Chapter 7

Mathematical Morphology with Noncommutative Symmetry Groups

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Mathematical morphology as originally developed by Matheron and Serra is a theory of set mappings, modeling binary image transformations, that are invariant under the group of Euclidean translations. Because this framework turns out to be too restricted for many practical applications, various generalizations have been proposed. First, the translation group may be replaced by an arbitrary commutative group. Second, one may consider more general object spaces, such as the set of all convex subsets of the plane or the set of gray-level functions on the plane, requiring a formulation in terms of complete lattices. So far, symmetry properties have been incorporated by assuming that the allowed image transformations are invariant under a certain commutative group of automorphisms on the lattice. In this chapter we embark on another generalization of mathematical morphology by dropping the assumption that the invariance group is commutative. To this end we consider an arbitrary homogeneous space (the plane with the Euclidean translation group is one example, the sphere with the rotation group another), that is, a set $\mathcal{X}$ on which a transitive but not necessarily commutative transformation group Γ is defined. As our object space we then take the Boolean algebra $\mathcal{P}(\mathcal{X})$ of all subsets of this homogeneous space. First we consider the case in which the transformation group is simply transitive or, equivalently, the basic set $\mathcal{X}$ is itself a group, so that we may study the Boolean algebra $\mathcal{P}(\Gamma)$. The general transitive case is subsequently treated by embedding the object space $\mathcal{P}(\mathcal{X})$ into $\mathcal{P}(\Gamma)$, using the results for the simply transitive case and translating the results back to $\mathcal{P}(\mathcal{X})$. Generalizations of dilations, erosions, openings, and closings are defined and several representation theorems are proved. For clarity of exposition as well as to emphasize the connection with classical Euclidean morphology, we have restricted ourselves to the case of Boolean lattices, which is appropriate for binary image transformations.

I. INTRODUCTION

Mathematical morphology was originally developed at the Paris School of Mines as a set-theoretical approach to image analysis [17,29]. It has a strong algebraic component, studying image transformations with a simple geometric interpretation and their decomposition and synthesis in terms of set operations. Other aspects are the probabilistic one, modeling (images of) samples of materials by random sets, and the integral geometric one, which is concerned with image functionals. Although the main object of our present study is the algebraic approach, we emphasize that our primary motivation comes from the geometric side, in the sense that various image transformations used in mathematical morphology today (dilations, erosions, openings, closings) have a straightforward geometric analogue in a more general context. It is then a natural question to ask whether a corresponding algebraic description can be found. From a practical point of view the importance of such an algebraic decomposition theory no doubt derives from the fact that it enables fast and efficient implementations on digital computers and special image analysis hardware. Because we will not deal with such questions here, we refer the reader to [7] for an elementary introduction to Euclidean morphology with emphasis on implementation.

In the original approach of Matheron and Serra [17,29], a two-dimensional image of, let us say, a planar section of a porous material is modeled as a subset X of the plane. In order to reveal the structure of the material, the image is probed by translating small subsets B, called *structuring elements*, of various forms and sizes over the image plane and recording the locations h where certain relations (e.g., "B_h included in X" or "B_h hits X") between the image X and the translate B_h of the structuring element B over the vector h are satisfied (see Figure 1a). In this way one can construct a large class of image transformations that are compatible with translations of the image plane or, to put it differently, are invariant under the Euclidean translation group. The underlying idea here is that the form or shape of objects in the image does not depend on the relative location with respect to an arbitrary origin and that therefore the transformations performed on the image should respect this. Notice that the basic object of study, the "object space," is not the reference space (the plane in our example) itself but the collection of subsets of this reference space and the transformations defined on this collection of subsets.

In practice, one encounters various situations where this framework is too restrictive. One of the earliest examples is mentioned in Serra's book [29, p. 17],

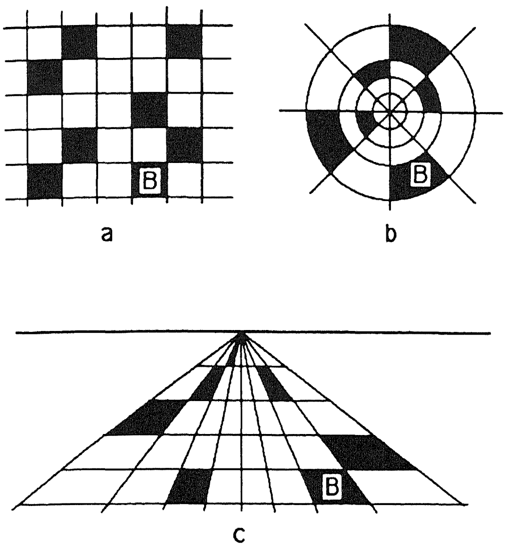

Figure 1. Copies (dark) of a structuring element B under (a) Euclidean translation, (b) rotations and scalar multiplication, and (c) perspective transformation.

where a photograph is shown of the trees in a forest, taken by putting the camera at ground level and aiming toward the sky. Such photographs are used to measure the amount of sunshine in the woods. The resulting image shows clear radial symmetry with intrinsic origin (the projection point of the zenith). It is clear that in this case we need image transformations that are adapted to the symmetries of this polar structure. It turns out that in fact one obtains a straightforward generalization of Euclidean morphology by replacing the Euclidean translations by an arbitrary *abelian* (*commutative*) group [11,25]. In the case of the example mentioned above, this would be the group generated by rotations and multiplications with respect to the origin. Here the size of the structuring element increases with increasing distance from the origin (Figure 1b). Another example occurs in the analysis of traffic scenes, where the goal is to recognize the shape of automobiles with a camera on a bridge overlooking a highway [3]. In this case the size of the structuring element has to be adapted according to the law of perspective (Figure 1c). It is not difficult to show that in this case there is again invariance under a commutative group. Notice that in the two examples just mentioned we have a variable structuring element as a function of position. This has been taken as the starting point by Serra and others to introduce arbitrary assignments of subsets to each point of the plane and define dilations and erosions accordingly, completely giving up invariance under a symmetry group (see also Section IV.A). However, in the examples just given the situation is different in the sense that there is a definite group connecting structuring elements at different locations, although their sizes differ. Actually, a metrical concept like "size" does not enter at all into the definition of the classical morphological operations. Only the group property of the Euclidean vector addition is involved, which explains why an extension to arbitrary groups is possible. In fact, we will argue that without a concept of invariance (under a group, or otherwise), one cannot even give a meaningful answer to the question when sets at different locations are "of the same shape" or not.

Instead of changing the symmetry group of the object space, one may generalize the object space itself. For example, instead of all subsets of the plane one may want to study a smaller collection, such as the open or closed sets or the convex sets. In that case the original approach is no longer valid because the union of an arbitrary collection of closed or convex sets is not necessarily closed or convex, the intersection of an arbitrary collection of open sets is not necessary open, etc. These difficulties can be overcome by taking as the object space a so-called *complete lattice*, that is, an ordered set $\mathcal{L}$ such that any subset of $\mathcal{L}$ has a supremum (smallest upper bound) and infimum (greatest lower bound), generalizing the set operations of union and intersection. This is the approach initiated by Serra and Matheron [30,31], as well as Heijmans [11]. A general study of this topic has been made by Heijmans and Ronse [12,27]. If one does not assume any invariance property one can only prove generalities. But again invariance under a group of automorphisms of the lattice may be introduced, as in [11,12,27], where so far the assumption made is always that the group is *commutative*. This enables a complete characterization of dilations, erosions, openings, closings, increasing transformations, etc. Another situation in which a lattice formulation is in order arises when one wants to go from binary images with their Boolean image algebra to gray-level images, that is, *functions*, defined on the basic reference space. Following Sternberg [32], one has introduced the so-called *umbras* to deal with this case [29,31,33]. After introducing an extra dimension for the function values, one performs the binary Euclidean operations in this enlarged space and translates the results back to the original space. However, for a mathematically satisfactory approach complete lattices are required; see Ronse [26].

In this chapter we want to generalize morphology by dropping the assumption that the invariance group is commutative. To this end we consider an arbitrary *homogeneous space*, a set $\mathcal{X}$ on which a transitive but not necessarily commutative group Γ of invertible transformations is defined. Here *transitive* means that for any pair of points in the set there is a transformation in the group that maps one point on the other. If this mapping is unique we say that the transformation group is *simply transitive* or *regular*. As the object space of interest from a morphological point of view, we take the Boolean algebra of all subsets of this homogeneous space.

We present two examples for basic motivation. First of all, one may extend Euclidean morphology in the plane by including rotations. This case has been extensively discussed in [24]. In many situations one does not want to distinguish between rotated versions of the same object. In that case it is appropriate to use the full Euclidean group of motions (the group generated by translations and rotations) as (noncommutative) invariance group. This is, for example, the basic assumption made in integral geometry to give a complete characterization (Hadwiger's theorem) of functionals of compact, convex sets in $\mathbf{R}^n$ [10]. As our second basic example we mention the sphere with its symmetry group of three-dimensional rotations, again a nonabelian group. Various motivations can be given here. First, the earth is spherical to a good approximation and this has to be taken into account when analyzing pictures taken by weather satellites. Second, pictures of virus particles show them to be nearly spherical with antibodies attached randomly to the surface, and a morphological description of the particle distribution on the surface is of interest. Third, from a theoretical point of view we observe that integral geometry and geometric probability on the sphere have been well investigated in the past [18,28]. Since there is a clear connection between these fields on the one hand and mathematical morphology on the other (see Serra [29, Chapters 4, 13]), it is of interest to develop morphology for the sphere as well. Here we can do no more than indicate how the sphere fits into our general framework, but clearly this case is important enough to warrant an in-depth study. A more detailed investigation of this case is presented in [23]. Another area of possible research is the question of how to take the projective geometry of the imaging process into account, because clearly the symmetry of a two-dimensional plane is not the same as the symmetry of the three-dimensional world of which it is a projection.

We first develop the theory for simply transitive transformation groups (all abelian transitive transformation groups fall in this category). It is easy to see that in this case there is a one-to-one correspondence between elements of $\mathscr{X}$ and those of Γ: let ω (the "origin") be an arbitrary point in $\mathscr{X}$ and associate to any $x \in \mathscr{X}$ the unique transformation in Γ that maps ω to x. Then a bijection between $\mathscr{X}$ and Γ is obtained. So in the simply transitive case we can assume without loss of generality that $\mathscr{X}$ coincides with the group Γ. This will be taken as the starting point in Section III, where we study the Boolean algebra $\mathscr{P}(\Gamma)$ of subsets of an arbitrary group Γ. Of course, this is precisely the situation in Euclidean morphology, where the group is that of the Euclidean translations, the only difference being that in the present case the group may be noncommutative.

Subsequently we will consider the general transitive case, enabling us to analyze the examples mentioned above (the Euclidean plane with the translation-rotation group, the sphere with the rotation group) as particular cases. It turns out that the general case can be handled by embedding the object space of interest (the set $\mathscr{P}(\mathscr{X})$ of subsets of $\mathscr{X}$) into another one (the set $\mathscr{P}(\Gamma)$ of subsets of Γ), which has a simply transitive transformation group. So the results for the latter case, although rather technical, have to be developed first in depth. The theory is illustrated by various examples. The possibility of an extension to non-Boolean lattices will be considered in future work.

Now some remarks about the organization of this chapter. In Section II we first review Euclidean morphology together with some lattice-theoretical concepts; then the concept of homogeneous spaces is introduced and we give some background material needed in the following. In Section III we generalize Euclidean morphology to the Boolean lattice of all subsets of an arbitrary group, ordered by set inclusion. In particular, we generalize the classical Minkowski set operations, as well as dilations, erosions, openings, and closings that are "translation invariant" in a generalized sense, that is, invariant under certain automorphisms induced by the transformation group. A complete characterization of these operations is given and we also prove a general representation theorem for translation-invariant mappings, generalizing earlier results of Matheron [17] and Banon and Barrera [1]. We point out the connection to the theory of residuated lattices and ordered semigroups [4,5]. Section IV then develops the general transitive case. Some interesting differences from the simply transitive case show up. We also introduce a concept of "shape" that explicitly depends on the symmetry group involved. Section V contains a discussion and we point out the possible relevance for some applications.

This chapter essentially contains the material of [21,22], apart from some minor additions and slight changes of notation. To enhance readability we have deferred all the proofs to an appendix. Definitions, theorems, etc. have been consecutively numbered in each section, e.g. Remark 2.1 is followed by Theorem 2.2, etc. The end of a remark is indicated by the symbol □.

II. PRELIMINARIES

In this section we first outline some elementary concepts and results from classical Euclidean morphology (Section II.A), followed by a few general lattice-theoretical concepts that are needed below (Section II.B). Then we introduce the concept of homogeneous spaces in Section II.C.

A. Euclidean Morphology

Let E be the Euclidean space $\mathbf{R}^n$ or the discrete grid $\mathbf{Z}^n$. By $\mathcal{P}(E)$ we denote the set of all subsets of E ordered by set-inclusion, henceforth called the object space. A binary image can be represented as a subset X of E. Now E is a commutative group under vector addition: we write $x + y$ for the sum of two vectors x and y, and $-x$ for the inverse of x. Then we can define the following elementary algebraic operations:

$$\text{Minkowski addition:} \qquad X \oplus A = \{x + a\colon x \in X,\ a \in A\} = \bigcup_{a \in A} X_a = \bigcup_{x \in X} A_x \tag{7.1}$$

$$\text{Minkowski subtraction:} \qquad X \ominus A = \bigcap_{a \in A} X_{-a} \tag{7.2}$$

where X_a is the translate of the set X along the vector a:

$$X_a = \{X + a\colon x \in X\} \tag{7.3}$$

Here we have followed the original definitions of Hadwiger [10], which is also the convention in [12,27,32,33]. Matheron [17] and Serra [29] use a slightly different definition for Minkowski subtraction; see Remark 2.1 below.

We collect some standard algebraic properties of Minkowski addition and subtraction [10]. Here E is the Euclidean space, o the origin of E, $\emptyset$ the empty set, X an arbitrary subset of E.

$$\begin{aligned}
X \oplus \{o\} &= X, \qquad X \ominus \{o\} = X \\
X \oplus \emptyset &= \emptyset, \qquad X \oplus E = E \\
X \ominus \emptyset &= E, \qquad \emptyset \ominus X = \emptyset, \qquad E \ominus X = E \\
X \oplus A &= A \oplus X \\
(X \oplus A) \oplus B &= X \oplus (A \oplus B) \\
(X \ominus A) \ominus B &= X \ominus (A \oplus B) \\
(X \cup Y) \oplus A &= (X \oplus A) \cup (Y \oplus A) \\
(X \cap Y) \ominus A &= (X \ominus A) \cap (Y \ominus A) \\
X \oplus (A \cup B) &= (X \oplus A) \cup (X \oplus B) \\
X \ominus (A \cup B) &= (X \ominus A) \cap (X \ominus B)
\end{aligned} \tag{7.4}$$

The transformations $\delta_A : X \mapsto X \oplus A$ and $\varepsilon_A : X \mapsto X \ominus A$ are called a *dilation* and an *erosion* by the structuring element A, respectively. There is a simple geometric interpretation of these operations:

$$\text{Dilation: } X \oplus A = \{h \in E\colon (\check{A})h \cap X \neq \emptyset\} \tag{7.5}$$

$$\text{Erosion: } X \ominus A = \{h \in E\colon A_h \subseteq X\} \tag{7.6}$$

where the *reflected* or *symmetric* set $\check{A}$ of A is defined by

$$\check{A} = \{-a\colon a \in A\} \tag{7.7}$$

There exists a *duality relation* with respect to set-complementation (X^c denotes the complement of the set X):

$$X \oplus A = (X^c \ominus \check{A})^c, \qquad \delta_A(X) = (\varepsilon_{\check{A}}(X^c))^c \tag{7.8}$$

that is, dilating an image by A gives the same result as eroding the background by $\check{A}$. To any mapping $\psi : \mathcal{P}(E) \to \mathcal{P}(E)$ we associate the *dual* mapping $\psi' : \mathcal{P}(E) \to \mathcal{P}(E)$ defined by $\psi'(X) = \{\psi(X^c)\}^c$. To avoid confusion with other forms of duality to be discussed below, we will refer to ψ' as the *Boolean dual* of ψ.

Remark 2.1. Matheron and Serra define the Minkowski subtraction of X by A as follows: $X \ominus A = \bigcap_{a \in A} X_a$. Then one has to write $X \ominus \check{A}$ in Eq. (7.6). The advantage of this definition is that the duality relation (7.8) does not involve a reflection of the structuring element. But it complicates the expression of adjunctions (see below), which is a notion persisting in lattices without complementation. □

Two characteristic properties of dilation are:

$$\text{Distributivity w.r.t. union:} \quad (\bigcup_{i \in I} X_i) \oplus A = \bigcup_{i \in I} (X_i \oplus A) \tag{7.9}$$

$$\text{Translation invariance:} \quad (X \oplus A)_h = X_h \oplus A \tag{7.10}$$

Similar properties hold for the erosion with intersection instead of union. A consequence of the distributivity property is that dilation and erosion are *increasing* mappings, that is, mappings such that for all $X, Y \in \mathcal{P}(E)$, $X \subseteq Y$ implies that $\psi(X) \subseteq \psi(Y)$.

Other important increasing transformations are the opening and closing by a structuring element A (the closing is defined slightly differently in [17,29]):

$$\text{Opening:} \quad X \bigcirc A := (X \ominus A) \oplus A = \bigcup_{h \in E} \{A_h : A_h \subseteq X\} \tag{7.11}$$

$$\text{Closing:} \quad X \bullet A := (X \oplus A) \ominus A = \bigcap_{h \in E} \{(\check{A}^c)_h : (\check{A}^c)_h \supseteq X\} \tag{7.12}$$

The opening is the union of all the translates of the structuring element that are included in the set X. Opening and closing are related by Boolean duality: $(X^c \bigcirc A)^c = X \bullet \check{A}$. A more general definition of dilations, erosions, openings, and closings will be given in the next subsection in the framework of complete lattices.

We end this review of Euclidean morphology by presenting a theorem by Matheron [17], which gives a characterization in the Euclidean case of translation-invariant increasing mappings.

Theorem 2.2. A mapping $\psi : \mathcal{P}(E) \to \mathcal{P}(E)$ is increasing and translation-invariant if and only if ψ can be decomposed as a union of erosions or, alternatively, as an intersection of dilations:

$$\psi(X) = \bigcup_{A \in \mathcal{V}(\psi)} X \ominus A = \bigcap_{A \in (\psi')} X \oplus \check{A}$$

where $\mathcal{V}(\psi) = \{A \in \mathcal{P}(E) : o \in \psi(A)\}$ is the kernel of ψ, and ψ' is the Boolean dual of ψ.

B. Lattice-Theoretical Concepts

The object spaces of interest in mathematical morphology are not restricted to Boolean algebras. For example, if one is interested in convex subsets of the plane or gray-level images, one has to introduce the notion of complete lattices. This approach has been initiated by Serra [30], Serra et al. [31], and Heijmans and Ronse [12,27]. Although the present generalization of mathematical morphology is confined to Boolean lattices, it is nevertheless advantageous to summarize a few lattice-theoretical concepts that will be needed below. The reader may want to skip this subsection at first reading and refer back to it later. For a full discussion, see [12,27]. A general introduction to lattice theory is given by Birkhoff [4].

A complete lattice $(\mathcal{L}, \leq)$ is a partially ordered set $\mathcal{L}$ with order relation $\leq$, a supremum or join operation written $\bigvee$, and an infimum or meet operation written $\bigwedge$, such that every (finite or infinite) subset of $\mathcal{L}$ has a supremum (smallest upper bound) and an infimum (greatest lower bound). In particular, there exist two universal bounds, the least element written $O_{\mathcal{L}}$ and the greatest element $I_{\mathcal{L}}$. In the case of the power lattice $\mathcal{P}(E)$ of all subsets of a set E, the order relation is set-inclusion $\subseteq$, the supremum is the union $\cup$ of sets, the infimum is the intersection $\cap$ of sets, the least element is the empty set $\emptyset$, and the greatest element is the set E itself. An *atom* is an element X of a lattice $\mathcal{L}$ such that for any $Y \in \mathcal{L}$, $O\mathcal{L} \leq Y \leq X$ implies that $Y = O\mathcal{L}$ or $Y = X$. A complete lattice $\mathcal{L}$ is called *atomic* if every element of $\mathcal{L}$ is the supremum of the atoms less than or equal to it. It is called *Boolean* if (1) it satisfies the distributivity laws $X \bigvee (Y \bigwedge Z) = (X \bigvee Y) \bigwedge (X \bigvee Z)$ and $X \bigwedge (Y \bigvee Z) = (X \bigwedge Y) \bigvee (X \bigwedge Z)$ for all $X, Y, Z \in \mathcal{L}$, and (2) every element X has a unique complement X^c, defined by $X \bigvee X^c = I_{\mathcal{L}}$, $X \bigwedge X^c = O_{\mathcal{L}}$. The power lattice $\mathcal{P}(E)$ is an atomic complete Boolean lattice, and conversely any atomic complete Boolean lattice has this form.

Since we are interested in image transformations, a main object of study is the set $\mathbb{O} := \mathcal{L}^{\mathcal{L}}$ of all maps (operators) on $\mathcal{L}$, that is, mappings $\psi: \mathcal{L} \to \mathcal{L}$. Operators are generally written in Greek letters, with γ, ϕ, δ, ε being reserved for openings, closings, dilations, and erosions. The identity operator $X \mapsto X$ is written $\mathrm{id}\mathcal{L}$. The composition of two operators ψ_1 and ψ_2 is defined by $\psi_1\psi_2(X) = \psi_1(\psi_2(X))$, $X \in \mathcal{L}$. Instead of $\psi\psi$ we write ψ^2.

The power lattice $\mathbb{O}$ inherits the complete lattice structure of $\mathcal{L}$. The ordering, supremum, and infimum in $\mathbb{O}$ are denoted by $\leq$, $\bigvee$, $\bigwedge$ as well, and for any subset $\mathfrak{Q} \subseteq \mathbb{O}$ they are defined by

$$\psi_1 \le \psi_2 \quad \Leftrightarrow \psi_1(X) \le \psi_2(X), \qquad \forall X \in \mathcal{L} \tag{7.13a}$$

$$(\bigvee \mathcal{Q})(X) = \bigvee_{\eta \in \mathcal{Q}} \eta(X), \qquad \forall X \in \mathcal{L} \tag{7.13b}$$

$$(\bigwedge \mathcal{Q})(X) = \bigwedge_{\eta \in \mathcal{Q}} \eta(X), \qquad \forall X \in \mathcal{L} \tag{7.13c}$$

In the case that $\mathcal{L}$ is itself a power lattice $\mathcal{P}(E)$ with ordering $\subseteq$, we will use the symbols $\subseteq$, $\cup$, and $\cap$ instead of $\le$, $\vee$, and $\wedge$ in (7.13).

Let $\mathcal{L}$ and $\tilde{\mathcal{L}}$ be complete lattices. A mapping $\psi : \mathcal{L} \to \tilde{\mathcal{L}}$ is called *increasing* (*isotone*, *order-preserving*) when $X \le Y \Rightarrow \psi(X) \le \psi(Y)$ for all X, $Y \in \mathcal{L}$, and *decreasing* (*antitone*, *order-reversing*) when $X \le Y \Rightarrow \psi(X) \ge \psi(Y)$ for all X, $Y \in \mathcal{L}$. An *automorphism* of $\mathcal{L}$ is a bijection $\psi : \mathcal{L} \to \mathcal{L}$ such that for any X, $Y \in \mathcal{L}$, $X \le Y$ if and only if $\psi(X) \le \psi(Y)$. When a group $\mathbf{T}$ is an automorphism group of both $\mathcal{L}$ and $\tilde{\mathcal{L}}$, a mapping $\psi : \mathcal{L} \to \tilde{\mathcal{L}}$ is called $\mathbf{T}$-*invariant* or a $\mathbf{T}$-*mapping* if it commutes with all $\tau \in \mathbf{T}$, that is, if $\psi(\tau(X)) = \tau(\psi(X))$ for all $X \in \mathcal{L}$, $\tau \in \mathbf{T}$. Accordingly, we will speak below of $\mathbf{T}$-dilations, $\mathbf{T}$-erosions, etc. If no invariance under a group is required, one may set $\mathbf{T} = \{\mathrm{id}_{\mathcal{L}}\}$. We will refer to the elements of $\mathbf{T}$ as *group translations* or $\mathbf{T}$-*translations*.

Next we give a general definition of dilations and erosions, which are examples of increasing mappings.

Definition 2.3. Let $\mathcal{L}$ and $\tilde{\mathcal{L}}$ be complete lattices. A *dilation* $\delta : \mathcal{L} \to \tilde{\mathcal{L}}$ is a mapping commuting with suprema. An *erosion* $\varepsilon : \mathcal{L} \to \tilde{\mathcal{L}}$ is a mapping commuting with infima. In other words, for any subset $\{X_i : i \in I\}$ of $\mathcal{L}$ it is true that

$$\delta(\bigvee_{i \in I} X_i) = \bigvee_{i \in I} \delta(X_i) \tag{7.14}$$

$$\varepsilon(\bigwedge_{i \in I} X_i) = \bigwedge_{i \in I} \varepsilon(X_i) \tag{7.15}$$

In particular, $\delta(O_{\mathcal{L}}) = O_{\tilde{\mathcal{L}}}$ and $\varepsilon(I_{\mathcal{L}}) = I_{\tilde{\mathcal{L}}}$.

The following definition, which applies only to mappings on a single lattice, generalizes the notion of Euclidean openings and closings.

Definition 2.4. A mapping $\psi : \mathcal{L} \to \mathcal{L}$ is called:

(a) *idempotent*, if $\psi^2 = \psi$;
(b) *extensive*, if for every $X \in \mathcal{L}$, $\psi(X) \ge X$;
(c) *antiextensive*, if for every $X \in \mathcal{L}$, $\psi(X) \le X$;
(d) a *closing*, if it is increasing, extensive, and idempotent;
(e) an *opening*, if it is increasing, antiextensive, and idempotent;
(f) an *involution*, if $\psi^2 = \mathrm{id}_{\mathcal{L}}$.

Of fundamental importance is the concept of adjunction.

Definition 2.5. Let $\varepsilon : \mathcal{L} \to \tilde{\mathcal{L}}$ and $\delta : \tilde{\mathcal{L}} \to \mathcal{L}$ be two mappings, where $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are complete lattices. Then the pair (ε, δ) is called an *adjunction between* $\mathcal{L}$ *and* $\tilde{\mathcal{L}}$, if for every $X \in \tilde{\mathcal{L}}$ and $Y \in \mathcal{L}$, the following equivalence holds:

$$\delta(X) \leq Y \Leftrightarrow X \leq \varepsilon(Y)$$

If $\tilde{\mathcal{L}}$ coincides with $\mathcal{L}$ we speak of an *adjunction on* $\mathcal{L}$.

The following properties of adjunctions are needed below. For the proof, see [8,12,27].

Lemma 2.6. Let $\mathcal{L}$ and $\tilde{\mathcal{L}}$ be complete lattices. Then:

(a) In an adjunction (ε, δ) between $\mathcal{L}$ and $\tilde{\mathcal{L}}$, $\varepsilon : \mathcal{L} \to \tilde{\mathcal{L}}$ is an erosion and $\delta : \tilde{\mathcal{L}} \to \mathcal{L}$ a dilation.
(b) For every dilation $\delta : \tilde{\mathcal{L}} \to \mathcal{L}$ there is a unique erosion $\varepsilon : \mathcal{L} \to \tilde{\mathcal{L}}$ such that (ε, δ) is an adjunction between $\mathcal{L}$ and $\tilde{\mathcal{L}}$; ε is given by $\varepsilon(Y) = \bigvee\{X \in \tilde{\mathcal{L}} : \delta(X) \leq Y\}$ and is called the upper adjoint of δ.
(c) For every erosion $\varepsilon : \mathcal{L} \to \tilde{\mathcal{L}}$ there is a unique dilation $\delta : \tilde{\mathcal{L}} \to \mathcal{L}$ such that (ε, δ) is an adjunction between $\mathcal{L}$ and $\tilde{\mathcal{L}}$; δ is given by $\delta(X) = \bigwedge\{Y \in \mathcal{L} : X \leq \varepsilon(Y)\}$ and is called the lower adjoint of ε.
(d) δ is **T**-invariant if and only if ε is **T**-invariant; if so, we call (ε, δ) a **T**-adjunction.
(e) For any adjunction between $\mathcal{L}$ and $\tilde{\mathcal{L}}$, we have $\delta\varepsilon \leq \mathrm{id}_{\mathcal{L}}$, $\varepsilon\delta \geq \mathrm{id}_{\tilde{\mathcal{L}}}$, $\delta\varepsilon\delta = \delta$, and $\varepsilon\delta\varepsilon = \varepsilon$. In particular, $\delta\varepsilon$ is an opening on $\mathcal{L}$ and $\varepsilon\delta$ is a closing on $\tilde{\mathcal{L}}$.
(f) Given two **T**-adjunctions (ε, δ) and (ε', δ'), $(\varepsilon'\varepsilon, \delta\delta')$ is a **T**-adjunction.
(g) If $(\varepsilon_j, \delta_j)$ is a **T**-adjunction for every $j \in J$, $(\bigwedge_{j\in J}\varepsilon_j, \bigvee_{j\in J}\delta_j)$ is a **T**-adjunction.

Definition 2.7. Let $\varepsilon : \mathcal{L} \to \tilde{\mathcal{L}}$ be an erosion with adjoint dilation $\delta : \tilde{\mathcal{L}} \to \mathcal{L}$. A *morphological opening* (*closing*) is an opening (closing) of the form $\delta\varepsilon$ $(\varepsilon\delta)$.

Next we recall some general properties of openings and closings. The supremum of a collection of openings is again an opening. The greatest opening on $\mathcal{L}$ is $\mathrm{id}_{\mathcal{L}}$, where the ordering of mappings is defined by (7.13a).

Definition 2.8. The *domain of invariance* of a mapping $\psi : \mathcal{L} \to \mathcal{L}$ is the set

$$\mathrm{Inv}(\psi) := \{X \in \mathcal{L} : \psi(X) = X\}$$

Openings are completely characterized by their domain of invariance: $\gamma_1 = \gamma_2 \Leftrightarrow \mathrm{Inv}(\gamma_1) = \mathrm{Inv}(\gamma_2)$.

Definition 2.9. Let B be an element of $\mathcal{L}$. The *structural* **T**-*opening* by the structuring element B is the mapping

$$\gamma_B^{\mathbf{T}}(X) = \bigvee\{\tau(B) : \tau \in \mathbf{T},\ \tau(B) \leq X\}, \qquad X \in \mathcal{L} \tag{7.16}$$

Similarly, the *structural closing* $\phi_B^{\mathbf{T}}$ by the structuring element B is defined by the formula

$$\phi_B^{\mathbf{T}}(X) = \bigwedge\{\tau(B): \tau \in \mathbf{T}, \tau(B) \geq X\}, \qquad X \in \mathcal{L} \tag{7.17}$$

As the name suggests, structural openings and closings are defined in terms of a single structuring element. Notice that (7.11) is a structural opening by the structuring element A and (7.12) is a structural closing by the structuring element $\check{A}^c$.

An important result is the following characterization of **T**-openings [12]:

Proposition 2.10. Let γ be a **T**-opening on $\mathcal{L}$. Then γ is a supremum of structural **T**-openings, that is,

$$\gamma(X) = \bigvee\{\gamma_B^{\mathbf{T}}(X): B \in \mathcal{B}\}, \qquad X \in \mathcal{L} \tag{7.18}$$

where $\mathcal{B}$ is the domain of invariance of γ. The subset $\mathcal{B} \subseteq \mathcal{L}$ in this formula may be replaced by any subset $\mathcal{B}'$ that generates $\mathcal{B}$ under **T**-translations and infinite suprema.

In the Euclidean case, a structural opening by B is also a morphological opening: $\gamma_B^{\mathbf{T}}(X) = \delta_B\varepsilon_B(X) = (X \ominus B) \oplus B$. The corresponding representation (7.18) of Euclidean openings on $\mathcal{P}(E)$ as a union of morphological openings was originally proved by Matheron [17].

C. Homogeneous Spaces

In this subsection we introduce the concept of a homogeneous space and give a brief account of prerequisites for later use. For a general introduction we refer the reader to [2,19,20,34].

Let $\mathcal{X}$ be a non-empty set. A bijection $\mathcal{X} \to \mathcal{X}$ is called a *permutation* of $\mathcal{X}$. By $\mathrm{Sym}_{\mathcal{X}}$ we denote the group of all permutations of $\mathcal{X}$. If $\mathcal{X}$ is a finite set of n elements, we write $\mathcal{S}_n$ instead of $\mathrm{Sym}_{\mathcal{X}}$. A subgroup Γ of $\mathrm{Sym}_{\mathcal{X}}$ is called a *permutation group* or *transformation group* on $\mathcal{X}$. We also say that Γ is a *group action* on $\mathcal{X}$ or that Γ *acts on* $\mathcal{X}$. Each element $g \in \Gamma$ is a mapping $\mathcal{X} \to \mathcal{X} : x \mapsto g(x)$, satisfying

$$(i) \quad gh(x) = g(h(x)), \qquad (ii) \quad e(x) = x$$

where e is the unit element of Γ (i.e., the identity mapping $x \mapsto x$, $x \in \mathcal{X}$), and gh denotes the product of two group elements g and h. The *inverse* of an element $g \in \Gamma$ will be denoted by g^{-1}. Usually we will also write gx instead of $g(x)$.

The permutation group Γ is called *transitive on* $\mathcal{X}$ if for each $x, y \in \mathcal{X}$ there is a $g \in \Gamma$ such that $gx = y$, and *simply transitive* or *regular* when this element g is *unique*. We will sometimes write *multitransitive* to mean "transitive, but not simply transitive" (the more natural phrase "multiply transitive" is avoided because it has a special technical meaning in group theory [20,34]).

Definition 2.11. A *homogeneous space* or a Γ-*set* is a pair $(\Gamma, \mathscr{X})$ where Γ is a permutation group acting transitively on $\mathscr{X}$.

Remark 2.12. In the references cited above, homogeneous spaces are defined in terms of group actions or, equivalently, permutation representations of an abstract group Γ on a set $\mathscr{X}$. If the representation of this abstract group is faithful, then Γ is isomorphic to a group of permutations on $\mathscr{X}$. In that case the above definition applies, which is more intuitive and sufficient for our purposes. □

The following result is standard in group theory [2,20,34].

Lemma 2.13. Any transitive abelian permutation group Γ is simply transitive.

Therefore our extension of mathematical morphology has to deal with two classes beyond the commutative case: the noncommutative simply transitive case and the noncommutative multitransitive case. The situation is summarized in Table 1.

If Γ acts on $\mathscr{X}$, the *stabilizer* or *isotropy group* of $x \in \mathscr{X}$ is the subgroup $\Gamma_x := \{g \in \Gamma : gx = x\}$. Stabilizers of different points form *conjugated* subgroups: $\Gamma_{gx} = g\Gamma_x g^{-1}$. Let $\omega \in \mathscr{X}$ be an arbitrary but fixed point of $\mathscr{X}$, henceforth called the *origin*. The stabilizer Γ_ω will be noted by Σ from now on:

$$\Sigma := \Gamma_\omega = \{g \in \Gamma : g\omega = \omega\} \tag{7.19}$$

Definition 2.14. The canonical projection π_ω is the mapping $\pi_\omega : \Gamma \mapsto \mathscr{X}$ given by $\pi_\omega(g) = g\omega$.

Define an equivalence relation on Γ as follows $g \sim h \Leftrightarrow h^{-1}g \in \Sigma$. So two elements g and h of the group are equivalent if $g\omega = h\omega$, that is, if they map the origin to the same point of $\mathscr{X}$. If g_x is an arbitrary element of Γ that maps the origin to x, then one easily sees that the collection of all elements equivalent to g_x is the subset $g_x\Sigma := \{g_x s : s \in \Sigma\}$ of Γ. This set is called a *left coset* with respect to the subgroup Σ. The collection of equivalence classes is called the *left coset space* associated to Σ and is denoted by Γ/Σ (read "Γ modulo Σ").

From the above it follows that there is a bijection between $\mathscr{X}$ and the coset space Γ/Σ: each point $x \in \mathscr{X}$ is identified with the coset $g_x\Sigma$. So instead of the pair $(\Gamma,\mathscr{X})$ we might as well study $(\Gamma,\Gamma/\Sigma)$. If Γ acts regularly on $\mathscr{X}$, the stabilizer reduces to the unit element of the group: $\Sigma = \{e\}$. So in that case we are left with the pair (Γ,Γ), which will be studied in Section III.

Table 1. Classification of Transformation Groups and the Associated Morphologies

	Simply Transitive	Multitransitive
Commutative	Euclidean morphology (Section II.A)	—
Noncommutative	Morphology on groups (Section III)	Morphology on homogeneous spaces (Section IV)

There are several ways in which a group acts on itself. Two standard transitive actions are introduced in the next definition.

Definition 2.15. Let Γ be a group and g be an arbitrary element of the group. The *left and right translations by* g are the mappings,

$$\lambda_g\colon \Gamma \to \Gamma, \qquad \lambda_g(h) = gh \text{ (left translation)}$$

$$\rho_g\colon \Gamma \to \Gamma, \qquad \rho_g(h) = hg \text{ (right translation)}$$

Notice that $\lambda_g\lambda_h = \lambda_{gh}$, $\rho_g\rho_h = \rho_{hg}$, so the permutation group $\Gamma^\lambda := \{\lambda_g : g \in \Gamma\}$ is isomorphic to Γ under the correspondence $g \leftrightarrow \lambda_g$, and the permutation group $\Gamma^\rho := \{\rho_g : g \in \Gamma\}$ is isomorphic to Γ under the correspondence $g \leftrightarrow \rho_g^{-1}$. The group Γ is a homogeneous space under both the actions Γ^λ and Γ^ρ. It is customary in group theory to use the expression "translations" for these standard actions on a group. Of course, this should not be confused with "translations" in the sense of the Euclidean parallel displacements, which form an abelian group.

In the following we present a number of examples of homogeneous spaces. In each example Γ denotes the group and $\mathcal{X}$ the corresponding Γ-set. The fourth and fifth examples will be considered in more detail in Section IV. A nice exposition of symmetry groups in nature can be found in the book by Weyl [36].

Example 2.16. $\mathcal{X}$ = Euclidean space $\mathbf{R}^n$, Γ = the Euclidean translation group. Γ is abelian, therefore simply transitive (the only translation having a point fixed is the zero translation).

Example 2.17. $\mathcal{X}$ = discrete space $\mathbf{Z}^n$; Γ = the discrete subgroup of the Euclidean translation group that is transitive on $\mathcal{X}$ and leaves the grid invariant. Again Γ is abelian, therefore simply transitive.

Example 2.18. $\mathcal{X} = \mathbf{R}^2\backslash\{0\}$; Γ = the rotation-multiplication group. Γ is abelian; see also [12,25].

Example 2.19. $\mathcal{X} = \mathbf{R}^n$ ($n \geq 2$); Γ = the Euclidean motion group $E^+(n)$ (proper Euclidean group, group of rigid motions), that is, the group generated by translations and rotations (see [24]). The subgroup leaving a point fixed is the set of all rotations around that point. Γ is not abelian. For drawing purposes we will replace $\mathcal{X}$ by a hexagonal grid and Γ by the subgroup $\mathcal{H}$ of Γ consisting of all motions leaving the grid invariant. We will refer to $\mathcal{H}$ as the "hexagonal" group in the following.

Example 2.20. $\mathcal{X}$ = the sphere S^2; Γ = the group SO(3) of rotations in 3-space (see [23]). The subgroup leaving a point fixed is the set of all rotations around an axis through that point and the center of the sphere. Γ is not abelian.

Example 2.21. $\mathcal{X}$ = finite set of n elements $\{1, 2, \ldots, n\}$; Γ = the symmetric group S_n (full permutation group) on n elements. The subgroup leaving an element fixed is the group S_{n-1} of all permutations of the remaining $n - 1$ elements. Γ is not abelian for $n \geq 3$.

Example 2.22. $\mathcal{X}$ = the vertices of a regular polyhedron; Γ = the discrete subgroup of SO(3) consisting of all rotations that leave the polyhedron invariant.

The group of the tetrahedron is $\mathcal{A}_4$, that of the cube and the octahedron is S_4, and that of the dodecahedron and icosahedron is $\mathcal{A}_5$. Here $\mathcal{A}_n$ denotes the alternating group on n points, that is, the subgroup of S_n containing only *even* permutations [2].

Example 2.23. Let E be a finite set of vertices of a graph. Let $\mathcal{X}$ be the complete graph generated by E and Γ the group of all permutations of E, which extends to a group acting on $\mathcal{X}$ in the obvious way [31]. Γ is in general not abelian.

III. THE SIMPLY TRANSITIVE CASE: MATHEMATICAL MORPHOLOGY ON NONCOMMUTATIVE GROUPS

Our aim in this section is to generalize Euclidean mathematical morphology as reviewed in Section II.A to the space $\mathcal{P}(\mathcal{X})$, where $\mathcal{X}$ is a homogeneous space under a group Γ acting *simply transitively* on $\mathcal{X}$. Because in this case there is a bijection between $\mathcal{X}$ and Γ, as explained in the last subsection, we can identify $\mathcal{X}$ with Γ without loss of generality. Hence in the remainder of this section we will study the power lattice $\mathcal{P}(\Gamma)$, that is, the set of subsets of Γ ordered by set-inclusion, where Γ is an arbitrary group. The classical Euclidean case corresponds to the case in which Γ is the abelian group of vector additions (translations). Our first step will be to find a generalization of the Minkowski operations, the main problem being how to overcome the noncommutativity of the group Γ (Section III.B). Subsequently we define generalized dilations and erosions invariant under Γ, followed by a discussion of adjunctions as well as openings and closings (Section III.C). Finally, we consider characterization theorems for the generalized morphological transformations (Section III.D). But first we will look at a pair of automorphism groups of $\mathcal{P}(\Gamma)$ that are essential in what follows.

A. Left and Right Translations on $\mathcal{P}(\Gamma)$

Let Γ be an arbitrary group. To be consistent with the notation in Section IV, we denote elements of Γ by g, h, k, etc., and subsets of Γ by the corresponding capitals G, H, K. The product of two group elements g and h is written gh. For a fixed $g \in \Gamma$, one can define the mappings $h \mapsto gh$ and $h \mapsto hg$ for any $h \in \Gamma$. These mappings are called *left translation by* g and *right translation by* g, respectively [2,20,34]. This definition can be trivially extended to subsets of the group as follows:

$$\text{left translation:} \quad \lambda_g\colon \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma), \quad \lambda_g(H) = \{gh\colon h \in H\} \tag{7.20a}$$

$$\text{right translation:} \quad \rho_g\colon \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma), \quad \rho_g(H) = \{hg\colon h \in H\} \tag{7.20b}$$

Instead of $\lambda_g(H)$ and $\rho_g(H)$ we will usually write gH and Hg. It is straightforward to check that the left translations on the lattice $\mathcal{P}(\Gamma)$ preserve unions, intersections, and complements:

$$g(G \cup H) = (gG) \cup (gH),$$
$$g(G \cap H) = (gG) \cap (gH), \qquad (gG)^c = gG^c$$

and similarly for right translations. So the sets $\Gamma^\lambda := \{\lambda_g : g \in \Gamma\}$ and $\Gamma^\rho := \{\rho_g : g \in \Gamma\}$ are both automorphism groups of $\mathcal{P}(\Gamma)$.

Remark 3.1. Notice that $\lambda_g\lambda_h = \lambda_{gh}$, $\rho_g\rho_h = \rho_{hg}$, so Γ_λ is isomorphic to Γ under the correspondence $g \leftrightarrow \lambda_g$, and Γ^ρ is isomorphic to Γ under the correspondence $g \leftrightarrow \rho_g^{-1}$. This is related to the concept of the *dual* Γ^* of a group Γ, which is obtained by defining a dual product "*" in Γ by $g * h = hg$. It is easy to see that the groups Γ^λ and Γ^ρ are dual. So we only need to give proofs for invariance with respect to left translations, say. The right-invariant counterparts then follow by group duality. For easy reference we nevertheless give most results in left- and right-invariant form. □

A simple yet fundamental observation is that left and right translations commute. Summarizing:

Lemma 3.2. Let Γ be a group. Then the groups Γ^λ and Γ^ρ of left and right translations are: (1) automorphism groups of the lattice $\mathcal{P}(\Gamma)$ and (2) isomorphic to Γ. Moreover, left and right translations commute: $\gamma_g\rho_h = \rho_h\gamma_g$ for all g, $h \in \Gamma$.

Finally, we define left and right translation-invariant mappings.

Definition 3.3. A mapping $\psi : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$ is called *left translation-invariant* when, for all $g \in \Gamma$, $\lambda_g\psi = \psi\lambda_g$ (i.e., $\psi(gG) = g\psi(G)$, $\forall G \in \mathcal{P}(\Gamma)$). Similarly, a mapping $\psi : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$ is called *right translation-invariant* when, for all $g \in \Gamma$, $\rho_g\psi = \psi\rho_g$ (i.e., $\psi(Gg) = (\psi(G))g$, $\forall G \in \mathcal{P}(\Gamma)$).

For brevity we will speak of *left-invariant* or *λ-mappings* and *right-invariant* or *ρ-mappings*.

B. Generalization of the Minkowski Operations

Since Γ is a group, we can use the group operation to define a multiplication on subsets of Γ, which leads to the generalization of the Minkowski addition.

Definition 3.4. Let G,H be subsets of the group Γ. The *product of G by H*, denoted by $G \overset{\Gamma}{\oplus} H$, is the subset of Γ defined by

$$G \overset{\Gamma}{\oplus} H = \{gh: g \in G, h \in H\} \tag{7.21a}$$
$$G \overset{\Gamma}{\oplus} \emptyset = \emptyset \overset{\Gamma}{\oplus} G = \emptyset \tag{7.21b}$$

Here we have explicitly indicated the dependence of the product $\overset{\Gamma}{\oplus}$ on the group Γ. It is immediate that, with e the unit element of Γ, $G \overset{\Gamma}{\oplus} \{e\} = \{e\} \overset{\Gamma}{\oplus} G = G$. Notice that, in general, the product operation is noncommutative, that is:

$$G \overset{\Gamma}{\oplus} H \neq H \overset{\Gamma}{\oplus} G \tag{7.22}$$

Remark 3.5. We notice in passing that $\mathcal{P}(\Gamma)$ is a *monoid* under the multiplication $\overset{\Gamma}{\oplus}$, that is, a semigroup with unit element $\{e\}$. Since $\mathcal{P}(\Gamma)$ is a complete lattice as well, and the multiplication $\overset{\Gamma}{\oplus}$ is distributive over unions (see the next proposition), we have an example here of a so-called *complete lattice-ordered monoid* or *cl-monoid*; see Birkhoff [4] or Blyth and Janowitz [5]. □

We can write (7.21a) in the alternative forms

$$G \overset{\Gamma}{\oplus} H = \bigcup_{g \in G} gH = \bigcup_{h \in H} Gh \tag{7.23}$$

The similarity with the Minkowski addition (7.1) is clear. Next we generalize the Minkowski subtraction.

Definition 3.6. Let G, H be subsets of the group Γ. The *left residual* of G by H, denoted by $G \overset{\lambda}{\ominus} H$, is the subset of Γ defined by

$$G \overset{\lambda}{\ominus} H = \{g \in \Gamma \colon gH \subseteq G\} \tag{7.24a}$$

The *right residual* of G by H, denoted by $G \overset{\rho}{\ominus} H$, is the subset of Γ defined by

$$G \overset{\rho}{\ominus} H = \{g \in \Gamma \colon Hg \subseteq G\} \tag{7.24b}$$

Remark 3.7. The above definition of residuals is standard in the theory of *residuated semigroups*. The left residual of G by H is characterized by the property that it is the largest subset K of Γ such that when multiplied on the right by H it is included in G:

(i) $(G \overset{\lambda}{\ominus} H) \overset{\Gamma}{\oplus} H \subseteq G$

(ii) $K \overset{\Gamma}{\oplus} H \subseteq G \Rightarrow K \subseteq G \overset{\lambda}{\ominus} H$

with a similar statement for right residuals; see Birkhoff [4] or Blyth and Janowitz [5]. Definition 3.6 also applies if Γ is just a semigroup instead of a group. Of course, the fact that we assume Γ to be a group enables us to derive more specific results. As far as notation is concerned, in residuation theory one usually writes GH, $G \cdot\!.\, H$, $G \,.\!\cdot H$ instead of $G \overset{\Gamma}{\oplus} H$, and $G \overset{\lambda}{\ominus} H$, and $G \overset{\rho}{\ominus} H$, respectively. With our choice of notation we maintain some resemblance to the symbols $\oplus$, $\ominus$ used in Euclidean morphology. □

Using the group nature of Γ, we easily derive the following equivalences:

$$\begin{aligned} gH \subseteq G &\Leftrightarrow gh \in G, \forall h \in H \Leftrightarrow g \in Gh^{-1}, \forall h \in H \\ &\Leftrightarrow g \in \bigcap_{h \in H} \mathrm{Gh}^{-1} \end{aligned}$$

Hence,

$$G \overset{\lambda}{\ominus} H = \bigcap_{h \in H} Gh^{-1}, \qquad G \overset{\rho}{\ominus} H = \bigcap_{h \in H} h^{-1}G \tag{7.25}$$

where the result for the right residual can be shown similarly. Both formulas reduce to the Euclidean Minkowski subtraction $G \ominus H$ if the group Γ is commutative, as a glance at Eq. (7.2) makes clear. Note that

$$\{g\} \overset{\Gamma}{\oplus} G = gG, \qquad G \overset{\Gamma}{\oplus} \{g\} = Gg \tag{7.26a}$$

$$G \overset{\lambda}{\ominus} \{g\} = Gg^{-1}, \qquad G \overset{\rho}{\ominus} \{g\} = g^{-1}G \tag{7.26b}$$

Next we state a number of algebraic properties of the set product and the residuals*, generalizing the formulas (7.4). For a proof of (a)–(f) in an abstract lattice-theoretical context, see [4,5].

Proposition 3.8. Let $G, H, K \subseteq \Gamma$ and $g, h, k \in \Gamma$. Then the following hold:

(a) $G \overset{\Gamma}{\oplus} (H \cup K) = (G \overset{\Gamma}{\oplus} H) \cup (G \overset{\Gamma}{\oplus} K)$ $\cup$-distributivity

$(G \cup H \overset{\Gamma}{\oplus} K = (G \overset{\Gamma}{\oplus} K) \cup (H \overset{\Gamma}{\oplus} K)$

(b) $(G \overset{\Gamma}{\oplus} H) \overset{\Gamma}{\oplus} K = G \overset{\Gamma}{\oplus} (H \overset{\Gamma}{\oplus} K)$ associativity

(c) $(G \cap H) \overset{\lambda}{\ominus} K = (G \overset{\lambda}{\ominus} K) \cap (H \overset{\lambda}{\ominus} K)$ $\cap$-distributivity

$(G \cap H) \overset{\rho}{\ominus} K = (G \overset{\rho}{\ominus} K) \cap (H \overset{\rho}{\ominus} K)$

(d) $G \overset{\lambda}{\ominus} (H \cup K) = (G \overset{\lambda}{\ominus} H) \cap (G \overset{\lambda}{\ominus} K)$

$G \overset{\rho}{\ominus} (H \cup K) = (G \overset{\rho}{\ominus} H) \cap (G \overset{\rho}{\ominus} K)$

(e) $G \overset{\Gamma}{\oplus} H \subseteq K \Leftrightarrow G \subseteq K \overset{\lambda}{\ominus} H \Leftrightarrow H \subseteq K \overset{\rho}{\ominus} G$

(f) $(G \overset{\lambda}{\ominus} H) \overset{\lambda}{\ominus} K = G \overset{\lambda}{\ominus} (K \overset{\Gamma}{\oplus} H)$ iteration

$(G \overset{\rho}{\ominus} H) \overset{\rho}{\ominus} K = G \overset{\rho}{\ominus} (H \overset{\Gamma}{\oplus} K)$

$(G \overset{\rho}{\ominus} H) \overset{\lambda}{\ominus} K = (G \overset{\lambda}{\ominus} K) \overset{\rho}{\ominus} H$

(g) $(gH) \overset{\Gamma}{\oplus} K = g(H \overset{\Gamma}{\oplus} K); H \overset{\Gamma}{\oplus} (Kg) = (H \overset{\Gamma}{\oplus} K)g$ Γ-invariance

$(gH) \overset{\lambda}{\ominus} K = g(H \overset{\lambda}{\ominus} K); (Hg) \overset{\rho}{\ominus} K = (H \overset{\rho}{\ominus} K)g$

(h) $H \overset{\lambda}{\ominus} (gK) = (H \overset{\lambda}{\ominus} K)g^{-1}; H \overset{\lambda}{\ominus} (Kg) = (Hg^{-1}) \overset{\lambda}{\ominus} K$

$H \overset{\rho}{\ominus} (gK) = (g^{-1}H) \overset{\rho}{\ominus} K; H \overset{\rho}{\ominus} (Kg) = g^{-1}(H \overset{\rho}{\ominus} K)$

As in the Euclidean case, there exists a duality by complementation. First we need some definitions.

Definition 3.9. Let G be a subset of Γ. The *inverted set* of G is the set $\check{G} = \{g^{-1} : g \in G\}$. The *complement* of G is the set $G^c = \{g \in \Gamma : g \notin G\}$. The complement of the inverted set is denoted by $\hat{G} := (\check{G})^c$.

* Proofs are given in the Appendix.

Notice that in the Euclidean case inversion of a set reduces to reflection of a set as defined in (7.7), hence our use of the same symbol.

Lemma 3.10. Let G, H be subsets of Γ. Then,

(a) $(G^c)^c = (\check{G})^\vee = (\hat{G})^\wedge = G$

(b) $(\check{G})^c = (G^c)^\vee$

(c) $(gG)^c = gG^c; (Gg)^c = G^c g$

(d) $(gG)^\vee = \check{G}g^{-1}; (Gg)^\vee = g^{-1}\check{G}$

(e) $(G \overset{\Gamma}{\oplus} H)^\vee = \check{H} \overset{\Gamma}{\oplus} \check{G}$

(f) $(G \cup H)^c = G^c \cap H^c, (G \cup H)^\vee = \check{G} \cup \check{H},$
$(G \cap H)^\vee = \check{G} \cap \check{H}, (G \cup H)^\wedge = \hat{G} \cap \hat{H}$

(g) $(G \overset{\Gamma}{\oplus} H)^c = G^c \overset{\lambda}{\ominus} \check{H} = H^c \overset{\rho}{\ominus} \check{G}$

(h) $(G \overset{\lambda}{\ominus} H)^\vee = \check{G} \overset{\rho}{\ominus} \check{H}$

(i) $(G \overset{\Gamma}{\oplus} H)^\wedge = \hat{G} \overset{\rho}{\ominus} H = \hat{H} \overset{\lambda}{\ominus} G$

By making use of duality by complementation one may derive pairs of equivalent results; for example, consider Proposition 3.8(g). Start with $(gH) \overset{\Gamma}{\oplus} K = g(H \overset{\Gamma}{\oplus} K)$. Take complements of both sides and use Lemma 3.10(g) to find $(gH^c) \overset{\lambda}{\ominus} \check{K} = g(H^c \overset{\lambda}{\ominus} \check{K})$. Since K and H ar arbitrary, we get $(gH) \overset{\lambda}{\ominus} K = g(H \overset{\lambda}{\ominus} K)$, which is the third item of Proposition 3.8(g). All this is completely analogous to the Euclidean case.

Remark 3.11. It is easy to show that the group product (7.23) has the following geometric interpretation, which is a straightforward generalization of the Euclidean result (7.5):

$$G \overset{\Gamma}{\oplus} H = \{k \in \Gamma: k\check{H} \cap G \neq \emptyset\} = \{k \in \Gamma: \check{G}k \cap H \neq \emptyset\}$$

The geometric interpretation of the residuals is implicit in Definition 3.6. □

C. Dilations, Erosions, Openings, and Closings

Now that we have generalized the Minkowski operations we are in a position to define various morphological transformations that are invariant under the group Γ. We start with a discussion of dilations and erosions.

Because of the noncommutativity of the set product (7.21) there are two possibilities for generalizing the dilation (7.5). We may consider, for a fixed $H \in \mathcal{P}(\Gamma)$, the mapping $G \mapsto G \overset{\Gamma}{\oplus} H$, as well as the mapping $G \mapsto H \overset{\Gamma}{\oplus} G$. This leads to the following definition.

Definition 3.12. Let $H \in \mathcal{P}(\Gamma)$. The *left dilation* $\bar{\delta}^{\lambda}_{H}$ and *right dilation* $\bar{\delta}^{\rho}_{H}$ by the structuring element H are the mappings: $\mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$ defined by

$$\tilde{\delta}^{\lambda}_H(G) = G \overset{\Gamma}{\oplus} H, \qquad \tilde{\delta}^{\rho}_H(G) = H \overset{\Gamma}{\oplus} G \tag{7.27}$$

That these mappings are dilations (i.e., commute with arbitrary unions; see Definition 2.3), is readily proved by extending Proposition 3.8(a) to distributivity with respect to infinite unions. The reason for the terminology is that left (right) dilations are left (right) translation invariant; see Proposition 3.8(g).

Next we show that left and right dilations can be decomposed in terms of the automorphisms of the lattice $\mathcal{P}(\Gamma)$. From (7.23) is it immediate that

$$\tilde{\delta}^{\lambda}_H(G) = \bigcup_{h\in H} Gh = \bigcup_{g\in G} gH \tag{7.28a}$$

$$\tilde{\delta}^{\rho}_H(G) = \bigcup_{h\in H} hG = \bigcup_{g\in G} Hg \tag{7.28b}$$

Defining the union and intersection of left and right translations pointwise (i.e., by the ordering inherited from $\mathcal{P}(\Gamma)$; see Section II.B), (7.28) can be written in operator form as

$$\tilde{\delta}^{\lambda}_H = \bigcup_{h\in H} \rho_h, \qquad \tilde{\delta}^{\rho}_H = \bigcup_{h\in H} \lambda_h \tag{7.29}$$

Since left and right translations commute, we see that $\tilde{\delta}^{\lambda}_H$ commutes with left translations and $\tilde{\delta}^{\rho}_H$ commutes with right translations. Below we will show that all left- and right-invariant dilations have this form. In a similar way we define left- and right-invariant erosions.

Definition 3.13. Let $H \in \mathcal{P}(\Gamma)$. The *left erosion* $\tilde{\varepsilon}^{\lambda}_H$ and *right erosion* $\tilde{\varepsilon}^{\rho}_H$ by the structuring element H are the mappings $\mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$ defined by

$$\tilde{\varepsilon}^{\lambda}_H(G) = G \overset{\lambda}{\ominus} H, \qquad \tilde{\varepsilon}^{\rho}_H(G) = G \overset{\rho}{\ominus} H \tag{7.30}$$

We also write λ-*dilation*/λ-*erosion* instead of left dilation/left erosion, with a similar convention for the right-invariant counterparts.

Again we decompose left and right erosions in terms of left and right translations. Just as there are two equivalent forms for the left and right dilation (7.28), one can derive two forms for the erosions. To see this take the complement of (7.28a), which by (7.27) equals the complement of $G \overset{\Gamma}{\oplus} H$:

$$\bigcap_{h\in H} G^c h = \bigcap_{g\in G} gH^c = (G \overset{\Gamma}{\oplus} H)^c = G^c \overset{\lambda}{\ominus} \check{H}$$

where we have used Lemma 3.10(g). Since this formula holds for arbitrary G, $H \in \mathcal{P}(\Gamma)$ we find (the proof for the right erosion is analogous)

$$\tilde{\varepsilon}^{\lambda}_H(G) = \bigcap_{h\in H} Gh^{-1} = \bigcap_{g\in G^c} g\hat{H} \tag{7.31a}$$

$$\tilde{\varepsilon}^{\rho}_H(G) = \bigcap_{h\in H} h^{-1}G = \bigcap_{g\in G^c} \hat{H}g \tag{7.31b}$$

where, as before, $\hat{H} = \check{H}^c$. In operator form,

$$\tilde{\varepsilon}_H^\lambda = \bigcap_{h \in H} \rho_h^{-1}, \qquad \tilde{\varepsilon}_H^\rho = \bigcap_{h \in H} \lambda_h^{-1} \tag{7.32}$$

The following lemma shows that as soon as we have proved a result for left-invariant dilations, there is a corresponding result for right-invariant dilations, as well as for left- or right-invariant erosions. First we need a definition.

Definition 3.14. Let $\psi : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$ be an arbitrary mapping. The *Boolean dual* ψ' of ψ is the mapping defined by $\psi'(G) = (\psi(G^c))^c$. The reflection $\check{\psi}$ of ψ is the mapping defined by $\check{\psi}(G) = (\psi(\check{G}))^{\vee}$. The *dual reflection* of ψ is the mapping $\hat{\psi}$ defined by $\hat{\psi}(G) = (\psi(\hat{G}))^{\wedge}$.

Lemma 3.15. Let $\psi : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$ be an *arbitrary mapping*. Then,

(a) $(\psi')' = (\check{\psi})^{\vee} = (\hat{\psi})^{\wedge} = \psi$.

(b) $(\check{\psi})' = (\psi')^{\vee}$.

(c) ψ is an increasing λ-mapping $\Leftrightarrow$ ψ' is an increasing λ-mapping; ψ is a dilation $\Leftrightarrow$ ψ' is an erosion. In particular, $(\tilde{\delta}_H^\lambda)' = \tilde{\varepsilon}_{\hat{H}}^\lambda$.

(d) ψ is right-invariant $\Leftrightarrow$ $\check{\psi}$ is left-invariant.
In particular, $(\lambda_h)^{\vee} = \rho_h^{-1}$, $(\tilde{\delta}_H^\lambda)^{\vee} = \tilde{\delta}_{\check{H}}^\rho$, $(\tilde{\varepsilon}_H^\lambda)^{\vee} = \tilde{\varepsilon}_{\check{H}}^\rho$.

(e) $(\tilde{\delta}_h^\rho)^{\vee} = \tilde{\varepsilon}_{\hat{H}}^\rho$.

Remark 3.16. Here is an example of how this lemma can be used. Suppose the following statement has been proved: ψ increasing $\Rightarrow$ ψ' increasing. To show the converse, apply this statement to ψ'. Then we find: ψ' increasing $\Rightarrow$ ψ'' increasing, but since the complementation operator is an involution ($\psi'' = \psi$) the proof is complete. In a similar way we can use results for left-invariant dilations to derive counterparts for right-invariant dilations (using $(\check{\psi})^{\vee} = \psi$) or for right-invariant erosions (using $(\hat{\psi})^{\wedge} = \psi$). □

Next we make a few remarks about adjunctions. By Proposition 3.8(e) we have the equivalences

$$\tilde{\delta}_H^\lambda(G) \subseteq K \Leftrightarrow G \subseteq \tilde{\varepsilon}_H^\lambda(K) \tag{7.33a}$$

$$\tilde{\delta}_H^\rho(G) \subseteq K \Leftrightarrow G \subseteq \tilde{\varepsilon}_H^\rho(K) \tag{7.33b}$$

We call $(\tilde{\varepsilon}_H^\lambda, \tilde{\delta}_H^\lambda)$ a *left-invariant adjunction* (λ*-adjunction*) and similarly we call $(\tilde{\varepsilon}_H^\rho, \tilde{\delta}_H^\rho)$ a *right-invariant adjunction* (ρ*-adjunction*). In particular all the properties of adjunctions as summarized in Lemma 2.6 hold for these adjunctions. So $\tilde{\varepsilon}_H^\rho$ is the supper adjoint of $\tilde{\delta}_H^\lambda$, $\tilde{\delta}_H^\lambda$ is the lower adjoint of $\tilde{\varepsilon}_H^\lambda$, etc. Lemma 3.15(c–e) expresses the relation between the duality by complementation, reflection and adjoint pairs.

From the properties of adjunctions (see Section II.B) we know that we can build so-called *morphological* openings and closings from dilations and erosions. In particular, the mappings $\tilde{\delta}^{\lambda}_{H}\tilde{\varepsilon}^{\lambda}_{H}$ and $\tilde{\delta}^{\rho}_{H}\tilde{\varepsilon}^{\rho}_{H}$ are left- and right-invariant morphological openings, and the mappings $\tilde{\varepsilon}^{\lambda}_{H}\tilde{\delta}^{\lambda}_{H}$ and $\tilde{\varepsilon}^{\rho}_{H}\tilde{\delta}^{\rho}_{H}$ are left- and right-invariant morphological closings. As in the Euclidean case, these mappings are also so-called *structural* openings and closings (see Definition 2.9). Explicitly:

Proposition 3.17. For all $G, H \in \mathcal{P}(\Gamma)$,

$$\tilde{\gamma}^{\lambda}_{H}(G) := \tilde{\delta}^{\lambda}_{H}\tilde{\varepsilon}^{\lambda}_{H}(G) = (G \overset{\lambda}{\ominus} H) \overset{\Gamma}{\oplus} H = \bigcup_{g\in\Gamma}\{gH: gH \subseteq G\}$$

$$\tilde{\gamma}^{\rho}_{H}(G) := \tilde{\delta}^{\rho}_{H}\tilde{\varepsilon}^{\rho}_{H}(G) = H \overset{\Gamma}{\oplus} (G \overset{\rho}{\ominus} H) = \bigcup_{g\in\Gamma}\{Hg: Hg \subseteq G\}$$

$$\tilde{\phi}^{\lambda}_{H}(G) := \tilde{\varepsilon}^{\lambda}_{H}\tilde{\delta}^{\lambda}_{H}(G) = (G \overset{\Gamma}{\oplus} H)\overset{\lambda}{\ominus}\mathrm{H} = \bigcap_{g\in\Gamma}\{g\hat{H}: g\hat{H} \supseteq G\}$$

$$\tilde{\phi}^{\rho}_{H}(G) := \tilde{\varepsilon}^{\rho}_{H}\tilde{\delta}^{\rho}_{H}(G) = (H \overset{\Gamma}{\oplus} G) \overset{\rho}{\ominus} H = \bigcap_{g\in\Gamma}\{\hat{H}g: \hat{H}g \supseteq G\}$$

This proposition contains the geometric interpretation of the morphological openings and closings. For example, the left-invariant opening $\tilde{\delta}^{\lambda}_{H}\tilde{\varepsilon}^{\lambda}_{H}(G)$ is the union of all left translates of H that are contained in G, etc. All this is completely analogous to the situation in Euclidean morphology. The following properties related to behavior under translations are immediate:

$$\tilde{\gamma}^{\lambda}_{H}(gG) = g\tilde{\gamma}^{\lambda}_{H}(G), \qquad \gamma^{\lambda}_{gH}(G) = \tilde{\gamma}^{\lambda}_{H}(G) \tag{7.34}$$

Similar properties can be proved for closings and the right-invariant counterparts of both by using the identities

$$(\tilde{\phi}^{\lambda}_{H})' = \gamma^{\lambda}_{H^c}, \qquad (\tilde{\gamma}^{\lambda}_{H})^{\vee} = \gamma^{\rho}_{\check{H}} \tag{7.35}$$

which follow from Lemma 3.15.

Summarizing the results so far, we have generalized the Minkowski operations and the associated dilations and erosions, forming adjoint pairs invariant under either left or right translations. Finally, we have constructed the morphological openings and closings that correspond to these adjunctions and provided a simple geometric interpretation for them. The questions we take up in the final section is whether all adjunctions, openings, and closings are of the form found above. Also, the representation theorem of Matheron for increasing translation-invariant mappings will be generalized.

D. Characterization Theorems

This section treats the representation theorems for adjunctions, openings and closings, as well as general translation-invariant mappings. We start with the characterization of adjunctions. Then follows a discussion of kernels of mappings $\psi : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$. Subsequently, we extend the results of [1] concerning a representation theorem for arbitrary translation-invariant mappings, obtaining decompositions of increasing or decreasing translation-invariant mappings as special cases. We end with a discussion of openings and closings.

The question whether all left- and right-invariant dilations and erosions have the forms (7.29) and (7.32), respectively, is answered by the following proposition.

Proposition 3.18. General form of adjunctions on $\mathcal{P}(\Gamma)$. A pair (ε, δ) of mappings from $\mathcal{P}(\Gamma)$ to itself is a left-invariant adjunction if and only if

$$\delta = \tilde{\delta}_H^\lambda = \bigcup_{h \in H} \rho_h, \qquad \varepsilon = \tilde{\varepsilon}_H^\lambda = \bigcap_{h \in H} \rho_h^{-1} \tag{7.36}$$

for some $H \in \mathcal{P}(\Gamma)$. A corresponding statement holds for right-invariant adjunctions.

Next we need to make a few remarks about kernels.

Definition 3.19. The *kernel* of a mapping $\psi : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$, denoted by $\mathcal{V}(\psi)$, is the family of subsets of Γ defined by (e denotes the unit element of the group Γ)

$$\mathcal{V}(\psi) = \{G \in \mathcal{P}(\Gamma) \colon e \in \psi(G)\}$$

Proposition 3.20. There is a 1–1 correspondence between subsets of the lattice $\mathcal{P}(\Gamma)$ and λ-mappings (ρ-mappings) $\psi : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$. More precisely, to any λ-mapping (ρ-mapping) ψ corresponds a family $\mathcal{B}$ of subsets of Γ, where $\mathcal{B}$ is the kernel of ψ. Conversely, to any subset $\mathcal{B} \subseteq \mathcal{P}(\Gamma)$ corresponds one λ-mapping ψ^λ defined by $\psi^\lambda(G) = \{h \in \Gamma : G \in h\mathcal{B}\}$ and one ρ-mapping ψ^ρ defined by $\psi^\rho(G) = \{h \in \Gamma : G \in \mathcal{B}h\}$, both with kernel $\mathcal{B}$.

Here we have used the notation $h\mathcal{B} = \{hB : B \in \mathcal{B}\}$, $\mathcal{B}h = \{Bh : B \in \mathcal{B}\}$. The proof is completely analogous to the Euclidean case (see, e.g., Matheron [17, chapter 8]) and is omitted here.

The following lemma shows the relation between the kernel of a mapping and that of its dual, reflection, and dual reflection, respectively. The proof is easy and left to the reader.

Lemma 3.21. Let $\psi : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$ be a mapping with kernel $\mathcal{V}(\psi)$. Then the kernels of the dual ψ', the reflection $\check{\psi}$, and the dual reflection $\hat{\psi}$ are given by

(a) $\mathcal{V}(\psi') = \{G \in \mathcal{P}(\Gamma) \colon G^c \notin \mathcal{V}(\psi)\}$
(b) $\mathcal{V}(\check{\psi}) = \{G \in \mathcal{P}(\Gamma) \colon \check{G} \in \mathcal{V}(\psi)\}$
(c) $\mathcal{V}(\hat{\psi}) = \{G \in \mathcal{P}(\Gamma) \colon \hat{G} \notin \mathcal{V}(\psi)\}$

In a recent paper, Banon and Barrera [1] generalized Matheron's theorem (Theorem 2.2) to arbitrary translation-invariant mappings (not necessarily increasing) on $\mathcal{P}(E)$, where E denotes Euclidean space. Following the simplified proof in [13], we extend this result here to the case $\mathcal{P}(\Gamma)$ with Γ a noncommutative group, getting as a by-product a generalization of Matheron's theorem. We only formulate the left translation-invariant case. The right translation-invariant case is obtained by left-right symmetry.

Define, for $F, G, H \in \mathcal{P}(\Gamma)$, the *left wedge transform* of G by the pair (F, H) by

$$\begin{aligned} G \overset{\lambda}{\oslash} (F, H) &:= \{g \in \Gamma \colon gF \subseteq G \subseteq gH\} \\ &= (G \overset{\lambda}{\ominus} F) \cap (G^c \overset{\lambda}{\ominus} H^c) \end{aligned}$$

where the second line follows from the definition Eq. (7.24a) of the left residual. In the Euclidean case, this operation is a slight modification of the hit-or-miss transform [29]. Clearly, the mapping $G \mapsto G \overset{\lambda}{\oslash} (F, H)$ is left translation-invariant. Two cases are of special interest:

1. $G \overset{\lambda}{\oslash} (F, \Gamma) = G \overset{\lambda}{\ominus} F$
2. $G \overset{\lambda}{\oslash} (\emptyset, H) = G^c \overset{\lambda}{\ominus} H^c$

Define also the "interval" $[F, H]$ between two arbitrary sets F and H as

$$[F, H] = \{G \in \mathcal{P}(\Gamma) : F \subseteq G \subseteq H\}$$

Clearly, $[F, H]$ and $G \overset{\lambda}{\oslash} (F, H)$ are both empty if $F \not\subseteq H$.

Definition 3.22. Let ψ be a mapping on $\mathcal{P}(\Gamma)$, with kernel $\mathcal{V}(\psi)$ given by Definition 3.19. The *bikernel* of ψ is defined by

$$\mathcal{W}(\psi) = \{(F, H) \in \mathcal{P}(\Gamma) \times \mathcal{P}(\Gamma) \colon [F, H] \subseteq \mathcal{V}(\psi)\}$$

If ψ is increasing and F is an element of $\mathcal{V}(\psi)$, then the whole interval $[F, H]$ is included in $\mathcal{V}(\psi)$ if $H \supseteq F$. Similarly, if ψ is decreasing and $H \in \mathcal{V}(\psi)$, then $[F, H]$ is included in $\mathcal{V}(\psi)$ if $F \subseteq H$. Hence

$$\psi \text{ increasing}, \qquad F \in \mathcal{V}(\psi) \Rightarrow (F, \Gamma) \in \mathcal{W}(\psi) \tag{7.37a}$$

$$\psi \text{ decreasing}, \qquad H \in \mathcal{V}(\psi) \Rightarrow (\emptyset, H) \in \mathcal{W}(\psi) \tag{7.37b}$$

Now we can state:

Theorem 3.23. Representation of translation-invariant mappings. The mapping $\psi : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$ is left translation-invariant if and only if

$$\psi(G) = \bigcup_{(F,H) \in \mathcal{W}(\psi)} G \overset{\lambda}{\oslash} (F, H) \tag{7.38}$$

Corollary 3.24. Representation of monotone translation-invariant mappings. If $\psi : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$ is an increasing λ-mapping it can be decomposed as a union of λ-erosions, or an intersection of λ-dilations:

$$\psi(G) = \bigcup_{F \in \mathcal{V}(\psi)} G \overset{\lambda}{\ominus} F = \bigcap_{F \in \mathcal{V}(\psi')} G \overset{\Gamma}{\oplus} \check{F} \tag{7.39a}$$

where ψ' is the Boolean dual of ψ. If $\psi : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$ is a decreasing λ-mapping, it can be similarly decomposed:

$$\psi(G) = \bigcup_{H \in \mathcal{V}(\psi)} G^c \overset{\lambda}{\ominus} H^c = \bigcap_{H \in \mathcal{V}(\psi')} G^c \overset{\Gamma}{\oplus} \hat{H} \tag{7.39b}$$

Recall from Section II.B that the domain of invariance of a mapping $\psi : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$ is the subset of $\mathcal{P}(\Gamma)$ defined by $\text{Inv}(\psi) = \{G \in \mathcal{P}(\Gamma) : \psi(G) = G\}$.

Theorem 3.25. Representation of openings. A mapping $\psi : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$ is a left-invariant opening if and only if ψ has the representation

$$\psi\ (G) = \bigcup_{H \in \mathcal{B}} \tilde{\gamma}_H^\lambda(G) \tag{7.40}$$

for some subset $\mathcal{B}$ of the lattice $\mathcal{P}(\Gamma)$, with $\gamma_H^\lambda(G) = (G \overset{\lambda}{\ominus} H) \overset{\Gamma}{\oplus} \text{H}$. Moreover, $\text{Inv}(\psi)$ is the class of sets generated by $\mathcal{B}$ under left translations and infinite unions, and any subset $\mathcal{B}$ which generates $\text{Inv}(\psi)$ in this way defines the same opening ψ.

IV. MATHEMATICAL MORPHOLOGY ON SPACES WITH A TRANSITIVE GROUP ACTION

In this section we study a homogeneous space $(\Gamma, \mathcal{X})$, where Γ is a group acting transitively on $\mathcal{X}$. The object space of interest is again the Boolean lattice $\mathcal{P}(\mathcal{X})$ of all subsets of $\mathcal{X}$, ordered by set inclusion. In the following we first informally sketch the basic idea of our construction of morphological operations on this homogeneous space with full invariance under the acting group Γ (Section IV.A). Then we outline in Section IV.B a general strategy of handling the transitive case by making use of the results for the simply transitive case developed above. In Section IV.C we define a "lift" and "projection" between the lattices $\mathcal{P}(\mathcal{X})$ and $\mathcal{P}(\Gamma)$ and state several properties of the associated operators π and ϑ (to be defined). Two examples are given that allow an easy visualization of the results. Section IV.D contains the characterization of set mappings on $\mathcal{P}(\mathcal{X})$ (adjunctions, openings/closings, translation-invariant mappings). Section IV.E briefly describes the situation where the group Γ has a subgroup Δ that acts transitively on $\mathcal{X}$. In this context we also introduce a concept of "shape" that explicitly takes into account the group which is involved.

A. The Basic Idea

It may be helpful to the reader if we first give some motivation and sketch the main ingredients entering into our generalization of mathematical morphology.

To fix ideas, let us consider the question of how morphological operations could be defined on the sphere S^2 with invariance under the rotation group SO(3) as the acting group Γ (see Example 2.20). By *invariance* we mean the requirement that a mapping $\psi : \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{X})$ satisfies

$$\psi(gX) = g\psi(X), \qquad \forall X \subseteq \mathcal{X} \tag{7.41}$$

for all $g \in \Gamma$. If ψ satisfies (7.41) it will be called Γ-*invariant*. That such mappings exist is easy to see. Let Y be a fixed subset of the sphere and define $\psi : S^2 \to S^2$ by

$$\psi(X) = \bigcup_{g\in\Gamma} \{gY: gY \subseteq X\} \tag{7.42}$$

That is, $\psi(X)$ is the union of all rotated copies of Y that are included in X. The reader will recognize this mapping as the generalization of the Euclidean opening (7.11). From the definition it is immediate that this mapping is Γ-invariant: opening the set X and then rotating the sphere will give the same result as first rotating the sphere and then performing the opening. All this is obvious from a geometric point of view. The question is whether corresponding algebraic definitions can be found which will give a decomposition of morphological operations into elementary ones, just as in the Euclidean case.

Let us start with an attempt to generalize the Minkowski operations. A first obvious possibility (see, e.g., [16]) to define a generalized Minkowski addition is to take a subset G of the group Γ (the "structuring element") and let it act on a subset X of $\mathcal{X}$ as follows:

$$X \bar{\oplus} G := \bigcup_{g\in G} gX \tag{7.43}$$

where

$$gX := \{gx: x \in X\} \tag{7.44}$$

is the image or "Γ-translate" of the subset X under the transformation g. Also, as an analogue of the Minkowski subtraction one could consider

$$X \bar{\ominus} G := \bigcap_{g\in G} g^{-1}X \tag{7.45}$$

It is not difficult to show that the mappings δ and ε defined by

$$\delta(X) = X \bar{\oplus} G, \qquad \varepsilon(X) = X \bar{\ominus} G \tag{7.46}$$

are a dilation and erosion, respectively, and that in fact the pair (ε, δ) forms an adjunction. However, what about the generalization of translation invariance? For the dilation we have

$$\delta(g_0X) = \bigcup_{g\in G} gg_0X$$

Now one can see a problem emerging: if we could interchange g and g_0 in the equation above the result would be $g_0\delta(X)$, since the group translations commute with unions (see the next subsection). But since Γ is a noncommutative group, this interchange will in general not be allowed. Therefore we look for another definition of dilations and erosions so that full invariance is obtained.

Now if the reader looks back to the previous section, the thought may occur that we should try to mimic the definition of *left-invariant* dilations and erosions: in other words the set X should somehow be "on the left" and the structuring elements "on the right." However, now we seem to be in trouble again since the group elements act from the left on subsets X. So let us try to find a solution in a "geometric" way.

First we recall the construction of dilations on $\mathcal{P}(\mathcal{X})$ without any invariance property, as given by Serra [31, Chapter 2, Proposition 2.1]: a mapping $\delta : \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{X})$ is a dilation if and only if there exists a function $\gamma : \mathcal{X} \to \mathcal{P}(\mathcal{X})$, called "structuring function," such that

$$\delta(X) = \bigcup_{x \in X} \gamma(x) \tag{7.47}$$

This statement can be interpreted as follows. Attach to each point x of $\mathcal{X}$ a subset $\gamma(x)$ of $\mathcal{X}$; that is, think of $\mathcal{X}$ as being completely "covered" by a collection of subsets of itself. Then the dilation $\delta(X)$ is the union of all the subsets that are attached to points of X. An illustration is given in Figure 2a for the case of a plane. It is easy to see that the erosion ε associated (by adjunction) to δ is given by

$$\varepsilon(X) = \{y \in \mathcal{X}: \gamma(y) \subseteq X\} \tag{7.48}$$

Next we assume that $\mathcal{X}$ is a homogeneous space under a group Γ that acts simply transitively on $\mathcal{X}$. Now we construct a dilation as above with the only difference that the subsets $\gamma(x)$ will be formed as translates (under Γ) of a *fixed*

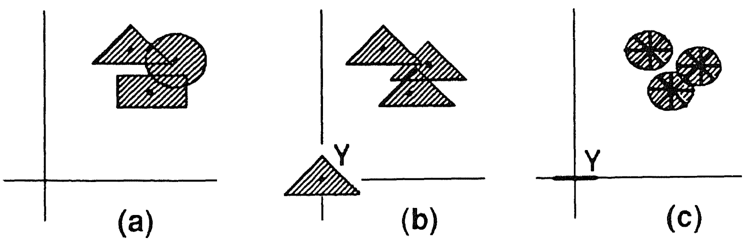

Figure 2. Dilations (hatched) of a subset X of the plane consisting of three points: (a) no invariance, (b) invariance under the Euclidean translation group, and (c) invariance under the Euclidean motion group.

structuring element $Y \subseteq \mathcal{X}$. To be more precise, let ω be an arbitrary point of $\mathcal{X}$, called the "origin" (see Section II.C). The choice of ω is immaterial (this is precisely what *homogeneous* means). Since Γ is simply transitive on $\mathcal{X}$, there exists for each $x \in \mathcal{X}$ a unique group element g_x that maps ω to x. Let Y be a fixed subset of $\mathcal{X}$. Then the mappings given by (7.47) and (7.48) with $\gamma(x) = g_x Y$ are a dilation and erosion, respectively, which in addition are Γ-invariant. For an illustration see Figure 2b, where we cover the plane with copies of a triangle and take the Euclidean translation group as the acting group Γ. More examples in this category were presented in Figure 1.

Finally, we return to the example of the sphere, which belongs to the multi-transitive case. Now there exists more than one rotation that "moves" Y from an initial position to an arbitrary point x. Following the construction above, we simply attach to x *all* the rotated sets gY, where g runs over the complete collection of rotations that move the origin to x, and repeat this process for all $x \in \mathcal{X}$. Now define

$$\delta(X) := \bigcup_{x \in X} \bigcup_{\{g \in \Gamma \,:\, g\omega = x\}} gY \tag{7.49}$$

It is plausible, and will be proved below, that in this way one indeed obtains a dilation δ that is rotation invariant. Moreover, we will show that all rotation-invariant dilations are of this form. The adjoint erosion of (7.49) is formed by associating to a subset X the collection of points $y \in \mathcal{X}$ such that $gY \subseteq X$ for *all* rotations $g \in \Gamma$ that move the origin to y. More details on the spherical case can be found in [23].

The construction sketched above for the sphere can in fact be generalized to any homogeneous space $(\Gamma, \mathcal{X})$, for example, to the plane with the translation-rotation group $E^+(3)$ as the acting group. For the latter case a sketch of the Γ-dilation by a structuring element Y consisting of a line segment is given in Figure 2c. From this figure it is clear that the line segment may be replaced by a disk, which is rotation invariant. More generally we will see below that dilations/erosions on any homogeneous space can be reduced to a form involving structuring elements that are invariant under all elements of Γ that leave the origin ω invariant.

In contrast to the Euclidean case, the dilations and erosions on the lattice $\mathcal{P}(\mathcal{X})$ constructed above are not the building blocks for other morphological transformations such as the opening (7.42). For this purpose we have to introduce dilations and erosions between the distinct lattices $\mathcal{P}(\mathcal{X})$ and $\mathcal{P}(\Gamma)$. Before we can explain this, we need to develop the ideas sketched above in full detail, to which we proceed now.

B. General Strategy

From now on we will write $\mathcal{L}$ instead of $\mathcal{P}(\mathcal{X})$ and $\tilde{\mathcal{L}}$ instead of $\mathcal{P}(\Gamma)$. It will be convenient to introduce a notation that clearly distinguishes between subsets of

$\mathscr{X}$ and subsets of Γ, as well as between mappings on $\mathscr{L}$ and mappings on $\tilde{\mathscr{L}}$; see Table 2.

Elements of the set $\mathscr{X}$ will be denoted by lower case letters x, y, z, x′, y′, z′; subsets of $\mathscr{X}$ by the corresponding capitals X, Y, Z, X′, Y′, Z′. For the group Γ we use the notation of the previous section, i.e. g, h, k, g′, h′, k′ for the group elements and G, H, K, G′, H′, K′ for the subsets of Γ. Mappings $\tilde{\mathscr{L}} \to \tilde{\mathscr{L}}$ will be denoted by a tilde, e.g., $\tilde{\psi}$, to distinguish the from mappings $\psi : \mathscr{L} \to \mathscr{L}$

As explained in Section II.C, the fact that Γ is a group acting on $\mathscr{X}$ means that each $g \in \Gamma$ defines a mapping $\mathscr{X} \to \mathscr{X} : x \mapsto gx$. This mapping can be extended to subsets of $\mathscr{X}$ as in Section III.A: define, for each $g \in \Gamma$, a mapping γ_g by

$$\gamma_g: \mathscr{L} \to \mathscr{L}, \qquad \gamma_g(X) := \{gx: x \in X\} \tag{7.50}$$

where instead of $\gamma_g(X)$ we usually write gX. We call the mapping (7.50) translation by g. On $\tilde{\mathscr{L}} = \mathscr{P}(\Gamma)$ we have the left and right translations,

$$\lambda_g: \tilde{\mathscr{L}} \to \tilde{\mathscr{L}}, \qquad \lambda_g(H) = gH := \{gh: h \in H\} \tag{7.51a}$$

$$\rho_g: \tilde{\mathscr{L}} \to \tilde{\mathscr{L}}, \qquad \rho_g(H) = Hg := \{hg: h \in H\} \tag{7.51b}$$

It is easy to check that γ_g commutes with unions, intersections, and complements on $\mathscr{L}$ for each $g \in \Gamma$, just as is the case for λ_g and ρ_g on $\tilde{\mathscr{L}}$. So the group Γ induces:

1. An automorphism group of $\mathscr{L} = \mathscr{P}(\mathscr{X})$, acting transitively on $\mathscr{X}$
2. Two mutually commuting automorphism groups of $\tilde{\mathscr{L}} = \mathscr{P}(\Gamma)$, acting *simply* transitively on Γ

In order to avoid a proliferation of symbols we will replace the notation (7.43) for the action of a subset G of Γ on a subset X of $\mathscr{X}$ by

$$GX := \bigcup_{g \in G} gX \tag{7.52}$$

We define a mapping on $\mathscr{L}$ to be *translation-invariant* or a Γ-*mapping* when it commutes with Γ-translations; see (7.41). On $\tilde{\mathscr{L}}$ we have left and right translation invariance; see Section III.A. We also need to study set mappings between $\mathscr{L}$ and $\tilde{\mathscr{L}}$. To define translation invariance for such mappings, we use *left* translations on

Table 2. Notation for the Elements and Subsets of $\mathscr{X}$ and Γ

Reference space	Elements	Subsets	Lattice
$\mathscr{X}$	x, y, z, x', y', z'	X, Y, Z, X', Y', Z'	$\mathscr{L} = \mathscr{P}(\mathscr{X})$
Γ	g, h, k, g', h', k'	G, H, K, G', H', K'	$\tilde{\mathscr{L}} = \mathscr{P}(\Gamma)$

$\tilde{\mathcal{L}}$. That is, a mapping $\psi : \mathcal{L} \to \tilde{\mathcal{L}}$ is called *translation-invariant* or *Γ-invariant* when, for all $g \in \Gamma$,

$$\psi\gamma_g = \lambda_g\psi, \quad \text{i.e.,} \quad \psi(gX) = g\psi(X) \qquad \forall X \in \mathcal{L} \tag{7.53}$$

and similarly for mappings $\psi : \tilde{\mathcal{L}} \to \mathcal{L}$.

Our strategy in defining dilations, erosions, etc. on the lattice $\mathcal{L}$ and proving representation theorems for such mappings is now as follows: given a mapping ψ on $\mathcal{L}$ we extend or "lift" it to a mapping $\tilde{\psi}$ on $\tilde{\mathcal{L}}$. Then we apply the results of Section III on $\tilde{\mathcal{L}}$ and finally "project" the results back to the original lattice $\mathcal{L}$; see Figure 3. We proceed by describing the lift and projection in detail.

C. Lift and Projection and Their Properties

We start by defining the projection of subsets of Γ to subsets of $\mathcal{X}$. To do this the canonical projection $\pi_\omega : \Gamma \to \mathcal{X}$ of Definition 2.14 is extended to subsets of $\mathcal{X}$ in a standard way.

Definition 4.1. The *projection* $\pi : \tilde{\mathcal{L}} \to \mathcal{L}$ is the mapping given by

$$\pi(G) = \{\pi_\omega(g) : g \in G\} = \{g\omega : g \in G\} \tag{7.54}$$

The preimage of a point $x \in \mathcal{X}$ under π_ω is the set

$$\vartheta_\omega(x) := \{g \in \Gamma : \pi_\omega(g) = x\} = g_x\Sigma \tag{7.55}$$

Here Σ is the stabilizer of the origin ω in $\mathcal{X}$ and $\vartheta_\omega(x)$ is the left coset associated to the point x, that is, the collection of all elements of the group Γ that map ω to x (see Section II.C). Again we can extend this mapping to subsets of $\mathcal{X}$.

Definition 4.2. The *lift* $\vartheta : \mathcal{L} \to \tilde{\mathcal{L}}$ is the mapping given by

$$\vartheta(X) = \bigcup_{x \in X} \vartheta_\omega(x) = \{g \in G : \pi_\omega(g) \in X\} \tag{7.56}$$

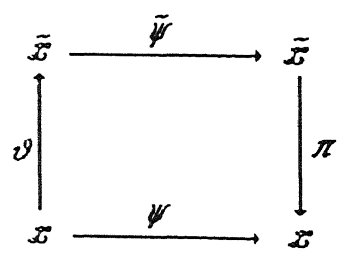

Figure 3. Relations between mappings on $\mathcal{L}$ and $\tilde{\mathcal{L}}$. π and ϑ are the projection (7.54) and the lift (7.56).

So $\vartheta(X)$ consists of all elements of Γ that project onto elements of X under π_ω.

For the subsequent discussion it will be useful to have a simple geometric picture of the lift and projection defined above. Such a graphical interpretation is introduced in the next two examples.

Example 4.3. The translation-rotation group on the plane [24]. Consider the plane $\mathscr{X} = \mathbf{R}^2$, acted upon by the translation-rotation group Γ; see Example 2.19. Elements of Γ are parameterized as $g_{t,\phi}$, with $t \in \mathbf{R}^2$, $\phi \in [0, 2\pi)$. The action is given by

$$g_{t,\phi}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} + \begin{pmatrix}t_1\\t_2\end{pmatrix}, \qquad x = \begin{pmatrix}x_1\\x_2\end{pmatrix} \in \mathbf{R}^2 \tag{7.57}$$

Let the origin ω be the point (0, 0). The stabilizer Σ is equal to the group $\mathscr{R}$ of rotations around the origin. If we define τ_y as the unique (Euclidean) translation that maps the origin to the point y, the lift of any point y is the coset $\tau_y\mathscr{R}$. To represent elements of Γ graphically, we associate to each point of the plane the collection of all unit vectors emanating from that point. A *pointer* p is a pair $(x, \mathbf{v})$, where $x \in \mathbf{R}^2$ and $\mathbf{v} \in S^1$ a unit vector. We call x the *base-point* of p. Define a *base-pointer* b to be the pair $(\omega, \mathbf{e}_1)$, where $\mathbf{e}_1 = (1, 0)$. Given any pointer $p = (x, \mathbf{v})$, there is a *unique* transformation in the group Γ which maps b to p (this can also be expressed by saying that Γ acts regularly on pairs of equidistant points). So, although Γ does not act regularly on $\mathscr{X}$, it acts regularly on the extended space $\tilde{\mathscr{X}} = \{(x, \mathbf{v}) : x \in \mathscr{X}, \mathbf{v} \in S^1\}$ of pointers. Hence, as we have seen in Section II.C, there is a 1–1 correspondence between $\tilde{\mathscr{X}}$ and Γ. Any pointer $p \in \tilde{\mathscr{X}}$ represents a unique element of Γ: if $p = (x, \mathbf{v})$, where $\mathbf{v} = (\cos\phi, \sin\phi)$, then this element is $g_{x,\phi}$. In this representation, $\mathscr{R}$ is the set of unit vectors attached to the origin and the left coset $\tau_y\mathscr{R}$ is the collection of all unit vectors attached to $y \in \mathbf{R}^2$. The reader may refer to Figure 4, where we use a hexagonal

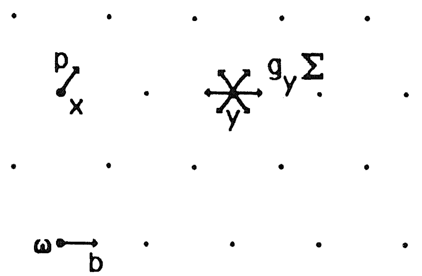

Figure 4. Representation of elements of the subgroup $\mathscr{H}$ of the Euclidean motion group. b, base-pointer; p, pointer with base-point x; $g_y\Sigma$, the collection of group elements that map the origin ω to y. Each pointer represents a unique group element.

lattice as the set $\mathscr{X}$ and where the group Γ is the hexagonal group $\mathscr{H}$ (see Example 2.19). The allowed rotations are over an integer multiple of $\pi/3$ and a coset is represented by the six unit vectors corresponding to the allowed rotations. On the hexagonal lattice, subsets of $\mathscr{X}$ are indicated by collections of heavy dots and subsets of $\mathscr{H}$ by heavy dots with one or more unit vectors attached to them.

Example 4.4. The rotation group on the sphere [23]. Consider the unit 2-sphere $\mathscr{X} = S^2$, acted upon by the three-dimensional rotation group $\Gamma = SO(3)$; see Example 2.20. As in the previous example, define a pointer p to be a pair $(x, \mathbf{v})$, where $x \in S^2$ and $\mathbf{v}$ a unit tangent vector at the base-point x. Choose the north pole $\mathcal{N}$ as the origin of the sphere, and define a base-pointer b to be an arbitrarily chosen (fixed) pointer with base-point $\mathcal{N}$. Then again any pointer p represents a unique rotation, i.e., the one that maps b to p. The stabilizer Σ and the left coset $g_x\Sigma$ are represented by the collection of unit tangent vectors attached to $\mathcal{N}$ and x, respectively; see Figure 5, where we have drawn six representative pointers belonging to $g_x\Sigma$.

Having introduced this geometric picture, it is now easy to visualize the action of the mappings ϑ and π defined above. Any subset G of Γ is represented by a set of pointers and π maps G to the set of base-points of the pointers in G (Figure 6a). Conversely, ϑ maps a subset X of $\mathscr{X}$ to the set of pointers in Γ that have their base-points in X (Figure 6b).

In a moment we will list several properties of the lift ϑ and the projection π that will enable us to settle the case of a transitive group on $\mathscr{X}$ by making use of the results for the simply transitive case. But first we need to introduce a special dilation and corresponding erosion.

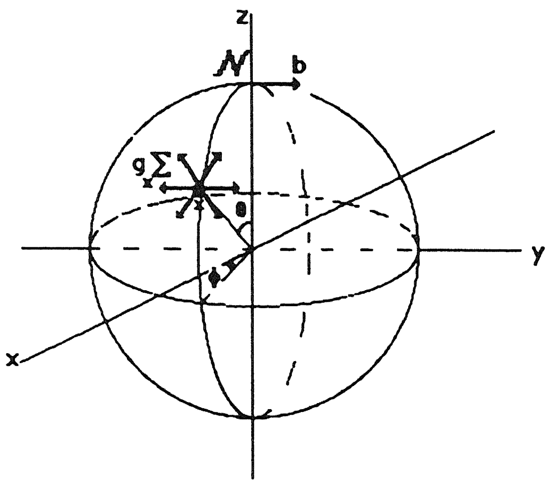

Figure 5. Representation of elements from the rotation group SO(3). b, base-pointer; $g_x\Sigma$, left coset representing all rotations that map $\mathcal{N}$ to x.

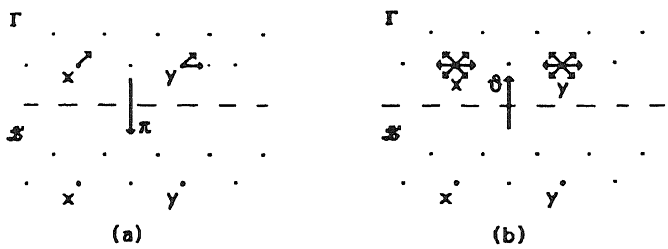

Figure 6. (a) Action of π on a subset of Γ; (b) action of θ on a subset of $\mathscr{X}$.

Definition 4.5. The dilation $\tilde{\delta}_\Sigma : \tilde{\mathscr{L}} \to \tilde{\mathscr{L}}$ and erosion $\tilde{\varepsilon}_\Sigma : \tilde{\mathscr{L}} \to \tilde{\mathscr{L}}$ by the stabilizer Σ are the mappings defined by

$$\tilde{\delta}_\Sigma(G) = G \overset{\Gamma}{\oplus} \Sigma, \qquad \tilde{\varepsilon}_\Sigma(G) = G \overset{\lambda}{\ominus} \Sigma \tag{7.58}$$

Here we have used the set-product and left residual on $\tilde{\mathscr{L}} = \mathscr{P}(\Gamma)$; see Section III.B. Clearly, $(\tilde{\varepsilon}_\Sigma, \tilde{\delta}_\Sigma)$ is a λ-adjunction on $\tilde{\mathscr{L}}$. Furthermore, we have

Lemma 4.6. The *adjunction* $(\tilde{\varepsilon}_\Sigma, \tilde{\delta}_\Sigma)$ satisfies

$$\tilde{\varepsilon}_\Sigma = \tilde{\varepsilon}_\Sigma^2 = \tilde{\delta}_\Sigma \tilde{\varepsilon}_\Sigma \tag{7.59a}$$

$$\tilde{\delta}_\Sigma = \tilde{\delta}_\Sigma^2 = \tilde{\varepsilon}_\Sigma \tilde{\delta}_\Sigma \tag{7.59b}$$

This lemma says that $\tilde{\varepsilon}_\Sigma$ is not only an erosion but also a (morphological) opening, and $\tilde{\delta}_\Sigma$ is not only a dilation but also a (morphological) closing.

The effect of the closing $\tilde{\delta}_\Sigma$ on a subset G of Σ is to make G "Σ-closed," that is, invariant under right multiplication by Σ. To put it differently, $\tilde{\delta}_\Sigma$ augments G by all group elements that are equivalent to some $g \in G$ (recall from Section II.C that two group elements $g, h \in \Gamma$ are called equivalent if $g\omega = h\omega$). Pictorially, any pointer $p = (x, \mathbf{v})$ is extended to the set of pointers $p\Sigma := \{(x, \mathbf{v}') : \mathbf{v}' \in S^1\}$. Similarly, the opening $\tilde{\varepsilon}_\Sigma$ extracts from a subset G of Γ all the "complete cosets" present in G, that is, the subset $G^* \subseteq G$ which is such that if $g \in G^*$, all the elements equivalent to g are also in G^*.

We also need to introduce a modified projection as follows.

Definition 4.7. Let π be the projection (7.54) and $\tilde{\varepsilon}_\Sigma$ the erosion (7.59a). Then $\pi_\Sigma : \tilde{\mathscr{L}} \to \mathscr{L}$ is the *modified projection* defined by

$$\pi_\Sigma = \pi \tilde{\varepsilon}_\Sigma \tag{7.60}$$

It follows from this definition that $\pi_\Sigma(G) = \{g\omega : g\Sigma \subseteq G\}$, so the projection π_Σ maps $G \in \tilde{\mathscr{L}}$ to the subset of $\mathscr{L}$ consisting of only those base-points of pointers in G to which a complete set of unit vectors is attached.

The next proposition contains a collection of properties of the operators ϑ and π.

Proposition 4.8. Let $\pi : \tilde{\mathcal{L}} \to \mathcal{L}$, $\vartheta : \mathcal{L} \to \tilde{\mathcal{L}}$, and $\pi_\Sigma : \tilde{\mathcal{L}} \to \mathcal{L}$ be defined by (7.54), (7.56), and (7.60), respectively. Then the following hold:

(a) π, ϑ, π_Σ are increasing and translation-invariant.
(b) ϑ commutes with unions and intersections.
(c) π commutes with unions, π_Σ commutes with intersections.
(d) $\vartheta = \tilde{\varepsilon}_\Sigma \vartheta = \tilde{\delta}_\Sigma \vartheta$.
(e) $\pi\vartheta = \mathrm{id}_{\mathcal{L}}$, $\vartheta\pi = \tilde{\delta}_\Sigma$; $\pi_\Sigma\vartheta = \mathrm{id}_{\mathcal{L}}$, $\vartheta\pi_\Sigma = \tilde{\varepsilon}_\Sigma$.
(f) $X \subseteq Y \Leftrightarrow \vartheta(X) \subseteq \vartheta(Y)$; $X \Uparrow Y \Leftrightarrow \vartheta(X) \Uparrow \vartheta(Y)$.
(g) $(\vartheta(X))^c = \vartheta(X^c)$, $(\pi(G))^c = \pi_\Sigma(G^c)$.
(h) (ϑ, π) forms an adjunction between $\mathcal{L}$ and $\tilde{\mathcal{L}}$: $\pi(G) \subseteq X \Leftrightarrow G \subseteq \vartheta(X)$.
(i) (π_Σ, ϑ) forms an adjunction between $\tilde{\mathcal{L}}$ and $\mathcal{L}$: $\vartheta(X) \subseteq G \Leftrightarrow X \subseteq \pi_\Sigma(G)$.

Here $X \Uparrow Y$ (X "hits" Y) is a shorthand notation for $X \cap Y \neq \emptyset$.

We discuss a few items of this proposition, which can be nicely illustrated by the pictorial representation introduced above; see [24].

Entry (c): the fact that π does not commute with intersections can easily be seen by considering two pointers $p = (x, \mathbf{v})$ and $q = (x, \mathbf{v}')$ with the same base-point x and $\mathbf{v} \neq \mathbf{v}'$. Clearly $\{p\} \cap \{q\} = \emptyset$, hence $\pi(\{p\} \cap \{q\}) = \emptyset$, but $\pi(\{p\}) \cap \pi(\{q\}) = \{x\} \cap \{x\} = \{x\}$. If we first open subsets of Γ by $\tilde{\varepsilon}_\Sigma$, we obtain complete cosets $p\Sigma$. Projecting sets of this form obviously commutes with intersections, which explains why π_Σ does.

Entry (d): the lift $\vartheta(X)$ of a subset $X \subseteq \mathcal{X}$ consists of complete cosets $p\Sigma$, hence is invariant under the closing by Σ as well as the opening by Σ.

Entry (e): lifting a subset X means pictorially adding a complete set of unit vectors to each point $x \subseteq X$. Projecting the result back by either π or π_Σ gives the original set X. Conversely, start with a pointer $p = (x, \mathbf{v})$: projection by π gives x, lifting this again by ϑ gives the coset $p\Sigma$; hence $\vartheta\pi$ equals the closing $\tilde{\delta}_\Sigma$ by Σ.

The next lemma gives us all the tools we need to derive properties of mappings on $\tilde{\mathcal{L}}$ from those on $\mathcal{L}$.

Lemma 4.9. The following hold:

(a) ψ is an increasing Γ-mapping on $\mathcal{L}$ $\Rightarrow$ $\vartheta\psi\pi$ and $\vartheta\psi\pi_\Sigma$ are increasing λ-mappings on $\tilde{\mathcal{L}}$.
(b) (ε, δ) is an adjunction on $\mathcal{L}$ $\Rightarrow$ $(\vartheta\varepsilon\pi_\Sigma, \vartheta\delta\pi)$ is an adjunction on $\tilde{\mathcal{L}}$.
(c) γ/ϕ is an opening/a closing on $\mathcal{L}$ $\Rightarrow$ $\vartheta\gamma\pi_\Sigma/\vartheta\phi\pi$ is an opening/a closing on $\tilde{\mathcal{L}}$.

Conversely,

(a′) $\tilde{\psi}$ is an increasing λ-mapping on $\tilde{\mathcal{L}}$ $\Rightarrow$ $\pi\tilde{\psi}\vartheta$ and $\pi_\Sigma\tilde{\psi}\vartheta$ are increasing Γ-mappings on $\mathcal{L}$.

(b′) $(\tilde{\varepsilon}, \tilde{\delta})$ is an adjunction on $\tilde{\mathcal{L}}$ $\Rightarrow$ $(\pi_\Sigma \tilde{\varepsilon} \vartheta, \pi \tilde{\delta} \vartheta)$ is an adjunction on $\mathcal{L}$.

(c′) $\tilde{\gamma}/\tilde{\phi}$ is an opening/a closing on $\tilde{\mathcal{L}}$ $\Rightarrow$ $\pi \tilde{\gamma} \vartheta / \pi_\Sigma \tilde{\phi} \vartheta$ is an opening/a closing on $\mathcal{L}$.

Remark 4.10. In a similar manner one can show that if (ε, δ) is an adjunction on $\mathcal{L}$, then $(\vartheta\varepsilon, \delta\pi)$ is an adjunction between $\mathcal{L}$ and $\tilde{\mathcal{L}}$ and $(\varepsilon\pi_\Sigma, \vartheta\delta)$ is an adjunction between $\tilde{\mathcal{L}}$ and $\mathcal{L}$. Similarly, if $(\varepsilon^\uparrow, \delta^\downarrow)$ is an adjunction between $\mathcal{L}$ and $\tilde{\mathcal{L}}$, then $(\varepsilon^\uparrow \pi_\Sigma, \vartheta\delta^\downarrow)$ and $(\pi_\Sigma \varepsilon^\downarrow, \delta^\downarrow \vartheta)$ are adjunctions on $\tilde{\mathcal{L}}$ and $\mathcal{L}$, respectively. And so on. □

D. Characterization Theorems

In this subsection we give a characterization of Γ-adjunctions, Γ-openings and Γ-closings, and finally of (increasing) Γ-mappings on $\mathcal{L}$, using the corresponding results on $\tilde{\mathcal{L}}$ of Section III.D.

First we consider adjunctions. As a preliminary we need

Definition 4.11. Let Γ be a permutation group on $\mathcal{X}$, with Σ the stabilizer of the origin ω in $\mathcal{X}$. A subset X of $\mathcal{X}$ is called Σ-invariant if $X = \overline{X}$, where $\overline{X} := \Sigma X = \bigcup_{s \in \Sigma} sX$. If X is not Σ-invariant, $\overline{X}$ is called the *Σ-invariant extension* of X.

Examples of Σ-invariant sets are given in [24] for the Euclidean motion group. Now we can state our first result.

Proposition 4.12. Representation of Γ-adjunctions. The pair (ε, δ) is a Γ-invariant adjunction on $\mathcal{P}(\mathcal{X})$ if and only if, for some $H \in \mathcal{P}(\Gamma)$, it is true that $\delta = \pi \tilde{\delta}_H \vartheta$ and $\varepsilon = \pi_\Sigma \tilde{\varepsilon}_H \vartheta$, that is,

$$\delta(X) = \pi[\vartheta(X) \overset{\Gamma}{\oplus} H], \qquad \varepsilon(X) = \pi_\Sigma[\vartheta(X) \overset{\lambda}{\ominus} H] \tag{7.61}$$

where $(\tilde{\varepsilon}_H, \tilde{\delta}_H)$ is the left-invariant adjunction with structuring element H on $\mathcal{P}(\Gamma)$ given in Proposition 3.18. Equivalently,

$$\delta(X) = \delta_Y^\Gamma(X) := \pi[\vartheta(X) \overset{\Gamma}{\oplus} \vartheta(Y)] = \pi\,[\vartheta(X) \overset{\Gamma}{\oplus} \vartheta(\overline{Y})] \tag{7.62a}$$

$$\varepsilon(X) = \varepsilon_Y^\Gamma(X) := \pi_\Sigma[\vartheta(X) \overset{\lambda}{\ominus} \vartheta(Y)] = \pi[\vartheta(X) \overset{\lambda}{\ominus} \vartheta(\overline{Y})] \tag{7.62b}$$

where $Y = \pi(H)$ and $\overline{Y}$ is the Σ-invariant extension of Y. In particular $(\varepsilon_Y^\Gamma, \delta_Y^\Gamma)$ is invariant under the substitution $Y \to \overline{Y}$.

Corollary 4.13. Geometric interpretation of Γ-adjunctions. The pair (ε, δ) is a Γ-invariant adjunction on $\mathcal{P}(\mathcal{X})$ if and only if, for some $Y \subseteq \mathcal{X}$,

$$\delta(X) = \delta_Y^\Gamma(X) = \bigcup_{g \in \vartheta(X)} gY \tag{7.63a}$$

$$= \{y \in \mathcal{X}\colon g\check{Y}^* \Uparrow X \text{ for some } g \in \vartheta_\omega(y)\}$$

$$\varepsilon(X) = \varepsilon_Y^\Gamma(X) = \bigcap_{g \in \check{\vartheta}(X^c)} g\,\hat{Y}^* \qquad (7.63b)$$
$$= \{y \in \mathscr{X}: gY \subseteq X \text{ for all } g \in \vartheta_\omega(y)\}$$

where $\check{Y}^* = \pi(\check{\vartheta}(Y))$ and $\hat{Y}^* = (\check{Y}^*)^c$. Here $\vartheta_\omega(y)$ is the coset representing y, that is, the collection of all group elements that map ω to y.

Here we have written $\check{\vartheta}(X)$ instead of $(\vartheta(X))^{\vee}$, and $Y \Uparrow X$ (read: Y hits X) is a shorthand notation for $Y \cap X \neq \emptyset$.

The proposition above shows that any dilation on $\mathscr{L}$ can be reduced to a dilation δ_Y^Γ involving a Σ-*invariant* structuring element Y; the same is true for erosions. Also, in (7.63a) we may replace "some" by "all": since $\check{Y}$ is Σ-invariant (easy to show), $g\check{Y}^*$ will hit X for *all* $g \in \vartheta_\omega(y)$ as soon as it hits X for *some* element g_y of this coset. For example, in the case of the Euclidean motion group acting on the plane, it would be natural to take for g_y the Euclidean translation τ_y that maps ω to y. Also, (7.63b) may equivalently be written as $\varepsilon_Y^\Gamma(X) = \{y \in \mathscr{X}: g_y\overline{Y} \subseteq X\}$.

A second consequence of the corollary is that the morphological opening $\varepsilon_Y^\Gamma\delta_Y^\Gamma$ and closing $\delta_Y^\Gamma\varepsilon_Y^\Gamma$ associated to an adjunction $(\varepsilon_Y^\Gamma, \delta_Y^\Gamma)$ with Y an arbitrary subset of $\mathscr{X}$ are also equivalent to the morphological opening or closing by the structuring element $\overline{Y}$. This raises the question of how to decompose Γ-openings that are not reducible to openings by a Σ-invariant structuring element. Consider the *structural* opening and closing by a subset Y of $\mathscr{X}$ defined by

$$\gamma_Y^\Gamma(X) = \bigcup_{g\in\Gamma}\{gY: gY \subseteq X\} \qquad (7.64a)$$

$$\phi_Y^\Gamma(X) = \bigcap_{g\in\Gamma}\{gY: gY \supseteq X\} \qquad (7.64b)$$

In other words, $\gamma_Y^\Gamma(X)$ is the union of all translates gY of Y that are included in X. For example, let X be a union of line segments of varying sizes in the plane and Y a line segment of size L with center at the origin. Take Γ equal to the translation-rotation group $\mathcal{M} := E^+(3)$. Then $\gamma_Y^{\mathcal{M}}(X)$ consists of the union of all segments in X of size L or larger, but $\delta_Y^{\mathcal{M}}\varepsilon_Y^{\mathcal{M}}(X) = \gamma_{\Sigma Y}^{\mathcal{M}}(X) = \emptyset$, since $\overline{Y}$ is a disk of radius $L/2$ and does not fit anywhere in X; see Figure 7.

So in general we can not build the opening γ_Y^Γ from a Γ-erosion ε_Y^Γ on $\mathscr{P}(\mathscr{X})$ followed by a Γ-dilation δ_Y^Γ on $\mathscr{P}(\mathscr{X})$. However, Theorem 2.7 of [5] guarantees that, given an opening or closing ψ on a lattice $\mathscr{L}$, there exists another lattice $\mathscr{L}'$ such that ψ can be decomposed into erosions and dilations between $\mathscr{L}$ and $\mathscr{L}'$. In the present case the situation is clarified by the next proposition.

Proposition 4.14. Decomposition of structural openings and closings. The structural opening $\gamma_Y^\Gamma : \mathscr{L} \to \mathscr{L}$ defined by (7.64a) is the projection of the Γ-opening $\tilde{\gamma}_{\vartheta(Y)}$ on $\tilde{\mathscr{L}}$, that is,

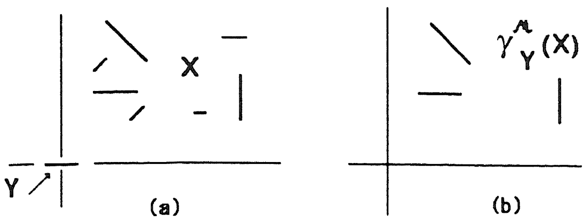

Figure 7. (a) X, a subset of the plane consisting of line segments; Y, a segment of size L at the origin. (b) opening $\gamma_Y^{\mathcal{M}}(X)$ by the Euclidean motion group $\mathcal{M}$. The opening $\delta_Y^{\mathcal{M}}\varepsilon_Y^{\mathcal{M}}(X) = \gamma_Y^{\mathcal{M}}(X)$ is empty.

$$\gamma_Y^\Gamma(X) = (\pi\tilde{\delta}_{\vartheta(Y)}\tilde{\varepsilon}_{\vartheta(Y)}\vartheta)(X) = \pi[\{\vartheta(X) \overset{\lambda}{\ominus} \vartheta(Y)\} \overset{\Gamma}{\oplus} \vartheta(Y)] \tag{7.65}$$

with $(\tilde{\varepsilon}_{\vartheta(Y)}, \tilde{\delta}_{\vartheta(Y)})$ the left-invariant adjunction $\tilde{\mathcal{L}}$ with structuring element $\vartheta(Y)$. Equivalently, γ_Y^Γ is the product of a Γ-erosion $\varepsilon^\uparrow : \mathcal{L} \to \tilde{\mathcal{L}}$ followed by a Γ-dilation $\delta^\downarrow : \tilde{\mathcal{L}} \to \mathcal{L}$, where $(\varepsilon^\uparrow, \delta^\downarrow) := (\tilde{\varepsilon}_{\vartheta(Y)}\vartheta, \pi\tilde{\delta}_{\vartheta(Y)})$ is a Γ-adjunction between $\mathcal{L}$ and $\tilde{\mathcal{L}}$.

Similarly, the structural closing $\phi_Y^\Gamma : \mathcal{L} \to \mathcal{L}$ defined by (7.64b) has the decomposition

$$\phi_Y^\Gamma(X) = (\pi_\Sigma\tilde{\varepsilon}_{\check{\vartheta}(Y^c)}\tilde{\delta}_{\check{\vartheta}(Y^c)}\vartheta)(X) = \pi[\{\vartheta(X) \overset{\Gamma}{\oplus} \check{\vartheta}(Y^c)\} \overset{\lambda}{\ominus} \check{\vartheta}(Y^c)] \tag{7.66}$$

Equivalently, ϕ_Y^Γ has the decomposition $\phi_Y^\Gamma = \varepsilon^\downarrow\delta^\uparrow$ where $(\varepsilon^\downarrow, \delta^\uparrow) := (\pi_\Sigma\tilde{\varepsilon}_{\check{\vartheta}(Y^c)}, \tilde{\delta}_{\check{\vartheta}(Y^c)}\vartheta)$ is a Γ-adjunction between $\tilde{\mathcal{L}}$ and $\mathcal{L}$

Remark 4.15. We could have introduced the following generalization of the Minkowski operations on $\mathcal{P}(\mathcal{X})$:

$$X \overset{\Gamma}{\oplus} Y := \pi[\vartheta(X) \overset{\Gamma}{\oplus} \vartheta(Y)], \qquad X \overset{\lambda}{\ominus} Y := \pi_\Sigma[\vartheta(X) \overset{\lambda}{\ominus} \vartheta(Y)]$$

where $\overset{\Gamma}{\oplus}$ and $\overset{\lambda}{\ominus}$ denote the set-product and left-residual on $\mathcal{P}(\Gamma)$, respectively. But since the basic building blocks for structural Γ-openings and Γ-closings are not the dilation $X \overset{\Gamma}{\oplus} Y$ and erosion $X \overset{\lambda}{\ominus} Y$ on $\mathcal{L}$, but dilations/erosions between $\mathcal{L}$ and $\tilde{\mathcal{L}}$, we have refrained from doing this. □

By Proposition 2.10 (see also [27]), every Γ-opening $\gamma : \mathcal{L} \to \mathcal{L}$ is a union of structural openings γ_Y^Γ, where Y ranges over a subset $\mathcal{Y} \subseteq \mathcal{L}$. Combining this with Proposition 4.14, we therefore can decompose any Γ-opening into Γ-openings of the form $\pi\tilde{\delta}_{\vartheta(Y)}\tilde{\varepsilon}_{\vartheta(Y)}\vartheta$. The formulation of the precise result, paralleling Theorem 3.25, is left to the reader.

Finally, we discuss the characterization of increasing Γ-mappings. We first need:

Definition 4.16. Let $\psi : \mathcal{L} \to \mathcal{L}$ be an arbitrary mapping. The *kernel* of ψ is the subset of $\mathcal{L}$ defined by $\mathcal{V}(\psi) = \{X \in \mathcal{L}: \omega \in \psi(X)\}$, where ω is the origin of $\mathcal{X}$.

Lemma 4.17. Let $\psi : \mathcal{L} \to \mathcal{L}$ be a mapping with lift $\tilde{\psi}: \tilde{\mathcal{L}} \to \tilde{\mathcal{L}}$ defined by $\tilde{\psi} = \vartheta\psi\pi$. Then,

$$\mathcal{V}(\tilde{\psi}) = \{G \in \tilde{\mathcal{L}}: \pi(G) \in \mathcal{V}(\psi)\}$$

Now we can state:

Theorem 4.18. Representation of increasing Γ-mappings. A mapping $\psi : \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{X})$ is an increasing Γ-mapping if and only if ψ is a union of projected erosions or, equivalently, an intersection of projected dilations:

$$\psi(X) = \bigcup_{Y \in \mathcal{V}(\psi)} \pi[\vartheta(X) \overset{\lambda}{\ominus} \vartheta(Y)] \tag{7.67a}$$

$$= \bigcap_{Y \in \mathcal{V}(\psi')} \pi_\Sigma[\vartheta(X) \overset{\Gamma}{\oplus} \check{\vartheta}(Y)] \tag{7.67b}$$

where $\mathcal{V}(\psi)$ is the kernel of ψ and ψ' is the Boolean dual of ψ.

It is important to notice that, in contrast to the situation in the simply transitive case (Corollary 3.24), ψ is in general *not* a union of Γ-erosions on $\mathcal{P}(\mathcal{X})$: $X \mapsto \pi[\vartheta(X) \overset{\lambda}{\ominus} \vartheta(Y)]$ is not an erosion since π is not.* Nor is ψ an intersection of Γ-dilations on $\mathcal{P}(\mathcal{X})$: $X \mapsto \pi_\Sigma[\vartheta(X) \overset{\Gamma}{\oplus} \check{\vartheta}(Y)]$ is not a dilation since π_Σ is not.* For example, the opening γ_Y^Γ cannot be written as a union of Γ-erosions ε_Z on $\mathcal{L}$; in the case of Figure 6, $\varepsilon_Z(X) = \varepsilon_{\check{Z}}(X)$ is empty for all $Z \neq \{\omega\}$ and equal to X when $Z = \{\omega\}$.

The above theorem can be extended to arbitrary translation-invariant mappings (see Theorem 3.23 of Section III.D). The result is given here without proof.

Define, for $F, G, H \in \tilde{\mathcal{L}}$ and $W, X, Y \in \mathcal{L}$,

$$G \overset{\lambda}{\circledS} (F,H) = \{g \in \Gamma : gF \subseteq G \subseteq gK\}$$

and

$$[F, H] = \{G \in \tilde{\mathcal{L}}: F \subseteq G \subseteq H\}, \qquad [W, Y] = \{X \in \mathcal{L}: W \subseteq X \subseteq Y\}$$

Then one can prove:

Theorem 4.19. Representation of Γ-mappings. The *mapping* $\psi : \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{X})$ is a Γ-mapping if and only if

$$\psi(X) = \bigcup_{[W,Y] \in \mathcal{W}(\psi)} \pi[\vartheta(X) \overset{\lambda}{\circledS} (\vartheta(W), \vartheta(Y))]$$

* An exception occurs when Y is Σ-invariant.

where $\mathcal{W}(\psi) = \{(W,Y) \in \mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{X})\colon [W,Y] \subseteq \mathcal{V}(\psi)\}$ with $\mathcal{V}(\psi)$ the kernel of ψ.

E. The Role of Symmetry Groups in Shape Description

In this final subsection we make a few succinct remarks that are relevant in the present context and on which further research is needed.

A first remark concerns the problem how to define "shape," which is known to present great difficulties and is a recurring theme in the image processing literature. Often, shape is defined as referring to those properties of geometric figurs which are invariant under the Euclidean similarity group [14]. Intuitively, one first has to bring figures to a standard location, orientation, and scale before being able to "compare" them. Now it is not necessary to restrict oneself to the similarity group, although in the absence of any form of group invariance there is no way at all to compare figures. In the present context the following definition seems appropriate.

Definition 4.20. Let $\mathcal{X}$ be a set, Γ a group acting on $\mathcal{X}$. Two subsets X, Y of $\mathcal{X}$ are said to have *the same shape with respect to* Γ, or *the same* Γ*-shape*, if they are Γ-equivalent, meaning that there is a $g \in \Gamma$ such that $Y = gX$. If no such $g \in \Gamma$ exist, X and Y are said to have *different* Γ*-shape*.

In essence this definition goes back to F. Klein's Erlanger Programm (1872), which considers geometry to be the study of transformation groups and the properties invariant under these groups [15]. So in Euclidean morphology, all translates of a set X by the Euclidean translation group $\mathcal{T}$ have the same $\mathcal{T}$-shape. Adding rotations to get the Euclidean motion group $\mathcal{M}$, rotated versions of X or its translates have the same $\mathcal{M}$-shape as X. Extreme cases are (1) $\Gamma = \{\mathrm{id}\}$, so that all sets have different shape, and (2) $\Gamma = \mathrm{Sym}_{\mathcal{X}}$, in which case all sets with the same cardinality have the same shape.

A related observation is the following: if the group Γ contains two subgroups Δ_1 and Δ_2 such that any $g \in \Gamma$ has a unique decomposition $g = d_1 d_2$ ($d_1 \in \Delta_1$, $d_2 \in \Delta_2$), where Δ_1 acts itself transitively on $\mathcal{X}$, we can accordingly decompose Γ-mappings. For example, in the case of the Euclidean motion group $\mathcal{M}$, each $g \in \mathcal{M}$ has the form $g = tr$, where $t \in \mathcal{T}$ and $r \in \mathcal{R}$, with $\mathcal{T}$ and $\mathcal{R}$ the two-dimensional translation and rotation group, respectively. It is easy to see that in this case every $\mathcal{M}$-dilation δ has the form $\delta(X) = \delta_B^{\mathcal{M}}(X) := X \oplus B$, where B is an $\mathcal{R}$-invariant structuring element and $\oplus$ denotes the Euclidean Minkowski addition (7.1). Also, we have the following decomposition of the structural $\mathcal{M}$-opening $\gamma_Y^{\mathcal{M}}$ by the structuring element Y:

$$\gamma_Y^{\mathcal{M}}(X) := \bigcup_{g \in \mathcal{M}} \{gY\colon gY \subseteq X\} = \bigcup_{r \in \mathcal{R}} \bigcup_{t \in \mathcal{T}} \{trY\colon trY \subseteq X\}$$
$$= \bigcup_{r \in \mathcal{R}} \gamma_{rY}^{\mathcal{T}}(X),$$

where $\gamma_{rY}^{\mathcal{T}}(X) = [X \ominus (rY)] \oplus (rY)$ is the Euclidean opening by the (not necessarily $\mathcal{R}$-invariant) structuring element rY. This shows that one can perform the opening $\gamma_Y^{\mathcal{M}}$ by carrying out the Euclidean openings $\gamma_{rY}^{\mathcal{T}}$ for all rotated versions of the structuring element Y and taking the union of the results. On the sphere such a decomposition cannot be found, since there is no proper subgroup of SO(3) that acts transitively on the sphere [2, Chapter 1.8].

A decomposition that is possible for any homogeneous space $(\Gamma, \mathcal{X})$ is the following. Consider the partitioning into cosets of the group Γ by the stabilizer Σ, and choose, for all $x \in \mathcal{X}$, a representative g_x for the coset associated to x. Let Δ denote the collection (in general not a group) of all representatives:

$$\Delta = \{g_x : x \in \mathcal{X}\}$$

Then the dilation and erosion by the structuring element Y can be written

$$\delta_Y^{\Gamma}(X) = \bigcup_{g \in \vartheta(X)} gY = \bigcup_{x \in X} \bigcup_{s \in \Sigma} g_x sY = \bigcup_{x \in X} g_x \bar{Y} =: \delta_{\bar{Y}}^{\Delta}(X) \tag{7.68a}$$

$$\begin{aligned} \varepsilon_Y^{\Gamma}(X) &= \{y \in \mathcal{X} : gY \subseteq X \; \forall g \in \vartheta_\omega(y)\} \\ &= \{y \in \mathcal{X} : g_y \bar{Y} \subseteq X\} =: \varepsilon_{\bar{Y}}^{\Delta}(X) \end{aligned} \tag{7.68b}$$

where, for any $Z \subseteq \mathcal{X}$,

$$\delta_Z^{\Delta}(X) := \bigcup_{x \in \mathcal{X}} g_x Z, \qquad \varepsilon_Z^{\Delta}(X) := \{y \in \mathcal{X} : g_y Z \subseteq X\} \tag{7.69}$$

So Γ-dilations/Γ-erosions by the structuring element Y are identical to dilations/erosions "with respect to Δ" by the structuring element $\bar{Y}$.

In the case of structural opening by Y we have

$$\begin{aligned} \gamma_Y^{\Gamma}(X) &= \bigcup_{g \in \Gamma} \{gY : gY \subseteq X\} = \bigcup_{s \in \Sigma} \bigcup_{x \in \mathcal{X}} \{g_x sY : g_x sY \subseteq X\} \\ &= \bigcup_{s \in \Sigma} \gamma_{sY}^{\Delta}(X) \end{aligned} \tag{7.70}$$

where γ_Z^{Δ} is the opening defined by

$$\gamma_Z^{\Delta}(X) := \bigcup_{x \in \mathcal{X}} \{g_x Z : g_x Z \subseteq X\} \tag{7.71}$$

In general (i.e., if Z is not Σ-invariant) neither the dilation δ_Z^{Δ}/erosion ε_Z^{Δ} nor the opening γ_Z^{Δ} possesses invariance properties, unless Δ is a group (i.e., a subgroup of Γ). In that case γ_Z^{Δ} is invariant under Δ-translations (e.g., $\delta_Z^{\Delta}(gX) = g\delta_Z^{\Delta}(X)$ for all $g \in \Delta$), but not under translations by elements $s \in \Sigma$. An example has been given above for the Euclidean motion group, where $\Delta = \mathcal{T}$ and $\Sigma = \mathcal{R}$.

V. DISCUSSION AND EXAMPLES

In this chapter we have generalized Euclidean morphology to arbitrary homogeneous spaces $(\mathscr{X}, \Gamma)$, where the group Γ acting on $\mathscr{X}$ is not necessarily commutative. The case where Γ acts simply transitively on $\mathscr{X}$, considered in Section III, leads to the study of transformations of subsets of an arbitrary group that are invariant under either left or right group translations. The general case where Γ acts transitively on $\mathscr{X}$ has been treated in Section IV by (1) mapping the subsets of $\mathscr{X}$ to subsets of Γ, (2) using the results for the simply transitive case, and (3) projecting back to the original space. The main result is that the scope of mathematical morphology is widened to situations where a noncommutative group is involved. Examples are the translation-rotation group acting on the plane, the rotation group acting on the sphere, or a subgroup of the symmetric group S_n acting on a finite set of n points. Although the emphasis of our work has clearly been on the mathematical framework, we want to finish by mentioning a few areas of possible practical relevance. As indicated in the introduction, we expect that other applications will be found as well.

A first example is that of the search for structures in a graph; see [31, p. 90]. As in Example 2.23, let E' be a set of vertices of a graph and E the complete graph generated by E'. Then $\mathscr{P}(E)$ is the set of subgraphs of E. The group Γ acting on E is generated by the set of all bijections of E' to itself. Let B be an arbitrary subgraph of E, which plays the role of the structuring element. Then the opening γ_B^Γ applied to a graph $X \subseteq E$ is the union of all the subgraphs of X that are isomorphic to B; see Figure 4.9 of [31].

A second example, which occurs in the problem of motion planning for robots, has been considered in great detail in [24]. The problem is to find a path for a robot moving in a plane E with obstacles. Since a robot has a finite size, one can find allowed positions of the (arbitrarily chosen) center of the robot by an erosion $\varepsilon_B^{\mathscr{T}}$ of the obstacle-free space E_{free}, where the structuring element B is the robot itself and $\mathscr{T}$ is the Euclidean translation group. Here we assume that only translations of the robot are allowed. Equivalently, one may perform the dilation $\delta_{\check{B}}^{\mathscr{T}}$ by the reflected set $\check{B}$ on the set E_{ob} of obstacles to find the *forbidden* positions of the center of the robot. In this connection we refer to related work by Ghosh [6], on spatial planning and other problems using the classical Minkowski operations (i.e., only Euclidean translations allowed).

If the robot has rotational degrees of freedom, one has to perform dilations with all rotated versions of the robot (Verwer [35]). In the framework of this chapter, the situation can be described as follows: given a set B (the robot), find the collection of all locations in the plane and all orientations of the robot at those locations such that the displaced robot fits into the obstacle-free space E_{free}. In the terminology of Section IV.C, this is precisely the erosion $\varepsilon_B^{\mathscr{M}\uparrow}(E_{\text{free}}) := \{(t, r) \in$

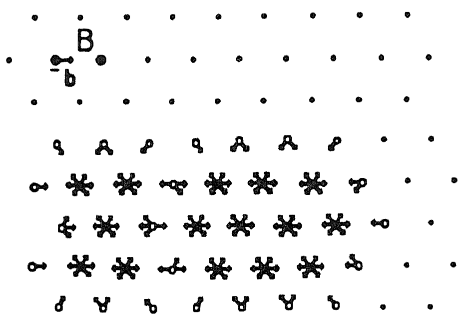

Figure 8. The forbidden positions and orientations for a robot with translational and rotational degrees of freedom are found by the dilation $\delta^{\mathcal{H}}_{B\uparrow}(E_{\mathrm{ob}})$ of the obstacles by the robot B, where $\mathcal{H}$ is the hexagonal group. Heavy dots, the obstacle space; arrows attached to heavy and open dots, the dilated set $\delta^{\mathcal{H}}_{B\uparrow}(E_{\mathrm{ob}})$ of forbidden states. The underlining in B indicates the origin and b is the base pointer (taken from [24]).

'positions' by taking the complement of $\varepsilon^{\mathcal{M}}_{\mathcal{M}} \uparrow (E_{\mathrm{free}})$, which equals the dilation (see [24]) $\delta^{\mathcal{M}}_{B} \uparrow (E_{\mathrm{ob}}) = \bigcup_{b \in E_{ob}} \tau_b \check{\vartheta}(B)$, where τ_b is the unique Euclidean translation that maps the origin to b. An example for the hexagonal grid is given in Figure 8.

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APPENDIX: PROOFS

Proof of Proposition 3.8. In cases where pairs of statements occur which differ only by left-right symmetry, we prove only one of them. In all proofs we use without comment that translations commute with unions and intersections.

(a) $$G \overset{\Gamma}{\oplus} (H \cup K) = \bigcup_{g \in G} g(H \cup K) = \bigcup_{g \in G} (gH \cup gK)$$
$$= (\bigcup_{g \in G} gH) \bigcup (\bigcup_{g \in G} gK) = (G \overset{\Gamma}{\oplus} H) \cup (G \overset{\Gamma}{\oplus} K),$$
which proves the left distributivity of the set product.

(b) Using that multiplication in a group is associative, we find
$$(G \overset{\Gamma}{\oplus} H) \overset{\Gamma}{\oplus} K = \bigcup_{g \in G, h \in H, k \in K} (gh)k = \bigcup_{g \in G, h \in H, k \in K} g(hk)$$
$$= G \overset{\Gamma}{\oplus} (H \overset{\Gamma}{\oplus} K)$$

(c) $(G \cap H) \overset{\lambda}{\ominus} K = \bigcap_{k \in K} (G \cap H)k^{-1} = \bigcap_{k \in K} (Gk^{-1}) \cap (Hk^{-1})$
$= (\bigcap_{k \in K} Gk^{-1}) \bigcap (\bigcap_{k \in K} Hk^{-1})$
$= (G \overset{\lambda}{\ominus} K) \cap (H \overset{\lambda}{\ominus} K).$

(d) $G \overset{\lambda}{\ominus} (H \cup K) = \bigcap_{m \in H \cup K} Gm^{-1} = (\bigcap_{m \in H} Gm^{-1}) \bigcap (\bigcap_{m \in K} Gm^{-1})$
$= (G \overset{\lambda}{\ominus} H) \cap (G \overset{\lambda}{\ominus} K).$

(e) $G \overset{\Gamma}{\oplus} H \subseteq K \Leftrightarrow \forall h \in H\colon Gh \subseteq K \Leftrightarrow \forall h \in H : G \subseteq Kh^{-1}$
$\Leftrightarrow G \subseteq \bigcap_{h \in H} Kh^{-1} = K \overset{\lambda}{\ominus} H$. Similarly,
$G \overset{\Gamma}{\oplus} H \subseteq K \Leftrightarrow \forall g \in G\colon gH \subseteq K \Leftrightarrow \forall g \in G : H \subseteq g^{-1}K$
$\Leftrightarrow H \subseteq \bigcap_{g \in G} g^{-1}K = K \overset{\rho}{\ominus} G.$

(f) $(G \overset{\lambda}{\ominus} H) \overset{\lambda}{\ominus} K = \bigcap_{k \in K} (G \overset{\lambda}{\ominus} H)k^{-1} = \bigcap_{k \in K} (\bigcap_{h \in H} Gh^{-1})k^{-1}$
$= \bigcap_{h \in H, k \in K} G(kh)^{-1} = \bigcap_{m \in K \overset{\Gamma}{\oplus} H} Gm^{-1} = G \overset{\lambda}{\ominus} (K \overset{\Gamma}{\oplus} H).$

In a similar way one proves that $(G \overset{\rho}{\ominus} H) \overset{\rho}{\ominus} K = G \overset{\rho}{\ominus} (H \overset{\Gamma}{\oplus} K)$. Finally, $(G \overset{\rho}{\ominus} H) \overset{\lambda}{\ominus} K = \bigcap_{h \in H, k \in K} h^{-1}Gk^{-1} = (G \overset{\lambda}{\ominus} K) \overset{\rho}{\ominus} H.$

(g) Follows from (b) and the identities (7.26).

(h) $H \overset{\lambda}{\ominus} (gK) = \bigcap_{m \in gK} Hm^{-1} = \bigcap_{k \in K} H(gk)^{-1} = \bigcap_{k \in K} (Hk^{-1})g^{-1}$
$= (\bigcap_{k \in K} Hk^{-1})g^{-1} = (H \overset{\lambda}{\ominus} K)g^{-1}$. The other results are proved similarly.

Proof of Lemma 3.10. We only prove (g)–(i). The other items are obvious.

(g) $(G \overset{\Gamma}{\oplus} H)^c = (\bigcup_{h \in H} Gh)^c = \bigcap_{h \in H} G^c h = G^c \overset{\lambda}{\ominus} \check{H}$. Also,
$(G \overset{\Gamma}{\oplus} H)^c = (\bigcup_{g \in G} gH)^c = \bigcap_{g \in G} gH^c = H^c \overset{\rho}{\ominus} \check{G}.$

(h) $(G \overset{\lambda}{\ominus} H)^{\vee} = (\bigcap_{h \in H} Gh^{-1})^{\vee} = \bigcap_{h \in H} h\check{G} = \check{G} \overset{\rho}{\ominus} \check{H}.$

(i) Follows from (e) and (g).

Proof of Lemma 3.15. Items (a) and (b) follow from Lemma 3.10(a,b).

(c) Let ψ be increasing. Then if $G \subseteq H$, $G^c \supseteq H^c$, so $\psi(G^c) \supseteq \psi(H^c)$. Therefore if $G \subseteq H$, then $\psi'(G) = (\psi(G^c))^c \subseteq (\psi(H^c))^c = \psi'(H)$, hence ψ' is increasing. The converse is proved similarly. Also, let ψ be left-invariant, $g \in \Gamma$, $H \subseteq \Gamma$. Then $\psi'(gH) = \{\psi((gH)^c)\}^c = \{\psi(gH^c)\}^c = \{g\psi(H^c)\}^c = g\{\psi(H^c)\}^c = g\psi'(H)$, hence ψ' is left-invariant. Next, let ψ be a dilation. Then $\psi'(\cap X_i) = \{\psi((\cap X_i)^c)\}^c = \{\psi(\cup X_i^c)\}^c = \{\cup \psi(X_i^c)\}^c = \cap\{\psi(X_i^c)\}^c = \cap \psi'(X_i)$, hence ψ' is an erosion; the reverse implication is proved similarly. Finally, $(\tilde{\delta}^{\lambda}_{H})'(G) = (\tilde{\delta}^{\lambda}_{H}(G^c))^c = (G^c \overset{\Gamma}{\oplus} H)^c = G \overset{\lambda}{\ominus} \check{H} = \tilde{\varepsilon}^{\lambda}_{\check{H}}(G)$, where we have used Lemma 3.10(g).

(d) Let ψ be right-invariant, $g \in \Gamma$, $H \subseteq \Gamma$. Then $\check{\psi}(gH) = \{\psi((gH)^{\vee})\}^{\vee} = \{\psi(\check{H}g^{-1})\}^{\vee} = \{\psi(\check{H})g^{-1}\}^{\vee} = g\{\psi(\check{H})\}^{\vee} = g\check{\psi}(H)$, where we used Lemma 3.10(d). So we have shown that if ψ is right-invariant, $\check{\psi}$ is left-invariant. The reverse statement is proved similarly.
Also,

$$(\lambda_g)^{\vee}(H) = \{\lambda_g(\check{H})\}^{\vee} = \{g\check{H}\}^{\vee} = Hg^{-1} = \rho_g^{-1}(H)$$

and

$$(\tilde{\delta}^{\lambda}_{H})^{\vee} = (\bigcup_{h\in H} \rho_h)^{\vee} = \bigcup_{h\in H} (\rho_h)^{\vee} = \bigcup_{h\in H} \lambda_h^{-1} = \tilde{\delta}^{\rho}_{\check{H}}$$

The result for the erosion follows in the same way.
(e) Follows from (c) and (d).

Proof of Proposition 3.17. We only prove the first and third formulas. From the defining Eq. (7.24a) of the left residual we have $(G \overset{\lambda}{\ominus} H) \overset{\Gamma}{\oplus} H = (\bigcup_{g\in\Gamma} \{g : gH \subseteq G\}) \overset{\Gamma}{\oplus} H = \bigcup_{g\in\Gamma}\{gH : gH \subseteq G\}$, which proves the result for the left-invariant opening. Using Boolean duality, we have $\bigcap_{g\in\Gamma}\{g\hat{H} : g\hat{H} \supseteq G\} = \bigcap_{g\in\Gamma}\{g\hat{H} : g(\hat{H})^c \subseteq G^c\} = (\bigcup_{g\in\Gamma}\{g\check{H} : g\check{H} \subseteq G^c\})^c = ((G^c \overset{\lambda}{\ominus} \check{H}) \overset{\Gamma}{\oplus} \check{H})^c = (G \overset{\Gamma}{\oplus} H) \overset{\lambda}{\ominus} H$, proving the third line.

Proof of Proposition 3.18. We have seen above that (7.36) is a λ-adjunction. Therefore it remains to prove the "only if" part. So assume that (ε, δ) is a λ-adjunction. Let $H = \delta(\{e\})$, where e is the unit element of Γ. Then, for each $g \in \Gamma$,

$$\delta(\{g\}) = \delta(\lambda_g\{e\}) = \lambda_g\delta(\{e\}) = \lambda_g(H)$$

Hence, for each $G \in \mathcal{P}(\Gamma)$,

$$\delta(G) = \delta(\bigcup_{g\in G} \{g\}) = \bigcup_{g\in G} \delta(\{g\}) = \bigcup_{g\in G} \lambda_g(H) = G \overset{\Gamma}{\oplus} H = \tilde{\delta}^{\lambda}_{H}(G)$$

proving that each left-invariant dilation has the form as in (7.36). To complete the proof, observe that if ε is a λ-erosion, then its lower adjoint δ is a λ-dilation, so $\delta = \tilde{\delta}^{\lambda}_{H}$ for some $H \in \mathcal{P}(\Gamma)$, whose unique upper adjoint is $\tilde{\varepsilon}^{\lambda}_{H}$. Hence $\varepsilon = \tilde{\varepsilon}^{\lambda}_{H}$.

Proof of Theorem 3.23. It is clear that ψ as given by (7.38) is a left-invariant mapping, since it is a union of such mappings. Conversely, let ψ be a left-invariant mapping. We show that ψ has the form (7.38). Given $G \in \mathcal{P}(\Gamma)$, let $Z = \bigcup_{(F,H) \in \mathcal{W}(\psi)} G \overset{\lambda}{\oslash} (F, H)$. We show that $\psi(G) = Z$.

(a) $\psi(G) \supseteq Z$: Let $g \in G \overset{\lambda}{\oslash} (F, H)$ for some $(F, H) \in \mathcal{W}(\psi)$. Then $gF \subseteq G \subseteq gH$, hence $F \subseteq g^{-1}G \subseteq H$ and so $g^{-1}G \in [F, H] \subseteq \mathcal{V}(\psi)$ by assumption on (F, H). It follows that $e \in \psi(g^{-1}G) = g^{-1}\psi(G)$, where e is the identity of Γ and we used left invariance of ψ. Therefore $g \in \psi(G)$, hence $\psi(G) \supseteq Z$.

(b) $\psi(G) \subseteq Z$: Let $g \in \psi(G)$. Then, using left invariance, $e \in g^{-1}\psi(G) = \psi(g^{-1}G)$, hence $g^{-1}G \in \mathcal{V}(\psi)$ and therefore $(g^{-1}G, g^{-1}G) \in \mathcal{W}(\psi)$. Combining this with the obvious fact that $G \overset{\lambda}{\oslash} (g^{-1}G, g^{-1}G) \supseteq \{g\}$, we conclude that $g \in Z$ and so $\psi(G) \subseteq Z$.

Proof of Corollary 3.24. By application of the above theorem to an increasing λ-mapping, and using (7.37a) combined with the obvious fact that $G \overset{\lambda}{\oslash} (F, H)$ is increasing in H, we have

$$\psi(G) = \bigcup_{F \in \mathcal{V}(\psi)} G \overset{\lambda}{\oslash} (F, \Gamma) = \bigcup_{F \in \mathcal{V}(\psi)} G \overset{\lambda}{\ominus} F$$

To prove the representation as an intersection of dilations, observe that the dual mapping ψ' of ψ is itself left-invariant and increasing; see Lemma 3.15. So, applying the decomposition just proved to ψ', we get

$$\psi'(G) = \bigcup_{F \in \mathcal{V}(\psi')} G \overset{\lambda}{\ominus} F$$

Now we take again the Boolean dual of ψ', using Lemma 3.10(g) and the fact that $\psi'' = \psi$ to find

$$\psi(G) = (\bigcup_{F \in \mathcal{V}(\psi')} G^c \overset{\lambda}{\ominus} F)^c = \bigcap_{F \in \mathcal{V}(\psi')} G \overset{\Gamma}{\oplus} \check{F}$$

This completes the proof for increasing λ-mappings. The proof for decreasing λ-mappings is analogous.

Proof of Theorem 3.25. We only have to prove the "only if" part, since a union of λ-openings is a λ-opening (see Section II.B). So assume that ψ is a λ-opening. Applying Proposition 2.10 of Section II.B with $\mathbf{T} = \Gamma^\lambda$, one finds that ψ has the form (7.40) with $\tilde{\gamma}_H^\lambda$ the structural λ-opening by the structuring element H. Since from Proposition 3.17, $\tilde{\gamma}_H^\lambda = \tilde{\delta}_H^\lambda \tilde{\varepsilon}_H^\lambda$, the proof is complete.

Proof of Lemma 4.6. Since Σ is a group, $\Sigma \overset{\Gamma}{\oplus} \Sigma = \Sigma = \Sigma \overset{\lambda}{\ominus} \Sigma$, so

$$G \overset{\lambda}{\ominus} \Sigma = G \overset{\lambda}{\ominus} (\Sigma \overset{\Gamma}{\oplus} \Sigma) = (G \overset{\lambda}{\ominus} \Sigma) \overset{\lambda}{\ominus} \Sigma$$

where we used Proposition 3.8(f). Also,

$$\forall s \in \Sigma, \quad \Sigma s = \Sigma \Rightarrow G \overset{\lambda}{\ominus} \Sigma = \bigcup_{s\in\Sigma} G \overset{\lambda}{\ominus} (\Sigma s) = \bigcup_{s\in\Sigma} (G \overset{\lambda}{\ominus} \Sigma)s$$
$$= (G \overset{\lambda}{\ominus} \Sigma) \overset{\Gamma}{\oplus} \Sigma$$

This proves (7.59a). To prove (7.59b), apply the Boolean duality relation Lemma 3.10(g) to (7.59a).

Proof of Proposition 4.8. We prove (a) to (i), but not necessarily in the stated order.

(a): Since π and ϑ are extensions of the point mappings π_ω and ϑ_ω to sets, they are increasing. So we will prove that π and ϑ are Γ-invariant, which then automatically implies that π_Σ is an increasing Γ-mapping, since $\tilde{\varepsilon}_\Sigma$ is increasing and translation-invariant. Let $g_0 \in \Gamma$; then, for any $G \in \mathcal{L}$, $\pi(g_0G) = \{g\omega : g \in g_0G\} = \{g_0g'\omega : g' \in G\} = g_0\{g'\omega : g' \in G\} = g_0\pi(G)$. Similarly, one proves the Γ-invariance of ϑ.

(b) and the first part of (c): These follow from the fact that π is the extension of a point mapping $\pi_\omega : \Gamma \to \mathcal{X}$ to subsets of Γ. The second part of (c) is proved below.

(e), first part: Using (a) and the fact that $\vartheta(\{\omega\}) = \Sigma$, one has for any $G \in \mathcal{L}$, $X \in \mathcal{L}$,

$$\pi\vartheta(X) = \pi\,\{g \in \Gamma : \pi_\omega(g) \in X\} = \{\pi_\omega(g) : g \in \Gamma,\ \pi_\omega(g) \in X\} = X$$
$$\vartheta\pi(G) = \vartheta\pi\left[\bigcup_{g\in G} \{g\}\right] = \bigcup_{g\in G} \vartheta(\{g\omega\}) = \bigcup_{g\in G} g\Sigma = G \overset{\Gamma}{\oplus} \Sigma = \tilde{\delta}_\Sigma(G)$$

(d): From (e), first part, we have $\vartheta = \vartheta\pi\vartheta = \tilde{\delta}_\Sigma\vartheta$, and, using Lemma 4.6, $\tilde{\varepsilon}_\Sigma\vartheta = \tilde{\varepsilon}_\Sigma\tilde{\delta}_\Sigma\vartheta = \tilde{\delta}_\Sigma\vartheta$.

(e), second part: From (d), $\pi_\Sigma\vartheta = \pi\tilde{\varepsilon}_\Sigma\vartheta = \pi\vartheta = \mathrm{id}_{\mathcal{L}}$, $\vartheta\pi_\Sigma = \vartheta\pi\tilde{\varepsilon}_\Sigma = \tilde{\delta}_\Sigma\tilde{\varepsilon}_\Sigma = \tilde{\varepsilon}_\Sigma$.

(c), second part: Since $\vartheta\pi_\Sigma = \tilde{\varepsilon}_\Sigma$, with $\tilde{\varepsilon}_\Sigma$ an erosion, we have, using (b), $\vartheta\pi_\Sigma(\bigcap_{i\in I} G_i) = \bigcap_{i\in I} \vartheta\pi_\Sigma(G_i) = \vartheta\left[\bigcap_{i\in I} \pi_\Sigma(G_i)\right]$ for any index set I. Operating on both sides of this quality by π and using $\pi\vartheta = \mathrm{id}_{\mathcal{L}}$, we get $\pi_\Sigma(\bigcap_{i\in I} G_i) = \bigcap_{i\in I} \pi_\Sigma(G_i)$.

(f,$\Rightarrow$): Follows from (a).

(f,$\Leftarrow$): Let $\vartheta(X) \subseteq \vartheta(Y)$. Then, since π is increasing and $\pi\vartheta = \mathrm{id}_{\mathcal{L}}$, $\pi\vartheta(X) \subseteq \pi\vartheta(Y)$, so $X \subseteq Y$.

(g): First, $(\vartheta(X))^c = \{g \in G: \pi_\omega(g) \in X\}^c = \{g \in G : \pi_\omega(g) \notin X\} = \{g \in G : \pi_\omega(g) \in X^c\} = \vartheta(X^c)$. Using this identity and the fact that $(G \overset{\Gamma}{\oplus} \Sigma)^c = G \overset{\lambda}{\ominus} \check{\Sigma}$

$= G \overset{\lambda}{\ominus} \Sigma$ (since Σ is a group, $\check{\Sigma} = \Sigma$), we find $(\pi(G))^c = \pi\vartheta[\pi(G)]^c = \pi[\{\vartheta\pi(G)\}^c] = \pi[\tilde{\delta}_\Sigma(G)]^c = \pi\tilde{\varepsilon}_\Sigma(G^c) = \pi_\Sigma(G^c)$.

(h,$\Rightarrow$): $\pi(G) \subseteq X \Rightarrow G \subseteq \vartheta\pi(G) \subseteq \vartheta(X)$, since $\vartheta\pi = \tilde{\delta}_\Sigma$ is a closing and ϑ is increasing.

(h,$\Leftarrow$): $G \subseteq \vartheta(X) \Rightarrow \pi(G) \subseteq \pi\vartheta(X) = X$, since π is increasing and $\pi\vartheta = \mathrm{id}_{\mathscr{L}}$.

(i,$\Rightarrow$): $\vartheta(X) \subseteq G \Rightarrow X = \pi_\Sigma\vartheta(X) \subseteq \pi_\Sigma(G)$, since π_Σ is increasing and $\pi_\Sigma\vartheta = \mathrm{id}_{\mathscr{L}}$.

(i,$\Leftarrow$): $X \subseteq \pi_\Sigma(G) \Rightarrow \vartheta(X) \subseteq \vartheta\pi_\Sigma(G) \subseteq G$, for $\vartheta\pi_\Sigma = \tilde{\varepsilon}_\Sigma$ is an opening and ϑ is increasing.

Proof of Lemma 4.9. (a) and (a′) are obvious. We prove (b) and the part of (c) and (c′) concerning openings. The other entries can be proved similarly.

(b): Let ε be an erosion on $\mathscr{L}$ with adjoint δ. Then $\vartheta\varepsilon\pi_\Sigma$ is an erosion on $\tilde{\mathscr{L}}$, and, since the adjoint of a product is the product of the adjoints in reverse order (see Lemma 2.6(f)), the corresponding dilation is $\vartheta\delta\pi$.

(c): Let γ be an opening. Then $\tilde{\gamma} := \vartheta\gamma\pi_\Sigma$ is increasing, antiextensive (since $\vartheta\gamma\pi_\Sigma \subseteq \vartheta\pi_\Sigma = \tilde{\varepsilon}_\Sigma \subseteq \mathrm{id}_{\tilde{\mathscr{L}}}$) and idempotent, for $\tilde{\gamma}^2 = \vartheta\gamma\pi_\Sigma\vartheta\gamma\pi_\Sigma = \vartheta\gamma^2\pi_\Sigma = \vartheta\gamma\pi_\Sigma = \tilde{\gamma}$. Hence $\tilde{\gamma}$ is an opening.

(c′): Let $\tilde{\gamma}$ be an opening. Then $\gamma := \pi\tilde{\gamma}\vartheta$ is increasing and antiextensive, since $\gamma = \pi\tilde{\gamma}\vartheta \subseteq \pi\vartheta = \mathrm{id}_{\mathscr{L}}$. Therefore $\gamma^2 \subseteq \gamma$, but also $\gamma^2 = \pi\tilde{\gamma}\vartheta\pi\tilde{\gamma}\vartheta = \pi\tilde{\gamma}\tilde{\delta}_\Sigma\tilde{\gamma}\vartheta \supseteq \pi\tilde{\gamma}^2\vartheta = \pi\tilde{\gamma}\vartheta = \gamma$, hence $\gamma^2 = \gamma$. So γ is an opening.

Proof of Proposition 4.12. We give the proof in operator form. Since $(\tilde{\varepsilon}_H, \tilde{\delta}_H)$ is a Γ-adjunction on $\tilde{\mathscr{L}}$, the pair (ε, δ) as defined in (7.61) forms a Γ-adjunction on $\mathscr{L}$ by Lemma 4.9(a′,b′). Conversely, let (ε, δ) form a Γ-adjunction on $\mathscr{L}$. Then by Lemma 4.9(a,b), the pair $(\vartheta\varepsilon\pi_\Sigma, \vartheta\delta\pi)$ is a Γ-adjunction on $\tilde{\mathscr{L}}$. Hence we know from Section III.D that $\tilde{\delta} := \vartheta\delta\pi = \tilde{\delta}_H$, $\tilde{\varepsilon} := \vartheta\varepsilon\pi_\Sigma = \tilde{\varepsilon}_H$ for some $H \in \tilde{\mathscr{L}}$, where $(\tilde{\varepsilon}_H, \tilde{\delta}_H)$ is the left-invariant adjunction (7.36) on $\tilde{\mathscr{L}}$ with structuring element H. Using that $\pi\vartheta = \pi_\Sigma\vartheta = \mathrm{id}_{\mathscr{L}}$, we find that $\delta = \pi\tilde{\delta}_H\vartheta$, $\varepsilon = \pi_\Sigma\tilde{\varepsilon}_H\vartheta$. This proves (7.61). Now $\pi = \pi\tilde{\delta}_\Sigma$, hence $\delta = \pi\tilde{\delta}_\Sigma\tilde{\delta}_H\vartheta = \pi\tilde{\delta}_{H\tilde{\oplus}\Sigma}\vartheta = \pi\tilde{\delta}_{\vartheta\pi(H)}\vartheta$, and $\vartheta = \tilde{\varepsilon}_E\vartheta$, hence $\varepsilon = \pi_\Sigma\tilde{\varepsilon}_H\tilde{\varepsilon}_\Sigma\vartheta = \pi_\Sigma\tilde{\varepsilon}_{H\tilde{\oplus}\Sigma}\vartheta = \pi_\Sigma\tilde{\varepsilon}_{\vartheta\pi(H)}\vartheta$. Writing Y instead of $\pi(H)$, we thus have found $\delta = \delta_Y^\Gamma := \pi\tilde{\delta}_{\vartheta(Y)}\vartheta$, $\varepsilon = \varepsilon_Y^\Gamma := \pi_\Sigma\tilde{\varepsilon}_{\vartheta(Y)}\vartheta$.

Since $\pi = \pi\tilde{\delta}_\Sigma$, $\vartheta = \tilde{\delta}_\Sigma\vartheta$, one has that $\delta = \pi\tilde{\delta}_\Sigma\tilde{\delta}_H\tilde{\delta}_\Sigma\vartheta = \pi\tilde{\delta}_{\Sigma\tilde{\oplus}H\tilde{\oplus}\Sigma}\vartheta = \pi\tilde{\delta}_{\vartheta(Y)}\vartheta$, and from $\pi_\Sigma = \pi_\Sigma\tilde{\varepsilon}_\Sigma$, $\vartheta = \tilde{\varepsilon}_\Sigma\vartheta$, one has $\varepsilon = \pi\tilde{\varepsilon}_\Sigma\tilde{\varepsilon}_H\tilde{\varepsilon}_\Sigma\vartheta = \pi\tilde{\varepsilon}_{\Sigma\tilde{\oplus}H\tilde{\oplus}\Sigma}\vartheta = \pi\tilde{\varepsilon}_{\vartheta(\check{Y})}\vartheta$, where as before $Y = \pi(H)$. It is clear that nothing changes when we replace Y by $\bar{Y}$. This completes the proof.

Proof of Corollary 4.13. From Proposition 4.12 we have that any adjunction has the form $(\varepsilon, \delta) = (\varepsilon_Y^\Gamma, \delta_Y^\Gamma)$ where

$$\delta_Y^\Gamma(X) = \pi[\vartheta(X) \overset{\Gamma}{\oplus} \vartheta(Y)] = \bigcup_{g\in\vartheta(X)} gY$$

$$\varepsilon_Y^\Gamma(X) = \pi_\Sigma[\vartheta(X) \overset{\lambda}{\ominus} \vartheta(Y)] = \bigcap_{g\in\vartheta(X^c)} g\hat{Y}^*$$

Here $\hat{Y}^* = \pi\Sigma[(\check{\vartheta}(Y))^c]$ and we have used the form of adjunctions on $\tilde{\mathcal{L}}$ in (7.28) and (7.32). Using Proposition 4.8g we also have $\hat{Y}^* = [\pi(\check{\vartheta}(Y))]^c = (\check{Y}^*)^c$ with $\check{Y}^* = \pi(\check{\vartheta}(Y))$. This proves the first part.

Second, using Proposition 4.12 again, $\delta_Y^\Gamma(X) = \pi[\vartheta(X) \overset{\Gamma}{\oplus} \vartheta(\bar{Y})] = \pi\{g \in \Gamma\colon g\check{\vartheta}(\Sigma Y) \Uparrow \vartheta(X)\} = \pi\{g \in \Gamma\colon g(\check{\vartheta}(Y) \overset{\Gamma}{\oplus} \Sigma) \Uparrow \vartheta(X)\} = \pi\{g \in \Gamma\colon g\vartheta(\pi\check{\vartheta}(Y)) \Uparrow \vartheta(X)\} = \pi\{g \in \Gamma\colon g\check{Y}^* \Uparrow X\} = \{y \in \mathcal{X}\colon g\check{Y}^* \Uparrow X \text{ for some } g \in \vartheta_\omega(y)\}$. Here we have made use of the geometric interpretation of dilations on $\mathcal{P}(\Gamma)$ (see Remark 3.11) and the obvious equivalence $\vartheta(X) \Uparrow \vartheta(Y) \Leftrightarrow X \Uparrow Y$. Finally, for the erosion we have $\varepsilon_Y^\Gamma(X) = \pi_\Sigma[\vartheta(X) \overset{\lambda}{\ominus} \vartheta(Y)] = \pi_\Sigma\{g \in \Gamma\colon gY \subseteq X\} = \{y \in \mathcal{X}\colon gY \subseteq X$ for all $g \in \vartheta_\omega(y)\}$.

Proof of Proposition 4.14. By explicit computation, we find

$$\begin{aligned}
\pi\tilde{\delta}_{\vartheta(Y)}\tilde{\varepsilon}_{\vartheta(Y)}\vartheta(X) &= \pi\,[(\vartheta(X) \overset{\lambda}{\ominus} \vartheta(Y)) \overset{\Gamma}{\oplus} \vartheta(Y)] = (\vartheta(X) \overset{\lambda}{\ominus} \vartheta(Y))Y \\
&= \left[\bigcup_{g\in\Gamma} \{g\colon g\vartheta(Y) \subseteq \vartheta(X)\}\right]Y = \left[\bigcup_{g\in\Gamma} \{g\colon gY \subseteq X\}\right]Y \\
&= \bigcup_{g\in\Gamma} \{gY\colon gY \subseteq X\}
\end{aligned}$$

Here we have used the notation (7.52) for the action of a subset G of Γ on a subset X or $\mathcal{X}$. This proves (7.65). Second, for the closing ϕ_Y^Γ we have

$$\begin{aligned}
\phi_Y^\Gamma(X) &= [\gamma_{Y^c}(X^c)]^c = [\pi\tilde{\delta}_{\vartheta(Y^c)}\tilde{\varepsilon}_{\vartheta Y^c}\vartheta(X^c)]^c \\
&= \pi_\Sigma\tilde{\varepsilon}_{\check{\vartheta}(Y^c)}\,\tilde{\delta}_{\check{\vartheta}(Y^c)}\,\vartheta(X) = \pi\tilde{\varepsilon}_{\check{\vartheta}(Y^c)}\tilde{\delta}_{\check{\vartheta}(Y^c)}\vartheta(X)
\end{aligned}$$

where we used Proposition 4.8(g) and Boolean duality for adjunctions on $\tilde{\mathcal{L}}$, as well as the fact that $\pi_\Sigma\tilde{\varepsilon}_{\check{\vartheta}(Y^c)} = \pi\tilde{\varepsilon}_{\check{\vartheta}(Y^c)}$, which follows from the Σ-invariance of $\check{\vartheta}(Y^c)$.

Proof of Lemma 4.17. $\mathcal{V}(\tilde{\psi}) = \{G \in \tilde{\mathcal{L}}\colon e \in \tilde{\psi}(G)\} = \{G \in \tilde{\mathcal{L}}\colon \pi(\{e\}) \in \psi(\pi(G))\} = \{G \in \tilde{\mathcal{L}}\colon \omega \in \psi(\pi(G))\} = \{G \in \tilde{\mathcal{L}}\colon \pi(G) \in \mathcal{V}(\psi)\}$, where we have used that $e \in \tilde{\psi}(G) = \vartheta\psi(\pi(G)) \Leftrightarrow \pi(\{e\}) \in \psi(\pi(G))$.

Proof of Theorem 4.18. Clearly, ψ as given by (7.67) is an increasing Γ-mapping. Conversely, assume that ψ is an increasing Γ-mapping. Then $\tilde{\psi} = \vartheta\psi\pi$ is an increasing λ-mapping (Lemma 4.9). Hence

$$\tilde{\psi}(G) = \bigcup_{H\in\mathcal{V}(\tilde{\psi})} \tilde{\varepsilon}_H(G)$$

where $\tilde{\varepsilon}_H(G) = G \overset{\lambda}{\ominus} H$ is the left-invariant erosion on $\mathcal{L}$. Now, from Proposition 4.8(e), $\psi = \pi\tilde{\psi}\vartheta$, so

$$\psi(X) = \pi\tilde{\psi}\vartheta(X) = \pi\left[\bigcup_{H\in\mathcal{V}(\tilde{\psi})} \tilde{\varepsilon}_H\vartheta(X)\right] = \bigcup_{H\in\mathcal{V}(\tilde{\psi})} \pi\tilde{\varepsilon}_H\vartheta(X)$$

Since $\tilde{\varepsilon}_H\vartheta = \tilde{\varepsilon}_H\tilde{\varepsilon}_\Sigma\vartheta = \tilde{\varepsilon}_{H\overset{\Gamma}{\oplus}\mathrm{Sq}} = ;_{\vartheta\pi(H)}\vartheta$, we have from Lemma 4.17

$$\psi(X) = \bigcup_{Y\in\mathcal{V}(\psi)} \bigcup_{H:\pi(H)=Y} \pi\tilde{\varepsilon}_{\vartheta\pi(H)}\vartheta(X) = \bigcup_{Y\in\mathcal{V}(\psi)} \pi\tilde{\varepsilon}_{\vartheta(Y)}\vartheta(X)$$

which completes the proof of (7.67a). Equation (7.67b) follows by applying (7.67a) to the Boolean dual ψ'.

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