Estimates of large failure times from the theory of stochasticly perturbed dynamical systems with and without feedback

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Summary: In this paper the asymptotic theory of stochastic exit for dynamical systems with random perturbations is related to the analysis of life time of a physical system. In case of feedback the control is such that the expected exit time is maximized.

1. Introduction

The life time analysis of physical systems is mostly seen as a pure statistical problem. In this paper we take a different approach and analyse the dynamics of a stochasticly perturbed system.

In section 2 we characterize the type of noise that may act upon a system and show, by the example of a nonlinear spring, the bounds in state space that have to be satisfied in order to have a proper functioning physical system.

In section 3 an asymptotic solution of the Fokker-Planck equation is presented, while in section 4 it is shown in which way this solution indicates the most likely type of failure of the system. Moreover, it is demonstrated how observed lifetimes for systems under strain (experiments) can be used to predict life times under regular conditions.

In section 5 we consider a linear control system and construct the feedback that maximizes the exit time.

Finally, in section 6 stochastic difference equations are formulated, which are used in the Monte Carlo simulation of stochastic dynamical systems.

2. Dynamical systems perturbed by noise

We consider a system given by n state variables depending continuously upon time and satisfying a system of coupled nonlinear differential equations of the type

(2.1) $$\frac{dx}{dt} = f(x), \quad x(t) = (x_1(t),\dots,x_n(t)).$$

Let for this system the origin x = 0 be a stable equilibrium point. Then we consider a bounded subdomain Ω of state space. This domain Ω contains the origin and no trajectories leave Ω for increasing t.

Example 1.1. A nonlinear spring satisfying

(2.2) $$M\frac{d^2z}{dt^2} = g(z, \frac{dz}{dt}), \quad g(0,0) = 0$$

can be transformed into (2.1):

(2.3) $$\frac{dx_1}{dt} = x_2$$

(2.3b) $$\frac{dx_2}{dt} = M^{-1}g(x_1,x_2).$$

Next we analyse the effect of small additive noise terms to the system (2.1). Thus, we will investigate a system of coupled stochastic differential equations of the type

(2.4) $$dX_i = f_i(X)dt + \varepsilon\sum_{j=1}^{m}\sigma_{ij}(X)R_j(t)dt, \quad i = 1,\dots,n$$

with $0 < \varepsilon << 1$. The noise terms $R_j(t)$, satisfying $E\{R_j(t)\} = 0$, are characterized by the autocorrelation function

(2.5) $$G(\tau) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} R(t)R(t+\tau)d\tau$$

or the spectral function

(2.6) $$S(\omega) = \int_{-\infty}^{\infty} e^{-i\omega\tau}\,G(\tau)d\tau,$$

see Gardiner [1].

For $G(\tau) = \delta(\tau)$ with $\delta(\tau)$ the Dirac delta function, we have a so-called white noise process: $S(\omega) = 1$ and all frequencies have equal intensity. In that case (2.4) is written as

$$(2.7) \qquad dX_i = f_i(X)dt + \varepsilon \sum_{j=1}^{m} \sigma_{ij}(X)dW_j, \quad i = 1,\dots,n,$$

where dW_j is the Wiener increment.

Another possibility is to consider the noise as the output of a damped linear system forced by white noise. For one component this is a so-called Ornstein-Uhlbeck process:

$$(2.8) \qquad dR = -\alpha R dt + \beta dW.$$

It is easily verified that

$$(2.9) \qquad G(\tau) = \frac{\beta^2}{2\alpha} e^{-\alpha|\tau|} \quad \text{and} \quad S(\omega) = \frac{\beta^2}{\alpha^2+\beta^2}.$$

Since in the spectral function the higher frequencies have lower intensities the process is called "red noise". It is noted that for $\alpha = \beta$ the red noise forces the dynamical system (2.4) with the same intensity as in the white noise case. Moreover, for $\alpha = \beta \to \infty$ R tends to the white noise process.

Example 2.2. We consider the nonlinear spring as part of the suspension of a car which is in a constant forward motion over a somewhat bumpy road, see fig. 1a. Let the spectral function of the random component of the force acting upon the spring be as depicted in fig. 1b. Then we may take the random force as a red noise process. By using $R = x_3/\varepsilon$ we write (2.3) as

$$(2.10a) \qquad dx_1 = x_2 dt,$$

$$(2.10b) \qquad dx_2 = \{M^{-1}g(x_1,x_2)+x_3\}dt,$$

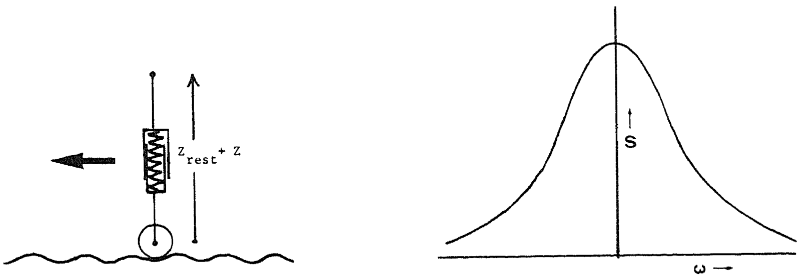
$$(2.10c) \qquad dx_3 = -\alpha x_3 dt + \varepsilon\alpha dW,$$

which is a system of the type (2.7).

From point of view of life time analysis of the spring we wish to have an estimate for the frequency by which the spring leaves the normal operation range. This range is determined by the requirements of bounded displacement and bounded acceleration:

(2.11a) $A < X_1 < B,$

(2.11b) $|M^{-1}g(X_1,X_2)+X_3| < C.$



(a) The physical system

(b) Spectral function of random force upon spring

Fig. 1. Influence of road upon spring

3. The Fokker-Planck equation

For the stochastic state variables $X_i(t)$ satisfying

(3.1a) $$dX_i = f_i(X)dt + \varepsilon \sum_{j=1}^{m} \sigma_{ij}\, dW_j,$$

(3.1b) $$X_i(0) = 0, \quad i = 1,\dots,n$$

the probability density $p(x,t)$ of being at x at time t satisfies the Fokker-Planck equation

(3.2) $$\frac{\partial p}{\partial t} = \tfrac{1}{2}\varepsilon^2 \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 p}{\partial x_i \partial x_j} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f_i(x)p, \quad p(x,0) = \delta(x)$$

with

$$a_{ij} = \sum_{k=1}^{m} \sigma_{ik}\sigma_{kj}^T.$$

It is noted that we took σ_{ij} independent of X, which is not an essential restriction for the method of analysis. It simplifies some of the computations.

For $t \to \infty$ a stationary distribution p(x) exists with

$$(3.3) \qquad M_\varepsilon p \equiv \tfrac{1}{2}\varepsilon^2 \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 p}{\partial x_i \partial x_j} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f_i(x)p = 0.$$

Making a WKB-Ansatz, we assume that asymptotically

$$(3.4) \qquad p(x) \approx w(x)e^{-Q(x)/\varepsilon^2}, \quad Q(0) = 0.$$

Substitution in (3.3) yields

$$(3.5) \qquad \tfrac{1}{2} \sum_{i,j=1}^{n} a_{ij} \frac{\partial Q}{\partial x_i} \frac{\partial Q}{\partial x_j} + \sum_{i=1}^{n} f_i(x) \frac{\partial Q}{\partial x_i} = 0,$$

which is the eikonal equation known from geometrical optics. Its solution is positive definite for $|x| > 0$. For small values of $|x|$ we take

$$(3.6) \qquad Q(x) \approx \tfrac{1}{2} \sum_{i,j=1}^{n} P_{ij} x_i x_j = \tfrac{1}{2} x^T P x$$

with $P = H^{-1}$ satisfying

$$(3.7) \qquad FH + HF^T + A = 0,$$

where

$$(3.8) \qquad A = \{a_{ij}\}_{n\times n}, \quad F = \left\{\frac{\partial f_i}{\partial x_j}\bigg|_{x=0}\right\}_{n\times n}.$$

To compute Q for larger values of x we have to integrate along rays in state space starting at a small sphere in the origin where (3.6)-(3.8) hold. The ray method is based upon the observation that we may write (3.5) as

$$(3.9) \qquad H(x,p) \equiv \tfrac{1}{2} \sum_{i,j=1}^{n} a_{ij} p_i p_j + \sum_{i=1}^{n} f_i(x)p_i = 0, \quad p_i = \frac{\partial Q}{\partial x_i} \quad .$$

and that along a ray

(3.10a) $$\frac{dx_i}{ds} = \frac{\partial H}{\partial p_i},$$

(3.10b) $$\frac{dp_i}{ds} = -\frac{\partial H}{\partial x_i},$$

(3.10c) $$\frac{dQ}{ds} = \tfrac{1}{2} \sum_{i=1}^{n} a_{ij}\, p_i\, p_j,$$

see Ludwig [4] and Grasman and Lankelma [2].

4. Expected exit time and life time

In this section we construct an asymptotic estimate for the expected time of residence in the domain Ω.

Let $Q(x)$ take its minimal value in $\partial\Omega$ at $\hat{x}$ and let

(4.1) $$K = \min_{\partial\Omega} Q(x) = Q(\hat{x}),$$

then $\hat{x} \in \partial\Omega$ is the most likely point of exit and for $\varepsilon \to 0$ the distribution of exit points tends to $\delta(x-\hat{x})$, see Appendix A. Thus, in case of failure of a system, we may conclude in this way about the most likely type of failure.

The expected time, needed to reach the boundary, is asymptotically

(4.2) $$T = \alpha e^{K/\varepsilon^2}, \quad 0 < \varepsilon << 1,$$

where α is determined by $w(x)$, which follows from higher order WKB-approximation, see Appendix B. The exit time is asymptotically the same for all points of Ω bounded away from $\partial\Omega$. This is understood from the fact that the drift towards the equilibrium is of a larger order of magnitude than the diffusion. More details about the problem of stochastic exit are found in Schuss [5].

Thus, we have found that

(4.3) $$\ell n T = K/\varepsilon^2 + O(1).$$

This formula can be used to find expected exit times for very small ε values from experiments with the physical system for larger ε, say $\varepsilon_{exp.}$, without any analytical knowledge of its dynamics. Finding an average exit time $T_{exp.}$ we obtain

(4.4) $$\ell nT = \frac{\varepsilon^2_{exp.}}{\varepsilon^2} \ell nT_{exp.}.$$

Carrying out experiments at two ε-values one may also eliminate α in in (4.2) and obtain a more accurate value of T(ε). The average frequency of leaving the domain of regular operation, $T(\varepsilon)^{-1}$, is a measure for the life time of the physical system. The present approach suggests that for life times a formula similar to (4.4) holds.

Example 4.1. For a bi-stable system the expected time of residence T_i in the domain of attraction of the two stable equilibria can be computed. Let us consider the motion of a point with unit mass at a surface V(x), see fig.2,

(4.5a) $$dX_1 = X_2 dt,$$

(4.5b) $$dX_2 = (1 - cX_2 - \frac{dV}{dx}(X_1))dt + \varepsilon dW.$$

Let Ω_i be the domain of attraction of $x^{(i)}$, i = 1,2 (for the deterministic system). Necessarily the minimum value of Q at $\partial\Omega^{(i)}$ is attained at the unstable equilibrium $x^{(0)}$. We have now

(4.6a) $$\ell nT_i = K_i/\varepsilon^2 + O(1),$$

(4.6b) $$K_i = \min_{\partial\Omega^{(i)}} Q(x) = Q(x^{(0)}).$$

From the theory it is deduced that the separatrix is most likely crossed at the saddle point $x^{(0)}$. It is also most likely that the system is then almost at rest. It will take a time

(4.7) $$\ell nT_0 = \ell n\ell n\varepsilon^{-1} + O(1)$$

to leave a neighborhood of $x^{(0)}$. Starting at $x = x^{(0)}$ the probability of arriving near $x^{(1)}$ or $x^{(2)}$ is fifty-fifty. Consequently, the bi-stable system may as well be modeled by a three states Markov chain with transition matrix

$$(4.8) \qquad M = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$$

and with residence times in those states as given above.

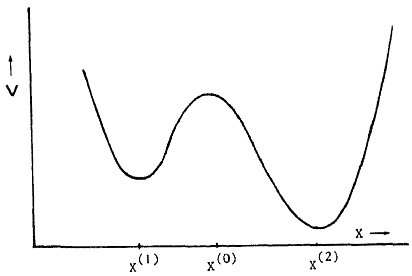


Fig.2. A bi-stable system

5. Linear control systems

Krtolica [3] investigates the linear control system

$$(5.1a) \qquad dX = FXdt + u(Y)dt + \varepsilon\sigma_V dV,$$

$$(5.1b) \qquad dY = CXdt + \varepsilon\sigma_W dW, \quad 0 < \varepsilon << 1,$$

where

$$X = (X_1,\ldots,X_n) \quad \text{and} \quad Y = (Y_1,\ldots,Y_m).$$

The vector type white noise processes are

$$dV = (dV_1,\ldots,dV_p), \quad dW = (dW_1,\ldots,dW_q).$$

Moreover, the pair (F,C) is assumed to be observable and (F,σ_V) is controllable.

We take the linear control law

$$(5.2) \qquad u\,dt = -R\,dY$$

and construct the matrix R that maximizes the expected residence time in $\Omega \subset \mathbb{R}^n$ with the origin contained in Ω.

For (5.1)-(5.2) we write

$$(5.3) \qquad dX = (F-RC)X\,dt + \varepsilon(\sigma_V dV - R\sigma_W dW)$$

or, replacing dV_i and dW_j by n Wiener increments dU_k,

$$(5.4) \qquad dX = (F-RC)X\,dt + \varepsilon\sigma dU$$

with

$$(5.5) \qquad \sigma^2 = \sigma_V\sigma_V^T + R\sigma_W\sigma_W^T R^T.$$

For the system (5.4) the stationary Fokker-Planck equation is solved, as in section 3. Since the system is linear the function $Q(x)$ of the WKB-approximation is of the form

$$(5.6) \qquad Q(x) = \tfrac{1}{2}\, x^T H^{-1} x$$

with H satisfying

$$(5.7a) \qquad (F-RC)H+H(F-RC)^T+RG_W R^T+G_V = 0,$$

$$(5.7b) \qquad G_V = \sigma_V\sigma_V^T, \quad G_W = \sigma_W\sigma_W^T.$$

We have to choose R such that the minimum value of Q at $\partial\Omega$ is maximized:

$$(5.8) \qquad K = \max_R \min_{\partial\Omega} Q(x).$$

We may interchange the minimization and maximization, as the boundary $\partial\Omega$ is independent of R. From results by Wonham [6] on Riccati equations, it is deduced that $Q(x)$ is at its maximum at $\partial\Omega$ for

$$R = PC^T G_W^{-1} \tag{5.9}$$

with P satisfying the Riccati equation

$$FP + PF^T - PC^T G_W^{-1} CP + G_V = 0. \tag{5.10}$$

6. Stochastic difference equations

Dynamical systems with small random perturbation can be simulated with the Monte Carlo method. To perform the simulation, the Wiener increment dW has to be replaced by a pseudo random generator G(t). Euler's method can then be applied giving the following system of stochastic difference equations

$$X_i(t+h) = X_i(t)+hf_i(X)+\varepsilon\sqrt{h}\, G_i(t), \quad i = 1,\dots,n. \tag{6.1}$$

The time step h gives an error in X of order O(h). We define the stochastic variable

$$\Delta X_i = X_i(t+h)-X_i(t). \tag{6.2}$$

This variable has first and second moments

$$E\{\Delta X_i(t)\} = hf_i(x)+\varepsilon\sqrt{h}\, E\{G_i(t)\} = hf_i(x), \tag{6.3a}$$

$$Var\{\Delta X_i(t)\} = \varepsilon^2 hE\{G_i^2(t)\} = \varepsilon^2 h. \tag{6.3b}$$

Consequently, in unit time the expectation of ΔX_i equals the local vector field f(x) while its variance equals ε^2. The average exit time over a large number of runs is approximated for ε small by the asymptotic expected exit time, which we computed in section 4.

Appendix A

We consider the singularly perturbed Dirichlet problem

(A1a) $$L_\varepsilon u \equiv \tfrac{1}{2}\varepsilon^2 \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} f_i(x) \frac{\partial u}{\partial x_i} = 0 \quad \text{in } \Omega$$

(A1b) $$u = h(x) \text{ at } \partial\Omega$$

with all trajectories of the deterministic system (2.1) entering Ω for increasing t on their way to the stable equilibrium $x = 0$.

Let for the system (3.1a) $q(\tilde{x},x)$ be the probability density of leaving Ω at $\tilde{x} \in \partial\Omega$, if started at $x \in \Omega$. Then we have that

(A2) $$\int_{\partial\Omega} q(\tilde{x},x)h(\tilde{x})\,dS = u(x).$$

From singular perturbation theory it is known that the asymptotic solution of (A1) has the form

(A3) $$u \approx \{h(x)-C\}e^{-\rho/\varepsilon^2} + C,$$

where ρ is the distance of x to $\partial\Omega$ and C an unknown constant. This constant is determined by the divergence theorem using the asymptotic solution $p(x)$ of the stationary Fokker-Planck equation, see (3.3)-(3.10):

(A4) $$\int_{\Omega} pL_\varepsilon u - uM_\varepsilon p\,dV = \int_{\partial\Omega} \tfrac{1}{2}\varepsilon^2 \left(p\frac{\partial u}{\partial n} - u\frac{\partial p}{\partial n}\right) + puf(x)\cdot\nu\,dS,$$

where ν is the outward normal at $\partial\Omega$ and $\partial\cdot/\partial n$ the co-normal derivative

(A5) $$\frac{\partial}{\partial n} = \sum_{i,j=1}^{n} a_{ij}\nu_i \frac{\partial}{\partial x_j}\,.$$

From (A4) we derive, using (A3),

(A6) $$C = \frac{\int_{\partial\Omega} pf(x)\cdot\nu h(x)\,dS}{\int_{\partial\Omega} pf(x)\cdot\nu\,dS}\,,$$

which, because of (3.4), is asymptotically equivalent with

(A7) $$C = h(\hat{x}),$$

where $\hat{x}$ is the point at $\partial\Omega$ with minimal Q.

Appendix B

The expected time $T(x;\varepsilon)$ needed to reach $\partial\Omega$, if starting at $x \in \Omega$, satisfies

(B1) $$L_\varepsilon T = -1 \quad \text{in} \quad \Omega, \qquad T = 0 \quad \text{at} \quad \partial\Omega$$

with the elliptic operator L_ε given in (A1a). Applying the divergence theorem, as we did in (A4), we obtain

(B2) $$\int_\Omega pL_\varepsilon T - TM_\varepsilon p dV = \int_{\partial\Omega} \tfrac{1}{2}\varepsilon^2 (p \frac{\partial T}{\partial n} - T \frac{\partial p}{\partial n}) + pTf\cdot\nu dS.$$

The formal asymptotic solution of (B1) has the form

(B3) $$T \approx -Ce^{-\rho/\varepsilon^2} + C,$$

where ρ is the distance of x to $\partial\Omega$ and C an unknown constant. Substitution of (3.4) and (B.3) in (B2) yields

(B4) $$C = \frac{\int_\Omega p\ dV}{-\frac{1}{2}\int_{\partial\Omega} pf\cdot\nu dS} .$$

The volume integral has its largest contribution from an ε-neighborhood of the origin, while the integral over $\partial\Omega$ has its minimal value at $\hat{x} \in \partial\Omega$, so that

(B5) $$C \approx e^{K/\varepsilon^2}.$$

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