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# Camera Placement in Integer Lattices

## (Extended Abstract)

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**Abstract**

We consider new techniques for studying an art gallery problem (the camera placement problem) in the infinite lattice $L^d$ of $d$ tuples of integers. A lattice point $A$ is visible from a camera $C$ (positioned at a vertex of $L^d$) if $A \neq C$ and if the line segment joining $A$ and $C$ crosses no other lattice vertex. By using a combination of probabilistic, combinatorial optimization and algorithmic techniques we can determine in polynomial time, for any given number $s \leq 5^d$ of cameras, the position they must occupy in the lattice $L^d$ in order to maximize their visibility. This improves previous results for $s \leq 3^d$ cameras.

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**CR Categories:** F.2.2, I.3.5
**Key Words and Phrases:** Art gallery problems, Camera placement problem, Density, Exchange method, Fast fourier transform, Integer lattice, Prime number, Riemann's zeta function, Variance, Visibility.

# 1 Introduction

The camera placement problem for integer lattices concerns the optimal placement of a fixed number of cameras on the $d$-dimensional integer lattice of $d$-tuples of integers in such a way that the visibility of these cameras is optimized. The problem was first proposed by the authors in [KP90a]. It is related to a still open art gallery problem first proposed by Moser in [Mos85] in 1966: "given a set $E$ of points in the plane how many guards located at points of $E$ are needed to see the unguarded points of $E$?" Little seems to be known on this problem except for the special case where the points of $E$ are located on the vertices of the finite integer lattice $L_n^2$. This case has been studied by Abbott [Abb74] who proved that if $f(n)$ denotes the number of guards which are necessary to see all the vertices of this integer lattice then

$$\frac{\ln n}{2 \ln \ln n} < f(n) < 4 \ln n.$$

By contrast the camera placement problem concerns the optimal placement of a fixed number of cameras regardless of whether or not these cameras can see all the vertices of the lattice. More formally the problem can be defined as follows. For a camera configuration $S$ let $d(S)$ (respectively, $u(S)$) denote the density of the set of lattice points visible by all

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(respectively, some) camera(s) in $S$. A simple "sieve" argument shows that the functions $d$ and $u$ are related by the following alternating-sum formula

$$u(S) = \sum_{E \subseteq S, E \neq \emptyset} (-1)^{|E|-1} \cdot d(E). \tag{1}$$

The problem of determining an optimal $s$-camera configuration, for fixed $s$, is the problem of optimizing the function $u(S)$ under the assumption $|S| = s$. This in turn requires the study of the function $d(S)$ (for which we can use a theorem of Rumsey [Rum66]), as well as the function $u(S)$ as given by the above alternating sum. It is clear that the main difficulty with the above optimization problem lies in the unwieldiness of the alternating sum (1).

For other interesting visibility problems on integer lattices the reader should consult [EGH89][problem 13.3], [Ham77][pages 39-41].

## 1.1 Assumptions of the Problem

To facilitate our study and in order to concentrate on the main issues at hand we make the following "idealized" assumptions: (1) the cameras are points which have "full vision" in the space concerned; for example, in two dimensions the angle of vision of each camera is 360 degrees (similarly in three dimensions), (2) the cameras can see the obstacles regardless of the distance from them, (3) the obstacles are in fact points which may only be located on the vertices of the integer lattice.

The first assumption is not very restrictive. If the cameras have limited vision then adjoining a sufficient number of cameras we can form a new full vision camera. However, if the visibility of the camera depends on the distance then although the nature of the problem will not change due to the symmetry of the integer lattice, the optimization problem may turn out to be significantly more complicated. Regarding the last restriction, the density aspects of our problem would change were we to assume that the obstacles are discs of radius $\geq r$, say. According to a theorem of Pólya [Pĺ8] no disc at a distance $\sqrt{\frac{1}{r^2} - 1}$ from the origin can ever be visible from it. (This comes from Pólya's solution of the so-called "Pólya's orchard problem", i.e. "How thick must the trunks of the trees in a regularly spaced circular orchard grow if they are to block completely the view from the center?" [PS76][Chapter 5, Problem 239], [All86].) See also [EGH89][problem 13.2] for generalizations of Pólya's orchard problem.

## 1.2 Results of the Paper

As it has already been argued the difficulty in our optimization problem lies mainly in the "complexity" of optimizing the alternating sum formula (1). In the sequel we develop techniques in order to overcome this problem. In a nutshell our main results are as follows.

By viewing the density (defined on the set of subsets of the lattice $L^d$) as a probability distribution we prove a reduction theorem for optimal configurations. This is kind of an "inductive formula" enabling us to reduce visibility with respect to any set $Q$ of primes to visibility with respect to a subset $Q'$ of $Q$, provided that $Q \setminus Q'$ is finite. This makes possible the derivation of the optimization theorem which not only gives a finer analysis of the density function but can also be used as a guide in our subsequent optimization analysis. With the aid of this theorem we are able to strongly conjecture that optimal configurations must be of minimal **variance**, where by variance we understand the sum $\sum |S \setminus (c_1^{j_1} \cup \cdots \cup c_k^{j_k})|^2$ where the $c_i^j$ range over the elements of $L/p_i$ and the $p_i$ range over the elements of the finite set of prime $p$ such that $|S/p| \neq |S|$ ($L/m$ is the set of equivalence classes under the equivalence relation $A \sim B$ if and only if $m|\gcd(A-B)$).

A careful analysis of the operator which associates to each configuration $S$ the family of numbers $(|S \setminus (c_1^{j_1} \cup \cdots \cup c_k^{j_k})|)_{(j_1,\cdots,j_k)}$ with the $c_i^{j_i}$ as above, allows us to show that a configuration is of minimal variance if and only if the cameras are "clustered" in equivalence classes of approximately equal size, that is $||S \cap c| - |S \cap c'|| \leq 1$ where $c$ and $c'$ range over the elements of $L/m$ for all $m$ square free.

We make a thorough analysis of the optimal configurations for any number of $s \leq 5^d$ cameras. We show that for almost all $s \leq 5^d$ an optimal configuration must be of minimal variance and for the remaining $s$ we show that optimal configurations are "near" a configuration of minimal variance. We develop an algorithm for computing in time polynomial in $s$ an optimal configuration for almost all $s \leq 5^d$. For the remaining $s$ we compute in polynomial time a small number of candidates to optimality which can then be compared by numerical evaluation of the density.

# 2 Preliminaries

In the sequel we give several basic definitions and results and establish notation that will be essential in our subsequent study. For any set $X \subseteq L^d$ of lattice points we define the density $D(X)$ of $X$ as the common value (if it exists) of the limit superior and limit inferior of the quotient $|X \cap L_n^d|/n^d$, as $n \to \infty$. It is easy to check that the density function is a finitely additive measure on those subsets of $L^d$ which have density. For our purposes it will be convenient to think of $D$ as a probability distribution on the subsets of $L^d$ which have density (theorems 3.1, 3.3).

Let $\mathcal{P} = \{2, 3, 5, \ldots\}$ be the set of prime numbers, $p$ ranges over $\mathcal{P}$ and $\mathcal{Q}$ over subsets of $\mathcal{P}$. Two points $A$ and $B$ are $p$-visible if $p$ is not a divisor of $\gcd(A-B)$. Two points $A$ and $B$ are $\mathcal{Q}$-visible if for all $p \in \mathcal{Q}$, $p$ is not a divisor of $\gcd(A-B)$. In particular two points $A, B$ which are $\mathcal{P}$-visible are visible in the geometric sense, i.e. the line segment joining $A$ and $B$ avoids all the lattice points but $A, B$. For $S$ a set of lattice points we use the following notations: $V_{\mathcal{Q}}(S)$ the set of points which are $\mathcal{Q}$-visible from each point of $S$, and $U_{\mathcal{Q}}(S)$ the set of points which are $\mathcal{Q}$-visible from some point of $S$. First we need a theorem of Rumsey [Rum66] which states that the above visibility sets $V_{\mathcal{Q}}(S)$ have a well defined density. In fact we have the following result.

**Theorem 2.1 ([Rum66])** *If $S$ is a finite set of points and $\mathcal{Q}$ is a set of primes then the set $V_{\mathcal{Q}}(S)$ has a density given by the formula*

$$D(V_{\mathcal{Q}}(S)) = \prod_{p \in \mathcal{Q}} \left(1 - \frac{|S/p|}{p^d}\right). \blacksquare$$

It is easy to see, using the principle of inclusion/exclusion, that also the visibility sets $U_{\mathcal{Q}}(S)$ have a density. Moreover if we define $d(\mathcal{Q}, S) := D(V_{\mathcal{Q}}(S))$ and $u(\mathcal{Q}, S) := D(U_{\mathcal{Q}}(S))$ then we have the following identity

$$u(\mathcal{Q}, S) = \sum_{E \subseteq S, E \neq \emptyset} (-1)^{|E|-1} d(\mathcal{Q}, E). \tag{2}$$

We call the above quantity the $\mathcal{Q}$-density of the configuration $S$ and denote it by $u(\mathcal{Q}, S)$. If $\mathcal{Q} = \mathcal{P}$ then in the above formulas we will usually omit mention of $\mathcal{P}$. A configuration $S$

consisting of $s$ points is called optimal if for any other $s$-point configuration $S'$ the density of $S$ exceeds the density of $S'$. We see easily that $u(\mathcal{Q}, S)$ depends only on the family of relations of $p$-visibility upon the set $S$ as $p$ runs over elements of $\mathcal{Q}$. In manipulating camera configurations we will make use of the following result concerning the realizability of a family of equivalence relations $(\sim_p)_{p\in\mathcal{Q}}$ on a set of cardinal $s$ by the relations of $p$-visibility on a set of $s$ lattice points $A_1, \cdots, A_s$.

**Theorem 2.2 (Realizability Theorem, [KP90a])** *Let $(\sim_p)_{p\in\mathcal{Q}}$ be a family of equivalence relations on the set $\{1, \cdots, s\}$ such that for $p$ sufficiently large $\sim_p$ is the identity relation and such that for any prime $p$ the cardinal of the quotient space $|\{1, \ldots, s\}/\sim_p|$ does not exceed $p^d$. Then there exist $s$ lattice points $A_1, \cdots, A_s$ in $L^d$ such that for every $p \in \mathcal{Q}$ and every $i, j \in \{1, \cdots, s\}$ $A_i$ and $A_j$ are $p$-visible if and only if $i \sim_p j$.*

Families $(\sim_p)_{p\in\mathcal{Q}}$ satisfying the hypothesis of the realizability theorem are called admissible systems (or families). The realizability theorem can be proved by induction on $s$ using the theorem of Rumsey. It is quite basic for our subsequent study. Not only does it allow us to "identify" the notions of camera configuration and admissible system but it is also essential in establishing the "legality" of the "algorithmic manipulations" we use (exchange method).

# 3 General Optimization Techniques

In the present section we combine the technique of admissible system developed in [KP90a] together with probabilistic techniques in order to obtain results characterizing optimal configurations. The key idea in overcoming the inherent complexity of optimizing $u(S)$ lies in the inductive formula for computing $u(S)$. We have the following theorem.

**Theorem 3.1 (Reduction Theorem)** *For any configuration $S$, any set $\mathcal{Q}$ of primes and any prime $p \in \mathcal{Q}$ the density $u(\mathcal{Q}, S)$ is given by the following formula*

$$u(\mathcal{Q}, S) = \sum_{c\in S/p} \frac{u(\mathcal{Q}\setminus p, S\setminus c)}{p^d} + \left(1 - \frac{|S/p|}{p^d}\right) \cdot u(\mathcal{Q}\setminus p, S)$$

Of particular importance for the optimization of $u(S)$ are the following two consequences of the reduction theorem. By repeated application of this theorem we can see that it admits the following generalization.

**Theorem 3.2 (Generalized Reduction Theorem)** *For any configuration $S$, any set $\mathcal{Q} \subseteq \mathcal{P}$, and any finite sequence of primes $p_0, \ldots, p_r \in \mathcal{Q}$ the density $u(\mathcal{Q}, S)$ is given by the following formula*

$$u(\mathcal{Q}, S) = \sum_{i=0}^{r} \sum_{c_i\in L/p_i} \frac{u(\mathcal{Q}\setminus \{p_0, \cdots, p_r\}, S\setminus (c_1\cup\cdots\cup c_k))}{p_0^d \cdots p_r^d}. \quad \blacksquare \tag{3}$$

The main interest in theorem 3.2 lies in a particular choice of the primes $p_0, \ldots, p_r$ which will facilitate the study of optimal configurations. To a given $S$ configuration we associate the (finite) set $\{p_0, p_1, \cdots, p_r\}$ of all primes $p$ such that $|S/p| \neq |S|$. The function $u(\mathcal{P}\setminus\{p_0, \cdots, p_r\}, E)$ will be of great importance in our subsequent optimality considerations and will be denoted by $u'(E)$, where $E \subseteq S$, i.e. $u'(E) := u(\mathcal{P} \setminus \{p_0, \cdots, p_r\}, E)$. The function $u'$ thus defined is also called the **reduced density** function of the configuration $S$.

**Theorem 3.3 (Optimization Theorem)** *Let $S$ be a configuration of $s$ cameras and let $u(E)$, and $u'(E)$ be the corresponding density and reduced density functions associated to $S$. Then for $E \subseteq S$ the function $u'(E)$ depends only on the size $|E|$ of the set $E$. Let $u'(e) = u'(E)$, where $e = |E|$. Then we can prove the following properties.*

1. *$u'(e)$ is strictly increasing as a function of $e$, where $1 \leq e \leq s$,*
2. *$u'(e)$ is concave, i.e. $u'(e+1) - u'(e)$ is strictly decreasing as a function of $e$,*
3. *the terms $|S \setminus (\bigcup_i c_i))|$ sum to a constant, $\sum_{c_i^{j_i} \in L/p_i} |S \setminus (\bigcup_i c_i^{j_i})| = s \prod_{p \in \{p_0, \cdots, p_r\}} (p^d - 1)$, i.e. the sum on the left-hand side is independent of the choice of the configuration and depends only on its size $s = |S|$ and of the set $\{p_0, \cdots, p_r\}$ of primes.* ▮

It is interesting to note that using theorems 3.1 and 3.3 we can obtain information on the relative placement of cameras in optimal configurations. In fact we can give simpler proofs of the following theorems from [KP90a].

**Theorem 3.4 (Finiteness Theorem)** *A necessary optimality condition for a configuration $S$ is that $\forall p \in \mathcal{P} \ \ |S/p| = \min\{|S|, p^d\}$.* ▮

**Theorem 3.5** *A configuration $S$ of size $\leq 3^d$ is optimal if and only if its variance is minimal.* ▮

## 3.1 An Optimization Strategy

The generalized reduction theorem in conjunction with the optimization theorem gives a fairly good indication on how to proceed in order to optimize the density $u(S)$. Let $\{p_1, \cdots, p_k\}$ be the set of primes $p$ such that $|S/p| \neq |S|$ and let $m$ be their product. Let $I = (j_1, \cdots, j_k)$ be a multi-index and let $b_I = |S \setminus \bigcup_i c_i^{j_i}|$ where $c_i^{j_i} \in S/p_i$ (the number of multi-indices is $m^d$). Then by the Generalized Reduction theorem we have

$$m^d u(S) = \sum_I u'(b_I). \tag{4}$$

The concavity of $u'$ and the fact that the terms $b_I$ sum to a constant suggest that for an optimal configuration the numbers $b_I$ must "differ from each other by a minimum amount". A classical measure of this deviation is the variance $\sum_I b_I^2$ of the numbers $b_I$. Because of the particular importance of this quantity we give the following definition

**Definition 3.1** *Let $S$ be a configuration and let $\{p_1, \cdots, p_k\}$ be the set of primes $p$ such that $|S/p| \neq |S|$. The sum $\sum_j |S \setminus \bigcup_i c_i^{j_i}|^2$ where $c_i^{j_i} \in S/p_i$ is called the* **variance** *of the configuration $S$.*

The previous considerations enable us to give the following conjecture.

**Conjecture 3.1** *An optimal configuration must be of minimal variance.*

## 4 Combinatorial Aspects

In the present section we give a characterization of the configurations for which the variance is minimal, in terms of the repartition of the cameras in the various classes of $L/n$ for $n$ square free integer. Our result is state in the following theorem

**Theorem 4.1** *The variance of the configuration $S$ is minimal if and only if for every square free integer $n$ and every $c$ and $c' \in L/n$ the cardinals of $c \cap S$ and $c' \cap S$ differ by at most one. In other words the cameras have to be clustered in equivalent classes of approximatively equal size.* ∎

It is now possible to introduce configurations which satisfy conjecture 3.1 using the above theorem. Suppose we have indexed the equivalent classes of $L/p$ with integers between 1 and $p^d$. So we can attach to each point $A$ of $L$ a sequence of integers which represent the various classes of $L/p$ at which $A$ belongs as the prime number $p$ increases: $p = 2, 3, 5, 7, \ldots$. Let $i$ be the operator of pointwise incrementation, i.e. $i(x_1, x_2, \ldots) = (x_1+1, x_2+1, \ldots)$. Let $\mathbf{1}$ be the sequence $(1, 1, 1, \ldots)$. For example we have $i(\mathbf{1}) = (2, 2, \ldots)$. Using the Realizability theorem the following sequences: $\mathbf{1}, i(\mathbf{1}), i^2(\mathbf{1}), \cdots, i^{s-1}(\mathbf{1})$, where the coordinates of each sequence are computed modulo $2^d, 3^d, \ldots$, lead to configurations of minimal variance.

## 5 Optimality for $s \leq 5^d$ Cameras

In this section we give optimality characterizations for the camera placement problem when the number of cameras is $s \leq 5^d$. It will be convenient to use the following notation. Let $L_1, \cdots, L_{2^d}$ be the $2^d$ classes of $L/2$ and $C_1, \cdots, C_{3^d}$ the $3^d$ classes of $L/3$. We use the abbreviations $l_i = |L_i \cap S|$, $c_j = |C_j \cap S|$, $a_{i,j} = |L_i \cap C_j \cap S|$. We recall that our conjecture is that optimal configurations must be of minimal variance. We begin by studying optimal configurations among the configurations of minimal variance.

### 5.1 Optimality among configurations of minimal variance

As we have previously shown configurations of minimal variance are characterized by $|l_i - l_j| \leq 1$, $|c_i - c_j| \leq 1$ and $|a_{i,j} - a_{k,l}| \leq 1$. From $\sum_i l_i = \sum_j c_j = \sum_{i,j} a_{i,j} = s$ we see that the above conditions are equivalent (possibly after renaming the indices) to

$$\begin{array}{lllllll} l_1 & = & \cdots & = & l_{i_0} & = & l+1 \\ l_{i_0+1} & = & \cdots & = & l_{2^d} & = & l \\ c_1 & = & \cdots & = & c_{j_0} & = & c+1 \\ c_{j_0+1} & = & \cdots & = & c_{3^d} & = & c \end{array} \qquad (5)$$

where $l = \lfloor \frac{s}{2^d} \rfloor$, $c = \lfloor \frac{s}{3^d} \rfloor$, $i_0 = s \bmod 2^d$ and $j_0 = s \bmod 3^d$, while $a_{i,j} = 1$ for exactly $s$ values of $(i, j)$.

Our first result is that the problem of finding an optimal configuration for $s \leq 5^d$ cameras is equivalent to solving a set partitioning problem [SM89, chap.13].

**Theorem 5.1** *Optimal configurations among configurations of minimal variance of size $\leq 5^d$ are characterized by the condition $\sum_{i>i_0, j>j_0} a_{i,j}$ is maximal. In particular if either $2^d | s$ or $3^d | s$ then any configuration of minimal variance has optimal visibility.* ∎

Surprisingly this optimization problem can be solved in time polynomial in $s$.

**Theorem 5.2** *The above optimization problem can be solved in time polynomial in s.* ▮

In the sequel we exhibit in the case $d = 2$ optimal configurations among configurations of minimal variance for $s$ cameras where $9 < s \leq 25$ and $s$ not divisible by 4 or 9. For $s \leq 9$ or multiple of 4 or 9 ($s = 12, 16, 18, 20$) the optimal configurations are exactly the configurations of minimal variance.

## 5.2 Proof of the conjecture for almost all $s \leq 5^d$.

By taking advantage of our main optimization theorem we give a complete description of the sizes of the traces on $S$ of the equivalent classes in $L/2$,$L/3$ and $L/6$.

**Theorem 5.3** *In an optimal configuration $S$ of size $s \leq 5^d$ the equivalence classes in $L/2$ and $L/3$ must satisfy $|l_i - l_j| \leq 1$ and $|c_i - c_j| \leq 1$. Furthermore for almost all values of $s$ the equivalence classes in $L/6$ must verify $|a_{i,j} - a_{k,l}| \leq 1$. In others words the variance of the configuration must be minimal for almost all values of $s$.*

**Proof (Outline).** We break up the proof of the theorem into several lemmas.

Let us say that two equivalence classes $L_i$ and $L_j$ in $L/3$, are **equipartitioned** (for $S$) if $|l_i - l_j| \leq 1$ and for every $k$, $|a_{i,k} - a_{j,k}| \leq 1$ (A similar definition holds for equivalence classes in $L/2$). Using the concavity of the reduced density we can prove the following two lemmas

**Lemma 5.1** *Two equivalence classes in $L/3$ (or $L/2$) which are of the same parity must be equipartitioned.* ▮

**Lemma 5.2** *Let $L_i$ and $L_j$ two equivalence classes of $L/3$ such that $l_i - l_j = \Delta l > 0$ is odd. Then for every $k$ we have $b_{j,k} - b_{i,k} \geq 0$ ($\Leftrightarrow \Delta l \geq a_{i,k} - a_{j,k}$). Furthermore the number of $k$ such that $b_{j,k} - b_{i,k} = 0$ is at least one more than the number of $k$ such that $b_{j,k} - b_{i,k}$ is an even non-null number (with equality only if $L_i$ and $L_j$ are equipartitioned). A similar result holds for equivalence classes in $L/3$.* ▮

As an application of lemma 5.1 we obtain a partition of $L/3$ and of $L/2$ in two parts of equipartitioned classes. Without loss of generality, there exist an $0 \leq i_0 < 2^d$ and a $0 \leq j_0 < 3^d$ such that

$$\begin{array}{lllllll} l_1 & = & \cdots & = & l_{i_0} & = & l + \Delta l \\ l_{i_0+1} & = & \cdots & = & l_{2^d} & = & l \\ c_1 & = & \cdots & = & c_{j_0} & = & c + \Delta c \\ c_{j_0+1} & = & \cdots & = & c_{3^d} & = & c \end{array}$$

where $\Delta l$ and $\Delta c$ are odd. Furthermore, if $l_i = l_j$ then for every $k$, $|a_{i,k} - a_{j,k}| \leq 1$, and if $c_i = c_j$ then for every $k$, $|a_{k,i} - a_{k,j}| \leq 1$. In fact we have a stronger result. Let $A$, $B$, $C$ and $D$ be the four blocks

$$A = \{(i,j) : i \leq i_0, j \leq j_0\} \quad B = \{(i,j) : i \leq i_0, j_0 < j\}$$

$$C = \{(i,j) : i_0 < i, j \leq j_0\} \quad D = \{(i,j) : i_0 < i, j_0 < j\}.$$

**Lemma 5.3** *In each block A,B,C and D we have $|a_{i,j} - a_{i',j'}| \leq 1$.* ▮

Let $\alpha$ and $\alpha + 1$ (resp. $\beta, \gamma, \delta$) be the two possible values of $a_{i,j}$ in the block $A$ (resp. $B, C, D$). Now using Lemmas 5.1, 5.2 and 5.3 we can show that

| (1) | (2) | (3) | (4) |
|---|---|---|---|
| $\Delta l = \alpha - \beta + 1$<br>$\Delta c = \alpha - \gamma + 1$ | $\Delta l = \alpha - \beta + 1$<br>$\Delta c = \beta - \delta + 1$or<br>$= \beta - \delta$or<br>$= \beta - \delta - 1$ | $\Delta l = \gamma - \delta + 1$<br>$\Delta c = \alpha - \gamma + 1$or<br>$= \alpha - \gamma$or<br>$= \alpha - \gamma - 1$ | $\Delta l = \gamma - \delta + 1$<br>$\Delta c = \beta - \delta + 1$ |

Figure 1: The four relations between $\alpha$,$\beta$,$\gamma$,$\delta$ and $\Delta l$,$\Delta c$.

**Lemma 5.4** *The numbers $\alpha, \beta, \gamma, \delta$ and $\Delta l$, $\Delta c$ must be related by one of the pair of relations depicted in figure 1.*

Using the above constraints on the repartition of the cameras combined with the observation that the number of cameras $s \leq 5^d$ is small when compared to the number $2^d \times 3^d$ of equivalence classes in $L/6$ we can prove that

**Lemma 5.5** $\Delta l = \Delta c = 1$ *and* $\alpha = \beta = \gamma = 0$. ∎

which completes the proof of the first part of our theorem. It remains to show the second part, that is $\delta = 0$ for almost all $s \leq 5^d$. Let $a$ be an integer $\geq 1$. We observe that in order to get $\delta \geq a$ we need at least $a \cdot (3^d - j_0)$ (resp. $a \cdot (2^d - i_0)$) cameras in the classes of $L/2$ (resp. $L/3$) of size $l$ (resp. $c$). That is

$$a \cdot (3^d - j_0) < l \quad \text{and} \quad a \cdot (2^d - i_0) < c. \tag{6}$$

But since in an optimal configuration $\Delta l = \Delta c = 1$, the numbers $s, c, l, i_0, j_0$ are given by

$$c = \lfloor \frac{s}{3^d} \rfloor \qquad j_0 = s \bmod 3^d \qquad l = \lfloor \frac{s}{2^d} \rfloor \quad \text{and} \quad i_0 = s \bmod 2^d.$$

Let us call problematic (of order $a$) the numbers $s \leq 5^d$ for which inequalities (6) are verified for $a$ but not for $a+1$. For example for $d = 2$ only $s = 23$ is problematic (of order 1), while for $d = 3$ the problematic numbers are $79, 102, 125$ (of order 1) and $103$ (of order 2). The following lemma shows that the number of problematic numbers is small compared to $5^d$.

**Lemma 5.6** *The ratio of problematic numbers is less than* $(\frac{5}{6})^d$. ∎

This last lemma ends the proof of our main theorem 5.3. ∎

### 5.3 The candidates to optimality for the remaining $s$

In this subsection we shall show that optimal configurations are near configurations of minimal variance in the sense that we can transform one to the other by a sequence of simple transformations. We suppose that $s$ is a problematic number of order $a$. Let us call $C(x)$ the set of configurations which verify the constraints proved in the previous subsection on optimal configurations (in the sequel we use the notations introduced in the previous subsection) that is $\Delta l = \Delta c = 1, \alpha = \beta = \gamma = 0$ and $\gamma = x$. Clearly configurations of minimal variance are in $C(0)$ and optimal configurations are in $C(x)$ for some $x \leq a$. Let $C = \cup_x C(x)$. We examine the following transformations:

- we pick a camera from $L_{i'} \cap C_j$ and put it in $L_i \cap C_j$ and similarly from $L_i \cap C_{j'}$ to $L_{i'} \cap C_{j'}$, which transform an element of $C$ into an element of $C$. There are two kinds of such transformations. Firstly $i, i' \leq i_0$ (or $i, i' > i_0$ or $j, j' \leq j_0$ or $j, j' > j_0$); in that case we can easily show that the visibility is unchanged. Secondly $i \leq i_0 < i'$ and $j \leq j_0 < j'$; in that case the only transformations are

$$\psi(\delta) : \begin{pmatrix} 0 & 1 \\ 1 & \delta \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & \delta+1 \end{pmatrix}$$

and its inverse. Our first result is that we can transform an optimal configuration in a configuration of minimal variance by a finite sequence of transformations of the second kind.

**Lemma 5.7** *Let $S_0$ be a configuration in $C(x+1)$. Then there exists a sequence of transformations $\psi(a_1), \cdots, \psi(a_t)$ such that for all $j$, $S_j := \psi^{-1}(a_j) \circ \cdots \circ \psi^{-1}(a_1)(S_0)$ is in $C(x+1)$ and such that $S_t$ is in $C(x)$.*

The above result shows clearly that if we could determine whether or not $\psi(x)$ is a visibility gain then we would also be able to characterize completely the optimal configurations. Unfortunately we are not able to prove that $\psi(x)$ is a visibility gain. Nevertheless we are now ready to generate the candidates to optimality. The idea is to compute for each value of $\delta \in \{1, \cdots, a\}$ the candidate(s) to optimality. The following theorem allows us to do that.

**Theorem 5.4** *Suppose that the optimal configuration is obtained for some value $x$ of $\delta$. Then the optimal configuration is obtained for $\sum_{i>i_0, j>j_0} a_{i,j}$ maximal among the configuration of $C(x)$. Such a configuration will be called a* **candidate** *to optimality.* ∎

As in the case of configuration of minimal variance the above instances of the set partitioning problem can be solved in time polynomial in $s$.

**Theorem 5.5** *The candidates to optimality can be computed in time polynomial in $s$.* ∎

In the sequel we exhibit the two candidates to optimality for $s = 23$ cameras in the plane ($d = 2$).

## 5.4 From almost to all ...

We return now to the problem of knowing if $\psi$ is a visibility gain or not. We have the following result

**Proposition 5.1** *Let $s$ be problematic of order at least 1 and let $b = s - l - c - 2$. The following assertions are equivalent*

1. *optimal configurations of size $s$ are of minimal variance*
2. *$\psi^{-1}(1)$ is a visibility gain*
3. $0 < \sum_{k \geq 1} (-1)^{k+1} \cdot \frac{k \cdot (k-1) \cdot P(k)}{(b+1)(b+2)(b+3)(b+4)} \cdot \binom{b+4}{k} \cdot d'(k)$
   *where* $P(k) = k^2 - (b+6) \cdot k + 6 - b - b^2$. ∎

### 5.5 ...by a numerical approach.

To decide between the candidates to optimality we show here how we can compute efficiently the numerical value of the density. The difficulty of a numerical evaluation is that the product

$$d'(k) = \prod_{p \neq 2,3} (1 - \frac{k}{p^d})$$

converges very slowly (a power of $1/N$ if we take $N$ terms). In order to speed up the convergence we use a technique developed by Vardi and Flajolet [VF90]. The idea is to use the formal expansion $1 - kx = \prod_{n=1}^{\infty}(1 - x^n)^{-a_n}$ where $-n \cdot a_n = \sum_{d \div n} \mu(\frac{n}{d}) \cdot k^d$ (see for example [Ber68, Chap.3]), to write

$$d'(k) = \prod_{p \neq 2,3} \prod_{n=1}^{\infty} (1 - \frac{1}{p^{d \cdot n}})^{-a_n} = \prod_{n=1}^{\infty} (\zeta(n) \cdot (1 - \frac{1}{2^d}) \cdot (1 - \frac{1}{3^d}))^{a_n}.$$

The nice thing about this formulae is that the convergence is in $O((\frac{k}{5^d})^N)$ where $N$ is the number of terms taken in the product.

This technique might allow us to decide that whether for $s$ cameras with $s$ problematic of order at least 1, optimal configurations are configurations of minimal variance.

## 6 Conclusion

In the present paper we devoted most of our effort in developing new "probabilistic" techniques based on the density measure of sets of lattice points. This enabled us to provide characterizations and computations of optimal camera configurations of size up to $5^d$ cameras, thus significantly extending the results of [KP90a] (which were for only up to $3^d$ cameras), and gave optimization techniques which are more easily amenable to the study of visibility problems in other lattice patterns (like, for example, brick and hexagonal tilings [KP90b], [GL87]).

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## 7 The optimal configuration for $d = 2$.

We exhibit optimal $s$-camera configurations in the plane when $9 < s \leq 25$ and $s$ is not divisible by either 4 or 9. For $s \leq 9$ or $4|s$ or $9|s$ (i.e. $s = 12, 16, 18, 20$) the optimal configurations are exactly the configurations of minimal variance. The camera configurations are represented as matrices with $3^d$ rows and $2^d$ columns, where the rows represent the sets $\{c \cap S : c \in L/3\}$, columns the sets $\{c \cap S : c \in L/2\}$, while boxes the sets $\{c \cap S : c \in L/6\}$. The various entries of the matrix give a complete description of the repartition of the cameras in the above mentioned classes. Using the realizability theorem 2.2 we can construct the corresponding optimal configuration in the plane lattice $L^2$.