# A New Short Proof of an Old Folk Theorem in Functional Differential Equations

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ABSTRACT

We use the theory of Weinstein-Aronszajn determinants to prove that the multiplicity of $z$ as a root of the characteristic equations equals the algebraic multiplicity of $z$ as an eigenvalue of the infinitesimal generator.

## 1. INTRODUCTION

Let $\zeta$ be an $n \times n$-matrix valued normalized bounded variation function. With the retarded functional differential equation

$$\frac{dx}{dt}(t) = \int_0^h d\zeta(\tau)x(t-\tau) \tag{1.1}$$

one can associate the characteristic equation

$$\det \Delta(z) = 0, \tag{1.2}$$

where

$$\Delta(z) = zI - \int_0^h d\zeta(\tau)e^{-z\tau}. \tag{1.3}$$

Zeros of the characteristic equation yield exponential solutions of (1.1) and higher order zeros yield polynomial-exponential solutions.

The definition

$$\big(T(t)\varphi\big)(\tau) = x(t+\tau;\varphi) \quad \text{for } -h \le \tau \le 0, \tag{1.4}$$

where $t \mapsto x(t;\varphi)$ denotes the unique solution of (1.1) corresponding to the initial condition

$$x(\tau) = \varphi(\tau) \quad \text{for } -h \le \tau \le 0 \tag{1.5}$$

yields a $\mathcal{C}_0$-semigroup of bounded linear operators on $\mathcal{C} = C\big([-h,0];\mathbf{C}^n\big)$. Its infinitesimal generator is given by (cf. Hale [6])

$$\mathcal{D}(A) = \{\varphi \in \mathcal{C}^1 : \dot\varphi(0) = \langle\zeta,\varphi\rangle\}, \qquad A\varphi = \dot\varphi, \tag{1.6}$$

where $\langle\zeta,\varphi\rangle$ denotes the functional $\int_0^h d\zeta(\tau)\varphi(-\tau)$. One easily verifies that $z \in \sigma(A)$ if and only if (1.2) holds. It is well known, but less easy to prove, that

THEOREM. *The algebraic multiplicity of $z$ as an eigenvalue of $A$ equals the multiplicity of $z$ as a root of the characteristic equation (1.2).*

The first proof of this theorem is due to Levinger [7]. A second proof was published by Kappel and Wimmer [9]. The aim of the present note is to show that the theorem actually is a straightforward consequence of the general theory of the Weinstein-Aronszajn determinant (Kato [10]). One can make this observation as soon as one realizes that (1.1) can be considered as a finite rank perturbation of the "trivial" functional differential equation

$$\frac{dx}{dt}(t) = 0. \tag{1.7}$$

The correct setting for such a perturbation point of view involves dual semigroups (Clément et al [1,2,3], Diekmann [5]) and the embedding of $\mathcal{C}$ into the space $\mathbf{C}^n \times L_\infty[-h,0]$. Here we shall exploit this space $\mathbf{C}^n \times L_\infty$ and some notation suggested by dual semigroup theory, but not the theory itself.

## 2. PROOF OF THE THEOREM

Let $D : L_\infty \to L_\infty$ be the unbounded operator with domain $\mathcal{D}\big(D\big) = \text{Lip}$, the set of equivalence classes containing a Lipschitz continuous function, and action

$$D\varphi = \dot\varphi. \tag{2.1}$$

For every $z \in \mathbf{C}$ we define a pseudo-inverse

$$\big(Ps(z-D)^{-1}\varphi\big)(\tau) = -\int_0^\tau e^{(\tau-\sigma)z}\varphi(\sigma)d\sigma. \tag{2.2}$$

Note that indeed $(z-D)Ps(z-D)^{-1} = I$, but that

$$\big(Ps(z-D)^{-1}(z-D)\varphi\big)(\tau) = \varphi(\tau) - e_z(\tau)\varphi(0) \quad \text{for } \varphi \in \mathcal{D}(D), \tag{2.3}$$

where

$$e_z(\tau) := e^{z\tau}. \tag{2.4}$$

Next we define the "unperturbed" operator $A_0^{\odot *} : \mathbf{C}^n \times L_\infty \to \mathbf{C}^n \times L_\infty$ by

$$\mathcal{D}(A_0^{\odot *}) = \left\{\begin{pmatrix}\alpha\\ \varphi\end{pmatrix} : \varphi \in \mathcal{D}(D), \alpha = \varphi(0)\right\}, \qquad A_0^{\odot *} = \begin{pmatrix}0 & 0\\ 0 & D\end{pmatrix}. \tag{2.5}$$

A straightforward calculation yields

$$\big(zI - A_0^{\odot *}\big)^{-1} = \begin{pmatrix} z^{-1} & 0 \\ z^{-1}e_z & Ps(z-D)^{-1}\end{pmatrix}. \tag{2.6}$$

We conclude that $z \in \rho(A_0^{\odot *})$ if and only if $z \neq 0$ and that $z = 0$ is an eigenvalue of algebraic multiplicity $n$, with eigenvectors $e_0\alpha$.

The perturbation has domain $\mathcal{D}(B) = \mathbf{C}^n \times \mathcal{C}$ (where $\mathcal{C}$ denotes the set of equivalence classes which contain a continuous function) and action

$$B = \begin{pmatrix}0 & \zeta\\ 0 & 0\end{pmatrix}, \tag{2.7}$$

where $\zeta$ denotes the functional $\varphi \mapsto \langle \zeta, \varphi\rangle$. Finally, we define

$$\mathcal{D}(A^{\odot *}) = \mathcal{D}(A_0^{\odot *}), \qquad A^{\odot *} = A_0^{\odot *} + B. \tag{2.8}$$

Note that $B$ is relatively bounded and has finite dimensional range. The identity

$$\big(zI - A^{\odot *}\big)^{-1} = \big(zI - A_0^{\odot *}\big)^{-1}\big(I - B\big(zI - A_0^{\odot *}\big)^{-1}\big)^{-1} \tag{2.9}$$

shows that we can analyse the spectrum of $A^{\odot *}$ by combining information about the spectrum of $A_0^{\odot *}$ and information concerning

$$\det\big(I - B\big(zI - A_0^{\odot *}\big)^{-1}\big).$$

This is exactly the key idea of the

WEINSTEIN-ARONSZAJN THEOREM (Kato [10] Theorem IV.6.2).

$$\widetilde{\nu}(z; A^{\odot *}) = \widetilde{\nu}(z; A_0^{\odot *}) + \nu\big(z; \det\big(I - B\big(zI - A_0^{\odot *}\big)^{-1}\big)\big), \tag{2.10}$$

where for any closed operator $T$

$$\widetilde{\nu}(z;T) = \begin{cases} 0 & \text{if } z \in \rho(T) \\ \dim \mathcal{R}\ \text{(spectral projection)} & \text{if } z \text{ is an isolated point of } \sigma(T) \\ \infty & \text{otherwise} \end{cases} \tag{2.11}$$

and for any meromorphic function $f$

$$\nu(z;f)=\begin{cases}k & \text{if } z \text{ is a zero of order } k \text{ of } f\\ -k & \text{if } z \text{ is a pole of order } k \text{ of } f\\ 0 & \text{otherwise.}\end{cases} \tag{2.12}$$

So, in particular

$$\widetilde{\nu}(z;A_0^{\odot *})=\begin{cases}0 & \text{for } z\neq 0\\ n & \text{for } z=0.\end{cases} \tag{2.13}$$

Combining (2.6) and (2.7) we find

$$I-B\big(zI-A_0^{\odot *}\big)^{-1}=\begin{pmatrix} I-z^{-1}\langle\zeta,e_z\rangle & -\langle\zeta,Ps(z-D)^{-1}\cdot\rangle\\ 0 & I\end{pmatrix} \tag{2.14}$$

and

$$\det\big(I-B\big(zI-A_0^{\odot *}\big)^{-1}\big)=\det\frac{\Delta(z)}{z}=z^{-n}\det\Delta(z). \tag{2.15}$$

COROLLARY.

$$\widetilde{\nu}(z;A^{\odot *})=\nu(z;\det\Delta(z)).$$

To conclude the proof it only remains to show the relationship between $A^{\odot *}$ and $A$.

Let $j:\mathcal{C}\to\mathbf{C}^n\times L_\infty$ be the embedding

$$j\varphi=\begin{pmatrix}\varphi(0)\\ \varphi\end{pmatrix}. \tag{2.16}$$

Since $\mathcal{D}\big(A_0^{\odot *}\big)\subset j\mathcal{C}$ the resolvent maps into $j\mathcal{C}$ and eigenvectors belong to $j\mathcal{C}$. So without loss of generality, we may restrict our spectral analysis to $j\mathcal{C}$. Now $A^{\odot *}j\varphi\in j\mathcal{C}$ if and only if $\varphi\in\mathcal{C}^1$ and $\dot\varphi(0)=\langle\zeta,\varphi\rangle$. Moreover, in that case $j^{-1}A^{\odot *}j\varphi=\dot\varphi$. It follows that $A$ is, modulo the embedding $j$, the part of $A^{\odot *}$ in $\mathcal{C}$.

## 3. REMARKS

(i) In Kaashoek and Verduyn Lunel [8], the authors develop a general procedure to construct characteristic matrix functions and use the idea of equivalence to prove the above multiplicity theorem for various classes of equations. For the operators appearing in the present paper the equivalence in [8] leads to the formulas mentioned next.

(ii) One can write

$$I-B\big(zI-A_0^{\odot *}\big)^{-1}=F(z)\begin{pmatrix} z^{-1}\Delta(z) & 0\\ 0 & I\end{pmatrix} \tag{3.1}$$

and

$$\big(zI-A_0^{\odot *}\big)^{-1}=E(z)\begin{pmatrix} z^{-1} & 0\\ 0 & I\end{pmatrix}, \tag{3.2}$$

where

$$F(z) = \begin{pmatrix} I & -\langle \zeta, Ps(z-D)^{-1} \cdot \rangle \\ 0 & I \end{pmatrix} \tag{3.3}$$

and

$$E(z) = \begin{pmatrix} I & 0 \\ e_z & Ps(z-D)^{-1} \end{pmatrix} \tag{3.4}$$

are regular operator valued functions. The formula

$$\left(zI - A^{\odot *}\right)^{-1} = E(z) \begin{pmatrix} \Delta^{-1}(z) & 0 \\ 0 & I \end{pmatrix} F(z)^{-1} \tag{3.5}$$

then clearly shows the equivalence of $\left(zI - A^{\odot *}\right)^{-1}$ and $\Delta^{-1}(z)$ and one can, among other things, derive the precise relationship between generalized eigenvectors of $A^{\odot *}$ and Jordan chains of $\Delta$ from that formula. (See Kaashoek and Verduyn Lunel [8].)

(iii) A spectral theory of unbounded operator matrices is currently being developed in Tübingen by Nagel [11] and others.

(iv) A related but somewhat different perturbation point of view is presented in the work of Desch and Schappacher [4].

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