# On Asymptotic Efficiency of the Cox Estimator

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Let $r_n$, $n=1,2,\cdots$ be a non-decreasing sequence of integers. Let

$$N^n = (N_t^n, F_t^n, P_{\alpha,\beta}^n) \quad \text{and} \quad A^n = (A_t^n, F_t^n, P_{\alpha,\beta}^n),$$

$$A_t^n = \int_0^1 col\{Y_s^{in} e^{\beta Z_s^{in}},\ i=1,\cdots,r_n\} d\alpha_s$$

be an $r_n$-variate counting process and its compensator, resp., defined on a stochastic basis which is $n$-th in the sequence

$$(\Omega^n, F^n, \{F_t^n,\ 0\leq t\leq 1\}, P_{\alpha,\beta}^n), \quad n=1,2,\cdots, \tag{1}$$

where $F_t^n$-predictable censoring factors $Y_t^{in}$ and the covariate processes $Z_t^{in}$ satisfy the conditions I-III (cf. [1], p.1105). The cumulative hazard rate $\alpha=\alpha_t$ is nuisance parameter, and $\beta$ is the scalar-valued parameter of interest to be estimated from an observed sample path of $N^n$. It is assumed that $\alpha_t$, $0\leq t\leq 1$, is a continuous nondecreasing function with $\alpha_1<\infty$, and that $\beta$ takes its values from a finite open set $\mathfrak{B}\in R^1$.

I (Asymptotic boundedness). The censoring factors $Y_t^{in}$, $i=1,\ldots,r_n$ take values 0 or 1 (so $N_t^{in}$ only jumps when $Y_t^{in}=1$), and for sufficiently large values of $n$

$$P_{\alpha,\beta}^n \text{ a.s.} \quad sup\{Y_t^{in}|Z_t^{in}|;\ 0\leq t\leq 1,\ i=1,\ldots,r_n\} < \infty$$

II (Asymptotic stability). Let $k_n$, $n=1,2,\cdots$ be a sequence of unboundedly increasing numbers. There exists a family $\{\phi^{(0)}(\beta),\ \beta\in\mathfrak{B}\}$ of deterministic functions $\phi^{(0)}=\phi_t^{(0)}$, $0\leq t\leq 1$ such that the difference

$$d_t^n(\beta) = \frac{1}{k_n}\sum_{i=1}^{r_n} Y_t^{in} e^{\beta Z_t^{in}} - \phi_t^{(0)}(\beta)$$

and its first two derivatives with respect to $\beta$ satisfy the following asymptotic relations: for each $\delta>0$

$$P_{\alpha,\beta}^n(sup\{\left|\frac{\partial^j}{\partial\beta^j}d_t(\beta)\right|,\ 0\leq t\leq 1\}>\delta) \to 0, \quad j=0,1,2,$$

as $n\to\infty$ uniformly in $\beta\in\mathfrak{B}$.

III (Asymptotic regularity). The function $\phi^{(0)}$ and its first two derivatives $\phi^{(1)}=(\partial/\partial\beta)\phi^{(0)}$ and $\phi^{(2)}=(\partial^2/\partial^2\beta)\phi^{(0)}$ are continuous in $\beta\in\mathfrak{B}$ uniformly in $t\in[0,1]$; they are bounded on $\mathfrak{B}\times[0,1]$ and $\phi^{(0)}$ is bounded away from zero. Besides,

$$\sigma^2 \equiv \sigma^2(\alpha,\beta) = \int_0^1 \{\phi^{(2)}(\beta) - \frac{[\phi^{(1)}(\beta)]^2}{\phi^{(0)}(\beta)}\} d\alpha > 0.$$

Define the Cox estimator $\hat{\beta}_n$ for $\beta$ by the condition

$$\sup_{\beta\in\mathfrak{B}} \int_0^1 \ln^T \Psi_s^n(\beta) d\mathbb{N}_s^n = \int_0^1 \ln^T \Psi_s^n(\hat{\beta}_n) d\mathbb{N}_s^n, \quad \ln \Psi^n = col\{\ln \Psi^{in},\ i=1,\ldots,r_n\} \tag{2}$$

with

$$\Psi_t^{in}(\beta) = Y_t^{in} e^{\beta Z_t^{in}} / \sum_{i=1}^{r_n} Y_t^{in} e^{\beta Z_t^{in}}.$$

Before characterising asymptotic properties of $\hat{\beta}_n$ we give the following definitions:

Definition 1. Let $H^n(\alpha,\beta) = (H_t^n(\alpha,\beta), F_t^n, P_{\alpha,\beta}^n)$ be an $r_n$-variate predictable process such that

$$\mathfrak{L}\{\kappa_n^{-\frac{1}{2}} \int_0^1 (H^n, S^n)^T dM^n \,\Big|\, P_{\alpha,\beta}^n\} \Rightarrow N(0,[c_{ij},\ i,j=1,2]) \tag{3}$$

where $M^n = M^n(\alpha,\beta) = N^n - A^n(\alpha,\beta)$, while for each $b \in R^1$ and $a \in L^2(\phi^{(0)} d\alpha)$ $S^n = S^n(a,b) = col\{bZ^{in}+a,\ i=1,\ldots,r_n\}$. Therefore concerning the second component solely the above requirement is met under the Conditions I-III with the limiting variance $c_{22} = \int_0^1 \{b^2\phi^{(2)} + 2ba\phi^{(1)} + a^2\phi^{(0)}\} d\alpha$. Also the limiting covariance matrix in (3) is nonsingular (with entries dependent on $\alpha$, $\beta$, $a$ and $b$, of course).

An estimator $\beta_n^* = \beta_n^*(H^n, c_{11})$ is called asymptotically linear and asymptotically normal if for certain $H^n$ and $c_{11}$ as above and for each $\delta > 0$

$$P_{\alpha,\beta}^n\{\,|(\kappa_n c_{11})^{\frac{1}{2}}(\beta_n^* - \beta) - (\kappa_n c_{11})^{-\frac{1}{2}} \int_0^1 H^n(\alpha,\beta) dM^n(\alpha,\beta)| > \delta\} \to 0 \tag{4}$$

Obviously, under $P_{\alpha,\beta}^n$ $\zeta^n = (\kappa_n c_{11})^{\frac{1}{2}}(\beta_n^* - \beta)$ has the standard normal limiting distribution.

Remark 1. Suppose that the filtration in (1) is minimal: $F_t^n = \sigma(\omega: N_s^n,\ s \leqslant t)$. Define on the $n$-th space of events the probability measure $P_{\alpha^n,\beta^n}^n$ giving to $N^n$ the compensator $A^n(\alpha^n,\beta^n)$ where $\beta^n = \beta + \kappa_n^{-\frac{1}{2}} b \in \mathfrak{B}$, $b \in R^1$ and $\alpha^n$ is a function of the same type as $\alpha$ such that $d\alpha^n/d\alpha = 1 + \kappa_n^{-\frac{1}{2}} a$, $a \in L^2(\phi^{(0)} d\alpha)$.

Proposition 6.2 in [2] allows us to apply here the third LeCam's lemma according to which the limiting distribution of $\zeta^n$ under $P_{\alpha^n,\beta^n}^n$ gets the bias equal to $c_{12} c_{11}^{-\frac{1}{2}}$.

Definition 2. Retain the special situation introduced in Remark 1. An estimator $\beta_R^n$ is called regular in Hajek's sense (at "point" $\alpha$ and $\beta$) if for some nondegenerate distribution function $G$ the following weak convergence takes place: for each $a$ and $b$ as above

$$\mathfrak{L}\{\kappa_n^{\frac{1}{2}}(\beta_R^n - \beta^n) \,|\, P_{\alpha^n,\beta^n}^n\} \Rightarrow G.$$

Remark 2. According to Remark 1 the estimator $\beta_n^*(H, c_{11})$ is Hajek's regular iff $c_{12} = bc_{11}$.

Definition 3. Remove now the condition that the filtration is minimal. As in this case Hajek's definition of regularity loses its meaning, we define the regular in wide sense asymptotically linear and asymptotically normal estimators $\beta_n^*(H^n, c_{11})$ by requiring that in (3) $c_{12} = bc_{11}$.

Now we formulate the statement about asymptotic optimality of $\hat{\beta}_n$ (for the proof consult [1,2]).

Theorem 1. Under the Conditions I-III

(1) $\hat{\beta}_n$ is asymptotically linear and asymptotically normal wide sense regular estimator $\hat{\beta}_n = \beta_n^*(\frac{\partial}{\partial\beta}\log \Psi^n, \sigma^2)$; it attains a lower bound for the asymptotic variances of such estimators $\beta_n^*(H^n, c_{11})$, for $c_{11} \leqslant \sigma^2$.

(2) Suppose in addition that $F_t^n = \sigma\{\omega: N_s^n, s \leqslant t\}$. Then $\hat{\beta}_n$ is Hajek's regular; it attains the lower bound for the risks of such estimators: for any continuous loss function $w$ allowing a polynomial majorant

$$\liminf_{n\to\infty} E_{\alpha,\beta}^n w((\kappa_n\sigma)^{\frac{1}{2}}(\beta_R^n - \beta)) \geqslant \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} w(x) e^{-\frac{1}{2}x^2} dx$$

$$= \lim_{n\to\infty} E_{\alpha,\beta}^n w((\kappa_n\sigma)^{1/2}(\hat{\beta}_n - \beta))$$

References

1. Andersen P.K., Gill R.D., Cox's regression model for counting processes: a large sample study. Ann. Statist., 1982, Vol. 10, No. 4, p. 1100-1120.

2. Dzhaparidze K., On asymptotic inference about intensity parameters of a counting process. in *Papers on semiparametric models at the ISI centenary session (with discussion)*, R.D. Gill and M. Voors (eds.), Report MS-R86XX, CWI, Amsterdam.