 1373, q+t = 1716, V(4,343):	0	1	3	2	20
q = 1381, q+t = 1726, V(4,345):	0	1	3	7	377
q = 1429, q+t = 1786, V(4,357):	0	1	3	2	44
q = 1453, q+t = 1816, V(4,363):	0	1	3	7	15
q = 1493, q+t = 1866, V(4,373):	0	1	3	2	16
q = 1549, q+t = 1936, V(4,387):	0	1	3	2	56
q = 1597, q+t = 1996, V(4,399):	0	1	3	7	133
q = 1613, q+t = 2016, V(4,403):	0	1	3	2	7
q = 1621, q+t = 2026, V(4,405):	0	1	3	7	68
q = 1637, q+t = 2046, V(4,409):	0	1	3	2	7
q = 1669, q+t = 2086, V(4,417):	0	1	3	7	138
q = 1693, q+t = 2116, V(4,423):	0	1	3	2	23
q = 1709, q+t = 2136, V(4,427):	0	1	3	2	10
q = 1733, q+t = 2166, V(4,433):	0	1	3	2	95
q = 1741, q+t = 2176, V(4,435):	0	1	3	2	53
q = 1789, q+t = 2236, V(4,447):	0	1	3	2	131
q = 1861, q+t = 2326, V(4,465):	0	1	3	2	10
q = 1877, q+t = 2346, V(4,469):	0	1	3	2	16
q = 1901, q+t = 2376, V(4,475):	0	1	3	2	10
q = 1933, q+t = 2416, V(4,483):	0	1	3	2	8
q = 1949, q+t = 2436, V(4,487):	0	1	3	2	5
q = 1973, q+t = 2466, V(4,493):	0	1	3	2	32
q = 1997, q+t = 2496, V(4,499):	0	1	3	2	58

**** m = 5 ****

q = 31, q+t = 37, V(5,6):	0	1	3	7	30	17
q = 41, q+t = 49, V(5,8):	0	1	3	22	14	18
q = 61, q+t = 73, V(5,12):	0	1	3	7	23	50
q = 71, q+t = 85, V(5,14):	0	1	3	9	25	54
q = 101, q+t = 121, V(5,20):	0	1	3	10	43	91
q = 131, q+t = 157, V(5,26):	0	1	3	6	48	15
q = 151, q+t = 181, V(5,30):	0	1	4	11	111	68
q = 181, q+t = 217, V(5,36):	0	1	3	7	34	169
q = 191, q+t = 229, V(5,38):	0	1	3	6	27	51
q = 211, q+t = 253, V(5,42):	0	1	3	6	76	95
q = 241, q+t = 289, V(5,48):	0	1	4	11	40	133
q = 251, q+t = 301, V(5,50):	0	1	4	13	30	175
q = 271, q+t = 325, V(5,54):	0	1	3	6	106	43
q = 281, q+t = 337, V(5,56):	0	1	3	7	93	178
q = 311, q+t = 373, V(5,62):	0	1	3	6	12	142

q = 331, q+t = 397, V(5,66):	0	1	3	7	31	61
q = 401, q+t = 481, V(5,80):	0	1	3	7	2	17
q = 421, q+t = 505, V(5,84):	0	1	3	6	25	317
q = 431, q+t = 517, V(5,86):	0	1	6	20	85	155
q = 461, q+t = 553, V(5,92):	0	1	3	7	41	312
q = 491, q+t = 589, V(5,98):	0	1	3	7	27	321
q = 521, q+t = 625, V(5,104):	0	1	3	6	27	33
q = 541, q+t = 649, V(5,108):	0	1	3	6	13	444
q = 571, q+t = 685, V(5,114):	0	1	4	13	30	395
q = 601, q+t = 721, V(5,120):	0	1	3	6	12	409
q = 631, q+t = 757, V(5,126):	0	1	3	7	17	233
q = 641, q+t = 769, V(5,128):	0	1	4	13	40	469
q = 661, q+t = 793, V(5,132):	0	1	3	7	15	31
q = 691, q+t = 829, V(5,138):	0	1	3	7	80	384
q = 701, q+t = 841, V(5,140):	0	1	3	6	10	113
q = 751, q+t = 901, V(5,150):	0	1	3	7	31	474
q = 761, q+t = 913, V(5,152):	0	1	3	7	23	127
q = 811, q+t = 973, V(5,162):	0	1	3	6	10	719
q = 821, q+t = 985, V(5,164):	0	1	3	7	26	467
q = 881, q+t = 1057, V(5,176):	0	1	3	7	15	222
q = 911, q+t = 1093, V(5,182):	0	1	4	13	23	157
q = 941, q+t = 1129, V(5,188):	0	1	3	6	33	732
q = 971, q+t = 1165, V(5,194):	0	1	4	13	40	215
q = 991, q+t = 1189, V(5,198):	0	1	3	7	27	446
q = 1021, q+t = 1225, V(5,204):	0	1	3	7	50	110
q = 1031, q+t = 1237, V(5,206):	0	1	3	7	34	886
q = 1051, q+t = 1261, V(5,210):	0	1	3	7	17	1034
q = 1061, q+t = 1273, V(5,212):	0	1	3	6	11	634
q = 1091, q+t = 1309, V(5,218):	0	1	3	7	44	284
q = 1151, q+t = 1381, V(5,230):	0	1	3	6	10	60
q = 1171, q+t = 1405, V(5,234):	0	1	3	7	31	901
q = 1181, q+t = 1417, V(5,236):	0	1	4	11	29	424
q = 1201, q+t = 1441, V(5,240):	0	1	3	6	10	418
q = 1231, q+t = 1477, V(5,246):	0	1	3	7	10	287
q = 1291, q+t = 1549, V(5,258):	0	1	3	7	15	946
q = 1301, q+t = 1561, V(5,260):	0	1	3	6	2	1133
q = 1321, q+t = 1585, V(5,264):	0	1	3	6	19	517
q = 1361, q+t = 1633, V(5,272):	0	1	3	7	15	302
q = 1381, q+t = 1657, V(5,276):	0	1	3	6	13	709
q = 1451, q+t = 1741, V(5,290):	0	1	3	6	13	577
q = 1471, q+t = 1765, V(5,294):	0	1	3	7	17	12
q = 1481, q+t = 1777, V(5,296):	0	1	3	6	14	298
q = 1511, q+t = 1813, V(5,302):	0	1	3	7	17	232
q = 1531, q+t = 1837, V(5,306):	0	1	3	6	12	389
q = 1571, q+t = 1885, V(5,314):	0	1	3	6	10	342
q = 1601, q+t = 1921, V(5,320):	0	1	3	6	2	74
q = 1621, q+t = 1945, V(5,324):	0	1	3	6	10	414
q = 1721, q+t = 2065, V(5,344):	0	1	3	6	31	236
q = 1741, q+t = 2089, V(5,348):	0	1	3	7	4	234
q = 1801, q+t = 2161, V(5,360):	0	1	3	7	2	1105
q = 1811, q+t = 2173, V(5,362):	0	1	4	13	66	1179
q = 1831, q+t = 2197, V(5,366):	0	1	3	6	12	731
q = 1861, q+t = 2233, V(5,372):	0	1	3	7	19	1701
q = 1871, q+t = 2245, V(5,374):	0	1	[illegible]	7	15	86
q = 1901, q+t = 2281, V(5,380):	0	1	[illegible]	7	10	58
q = 1931, q+t = 2317, V(5,386):	0	1	3	7	10	1083
q = 1951, q+t = 2341, V(5,390):	0	1	3	7	12	625
q = 2011, q+t = 2413, V(5,402):	0	1	4	11	21	369

**** m = 6 ****

q = 31,	q+t = 36,	V(6,5):	0	1	7	30	12	21	15
q = 43,	q+t = 50,	V(6,7):	0	1	3	16	35	26	36
q = 67,	q+t = 78,	V(6,11):	0	1	3	14	7	24	27
q = 79,	q+t = 92,	V(6,13):	0	1	3	7	55	47	34
q = 103,	q+t = 120,	V(6,17):	0	1	3	2	14	99	29
q = 127,	q+t = 148,	V(6,21):	0	1	4	13	66	93	45
q = 139,	q+t = 162,	V(6,23):	0	1	3	2	31	128	58
q = 151,	q+t = 176,	V(6,25):	0	1	3	2	107	142	149
q = 163,	q+t = 190,	V(6,27):	0	1	3	2	54	89	16
q = 199,	q+t = 232,	V(6,33):	0	1	3	2	23	49	64
q = 211,	q+t = 246,	V(6,35):	0	1	3	2	22	114	111
q = 223,	q+t = 260,	V(6,37):	0	1	4	13	39	216	147
q = 271,	q+t = 316,	V(6,45):	0	1	3	2	7	53	168
q = 283,	q+t = 330,	V(6,47):	0	1	3	6	13	33	124
q = 307,	q+t = 358,	V(6,51):	0	1	3	8	18	215	91
q = 331,	q+t = 386,	V(6,55):	0	1	3	2	8	147	89
q = 367,	q+t = 428,	V(6,61):	0	1	3	2	13	311	84
q = 379,	q+t = 442,	V(6,63):	0	1	3	2	5	346	300
q = 439,	q+t = 512,	V(6,73):	0	1	4	14	25	184	45
q = 463,	q+t = 540,	V(6,77):	0	1	3	2	7	18	133
q = 487,	q+t = 568,	V(6,81):	0	1	3	2	8	334	91
q = 499,	q+t = 582,	V(6,83):	0	1	3	8	23	376	474
q = 523,	q+t = 610,	V(6,87):	0	1	3	2	8	502	266
q = 547,	q+t = 638,	V(6,91):	0	1	3	2	7	434	281
q = 571,	q+t = 666,	V(6,95):	0	1	3	2	5	59	192
q = 607,	q+t = 708,	V(6,101):	0	1	3	2	5	128	324
q = 619,	q+t = 722,	V(6,103):	0	1	3	2	7	17	264
q = 631,	q+t = 736,	V(6,105):	0	1	3	2	5	86	411
q = 643,	q+t = 750,	V(6,107):	0	1	3	10	24	179	117
q = 691,	q+t = 806,	V(6,115):	0	1	3	6	12	30	361
q = 727,	q+t = 848,	V(6,121):	0	1	4	11	16	445	29
q = 739,	q+t = 862,	V(6,123):	0	1	3	6	17	68	360
q = 751,	q+t = 876,	V(6,125):	0	1	3	2	5	59	189
q = 787,	q+t = 918,	V(6,131):	0	1	3	2	8	67	482
q = 811,	q+t = 946,	V(6,135):	0	1	3	6	11	465	66
q = 823,	q+t = 960,	V(6,137):	0	1	3	2	5	25	350
q = 859,	q+t = 1002,	V(6,143):	0	1	3	2	5	139	271
q = 883,	q+t = 1030,	V(6,147):	0	1	3	2	11	50	288
q = 907,	q+t = 1058,	V(6,151):	0	1	3	2	7	393	846
q = 919,	q+t = 1072,	V(6,153):	0	1	4	11	16	200	231
q = 967,	q+t = 1128,	V(6,161):	0	1	3	2	7	27	311
q = 991,	q+t = 1156,	V(6,165):	0	1	3	2	8	352	96
q = 1039,	q+t = 1212,	V(6,173):	0	1	3	2	5	14	422
q = 1051,	q+t = 1226,	V(6,175):	0	1	3	6	11	247	329
q = 1063,	q+t = 1240,	V(6,177):	0	1	3	2	5	315	567
q = 1087,	q+t = 1268,	V(6,181):	0	1	3	2	5	561	861
q = 1123,	q+t = 1310,	V(6,187):	0	1	3	2	5	384	786
q = 1171,	q+t = 1366,	V(6,195):	0	1	3	2	5	70	392
q = 1231,	q+t = 1436,	V(6,205):	0	1	3	2	5	122	559
q = 1279,	q+t = 1492,	V(6,213):	0	1	3	2	5	46	747
q = 1291,	q+t = 1506,	V(6,215):	0	1	3	2	5	21	1257
q = 1303,	q+t = 1520,	V(6,217):	0	1	3	2	9	91	861
q = 1327,	q+t = 1548,	V(6,221):	0	1	4	11	2	159	1119
q = 1399,	q+t = 1632,	V(6,233):	0	1	4	11	30	7	229
q = 1423,	q+t = 1660,	V(6,237):	0	1	4	11	26	488	436
q = 1447,	q+t = 1688,	V(6,241):	0	1	3	2	5	72	1226

q = 1459, q+t = 1702, V(6,243):	0	1	3	6	17	78	522
q = 1471, q+t = 1716, V(6,245):	0	1	4	11	3	39	1184
q = 1483, q+t = 1730, V(6,247):	0	1	3	2	5	277	690
q = 1531, q+t = 1786, V(6,255):	0	1	3	2	13	41	1451
q = 1543, q+t = 1800, V(6,257):	0	1	3	2	7	150	1116
q = 1567, q+t = 1828, V(6,261):	0	1	3	2	5	15	562
q = 1579, q+t = 1842, V(6,263):	0	1	3	6	11	40	200
q = 1627, q+t = 1898, V(6,271):	0	1	3	6	12	334	1072
q = 1663, q+t = 1940, V(6,277):	0	1	3	2	5	109	217
q = 1699, q+t = 1982, V(6,283):	0	1	3	6	12	369	1269
q = 1723, q+t = 2010, V(6,287):	0	1	3	6	12	21	1169
q = 1747, q+t = 2038, V(6,291):	0	1	3	2	5	142	1186
q = 1759, q+t = 2052, V(6,293):	0	1	3	2	9	106	1618
q = 1783, q+t = 2080, V(6,297):	0	1	3	2	7	37	1024
q = 1831, q+t = 2136, V(6,305):	0	1	4	13	2	115	613
q = 1867, q+t = 2178, V(6,311):	0	1	3	2	9	32	638
q = 1879, q+t = 2192, V(6,313):	0	1	3	2	7	53	911
q = 1951, q+t = 2276, V(6,325):	0	1	3	2	5	14	1842
q = 1987, q+t = 2318, V(6,331):	0	1	3	2	7	212	877
q = 1999, q+t = 2332, V(6,333):	0	1	4	11	2	59	882
q = 2011, q+t = 2346, V(6,335):	0	1	3	2	5	195	247

**** m = 7 ****

q = 43, q+t = 49, V(7,6):	0	1	12	27	37	16	30	35
q = 71, q+t = 81, V(7,10):	0	1	3	45	9	50	28	16
q = 113, q+t = 129, V(7,16):	0	1	3	7	82	72	93	39
q = 127, q+t = 145, V(7,18):	0	1	3	6	97	114	99	26
q = 197, q+t = 225, V(7,28):	0	1	3	6	107	187	82	12
q = 211, q+t = 241, V(7,30):	0	1	3	7	50	2	69	93
q = 239, q+t = 273, V(7,34):	0	1	3	6	10	153	234	80
q = 281, q+t = 321, V(7,40):	0	1	3	7	34	79	184	132
q = 337, q+t = 385, V(7,48):	0	1	3	6	16	82	184	30
q = 379, q+t = 433, V(7,54):	0	1	3	7	12	301	95	130
q = 421, q+t = 481, V(7,60):	0	1	3	7	16	38	397	218
q = 449, q+t = 513, V(7,64):	0	1	3	6	2	423	366	141
q = 463, q+t = 529, V(7,66):	0	1	3	6	20	57	110	82
q = 491, q+t = 561, V(7,70):	0	1	3	7	2	401	9	37
q = 547, q+t = 625, V(7,78):	0	1	3	6	11	19	450	147
q = 617, q+t = 705, V(7,88):	0	1	3	7	2	259	237	497
q = 631, q+t = 721, V(7,90):	0	1	4	11	16	200	560	529
q = 659, q+t = 753, V(7,94):	0	1	3	6	2	407	544	168
q = 673, q+t = 769, V(7,96):	0	1	4	11	16	61	485	536
q = 701, q+t = 801, V(7,100):	0	1	3	6	14	130	196	174
q = 743, q+t = 849, V(7,106):	0	1	3	6	2	588	607	153
q = 757, q+t = 865, V(7,108):	0	1	3	7	15	49	455	732
q = 827, q+t = 945, V(7,118):	0	1	3	6	10	136	18	740
q = 883, q+t = 1009, V(7,126):	0	1	3	7	15	137	59	429
q = 911, q+t = 1041, V(7,130):	0	1	3	7	2	175	662	622
q = 953, q+t = 1089, V(7,136):	0	1	4	11	16	252	710	317
q = 967, q+t = 1105, V(7,138):	0	1	3	6	11	370	836	845
q = 1009, q+t = 1153, V(7,144):	0	1	3	7	15	31	973	922
q = 1051, q+t = 1201, V(7,150):	0	1	3	7	2	12	336	684
q = 1093, q+t = 1249, V(7,156):	0	1	3	7	14	122	52	257
q = 1163, q+t = 1329, V(7,166):	0	1	4	11	16	212	754	190
q = 1289, q+t = 1473, V(7,184):	0	1	3	7	15	200	617	1204
q = 1303, q+t = 1489, V(7,186):	0	1	3	6	12	170	79	139
q = 1373, q+t = 1569, V(7,196):	0	1	3	6	2	30	527	294
q = 1429, q+t = 1633, V(7,204):	0	1	3	6	2	217	725	458
q = 1471, q+t = 1681, V(7,210):	0	1	3	6	2	8	1130	989

q = 1499, q+t = 1713, V(7,214):	0	1	3	7	15	110	1313	783	
q = 1583, q+t = 1809, V(7,226):	0	1	3	7	15	50	774	1438	
q = 1597, q+t = 1825, V(7,228):	0	1	3	7	15	91	607	945	
q = 1667, q+t = 1905, V(7,238):	0	1	3	6	2	121	30	1182	
q = 1709, q+t = 1953, V(7,244):	0	1	4	11	28	63	397	199	
q = 1723, q+t = 1969, V(7,246):	0	1	3	6	11	59	1525	1037	
q = 1877, q+t = 2145, V(7,268):	0	1	3	7	15	55	1852	1681	
q = 1933, q+t = 2209, V(7,276):	0	1	3	6	11	17	816	485	
q = 2003, q+t = 2289, V(7,286):	0	1	4	11	16	97	593	618	
q = 2017, q+t = 2305, V(7,288):	0	1	3	7	2	22	1961	1493	

**** m = 8 ****

q = 73, q+t = 82, V(8,9):	0	1	20	70	23	59	3	8	19
q = 89, q+t = 100, V(8,11):	0	1	6	56	22	35	47	23	60
q = 137, q+t = 154, V(8,17):	0	1	3	2	133	126	47	109	74
q = 233, q+t = 262, V(8,29):	0	1	4	11	94	60	85	16	198
q = 281, q+t = 316, V(8,35):	0	1	3	6	32	37	271	266	171
q = 313, q+t = 352, V(8,39):	0	1	3	7	67	135	72	197	145
q = 409, q+t = 460, V(8,51):	0	1	3	2	5	295	124	54	353
q = 457, q+t = 514, V(8,57):	0	1	3	2	12	333	363	154	340
q = 521, q+t = 586, V(8,65):	0	1	3	2	5	509	443	183	18
q = 569, q+t = 640, V(8,71):	0	1	3	2	5	179	142	337	47
q = 601, q+t = 676, V(8,75):	0	1	6	20	2	89	220	395	30
q = 617, q+t = 694, V(8,77):	0	1	3	8	5	242	354	371	321
q = 761, q+t = 856, V(8,95):	0	1	3	2	5	89	740	30	61
q = 809, q+t = 910, V(8,101):	0	1	3	2	5	539	13	216	72
q = 857, q+t = 964, V(8,107):	0	1	3	2	5	85	794	148	646
q = 937, q+t = 1054, V(8,117):	0	1	4	11	16	114	686	107	597
q = 953, q+t = 1072, V(8,119):	0	1	3	2	5	49	26	629	98
q = 1033, q+t = 1162, V(8,129):	0	1	3	6	11	39	992	141	701
q = 1049, q+t = 1180, V(8,131):	0	1	3	6	11	34	768	675	801
q = 1097, q+t = 1234, V(8,137):	0	1	3	6	11	20	155	930	262
q = 1129, q+t = 1270, V(8,141):	0	1	3	2	12	80	713	257	653
q = 1193, q+t = 1342, V(8,149):	0	1	3	6	11	47	985	664	768
q = 1289, q+t = 1450, V(8,161):	0	1	4	11	2	107	849	356	411
q = 1321, q+t = 1486, V(8,165):	0	1	3	7	15	62	1294	176	38
q = 1433, q+t = 1612, V(8,179):	0	1	4	13	2	67	365	728	982
q = 1481, q+t = 1666, V(8,185):	0	1	3	6	11	17	1419	793	1429
q = 1609, q+t = 1810, V(8,201):	0	1	4	13	32	74	640	507	689
q = 1657, q+t = 1864, V(8,207):	0	1	3	7	2	17	1214	1555	1537
q = 1721, q+t = 1936, V(8,215):	0	1	4	11	7	25	471	242	949
q = 1753, q+t = 1972, V(8,219):	0	1	3	6	11	56	83	770	1506
q = 1801, q+t = 2026, V(8,225):	0	1	4	14	3	134	1419	1339	985
q = 1913, q+t = 2152, V(8,239):	0	1	4	11	3	34	540	553	434
q = 1993, q+t = 2242, V(8,249):	0	1	3	2	8	15	1339	1914	630