BIT 25 (1985), 380–385

ON M-FUNCTIONS AND NONLINEAR RELAXATION METHODS

E. J. VAN ASSELT

Department of Numerical Mathematics, Mathematisch Centrum, Kruislaan 413, 1098 SJ Amsterdam, The Netherlands

Abstract.

Globally convergent nonlinear relaxation methods are considered for a class of nonlinear boundary value problems (BVPs), where the discretizations are continuous M-functions.

It is shown that the equations with one variable occurring in the nonlinear relaxation methods can always be solved by Newton's method combined with the Bisection method. The nonlinear relaxation methods are used to get an initial approximation in the domain of attraction of Newton's method. Numerical examples are given.

1980 Mathematics subject classification: 65 H 10, 65 N 10.
Keywords and Phrases: nonlinear relaxation methods, Newton-Bisection method, M-functions.

1. Introduction.

In section 2 we introduce nonlinear singularly perturbed elliptic BVP in 2 dimensions, where the discretizations with the first order Osher-Engquist scheme (cf. [5]) are continuous M-functions. The solutions of the discrete equations are unique, and with the theory of M-functions it follows that the nonlinear Jacobi (NLJAC) and the nonlinear SOR (NLSOR) process are globally convergent (cf. [4]).

In section 3 we show that the equations with one variable occurring in NLJAC and NLSOR can always be solved by Newton's method combined with the bisection method. We will give a 2-D example with an initial approximation for which NLJAC with Newton's method (NLJAC-N) does not converge, whereas NLJAC with Newton's method combined with bisection (NLJAC-NB) does.

In section 4 we use NLSOR-NB to get an initial approximation in the domain of attraction of Newton's method for the whole system of equations (NEWT).

2. A class of nonlinear BVPs.

We consider the following class of nonlinear BVPs in two dimensions:

$$(2.1)\quad N_\varepsilon u \equiv -\varepsilon_1 \partial^2 u/\partial x^2 - \varepsilon_2 \partial^2 u/\partial y^2 + a_1(u)\partial u/\partial x + a_2(u)\partial u/\partial y + g(u,x,y) = 0, \text{ on } \Omega = \{(x,y) | 0 < x < 1, 0 < y < 1\}, \text{ and}$$

Received January 1984.

(2.2) $u(x, y) = b(x, y)$ on $\delta\Omega = \bar{\Omega}\backslash\Omega$, with $\varepsilon = (\varepsilon_1, \varepsilon_2)$, $\varepsilon_1, \varepsilon_2 > 0$;

$a_1, a_2 \in C^1(\mathbb{R})$, $g \in C^2(\mathbb{R} \times \Omega)$, $(\partial/\partial u)g(u, x, y) \geqq \mu > 0$ for all $(x, y) \in \Omega$,

$u \in C^2(\Omega)$ and $b \in C(\delta\Omega)$.

For the discretization of this problem we introduce G_h, a uniform rectangular grid with mesh-size $h = (h_1, h_2)$, $h_1 = 1/m$, $h_2 = 1/n$ and meshpoints $x_{ij}^h = (ih_1, jh_2)$, $0 \leqq i \leqq m$, $0 \leqq j \leqq n$, and use the one sided Osher-Engquist difference scheme (cf. [5]):

$$
\begin{aligned}
(2.3)\quad (N_{\varepsilon,h}u^h)_{ij} \equiv\ & -\varepsilon_1 h_1^{-2}(u_{i+1\,j}^h - 2u_{ij}^h + u_{i-1\,j}^h) - \varepsilon_2 h_2^{-2}(u_{ij+1}^h - 2u_{ij}^h + u_{ij-1}^h) \\
& + h_1^{-1}(\Delta_x f_{1^-}(u_{ij}^h) + \nabla_x f_{1^+}(u_{ij}^h)) \\
& + h_2^{-1}(\Delta_y f_{2^-}(u_{ij}^h) + \nabla_y f_{2^+}(u_{ij}^h)) + g(u_{ij}^h, x_{ij}^h) \\
=\ & 0 \text{ for all } x_{ij}^h \in G_h \cap \Omega,
\end{aligned}
$$

(2.4) $(N_{\varepsilon,h}u^h)_{ij} \equiv u_{ij}^h = b(x_{ij}^h)$ for all $x_{ij}^h \in G_h \cap \delta\Omega$, where for $m = 1, 2$:

$$
f_{m^+}(u_{ij}^h) = \int^{u_{ij}^h} \max(0, a_m(s))ds, \qquad f_{m^-}(u_{ij}^h) = \int^{u_{ij}^h} \min(0, a_m(s))ds,
$$

$$
\Delta_x u_{ij}^h = u_{i+1\,j}^h - u_{ij}^h, \qquad \nabla_x u_{ij}^h = u_{ij}^h - u_{i-1\,j}^h,
$$

$$
\Delta_y u_{ij}^h = u_{ij+1}^h - u_{ij}^h, \qquad \nabla_y u_{ij}^h = u_{ij}^h - u_{ij-1}^h.
$$

Denoting the space of gridfunctions on G_h with GF_h, and extending the results of Lorentz (cf. [3]) to 2 dimensions, we can easily prove that $N_{\varepsilon,h}$ is a continuous M-function from GF_h onto GF_h.

In the following section we use the property that M-functions are strictly diagonally isotone (cf. [4]).

3. The nonlinear relaxation methods.

In this section we consider NLJAC and NLSOR, and give a 2-D example for which NLJAC does not converge when the resulting scalar nonlinear equations are solved with Newton's method only, while NLJAC converges when we use Newton's method combined with bisection.

Let $F(x) = 0$ with $F = (f_1(x), \ldots, f_n(x))$, $x = (x_1, \ldots, x_n)$ be a system of n nonlinear equations. Let $x^{(0)}$ be an initial approximation to the solution, then $x^{(k+1)}$ is obtained by solving for x_i the ith equation:

(3.1) $f_i(x_1^{(k)}, \ldots, x_{i-1}^{(k)}, x_i, x_{i+1}^{(k)}, \ldots, x_n^{(k)}) = 0$ for NLJAC, or

(3.2) $f_i(x_1^{(k+1)}, \ldots, x_{i-1}^{(k+1)}, x_i, x_{i+1}^{(k)}, \ldots, x_n^{(k)}) = 0$ for NLSOR, and setting

(3.3) $x_i^{(k+1)} = x_i^{(k)} + \omega(x_i - x_i^{(k)}), \quad \omega \in (0, 1], \quad i = 1, 2, \ldots, n.$

For our class of equations F is a continuous M-function, and both NLJAC and NLSOR converge globally to the unique solution of (2.3), (2.4), provided the (generally nonlinear) equations (3.1), (3.2) for which a unique solution exists can be solved (cf. [4], Theorem 13.5.9).

For M-functions $df_i/dx_i > 0$. This implies that for a positive (negative) function value of an iterant $x_{i(r)}$ the next Newton iterant $x_{i(r+1)}$ is always smaller (greater) than $x_{i(r)}$, and we can start with only a single initial approximation. When the function values of the subsequent iterants never change sign only Newton iteration is applied, and we have monotone convergence. However if the function value of an iterant $x_{i(r)}$ changes sign we have an interval round the solution: $I_{i(r)} = (\min(x_{i(r-1)}, x_{i(r)}), \max(x_{i(r-1)}, x_{i(r)})$. This interval is adapted in each iteration step. If $I_{i(r+1)} \subset I_{i(r)}$, Newton's method is applied, if this is not the case one step of the bisection method is applied.

As the bisection method always converges the above-mentioned combination always converges.

This process is described in the following Algol-like procedure NEWTON BISECTION. The procedure takes the current value of x as an initial value and delivers in x an approximate solution to $f(x) = 0$, so that $|f(x)| < \text{tol}$.

(3.4)
```
comment f is the function whose zero should be determined,
    df the derivative (df > 0) and tol a given tolerance;
procedure newton_bisection:
begin integer sgn = sign(f(x)); real xold, xnew, xneg, xpos;
xnew := x;
while abs (f(xnew)) > tol and sign(f(xnew)) = sgn
do xold := xnew; xnew := xold-f(xold)/df(xold) od;
if sign(f(xnew)) > 0 then xpos := xnew; xneg := xold
                     else xneg := xnew; xpos := xold fi;
xold := xnew;
while abs(f(xnew)) > tol
do while abs(xnew-xold) < xpos-xneg and abs(f(xnew)) > tol
   do if sign(f(xnew)) > 0 then xpos := xnew; else xneg := xnew fi;
      xold := xnew; xnew := xold-f(xold)/df(xold) od;
   if abs(f(xnew)) > tol then xnew := (xneg+xpos)*0.5 fi od;
x := xnew
end
```

Both NLJAC-NB and NLSOR-NB converge to the solution and in the course of these processes in each point we will have a better initial approximation for the Newton-Bisection process. This means that for a given tolerance TOL in the course of the NLJAC-NB and NLSOR-NB process the required number of Newton-Bisection iteration steps decreases (cf. Table 1, 2, 3). This phenomenon is used in section 4, where NLJAC-NB and NLSOR-NB are used to get an initial approximation for Newton's method.

In example (3.5) NLJAC-N does not converge, whereas NLJAC-NB does.

(3.5) EXAMPLE.

We consider (2.1)–(2.4) with $\varepsilon_1 = \varepsilon_2 = 10^{-6}$, $h_1 = h_2 = 1/32$, $a_1(u) = a_2(u) \equiv$ $\equiv \arctan(4u)$, $g(u, x, y) \equiv u$ and b defined by

$$\begin{aligned}
b(0, y) &= 0 && 0 \leqq y \leqq 1 &,\\
b(1, y) &= -1 && 0 \leqq y \leqq 1 - 10^{-3},\\
b(1, y) &= 10^3 y - 10^3 && 1 - 10^{-3} \leqq y \leqq 1 &,\\
b(x, 0) &= -1 && 10^{-3} \leqq x \leqq 1 &,\\
b(x, 0) &= -10^3 x && 0 \leqq x \leqq 10^{-3} &,\\
b(x, 1) &= 0 && 0 \leqq x \leqq 1 &.
\end{aligned}$$

For the gridfunction u^h in (2.3), (2.4) we take the lexicographical ordering (i.e. $u^h = (\ldots, u^h_{ij}, u^h_{ij+1}, \ldots, u^h_{i+1j}, u^h_{i+1j+1})$). As initial approximation for NLJAC-N ($\omega = 1$) and NLJAC-NB ($\omega = 1$) we take the gridfunction $u^{(0)}$ defined by $u^{(0)}_{ij} \equiv -1$ on $G_h \cap \Omega$ and $u^{(0)}_{ij} \equiv b$ on $G_h \cap \delta\Omega$.

NLJAC-N does not converge, while we need $k = 4$ NLJAC-NB steps to obtain $\|N_{\varepsilon, h} u^{(k)}\| \leqq 10^{-12}$ with $\|\cdot\|$ the maximum norm.

For NLJAC-NB in Table 1 we give the maximum number of Newton-Bisection iterations for TOL $= 10^{-8}$ (cf. (3.4)).

Table 1. *The maximum number of Newton-Bisection iterations per NLJAC-NB iteration step i.* ($TOL = 10^{-8}$).

i	1	2	3	4
N	42	2	1	1

4. NLJAC-NB and NLSOR-NB combined with NEWT.

In this section we use NLJAC-NB and NLSOR-NB to get an initial approximation within the domain of attraction of NEWT.

There are other methods which use time-steps combined with NEWT (cf. [1]). We consider here NLJAC-NB and NLSOR-NB because these can be used as

relaxation processes in Multigrid methods (cf. [6]). Moreover the number of Newton-Bisection steps can be used as a criterion to switch from NLJAC-NB or NLSOR-NB to NEWT, as the number of Newton-Bisection iterations decreases in the course of these processes.

So instead of trying NEWT after each NLJAC-NB or NLSOR-NB iteration step we try it when the maximum number of Newton-Bisection iterations is sufficiently small.

In the following 1-D example we have an initial approximation for which NEWT does not converge while NLJAC-NB and NLSOR-NB do.

We also show that the combination NLSOR-NB with NEWT is cheaper than NLSOR-NB alone.

(4.1) EXAMPLE.

Consider the 1-D problem

$$(4.2) \qquad N_\varepsilon u = -\varepsilon u'' + (u^2 - 1/4)u' + u = 0 \qquad \text{on } [0, 1] \text{ with } \varepsilon = 10^{-6},$$

$u(0) = 1$, $u(1) = -1$, discretized by the Osher-Engquist scheme (cf. (2.3), (2.4)) on a uniform grid with mesh size 1/20 (cf. [1], [2]).

NEWT with initial approximation $u^{(0)}$ defined by $u_j^{(0)} = 1 - 2x_j$, $j = 0, \ldots, 20$, does not converge while NLJAC-NB and NLSOR-NB ($\omega = 1$) do. We need $k = 58$ NLJAC-NB or $k = 39$ NLSOR-NB steps for $\|N_{\varepsilon,h}u^{(k)}\| \leqq 10^{-6}$, with $\|.\|$ the maximum norm.

For NLJAC-NB and NLSOR-NB respectively in Table 2 and Table 3 we give the maximum number of Newton-Bisection iterations with TOL $= 10^{-8}$ (cf. (3.4)).

Table 2. *The maximum number of Newton-Bisection iterations N per NLJAC-NB iteration step i.* ($TOL = 10^{-8}$).

i	1	2	3–5	6	7–8	9–17	18–38	39–58
N	7	6	4	5	4	3	2	1

Table 3. *The maximum number of Newton-Bisection iterations N per NLSOR-NB iteration step i.* ($TOL = 10^{-8}$).

i	1	2–3	4–5	6	7	8	9–12	13–20	21–29	30–39
N	7	5	4	5	4	3	4	3	2	1

When we try NEWT after the maximum number of Newton-Bisection iterations in NLSOR-NB becomes 2, we need 21 NLSOR-NB steps and 2 NEWT steps for $\|N_{\varepsilon,h}u\| \leqq 10^{-6}$, with $\|.\|$ the maximum norm.

An operations and function evaluations count shows that for 1 NEWT step the number of operations and function evaluations per gridpoint is less than

for 2 NLSOR-NB steps, with one or more Newton-Bisection iterations per grid-point. Hence for this example the combination NLSOR-NB and NEWT is more than 35% cheaper than NLSOR-NB solely.

5. Conclusions.

For continuous M-functions the equations with one variable occurring in the globally convergent nonlinear relaxation methods NLJAC and NLSOR can always be solved by Newton's method combined with the Bisection method, and we need only a single initial approximation. Newton's method alone is not always sufficient.

The nonlinear relaxation methods can be used to get an initial approximation for Newton's method for the whole system of equations, where the decreasing number of Newton-Bisection iterations can be used as switching criterion.

Acknowledgements.

The author wishes to thank dr. P. W. Hemker and drs. P. M. de Zeeuw (Mathematical Centre, Amsterdam) for reading the manuscript.

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