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# A New Coalgebraic Semantics for Positive Modal Logic

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ABSTRACT

Positive Modal Logic is the restriction of the modal local consequence relation defined by the class of all Kripke models to the propositional negation-free modal language. The class of *positive modal algebras* is the one canonically associated with PML according to the theory of Abstract Algebraic Logic. In [4], a Priestley-style duality is established between the category of positive modal algebras and the category of $\mathbf{K}^+$-*spaces*. In this paper, we establish a categorical equivalence between the category $\mathbf{K}^+$ of $\mathbf{K}^+$-spaces and the category $\mathbf{Coalg(V)}$ of coalgebras of a suitable endofunctor $\mathbf{V}$ on the category of Priestley spaces.

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## 1. Introduction

Positive Modal Logic (PML) was introduced by Dunn in [7], and it is the restriction of the modal local consequence relation defined by the class of all Kripke models to the propositional negation-free modal language, whose connectives are $\wedge, \vee, \Box, \Diamond, \top, \bot$. In [13], Jansana shows that the class of *positive modal algebras* (see Definition 2.2.1 below) introduced in [7] is the one canonically associated with PML according to the theory of the algebraization of logics developed in [10]. Intuitively, this means that positive modal algebras are to PML what Boolean algebras with operators are to the normal modal logic $K$ (and its associated local consequence relation). In [4], a Priestley-style duality is established between the category of positive modal algebras and the category of $\mathbf{K}^+$-*spaces*, which are structures based on Priestley spaces (see Definition 2.6.1 below).

In this paper, we establish a categorical equivalence between the category $\mathbf{K}^+$ of $\mathbf{K}^+$-spaces introduced in [4] and the category $\mathbf{Coalg(V)}$ of coalgebras of a suitable endofunctor $\mathbf{V}$ on the category $\mathbf{Pri}$ of Priestley spaces. Such equivalence is built along the lines of the equivalence of categories presented in [15], between descriptive general frames for the normal modal logic $K$ and the coalgebras of the *Vietoris endofunctor* on the category of Stone spaces.

The category $\mathbf{Coalg(V)}$ obtained in this way provides a new coalgebraic semantics for PML, the standard one being the well-known representation of Kripke frames as coalgebras of the covariant powerset endofunctor $\mathcal{P}$ on the category $\mathbf{Set}$ of sets and set maps. PML and $K$ have the same Kripke semantics (hence, they have the same standard coalgebraic semantics), but different algebraic semantics (positive modal algebras and Boolean algebras with operators respectively). The new semantics for PML presented here and the the one for $K$ given in [15] are capable to reflect this difference in a context of coalgebras. More in general, the categorical equivalences and dualities involved in the process of associating the new coalgebraic semantics with the two logics imply that the total amount of information about PML (and $K$ respectively) carried by the class of positive modal algebras (Boolean algebras with operators) is translated to the new coalgebraic semantics.

This report is organized as follows: In Section 2 the basic notions are presented, together with facts about them. Section 3 is about the definition of the *Vietoris endofunctor* $\mathbf{V}$ on Priestley spaces. The equivalence between $\mathbf{K}^+$ and $\mathbf{Coalg}(\mathbf{V})$ is established in Section 4. In Section 5, a question which involves connections between Heyting algebras and the framework introduced here is answered for the negative. Finally, some open problems are listed in Section 6.

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## 2. Preliminaries

### *2.1 Coalgebras*

**Definition 2.1.1. ($T$-coalgebra)** *For every category $\mathcal{C}$ and every endofunctor $T$ on $\mathcal{C}$, a $T$-*coalgebra *is an arrow in $\mathcal{C}(X, T(X))$, where $X$ is an object in $\mathcal{C}$.*

**Definition 2.1.2. ($T$-coalgebra morphism)** *Let $T$ be an endofunctor on $\mathcal{C}$, and $\rho : X \longrightarrow T(X)$ and $\sigma : Y \longrightarrow T(Y)$ be $T$-coalgebras. A $T$-*coalgebra morphism *is an arrow $f \in \mathcal{C}(X, Y)$ such that the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \rho\downarrow & & \sigma\downarrow \\ T(X) & \xrightarrow{T(f)} & T(Y). \end{array}$$

### *2.2 Positive modal algebras*

**Definition 2.2.1. (Positive modal algebra)** $\mathcal{A} = \langle A, \wedge, \vee, \Box, \Diamond, 0, 1\rangle$ *is a* positive modal algebra *iff $\langle A, \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice and $\Box$ and $\Diamond$ are unary operations such that*

| | |
|---|---|
| *1.* $\Box(a \wedge b) = \Box a \wedge \Box b$ | *2.* $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$ |
| *3.* $\Box a \wedge \Diamond b \leq \Diamond(a \wedge b)$ | *4.* $\Box(a \vee b) \leq \Box a \vee \Diamond b$ |
| *5.* $\Box 1 = 1$ | *6.* $\Diamond 0 = 0$. |

For every partial order $\langle X, \leq\rangle$, let $\mathcal{P}_{\leq}(X)$ be the collection of the $\leq$*-increasing* subsets of $X$, i.e. those subsets $Y \subseteq X$ such that if $x \leq y$ and $x \in Y$ then $y \in Y$. It holds that $\langle \mathcal{P}_{\leq}(X), \cap, \cup, \emptyset, X\rangle$ is a bounded distributive lattice.

A *frame* [4] is a structure $\langle X, \leq, R\rangle$ such that $X \neq \emptyset$, $\leq$ is a preorder on $X$ (i.e. it is reflexive and transitive) and $R \subseteq X \times X$ such that $(\leq \circ R) \subseteq (R \circ \leq)$ and $(\leq^{-1} \circ R) \subseteq (R \circ \leq^{-1})$. Let us define $R_\Box = (R \circ \leq)$ and $R_\Diamond = (R \circ \leq^{-1})$. For every relation $S \subseteq X \times X$ and every $Y \subseteq X$, let

$$\Box_S(Y) = \{x \in X \mid S[x] \subseteq Y\}$$
$$\Diamond_S(Y) = \{x \in X \mid S[x] \cap Y \neq \emptyset\}.$$

**Example 2.2.2.** *For every frame $\langle X, \leq, R\rangle$, $\langle \mathcal{P}_{\leq}(X), \cap, \cup, \Box_{R_\Box}, \Diamond_{R_\Diamond}\emptyset, X\rangle$ is a positive modal algebra.*

### *2.3 The category* **Pri** *of Priestley spaces*

**Definition 2.3.1. (Priestley space)** (cf. [6]) *A* Priestley space *is a structure $\mathbf{X} = \langle X, \leq, \tau\rangle$ such that $\langle X, \leq\rangle$ is a partial order, $\langle X, \tau\rangle$ is a compact topological space which is* totally order-disconnected, *i.e. for every $x, y \in X$, if $x \not\leq y$ then $x \in U$ and $y \notin U$ for some clopen increasing set $U$.*

**Example 2.3.2.** *If $\mathcal{A} = \langle A, \wedge, \vee\rangle$ is a finite lattice, then $\langle A, \leq, \mathcal{P}(A)\rangle$ is a Priestley space.*

**Example 2.3.3.** *If $\mathbf{X} = \langle X, \tau\rangle$ is a Stone space, then $\langle X, =, \tau\rangle$ is a Priestley space.*

**Lemma 2.3.4.** *Let $\mathbf{X} = \langle X, \leq, \tau\rangle$ be a compact ordered topological space, and let $\mathcal{B}$ be a collection of clopen subsets such that for every $x, y \in X$, if $x \not\leq y$ then $x \in B$ and $y \notin B$ for some $B \in \mathcal{B}$. Then*

1. *$\mathbf{X}$ is Hausdorff.*
2. *$\mathcal{C} = \mathcal{B} \cup \{(X \setminus B) \mid B \in \mathcal{B}\}$ is a subbase of $\tau$.*
3. *$\mathbf{X}$ is 0-dimensional, hence $\langle X, \tau\rangle$ is a Stone space.*

*Proof.* 1. Let $x, y \in X$. If $x \neq y$, then, as $\leq$ is antisymmetric, we can assume that $x \not\leq y$. Then $x \in B$ and $y \notin B$ for some $B \subseteq \mathcal{B}$. As $B$ is clopen, then $B$ and $(X \setminus B)$ are open subsets of $X$ such that $x \in B$, $y \in (X \setminus B)$ and $B \cap (X \setminus B) = \emptyset$.
2. Let $A \in \tau$, and let $x \in A$. We have to show that there exist $D_1, \ldots, D_n \in \mathcal{C}$ such that $x \in \bigcap_{i=1}^n D_i \subseteq A$. As $x \in A$, then $x \neq y$ for every $y \in (X \setminus A)$, and as $\leq$ is antisymmetric, then either $x \not\leq y$ or $y \not\leq x$, hence for every $y \in (X \setminus A)$ there exists $C_y \in \mathcal{C}$ such that $x \notin C_y$ and $y \in C_y$, and so $\{C_y \mid y \in (X \setminus A)\}$ forms an open cover of $(X \setminus A)$. As $\mathbf{X}$ is compact and $(X \setminus A)$ is closed, then $(X \setminus A)$ is compact, hence there exist $y_1, \ldots, y_n \in (X \setminus A)$ such that $(X \setminus A) \subseteq \bigcup_{i=1}^n C_i$. Hence $(X \setminus C_i) \in \mathcal{C}$, $i = 1, \ldots, n$ and $x \in \bigcap_{i=1}^n (X \setminus C_i) \subseteq A$.
3. Item 2 implies that $\mathbf{X}$ has a subbase of clopen subsets. ☐

**Corollary 2.3.5.** *For every Priestley space $\mathbf{X} = \langle X, \leq, \tau\rangle$, $\mathbf{X}$ is Hausdorff, 0-dimensional and*

$$\{U \mid U \text{ clopen and increasing}\} \cup \{(X \setminus U) \mid U \text{ clopen and increasing}\}$$

*is a subbase of $\tau$.*

So Priestley spaces can be thought of as Stone spaces with a designated partial order.

**Proposition 2.3.6.** *For every Priestley space $\mathbf{X} = \langle X, \leq, \tau\rangle$, $\leq$ is a closed subset of $X \times X$ with the product topology.*

*Proof.* Assume that $\langle x, y\rangle \notin \leq$. We have to show that $\langle x, y\rangle \in A$ and $A \cap \leq = \emptyset$ for some open $A \subseteq X \times X$. By total order-disconnectedness, $x \in U$ and $y \in (X \setminus U)$ for some clopen increasing $U \subseteq X$. Let $A = U \times (X \setminus U)$. As $U$ is clopen in $\mathbf{X}$, then $A$ is open in the product topology, and $\langle x, y\rangle \in A$. If $\langle u, v\rangle \in A$, then $u \in U$ and $v \notin U$, hence $u \not\leq v$, for if $u \leq v$, then, as $U$ is increasing, $v \in U$, contradiction. Hence $A \cap \leq = \emptyset$. ☐

For every partial order $\mathbf{X} = \langle X, \leq\rangle$ let us denote $x\!\uparrow = \{y \in X \mid x \leq y\}$ and $x\!\downarrow = \{y \in X \mid y \leq x\}$ for every $x \in X$. For every topological space $\mathbf{X}$, let $K(\mathbf{X})$ be the set of the closed subsets of $\mathbf{X}$.

**Lemma 2.3.7.** *For every Priestley space $\mathbf{X} = \langle X, \leq, \tau\rangle$ and every $F \in K(\mathbf{X})$, $\bigcup_{y \in F} y\!\uparrow$ and $\bigcup_{y \in F} y\!\downarrow$ are closed subsets of $\mathbf{X}$.*

*Proof.* In order to show that $\bigcup_{y \in F} y\!\uparrow \in K(\mathbf{X})$, assume that $x \notin \bigcup_{y \in F} y\!\uparrow$, and show that $x \in A$ and $A \cap \bigcup_{y \in F} y\!\uparrow = \emptyset$ for some $A \in \tau$.
If $x \notin \bigcup_{y \in F} y\!\uparrow$, then for every $y \in F$, $x \notin y\!\uparrow$, i.e. $y \not\leq x$. Then by total order-disconnectedness, for every $y \in F$ there exists a clopen increasing subset $U_y$ such that $y \in U_y$ and $x \notin U_y$. Therefore $F \subseteq \bigcup_{y \in F} U_y$, and as $F$ is compact, for $F$ is a closed subset of the compact space $\mathbf{X}$, then $F \subseteq \bigcup_{i=1}^n U_{y_i}$ for some $y_1, \ldots, y_n \in F$. Let $A = X \setminus \bigcup_{i=1}^n U_{y_i}$. It holds that $A$ is an open and decreasing subset of $X$, $x \in A$ and $A \cap F = \emptyset$. Let us show that $A \cap \bigcup_{y \in F} y\!\uparrow = \emptyset$. Suppose that $z \in A \cap \bigcup_{y \in F} y\!\uparrow$ for some $z \in X$. Then $z \in A$ and $y_0 \leq z$ for some $y_0 \in F$, and as $A$ is decreasing, then $y_0 \in A$, hence $y_0 \in A \cap F = \emptyset$, contradiction. The proof that $\bigcup_{y \in F} y\!\downarrow \in K(\mathbf{X})$ is similar. ☐

**Corollary 2.3.8.** *For every Priestley space* $\mathbf{X} = \langle X, \leq, \tau \rangle$ *and for every* $x \in X$, $x{\uparrow} = \{y \in X \mid x \leq y\}$ *and* $x{\downarrow} = \{y \in X \mid y \leq x\}$ *are closed subsets of* $X$.

*Proof.* As $\mathbf{X}$ is Hausdorff, then $\{x\}$ is closed for every $x \in X$. □

**Definition 2.3.9. (Arrows in Pri)** *An arrow in the category* **Pri** *is a continuous and monotone map between Priestley spaces.*

*2.4 The closed and convex subsets*

**Lemma 2.4.1.** *Let* $\langle X, \leq \rangle$ *be a partial order, then the following are equivalent for every* $F \subseteq X$:

1. $F = U \cap V$ *for some subsets* $U, V$ *of* $X$, $U$ *increasing and* $V$ *decreasing.*
2. $F = \bigcup_{x,y \in F}(x{\uparrow} \cap y{\downarrow})$.
3. *If* $x, y \in F$ *and* $x \leq y$, *then* $z \in F$ *for every* $z$ *such that* $x \leq z \leq y$.

**Definition 2.4.2. (Convex subset)** *A subset* $F$ *of a partial order* $\langle X, \leq \rangle$ *is* convex *iff* $F$ *satisfies any of the conditions of Lemma 2.4.1.*

For every ordered topological space $\mathbf{X} = \langle X, \leq, \tau \rangle$ let us denote $\mathcal{F}_{\mathbf{X}}$ the collection of the closed and convex subsets of $\mathbf{X}$. The collection $\mathcal{F}_{\mathbf{X}}$ will play an important role in the definition of the equivalence.

*2.5 The Vietoris endofunctor* **K** *on Stone spaces*

**Definition 2.5.1. (The Vietoris space)** (cf. [14]) *Let* $\mathbf{X} = \langle X, \tau \rangle$ *be a topological space. The* Vietoris space associated with $\mathbf{X}$ *is the topological space* $\mathbf{K}(\mathbf{X}) = \langle K(\mathbf{X}), \tau_V \rangle$, *where* $K(\mathbf{X})$ *is the collection of the closed subsets of* $\mathbf{X}$, *and the topology* $\tau_V$ *is the one generated by taking the following collection as a subbase:*

$$\{t(A) \mid A \in \tau\} \cup \{m(A) \mid A \in \tau\},$$

*and for every* $A \in \tau$, $t(A) = \{F \in K(\mathbf{X}) \mid F \subseteq A\}$ *and* $m(A) = \{F \in K(\mathbf{X}) \mid F \cap A \neq \emptyset\}$.

**Lemma 2.5.2.** *For every topological space* $\mathbf{X} = \langle X, \tau \rangle$, *every collection* $\{A_i \mid i \in I\} \subseteq \tau$ *and every clopen subset* $U$ *of* $X$,

1. $m(\bigcup_{i \in I} A_i) = \bigcup_{i \in I} m(A_i)$.
2. $t(\bigcap_{i \in I} A_i) = \bigcap_{i \in I} t(A_i)$.
3. $m(X \setminus U) = K(\mathbf{X}) \setminus t(U)$, *hence* $t(U)$ *is a clopen subset of* $\mathbf{K}(\mathbf{X})$.
4. $t(X \setminus U) = K(\mathbf{X}) \setminus m(U)$ *hence* $m(U)$ *is a clopen subset of* $\mathbf{K}(\mathbf{X})$.

*Proof.* 1. For every $F \in K(\mathbf{X})$, $F \in m(\bigcup_{i \in I} A_i)$ iff $F \cap \bigcup_{i \in I} A_i \neq \emptyset$, iff $F \cap A_i \neq \emptyset$ for some $i \in I$, iff $F \in m(A_i)$ for some $i \in I$, iff $F \in \bigcup_{i \in I} m(A_i)$.
2. For every $F \in K(\mathbf{X})$, $F \in t(\bigcap_{i \in I} A_i)$ iff $F \subseteq \bigcap_{i \in I} A_i$, iff $F \subseteq A_i$ for every $i \in I$, iff $F \in t(A_i)$ for every $i \in I$, iff $F \in \bigcap_{i \in I} t(A_i)$.
3. For every $F \in K(\mathbf{X})$, $F \in m(X \setminus U)$ iff $F \cap (X \setminus U) \neq \emptyset$, iff $F \not\subseteq U$, iff $F \in K(\mathbf{X}) \setminus t(U)$.
4. For every $F \in K(\mathbf{X})$, $F \in t(X \setminus U)$ iff $F \subseteq (X \setminus U)$, iff $F \cap U = \emptyset$, iff $F \in K(\mathbf{X}) \setminus m(U)$. □

**Proposition 2.5.3.** *For every topological space* $\mathbf{X} = \langle X, \tau \rangle$,

1. *if* $\mathbf{X}$ *is compact and Hausdorff, then so is* $\mathbf{K}(\mathbf{X})$.
2. *if* $\mathbf{X}$ *is 0-dimensional, then so is* $\mathbf{K}(\mathbf{X})$.
3. *if* $\mathbf{X}$ *is a Stone space, then so is* $\mathbf{K}(\mathbf{X})$.

*Proof.* See [15]. □

The assignment $\mathbf{X} \mapsto \mathbf{K}(\mathbf{X})$ can be extended to an endofunctor on the category **St** of Stone spaces and their continuous maps as follows ([14, 15]): For every $f \in Hom_{\mathbf{St}}(\mathbf{X}, \mathbf{Y})$ and every $F \in K(\mathbf{X})$, $\mathbf{K}(f)(F) := f[F]$. $\mathbf{K}$ is the *Vietoris endofunctor on Stone spaces.*

*2.6 The category $\mathbf{K}^+$ of $\mathbf{K}^+$-spaces*

**Definition 2.6.1. ($\mathbf{K}^+$-space)** (cf. def. 3.5 of [4]) *A* $\mathbf{K}^+$-space *is a structure* $\mathcal{G} = \langle X, \leq, R, \mathcal{A}\rangle$ *such that* $\leq$ *is a partial order on* $X$, $\mathcal{A}$ *is a sublattice of* $\langle \mathcal{P}_{\leq}(X), \cap, \cup, \emptyset, X\rangle$ *and* $R$ *is a binary relation on* $X$ *such that the following conditions are satisfied:*

- *D1. The space* $\mathbf{X}_{\mathcal{G}} = \langle X, \leq, \tau_{\mathcal{A}}\rangle$, *where* $\tau_{\mathcal{A}}$ *is the topology defined by taking* $\{U \mid U \in \mathcal{A}\} \cup \{(X \setminus U) \mid U \in \mathcal{A}\}$ *as a subbase, is a Priestley space such that* $\mathcal{A}$ *is the collection of the clopen increasing subsets of* $\tau_{\mathcal{A}}$.
- *D2.* $\mathcal{A}$ *is closed under the operations* $\Box_R$ *and* $\Diamond_R$.
- *D3. For every* $x \in X$, $R[x]$ *is a closed subset of* $\mathbf{X}_{\mathcal{G}}$.
- *D4. For every* $x \in X$, $R[x] = (R\circ \leq)[x] \cap (R\circ \leq^{-1})[x]$.

Let us recall that for every $\mathbf{K}^+$-space $\mathcal{G}$, the collection of the closed and convex subsets of $\mathbf{X}_{\mathcal{G}}$ is

$$\begin{aligned}\mathcal{F}_{\mathbf{X}_{\mathcal{G}}} &= \{F \in K(\mathbf{X}_{\mathcal{G}}) \mid F = U \cap V \text{ for some } U \in \mathcal{P}_{\leq}(X), V \in \mathcal{P}_{\leq^{-1}}(X)\} \\ &= \{F \in K(\mathbf{X}_{\mathcal{G}}) \mid F = \textstyle\bigcup_{x,y \in F}(x\uparrow \cap\, y\downarrow)\}.\end{aligned}$$

**Lemma 2.6.2.** *Conditions D3 and D4 are equivalent to the statement that for every* $x \in X$, $R[x] \in \mathcal{F}_{\mathbf{X}_{\mathcal{G}}}$.

*Proof.* For every $x \in X$, $(R\circ \leq)[x] \cap (R\circ \leq^{-1})[x] = \bigcup_{u,v \in R[x]}(u\uparrow \cap\, v\downarrow)$. Indeed, for every $z \in X$,

$$\begin{aligned} z \in (R\circ \leq)[x] \cap (R\circ \leq^{-1})[x] &\Leftrightarrow xRu \leq z \text{ and } xRv \leq^{-1} z \text{ for some } u, v \in X \\ &\Leftrightarrow u \leq z \leq v \text{ for some } u, v \in R[x] \\ &\Leftrightarrow z \in \textstyle\bigcup_{u,v \in R[x]}(u\uparrow \cap\, v\downarrow). \end{aligned}$$

□

**Lemma 2.6.3.** (cf. Prop 3.6 of [4]) *For every* $\mathbf{K}^+$*-space* $\mathcal{G} = \langle X, \leq, R, \mathcal{A}\rangle$,

$$(\leq \circ R) \subseteq (R\circ \leq) \quad \text{and} \quad (\leq^{-1} \circ R) \subseteq (R\circ \leq^{-1}).$$

**Definition 2.6.4. (Morphism in $\mathbf{K}^+$)** (cf. def. 3.8 of [4]) *For all* $\mathbf{K}^+$*-spaces* $\mathcal{G}_i = \langle X_i, \leq_i, R_i, \mathcal{A}_i\rangle$ $i = 1, 2$, *a map* $f : X_1 \longrightarrow X_2$ *is a* bounded morphism *between* $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *iff the following conditions are satisfied:*

- *B1.* $f$ *is order-preserving.*
- *B2. For every* $x, y \in X_1$, *if* $\langle x, y\rangle \in R_1$ *then* $\langle f(x), f(y)\rangle \in R_2$.
- *B3. If* $\langle f(x), y'\rangle \in R_2$, *then* $f(z_1) \leq y' \leq f(z_2)$ *for some* $z_1, z_2 \in R_1[x]$.
- *B4. For every* $U' \in \mathcal{A}_2$, $f^{-1}[U'] \in \mathcal{A}_1$.

**Lemma 2.6.5.** *Let* $\mathcal{G}_i = \langle X_i, \leq_i, R_i, \mathcal{A}_i\rangle$ $i = 1, 2$ *be* $\mathbf{K}^+$*-spaces. If* $f : X_1 \longrightarrow X_2$ *is a bounded morphism between* $\mathcal{G}_1$ *and* $\mathcal{G}_2$, *then* $f$ *is a continuous and order-preserving map between* $\mathbf{X}_{\mathcal{G}_1}$ *and* $\mathbf{X}_{\mathcal{G}_2}$.

*Proof.* The map $f$ is order-preserving by condition B1 of Definition 2.6.4. In order to show that $f$ is continuous, it is enough to show that for every clopen increasing subset $U$ of $\mathbf{X}_{\mathcal{G}_2}$, $f^{-1}[U]$ is a clopen subset of $\mathbf{X}_{\mathcal{G}_2}$. If $U$ is a clopen increasing subset of $\mathbf{X}_{\mathcal{G}_2}$, then by condition D1 of 2.6.1 $U \in \mathcal{A}_2$, and so by B4 $f^{-1}[U] \in \mathcal{A}_1$, hence $f^{-1}[U]$ is a clopen subset of $\mathbf{X}_{\mathcal{G}_1}$. □

## 3. The Vietoris endofunctor V on Pri

In this section, we are going to define an endofunctor $\mathbf{V}$ on the category of Priestley spaces, in such a way that the categories $\mathbf{K}^+$ and $\mathbf{Coalg}(\mathbf{V})$ will turn out to be equivalent. The starting points are: a) the intuition that every Priestley space is a Stone space with a designated partial order, and b) the fact that the Vietoris construction is functorial on the category of Stone spaces. For every Priestley space $\mathbf{X} = \langle X, \leq, \tau\rangle$, $\langle X, \tau\rangle$ is a Stone space, so we can consider the associated Vietoris space $\langle K(X), \tau_V\rangle$, which is a Stone space. So the question is: Can we endow $\langle K(X), \tau_V\rangle$ with a partial order $\leq^*$, in such a way that

1. $\langle K(X), \leq^*, \tau_V\rangle$ is a Priestley space, and
2. such construction can be used to define an endofunctor on Priestley spaces that extends in a natural way the Vietoris endofunctor $\mathbf{K}$ on Stone spaces?

Our candidate for $\leq^*$ is the *Egli-Milner power order* $\leq^{\mathcal{P}}$ [3, 16]. We will see that such order does not meet all the requirements, i.e. for every Priestley space $\langle X, \leq, \tau\rangle$, the space $\langle K(X), \leq^{\mathcal{P}}, \tau_V\rangle$ is not in general a Priestley space, and the only condition that fails is the antisymmetry of $\leq^{\mathcal{P}}$ (see Example 3.2.5 below), but this is the first step of the right construction.

The Vietoris space endowed with $\leq^{\mathcal{P}}$ is an instance of a more general construction called the *Vietoris power space* (cf. def 2.36 of [3]).

### *3.1 The Egli-Milner power order*

**Definition 3.1.1. (The Egli-Milner power order)**(cf def 2.30 of [3]) *For every set $X$ and every preorder $\leq$ on $X$, the* Egli-Milner power order of $\leq$ *is the relation $\leq^{\mathcal{P}} \subseteq \mathcal{P}(X) \times \mathcal{P}(X)$ defined as follows: For every $Y, Z \subseteq X$,*

$$Y \leq^{\mathcal{P}} Z \quad \textit{iff} \quad (\forall y \in Y \exists z \in Z. y \leq z) \ \&\ (\forall z \in Z \exists y \in Y. y \leq z).$$

Clearly, if $\leq$ is the identity, then $\leq^{\mathcal{P}}$ is the identity too. The Egli-Milner power order behaves well w.r.t. the order-preserving maps and the binary relations satifying certain conditions that we met already in 2.6.3, as the next two lemmas show:

**Lemma 3.1.2.** *For every order-preserving map $f : \langle X_1, \leq_1\rangle \longrightarrow \langle X_2, \leq_2\rangle$ between partial orders and every $Z, W \subseteq X$,*

$$\textit{if } Z \leq_1^{\mathcal{P}} W \textit{ then } f[Z] \leq_2^{\mathcal{P}} f[W].$$

*Proof.* If $z' \in f[Z]$, then $z' = f(z)$ for some $z \in Z$, and as $Z \leq_1^{\mathcal{P}} W$, then $z \leq_1 w$ for some $w \in W$, hence $z' = f(z) \leq_2 f(w) \in f[W]$.

If $w' \in f[W]$, then $w' = f(w)$ for some $w \in W$, and as $Z \leq_1^{\mathcal{P}} W$, then $z \leq_1 w$ for some $z \in z$, hence $f(z) \in f[Z]$ and $f(z) \leq_2 f(w) = w'$. □

**Lemma 3.1.3.** *For every partial order $\langle X, \leq\rangle$ and every binary relation $R$ on $X$, the following are equivalent:*

1. *For every $x, y \in X$, if $x \leq y$ then $R[x] \leq^{\mathcal{P}} R[y]$.*
2. *$(\leq \circ R) \subseteq (R \circ \leq)$ and $(\leq^{-1} \circ R) \subseteq (R \circ \leq^{-1})$.*

*Proof.* (1. ⇒ 2.) If $\langle x, y\rangle \in (\leq \circ R)$, then $x \leq z_1$ and $z_1 R y$ for some $z_1 \in X$, then $y \in R[z_1]$ and by assumption $R[x] \leq^{\mathcal{P}} R[z_1]$, so in particular $z_2 \in R[x]$ such that $z \leq y$, i.e. $\langle x, y\rangle \in (R\circ \leq)$. The proof of the second inclusion is similar.

(2. ⇒ 1.) Assume that $x \leq y$, and prove that a) for every $z \in R[x]$ there exists $w \in R[y]$ such that $z \leq w$, and b) for every $w \in R[y]$ there exists $z \in R[x]$ such that $z \leq w$.

a) If $z \in R[x]$ then $y \leq^{-1} x$ and $xRz$, so $\langle y, z\rangle \in \ (\leq^{-1} \circ R) \subseteq (R \circ \leq^{-1})$, hence $yRw$ and $w \leq^{-1} z$ for some $w \in X$, i.e. $w \in R[y]$ and $z \leq w$. The proof of b) is similar. □

*3.2 The Vietoris power space*

**Definition 3.2.1. (Vietoris power space)** *Let* $\mathbf{X} = \langle X, \leq, \tau\rangle$ *be a Priestley space. The* Vietoris power space of $\mathbf{X}$ *is the ordered topological space* $\langle K(\mathbf{X}), \leq^{\mathcal{P}}, \tau_V\rangle$, *where* $\leq^{\mathcal{P}}$ *is the restriction of the Egli-Milner power order to* $K(\mathbf{X}) \times K(\mathbf{X})$.

**Lemma 3.2.2.** *For every ordered topological space* $\mathbf{X} = \langle X, \leq, \tau\rangle$ *and every* $A \in \tau$, *if* $A$ *is* $\leq$*-increasing, then* $m(A)$ *and* $t(A)$ *are* $\leq^{\mathcal{P}}$*-increasing.*

*Proof.* Assume that $F \in m(A)$ and $F \leq^{\mathcal{P}} G$, and show that $G \in m(A)$, i.e. that $G \cap A \neq \emptyset$. As $F \in m(A)$, then $x \in F \cap A$ for some $x \in X$, and $F \leq^{\mathcal{P}} G$ implies that $x \leq y$ for some $y \in G$. As $x \in A$ and $A$ is increasing, then $y \in A$, hence $y \in G \cap A$.
Assume that $F \in t(A)$ and $F \leq^{\mathcal{P}} G$, and show that $G \in t(A)$, i.e. that $G \subseteq A$. Let $y \in G$. As $F \in t(A)$, then $F \subseteq A$, and $F \leq^{\mathcal{P}} G$ implies that $x \leq y$ for some $x \in F$. As $x \in A$ and $A$ is increasing, then $y \in A$. □

The Egli-Milner power order enjoys a property that is going to be crucial for us, and it is stated in item 2 of the next lemma:

**Lemma 3.2.3.** *For every Priestley space* $\mathbf{X} = \langle X, \leq, \tau\rangle$,

1. *for every* $F, G \in K(\mathbf{X})$, *if* $F \not\leq^{\mathcal{P}} G$, *then there exists a clopen increasing* $U \subseteq X$ *such that either* $F \in m(U)$ *and* $G \notin m(U)$, *or* $F \in t(U)$ *and* $G \notin t(U)$.
2. $\leq^{\mathcal{P}}$ *is a closed subset of* $K(\mathbf{X}) \times K(\mathbf{X})$ *with the product topology.*

*Proof.* 1. If $F \not\leq^{\mathcal{P}} G$, then either a) there exists $z \in F$ such that for every $w \in G$ $z \not\leq w$, or b) there exists $w \in G$ such that for every $z \in F$ $z \not\leq w$.
If a), then, as $\mathbf{X}$ is totally order-disconnected, for every $w \in G$ there exists a clopen increasing $U_w \subseteq X$ such that $z \in U_w$ and $w \notin U_w$. Therefore $G \subseteq \bigcup_{w\in G}(X \setminus U_w)$, i.e. the subsets $(X \setminus U_w)$ form an open covering of $G$, and as $G$ is compact, for it is a closed subset of the compact space $\mathbf{X}$, then $G \subseteq \bigcup_{i=1}^{n}(X \setminus U_{w_i})$ for some $w_1, \ldots, w_n \in G$. Let $U = \bigcap_{i=1}^{n} U_{w_i}$. $U$ is clopen increasing, $z \in F \cap U$ and $G \cap U = \emptyset$, hence $F \in m(U)$ and $G \notin m(U)$.
If b), then, as $\mathbf{X}$ is totally order-disconnected, for every $z \in F$ there exists a clopen increasing $U_z \subseteq X$ such that $z \in U_z$ and $w \notin U_z$. Therefore $F \subseteq \bigcup_{z\in F} U_z$, i.e. the subsets $U_z$ form an open covering of $F$, and as $F$ is compact, then $F \subseteq \bigcup_{i=1}^{n} U_{z_i}$ for some $z_1, \ldots, z_n \in F$. Let $U = \bigcup_{i=1}^{n} U_{z_i}$. $U$ is clopen increasing, $F \subseteq U$ and $w \in (G \setminus U)$, hence $F \in t(U)$ and $G \notin t(U)$.

2. Let $\langle F, G\rangle \notin \leq^{\mathcal{P}}$. We have to show that $\langle F, G\rangle \in \mathcal{U}$ and $\mathcal{U} \cap \leq^{\mathcal{P}} = \emptyset$ for some open subset $\mathcal{U} \in K(\mathbf{X}) \times K(\mathbf{X})$.
As $F \not\leq^{\mathcal{P}} G$, then by item 1. of this Lemma, there exists a clopen increasing $U \subseteq X$ such that either a) $F \in m(U)$ and $G \notin m(U)$, or b) $F \in t(U)$ and $G \notin t(U)$.
If a), then take $\mathcal{U} = t(U) \times (K(\mathbf{X}) \setminus t(U))$. $\langle F, G\rangle \in \mathcal{U}$. Let us show that if $\langle F', G'\rangle \in \mathcal{U}$, then $F' \not\leq^{\mathcal{P}} G'$. As $\langle F', G'\rangle \in \mathcal{U}$, then $F' \in t(U)$, i.e. $F' \subseteq U$, and $G' \notin t(U)$, i.e. $G' \not\subseteq U$, hence there exists $w \in (G' \setminus U)$. Let us show that $z \not\leq w$ for every $z \in F'$: if $z \in F' \subseteq U$ and $z \leq w$, then, as $U$ is increasing, $w \in U$, contradiction. Therefore $F' \not\leq^{\mathcal{P}} G'$.
If b), then take $\mathcal{U} = m(U) \times (K(\mathbf{X}) \setminus m(U))$. $\langle F, G\rangle \in \mathcal{U}$. Let us show that if $\langle F', G'\rangle \in \mathcal{U}$, then $F' \not\leq^{\mathcal{P}} G'$. As $\langle F', G'\rangle \in \mathcal{U}$, then $F' \in m(U)$, i.e. $F' \cap U \neq \emptyset$, hence there exists $z \in (F' \setminus U)$ and $G' \notin t(U)$, i.e. $G' \cap U = \emptyset$. Let us show that $z \not\leq w$ for every $w \in G'$: if $w \in G' \subseteq U$ and $z \leq w$, then, as $U$ is increasing, $w \in U$, contradiction. Therefore $F' \not\leq^{\mathcal{P}} G'$. □

**Corollary 3.2.4.** *For every Priestley space* $\mathbf{X} = \langle X, \leq, \tau\rangle$, *the Vietoris power space of* $\mathbf{X}$ *is totally order-disconnected, and the collection*

$$\{m(U), t(U) \mid U \tau\text{-clopen, } U \text{ increasing or decreasing}\}$$

*is a subbase of* $\tau_V$.

*Proof.* The total order-disconnectedness immediately follows from item 1 of the lemma above, and from the fact that if $U \subseteq X$ is clopen increasing, then $m(U)$ and $t(U)$ are clopen increasing in the Vietoris power space of $\mathbf{X}$. The second part of the statement immediately follows from item 1 of the lemma above and from Lemma 2.3.4. □

It holds that if $\leq$ is a preorder on a set $X$, then $\leq^{\mathcal{P}}$ is a preorder on $\mathcal{P}(X)$, but if $\leq$ is a partial order, then $\leq^{\mathcal{P}}$ might not be a partial order: The following one is an example of a Priestley space $\mathbf{X}$ such that $\leq^{\mathcal{P}}$ is not antisymmetric on $K(\mathbf{X})$.

**Example 3.2.5.** *Let us consider a four element chain* $0 < a < b < 1$, *which is a finite (distributive) lattice. By example 2.3.2 this chain is a Priestley space if it is endowed with the discrete topology. The subsets* $F = \{0, a, 1\}$ *and* $G = \{0, b, 1\}$ *are distinct closed subsets which share the maximum and the minimum, and so* $F \leq^{\mathcal{P}} G$ *and* $G \leq^{\mathcal{P}} F$.

Therefore the Vietoris power space of a Priestley space is not in general a Priestley space, and the only condition that fails is the antisymmetry of $\leq^{\mathcal{P}}$.

*3.3 The action of* $\mathbf{V}$ *on the objects of* $\mathbf{Pri}$

For every preorder $\langle X, \leq \rangle$, we can consider the equivalence relation $\equiv \subseteq \mathcal{P}(X) \times \mathcal{P}(X)$ defined as follows: For every $Y, Z \subseteq X$,

$$Y \equiv Z \quad \text{iff} \quad Y \leq^{\mathcal{P}} Z \text{ and } Z \leq^{\mathcal{P}} Y.$$

The Vietoris endofunctor $\mathbf{V}$ on $\mathbf{Pri}$ will associate every Priestley space with the $\equiv$-quotient of its Vietoris power space.

**Definition 3.3.1. ($\mathbf{V}(\mathbf{X})$)** *For every Priestley space* $\mathbf{X} = \langle X, \leq, \tau \rangle$, $\mathbf{V}(\mathbf{X}) = \langle K(\mathbf{X})_{\equiv}, \leq^{\mathcal{P}}_{\equiv}, \tau_{V\equiv} \rangle$, *and:*

1. $K(\mathbf{X})_{\equiv} = \{[F] \mid F \in K(\mathbf{X})\}$, *and for every* $F \in K(\mathbf{X})$, $[F] = \{G \in K(\mathbf{X}) \mid F \equiv G\}$.
2. *For every* $[F], [G] \in K(\mathbf{X})_{\equiv}$,
$$[F] \leq^{\mathcal{P}}_{\equiv} [G] \textit{ iff } F' \leq^{\mathcal{P}} G' \textit{ for some } F' \in [F] \textit{ and } G' \in [G].$$
3. $\tau_{V\equiv} = \{\mathcal{X} \subseteq K(\mathbf{X})_{\equiv} \mid \pi^{-1}[\mathcal{X}] \in \tau_V\}$, *and* $\pi : K(\mathbf{X}) \longrightarrow K(\mathbf{X})_{\equiv}$ *is the canonical projection.*

**Lemma 3.3.2.** *For every Priestley space* $\mathbf{X} = \langle X, \leq, \tau \rangle$,

1. *for every* $F, G \in K(\mathbf{X})$, $[F] \leq^{\mathcal{P}}_{\equiv} [G]$ *iff* $F \leq^{\mathcal{P}} G$, *hence* $\leq^{\mathcal{P}}_{\equiv}$ *is a partial order.*
2. *The canonical projection* $\pi : K(\mathbf{X}) \longrightarrow K(\mathbf{X})_{\equiv}$ *is a continuous and order-preserving map.*
3. *for every* $F \in K(\mathbf{X})$, $[F]$ *is a closed subset of the Vietoris power space of* $\mathbf{X}$.
4. *For every* $U$ *clopen increasing or clopen decreasing subset of* $X$,
$$\pi^{-1}[\pi[t(U)]] = t(U) \textit{ and } \pi^{-1}[\pi[m(U)]] = m(U),$$
*hence* $\pi[t(U)]$ *and* $\pi[m(U)]$ *are clopen increasing subsets of* $\mathbf{V}(\mathbf{X})$.
5. *If* $U_i, V_j \subseteq X$ *are clopen increasing* $i = 1, \ldots, n$ *and* $j = 1, \ldots, m$ *and* $A = (\bigcap_{i=1}^{n} m(U_i)) \cap (\bigcap_{j=1}^{m} t(V_j))$, *then* $\pi^{-1}[\pi[A]] = A$, *hence* $\pi[A]$ *is a clopen increasing subset of* $\mathbf{V}(\mathbf{X})$.

*Proof.* 1. If $[F] \leq^{\mathcal{P}}_{\equiv} [G]$, then $F' \leq^{\mathcal{P}} G'$ for some $F' \in [F]$ and $G' \in [G]$, hence $F \leq^{\mathcal{P}} F' \leq^{\mathcal{P}} G' \leq^{\mathcal{P}} G$, and as $\leq^{\mathcal{P}}$ is transitive, then $F \leq^{\mathcal{P}} G$. The converse implication holds by definition. Let us show that $\leq^{\mathcal{P}}_{\equiv}$ is antisymmetric: If $[F] \leq^{\mathcal{P}}_{\equiv} [G]$ and $[G] \leq^{\mathcal{P}}_{\equiv} [F]$, then by the first part of the statement $F \leq^{\mathcal{P}} G$ and $G \leq^{\mathcal{P}} F$, hence $F \equiv G$, and so $[F] = [G]$.

2. It immediately follows from the definitions of quotient topology and of $\leq^{\mathcal{P}}_{\equiv}$.

3. As $\mathbf{K}(\mathbf{X})$ is Hausdorff, then for every $F \in K(\mathbf{X})$, $\{F\}$ is a closed subset of $\mathbf{K}(\mathbf{X}) = \langle K(\mathbf{X}), \tau_V\rangle$, hence $K(\mathbf{X}) \times \{F\}$ is a closed subset of of the product space $\mathbf{K}(\mathbf{X}) \times \mathbf{K}(\mathbf{X})$. By item 2 of Lemma 3.2.3, $\leq^{\mathcal{P}}$ is a closed subset of $\mathbf{K}(\mathbf{X}) \times \mathbf{K}(\mathbf{X})$, hence $\leq^{\mathcal{P}^{-1}}$ is a closed subset of $\mathbf{K}(\mathbf{X}) \times \mathbf{K}(\mathbf{X})$, therefore $\equiv\, = \,\leq^{\mathcal{P}} \cap \leq^{\mathcal{P}^{-1}}$ is a closed subset of $\mathbf{K}(\mathbf{X}) \times \mathbf{K}(\mathbf{X})$ too. Hence the subset $\mathcal{X} = \{\langle G, F\rangle \mid G \equiv F\}$ is a closed subset of $\mathbf{K}(\mathbf{X}) \times \mathbf{K}(\mathbf{X})$, for $\mathcal{X} = \,\equiv \cap\, (K(\mathbf{X}) \times \{F\})$. Let $p_1 : \mathbf{K}(\mathbf{X}) \times \mathbf{K}(\mathbf{X}) \longrightarrow \mathbf{K}(\mathcal{X})$ be the first projection. As the restriction of $p_2$ to $K(\mathbf{X}) \times \{F\}$ is injective, then $p_2 : K(\mathbf{X}) \times \{F\} \longrightarrow K(\mathbf{X})$ is a homeomorphism, and $[F] = \{G \in K(\mathbf{X}) \mid G \equiv F\} = p_2[\mathcal{X}]$, so $[F]$ is closed.

4. Assume that $U$ is clopen increasing, and let us show that $\pi^{-1}[\pi[t(U)]] \subseteq t(U)$: If $F \in \pi^{-1}[\pi[t(U)]]$ then $\pi(F) \in \pi[t(U)]$, i.e. there exists $F' \in K(\mathbf{X})$ such that $F' \equiv F$ and $F' \in t(U)$, i.e. $F \leq^{\mathcal{P}} F'$, $F' \leq^{\mathcal{P}} F$ and $F' \subseteq U$. Let us show that $F \in t(U)$, i.e. that $F \subseteq U$: If $x \in F$, then, as $F' \leq^{\mathcal{P}} F$, there exists $w \in F' \subseteq U$ such that $w \leq x$, and as $U$ is increasing, then $x \in U$. The converse inclusion always holds.

Assume that $U$ is clopen increasing, and let us show that $\pi^{-1}[\pi[m(U)]] \subseteq m(U)$: If $F \in \pi^{-1}[\pi[m(U)]]$ then $\pi(F) \in \pi[m(U)]$, i.e. there exists $F' \in K(\mathbf{X})$ such that $F' \equiv F$ and $F' \in m(U)$, i.e. $F \leq^{\mathcal{P}} F'$, $F' \leq^{\mathcal{P}} F$ and $w \in F' \cap U$ for some $w \in X$. Let us show that $F \in m(U)$, i.e. that $F \cap U \neq \emptyset$: As $w \in F'$ and $F' \leq^{\mathcal{P}} F$, then there exists $x \in F$ such that $w \leq x$, and as $U$ is increasing, then $x \in U$. The converse inclusion always holds. The cases in which $U$ is clopen decreasing are similar.

5. If $A = (\bigcap_{i=1}^n m(U_i)) \cap (\bigcap_{j=1}^m t(V_j))$, then

$$\begin{array}{rcl}\pi[A] & = & \pi[(\bigcap_{i=1}^n m(U_i)) \cap (\bigcap_{j=1}^m t(V_j))] \\ & = & \pi[(\bigcap_{i=1}^n \pi^{-1}[\pi[m(U_i)]]) \cap (\bigcap_{j=1}^m \pi^{-1}[\pi[t(V_j)]])] \\ & = & \pi[\pi^{-1}[(\bigcap_{i=1}^n \pi[m(U_i)]) \cap (\bigcap_{j=1}^m \pi[t(V_j)])]] \\ & = & (\bigcap_{i=1}^n \pi[m(U_i)]) \cap (\bigcap_{j=1}^m \pi[t(V_j)]),\end{array} \tag{3.1}$$

hence

$$\begin{array}{rcl}\pi^{-1}[\pi[A]] & = & \pi^{-1}[(\bigcap_{i=1}^n \pi[m(U_i)]) \cap (\bigcap_{j=1}^m \pi[t(V_j)])] \\ & = & (\bigcap_{i=1}^n \pi^{-1}[\pi[m(U_i)]]) \cap (\bigcap_{j=1}^m \pi^{-1}[\pi[t(V_j)]]) \\ & = & (\bigcap_{i=1}^n m(U_i)) \cap (\bigcap_{j=1}^m t(V_j)) \\ & = & A.\end{array}$$

□

For every Priestley space $\mathbf{X} = \langle X, \leq, \tau\rangle$, let us denote

$$\mathcal{B}_{\mathbf{X}} = \{\pi[(\bigcap_{i=1}^n m(U_i)) \cap (\bigcap_{j=1}^m t(V_j))] \mid U_i, V_j \subseteq X \text{ clopen increasing}\}.$$

**Lemma 3.3.3.** *For every Priestley space* $\mathbf{X} = \langle X, \leq, \tau\rangle$*,*

1. *for every* $[F], [G] \in K(\mathbf{X})_{\equiv}$*, if* $[F] \not\leq^{\mathcal{P}}_{\equiv} [G]$*, then* $[F] \in B$ *and* $[G] \notin B$ *for some* $B \in \mathcal{B}_{\mathbf{X}}$*.*
2. $\mathcal{B}_{\mathbf{X}} \cup \{(K(\mathbf{X})_{\equiv} \setminus \mathcal{U}) \mid \mathcal{U} \in \mathcal{B}_V\}$ *is a subbase of the topology of* $\mathbf{V}(\mathbf{X})$*.*
3. $\mathbf{V}(\mathbf{X})$ *is totally order-disconnected.*

*Proof.* 1. Let $[F], [G] \in K(\mathbf{X})_{\equiv}$, assume that $[F] \not\leq^{\mathcal{P}}_{\equiv} [G]$. We have to show that $[F] \in B$ and $[G] \notin B$ for some $B \in \mathcal{B}_{\mathbf{X}}$.

If $[F] \not\leq^{\mathcal{P}}_{\equiv} [G]$, then $F \not\leq^{\mathcal{P}} G'$ for every $G' \in [G]$, and so, by item 1 of Lemma 3.2.3, $F' \in \mathcal{U}_{G'}$ and $G' \notin \mathcal{U}_{G'}$ for some clopen increasing subset $\mathcal{U}_{G'}$ of the Vietoris power space of $\mathbf{X}$ which is either of the form $m(U)$ or of the form $t(U)$ for some clopen increasing $U \subseteq X$.

Therefore $[G] \subseteq \bigcup_{G' \in [G]}(K(\mathbf{X}) \setminus \mathcal{U}_{G'})$, and as, by item 3 of Lemma 3.3.2, $[G]$ is a closed subset of the Vietoris power space of $\mathbf{X}$, then $[G]$ is compact, and so $[G] \subseteq \bigcup_{i=1}^{n}(K(\mathbf{X}) \setminus \mathcal{U}_{G'_i})$ for some $G'_1, \ldots, G'_n \in [G]$. Let $\mathcal{U} = \bigcap_{i=1}^{n} \mathcal{U}_{G'_i}$ and $B = \pi[\mathcal{U}]$. As $F \in \mathcal{U}$, then $\pi(F) \in B$. It holds that $[G] \cap \mathcal{U} = \emptyset$, and so $\pi(G) \notin B$.

2. It immediately follows from item 1 of this Lemma and item 2 of Lemma 2.3.4.

3. By item 5 of Lemma 3.3.2, every element in $\mathcal{B}_{\mathbf{X}}$ is a clopen increasing subset of $\mathbf{V}(\mathbf{X})$. Then the statement immediately follows from item 1 of this lemma. □

**Proposition 3.3.4.** *For every Priestley space $\mathbf{X} = \langle X, \leq, \tau \rangle$, $\mathbf{V}(\mathbf{X})$ is a Priestley space.*

*Proof.* The relation $\leq_{\equiv}^{\mathcal{P}}$ is a partial order (see item 1 of Lemma 3.3.2). As $\mathbf{X}$ is compact, then $\mathbf{K}(\mathbf{X}) = \langle K(\mathbf{X}), \tau_V \rangle$ is compact, so $\mathbf{V}(\mathbf{X})$ is compact, for it is the quotient space of a compact space, moreover $\mathbf{V}(\mathbf{X})$ is totally order-disconnected (see item 3 of Lemma 3.3.3). □

*3.4 The action of $\mathbf{V}$ on the morphisms of $\mathbf{Pri}$*

**Definition 3.4.1. ($\mathbf{V}(f)$)** *Let $\mathbf{X}_i = \langle X_i, \leq_i, \tau_i \rangle$ be Priestley spaces, $i = 1, 2$. For every continuous and order-preserving map $f : X_1 \longrightarrow X_2$, the map $\mathbf{V}(f) : K(\mathbf{X}_1)_{\equiv_1} \longrightarrow K(\mathbf{X}_2)_{\equiv_2}$ is given by the assignment $[F] \mapsto [f[F]]$ for every $F \in K(\mathbf{X}_1)$.*

**Lemma 3.4.2.** *For every continuous and order-preserving map $f : \mathbf{X}_1 \longrightarrow \mathbf{X}_2$ of Priestley spaces, and for every $U$ clopen increasing subset of $X_2$, if $\pi : K(\mathbf{X}_1) \longrightarrow K(\mathbf{X}_1)_{\equiv_1}$ is the canonical projection, then*

1. $\mathbf{V}(f)^{-1}[\pi[m(U)]] = \pi[\mathbf{K}(f)^{-1}[m(U)]]$.
2. $\pi^{-1}[\pi[\mathbf{K}(f)^{-1}[m(U)]]] = \mathbf{K}(f)^{-1}[m(U)]$, *hence* $\pi[\mathbf{K}(f)^{-1}[[m(U)]]$ *is a clopen subset of* $\mathbf{V}(\mathbf{X}_2)$.
3. $\mathbf{V}(f)^{-1}[\pi[t(U)]] = \pi[\mathbf{K}(f)^{-1}[t(U)]]$.
4. $\pi^{-1}[\pi[\mathbf{K}(f)^{-1}[t(U)]]] = \mathbf{K}(f)^{-1}[t(U)]$, *hence* $\pi[\mathbf{K}(f)^{-1}[t(U)]]$ *is a clopen subset of* $\mathbf{V}(\mathbf{X}_2)$.

*Proof.* 1. For every $F \in K(\mathbf{X}_1)$,

$$\begin{array}{rcl} [F] \in \mathbf{V}(f)^{-1}[\pi[m(U)]] & \Leftrightarrow & [f[F]] \in \pi[m(U)] \\ & \Leftrightarrow & \pi(f[F]) \in \pi[m(U)] \\ & \Leftrightarrow & f[F] \in \pi^{-1}[\pi[m(U)]] \\ & \Leftrightarrow & f[F] \in m(U) \\ & \Leftrightarrow & \mathbf{K}(f)(F) \in m(U) \\ & \Leftrightarrow & F \in \mathbf{K}(f)^{-1}[m(U)] \\ & \Leftrightarrow & [F] \in \pi[\mathbf{K}(f)^{-1}[m(U)]]. \end{array}$$

2. For every $F \in K(\mathbf{X}_1)$,

$$\begin{array}{rcll} F \in \pi^{-1}[\pi[\mathbf{K}(f)^{-1}[t(U)]]] & \Leftrightarrow & [F] \in \pi[\mathbf{K}(f)^{-1}[t(U)]] & \\ & \Leftrightarrow & F' \in \mathbf{K}(f)^{-1}[t(U)] \text{ for some } F' \in [F] & \\ & \Leftrightarrow & \mathbf{K}(f)(F') \in t(U) \text{ for some } F' \in [F] & \\ & \Leftrightarrow & f[F'] \in t(U) \text{ for some } F' \in [F] & \\ & \Rightarrow & f[F] \in t(U) & (*) \\ & \Leftrightarrow & \mathbf{K}(f)(F) \in t(U) & \\ & \Leftrightarrow & F \in \mathbf{K}(f)^{-1}[t(U)]. & \end{array}$$

Let us show the implication marked with $(*)$: As $F' \in [F]$ then $F' \equiv_1 F$, hence in particular $F' \leq_1^{\mathcal{P}} F$, and so $f[F'] \leq^{\mathcal{P}} f[F]$ by Lemma 3.1.2. As $U$ is $\leq_2$-increasing, then by Lemma 3.2.2 $t(U)$ is $\leq_2^{\mathcal{P}}$-increasing, so $f[F'] \in t[U]$ implies that $f[F] \in t[U]$. The converse inclusion always holds.

As $U$ is a clopen subset of $\mathbf{X}_2$, then $t(U)$ is a clopen subset of $\mathbf{K}(\mathbf{X}_2)$, and as the map $\mathbf{K}(f) : \mathbf{K}(\mathbf{X}_1) \longrightarrow \mathbf{K}(\mathbf{X}_2)$ is continuous, then $\mathbf{K}(f)^{-1}[t(U)] = \pi^{-1}[\pi[\mathbf{K}(f)^{-1}[t(U)]]]$ is a clopen subset of $\mathbf{X}_1$, hence $\pi[\mathbf{K}(f)^{-1}[t(U)]]$ is a clopen subset of $\mathbf{V}(\mathbf{X}_1)$. The proof involving $m(U)$ is similar. □

**Proposition 3.4.3.** *Let $\mathbf{X}_i = \langle X_i, \leq_i, \tau_i\rangle$ be Priestley spaces, $i = 1, 2$. For every continuous and order-preserving map $f : \mathbf{X}_1 \longrightarrow \mathbf{X}_2$, the map $\mathbf{V}(f) : K(\mathbf{X}_1)_{\equiv_1} \longrightarrow K(\mathbf{X}_2)_{\equiv_2}$, given by the assignment $[F] \mapsto [f[F]]$ for every $F \in K(\mathbf{X}_1)$, is continuous and order-preserving.*

*Proof.* As $f$ is order-preserving, then Lemma 3.1.2 implies that for every $F, G \in K(\mathbf{X})$, if $F \equiv_1 G$ then $f[F] \equiv_2 f[G]$, hence $\mathbf{V}(f)$ is well defined.

Lemma 3.1.2 also implies that $\mathbf{V}(f)$ is order-preserving, for if $[F]\leq^{\mathcal{P}}_{1\ \equiv_2}[G]$, then by item 1 of Lemma 3.3.2, $F \leq^{\mathcal{P}}_1 G$, hence $f[F] \leq^{\mathcal{P}}_2 f[G]$, and so by item 2 of Lemma 3.3.2, $[f[F]]\leq^{\mathcal{P}}_{2\ \equiv_2}[f[G]]$.

In order to show that $\mathbf{V}(f)$ is continuous, by item 2 of Lemma 3.3.3, it is enough to show that for every $\mathcal{U} \in \mathcal{B}_{\mathbf{X}_2}$, $\mathbf{V}(f)^{-1}[\mathcal{U}]$ is a clopen subset of $\mathbf{V}(\mathbf{X}_1)$. As $\mathcal{U} \in \mathcal{B}_{\mathbf{X}_2}$, then $\mathcal{U} = \pi[(\bigcap_{i=1}^n m(U_i)) \cap (\bigcap_{j=1}^m t(V_j))]$ for some clopen increasing subsets $U_i, V_j$ of $X_2$, hence

$$\begin{array}{rcll}
\mathbf{V}(f)^{-1}[\mathcal{U}] & = & \mathbf{V}(f)^{-1}[\pi[(\bigcap_{i=1}^n m(U_i)) \cap (\bigcap_{j=1}^m t(V_j))]] & \\
& = & \mathbf{V}(f)^{-1}[(\bigcap_{i=1}^n \pi[m(U_i)]) \cap (\bigcap_{j=1}^m \pi[t(V_j)])] & ((1)) \\
& = & (\bigcap_{i=1}^n \mathbf{V}(f)^{-1}[\pi[m(U_i)]]) \cap (\bigcap_{j=1}^m \mathbf{V}(f)^{-1}[\pi[t(V_j)]]) & \\
& = & (\bigcap_{i=1}^n \pi[\mathbf{K}(f)^{-1}[m(U_i)]]) \cap (\bigcap_{j=1}^m \pi[\mathbf{K}(f)^{-1}[t(V_j)]]), & (3.4.2)
\end{array}$$

and as $\pi[\mathbf{K}(f)^{-1}[m(U_i)]]$ and $\pi[\mathbf{K}(f)^{-1}[t(V_j)]]$ are clopen by item 2 of Lemma 3.4.2, then $\mathbf{V}(f)^{-1}[\mathcal{U}]$ is clopen. □

## 4. The equivalence between $\mathbf{K}^+$ and $\mathbf{Coalg}(\mathbf{V})$

### *4.1 From $\mathbf{K}^+$ to $\mathbf{Coalg}(\mathbf{V})$*

Let $\mathcal{G} = \langle X, \leq, R, \mathcal{A}\rangle$ be a $\mathbf{K}^+$-space, then the space $\mathbf{X}_{\mathcal{G}}$ associated with $\mathcal{G}$ is a Priestley space. Then we can consider the following map:

$$\begin{array}{rcl}
\rho_{\mathcal{G}} : \mathbf{X}_{\mathcal{G}} & \longrightarrow & K(\mathbf{X}_{\mathcal{G}})_{\equiv} \\
x & \longmapsto & \pi(R[x]).
\end{array}$$

As $\mathcal{G}$ is a $\mathbf{K}^+$-space, then $R[x] \in K(\mathbf{X})$ for every $x \in X$, so $\rho_{\mathcal{G}}$ is of the right type.

**Lemma 4.1.1.** *For every $\mathbf{K}^+$-space $\mathcal{G} = \langle X, \leq, R, \mathcal{A}\rangle$ and every clopen increasing subset $U \in \tau_{\mathcal{A}}$ $\rho_{\mathcal{G}}^{-1}[\pi[t(U)]] = \Box_R(U)$ and $\rho_{\mathcal{G}}^{-1}[\pi[m(U)]] = \Diamond_R(U)$.*

*Proof.* We just show the first equality, the proof of the other one being similar:

$$\begin{array}{rcl}
\rho_{\mathcal{G}}^{-1}[\pi[t(U)]] & = & \{x \in X \mid \rho_{\mathcal{G}}(x) \in \pi[t(U)]\} \\
& = & \{x \in X \mid \pi[R[x]] \in \pi[t(U)]\} \\
& = & \{x \in X \mid R[x] \in \pi^{-1}[\pi[t(U)]]\} \\
& = & \{x \in X \mid R[x] \in t(U)\} \\
& = & \{x \in X \mid R[x] \subseteq U\} \\
& = & \Box_R(U).
\end{array}$$

□

**Proposition 4.1.2.** *For every $\mathbf{K}^+$-space $\mathcal{G} = \langle X, \leq, R, \mathcal{A}\rangle$ the map $\rho_{\mathcal{G}}$ is a continuous and order-preserving map between Priestley spaces.*

*Proof.* Let us show that $\rho_{\mathcal{G}}$ is order preserving, so assume that $x \leq y$. As $\mathcal{G}$ is a $\mathbf{K}^+$-space, then by Lemma 2.6.3 $(\leq \circ R) \subseteq (R\circ \leq)$ and $(\leq^{-1} \circ R) \subseteq (R\circ \leq^{-1})$, hence by Lemma 3.1.3, $R[x] \leq^{\mathcal{P}} R[y]$, and as $\pi$ is order-preserving (see item 2 of Lemma 3.3.2), then $\rho_{\mathcal{G}}(x) = \pi(R[x]) \leq^{\mathcal{P}}_{\equiv} \pi(R[y]) = \rho_{\mathcal{G}}(y)$.

In order to show that $\rho_{\mathcal{G}}$ is continuous, by item 2 of Lemma 3.3.3 it is sufficient to show that for every $B \in \mathcal{B}_V$, $\rho_{\mathcal{G}}^{-1}[B]$ is a clopen subset of $\mathbf{X}_{\mathcal{G}}$. If $B \in B_V$, then $B = \pi[(\bigcap_{i=1}^n m(U_i)) \cap (\bigcap_{j=1}^m t(V_j))]$ for some $U_i, V_j \subseteq X$ clopen increasing. Then

$$\begin{array}{rcll} \rho_{\mathcal{G}}^{-1}[B] & = & \rho_{\mathcal{G}}^{-1}[\pi[(\bigcap_{i=1}^{n} m(U_i)) \cap (\bigcap_{j=1}^{m} t(V_j))]] & \\ & = & \rho_{\mathcal{G}}^{-1}[(\bigcap_{i=1}^{n} \pi[m(U_i)]) \cap (\bigcap_{j=1}^{m} \pi[t(V_j)])] & (3.3.2\ (1)) \\ & = & (\bigcap_{i=1}^{n} \rho_{\mathcal{G}}^{-1}[\pi[m(U_i)]]) \cap (\bigcap_{j=1}^{m} \rho_{\mathcal{G}}^{-1}[\pi[t(V_j)]]) & \\ & = & (\bigcap_{i=1}^{n} \Diamond_R(U_i)) \cap (\bigcap_{j=1}^{m} \Box_R(V_j)). & (4.1.1) \end{array}$$

As $U_i, V_j \subseteq X$ clopen increasing and $\mathcal{G}$ is a $\mathbf{K}^+$-space, then the collection of clopen increasing subsets of $\mathbf{X}_{\mathcal{G}}$ coincides with $\mathcal{A}$, and $\mathcal{A}$ is closed under $\Box_R$ and $\Diamond_R$, hence $\Diamond_R(U_i)$ and $\Box_R(V_j)$ are clopen increasing, and so $\rho_{\mathcal{G}}^{-1}[B]$ is clopen. □

**Proposition 4.1.3.** *For every bounded morphism of $\mathbf{K}^+$-spaces $f : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$, $f$ is a $\mathbf{V}$-coalgebra morphism between $\rho_{\mathcal{G}_1}$ and $\rho_{\mathcal{G}_2}$.*

*Proof.* By Lemma 2.6.5, $f$ is a continuous and order-preserving map between $\mathbf{X}_{\mathcal{G}_1}$ and $\mathbf{X}_{\mathcal{G}_2}$. Let us show that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{X}_{\mathcal{G}_1} & \xrightarrow{f} & \mathbf{X}_{\mathcal{G}_2} \\ \rho_{\mathcal{G}_1}\downarrow & & \rho_{\mathcal{G}_2}\downarrow \\ \mathbf{V}(\mathbf{X}_{\mathcal{G}_1}) & \xrightarrow{\mathbf{V}(f)} & \mathbf{V}(\mathbf{X}_{\mathcal{G}_2}). \end{array}$$

For every $x \in X_1$, $\mathbf{V}(f)(\rho_{\mathcal{G}_1}(x)) = \mathbf{V}(f)([R_1[x]]) = [f[R_1[x]]]$, and $\rho_{\mathcal{G}_2}(f(x)) = [R_2[f(x)]]$. So in order to show that the diagram above commutes, it is enough to show that $f[R_1[x]] \equiv R_2[f(x)]$, i.e. that a) for every $y' \in f[R_1[x]]$, $z'_1 \leq y' \leq z'_2$ for some $z'_1, z'_2 \in R_2[f(x)]$, and b) for every $z' \in R_2[f(x)]$, $y'_1 \leq z' \leq y'_2$ for some $y'_1, y'_2 \in f[R_1[x]]$.

a) If $y' \in f[R_1[x]]$, then $f(y) = y'$ for some $y \in R_1[x]$. By B2, $xR_1y$ implies that $f(x)R_2y'$, hence by B3, there exist $z_1, z_2 \in R_1[x]$ such that $f(z_1) \leq y' \leq f(z_2)$, then take $z'_i = f(z_i)$, $i = 1, 2$: as $z_i \in R_1[x]$, then by B2, $f(z_i) \in R_2[f(x)]$.

b) If $z' \in R_2[f(x)]$, then $f(x)R_2z'$, hence by B3, there exist $y_1, y_2 \in R_1[x]$ such that $f(y_1) \leq z' \leq f(y_2)$, then take $y'_i = f(y_i)$, $i = 1, 2$: as $y_i \in R_1[x]$, then $f(y_i) \in f[R_1[x]]$. □

*4.2 The Egli-Milner order on convex subsets*

**Lemma 4.2.1.** *For every Priestley space $\mathbf{X} = \langle X, \leq, \tau \rangle$,*

1. *the restriction of $\leq^{\mathcal{P}}$ to $(\mathcal{F}_{\mathbf{X}} \times \mathcal{F}_{\mathbf{X}})$ is antisymmetric, hence if $F, F' \in \mathcal{F}_{\mathbf{X}}$ and $F \equiv F'$, then $F = F'$.*

2. *For every $F \in K(\mathbf{X})$, there exists $F^+ \in \mathcal{F}_{\mathbf{X}}$ such that $F \equiv F^+$.*

3. *For every $F \in K(\mathbf{X})$, there exists a unique $F^+ \in \mathcal{F}_{\mathbf{X}}$ such that $F \equiv F^+$.*

4. *For every $F \in \mathcal{F}_{\mathbf{X}}$, $G \subseteq F$ for every $G \in [F]$.*

*Proof.* 1. Let $F, G \in \mathcal{F}_{\mathbf{X}}$, assume that $F \leq^{\mathcal{P}} G$ and $G \leq^{\mathcal{P}} F$, and let us show that $F = G$. Suppose $F \neq G$, then we can assume that $x \in (F \setminus G)$ for some $x \in X$. As $x \in F$ and $F \leq^{\mathcal{P}} G$ and $G \leq^{\mathcal{P}} F$, then $y_0 \leq x \leq z_0$ for some $y_0, z_0 \in G$, i.e. $x \in (y_0{\uparrow} \cap z_0{\downarrow}) \subseteq \bigcup_{y,z \in G}(y{\uparrow} \cap z{\downarrow}) = G$, contradiction.

2. For every $F \in K(\mathbf{X})$, let $F^+ = \bigcup_{x,y \in F}(x{\uparrow} \cap y{\downarrow})$. $F^+ \in \mathcal{F}_{\mathbf{X}}$ because $F^+ = (\bigcup_{x \in F} x{\uparrow}) \cap (\bigcup_{y \in F} x{\downarrow})$, $(\bigcup_{x \in F} x{\uparrow}) \in \mathcal{P}_{\leq}(X)$, $(\bigcup_{y \in F} x{\downarrow}) \in \mathcal{P}_{\leq^{-1}}(X)$ and $(\bigcup_{x \in F} x{\uparrow}), (\bigcup_{y \in F} x{\downarrow}) \in K(\mathbf{X})$ by Lemma 2.3.7. Let us show that $F \equiv F^+$. As $F \subseteq F^+$, then for every $x \in F$ there exist $y, z \in F \subseteq F^+$ such that $y \leq x \leq z$: take $y = x = z$. If $z \in F^+ = \bigcup_{x,y \in F}(x{\uparrow} \cap y{\downarrow})$, then $x_0 \leq z \leq y_0$ for some $x_0, y_0 \in F$.

3. If $F_1, F_2 \in \mathcal{F}_{\mathbf{X}}$ and $F_i \equiv F$ for $i = 1, 2$, then $F_1 \equiv F_2$, so by item 1 of this lemma $F_1 = F_2$.

4. Assume that $F \in \mathcal{F}_{\mathbf{X}}$ and $G \in [F]$. If $x \in G$, then as $G \equiv F$, $y_1 \leq x \leq y_2$ for some $y_1, y_2 \in F$, and as $F$ is convex, then $x \in F$. □

*4.3 From* $\mathbf{Coalg}(\mathbf{V})$ *to* $\mathbf{K}^+$

Let $\rho : \mathbf{X} \longrightarrow \mathbf{V}(\mathbf{X})$ be a $\mathbf{V}$-coalgebra, so $\mathbf{X} = \langle X, \leq, \tau\rangle$ is a Priestley space, and the collection $\mathcal{A}_\tau$ of the clopen increasing subsets of $\tau$ is a sublattice of $\langle \mathcal{P}_\leq(X), \cap, \cup, \emptyset, X\rangle$. For every $x \in X$, $\rho(x) \in K(\mathbf{X})_\equiv$, hence $\rho(x) = \pi(F) = [F]$ for some $F \in K(\mathbf{X})$. By item 3 of Lemma 4.2.1, there exists a unique $F^+ \in [F] \cap \mathcal{F}_{\mathbf{X}_\mathcal{G}} = \rho(x) \cap \mathcal{F}_{\mathbf{X}_\mathcal{G}}$. Let us define $R_\rho \subseteq X \times X$ by putting $R_\rho[x] = F^+$ for every $x \in X$.

Then we can associate $\rho$ with $\mathcal{G}_\rho = \langle X, \leq, R_\rho, \mathcal{A}_\tau\rangle$.

**Lemma 4.3.1.** *For every* $\mathbf{V}$*-coalgebra* $\rho : \mathbf{X} \longrightarrow \mathbf{V}(\mathbf{X})$*,*

1. *for every* $x \in X$*,* $\rho(x) = [R_\rho[x]]$*.*
2. *For every open increasing* $U \subseteq \mathbf{X}$*,* $\Box_{R_\rho}(U) = \rho^{-1}[\pi[t(U)]]$*.*
3. *For every open* $U \subseteq \mathbf{X}$*,* $\Diamond_{R_\rho}(U) = \rho^{-1}[\pi[m(U)]]$*.*

*Proof.* 1. By definition, $R_\rho[x] \in \pi^{-1}[\rho(x)]$ for every $X \in X$, hence $[R_\rho[x]] = \pi(R_{\rho(x)}) = \rho(x)$.

2. For every $x \in X$,

$$\begin{array}{rcl} x \in \Box_{R_\rho}(U) & \Leftrightarrow & R_\rho[x] \subseteq U \\ & \Leftrightarrow & R_\rho[x] \in t(U) \\ & \Rightarrow & \pi(R_\rho[x]) \in \pi[t(U)] \\ & \Leftrightarrow & \rho(x) \in \pi[t(U)] \\ & \Leftrightarrow & x \in \rho^{-1}[\pi[t(U)]]. \end{array}$$

Let us show that if $\rho(x) \in \pi[t(U)]$, then $R_\rho[x] \in t(U)$. If $\rho(x) \in \pi[t(U)]$, then $\rho(x) = [F]$ for some $F \in t(U)$. By definition, $R_\rho[x] = F^+ = \bigcup_{u,v \in F}(u{\uparrow} \cap v{\downarrow}) = (\bigcup_{u\in F} u{\uparrow}) \cap (\bigcup_{v \in F} v{\downarrow})$. As $F \in t(U)$, then $F \subseteq U$, and as $U$ is increasing, then $(\bigcup_{u\in F} u{\uparrow}) \subseteq U$. Therefore $R_\rho[x] = F^+ = (\bigcup_{u\in F} u{\uparrow}) \cap (\bigcup_{v\in F} v{\downarrow}) \subseteq (\bigcup_{u\in F} u{\uparrow}) \subseteq U$, i.e $R_\rho[x] \in t(U)$.

3. For every $x \in X$,

$$\begin{array}{rcl} x \in \Diamond_{R_\rho}(U) & \Leftrightarrow & R_\rho[x] \cap U \neq \emptyset \\ & \Leftrightarrow & R_\rho[x] \in m(U) \\ & \Rightarrow & \pi(R_\rho[x]) \in \pi[m(U)] \\ & \Leftrightarrow & \rho(x) \in \pi[m(U)] \\ & \Leftrightarrow & x \in \rho^{-1}[\pi[m(U)]]. \end{array}$$

Let us show that if $\rho(x) \in \pi[m(U)]$, then $R_\rho[x] \in m(U)$. If $\rho(x) \in \pi[m(U)]$, then $\rho(x) = [F]$ for some $F \in m(U)$. By definition, $R_\rho[x] = F^+ \in \mathcal{F}_\mathbf{X}$, hence by item 4 of Lemma 4.2.1 and so $F \subseteq R_\rho[x]$, hence $R_\rho[x] \cap U \supseteq F \cap U \neq \emptyset$, i.e. $R_\rho[x] \in m(U)$. □

**Proposition 4.3.2.** *For every* $\mathbf{V}$*-coalgebra* $\rho : \mathbf{X} \longrightarrow \mathbf{V}(\mathbf{X})$*,* $\mathcal{G}_\rho = \langle X, \leq, R_\rho, \mathcal{A}_\tau\rangle$ *is a* $\mathbf{K}^+$*-space.*

*Proof.* By construction, $\mathcal{A}_\tau$ is a sublattice of $\langle \mathcal{P}_\leq(X), \cap, \cup, \emptyset, X\rangle$, and for every $x \in X$, $R_\rho[x] \in \mathcal{F}_{\mathbf{X}_g}$, which implies, by Lemma 2.6.2, that $R_\rho$ verifies conditions D3 and D4 of the definition of $\mathbf{K}^+$-space. So the only thing we have to show is that $\mathcal{A}_\tau$ is closed under $\Box_{R_\rho}$ and $\Diamond_{R_\rho}$, i.e. that for every clopen increasing $U \subseteq \mathbf{X}$, $\Box_{R_\rho}(U)$ and $\Diamond_{R_\rho}(U)$ are clopen increasing. By items 2 and 3 of Lemma 4.3.1, $\Box_{R_\rho}(U) = \rho^{-1}[\pi[t(U)]]$, and $\Diamond_{R_\rho}(U) = \rho^{-1}[\pi[m(U)]]$. As $\rho$ is a $\mathbf{V}$-coalgebra, then $\rho$ is a continuous and order-preserving map, and as, by item 4 of Lemma 3.3.2, $\pi[t(U)]$ and $\pi[m(U)]$ are clopen increasing subsets of $\mathbf{V}(\mathbf{X})$, then $\rho^{-1}[\pi[t(U)]]$ and $\rho^{-1}[\pi[m(U)]]$ are clopen increasing subsets of $\mathbf{X}$. □

**Proposition 4.3.3.** *For every* $\mathbf{V}$*-coalgebra morphism* $f : \rho_1 \longrightarrow \rho_2$*,* $f$ *is a bounded morphism between* $\mathcal{G}_{\rho_1}$ *and* $\mathcal{G}_{\rho_2}$*.*

*Proof.* Let $\rho_i : \mathbf{X}_i \longrightarrow \mathbf{V}(\mathbf{X}_i)$, $i = 1, 2$. By assumption, $f : \mathbf{X}_1 \longrightarrow \mathbf{X}_2$ is a continuous and order-preserving map, such that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{X}_1 & \xrightarrow{f} & \mathbf{X}_2 \\ \rho_1 \big\downarrow & & \rho_2 \big\downarrow \\ \mathbf{V}(\mathbf{X}_1) & \xrightarrow{\mathbf{V}(f)} & \mathbf{V}(\mathbf{X}_2). \end{array}$$

Let $\mathcal{G}_{\rho_i} = \langle X_i, \leq_i, R_{\rho_i}, \mathcal{A}_i \rangle$, $i = 1, 2$. As $f$ is order preserving, then $f$ meets condition $B1$ of Definition 2.6.4. Let us show that B4 holds: If $U \in \mathcal{A}_2$, then $U$ is a clopen increasing subset of $\mathbf{X}_2$, and as $f$ is continuous, then $f^{-1}[U]$ is a clopen subset of $\mathbf{X}_1$. If $x \in f^{-1}[U]$ and $x \leq_1 y$, then $f(x) \in U$ and $f(x) \leq_2 f(y)$, and as $U$ is increasing, then $f(y) \in U$, i.e. $y \in f^{-1}[U]$, so $f^{-1}[U]$ is also increasing, hence $f^{-1}[U] \in \mathcal{A}_1$.

It holds that for every $x \in X_i$, $\rho_i(x) = [R_{\rho_i}[x]]$, $i = 1, 2$, so the commutativity of the diagram implies that $[R_{\rho_2}[f(x)]] = \rho_2(f(x)) = \mathbf{V}(f)(\rho_1(x)) = [f[R_{\rho_1}[x]]]$, hence $R_{\rho_2}[f(x)] \equiv_2 f[R_{\rho_1}[x]]$. Let us show B3: If $y' \in R_{\rho_2}[f(x)]$, then, as $R_{\rho_2}[f(x)] \leq_2^{\mathcal{P}} f[R_{\rho_1}[x]]$, there exist $z_1, z_2 \in R_{\rho_1}[x]$ such that $f(z_1) \leq_2 y' \leq_2 f(z_2)$. Finally, let us show B2: As $R_{\rho_2}[f(x)] \in \mathcal{F}_{\mathbf{X}_2}$ and $f[R_{\rho_1}[x]] \equiv_2 R_{\rho_2}[f(x)]$, then by item 4 of Lemma 4.2.1, $f[R_{\rho_1}[x]] \subseteq R_{\rho_2}[f(x)]$. Hence, if $y \in R_{\rho_1}[x]$, then $f(y) \in f[R_{\rho_1}[x]] \subseteq R_{\rho_2}[f(x)]$, and so $f(x) R_{\rho_2} f(y)$. □

*4.4 Equivalence*

**Proposition 4.4.1.** *For every $\mathbf{K}^+$-space $\mathcal{G}$ and every $\mathbf{V}$-coalgebra $\rho$, $\mathcal{G}_{\rho_\mathcal{G}} = \mathcal{G}$ and $\rho_{\mathcal{G}_\rho} = \rho$.*

*Proof.* If $\mathcal{G} = \langle X, \leq, R, \mathcal{A} \rangle$, then by spelling out the definitions involved, we have that $\mathcal{G}_{\rho_\mathcal{G}} = \langle X, \leq, R_{\rho_\mathcal{G}}, \mathcal{A} \rangle$, and for every $x \in X$ $R_{\rho_\mathcal{G}}[x] \in \rho_\mathcal{G}(x) = [R[x]]$, hence $R_{\rho_\mathcal{G}}[x] \equiv R[x]$, and since both sets are closed and convex, then by item 1 of lemma 4.2.1 $R_{\rho_\mathcal{G}}[x] = R[x]$.

If $\rho : \mathbf{X} \longrightarrow \mathbf{V}(\mathbf{X})$, then by spelling out the definitions involved we have that $\mathbf{X}_{\mathcal{G}_\rho} = \mathbf{X}$, hence $\rho_{\mathcal{G}_\rho} : \mathbf{X} \longrightarrow \mathbf{V}(\mathbf{X})$, and for every $x \in X$ $\rho_{\mathcal{G}_\rho}(x) = [R_\rho[x]] = \rho(x)$. □

## 5. A negative result about Heyting algebras

It is well-known that the class of Heyting algebras (see Definition 5.1.1 below) is the one canonically associated with the intuitionistic propositional logic. Such class of algebras and the homomorphism between its members form a category, that is dually equivalent to the category $\mathbf{E}$ (see Definition 5.2.1 below) of ordered Stone spaces $\langle X, \leq, \tau \rangle$ such that the assignment $x \mapsto x\uparrow$ defines a coalgebra of the Vietoris endofunctor $\mathbf{K}$ on Stone spaces. The spaces of the category $\mathbf{E}$ can be characterized as special Priestley spaces (see Proposition 5.2.5 below). So a natural questions that can be asked is whether for every space in $\mathbf{E}$ the assignment $x \mapsto \pi(x\uparrow)$ defines a coalgebra of the endofunctor $\mathbf{V}$ on Priestley spaces, so that $\mathbf{E}$ can be characterized as a subcategory of $\mathbf{Coalg}(\mathbf{V})$. We will give a negative answer to such question.

*5.1 Heyting algebras*

**Definition 5.1.1. (Heyting algebra)** *An algebra $\mathcal{A} = \langle A, \wedge, \vee, \rightarrow, 0, 1 \rangle$ is a* Heyting algebra *iff $\langle A, \wedge, \vee, 0, 1 \rangle$ is a bounded distributive lattice and $\rightarrow$ is* the residuum of $\wedge$, *i.e. it is a binary operation such that for every $a, b, c \in A$,*

$$(a \wedge c) \leq b \text{ iff } c \leq (a \rightarrow b).$$

An *intuitionistic frame* [5] is a poset, i.e. a structure $\langle X, \leq \rangle$, such that $X \neq \emptyset$ and $\leq$ is a reflexive, antisymmetric and transitive binary relation on $X$. Let $\mathcal{P}_\leq(X)$ be the collection of the $\leq$-increasing subsets of $X$. For every relation $S \subseteq X \times X$ and every $Y, Z \subseteq X$, let

$$\begin{array}{rcl}\Box_S(Y) & = & \{x \in X \mid S[x] \subseteq Y\} \\ \\ Z \Rightarrow_S Y & = & \Box_S((X \setminus Z) \cup Y) \\ & = & \{x \in X \mid \forall y \in X(x \leq y \& y \in Z \Rightarrow y \in Y\}.\end{array}$$

**Lemma 5.1.2.** *For every poset $\langle X, \leq\rangle$ and every $A, B \in \mathcal{P}_{\leq}(X)$, $A \Rightarrow_{\leq} B \in \mathcal{P}_{\leq}(X)$.*

*Proof.* Assume that $x \in (A \Rightarrow_{\leq} B)$ and $x \leq y$. Then for every $z \in A$, if $y \leq z$, then $x \leq z$, and so $z \in B$. This shows that $y \in (A \Rightarrow_{\leq} B)$. □

**Example 5.1.3.** *For every intuitionistic frame $\langle X, \leq\rangle$, $\langle \mathcal{P}_{\leq}(X), \cap, \cup, \Rightarrow_{\leq}, \emptyset, X\rangle$ is a Heyting algebra.*

*Proof.* For every partial order $\langle X, \leq\rangle$, it holds that $\langle \mathcal{P}_{\leq}(X), \cap, \cup, \emptyset, X\rangle$ is a bounded distributive lattice. Let us show that for every $A, B, C \in \mathcal{P}_{\leq}(X)$,

$$(A \cap C) \subset B \text{ iff } C \subseteq (A \Rightarrow_{\leq} B).$$

($\Rightarrow$) Let $c \in C$,and let us show that $c \in A \Rightarrow_{\leq} B$, i.e. that if $c \leq y$ and $y \in A$, then $y \in B$. As $c \leq y$, $c \in C$ and $C$ is $\leq$-increasing, then $y \in C$, so $y \in A \cap C \subseteq B$.
($\Leftarrow$) If $x \in A \cap C \subseteq C \subseteq A \Rightarrow_{\leq} B$, then for every $y \in A$ such that $x \leq y$, $y \in B$. Then take $y = x$. □

*5.2 The category* **E** *of Èsakia spaces*

**Definition 5.2.1. (Èsakia space)** (cf. def. 1 of [9]) *An* Èsakia space $\mathbf{X} = \langle X, \leq, \tau\rangle$ *is an ordered Stone space such that the assignment $x \mapsto x{\uparrow}$ defines a continuous map $\rho : \langle X, \tau\rangle \to \langle K(\mathbf{X}), \tau_V\rangle$.*

**Definition 5.2.2. (Strongly isotone map)** (cf. def. 2 of [9]) *Let $\langle X, \leq\rangle$ and $\langle Y, \leq'\rangle$ be pre-ordered sets. A map $f : X \to Y$ is* strongly isotone *iff*

$$\forall x \in X\ \forall y \in Y\ \ (f(x) \leq' y\ \Leftrightarrow\ \exists x'(x \leq x'\ \&\ f(x') = y)).$$

Clearly, if $f$ is strongly isotone then it is monotone, moreover the composition of strongly isotone maps is strongly isotone.

**Theorem 5.2.3.** (cf. theor 3 of [9]) *The category of Èsakia spaces and strongly isotone and continuous maps is dually equivalent to the category of Heyting algebras and their homomorphisms.*

**Lemma 5.2.4.** *For every ordered space $\mathbf{X} = \langle X, \leq, \tau\rangle$ such that $x{\uparrow} \in K(\mathbf{X})$ for every $x \in X$ and every open subset $A$,*

$$\Diamond_{\leq}(A) = A{\downarrow} = \rho^{-1}[m(A)],$$

*where $\rho(x) = x{\uparrow}$ for every $x \in X$.*

*Proof.* For every open subset $A$,

$$\begin{array}{rcl}A{\downarrow} & = & \bigcup_{y \in A} y{\downarrow} \\ & = & \{x \in X \mid \exists y(y \in A\ \&\ x \leq y)\} \\ & = & \{x \in X \mid x{\uparrow} \cap A \neq \emptyset\} \\ & = & \Diamond_{\leq}(A). \\ \\ \rho^{-1}[m(A)] & = & \{x \in X \mid \rho(x) \cap A \neq \emptyset\} \\ & = & \{x \in X \mid x{\uparrow} \cap A \neq \emptyset\} \\ & = & \Diamond_{\leq}(A).\end{array}$$

□

The next proposition is considered folklore:

**Proposition 5.2.5.** *The following are equivalent for every ordered Stone space* $\mathbf{X} = \langle X, \leq, \tau\rangle$*:*

1. $\mathbf{X}$ *is an Èsakia space.*

2. $\mathbf{X}$ *is a Priestley space such that for every clopen subset* $U$, $U\downarrow$ *is clopen.*

*Proof.* (1. $\Rightarrow$ 2.) If $U$ is a clopen subset of $X$, then $m(U)$ is a clopen subset of $K(\mathbf{X})$, and as by assumption $\rho$ is continuous, then $\rho^{-1}[m(U)]$ is a clopen subset of $X$, i.e. by Lemma 5.2.4 $U\downarrow$ is clopen.
Let us show that $\mathbf{X}$ is totally order-disconnected: It is enough to show that if $x, y \in X$ and $x \not\leq y$, then $y \in U$ and $x \notin U$ for some clopen decreasing subset $U$ of $X$. If $x, y \in X$ and $x \not\leq y$, then $y\uparrow \not\subseteq x\uparrow$, i.e. $y\uparrow \cap (X \setminus x\uparrow) \neq \emptyset$, i.e. $y\uparrow \in m(X \setminus x\uparrow)$, i.e $y \in \rho^{-1}[m(X \setminus x\uparrow)] = (X \setminus x\uparrow)\downarrow$. As $\rho$ is continuous by assumption and $m(X \setminus x\uparrow)$ is an open subset of $\langle K(\mathbf{X}), \tau_V\rangle$, $(X \setminus x\uparrow)\downarrow$ is an open subset of $X$, and it is decreasing. As $\mathbf{X}$ is a Stone space, $y \in V \subseteq (X \setminus x\uparrow)\downarrow$ for some clopen subset $V$ of $X$. Then take $U = V\downarrow$: As $V$ is clopen, then $U$ is clopen, moreover $y \in U$ and $U \subseteq (X \setminus x\uparrow)\downarrow$, hence $x \notin U$.
(2. $\Rightarrow$ 1.) By Corollary 3.2.4, in order to show that $\rho$ is continuous, it is enough to show that for every $U$ clopen subset of $X$, $\rho^{-1}[m(U)]$ and $\rho^{-1}[t(U)]$ are clopen. As $U$ is clopen, then by assumption $\rho^{-1}[m(U)] = U\downarrow$ is clopen, moreover $(X \setminus U)$ is clopen, hence $(X \setminus U)\downarrow$ is clopen, and the following holds:

$$\begin{aligned}\rho^{-1}[t(U)] &= \rho^{-1}[K(\mathbf{X}) \setminus m(X \setminus U)] \\ &= X \setminus \rho^{-1}[m(X \setminus U)] \\ &= X \setminus ((X \setminus U)\downarrow).\end{aligned}$$

□

**Proposition 5.2.6.** *The following are equivalent for every ordered Stone space* $\mathbf{X} = \langle X, \leq, \tau\rangle$*:*

1. $\mathbf{X}$ *is a Priestley space such that for every clopen increasing subset* $U$, $U\downarrow$ *is clopen increasing.*

2. *The general frame* $\mathcal{G}_{\mathbf{X}} = \langle X, \leq, \leq, \mathcal{A}_\tau\rangle$*, where* $\mathcal{A}_\tau$ *is the algebra of the clopen increasing subsets of* $\mathbf{X}$*, is a* $\mathbf{K}^+$*-space.*

3. $\mathbf{X}$ *is a Priestley space such that the map* $\rho : \mathbf{X} \to \mathbf{V}(\mathbf{X})$ *given by* $\rho(x) = \pi[x\uparrow]$ *is a* $\mathbf{V}$*-coalgebra.*

4. $\mathbf{X}$ *is a Priestley space such that the map* $\rho' : \mathbf{X} \to \langle K(\mathbf{X}), \leq^{\mathcal{P}}, \tau_V\rangle$ *given by* $\rho'(x) = x\uparrow$ *is continuous and order-preserving.*

*Proof.* (2. $\Rightarrow$ 1.) If $\mathcal{G}_{\mathbf{X}} = \langle X, \leq, \leq, \mathcal{A}_\tau\rangle$ is a $\mathbf{K}^+$-space, and $\mathcal{A}_\tau$ is the algebra of the clopen increasing subsets of $\mathbf{X}$, then $\mathbf{X} = \mathbf{X}_{g_{\mathbf{X}}}$ is a Priestley space, and $\mathcal{A}_\tau$ is closed under $\diamond_\leq$, i.e. for every clopen increasing subset $U$ of $X$ $U\downarrow = \diamond_\leq(U)$ is clopen increasing.
(1. $\Rightarrow$ 2.) D1 holds because $\mathbf{X}_{\mathcal{G}_{\mathbf{X}}} = \mathbf{X}$, and so D3 holds by Corollary 2.3.8. For every $x \in X$, $x\uparrow = (\leq \circ \leq)[x]$, and $x\uparrow \subseteq (\leq \circ \leq^{-1})[x]$, hence $x\uparrow = (\leq \circ \leq)[x] \cap (\leq \circ \leq^{-1})[x]$, which is D4. For every clopen increasing subset $U$ of $X$, $\diamond_\leq(U) = U\downarrow$, which is clopen increasing by assumption, and $\Box_\leq(U) = X \setminus \diamond_\leq(X \setminus U) = X \setminus ((X \setminus U)\downarrow) = X \setminus (X \setminus U) = U$.
(2. $\Rightarrow$ 3.) It immediately follows from Proposition 4.1.2, since $\rho$ is $\rho_{\mathcal{G}_{\mathbf{X}}}$.
(3. $\Rightarrow$ 2.) It immediately follows from Proposition 4.3.2, since $\mathcal{G}_{\mathbf{X}}$ is $\mathcal{G}_\rho$.
(3. $\Rightarrow$ 4.) In order to show that $\rho'$ is continuous, by Corollary 3.2.4 it is enough to show that ${\rho'}^{-1}[t(U)]$ and ${\rho'}^{-1}[m(U)]$ are clopen for every $U$ clopen increasing or clopen decreasing subset of $\mathbf{X}$. By item 4 of Lemma 3.3.2, ${\rho'}^{-1}[t(U)] = {\rho'}^{-1}[\pi^{-1}[\pi[t(U)]]] = \rho[\pi[t(U)]]$ which is clopen, for $\pi[t(U)]$ is a clopen subset of $\mathbf{V}(\mathbf{X})$ and $\rho$ is continuous by assumption. The proof that ${\rho'}^{-1}[m(U)]$ is a clopen subset is similar.
(4. $\Rightarrow$ 3.) The canonical projection $\pi : \mathbf{X} \to \mathbf{V}(\mathbf{X})$ is continuous and order-preserving (see item 2 of Lemma 3.3.2), $\rho'$ is continuous and order-preserving by assumption, hence $\rho = \pi \circ \rho'$ is continuous and order-preserving. □

The equivalence between items 3 and 4 of the proposition above implies that not for every Èsakia space $\mathbf{X}$ the map $\rho : \mathbf{X} \to \mathbf{V}(\mathbf{X})$ given by $\rho(x) = \pi[x\uparrow]$ is a $\mathbf{V}$-coalgebra, because the map $\rho' : \mathbf{X} \to \langle K(\mathbf{X}), \leq^{\mathcal{P}}, \tau_V \rangle$ given by $\rho'(x) = x\uparrow$ might not be order-preserving:

**Example 5.2.7.** *Let us consider the space $\mathbf{X} = \langle X, \leq, \tau \rangle$, where $X = \{a, b, c\}$, $\leq = \{\langle a, b \rangle, \langle a, c \rangle\} \cup \Delta$ and $\tau$ is the discrete topology. It is easy to see that $\mathbf{X}$ is a Priestley space such that for every clopen subset $U$, $U\downarrow$ is clopen, and so $\mathbf{X}$ is an Èsakia space. By Lemma 3.1.3, the map $\rho' : \mathbf{X} \to \langle K(\mathbf{X}), \leq^{\mathcal{P}}, \tau_V \rangle$ given by $\rho'(x) = x\uparrow$ is order-preserving iff $(\leq \circ \leq^{-1}) \subseteq (\leq^{-1} \circ \leq)$, i.e. for every $x, y \in X$ such that $z \leq x$ and $z \leq y$ for some $z \in X$, there exists $z' \in X$ such that $x \leq z'$ and $y \leq z'$. Clearly, this condition does not hold for $b, c \in X$.*

## 6. Open problems

**Closed and convex subsets.** In order to be able to define the correspondence from $\mathbf{Coalg}(\mathbf{V})$ to $\mathbf{K}^+$, we relied on the fact that the $\equiv$-equivalence classes of any Priestley space $\mathbf{X} = \langle X, \leq, \tau \rangle$ can be identified with the closed and convex subsets of $\mathbf{X}$ (see Lemma 4.2.1). So a natural alternative way of defining $\mathbf{V}(\mathbf{X})$ would be to consider the space $\langle \mathcal{F}_{\mathbf{X}}, \leq^{\mathcal{P}}, \tau'_V \rangle$, where $\mathcal{F}_{\mathbf{X}}$ is the set of the closed and convex subsets of $\mathbf{X}$, $\leq^{\mathcal{P}}$ is the Egli-Milner power order restricted to $\mathcal{F}_{\mathbf{X}} \times \mathcal{F}_{\mathbf{X}}$, and $\tau'_V$ is the topology defined by taking all the subsets of the form $m(A) = \{F \in \mathcal{F}_{\mathbf{X}} \mid F \cap A \neq \emptyset\}$, $t(A) = \{F \in \mathcal{F}_{\mathbf{X}} \mid F \subseteq A\}$ for every $A \in \tau$, as a subbase. Such definition would be more desirable in many respects, for example it would make the connection with analogous constructions on *spectral spaces* more transparent, but at the moment we do not have proof that, for every Priestley space $\mathbf{X}$, the space $\langle \mathcal{F}_{\mathbf{X}}, \leq^{\mathcal{P}}, \tau'_V \rangle$ is compact. A sufficient condition for the compactness of such space is that the $\mathcal{F}_{\mathbf{X}}$ is a closed subset of $\langle K(\mathbf{X}), \tau_V \rangle$. Notice that such condition is not implied by the facts stated in Lemma 4.2.1, however such facts would imply that the $\equiv$-quotient space $\mathbf{V}(\mathbf{X})$ is homeomorphic to the space $\langle \mathcal{F}_{\mathbf{X}}, \leq^{\mathcal{P}}, \tau'_V \rangle$ under the hypothesis that $\mathcal{F}_{\mathbf{X}}$ is a closed subset of $\langle K(\mathbf{X}), \tau_V \rangle$.

**The old and the new semantics.** Coalgebras of the Vietoris endofunctor on **Pri** are endowed with a notion of bisimulation. The relation between such notion and the standard one is still to be investigated. More in general, the specific features of $\mathbf{Coalg}(\mathbf{V})$ as a semantics for PML are to be studied, and a general characterization of the logics which admit an analogous semantics should be matter of further investigation.

**Coalgebras for endofunctors on Priestley spaces.** In [12], a special class of endofunctors on **Set** is defined, namely the class of *Kripke polinomial functors*. Such class of functors is inductively defined using a formal grammar which includes the covariant powerset functor $\mathcal{P}$, and a soundness and completeness theorem is given for the coalgebraic modal logics associated with coalgebras of Kripke polinomial functors. In [15] an analogous class of endofunctors on the category of Stone spaces is defined using a formal grammar which includes the Vietoris endofunctor $\mathbf{K}$. A possible development of this work would be defining an analogous class of endofunctors on **Pri**, in which the role of $\mathcal{P}$ or $\mathbf{K}$ would be played by the endofunctor $\mathbf{V}$, and studying the associated coalgebraic (positive) modal logics. A further step in such research project would be studying the connections between such constructions and the framework presented by Abramsky in [1].

**Dual equivalence.** Given an endofunctor $\mathbf{H}$ on a category $\mathcal{C}$, the category $\mathbf{Alg}(\mathbf{H})$ of the $\mathbf{H}$-algebras is dually equivalent to the category $\mathbf{Coalg}(\mathbf{H}^{\mathbf{op}})$ of the $\mathbf{H}^{\mathbf{op}}$-coalgebras. As **Pri** is equivalent to $\mathbf{BDL}^{\mathbf{op}}$, where **BDL** is the category of bounded distributive lattices and their homomorphisms, and the category **PMA** of positive modal algebras and their homomorphisms is dually equivalent to $\mathbf{K}^+$, then, as a consequence of the equivalence of categories established in Section 4, the following chain of categorical equivalences holds for some endofunctor $\mathbf{H}$ on **BDL**:

$$\mathbf{PMA}^{\mathbf{op}} \backsimeq \mathbf{K}^+ \backsimeq \mathbf{Coalg}(\mathbf{V}) \backsimeq \mathbf{Coalg}(\mathbf{H}^{\mathbf{op}}) \backsimeq \mathbf{Alg}(\mathbf{H})^{\mathbf{op}},$$

hence **PMA** $\simeq$ **Alg**(**H**) for some endofunctor **H** on **BDL**. This is analogous to the case treated in [15] (i.e. the category **BAO** of Boolean algebras with operators is equivalent to the category **Alg**(**G**) of the **G**-algebras, for some endofunctor **G** on Boolean algebras), and from the existence of the initial object in **Alg**(**H**) we can deduce the existence of the final object in **Coalg**(**V**). The endofunctor **H** and its connections with **V** are worth further investigation, starting with its actual definition.

**Èsakia spaces.** As we saw in section 5, Èsakia spaces and strongly isotone and continuous maps form a subcategory **E** of the category **Pri** of Priestley spaces and monotone and continuous maps, so a natural question that arises is whether for every Èsakia space **X**, **V**(**X**) is an Èsakia space. If this is the case, then analogous constructions and facts could be extended to **E**.

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