Seven Criteria for Integer Sequences Being Graphic

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ABSTRACT

Seven criteria for integer sequences being graphic are listed. Being graphic means that there is a simple graph with the given integer sequence as degree sequence. One of the criteria leads to a new and constructive proof of the well-known criterion of Erdös–Gallai.

1. INTRODUCTION

Let $(d_1, \ldots, d_n)$ be a nonincreasing sequence of positive integers with even sum. The sequence $(d_1, \ldots, d_n)$ is called graphic iff there is a simple graph (without loops and multiple edges) that has $(d_1, \ldots d_n)$ as degree sequence. In this paper seven criteria for such an integer sequence being graphic are listed; one of these is well known and due to Erdös and Gallai [5]. Proofs of the Erdös–Gallai Criterion can be found in Berge [1] and in Harary [10]. Harary's proof is rather lengthy and Berge's proof uses flows in networks. Recently, Choudum [4] has given a different proof which, in our opinion, is not very appealing either. Using the recently discovered Hässelbarth Criterion we are able to give a new and elegant proof.

2. THE SEVEN CRITERIA

In Theorem 1 it will be shown that the following conditions (A)–(G) are all equivalent to "$(d_1, \ldots d_n)$ is graphic."

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A. The Ryser Criterion. See Bondy and Murty [3] and Ryser [12]. A sequence $(a_1, \ldots, a_p; b_1, \ldots, b_n)$ is called bipartite-graphic iff there is a simple bipartite graph such that one component has degree sequence $(a_1, \ldots, a_p)$ and the other one has $(b_1, \ldots, b_n)$. Define $f = \max\{i \mid d_i \geq i\}$ and $\tilde{d}_i = d_i + 1$ if $i \in \langle f \rangle (=\{1, \ldots, f\})$ and $\tilde{d}_i = d_i$ otherwise. The criterion can be stated as follows:

$$\text{The integer sequence } (\tilde{d}_1, \ldots, \tilde{d}_n; \tilde{d}_1, \ldots, \tilde{d}_n) \text{ is bipartite-graphic.} \tag{A}$$

B. The Berge Criterion. See, e.g., Berge [1]. Define $(\bar{d}_1, \ldots, \bar{d}_n)$ as follows: For $i \in \langle n \rangle$, $\bar{d}_i$ is the ith column sum of the (0,1)-matrix, which has for each k the d_k leading terms in row k equal to 1 except for the (k,k)th term that is 0 and also the remaining entries are 0. For example, if $d_1 = 3$, $d_2 = 2$, $d_3 = 2$, $d_4 = 2$, $d_5 = 1$, then $\bar{d}_1 = 4$, $\bar{d}_2 = 3$, $\bar{d}_3 = 2$, $\bar{d}_4 = 1$, $\bar{d}_5 = 0$, and the (0,1)-matrix becomes

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The criterion is

$$\sum_{i=1}^{k} d_i \leq \sum_{i=1}^{k} d_i \quad \text{for each } k \in \langle n \rangle . \tag{B}$$

C. The Erdös–Gallai Criterion. See, e.g., Bondy and Murty [3].

$$\sum_{i=1}^{k} d_i \leq (k)(k-1) + \sum_{j=k+1}^{n} \min\{k, d_j\} \quad \text{for each } k \in \langle n \rangle . \tag{C}$$

D. The Fulkerson–Hoffman–McAndrew Criterion. See [5] and Grünbaum [6].

$$\sum_{i=1}^{k} d_i \leq (k)(n-m-1) + \sum_{i=n-m+1}^{n} d_i \quad \text{for each } k \in \langle n \rangle, \quad m \geq 0 \text{ and } k + m \leq n . \tag{D}$$

E. The Bollobás Criterion. See [2].

$$\sum_{i=1}^{k} d_i \leq \sum_{i=k+1}^{n} d_i + \sum_{i=1}^{k} \min\{d_i, k-1\} \quad \text{for each } k \in \langle n \rangle . \tag{E}$$

F. The Grünbaum Criterion. See Grünbaum [7].

$$\sum_{i=1}^{k} \max\{k - 1, d_i\} \leq (k)(k - 1) + \sum_{i=k+1}^{n} d_i \quad \text{for each } k \in \langle n \rangle . \qquad \text{(F)}$$

G. The Hässelbarth Criterion. See Hässelbarth [11]. Define $(d_1^*, \ldots, d_n^*)$ as follows: For $i \in \langle n \rangle$, d_i^* is the ith column sum of the $(0,1)$-matrix in which the d_i leading terms in row i are 1's, and the remaining entries are 0's. The criterion is

$$\sum_{i=1}^{k} d_i \leq \sum_{i=1}^{k} (d_i^* - 1) \quad \text{for each } k \in \langle f \rangle$$

with $f = \max\{i \,|\, d_i \geq i\}$. (G)

We call (G) the Hässelbarth Criterion because it was first described in Hässelbarth [11]. However, the form in which it is described in [11] is rather hidden, and there is also no proof. We will show that (G) is in fact equivalent to criteria (A)–(F). Hässelbarth's Criterion is concise and clear, and gives rise to an elegant proof of the Erdös–Gallai Criterion.

3. THE MAIN THEOREM

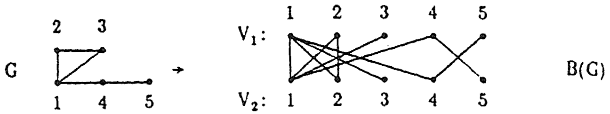
Theorem 1. Let $(d_i, \ldots d_n)$ be a positive integer sequence with even sum. Then the following holds:

Each of the criteria (A)–(G) is equivalent to the statement that $(d_1, \ldots d_n)$ is graphic.

Proof. To prove the theorem we go through the following implications cycle:

$$\text{(Graphic)} \overset{1}{\Rightarrow} \text{(A)} \overset{2}{\Rightarrow} \text{(B)} \overset{3}{\Rightarrow} \text{(C)} \overset{4}{\Rightarrow} \text{(D)} \overset{5}{\Rightarrow} \text{(E)} \overset{6}{\Rightarrow} \text{(F)} \overset{7}{\Rightarrow} \text{(G)} \overset{8}{\Rightarrow} \text{(Graphic)}.$$

1. Let $G = (V, E)$ be the simple graph that realizes $(d_1, \ldots, d_n)$. Starting with G, it is possible to construct a bipartite graph $B(G)$ with components V_1 and V_2 where $V_1 = V_2\ (= V)$, and edge set F defined as follows: If the edge joining i and j with $i, j \in V$ is in E, the edges joining $i \in V_1$ and $j \in V_2$ as well as the one joining $j \in V_1$ and $i \in V_2$ are in F. Moreover, the edges joining $i \in V_1$ and $j \in V_2$ for $i \in \langle f \rangle$ are also taken in F. For example,



The bipartite graph $B(G)$ has in both components degree sequence $(d_1 + 1, \ldots, d_f + 1, d_{f+1}, \ldots, d_n)$. So the Ryser Criterion holds.

2. Let the sequence $(\tilde{d}_1, \ldots, \tilde{d}_n; \bar{d}_1, \ldots, \bar{d}_n)$, as defined under A, be bipartite-graphic. By a well-known theorem of Ryser (see, e.g., Bondy and Murty [3], p. 210, and Ryser [12]), it follows that

$$\sum_{i=1}^{k} \tilde{d}_i \leq \sum_{i=1}^{k} \tilde{d}_i^* \quad \text{for each } k \in \langle n \rangle,$$

where d_i^* is defined as in (G). Hence $\tilde{d}_i^* = \bar{d}_i + 1$ for each $i \in \langle f \rangle$, and $\tilde{d}_i = \bar{d}_i$ for each $i \in \langle n \rangle \backslash \langle f \rangle$.

Moreover, $\tilde{d}_i = d_i + 1$ for each $i \in \langle f \rangle$, and $\tilde{d}_i = d_i$ for each $i \in \langle n \rangle \backslash \langle f \rangle$. If $k \leq f$, we find

$$\sum_{i=1}^{k} (d_i + 1) = \sum_{i=1}^{k} \tilde{d}_i \leq \sum_{i=1}^{k} \tilde{d}_i^* = \sum_{i=1}^{k} (\bar{d}_i + 1),$$

hence $\Sigma_{i=1}^{k} d_i \leq \Sigma_{i=1}^{k} \bar{d}_i$.

If $k \geq f + 1$ we find

$$\sum_{i=1}^{k} d_i + f = \sum_{i=1}^{k} \tilde{d}_i \leq \sum_{i=1}^{k} \tilde{d}_i^* = \sum_{i=1}^{k} \bar{d}_i + f, \quad \text{so that } \sum_{i=1}^{k} d_i \leq \sum_{i=1}^{k} \bar{d}_i .$$

Therefore, $\Sigma_{i=1}^{k} d_i \leq \Sigma_{i=1}^{k} \bar{d}_i$ for each $k \in \langle n \rangle$.

3. Consider the (0,1)-matrix corresponding to $(\bar{d}_1, \ldots, \bar{d}_n)$ as defined in (B). Take any $k \in \langle n \rangle$. Then $\Sigma_{i=1}^{k} \bar{d}_i$ is the number of 1's in the first k columns. In this (0,1)-matrix all diagonal elements are 0, which means that in the submatrix consisting of the first k rows and columns at most $k^2 - k$ entries are 1. On the other hand, each row j has precisely $\min\{k, d_j\}$ 1's on the first k positions. Hence, for each $k \in \langle n \rangle$, we find

$$\sum_{i=1}^{k} d_i \leq \sum_{i=1}^{k} \bar{d}_i \leq (k)(k - 1) + \sum_{j=k+1}^{n} \min\{k, d_j\}.$$

4. For each $k \in \langle n \rangle$, $m \geq 0$ and $n - m \geq k$, it follows that

$$\begin{aligned} \sum_{i=1}^{k} d_i &\leq (k)(k - 1) + \sum_{j=k+1}^{n} \min\{k, d_j\} \\ &\leq (k)(k - 1) + \sum_{j=k+1}^{n-m} \min\{k, d_j\} + \sum_{j=n-m}^{n} \min\{k, d_j\} \end{aligned}$$

$$\leq (k)(k-1) + (k)(n-m-k) + \sum_{j=n-m+1}^{n} \min\{k, d_j\}$$

$$\leq (k)(n-m-1) + \sum_{j=n-m+1}^{n} d_j .$$

5. Take any $k \in \langle n \rangle$. If $d_1, \ldots, d_k \leq k-1$, then substitute $n - m = k$, and we are done. So, we may assume that $d_1, \ldots, d_l \geq k-1$ and $d_{l+1}, \ldots, d_k < k-1$ for some $l < k$. Then the following holds:

$$\sum_{i=1}^{k} d_i = \sum_{i=1}^{l} + \sum_{i=l+1}^{k} d_i \leq \text{(using (D))}$$

$$\leq (l)(n-m-1) + \sum_{i=n-m+1}^{n} d_i + \sum_{i=l+1}^{k} d_i = \text{(using } m = n-k\text{)}$$

$$= (l)(k-1) + \sum_{i=k+1}^{n} d_i + \sum_{i=l+1}^{k} d_i$$

$$= \sum_{i=k+1}^{n} d_i + \sum_{i=1}^{l} (k-1) + \sum_{i=l+1}^{n} d_i$$

$$= \sum_{i=k+1}^{n} d_i + \sum_{i=1}^{l} \min\{d_i, k-1\} + \sum_{i=l+1}^{k} \min\{d_i, k-1\}$$

$$= \sum_{i=k+1}^{n} d_i + \sum_{i=1}^{k} \min\{d_i, k-1\}.$$

6. For each $k \in \langle n \rangle$ the following holds:

$$\sum_{i=1}^{k} d_i \leq \sum_{i=k+1}^{n} d_i + \sum_{i=1}^{k} \min\{d_i, k-1\}$$

$$\Leftrightarrow \sum_{i=1}^{k} d_i + \sum_{i=1}^{k} \max\{-d_i, -k+1\} \leq \sum_{i=k+1}^{n} d_i$$

$$\Leftrightarrow \sum_{i=1}^{k} \max\{0, d_i - k + 1\} \leq \sum_{i=k+1}^{n} d_i$$

$$\Leftrightarrow \sum_{i=1}^{k} [\max\{k-1, d_i\} - k + 1] \leq \sum_{i=k+1}^{n} d_i$$

$$\Leftrightarrow \sum_{i=1}^{k} \max\{k-1, d_i\} \leq (k)(k-1) + \sum_{i=k+1}^{n} d_i .$$

7. Suppose to the contrary that there is an index $k \in \langle f \rangle$ such that

$$\sum_{i=1}^{k} d_i > \sum_{i=1}^{k} (d_i^* - 1).$$

Note that $f \leq n - 1$. If $f = n - 1$, then $d_i = n - 1$ for each $i \in \langle n \rangle$ and (G) is trivially true. So we may assume that $f \leq n - 2$. There now exists an index $m \in \{k, \ldots, n\}$ such that $d_i \geq k$ for $i \leq m$ and $d_i < k$ for $i \geq m + 1$.

We then have,

$$\sum_{i=1}^{k} d_i > \sum_{i=1}^{k} (d_i^* - 1) = (k)(m - 1) + \sum_{i=m+1}^{n} d_i \geq \text{(using (G))}$$

$$\geq (k)(m - 1) - (m)(m - 1) + \sum_{i=1}^{m} \max\{m - 1, d_i\}.$$

Hence,

$$\sum_{i=1}^{m} \max\{m - 1, d_i\} < (m - k)(m - 1) + \sum_{i=1}^{k} d_i$$

$$= \sum_{i=1}^{k} d_i + \sum_{i=k+1}^{m} (m - 1) \left(\text{with } \sum_{i=m+1}^{m} (m - 1) \text{ defined } 0\right)$$

$$\leq \sum_{i=1}^{k} \max\{m - 1, d_i\} + \sum_{i=k+1}^{m} \max\{m - 1, d_i\}$$

$$= \sum_{i=1}^{m} \max\{m - 1, d_i\}.$$

This is a contradiction. So Hässelbarth's Criterion holds.

8. Using Hässelbarth's Criterion we finally show that the integer sequence $(d_1, \ldots, d_n)$ is graphic.

   Define for each $i \in \langle f \rangle$

$$a_i = d_i - i + 1 \quad \text{and} \quad b_i = d_i^* - i\,.$$

Hässelbarth's Criterion can then be rewritten as follows:

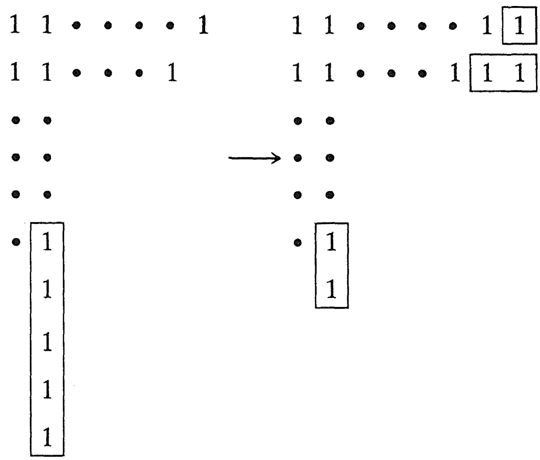
$$\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i \quad \text{for each } k \in \langle f \rangle.$$

Our first objective is to transform the (0,1)-matrix A, as defined under (G), into a symmetric (0,1)-matrix A'' with 0's on the main diagonal. Obviously, A'' corresponds with a simple graph. In general, however, this graph does not satisfy the original degree sequence $(d_1, \ldots, d_n)$. So we need a transformation that results in a symmetric matrix A^* with row sum vector the original degree sequence. The first step, in obtaining A'', is the following algorithm; it results, generally, in a nonsymmetric matrix A'.

$k := f$
WHILE $k > 1$ DO
BEGIN
IF $a_k = b_k$ THEN $a'_k = a_k$
IF $a_k < b_k$ THEN
BEGIN $a'_k := a_k + \lfloor \frac{1}{2}(b_k - a_k) \rfloor$
IF $b_k - a_k$ odd THEN $a_{k-1} := a_{k-1} + 1$
END (1)
IF $a_k > b_k$ THEN
BEGIN $a'_k := b_k$ and $a_{k-1} := a_{k-1} + a_k - b_k$
END (2)
$k := k - 1$
END
$a'_1 := a_1 + \frac{1}{2}(b_1 - a_1)$
Define $b'_i = a'_i$ for $i \in \langle f \rangle$ (3)

The above algorithm needs the following remarks:

(1) This step can be visualized as follows



(2) In this step $a_k - b_k$ 1's from row k are lifted to row $k - 1$.
(3) The fact that $b_1 - a_1$ (with a_1 the number of 1's in row 1 after applying the $f - 1$ steps of the algorithm!) is nonnegative and even, is the key of the algorithm. The proof is as follows.

Suppose, there is a maximal index $\alpha \geq 2$ such that $b_i \leq a_i$ for each $i = 2, \ldots, \alpha$. Hence, for the rows $\alpha, \alpha - 1, \ldots, 2$ only loop (2) of the algorithm is used. Row α receives at most one 1 from column $\alpha + 1$, namely in case $b_{\alpha+1} - a_{\alpha+1}$ is odd. Therefore, $a_\alpha := a_\alpha(+\,1)$. The final step, when executing loop (2), implies that

$$a_1 := \sum_{i=1}^{\alpha} a_i - \sum_{i=2}^{\alpha} b_i(+\,1),$$

whereas the a_i's in the right-hand side of this expression are the original a_i's of the matrix A.

The Hässelbarth Criterion implies that this new a_1 is $\leq b_1$. This can be seen as follows.

In case $b_{\alpha+1} - a_{\alpha+1}$ is even, we do not have the extra +1, hence, $a_1 \leq b_1$ is nothing more than the Hässelbarth Criterion for $k = \alpha$. In case $b_{\alpha+1} - a_{\alpha+1}$ is odd, we have to show that $1 + \Sigma_{i=1}^{\alpha} a_i - \Sigma_{i=2}^{\alpha} b_i \leq b_1$. Suppose, to the contrary, that $1 + \Sigma_{i=1}^{\alpha} a_i - \Sigma_{i=2}^{\alpha} b_i > b_1$. As, on the other hand, $\Sigma_{i=1}^{\alpha} a_i - \Sigma_{i=2}^{\alpha} b_i \leq b_1$, it follows that $b_1 = \Sigma_{i=1}^{\alpha} a_i - \Sigma_{i=2}^{\alpha} b_i$. But this implies that $1 + [\Sigma_{i=1}^{\alpha} a_i - \Sigma_{i=2}^{\alpha} b_i] + b_1$ is not even, which contradicts the facts that the total number of 1's is even and that $a_2' = b_2', \ldots, a_f' = b_f'$. Hence, $a_1 \leq b_1$.

If there does not exist such a smallest α, then $b_2 \geq a_2$ (with a_2 after applying $f - 2$ times of the algorithm) and hence, $a_1 \leq b_1$ (with a_1 after applying $f - 1$ steps of the algorithm) according to the Hässelbarth Criterion.

Needless to say that $b_1 - a_1$ (with a_1 after applying $f - 1$ steps of the algorithm) is even, so that a_1' is well defined.

For $a_i' > 0$, A' has $a_i' + i + 1$ 1's in the first $a_i' + i - 1$ positions of row i $(i \in \langle f \rangle)$, $b_j' + j$ 1's in the first $b_j' + j$ positions in column j $(j \in \langle f \rangle)$, and zeros elsewhere.

The (0,1)-matrix A'' results now as follows: A number 1 on the main diagonal in row i is replaced by the number 0 and the 0 following the 1 in position $a_i' + i - 1$ in row i is replaced by the number 1; $i \in \langle f \rangle$.

The so-formed matrix $A'' = \{a_{ij}''\}$ is symmetric and has zeros on its main diagonal. Let the row sum vector of A'' be $(d_1', \ldots, d_n')$.

The matrix A^* is now formed in the following way:

If $(d_1', \ldots, d_n') \neq (d_1, \ldots, d_n)$, then consider the sequence $d_1' - d_1, \ldots, d_n' - d_n$.

As loops (1) and (2) of the algorithm "lift" 1's to rows with lower indices, there is a greatest index s and a smallest index t, such that

$$d_s' \geq d_s + 1, \quad d_t' \leq d_t - 1, \quad \text{and} \quad s < t.$$

As $d_s' \geq d_s + 1 \geq d_t + 1 \geq d_t' + 2$, there is an index k with $s \neq k \neq t$ and

$$a_{sk}'' = a_{ks}'' = 1, \qquad a_{kt}'' = a_{tk}'' = 0.$$

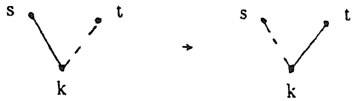
The matrix A'' is now changed into the matrix $\{a_{ij}^*\}$ satisfying $a_{ij}^* = a_{ij}''$ for each i and j, except that

$$a_{sk}^* = a_{ks}^* = 0 \quad \text{and} \quad a_{kt}^* = a_{tk}^* = 1;$$

i.e.,

$$\begin{array}{c|ccc} & s & k & t \\ \hline s & & 1 & \\ k & 1 & & 0 \\ t & & 0 & \end{array} \longrightarrow \begin{array}{c|ccc} & s & k & t \\ \hline s & & 0 & \\ k & 0 & & 1 \\ t & & 1 & \end{array}$$

or for the corresponding graph,



This procedure is repeated until the row sums of the resulting matrix A^* are $d_1, \ldots, d_n$. Note that the number of steps in this procedure is equal to the number of 1's "lifted" by the algorithm that transforms A into A'. The graph corresponding to A^* is a realization of the sequence $(d_1, \ldots, d_n)$. This completes the proof.

References

[1] C. Berge, *Graphs and Hypergraphs.* North Holland, Amsterdam, and American Elsevier, New York (1973).

[2] B. Bollobás, *Extremal Graph Theory.* Academic Press, New York (1978).

[3] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications.* American Elsevier, New York (1976).

[4] S.A. Choudum, A simple proof of the Erdös-Gallai theorem on graph sequences. *Bull. Austr. Math. Soc.* **33** (1986) 67–70.

[5] P. Erdös and T. Gallai, Graphs with given degree of vertices. *Mat. Lapok* **11** (1960) 264–274.

[6] D.R. Fulkerson, A.J. Hofman, and M.H. McAndrew, Some properties of graphs with multiple edges. *Can. J. Math.* **17** (1965) 166–177.

[7] B. Grünbaum, Graphs and complexes. Report of the University of Washington, Seattle, Math., 572B, 1969 (private communication).

[8] S.L. Hakimi, On realizability of a set of integers as degrees of the vertices of a linear graph I. *J. Soc. Indust. Appl. Math.* **10** (1962) 496–506.

[9] S.L. Hakimi, On realizability of a set of integers as degree of the vertices of a linear graph II. Uniqueness. *J. Soc. Indust. Appl. Math.* **11** (1963) 135–147.

[10] F. Harary, *Graph Theory.* Addison–Wesley, Reading, MA (1969).

[11] W. Hässelbarth, Die Verzweigtheit von Graphen. *Match.* **16** (1984) 3–17.

[12] H.J. Ryser, Combinatorial properties of matrices of zeros and ones. *Can. J. Math.* **9** (1957) 371–377.