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Fuzzy Logic and Mathematical Morphology

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ABSTRACT
In this report we investigate the general theory of grey-scale morphology within the framework of complete lattices and fuzzy logic. This includes grey-scale granulometries, hit-or-miss operators for grey-scale images, rank operators, and connected operators. We also show that the Matheron's representation theory does not hold for general grey-scale images and we present some results related to the representation theory. Besides these, in this report, we put forward a new approach to fuzzy morphology through the extension of infimum, supremum, and conjunction.

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1. Introduction

Mathematical morphology was founded in the early sixties by Matheron [10] and Serra [13] as a novel geometry-based technique for image processing and analysis. Originally, mathematical morphology was developed for binary images and used simple concepts from set theory and geometry such as set inclusion, intersection, union, complementation, and translation. This resulted in a collection of tools, called *morphological operators*, which are eminently suited for the analysis of shape and structure in binary images [16]. The most well-known of these operators are erosion and dilation. In binary morphology, a series of theories, such as the well-known Matheron's representation theorem, hit-or-miss transform, granulometry, and morphological filters, have been developed.

Soon thereafter, mathematical morphology was extended to grey-scale images. Such an extension requires rules for the 'combination' of different grey-values. In the binary case, the set paradigm leads in a natural way to 'combinations' based on concepts from Boolean logic. In the grey-scale case, the set paradigm is no longer valid, and as a consequence it is not a priori clear which 'combination mechanism' should be used. Furthermore, many theories, for instance, grey-scale granulometry, connectivity and connected operators, haven't been developed.

In our first report [5], we founded morphological operators within the framework from fuzzy logic. In that report, a large class of morphological operators for grey-scale images was conducted, and the classical grey-scale Minkowski addition and subtraction are the special case within fuzzy logic framework.

The first author using concepts from fuzzy logic in mathematical morphology is Goetcherian [6]. Since then, several authors have followed this approach, for example Sinha and Dougherty [14, 15], Bloch and Maitre [1, 2], and De Baets, Nachtegael and Kerre [4, 3, 11]. An excellent source is the recent volume edited by Kerre and Nachtegael [8].

This report is the sequel of our first one. In this report, we investigate granulometry, hit-or-miss operator, rank operator, connectivity, and the Matheron's representation theorem for grey-scale images. Besides these, we put forward a new approach to fuzzy morphological operators by the extension of infimum and supremum operators as well as conjunction. This report just is an attempt to develop these theories, and contains some preliminary ideas which need to be worked out in much greater detail in the future.

2. Grey-scale Mathematical Morphology

Let U be a nonempty set called a universe, let $\mathcal{P}(U)$ be the family of all subsets of U, and let $\mathcal{F}(U)$ be the family of all fuzzy subsets (fuzzy sets) on U. In general, we take $U = \mathbb{R}^d$, the d-dimensional Euclidean space, in which case a subset X of U represents a continuous binary image on U, or we take $U = \mathbb{Z}^d$, in which case $X \subseteq U$ represents a discrete binary image on U.

In this section, we briefly recall some primary concepts and results in Mathematical morphology.

2.1 Binary Morphology

In set morphology, dilation δ and erosion ε are defined from the Minkowski addition and subtraction as follows

$$\delta_A(X) = X \oplus A \quad \text{and} \quad \varepsilon_A(X) = X \ominus A .$$

for any image $X \in \mathcal{P}(U)$. The set A is called *structuring element* in the morphological literature.

The most important relation between dilation and erosion is

$$Y \oplus A \subseteq X \iff Y \subseteq X \ominus A , \quad X , Y , A \in \mathcal{P}(U) ,$$

which is called the *adjunction* relation.

Let $A \in \mathcal{P}(U)$ be a structuring element, for any binary image $X \in \mathcal{P}(U)$, the opening and closing of image X by A are, respectively, defined as follows:

$$\alpha_A(X) = X \circ A = (X \ominus A) \oplus A \;\text{ and }\; \beta_A(X) = X \bullet A = (X \oplus A) \ominus A .$$

2.2 Morphology on Complete Lattices

In mathematical morphology, the framework of complete lattices is a right implement.

2.1. Definition. Suppose that $\mathcal{L}$ and $\mathcal{M}$ are complete lattices, and that J is an index set. An operator $\delta : \mathcal{M} \to \mathcal{L}$ is called a *dilation* if $\delta(\bigvee_{i\in J} Y_i) = \bigvee_{i\in J} \delta(Y_i)$ for every collection $\{Y_i \mid i \in J\} \subseteq \mathcal{M}$. An operator $\varepsilon : \mathcal{L} \to \mathcal{M}$ is called an *erosion* if $\varepsilon(\bigwedge_{i\in J} X_i) = \bigwedge_{i\in J} \varepsilon(X_i)$ for every collection$\{X_i \mid i \in J\} \subseteq \mathcal{L}$. Note in particular, by choosing J to be the empty set, we get $\delta(\bot) = \bot$ and $\varepsilon(\top) = \top$, where $\bot$ and $\top$ are the least and greatest elements, respectively. Two operators $\varepsilon : \mathcal{L} \to \mathcal{M}$ and $\delta : \mathcal{M} \to \mathcal{L}$ are said to form an *adjunction* if and only if

$$\delta(Y) \le X \iff Y \le \varepsilon(X)$$

for any $X \in \mathcal{L}$ and $Y \in \mathcal{M}$. In this case, we say that (ε , δ) is an *adjunction* between $\mathcal{L}$ and $\mathcal{M}$.

2.2. Proposition. *Let $\mathcal{L}$ and $\mathcal{M}$ be two complete lattices, and let (ε , δ) be an adjunction between $\mathcal{L}$ and $\mathcal{M}$, then ε is an erosion and δ is a dilation.*

2.3. Proposition. *If $\delta : \mathcal{M} \to \mathcal{L}$ is a dilation, then there exists a unique erosion $\varepsilon : \mathcal{L} \to \mathcal{M}$ such that (ε , δ) is an adjunction between $\mathcal{L}$ and $\mathcal{M}$. Dually, if $\varepsilon : \mathcal{L} \to \mathcal{M}$ is an erosion, then there exists a unique dilation $\delta : \mathcal{M} \to \mathcal{L}$ such that (ε , δ) is an adjunction between $\mathcal{L}$ and $\mathcal{M}$.*

2.4. Proposition. *Assume that $(\varepsilon_1 , \delta_1)$ is an adjunction between complete lattices $\mathcal{L}$ and $\mathcal{M}$, and that $(\varepsilon_2 , \delta_2)$ is an adjunction between complete lattices $\mathcal{M}$ and $\mathcal{N}$, then $(\varepsilon_2\varepsilon_1 , \delta_1\delta_2)$ is an adjunction between $\mathcal{L}$ and $\mathcal{N}$.*

2.5. Proposition. *Let $(\varepsilon_i, \delta_i)$ be an adjunction between complete lattices $\mathcal{L}$ and $\mathcal{M}$ for any $i \in J$, then $(\bigwedge_{i\in J} \varepsilon_i, \bigvee_{i\in J} \delta_i)$ is an adjunction between $\mathcal{L}$ and $\mathcal{M}$ as well.*

2.6. Definition. Given a complete lattice $\mathcal{L}$, an operator $\alpha : \mathcal{L} \to \mathcal{L}$ is called an *opening* if α is increasing (that is, $X_1 \leq X_2$ implies that $\alpha(X_1) \leq \alpha(X_2)$), idempotent (that is $\alpha(\alpha(X)) = \alpha(X)$, or briefly $\alpha^2 = \alpha$), and anti-extensive (that is $\alpha(X) \leq X$, for every $X \in \mathcal{L}$). Dually, an operator β is called a *closing* if it satisfies the first two properties and also extensive (that is $X \leq \beta(X)$, for every $X \in \mathcal{L}$).

2.7. Proposition. *If (ε, δ) is an adjunction between complete lattices $\mathcal{L}$ and $\mathcal{M}$, then $\varepsilon\delta$ is a closing on $\mathcal{M}$, whereas $\delta\varepsilon$ is an opening on $\mathcal{L}$.*

2.3 Fuzzy Logical Operations

In this subsection, we review some basic concepts from fuzzy logic which are important for the sequel of this report. Special attention should be paid to the conjunction C and implication I, and the interpretation of how these two notions can be paired by means of the adjunction relation discussed in the previous subsection. There is a huge literature on fuzzy logic and fuzzy set theory (see e.g. [9, 12, 17]), and it should be clear that the discussion presented here is far from complete.

In fuzzy logic, the operations C and I are extended from the Boolean domain $\{0,1\}\times\{0,1\}$ to the rectangle $[0,1] \times [0,1]$.

2.8. Definition. A mapping $C: [0,1]\times[0,1] \to [0,1]$ is called a *fuzzy conjunction* (briefly, *conjunction*) if it is increasing in both arguments and satisfies the boundary conditions of classical logic

$$C(0,0) = C(1,0) = C(0,1) = 0 \text{ and } C(1,1) = 1. \tag{2.2.1}$$

A mapping $I: [0,1] \times [0,1] \to [0,1]$ is called a *fuzzy implication* (briefly, *implication*) if it is decreasing in the first argument, increasing in the second, and coincides with the classical implication

$$I(0,0) = I(0,1) = I(1,1) = 1 \text{ and } I(1,0) = 0. \tag{2.2.2}$$

2.9. Definition. An implication I and a conjunction C are said to be *adjoint* (on $[0,1]$) if

$$C(a,t) \leq s \iff t \leq I(a,s) \tag{2.2.3}$$

for all $a, s, t \in [0,1]$.

Thus an implication I and a conjunction C are adjoint if for every $a \in [0,1]$, the pair $(I(a,\cdot), C(a,\cdot))$ forms an adjunction on $[0,1]$, (briefly, we say that the pair (I, C) is an adjunction). This means in particular that $I(a,\cdot)$ is an erosion on $[0,1]$, or alternatively, *continuous from the right*, and that $C(a,\cdot)$ is a dilation, or alternatively, *continuous from the left*.

In the context of grey-scale morphology, the grey-value set usually is a discrete point set, for instance, $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, +\infty\}$, or $\{a_0, a_1, \ldots, a_N\}$.

Let us firstly present the definition of conjunctions and implications on arbitrary complete lattices.

2.10. Definition. Let $\mathcal{L}$ be a complete lattice, the least and greatest elements of $\mathcal{L}$ are denoted by $\perp$ and $\top$, respectively. A conjunction C on $\mathcal{L}$ is defined as an operator from $\mathcal{L} \times \mathcal{L}$ to $\mathcal{L}$, satisfying that C is increasing in both arguments, and

$$C(\perp,\perp) = C(\perp,\top) = C(\top,\perp) = \perp \text{ and } C(\top,\top) = \top,$$

whereas, an implication I on $\mathcal{L}$ is decreasing in the first argument, increasing in the second and satisfying that

$$I(\bot\,,\bot) = I(\bot\,,\top) = I(\top\,,\top) = \top \quad \text{and} \quad I(\top\,,\bot) = \bot\,.$$

An implication I and a conjunction C on $\mathcal{L}$ are said to be adjoint if

$$C(a\,,t) \le s \iff t \le I(a\,,s)\,, \qquad a\,,s\,,t \in \mathcal{L}\,.$$

2.11. Proposition. *Let $\mathcal{L}$ be a complete lattice, and let $\sigma:\ [0\,,1] \to \mathcal{L}$ be a continuous increasing mapping such that $\sigma(0) = \bot$ and $\sigma(1) = \top$. I and C be two functions from $[0\,,1] \times [0\,,1] \to [0\,,1]$, respectively. For any $s\,,t \in \mathcal{L}$, define $C_\sigma(s\,,t) = \sigma(C(\sigma^{-1}(s)\,,\sigma^{-1}(t)))$ and $I_\sigma(s\,,t) = \sigma(I(\sigma^{-1}(s)\,,\sigma^{-1}(t)))$, then the following assertions hold.*

(1) I is an implication (C is a conjunction) if and only if I_σ is an implication (C_σ is a conjunction).

(2) $(I\,,C)$ is adjoint on $[0\,,1]$ if and only if $(I_\sigma\,,C_\sigma)$ is adjoint on $\mathcal{L}$

2.4 Grey-scale Morphology

Let U be the universe discoursed, $\mathcal{T}$ be the grey-scale value set of images on U, and let $\mathcal{T}^U = \{F \mid F : U \to \mathcal{T}\}$ be the family of all mappings in which every mapping F represents a grey-scale image on U. If $\mathcal{T}$ is a complete lattice with a partial ordering $\le$, then $\mathcal{T}^U$ is also a complete lattice with a partial ordering, also denoted by $\le$, defined as, for any $F_1\,,F_2 \in \mathcal{T}^U$,

$$F_1 \le F_2 \iff F_1(x) \le F_2(x)\,, \quad \forall\, x \in U\,.$$

We denote $\mathcal{T}^U$ by $\mathcal{F}(U)$ which can be regarded as the family of all fuzzy subsets on U. From this viewpoint, a grey-scale image F on U is a fuzzy subset $F \in \mathcal{F}(U)$. A fuzzy subset (or fuzzy set) F is uniquely determined by its membership function $\mu_F(x)$, briefly, denoted by $F(x)$, $x \in U$.

We may extend the set relations inclusion and intersection of two crisp sets to fuzzy case.

Let $F\,,G \in \mathcal{F}(U)$, we denote by $|G \subseteq F|$ the degree of fuzzy set G included in fuzzy set F, then

$$\begin{aligned} |G \subseteq F| &:= \text{a quantity concerning } I(G(y)\,,F(y))\,, \text{ for all } y \in U \\ &:= \bigwedge\{I(G(y)\,,F(y)) \mid y \in U\} \\ &:= \bigwedge_{y \in U} I(G(y)\,,F(y))\,, \end{aligned}$$

where I is a fuzzy implication.

The degree of erosion of the grey-scale image F by G, called a structuring element (a structuring function) at point x should be $|G_x \subseteq F|$, here G_x is a fuzzy set, denoting the translation of fuzzy set G along x, $G_x(y) = G(y-x)\,,\ y \in U$.

Analogously, we denote by $|G \Uparrow F|$ the degree of fuzzy set G hitting fuzzy set F, then

$$\begin{aligned} |G \Uparrow F| &:= \text{a quantity concerning } C(G(y)\,,F(y))\,, \text{ for some } y \in U \\ &:= \bigvee\{C(G(y)\,,F(y)) \mid y \in U\} \\ &:= \bigvee_{y \in U} C(G(y)\,,F(y))\,, \end{aligned}$$

where C is a fuzzy conjunction.

The degree of dilation of the grey-scale image F by the structuring function G at point x should be $|\check{G}_x \Uparrow F|$, here $\check{G}$ and $\check{G}_x = (\check{G})_x$ mean that $\check{G}(y) = G(-y)$ and $\check{G}_x(y) = G(x-y)\,,\ y \in U$, respectively.

2.12. Definition. Let $G \in \mathcal{F}(U)$ be a structuring element (structuring function), $F \in \mathcal{F}(U)$ be a grey-scale image. Suppose that implication I and conjunction C form an adjunction, define *dilation* and *erosion* of the grey-scale image F by G, separately as follows

$$\mathcal{D}_G(F)(x) = (F \oplus_C G)(x) = \bigvee_{y \in U} C(G(x-y)\,, F(y))\,,$$

$$\mathcal{E}_G(F)(x) = (F \ominus_C G)(x) = \bigwedge_{y \in U} I(G(y-x)\,, F(y))\,,$$

for every $x \in U$.

2.13. Proposition. *Let I be an implication and C be a conjunction on [0, 1], then $(I\,, C)$ is adjoint on [0, 1] if and only if $(\mathcal{E}_G\,, \mathcal{D}_G)$ is an adjunction on $\mathcal{F}(U)$ for any $G \in \mathcal{F}(U)$.*

2.14. Proposition. *If $(I\,, C)$ is an adjunction, then*

$$\beta_G(F)(x) = (F \bullet_C G)(x) = \mathcal{E}_G(\mathcal{D}_G(F))(x)$$

is an adjunctional closing, and

$$\alpha_G(F)(x) = (F \circ_C G)(x) = \mathcal{D}_G(\mathcal{E}_G(F))(x)$$

is an adjunctional opening. That means, for a fixed $G \in \mathcal{F}(U)$, and any $F\,, F_1\,, F_2 \in \mathcal{F}(U)$, if $F_1 \leq F_2$, then

$$\beta_G(F_1) \leq \beta_G(F_2)\,,\ \alpha_G(F_1) \leq \alpha_G(F_2)\,,$$
$$\beta_G(\beta_G(F)) = \beta_G(F)\,,\ \alpha_G(\alpha_G(F)) = \alpha_G(F)\,,$$
$$\alpha_G(F) \leq F \leq \beta_G(F)\,.$$

It is easy to check that all of the morphological operators, $\mathcal{E}_G$ and $\mathcal{D}_G$, as well as α_G and β_G are translation invariant.

2.15. Proposition. *Let $(I\,, C)$ be an adjunction, if C is continuous from the left for the first argument, then for any $F \in \mathcal{F}(U)$ and $\{G_i\}_{i \in J} \subseteq \mathcal{F}(U)$,*

$$\mathcal{D}_{\bigcup_{i \in J} G_i}(F) = \bigcup_{i \in J} \mathcal{D}_{G_i}(F) \quad \textit{and} \quad \mathcal{E}_{\bigcup_{i \in J} G_i}(F) = \bigcap_{i \in J} \mathcal{E}_{G_i}(F)\,,$$

where $(\bigcup_{i \in J} X_i)(x) = \sup_{i \in J} X_i(x)$ and $(\bigcap_{i \in J} X_i)(x) = \inf_{i \in J} X_i(x)$, $\{X_i\}_{i \in J} \subseteq \mathcal{F}(U)$, $x \in U$.

2.16. Proposition. *Let $(I\,, C)$ be an adjunction, if conjunction C is commutative and associative, then $(F \oplus_C G_1) \oplus_C G_2 = F \oplus_C (G_1 \oplus_C G_2)$ and $(F \ominus_C G_1) \ominus_C G_2 = F \ominus_C (G_1 \oplus_C G_2)$ for any $F\,,\ G_1\,,\ G_2 \in \mathcal{F}(U)$.*

This proposition shows that structuring elements are decomposable.

Let the grey-value set $\mathcal{T} = \{0\,, 1\}$, which means that the grey-scale images on U are actually binary images. In this case, $\mathcal{T}^U$ and $\mathcal{P}(U)$ are isomorphic. For this reason, we don't distinguish between the elements in $\mathcal{T}^U$ and that in $\mathcal{P}(U)$.

2.17. Proposition. *Let $(I\,, C)$ be an adjunction, $F\,, G \in \mathcal{P}(U)$, then*

$$\mathcal{D}_G(F) = \delta_G(F) \quad \textit{and} \quad \mathcal{E}_G(F) = \varepsilon_G(F)\,.$$

2.18. Proposition. *Let $(I\,, C)$ be an adjunction, then $C(1\,, t) = t$ for every $t \in [0\,, 1]$ if and only if for any $F \in \mathcal{F}(U)$, $G \in \mathcal{P}(U)$, and any $x \in U$,*

$$\mathcal{D}_G(F)(x) = \bigvee_{y \in \check{G}_x} F(y) \quad \textit{and} \quad \mathcal{E}_G(F)(x) = \bigwedge_{y \in G_x} F(y)\,.$$

3. Granulometry

In the processing of images, granulometry is an important concept and implement. In this report, we will investigate granulometry for grey-scale images. This work is just a beginning in this area.

3.1. Definition. Let $G \in \mathcal{F}(U)$ be a fuzzy set, $\{G_\alpha\}_{\alpha\in(0,1)}$ be a family of crisp sets, if

(1) $0 < \alpha < \beta < 1 \Rightarrow G_\beta \subseteq G_\alpha$,

(2) $G(x) = \bigvee\{\alpha \in (0,1) \mid x \in G_\alpha\},\ x \in U$,

then we call the family $\{G_\alpha\}_{\alpha\in(0,1)}$ of sets a *set representation* of fuzzy set G.

3.2. Definition. Let $G \in \mathcal{F}(U)$, $\alpha \in [0,1]$, $[G]_\alpha = \{x \in U \mid G(x) \geq \alpha\}$ and $[G]^\alpha = \{x \in U \mid G(x) > \alpha\}(\alpha \neq 1)$ are called the cut set and strictly cut set of fuzzy set G at level α, respectively.

3.3. Proposition. *The families of cut sets $\{[G]_\alpha\}_{\alpha\in[0,1]}$ and strictly cut sets $\{[G]^\alpha\}_{\alpha\in[0,1)}$ are the set representations of fuzzy set G.*

3.4. Proposition. *$\{G_\alpha\}_{\alpha\in(0,1)}$ is a set representation of fuzzy set G if and only if for all $\alpha \in (0,1)$,*

$$[G]^\alpha \subseteq G_\alpha \subseteq [G]_\alpha .$$

Proof. $\Rightarrow$: If $\{G_\alpha\}_{\alpha\in(0,1)}$ is a set representation of fuzzy set G, then $\forall\,\alpha \in (0,1)$,

$$\begin{aligned} x \in [G]^\alpha &\Rightarrow G(x) > \alpha \\ &\Rightarrow \bigvee\{r \in (0,1) \mid x \in G_r\} > \alpha \\ &\Rightarrow \exists\beta \in (\alpha,1) \text{ such that } x \in G_\beta \\ &\Rightarrow x \in G_\alpha \\ &\Rightarrow G(x) = \bigvee\{r \in (0,1) \mid x \in G_r\} \geq \alpha . \end{aligned}$$

$\Leftarrow$: Suppose that there exists a family of sets $\{G_\alpha\}_{\alpha\in(0,1)}$ such that $[G]^\alpha \subseteq G_\alpha \subseteq [G]_\alpha$. If $0 < \alpha < \beta < 1$, then $x \in G_\beta \Rightarrow x \in [G]_\beta \Rightarrow G(x) \geq \beta > \alpha \Rightarrow x \in [G]^\alpha \Rightarrow x \in G_\alpha$. Thus, $G_\beta \subseteq G_\alpha$.

We define fuzzy sets $\alpha[G]_\alpha$, $\alpha[G]^\alpha$ and αG_α as, respectively, for every $x \in U$,

$$\alpha[G]_\alpha(x) = \begin{cases} \alpha, & x \in [G]_\alpha, \\ 0, & \text{otherwise}. \end{cases}$$

$$\alpha[G]^\alpha(x) = \begin{cases} \alpha, & x \in [G]^\alpha, \\ 0, & \text{otherwise}. \end{cases}$$

$$\alpha G_\alpha(x) = \begin{cases} \alpha, & x \in G_\alpha, \\ 0, & \text{otherwise}. \end{cases}$$

By the condition $[G]^\alpha \subseteq G_\alpha \subseteq [G]_\alpha$, we have $\alpha[G]^\alpha \subseteq \alpha G_\alpha \subseteq \alpha[G]_\alpha$. Thus, by Decomposition Theorem

$$G = \bigvee_{\alpha\in[0,1)} \alpha[G]^\alpha = \bigvee_{\alpha\in[0,1]} \alpha[G]_\alpha .$$

It's natural that

$$G = \bigvee_{\alpha\in[0,1)} \alpha[G]^\alpha \subseteq \bigvee_{\alpha\in(0,1)} \alpha G_\alpha \subseteq \bigvee_{\alpha\in[0,1]} \alpha[G]_\alpha = G .$$

That is, for any $x \in U$,

$$G(x) = \bigvee_{\alpha\in(0,1)} (\alpha G_\alpha)(x) = \bigvee\{\alpha \in (0,1) : x \in G_\alpha\} .$$

□

3.5. Proposition. *If the family of crisp sets* $\{G_\alpha\}_{\alpha\in(0,1)}$ *is the set representation of fuzzy set* G *as well as fuzzy set* H*, then* $G = H$*.*

Extension Principle Let $f : U^2 \to R$ be a mapping, C be a conjunction, and $G_1, G_2 \in \mathcal{F}(U)$ be two fuzzy sets, then $f(G_1, G_2)$ is a fuzzy set on R. Moreover, for every $x \in R$,

$$f(G_1, G_2)(x) = \bigvee_{f(x_1,x_2)=x,\,(x_1,x_2)\in U^2} C(G_1(x_1), G_2(x_2)).$$

If for a $x \in R$, there doesn't exist $(x_1, x_2) \in U^2$ such that $f(x_1, x_2) = x$, then $f(G_1, G_2)(x) = 0$.

3.6. Proposition. *Let* $f : U^2 \to R$ *be a non-decreasing mapping,* C *be a conjunction satisfying* $C(a, b) \le \min(a, b)$ *for any* $a, b \in [0, 1]$*,* $G_1, G_2 \in \mathcal{F}(U)$ *be two fuzzy sets, then*

(1) $[f(G_1, G_2)]^\alpha = f([G_1]^\alpha, [G_2]^\alpha) = \{f(s, t) \mid s \in [G_1]^\alpha,\ t \in [G_2]^\alpha\},\ \alpha \in [0, 1)$.

(2) $[f(G_1, G_2)]_\alpha \supseteq f([G_1]_\alpha, [G_2]_\alpha) = \{f(s, t) \mid s \in [G_1]_\alpha,\ t \in [G_2]_\alpha\},\ \alpha \in [0, 1]$.

(3) If $\{(G_1)_\alpha\}_{\alpha\in(0,1)}$ *and* $\{(G_2)_\alpha\}_{\alpha\in(0,1)}$ *are the set representations of fuzzy sets* G_1 *and* G_2*, respectively, then* $\{f((G_1)_\alpha, (G_2)_\alpha)\}_{\alpha\in(0,1)}$ *is a set representation of fuzzy set* $f(G_1, G_2)$*, where* $f((G_1)_\alpha, (G_2)_\alpha) = \{f(s, t) \mid s \in (G_1)_\alpha, t \in (G_2)_\alpha\}$*.*

Proof. (1) For all $\alpha \in [0, 1)$, $x \in R$,

$$\begin{aligned}
x \in [f(G_1, G_2)]^\alpha &\iff f(G_1, G_2)(x) > \alpha \\
&\iff \bigvee_{f(x_1,x_2)=x,\,(x_1,x_2)\in U^2} C(G_1(x_1), G_2(x_2)) > \alpha \\
&\iff \exists (s, t) \in U^2,\ \ni f(s, t) = x\ \&\ C(G_1(s), G_2(t)) > \alpha \\
&\iff f(s, t) = x\ \&\ G_1(s) > \alpha\ \&\ G_2(t) > \alpha \\
&\iff f(s, t) = x\ \&\ s \in [G_1]^\alpha\ \&\ t \in [G_2]^\alpha \\
&\iff f(s, t) = x \in f([G_1]^\alpha, [G_2]^\alpha).
\end{aligned}$$

The proof of (2) is analogous.
(3) It's sufficient to prove the conclusion

$$[f(G_1, G_2)]^\alpha \subseteq f((G_1)_\alpha, (G_2)_\alpha) \subseteq [f(G_1, G_2)]_\alpha.$$

In fact, for all $\alpha \in (0, 1)$, $[G_1]^\alpha \subseteq (G_1)_\alpha \subseteq [G_1]_\alpha$ and $[G_2]^\alpha \subseteq (G_2)_\alpha \subseteq [G_2]_\alpha$. In term of the monotonicity of mapping f and (1), (2), we have

$$\begin{aligned}
[f(G_1, G_2)]^\alpha &= f([G_1]^\alpha, [G_2]^\alpha) \\
&\subseteq f((G_1)_\alpha, (G_2)_\alpha) \\
&\subseteq f([G_1]_\alpha, [G_2]_\alpha) \\
&\subseteq [f(G_1, G_2)]_\alpha.
\end{aligned}$$

□

3.7. Definition. Let $G \in \mathcal{F}(U)$ be a fuzzy set, for any $x, y \in U$, $r \in [0, 1]$, if $G(rx + (1 - r)y) \ge \min(G(x), G(y))$, then G is said to be a convex fuzzy set.

Note that the membership function of a convex fuzzy set is not always a (up-)convex function. For example, a fuzzy set characterized by its membership function $F(x) = e^{-x^2}$ is a convex fuzzy set, but $F(x)$ is not a convex function, nor a concave function. The fuzzy set of 'young people' is a convex fuzzy set.

3.8. Proposition. *The following three statements are equivalent:*

(1) $G \in \mathcal{F}(U)$ is a convex fuzzy set.

(2) For any $\alpha \in [0,1]$, $[G]_\alpha$ is an interval.

(3) For any $\alpha \in [0,1)$, $[G]^\alpha$ is an interval.

Proof. (1) ⇒ (2): If G is a convex fuzzy set, then $\forall \alpha \in [0,1]$, for any $x, y \in [G]_\alpha$, and any $r \in [0,1]$, $G(rx + (1-r)y) \geq \min(G(x), G(y)) \geq \min(\alpha, \alpha) = \alpha$. That is to say, $rx + (1-r)y \in [G]_\alpha$. Thus, $[G]_\alpha$ is an interval.

(2) ⇒ (3): Let $\alpha \in [0,1)$, any $r \in [0,1]$, and any $x, y \in [G]^\alpha$, then $G(x) > \alpha$, $G(y) > \alpha$. So, there exists $\varepsilon_0 > 0$ ($\varepsilon_0 \leq 1-\alpha$) such that $G(x) \geq \alpha + \varepsilon_0$ and $G(y) \geq \alpha + \varepsilon_0$. Thus, $x, y \in [G]_{\alpha+\varepsilon_0}$. Since $[G]_{\alpha+\varepsilon_0}$ is an interval, $rx + (1-r)y \in [G]_{\alpha+\varepsilon_0}$. That is $G(rx + (1-r)y) \geq \alpha + \varepsilon_0 > \alpha$. $rx + (1-r)y \in [G]^\alpha$, e.g., $[G]^\alpha$ is an interval.

(3) ⇒ (1): $\forall x, y \in U$, $\forall r \in [0,1]$, let $\alpha = \min(G(x), G(y))$. If $\alpha = 0$, then $G(rx + (1-r)y) \geq 0 = \alpha = \min(G(x), G(y))$. If $\alpha > 0$, then $\forall \varepsilon > 0$ ($\varepsilon \leq \alpha$), we have $G(x) \geq \alpha > \alpha - \varepsilon$, and $G(y) \geq \alpha > \alpha - \varepsilon$. That is, $x, y \in [G]^{\alpha-\varepsilon}$. By the fact that $[G]^{\alpha-\varepsilon}$ is an interval, so, $rx + (1-r)y \in [G]^{\alpha-\varepsilon}$, e.g., $G(rx + (1-r)y) > \alpha - \varepsilon$. By the arbitrary constant of ε, we obtain the conclusion $G(rx + (1-r)y) \geq \alpha = \min(G(x), G(y))$ which means that G is a convex fuzzy set. □

3.9. Proposition. *If G is a convex fuzzy set on U, then for any $s > 0$, sG is also a convex fuzzy set on U.*

Proof. For any $x, y \in U$ and any $r \in [0,1]$,

$$sG(rx + (1-r)y) = G(rx/s + (1-r)y/s) \geq \min(G(x/s), G(y/s)) = \min(sG(x), sG(y)),$$

which means sG is a convex fuzzy set. □

3.10. Proposition. *Let G be a convex fuzzy set, $\alpha \in [0,1]$, $s > 0$, then $[G]_\alpha = [a, b] \iff [sG]_\alpha = [sa, sb]$; $[G]^\alpha = (a, b) \iff [sG]^\alpha = (sa, sb), (\alpha \neq 1)$.*

Proof. $x \in [a,b] = [G]_\alpha \iff G(x) \geq \alpha \iff sG(sx) = G(x) \geq \alpha \iff sx \in [sG]_\alpha = [a', b'] \iff sx \in [sG]_\alpha = [sa, sb]$.

By analogy, the second statement can be proved. □

3.11. Definition. Let $[a,b], [c,d]$ be two closed intervals, the Minkowski addition and subtraction between two intervals $[a,b]$ and $[c,d]$ are, respectively, as follows

$$[a,b] + [c,d] = [a+c, b+d],$$

and

$$[a,b] - [c,d] = [a-c, b-d].$$

The Minkowski addition and subtraction between open intervals can be defined analogously.

3.12. Proposition. *Let $G \in \mathcal{F}(U)$ be a convex fuzzy set, $s > 0$, $t > 0$, then*

(1) $\forall \alpha \in [0,1]$, $[sG]_\alpha + [tG]_\alpha = [(s+t)G]_\alpha$.

(2) $\forall \alpha \in [0,1)$, $[sG]^\alpha + [tG]^\alpha = [(s+t)G]^\alpha$.

3.13. Proposition. *If $G \in \mathcal{F}(U)$ is a convex fuzzy set, C is a conjunction satisfying, for any $a, b \in [0,1]$, $C(a,b) \leq \min(a,b)$, then, for any $s > 0$, $t > 0$, $sG \oplus_C tG = tG \oplus_C sG = (s+t)G$.*

Proof. In Extension Principle, let $f(x_1, x_2) = x_1 + x_2$, then, for any $s > 0$, $t > 0$, $f(sG, tG) = sG \oplus_C tG$.

Let $\{(sG)_\alpha\}_{\alpha \in (0,1)}$ and $\{(tG)_\alpha\}_{\alpha \in (0,1)}$ be the set representations of fuzzy sets sG and tG, respectively, then $\{(sG)_\alpha \oplus (tG)_\alpha\}_{\alpha \in (0,1)}$ should be the set representation of fuzzy set $sG \oplus_C tG$, where $\oplus$ is the Minkowski addition. That is, for any $\alpha \in (0,1)$,

$$[sG]^\alpha \subseteq (sG)_\alpha \subseteq [sG]_\alpha, \ [tG]^\alpha \subseteq (tG)_\alpha \subseteq [tG]_\alpha, \tag{3.1}$$

and

$$[sG\oplus_C tG]^\alpha \subseteq (sG)_\alpha \oplus (tG)_\alpha \subseteq [sG\oplus_C tG]_\alpha .$$

By (3.1), we obtain

$$[sG]^\alpha \oplus [tG]^\alpha \subseteq (sG)_\alpha \oplus (tG)_\alpha \subseteq [sG]_\alpha \oplus [tG]_\alpha .$$

Since $[sG]_\alpha\,, [sG]^\alpha\,, [tG]_\alpha$ and $[tG]^\alpha$ are intervals,

$$[sG]_\alpha \oplus [tG]_\alpha = [sG]_\alpha + [tG]_\alpha = [(s+t)G]_\alpha\,,$$

$$[sG]^\alpha \oplus [tG]^\alpha = [sG]^\alpha + [tG]^\alpha = [(s+t)G]^\alpha .$$

Thus

$$[(s+t)G]^\alpha \subseteq (sG)_\alpha \oplus (tG)_\alpha \subseteq [(s+t)G]_\alpha\,,$$

which means that the family $\{(sG)_\alpha \oplus (tG)_\alpha\}_{\alpha\in(0,1)}$ is the set representation of fuzzy set $(s+t)G$. So, $sG\oplus_C tG = (s+t)G$.

Since the Minkowski addition $\oplus$ is commutative, $tG\oplus_C sG = (s+t)G$. □

3.14. Proposition. *Let implication I and conjunction C be adjoint and let C be commutative and associative satisfying the condition in Proposition 3.13, $G \in \mathcal{F}(U)$ is a convex fuzzy set, $s > 0$, $t > 0$, then*

$$\mathcal{D}_{sG}\mathcal{D}_{tG} = \mathcal{D}_{(s+t)G} \quad \textit{and} \quad \mathcal{E}_{sG}\mathcal{E}_{tG} = \mathcal{E}_{(s+t)G}\,.$$

Proof. For any $F \in \mathcal{F}(U)$, $\mathcal{D}_{sG}(F) = F\oplus_C sG\,,\ \mathcal{D}_{tG}(F) = F\oplus_C tG$.

$$\begin{aligned}
\mathcal{D}_{sG}\mathcal{D}_{tG}(F) &= \mathcal{D}_{sG}(\mathcal{D}_{tG}(F))\\
&= (F\oplus_C tG)\oplus_C sG\\
&= F\oplus_C(tG\oplus_C sG)\\
&= F\oplus_C(sG\oplus_C tG)\\
&= F\oplus_C(s+t)G\\
&= \mathcal{D}_{(s+t)G}(F)\,.
\end{aligned}$$

The second statement can be proved analogously. □

3.15. Definition. A family of opening $\{\alpha_r\}_{r>0}$ on $\mathcal{F}(U)$ is called a granulometry if

$$\alpha_r\alpha_s = \alpha_s\alpha_r = \alpha_s, \quad s \geq r > 0\,.$$

A granulometry $\{\alpha_r\}_{r>0}$ is called a Minkowski granulometry if for any $r > 0$, the opening α_r is translation invariant, and $\alpha_{rG}(rF) = r\alpha_1(F)$ for any $F \in \mathcal{F}(U)$.

3.16. Proposition. *Let implication I and conjunction C form an adjunction; In addition, if C satisfies the conditions in Proposition 3.14, and G is a convex fuzzy set, then for any $r > 0\,, \{\alpha_r = \mathcal{D}_{rG}\mathcal{E}_{rG}\}_{r>0}$ define a granulometry. Moreover, $\{\alpha_r\}_{r>0}$ is a Minkowski granulometry.*

Proof. In order to prove the fact that $\{\alpha_r\}_{r>0}$ is a granulometry, it's sufficient to prove that, for any $s \geq r > 0$, and any convex fuzzy set $G \in \mathcal{F}(U)$, $\alpha_r\alpha_s = \alpha_s\alpha_r = \alpha_s$.

By the adjunction of $(\mathcal{E}_{rG}\,, \mathcal{D}_{rG})$ for any $r > 0$, we have $\mathcal{D}_{rG}\mathcal{E}_{rG}\mathcal{D}_{rG} = \mathcal{D}_{rG}$. Thus, for any $s > r > 0$, we have $\mathcal{D}_{sG} = \mathcal{D}_{rG}\mathcal{D}_{(s-r)G}$. Thus

$$\alpha_r\alpha_s = \mathcal{D}_{rG}\mathcal{E}_{rG}\mathcal{D}_{sG}\mathcal{E}_{sG} = \mathcal{D}_{rG}\mathcal{E}_{rG}\mathcal{D}_{rG}\mathcal{D}_{(s-r)G}\mathcal{E}_{sG}$$

$$= \mathcal{D}_{rG}\mathcal{D}_{(s-r)G}\mathcal{E}_{sG} = \mathcal{D}_{sG}\mathcal{E}_{sG} = \alpha_s\,.$$

Analogously, the assertion $\alpha_s\alpha_r = \alpha_s$ is also true.

If $s = r > 0$, by the idempotence of opening, it's natural that $\alpha_r\alpha_s = \alpha_s\alpha_r = \alpha_s$.

It's not difficult to check that the granulometry $\{\alpha_r\}_{r>0}$ satisfies the conditions of Minkowski granulometry.

In fact, for any $F \in \mathcal{F}(U)$, $h \in U$. By the definition of translation F_h of fuzzy set F along h, $(F_h(x) = F(x-h),\ x \in U)$, we have

$$\alpha_r(F_h) = (\mathcal{D}_{rG}\mathcal{E}_{rG})(F_h) = \mathcal{D}_{rG}(\mathcal{E}_{rG}(F_h))$$

$$= \mathcal{D}_{rG}((\mathcal{E}_{rG}(F))_h) = (D_{rG}(\mathcal{E}_{rG}(F)))_h = (\alpha_r(F))_h\,,$$

and for any $x \in U$,

$$\begin{aligned}
\alpha_r(rF)(x) &= \mathcal{D}_{rG}\mathcal{E}_{rG}(rF)(x) = \mathcal{D}_{rG}(\mathcal{E}_{rG}(rF)(x)) \\
&= \bigvee_{y\in U} C(rG(x-y)\,,\ \bigwedge_{z\in U} I(rG(z-y), rF(z))) \\
&= \bigvee_{y\in U} C(G(x/r-y/r)\,,\ \bigwedge_{z\in U} I(G(z/r-y/r), F(z/r))) \\
&= \bigvee_{y'\in U} C(G(x/r-y')\,,\ \bigwedge_{z'\in U} I(G(z'-y')\,, F(z'))) \\
&= \mathcal{D}_G\mathcal{E}_G(F)(x/r) \\
&= \alpha_1(F)(x/r) \\
&= r\alpha_1(F)(x)\,.
\end{aligned}$$

This means that $\{\alpha_r\}_{r>0}$ is a Minkowski granulometry. □

Note that the sufficient condition such that $\{\alpha_r\}_{r>0}$ to be a granulometry needs to be worked out further.

4. Hit-or-Miss Operator

4.1 Hit-or-Miss Transformation for Binary Images

4.1. Definition. Suppose that $X \in \mathcal{P}(U)$ is a binary image, $A\,, B \in \mathcal{P}(U)$ are two structuring elements satisfying the condition $A \cap B = \emptyset$, the Hit-or-Miss transformation of image X by structuring element pair $(A\,, B)$ is as follows:

$$X \circledast (A\,, B) = \{h \mid A_h \subseteq X \ \text{ and } \ B_h \subseteq X^c\}\,.$$

It is easy to see that

$$X \circledast (A\,, B) = \varepsilon_A(X) \cap \varepsilon_B(X^c)\,.$$

Following this definition of hit-or-miss transform, the *thickening* and the *thinning* of binary image X by the structuring element pair $(A\,, B)$ can be defined as, respectively,

$$X \odot (A\,, B) = X \cup (X \circledast (A\,, B))\,,$$

and

$$X \circledcirc (A\,, B) = X \setminus (X \circledast (A\,, B))\,.$$

The following properties of Hit-or-Miss transformation hold.

4.2. Proposition. *Let $X\,, A\,, B \in \mathcal{P}(U)$, $h \in U$, then*

$$X \circledast (A, \emptyset) = \varepsilon_A(X)\,,$$

$$X \circledast (\emptyset\,, B) = \varepsilon_B(X^c)\,,$$

$$X^c \circledast (A\,, B) = X \circledast (B\,, A)\,,$$

$$X_h \circledast (A\,, B) = (X \circledast (A\,, B))_h\,,$$

$$X \circledast (A_h\,, B_h) = (X \circledast (A\,, B))_{-h}\,.$$

$$X^c \odot (A\,, B) = (X \circledcirc (B\,, A)^c\,.$$

4.2 Hit-or-Miss Transformation for Grey-scale Images

4.3. Definition. Let ν be a negation, and $G, H \in \mathcal{F}(U)$, for every $F \in \mathcal{F}(U)$, the *Hit-or-Miss* transformation of grey-scale image F with respect to the structuring function pair $\langle G, H\rangle$ is defined as

$$F \circledast \langle G, H\rangle(x) = C(\mathcal{E}_G(F)(x), \mathcal{E}_H(F^*)(x))$$

for every $x \in U$, where C is a given conjunction.

That means that the degree of a point $x \in U$ belonging to the Hit-or-Miss transformed set $F \circledast \langle G, H\rangle$ equals to the value of conjunction at the point (s, t), here s and t are the degree of G 'being included in' F and the degree of H 'being excluded from' F, respectively.

There is some difference from that in crisp case. When $G \cap H \neq \emptyset$, the Hit-or-Miss transformed set $F \circledast \langle G, H\rangle$ may be non-empty set.

4.4. Proposition. *Let (I, C) be an adjoint satisfying $C(s, s) = s$ for any $s \in [0, 1]$, then for any $s, t, r \in [0, 1]$, we have*

$$I(s, t) \geq r \quad iff \quad s \geq r \Rightarrow t \geq r .$$

Proof. By the fact that $I(s, t) \geq r \iff C(s, r) \leq t$ for any $s, t, r \in [0, 1]$. So, if $I(s, t) \geq r$ and $s \geq r$, then $t \geq C(s, r) \geq C(r, r) = r$.

On the other hand, since for any $s, t, r \in [0, 1]$, from $s \geq r$, we have $t \geq r$, therefore $t \geq s = C(s, s) \geq C(s, r)$. Hence, $I(s, t) \geq r$. □

Let $F \in \mathcal{F}(U)$ be a grey-scale image, $\alpha \in [0, 1]$, the thresholding set $\mathrm{X}(F, \alpha)$, and strong thresholding set $\mathrm{X}(F, \bar{\alpha})$, by simplicity denoted by $[F]_\alpha$, and by $[F]^\alpha$, respectively, of F at level α are

$$[F]_\alpha = \{x \in U \mid F(x) \geq \alpha\},$$

and

$$[F]^\alpha = \{x \in U \mid F(x) > \alpha\}.$$

4.5. Proposition. *Let ν be a negation, and let (I, C) be adjoint, if for any $s, t, r \in [0, 1]$, $C(s, s) = s$, and when $C(s, t) \geq r$, $\min(s, t) \geq r$, then for any $F, G, H \in \mathcal{F}(U)$, and $x \in U$,*

$$F \circledast \langle G, H\rangle(x) = \bigvee\{\alpha \in [0, 1] \mid x \in [F]_\alpha \otimes ([G]_\alpha, [H]_\alpha)\},$$

where

$$[F]_\alpha \otimes ([G]_\alpha, [H]_\alpha) = \varepsilon_{[G]_\alpha}([F]_\alpha) \cap \varepsilon_{[H]_\alpha}(([F]^{\nu(\alpha)})^c), \qquad \alpha \in [0, 1].$$

Proof. It is sufficient to prove that, for any $\alpha \in [0, 1]$, $[F \circledast \langle G, H\rangle]_\alpha = [F]_\alpha \otimes ([G]_\alpha, [H]_\alpha)$.

Let $\alpha \in [0, 1]$, for any $x \in U$,

$$\begin{aligned}
& x \in [F \circledast \langle G, H\rangle]_\alpha \\
\iff & C(\mathcal{E}_G(F)(x), \mathcal{E}_H(F^*)(x)) \geq \alpha \\
\iff & \mathcal{E}_G(F)(x) \geq \alpha \quad \& \quad \mathcal{E}_H(F^*)(x) \geq \alpha \\
\iff & \bigwedge_{y \in U} I(G(y-x), F(y)) \geq \alpha \quad \& \quad \bigwedge_{z \in U} I(H(z-x), F^*(z)) \geq \alpha \\
\iff & \forall y \in U,\ I(G(y-x), F(y)) \geq \alpha \quad \& \quad \forall z \in U,\ I(H(z-x), F^*(z)) \geq \alpha \\
\iff & \forall y \in U,\ G_x(y) \geq \alpha \Rightarrow F(y) \geq \alpha \quad \& \quad \forall z \in U,\ H_x(z) \geq \alpha \Rightarrow F^*(z) \geq \alpha \\
\iff & \forall y \in U,\ y \in [G_x]_\alpha \Rightarrow y \in [F]_\alpha \quad \& \quad \forall z \in U,\ z \in [H_x]_\alpha \Rightarrow z \in [F^*]_\alpha \\
\iff & [G_x]_\alpha \subseteq [F]_\alpha \quad \& \quad [H_x]_\alpha \subseteq [F^*]_\alpha \\
\iff & ([G]_\alpha)_x \subseteq [F]_\alpha \quad \& \quad ([H]_\alpha)_x \subseteq [F^*]_\alpha = ([F]^{\nu(\alpha)})^c \\
\iff & x \in \varepsilon_{[G]_\alpha}([F]_\alpha) \quad \& \quad x \in \varepsilon_{[H]_\alpha}(([F]^{\nu(\alpha)})^c) \\
\iff & x \in [F]_\alpha \otimes ([G]_\alpha, [H]_\alpha)
\end{aligned}$$

□

Note that there is slight difference between $[F]_\alpha \otimes ([G]_\alpha\,, [H]_\alpha)$ in Proposition 4.5 and the binary Hit-or-Miss transformation $[F]_\alpha \circledast ([G]_\alpha\,, [H]_\alpha) = \varepsilon_{[G]_\alpha}([F]_\alpha) \cap \varepsilon_{[H]_\alpha}(([F]_\alpha)^c)$. This fact also shows that grey-scale Hit-or-Miss cannot be generated by binary Hit-or-Miss via the thresholding set (binary image) of a grey-scale image. Even though, grey-scale Hit-or-Miss still possesses properties similar to that for binary Hit-or-Miss transformation.

4.6. Proposition. *Let ν be a negation, and let $F\,, G\,, H \in \mathcal{F}(U)$, $h \in U$, then*

$$F \circledast \langle G\,, \emptyset\rangle = F \ominus_C G = \mathcal{E}_G(F)\,,$$

$$F \circledast \langle \emptyset\,, H\rangle = \nu(F) \ominus_C H = \mathcal{E}_H(F^*)\,,$$

$$F^* \circledast \langle G\,, H\rangle = F \circledast \langle H\,, G\rangle\,,$$

$$F_h \circledast \langle G\,, H\rangle = (F \circledast \langle G\,, H\rangle)_h\,,$$

$$F \circledast \langle [G]_h\,, [H]_h\rangle = (F \circledast \langle G\,, H\rangle)_{-h}\,.$$

The theory of thinning and skeleton for grey-scale images is to be developed.

5. About Matheron's Representation Theory

Matheron's Representation Theorem plays an important role in the theory of classical mathematical morphology.

Matheron's Representation Theorem Every increasing and translation invariant operator Φ can be represented as the union of erosions ε_A, e.g.,

$$\Phi = \bigcup_{A \in \mathcal{V}(\Phi)} \varepsilon_A \quad \text{or} \quad \Phi(X) = \bigcup_{A \in \mathcal{V}(\Phi)} \varepsilon_A(X)\,, \quad X \in \mathcal{P}(U)\,,$$

where $\mathcal{V}(\Phi) = \{B \in \mathcal{P}(U) \mid 0 \in \Phi(B)\}$ is the kernel of operator Φ.

In grey-scale morphology for fuzzy sets, Matheron's representation theorem doesn't work. Let us give a counter-example.

Assume that $U = (-\infty, \infty)$, and that implication I and conjunction C form an adjunction. For every $x \in U$, let $F(x) \equiv 1\,,\ x \in U$, and

$$G(x) = \begin{cases} r < 1\,, & x \in [0,1]\,, \\ 0\,, & x \notin [0,1]\,, \end{cases}$$

then for every $x \in U$, and any $H \in \mathcal{F}(U)$,

$$(F \ominus_C H)(x) = \bigwedge_{y \in U} I(H(y-x), F(y)) \equiv 1\,.$$

But, for every $x \in U$,

$$\begin{aligned} (F \circ_C G)(x) &= \bigvee_{y \in U} C(G(x-y), \bigwedge_{z \in U} I(G(z-y), F(z))) \\ &= \bigvee_{y \in U} C(G(x-y), 1) = C(r, 1) < 1\,. \end{aligned}$$

That means that fuzzy opening which is increasing and translation invariant operator cannot be represented as the union of fuzzy erosions.

Let Ψ be an increasing, translation invariant operator on $\mathcal{F}(U)$, define the kernel of Ψ as

$$\mathcal{V}(\Psi) = \{F \in \mathcal{F}(U) \mid \Psi(F)(0) > 0\}\,,$$

and define, $t \in (0\,,1]$

$$\mathcal{V}_t(\Psi) = \{F \in \mathcal{F}(U) \mid \Psi(F)(0) \geq t\}\,,$$

then $\mathcal{V}_t(\Psi)$ is a decreasing set family with respect to t. That means that

$$s \le t \Longrightarrow \mathcal{V}_t(\Psi) \subseteq \mathcal{V}_s(\Psi)\,,$$

$$\mathcal{V}_t(\Psi) = \bigcap_{s \le t} \mathcal{V}_s(\Psi)\,.$$

Furthermore,

$$\mathcal{V}(\Psi) = \bigcup_{t \in (0\,,1]} \mathcal{V}_t(\Psi)\,.$$

If Φ and Ψ are two increasing, translation invariant operators, then

$$\Phi \subseteq \Psi \iff \mathcal{V}_t(\Phi) \subseteq \mathcal{V}_t(\Psi)\,, \quad t \in (0\,,1]\,,$$

where $\Phi \subseteq \Psi$ means that for any $F \in \mathcal{F}(U)$, $\Phi(F) \subseteq \Psi(F)$.

5.1. Proposition. *Let $(I\,,C)$ be an adjunction, if $C(s\,,1) = s$ for any $s \in [0\,,1]$, then the equivalence relation*

$$s \le t \iff I(s\,,t) = 1$$

holds for any $s\,,t \in [0\,,1]$. Furthermore, the equivalence relation

$$G \subseteq F \iff \mathcal{E}_G(F)(0) = 1$$

holds. Meanwhile,

$$\mathcal{V}_t(\Psi) = \bigcup_{G \in \mathcal{V}_t(\Psi)} \mathcal{V}_1(\mathcal{E}_G)\,,$$

where

$$\begin{aligned}\mathcal{V}_t(\mathcal{E}_G) &= \{F \in \mathcal{F}(U) \mid \mathcal{E}_G(F)(0) \ge t\} \\ &= \{F \in \mathcal{F}(U) \mid I(G(y)\,,F(y)) \ge t\,, y \in U\}\,,\end{aligned}$$

and

$$\mathcal{V}_1(\mathcal{E}_G) = \{F \in \mathcal{F}(U) \mid G \subseteq F\}\,.$$

Proof. For any $s\,,t \in [0\,,1]$, if $s \le t$, then for any $r \in [0\,,1]$, $C(s\,,r) \le C(s\,,1) = s \le t$. Thus $I(s\,,t) = \bigvee\{r \in [0\,,1] \mid C(s\,,r) \le t\} = 1$.

On the other hand, if $I(s\,,t) = 1$, then $s = C(s\,,1) = C(s\,,I(s\,,t)) = C(s\,,\bigvee\{r \in [0\,,1] \mid C(s\,,r) \le t\}) = \bigvee\{C(s\,,r) \mid C(s\,,r) \le t\} = t$.

The second equivalence relation can be proved straightforwardly.

Now, we prove the third equality.

For any $H \in \mathcal{V}_t(\Psi)$, we have that $H \in \mathcal{V}_1(\mathcal{E}_H) = \{F \in \mathcal{F}(U) \mid H \subseteq F\} \subseteq \bigcup_{G \in \mathcal{V}_t(\Psi)} \mathcal{V}_1(\mathcal{E}_G)$. Thus $\mathcal{V}_t(\Psi) \subseteq \bigcup_{G \in \mathcal{V}_t(\Psi)} \mathcal{V}_1(\mathcal{E}_G)$.

On the other hand, when $H \in \mathcal{V}_t(\Psi)$, $\Psi(H)(0) \ge t$, By the increasingness of Ψ, for any F, if $H \subseteq F$, then $\Psi(F)(0) \ge \Psi(H)(0) \ge t$. So, $F \in \mathcal{V}_t(\Psi)$, and $\{F \in \mathcal{F}(U) \mid H \subseteq F\} \subseteq \mathcal{V}_t(\Psi)$. But, $\{F \in \mathcal{F}(U) \mid H \subseteq F\} = \mathcal{V}_1(\mathcal{E}_H)$. Therefore, $\bigcup_{G \in \mathcal{V}_t(\Psi)} \mathcal{V}_1(\mathcal{E}_G) \subseteq \mathcal{V}_t(\Psi)$. □

5.2. Proposition. *For every $F \in \mathcal{F}(U)$, let*

$$\Phi(F) = \bigcup_{G \in \mathcal{V}_k(\Psi)} \mathcal{E}_G(F)\,,$$

then

$$\mathcal{V}_t(\Phi) = \bigcup_{G \in \mathcal{V}_k(\Psi)} \mathcal{V}_t(\mathcal{E}_G)$$

for any $k\,,t \in (0\,,1]$. Moreover, when $k = 1$,

$$\mathcal{V}_1(\Phi) = \mathcal{V}_1(\Psi)\,.$$

Proof. $F \in \mathcal{V}_t(\Phi)) \iff \Phi(F)(0) \ge t \iff (\bigcup_{G \in \mathcal{V}_k(\Psi)} \mathcal{E}_G(F))(0) \ge t \iff$ there exists a $H \in \mathcal{V}_k(\Psi)$ such that $\mathcal{E}_H(F)(0) \ge t \iff F \in \mathcal{V}_t(\mathcal{E}_H) \iff F \in \bigcup_{G \in \mathcal{V}_k(\Psi)} \mathcal{E}_G(F)$.

Since $\mathcal{V}_1(\Psi) = \bigcup_{G \in \mathcal{V}_1(\Psi)} \mathcal{V}_1(\mathcal{E}_G)$ and $\mathcal{V}_1(\Phi) = \bigcup_{G \in \mathcal{V}_k(\Psi)} \mathcal{V}_1(\mathcal{E}_G) = \bigcup_{G \in \mathcal{V}_1(\Psi)} \mathcal{V}_1(\mathcal{E}_G)$, the last equality holds. □

6. Rank Operator

Let U be a finite universe, $F \in \mathcal{F}(U)$ be a fuzzy subset, and $G = \{(a_i, G(a_i)) \mid i = 1, 2, \ldots, n\}$ be a finite fuzzy subset on U. Assume that I and C form an adjunction are implication and conjunction, respectively, and $C(1, t) = t$ for every $t \in [0, 1]$.

For every $x \in U$, let $u_i = C(\check{G}_x(b_i), F(b_i)) = C(G(a_i), F(x - a_i))$ $(b_i = x - a_i)$ and $v_i = I(G_x(b_i), F(b_i)) = I(G(a_i), F(x + a_i))$ $(b_i = x + a_i)$.

For every $x \in U$, $1 \le k \le n$, we define rank operators

$$r^C_{G,k}(F)(x) := \text{the } k\text{th largest number among } u_1, u_2, \ldots, u_n,$$

$$r^I_{G,k}(F)(x) := \text{the } k\text{th largest number among } v_1, v_2, \ldots, v_n,$$

then $r^C_{G,k}(F)$ and $r^I_{G,k}(F)$ are non-increasing with respect to k, $k = 1, 2, \ldots, n$. That is

$$r^C_{G,n}(F) \le r^C_{G,n-1}(F) \le \ldots \le r^C_{G,2}(F) \le r^C_{G,1}(F),$$

and

$$r^I_{G,n}(F) \le r^I_{G,n-1}(F) \le \ldots \le r^I_{G,2}(F) \le r^I_{G,1}(F).$$

Thus

$$r^C_{G,1}(F) = \bigvee_i u_i = \mathcal{D}_G(F) = F \oplus_C G,$$

$$r^I_{G,n}(F) = \bigwedge_i v_i = \mathcal{E}_G(F) = F \ominus_C G.$$

When G is a crisp set, $u_i = F(x - a_i)$ and $v_i = F(x + a_i)$, $i = 1, 2, \ldots, n$. So,

$$r^C_{G,1}(F)(x) = \bigvee_i u_i = \bigvee_i F(x - a_i) = F \oplus_C G,$$

and

$$r^I_{G,n}(F)(x) = \bigwedge_i v_i = \bigwedge_i F(x + a_i) = F \ominus_C G.$$

Furthermore, if G is symmetrical and $0 \in G$, then $u_i = v_i$ for every $i = 1, 2, \ldots, n$. So, $r^C_{G,k}(F) = r^I_{G,k}(F)$ for every $k = 1, 2, \ldots, n$.

7. Connectivity and Connected Operators

Let U be a nonempty set, $\mathcal{P}(U)$ be the power set of U, and let $\mathcal{L}$ be a complete lattice, whose least element is denoted by $\perp$, and whose greatest element by $\top$. Let $\mathcal{L}^U$ be the family of all L-fuzzy subsets (or L-fuzzy sets) on U, that is $\mathcal{L}^U = \{\mathcal{F} \mid \mathcal{F} : U \to \mathcal{L}\}$. Any grey-scale image $\mathcal{F}$ on U can be regarded as a L-fuzzy set $\mathcal{F} \in \mathcal{L}^U$, whose membership function is written as $\mathcal{F}(x) \in \mathcal{L}, x \in U$. Let $\mathcal{L}^U_p$ denote the family of all L-fuzzy points on U, e.g.,

$$\mathcal{L}^U_p = \{f = f_{x,\,\lambda} \mid x \in U,\ \lambda \in \mathcal{L}\},$$

where $f = f_{x,\,\lambda}$ is a L-fuzzy point satisfying $f_{x,\,\lambda}(y) = \begin{cases} \lambda, & y = x \\ \perp, & \text{otherwise} \end{cases}$ for any $y \in U$.

1. Every L-fuzzy point in U is a special L-fuzzy set on U, so $\mathcal{L}^U_p \subseteq \mathcal{L}^U$.
2. $\mathcal{L}^U_p$ is a sup-generating family of $\mathcal{L}^U$. Let $\mathcal{F} \in \mathcal{L}^U$ be a L-fuzzy set, then for any $x \in U$,

$$\mathcal{F}(x) = \bigvee\{f_{x,\,\lambda} \in \mathcal{L}^U_p \mid \lambda \le \mathcal{F}(x),\ \lambda \in \mathcal{L}\}. \tag{7.1}$$

3. If $\mathcal{L} = [0, 1]$, then L-fuzzy sets and L-fuzzy points are, respectively, ordinary fuzzy sets and fuzzy points.

Let $\mathcal{C}$ satisfying $\mathcal{L}^U_p \subseteq \mathcal{C} \subseteq \mathcal{L}^U$ be a sub-family of L-fuzzy sets on U. If $\mathcal{C}$ satisfies

(1) $f = \mathbf{0} \in \mathcal{C}$, where $\mathbf{0}$ means that $\mathbf{0}(y) = \begin{cases} \top, & y = 0 \\ \bot, & \text{otherwise} \end{cases}$ for any $y \in U$,

(2) If $\mathcal{F}_i \in \mathcal{C}$, for any $i \in I$, and $\bigwedge_{i \in I} \mathcal{F}_i \neq \mathbf{0}$, then $\bigvee_{i \in I} \mathcal{F}_i \in \mathcal{C}$,

then $\mathcal{C}$ is called a connectivity.

In the sequel of this section, we call a L-fuzzy set briefly a fuzzy set if there is no confusion occurred.

7.1 Path-Connectivity

Given a fuzzy set $\mathcal{F} \in \mathcal{L}^U$, which denotes a grey-scale image on U. For every fuzzy subset $F \subseteq \mathcal{F}$, which represents a sub-image of image $\mathcal{F}$, and which may be a fuzzy point such as $f_{x,\,\lambda}$ or a fuzzy set generated by means of (7.1).

7.1. Definition. The degree of connectivity of F at points $x\,, y \in U$ is defined as

$$D_F(x\,,\ y) = \sup_{L_{x\,y}} \inf_{z \in L_{x\,y}} F(z)\,,$$

where $L_{x\,y}$ denotes a path from point x to y.

For any $x\,,\, y \in U$ we always have,

$$D_F(x\,,\ y) \leq \min(F(x)\,,\ F(y))\,.$$

For a $\lambda \in \mathcal{L}$, if $D_F(x\,,\ y) \geq \lambda$, then we say that the fuzzy set (grey-scale image) F is λ-connected from point x to y. If F is λ-connected for any two points in U, we say that F is λ-connected in U. In general, for a given $\lambda \in \mathcal{L}$, an image F is not always λ-connected in the whole area U. However, it may be true in some parts (zones) of U, respectively. Let $P_\bot(F) = \{x \in U \mid F(x) = \bot\}$, then for any $x\,,\, y \in P_\bot(F)\,,\ D_F(x\,,y) = \bot$. The area $P_\bot(F)$ is called the background of image F.

For any $F \subseteq \mathcal{F}$, D_F is a fuzzy relation from $U \times U$ to $\mathcal{L}$. Moreover, it is obvious that D_F is symmetric. For any $x\,,\, y\,,\, z \in U$, let $\min(D_F(x\,,\,y)\,,\, D_F(y\,,\,z)) = \lambda$. If there exists at least one point $p \in \{x\,,\,y\,,\,z\}$ such that $p \in P_\bot(F)$, then $D_F(x\,,\,z) \geq \lambda = \bot$. Otherwise, there exist paths $L_{x\,y}$ and $L_{y\,z}$, respectively, such that for any $z_1 \in L_{x\,y}$ and for any $z_2 \in L_{y\,z}$, $F(z_1) \geq \lambda$, $F(z_2) \geq \lambda$. Therefore, for any $z_3 \in L_{x\,y\,z}$, $F(z_3) \geq \lambda$. So, $D_F(x\,,\,z) \geq \lambda$, which means that D_F is transitive.

Let I_U be an identity relation on U, e.g., for any $x \in U$, $I_U(x\,,\,x) = \top$, then the new relation $\mathcal{D}_F = D_F \cup I_U$ is a fuzzy equivalence relation on U. By $\mathcal{D}_F$, given different thresholds $\lambda \in \mathcal{L}$, U can be partitioned into different kinds of set-families (partitions of U) $\mathcal{P}_\lambda(F)$, called flat zones partitions.

(1) When $\lambda = \bot$, $\mathcal{P}_\lambda(F) = \{U\}$, which is the coarsest partition of U for any $F \subseteq \mathcal{F}$.

(2) When $\lambda = \top$, $\mathcal{P}_\lambda(F)$ is the finest partition of U for any $F \subseteq \mathcal{F}$ and $F \neq \emptyset$. At this time, $\mathcal{P}_\lambda(F)$ is the partition of binary image $[F]_\lambda = [F]_\top = \begin{cases} \top, & F(x) = \top\,, \\ \bot, & F(x) < \top\,. \end{cases}$

(3) When $\lambda_1 \leq \lambda_2$, the partition $\mathcal{P}_{\lambda_1}(F)$ is coarser than $\mathcal{P}_{\lambda_2}(F)$. That is, for every $P_{\lambda_2} \in \mathcal{P}_{\lambda_2}(F)$, there exists a unique set (class) $P'_{\lambda_1} \in \mathcal{P}_{\lambda_1}(F)$ such that $P_{\lambda_2} \subseteq P'_{\lambda_1}$. That means that every class (set) in $\mathcal{P}_{\lambda_2}(F)$ is a subclass (subset) of a certain class in $\mathcal{P}_{\lambda_1}(F)$.

(4) $F(x) \geq \sup\{\lambda \in \mathcal{L} \mid x \in P_\lambda \in \mathcal{P}_\lambda(F)\,,\, P_\lambda \cap P\bot = \emptyset\}\,,\quad x \in U$.
$[x \in P_\lambda \in \mathcal{P}_\lambda(F) \Longrightarrow \forall y \in P_\lambda, D_F(x\,,\,y) \geq \lambda \Longrightarrow F(x) \geq \lambda \Longrightarrow x \in [F]_\lambda \Longrightarrow P_\lambda \subseteq [F]_\lambda \Longrightarrow \bigvee \lambda P_\lambda(x) = \sup\{\lambda \mid x \in P_\lambda\} \leq \bigvee \lambda [F]_\lambda(x) = F(x)]$

In the sequel of this section, we always assume that $\lambda > \perp$.

For a given $F \in \mathcal{F}$ and $\lambda \in \mathcal{L}$, $\mathcal{P}_\lambda(F)$ possesses the following properties.

- For any $P_\lambda \in \mathcal{P}_\lambda(F) \Longrightarrow P_\lambda \neq \emptyset$.
- For any $(P_\lambda)_1\,,\ (P_\lambda)_2 \in \mathcal{P}_\lambda(F) \Longrightarrow (P_\lambda)_1 = (P_\lambda)_2$ or $(P_\lambda)_1 \cap (P_\lambda)_2 = \emptyset$.
- $U = \bigcup \mathcal{P}_\lambda(F) = \bigcup \{P_\lambda \mid P_\lambda \in \mathcal{P}_\lambda(F)\}$.
- For any $x \in P_\perp$, there exists one and only one set $P_\lambda \in \mathcal{P}_\lambda(F)$ such that $P_\lambda = \{x\}$.
- $P_\perp(F) = \bigcup_{x \in P_\lambda\,, F(x) = \perp} P_\lambda = \{x \in U \mid x \in P_\lambda \in \mathcal{P}_\lambda(F)\,,\ F(x) = \perp\}$.

Let $(P_\lambda)_\perp(F) = \{P_\lambda \in \mathcal{P}_\lambda(F) \mid x \in P_\lambda\,,\ F(x) = \perp\}$. For any class $P_\lambda \in \mathcal{P}_\lambda(F) \setminus (P_\lambda)_\perp(F)$, we have for any $x\,, y \in P_\lambda$, $\mathcal{D}_F(x\,,\ y) \geq \lambda$, which means that F is connected in P_λ, or F is λ-connected in P_λ, or P_λ is a singleton $\{x\}$ such that $F(x) > \perp$. Let

$$\gamma_{x\,,\ \lambda}(F) = \{P_\lambda \in \mathcal{P}_\lambda(F) \setminus (P_\lambda)_\perp(F) \mid F \text{ is } \lambda\text{-connected in } P_\lambda\,, x \in P_\lambda\}\,,$$

then $\gamma_{x\,,\ \lambda}(F)$ is a connected component of grey-scale image F at level λ.

$\gamma_{x\,,\ \lambda}$ possesses the following properties.

1. For any $x \in U$, $\gamma_{x\,,\ \lambda}(\mathbf{0}) = \begin{cases} \{0\}\,, & \text{if} \quad \lambda > \perp\,, \\ U\,, & \text{if} \quad \lambda = \perp\,, \end{cases}$ and $\quad P_\perp(\mathbf{0}) = U \setminus \{0\}$.

2. Let $f_{y\,,\ \mu} \in \mathcal{L}_p^U$, then $\gamma_{x\,,\ \lambda}(f_{y\,,\ \mu}) = \begin{cases} \{x\} = \{y\}\,, & \text{if} \quad y = x \;\&\; \mu \geq \lambda \geq \perp\,, \\ U\,, & \text{if} \quad y = x \;\&\; \mu = \lambda = \perp\,, \\ \emptyset\,, & \text{otherwise,} \end{cases}$ and

$P_\perp(f_{y\,,\ \mu}) = U \setminus \{y\}$.

3. For a fixed $\lambda \in \mathcal{L}$, and any $x\,, y \in U$, $\gamma_{x\,,\ \lambda}(F) = \gamma_{y\,,\ \lambda}(F)$ or $\gamma_{x\,,\ \lambda}(F) \cap \gamma_{y\,,\ \lambda}(F) = \emptyset$.

4. For a fixed $x \in U$ and $F \subseteq \mathcal{F}$, $\gamma_{x\,,\ \lambda}(F)$ is decreasing set function with respect to λ. That means that when $\lambda_1 \leq \lambda_2\,,\ \gamma_{x\,,\ \lambda_2}(F) \subseteq \gamma_{x\,,\ \lambda_1}(F)$.

5. For a fixed $x \in U$ and $\lambda \in \mathcal{L}$, if $F_1 \subseteq F_2 \subseteq \mathcal{F}$, then $P_\perp(F_2) \subseteq P_\perp(F_1)$, and $\gamma_{x\,,\ \lambda}(F_1) \subseteq \gamma_{x\,,\ \lambda}(F_2)$. That is, every set in $\mathcal{P}_\lambda(F_1) \setminus (P_\lambda)_\perp(F_1)$ is a subset of a set in $\mathcal{P}_\lambda(F_2) \setminus (P_\lambda)_\perp(F_2)$.

6. Let $\mathcal{L}^U \mid \mathcal{F} = \{F \mid F \in \mathcal{L}^U\,, F \subseteq \mathcal{F}\}$. Given a $x \in U$ and a $\lambda \in \mathcal{L} \setminus \{\perp\}$, a mapping $\mathbf{F} : \mathcal{L}^U \mid \mathcal{F} \to \mathcal{L}$ given by

$$\mathbf{F}(F) = F(\gamma_{x\,,\ \lambda}(F))$$

or concretely, for any $y \in U$,

$$\mathbf{F}(F)(y) = \begin{cases} \lambda\,, & y \in \gamma_{x\,,\ \lambda}(F)\,, \\ \perp\,, & \text{otherwise}\,, \end{cases}$$

then it is easy to show that $\mathbf{F}(\cdot) = \cdot(\gamma_{x\,,\ \lambda}(\cdot))$ is an opening. That means that

- $\mathbf{F}(\cdot)$ is increasing: $F_1 \subseteq F_2 \Longrightarrow \mathbf{F}(F_1) \subseteq \mathbf{F}(F_2)$.
- $\mathbf{F}(\cdot)$ is idempotent: $\mathbf{F}^2(F) = \mathbf{F}(F)$.
- $\mathbf{F}(\cdot)$ is anti-extensive: $\mathbf{F}(F) \subseteq F$.

For the opening operator $\mathbf{F}(\cdot) = \cdot(\gamma_{x\,,\ \lambda}(\cdot))$, and $\mathcal{F} \in \mathcal{L}^U$, let

$$f_{x\,,\ \lambda} = \mathbf{F}(\mathcal{F}) = \mathcal{F}(\gamma_{x\,,\ \lambda}(\mathcal{F}))\,, \qquad x \in U\,,\ \lambda \in \mathcal{L}\,,$$

then $\{f_{x\,,\ \lambda}\}$, together with $\mathbf{0}$, forms a connectivity of $\mathcal{F}$.

7.2 Connected Operator

Let $\mathcal{F} \in \mathcal{L}^U$, an operator $\Phi : \mathcal{L}^U \mid \mathcal{F} \to \mathcal{L}^U$ is called connected (connected operator) if the partition $[\mathcal{P}_\lambda(\Phi(F)) \setminus (P_\lambda)_\perp(\Phi(F))] \cup \{P_\perp(\Phi(F))\}$ is coarser than the partition $[\mathcal{P}_\lambda(F) \setminus (P_\lambda)_\perp(F)] \cup \{P_\perp(F)\}$ for $F \subseteq \mathcal{F}$ and any $\lambda \in \mathcal{L}$, $\lambda > \perp$.

7.2. Proposition. *An anti-extensive operator Φ is a connected operator if and only if*

$$\overline{\mathcal{D}}_{\Phi(F)}(x\,,\ y) \geq \overline{\mathcal{D}}_F(x\,,\ y) \qquad x\,, y \in U \tag{7.2}$$

where

$$\overline{\mathcal{D}}_F(x\,, y) = \begin{cases} \top\,, & x\,, y \in P_\perp(F)\,, \\ \mathcal{D}_F(x\,, y)\,, & \textit{otherwise}\,. \end{cases}$$

Proof. If $\Phi(F) \subseteq F$, then by the fifth result of the above subsection, we have, $P_\perp(F) \subseteq P_\perp(\Phi(F))$, and also, every class in $\mathcal{P}_\lambda(\Phi(F)) \setminus (P_\lambda)_\perp(\Phi(F))$ is a subclass of a certain class in $\mathcal{P}_\lambda(F) \setminus (P_\lambda)_\perp(F)$ for any $\lambda \in \mathcal{L}$, $\lambda > \perp$.

$\Leftarrow$: If (7.2) is true, then for any class $P_\lambda \in \mathcal{P}_\lambda(F)$, if $P_\lambda \subseteq P_\perp(F)$, then $P_\lambda \subseteq P_\perp(\Phi(F))$; If $P_\lambda \in \mathcal{P}_\lambda(F) \setminus (P_\lambda)_\perp(F)$ and $D_F(x\,, y) \geq \lambda$ for any $x\,, y \in P_\lambda$, then $D_{\Phi(F)}(x\,, y) \geq \lambda$. So, there exists a class $P_\lambda^{'} \in \mathcal{P}_\lambda(\Phi(F)) \setminus (P_\lambda)_\perp(\Phi(F))$ such that $P_\lambda \subseteq P_\lambda^{'}$; Otherwise, $D_{\Phi(F)}(x\,, y) < \lambda$. In this case, $P_\lambda \subseteq P_\perp(\Phi(F))$. Therefore, the partition $[\mathcal{P}_\lambda(\Phi(F)) \setminus (P_\lambda)_\perp(\Phi(F))] \cup \{P_\perp(\Phi(F))\}$ is coarser than the partition $[\mathcal{P}_\lambda(F) \setminus (P_\lambda)_\perp(F)] \cup \{P_\perp(F)\}$, which implies that Φ is connected operator.

$\Rightarrow$: If Φ is a connected operator, then for any $F \subseteq \mathcal{F}$, any $\lambda \in \mathcal{L}$, and for any class $P_\lambda \in [\mathcal{P}_\lambda(F) \setminus (P_\lambda)_\perp(F)] \cup \{P_\perp(F)\}$, there exists a class $P_\lambda^{'} \in [\mathcal{P}_\lambda(\Phi(F)) \setminus (P_\lambda)_\perp(\Phi(F))] \cup \{P_\perp(\Phi(F))\}$, such that $P_\lambda \subseteq P_\lambda^{'}$.

For any $x\,, y \in U$, if $x\,, y \in P_\perp(F)$ then $x\,, y \in P_\perp(\Phi(F))$. So, $\overline{\mathcal{D}}_{\Phi(F)}(x\,,\ y) = 1 \geq \overline{\mathcal{D}}_F(x\,,\ y) = 1$. If $x\,, y \in P_\lambda \in \mathcal{P}_\lambda(F) \setminus (P_\lambda)_\perp(F)$, there exists a set $P_\lambda^{'} \in [\mathcal{P}_\lambda(\Phi(F)) \setminus (P_\lambda)_\perp(\Phi(F))] \cup \{P_\perp(\Phi(F))\}$ such that $P_\lambda \subseteq P_\lambda^{'}$. If $P_\lambda^{'} \subseteq P_\perp(\Phi(F))$, then $\overline{\mathcal{D}}_{\Phi(F)}(x\,, y) = 1 \geq \overline{\mathcal{D}}_F(x\,, y)$. If $P_\lambda^{'} \in \mathcal{P}_\lambda(\Phi(F)) \setminus (P_\lambda)_\perp(\Phi(F))$, then $\overline{\mathcal{D}}_{\Phi(F)}(x\,, y) \geq \lambda$. So, $\overline{\mathcal{D}}_{\Phi(F)}(x\,,\ y) \geq \overline{\mathcal{D}}_F(x\,,\ y)$. If for the x and y, one is in $P_\perp(F)$ and another is in $P_\lambda \in \mathcal{P}_\lambda(F) \setminus (P_\lambda)_\perp(F)$, then $D_F(x\,, y) = \perp$, and moreover $\overline{\mathcal{D}}_F(x\,, y) = \perp \leq \overline{\mathcal{D}}_{\Phi(F)}(x\,, y)$. Therefore, (7.2) holds. □

In this case, the operator Φ only changes some connected components into background or remove some connected components and leave other parts unchanged.

It's natural that the opening $\mathbf{F}(\cdot) = \cdot(\gamma_{x,\ \lambda}(\cdot))$ defined above is a connected operator.

7.3 Function Connectivity

Given a relation R on $\mathcal{L}$, we say that F satisfies relation R, which means that for any $x\,, y \in U$, $(F(x)\,,\ F(y)) \in R$. Let

$$F_R = \{F \in \mathcal{L}^U \mid F \,\text{satisfies relation}\ R\ \text{on}\ \mathcal{L}\}\,.$$

Since $\mathcal{L}$ is a complete lattice, F_R with a certain partial ordering also forms a complete lattice. For example, if R is an identity relation, then F_R is a family of constant functions.

Assuming that $\mathcal{C}$ is a (crisp) connectivity class on $\mathcal{P}(U)$, $F \in F_R$, let

$$G_{C,\ F}(y) = \begin{cases} F(y)\,, & y \in C \in \mathcal{C} \\ \perp\,, & \text{otherwise}\,, \end{cases}$$

then the collection $\{G_{C\,,F}\}_{C\in\mathcal{C}}$, together with the constant function $G \equiv \mathbf{0}$ forms a connectivity.

Taking different fuzzy relations R, there are different kinds of connectivities for a given fuzzy set F.

Let $F \in \mathcal{L}^U$, $\lambda \in \mathcal{L}$,

$$X(F\,,\ \lambda) = \{x \in U \mid F(x) \geq \lambda\}$$

is the threshold set of F at level λ.

Let $\mathcal{C}$ be a connectivity class, $F \in F_R$, then

$$G_{C\,,\;F\,,\;\lambda}(y) = \begin{cases} \lambda\,, & y \in C \cap X(F\,,\;\lambda) \\ \perp\,, & \text{otherwise} \end{cases}\,,\; C \in \mathcal{C}\,,\;\lambda \in \mathcal{L}\,,$$

together with the constant function $G \equiv \mathbf{0}$ forms a connectivity.

For any $x \in U$ and $\lambda \in \mathcal{L}$,

$$\gamma_x(X(F\,,\;\lambda)) = \bigcup\{C \in \mathcal{C} \mid x \in C\,,\; C \subseteq X(F\,,\;\lambda)\}$$

is a connected component of U. Also, it is a partition of set U.

Let

$$\rho(X(F\,,\;\lambda)) = \bigcup\{C \in \mathcal{C} \mid C \subseteq \gamma_x(X(F\,,\;\lambda))\,,\; C \cap X(F\,,\;\lambda) \neq \emptyset\}\,,$$

then $X(F\,,\;\lambda) \bigcap \rho(X(F\,,\;\lambda))$ is coarser than $\{\gamma_x(X(F\,,\;\lambda))\}$.

8. Fuzzy Morphology

In Section 2, we fuzzified the inclusion relation and the intersection relation of two sets, respectively. There, we dealt with the arbitrary variant family by means of common inf-operation and sup-operation, respectively.

Let $F\,, G \in \mathcal{F}(U)$, and $H(y) = I(G(y)\,, F(y))$, the degree of fuzzy set G being included in fuzzy set F is

$$|G \subseteq F| = \bigwedge_{y \in U} I(G(y)\,, F(y)) = \bigwedge_{y \in U} H(y)\,.$$

In this expression we want to replace the infimum or $\wedge$ by another operator INF or $\sqcap$ which is compatible with the respective conjunction C, e.g.,

$$|G \subseteq F| = \mathrm{INF}(H) = \sqcap_{y \in U} H(y)\,.$$

8.1. Definition. An extended operator INF is a function mapping from $\mathcal{F}(U)$ to the unit interval $[0\,,1]$ satisfying the following properties:

(1) $H \in \mathcal{F}(U)$, $H \equiv 1 \iff \mathrm{INF}(H) = 1$.

(2) $H \in \mathcal{F}(U)$, and $H(u) = 0$ for some $u \in U \Longrightarrow \mathrm{INF}(H) = 0$.

(3) $H_1 \subseteq H_2 \Longrightarrow \mathrm{INF}(H_1) \leq \mathrm{INF}(H_2)$.

(4) If $H(y) = \begin{cases} t_1 \in [0\,,1]\,, & y = y_1 \\ t_2 \in [0\,,1]\,, & y = y_2 \\ 1\,, & y \in U \setminus \{y_1\,, y_2\} \end{cases}$

then $\mathrm{INF}(H) = C(t_1, t_2)$, where C is a given fuzzy conjunction.

(5) For every bijection $\tau : U \to U$, $\mathrm{INF}(\tau(H)) = \mathrm{INF}(H)$. Here $\tau(H)(y) = H(\tau(y))$.

We may define an operator SUP, or denoted by $\sqcup$ which is an extension of sup-operation.

8.2. Definition. Let ν be a negation, INF be an extended operator from a conjunction C. Let

$$D(s\,,t) = \nu(C(\nu(s)\,,\nu(t)))\,,$$

for all $s\,,t \in [0\,,1]$, then D is a disjunction, and the extension of disjunction D is the extended operator SUP.

8.1 Examples

In this subsection, we give some examples for the extended function operators INF and the respective conjunctions.

(1) When the universe U is a finite point set, every associative conjunction C can be extended to the extended operator INF.

(2) The inf-operation is a particular function operator INF, and the respective conjunction C is the min-operation.

(3) For every $F \in \mathcal{F}(U)$, define

$$\mathrm{INF}(F) = \prod_{x \in U} S(F(x)/2)\,,$$

where the function S is defined as

$$S(x) = (-x \ln x - (1-x)\ln(1-x))/\ln 2\,.$$

The respective conjunction is

$$C(s\,,t) = S(s/2)S(t/2)\,.$$

When the universe U is finite, this definition of INF makes sense.

(4) Let $\theta : [0\,,1] \to \overline{R^+} = [0\,,+\infty]$ be a decreasing and continuous mapping satisfying $\theta(1) = 0\,,\theta(0) = +\infty$.

If U is a continuous space: $U = \overline{\mathbb{R}}$, define INF as follows

$$\mathrm{INF}(F) = \theta^{-1}(\int_U \theta(F(y))\,\mathrm{d}\sigma)\,.$$

If U is discrete, e.g., $U = \overline{\mathbb{Z}}$, define INF as follows

$$\mathrm{INF}(F) = \theta^{-1}(\sum_{y=-\infty}^{+\infty} \theta(F(y)))\,.$$

Meanwhile, the respective conjunction and implication are respectively

$$C(s\,,t) = \theta^{-1}(\theta(s) + \theta(t))\,,$$

$$I(s\,,t) = \begin{cases} 1\,, & s \le t\,, \\ \theta^{-1}(\theta(t) - \theta(s))\,, & s > t\,. \end{cases}$$

At this time, for any $F\,,G \in \mathcal{F}(U)$,

$$|F \subseteq F| = \sqcap_{y \in U} I(F(y)\,,F(y)) = 1\,,$$

and if conjunction C satisfies the condition $C \le \min$, and $(F \cap G)(y) = C(F(y)\,,G(y))\,,y \in U$, then

$$|F \cap G \subseteq F| = \sqcap_{y \in U} I(C(F(y)\,,G(y))\,,F(y)) = 1\,.$$

From these examples, we know that the extended operator INF should be the extension of a conjunction C.

8.2 Commutative and Associative Conjunction

Assume that INF or $\sqcap$ is an extension of a certain conjunction C, and that SUP or $\sqcup$ is the dual operator of INF or $\sqcap$.

8.3. Definition. Let $\sqcap$ be an extension of conjunction C such that for any mapping $H : U^2 \to [0\,,1]$,

$$\sqcap_{y \in U} \sqcap_{x \in U} H(x\,,y) = \sqcap_{x \in U} \sqcap_{y \in U} H(x\,,y)\,,$$

then we said that $\sqcap$ is commutative and associative.

If $\sqcap$ is commutative and associative, then for any family of fuzzy sets $\{F_v\}_{v\in U} \subseteq \mathcal{F}(U)$, we have

$$(\sqcap_{v\in U} F_v)(y) = \sqcap_{y\in U} H(y\,,v) = \sqcap_{v\in U} F_v(y)\,.$$

8.4. Proposition. *Let $(I\,,C)$ be an adjunction, and $C(a\,,1) = a$ for any $a \in [0\,,1]$, then*

$$|F \subseteq F| = 1\,,$$

and

$$|F_1 \subseteq F_2| = 1 \iff F_1 \subseteq F_2\,.$$

for any $F\,,F_1\,,F_2 \in \mathcal{F}(U)$.

Proof. Since for any $a \in [0\,,1]$, $C(a\,,1) = a$, for any $t \in [0\,,1]$,

$$I(t\,,t) = \bigvee\{r \in [0\,,1] \mid C(t\,,r) \leq t\} = 1\,.$$

Therefore, $|F \subseteq F| = \sqcap_{y\in U} I(F(y)\,,F(y)) = 1$.

For the proof of the second assertion,

$\Leftarrow$: If $a \leq b$, then $I(a\,,b) = \bigvee\{r \in [0\,,1] \mid C(a\,,r) \leq b\} = \bigvee\{r \in [0\,,1] \mid C(a\,,r) \leq C(a\,,1) \leq b\} = 1$. Therefore, when $F_1 \subseteq F_2$, for any $y \in U$, $I(F_1(y)\,,F_2(y)) = 1$. Hence, $|F_1 \subseteq F_2| = \sqcap_{y\in U} I(F_1(y)\,,F_2(y)) = 1$.

$\Rightarrow$: By the condition of $|F_1 \subseteq F_2| = 1$, we have for any $y \in U$, $I(F_1(y)\,,F_2(y)) = 1$. So, for any $y \in U$, $F_1(y) = C(F_1(y)\,,1) = C(F_1(y)\,,I(F_1(y)\,,F_2(y))) = C(F_1(y)\,,\bigvee\{r \in [0\,,1] \mid C(F_1(y)\,,r) \leq F_2(y)\}) = \bigvee C(F_1(y), \{r \in [0\,,1] \mid C(F_1(y)\,,r) \leq F_2(y)\}) = \bigvee\{C(F_1(y)\,,r) \mid C(F_1(y)\,,r) \leq F_2(y)\} = F_2(y)$. □

8.5. Definition. Let C be a conjunction and D be its dual disjunction, if for any $s\,,t_1\,,t_2 \in [0\,,1]$,

$$D(s\,,C(t_1\,,t_2)) = C(D(s\,,t_1)\,,D(s\,,t_2))\,,$$

and

$$C(s\,,D(t_1\,,t_2)) = D(C(s\,,t_1)\,,C(s\,,t_2))\,,$$

then C and D are called distributive.

8.6. Proposition. *Let implication I and conjunction C form an adjunction. If C and D can be distributive, then for arbitrary index set J,*

$$C(s\,,\sqcup_{j\in J} t_j) = \sqcup_{j\in J} C(s\,,t_j) \quad \textit{and} \quad I(s\,,\sqcap_{j\in J} t_j) = \sqcap_{j\in J} I(s\,,t_j)\,, \tag{8.1}$$

hold for any $s \in [0\,,1]$ and any family $\{t_j\}_{j\in J} \subseteq [0\,,1]$.

If C and D are distributive, and $C \leq \min$, then $(I\,,C)$ is an adjunction if and only if (8.1) holds.

8.3 Fuzzy Adjunction

Let U and V be two nonempty sets, $\mathcal{F}(U)$ and $\mathcal{F}(V)$ be the families of all fuzzy subsets on U and V, respectively.

8.7. Definition. Let $\mathcal{E}$ be a mapping from $\mathcal{F}(U)$ to $\mathcal{F}(V)$, and $\mathcal{D}$ be a mapping from $\mathcal{F}(V)$ to $\mathcal{F}(U)$. If

$$|\mathcal{D}(G) \subseteq F| = |G \subseteq \mathcal{E}(F)|$$

for any $F \in \mathcal{F}(U)$ and $G \in \mathcal{F}(V)$, we call the pair $(\mathcal{E}\,,\mathcal{D})$ a fuzzy adjunction between $\mathcal{F}(U)$ and $\mathcal{F}(V)$. If $U = V$, we call the pair $(\mathcal{E}\,,\mathcal{D})$ a fuzzy adjunction on $\mathcal{F}(U)$.

8.8. Definition. An operator $\mathcal{E} : \mathcal{F}(U) \to \mathcal{F}(V)$ is called a fuzzy erosion if

$$\mathcal{E}(\sqcap_{v\in U} F_v) = \sqcap_{v\in U}\mathcal{E}(F_v)$$

for every family $\{F_v\}_{v\in U} \subseteq \mathcal{F}(U)$. An operator $\mathcal{D} : \mathcal{F}(V) \to \mathcal{F}(U)$ is called a fuzzy dilation if

$$\mathcal{D}(\sqcup_{v\in U} F_v) = \sqcup_{v\in U}\mathcal{D}(F_v)$$

for every family $\{F_v\}_{v\in U} \in \mathcal{F}(V)$.

In the sequel of this section, we assume that $U = V$.

8.9. Proposition. *Let $(I\,,C)$ be an adjunction, and let the extended conjunction $\sqcap$ be commutative and associative, C and D be distributive, then*

$$\mathcal{E}_G(F)(x) = \sqcap_{y\in U} I(G(y-x)\,, F(y))$$

is an erosion, and

$$\mathcal{D}_G(F)(x) = \sqcup_{y\in U} C(G(x-y)\,, F(y))$$

is a dilation for any $G \in \mathcal{F}(U)$ and $x \in U$.

Proof. Let $\{F_v\}_{v\in U} \subseteq \mathcal{F}(U)$ be a family of fuzzy sets, then for any $G \in \mathcal{F}(U)$, and any $x \in U$,

$$\begin{aligned} \sqcap_{v\in U} I(G(y-x)\,, \sqcap_{v\in U} F_v(y)) &= \sqcap_{y\in U} \sqcap_{v\in U} I(G(y-x)\,, F_v(y)) \\ &= \sqcap_{v\in U} \sqcap_{y\in U} I(G(y-x)\,, F_v(y))\,. \end{aligned}$$

Therefore, $\mathcal{E}_G(F)$ is an erosion.
Similar proof to the dilation. □

8.10. Proposition. *Let I be an implication and C be a conjunction, and $C \le \min$, then $(I\,,C)$ is adjoint if and only if $(\mathcal{E}_G\,,\mathcal{D}_G)$ is a fuzzy adjunction for any $G \in \mathcal{F}(U)$.*

Proof. If $C \le \min$, then $D \ge \max$.
⇒: For any $a\,,s\,,t \in [0\,,1]$, we have $C(a\,,t) \le s \iff t \le I(a\,,s)$. So for any $F\,,G\,,H \in \mathcal{F}(U)$,

$$\begin{aligned} \mathcal{D}_G(F) \le H &\iff \forall\, x \in U\,,\ \mathcal{D}_G(F)(x) \le H(x) \\ &\iff \forall\, x \in U\,, \sqcup_{y\in U} C(G(x-y)\,, F(y)) \le H(x) \\ &\iff \forall\, x \in U\,,\ \forall\, y \in U\,,\ C(G(x-y)\,, F(y)) \le H(x) \\ &\iff \forall\, x \in U\,,\ \forall\, y \in U\,,\ F(y) \le I(G(x-y)\,, H(x)) \\ &\iff \forall\, y \in U\,,\ F(y) \le \sqcap_{x\in U} I(G(x-y)\,, H(x)) = \mathcal{E}_G(H)(y) \\ &\iff F \le \mathcal{E}_G(H)\,. \end{aligned}$$

⇐: Given $a\,,s\,,t \in [0\,,1]$, define the constant functions $G \equiv a\,, F \equiv s\,,$ and $H \equiv t$. Then $\mathcal{D}_G(F)(x) = \sqcup_{y\in U} C(G(x-y)\,, F(y)) = C(a\,,s)$ and $\mathcal{E}_G(H)(x) = \sqcap_{y\in U} I(G(y-x)\,, H(y)) = I(a\,,t)$, for every $x \in U$. Therefore, $C(a\,,s) \le t \iff \mathcal{D}_G(F) \le H \iff F \le \mathcal{E}_G(H) \iff s \le I(a\,,t)$. □

8.11. Proposition. *Let $(I\,,C)$ be an adjunction, C and I be exchangeable, $C(a\,,1) = a$ for $a \in [0\,,1]$, and the extended conjunction $\sqcap$ be commutative and associative, then $\mathcal{E}$ is an erosion. Furthermore, if C is continuous from the left with respect to the first argument, then $\mathcal{D}$ is a dilation.*

Proof. For any family $\{F_v\}_{v\in U} \subseteq \mathcal{F}(U)$, and $G \in \mathcal{F}(U)$,

$$\begin{aligned}
|G \subseteq \mathcal{E}(\sqcap_{v\in U} F_v)| &= |\mathcal{D}(G) \subseteq \sqcap_{v\in U} F_v| \\
&= \sqcap_{u\in U} I(\mathcal{D}(G)(u)\,, \sqcap_{v\in U} F_v(u)) \\
&= \sqcap_{u\in U} \sqcap_{v\in U} I(\mathcal{D}(G)(u)\,, F_v(u)) \\
&= \sqcap_{v\in U} \sqcap_{u\in U} I(\mathcal{D}(G)(u)\,, F_v(u)) \\
&= \sqcap_{v\in U} |\mathcal{D}(G) \subseteq F_v| \\
&= \sqcap_{v\in U} |G \subseteq \mathcal{E}(F_v)| \\
&= \sqcap_{v\in U} \sqcap_{u\in U} I(G(u)\,, \mathcal{E}(F_v)(u)) \\
&= \sqcap_{u\in U} \sqcap_{v\in U} I(G(u)\,, \mathcal{E}(F_v)(u)) \\
&= \sqcap_{u\in U} I(G(u)\,, \sqcap_{v\in U} \mathcal{E}(F_v)(u)) \\
&= |G \subseteq \sqcap_{v\in U} \mathcal{E}(F_v)|\,.
\end{aligned}$$

Taking $G = \mathcal{E}(\sqcap_{v\in U} F_v)$, we have

$$|\mathcal{E}(\sqcap_{v\in U} F_v) \subseteq \sqcap_{v\in U} \mathcal{E}(F_v)| = 1\,,$$

and taking $G = \sqcap_{v\in U} \mathcal{E}(F_v)$, we have

$$|\sqcap_{v\in U}\, \mathcal{E}(F_v) \subseteq \mathcal{E}(\sqcap_{v\in U} F_v)| = 1\,.$$

By Proposition 8.4, we have that

$$\mathcal{E}(\sqcap_{v\in U} F_v) = \sqcap_{v\in U} \mathcal{E}(F_v)\,.$$

The second assertion van be proved analogously. □

8.12. Proposition. *Let $(\mathcal{E}\,, \mathcal{D})$ be an adjunction, $C(a\,,1) = a$ for $a \in [0\,,1]$, and the extended conjunction $\sqcap$ be commutative and associative, then for any $F\,, G \in \mathcal{F}(U)$,*

$$|\mathcal{D}\mathcal{E}(F) \subseteq F| = 1 \quad \textit{and} \quad |G \subseteq \mathcal{E}\mathcal{D}(G)| = 1\,.$$

$$\mathcal{E}\mathcal{D}\mathcal{E} = \mathcal{E} \qquad \textit{and} \qquad \mathcal{D}\mathcal{E}\mathcal{D} = \mathcal{D}\,.$$

Proof. Since $(\mathcal{E}\,, \mathcal{D})$ is an adjunction, then for any $F\,, G \in \mathcal{F}(U)$, $|\mathcal{D}(G) \subseteq F| = |G \subseteq \mathcal{E}(F)|$. Taking $G = \mathcal{E}(F)$ we have

$$|\mathcal{D}\mathcal{E}(F) \subseteq F| = |\mathcal{E}(F) \subseteq \mathcal{E}(F)| = 1\,.$$

On the other hand, taking $F = \mathcal{D}(G)$ yields

$$|G \subseteq \mathcal{E}\mathcal{D}(G)| = |\mathcal{D}(G) \subseteq \mathcal{D}(G)| = 1\,.$$

Similarly, we have that

$$|\mathcal{E}(F) \subseteq \mathcal{E}\mathcal{D}\mathcal{E}(F)| = 1 \;\text{ and }\; |\mathcal{D}\mathcal{E}\mathcal{D}(G) \subseteq \mathcal{D}(G)| = 1\,.$$

By the increasingness of $\mathcal{E}$ and $\mathcal{D}$, we obtain that

$$\mathcal{E}\mathcal{D}\mathcal{E}(F) \subseteq \mathcal{E}(F) \;\text{ and }\; \mathcal{D}(G) \subseteq \mathcal{D}\mathcal{E}\mathcal{D}(G)\,.$$

Combination of these results yields

$$\mathcal{E}\mathcal{D}\mathcal{E} = \mathcal{E} \qquad \text{and} \qquad \mathcal{D}\mathcal{E}\mathcal{D} = \mathcal{D}\,.$$

□

Figure 1: Original 256×256 grey-scale image.

9. Experimental Results

In this section we present some experiments showing the differences between basic morphological operators using different conjunctions. We compare the outcomes of the operators based on fuzzy logic with the corresponding flat operators using a crisp structuring element. Our input image is depicted in Fig. 1. The structuring function G used for the 'fuzzy' operators is represented by the matrix

$$G = \frac{1}{20} * \begin{pmatrix} 0 & 2 & 4 & 5 & 4 & 2 & 0 \\ 2 & 6 & 9 & 10 & 9 & 6 & 2 \\ 4 & 9 & 13 & 15 & 13 & 9 & 4 \\ 5 & 10 & 15 & 20 & 15 & 10 & 5 \\ 4 & 9 & 13 & 15 & 13 & 9 & 4 \\ 2 & 6 & 9 & 10 & 9 & 6 & 2 \\ 0 & 2 & 4 & 5 & 4 & 2 & 0 \end{pmatrix}$$

Observe that this matrix approximates a cone in the sense that an entry is approximately given by $1 - \frac{1}{4}(i^2 + j^2)^{\frac{1}{2}}$, where (i, j) are the coordinates of the entry relative to the center of the matrix. This approximation is based on the 5-7-11 chamfer distance [7]. The binary structuring element is obtained by thresholding the (fuzzy) structuring function at level 0.5, and is given by

$$A = \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & 1 & 1 & \cdot \\ 1 & 1 & 1 & 1 & 1 \\ \cdot & 1 & 1 & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \end{pmatrix}$$

In Fig. 2 we show the dilation, erosion, closing and opening for the flat structuring element A, as well as the various conjunctions. The images in columns represent dilation, erosion, closing and opening, respectively. The first row is the images using flat structuring element A. The last five rows stand for the operators using the structuring function G in combination with the Gödel-Brouwer conjunction, the Lukasiewicz conjunction, the Kleene-Dienes conjunction, the Reichenbach conjunction and the Hamacher conjunction, respectively.

In Fig. 3 and in Fig. 4 we show the differences between dilation, erosion, closing , opening, and the original image, respectively. The interpretation of the rows is same as that in Fig. 2, the first column stands for the dilation minus erosion, the second is the dilation minus original, and the third is the original minus erosion in Fig. 3, and in Fig. 4, the columns represent the closing minus opening, the closing minus original, and the original minus opening.

The fuzzy nature of the operators is clearly reflected by the images, especially close to the edges. Furthermore, there are various differences (again near the edges) among these operators, indicating that the particular choice of the conjunction does have a serious impact on the results.

10. Conclusion

In this report, within the framework of complete lattices and fuzzy logic, We have investigated the general theory of grey-scale morphology, which includes grey-scale granulometry, hit-or-miss operator for grey-scale images, rank operator, and connectivity and connected operators. We also gave an example to show that the Matheron's representation theory doesn't hold for general grey-scale images and presented some results related to the representation theory.

Many results of this report have to be considered as a hint and the beginning towards a general theory of grey-scale morphology using concepts from fuzzy logic. Especially, the content in Section 8 is very different from the existing approaches to fuzzy morphology. We believe that the rich theory will be founded through the first step work in grey-scale morphology and fuzzy morphology.

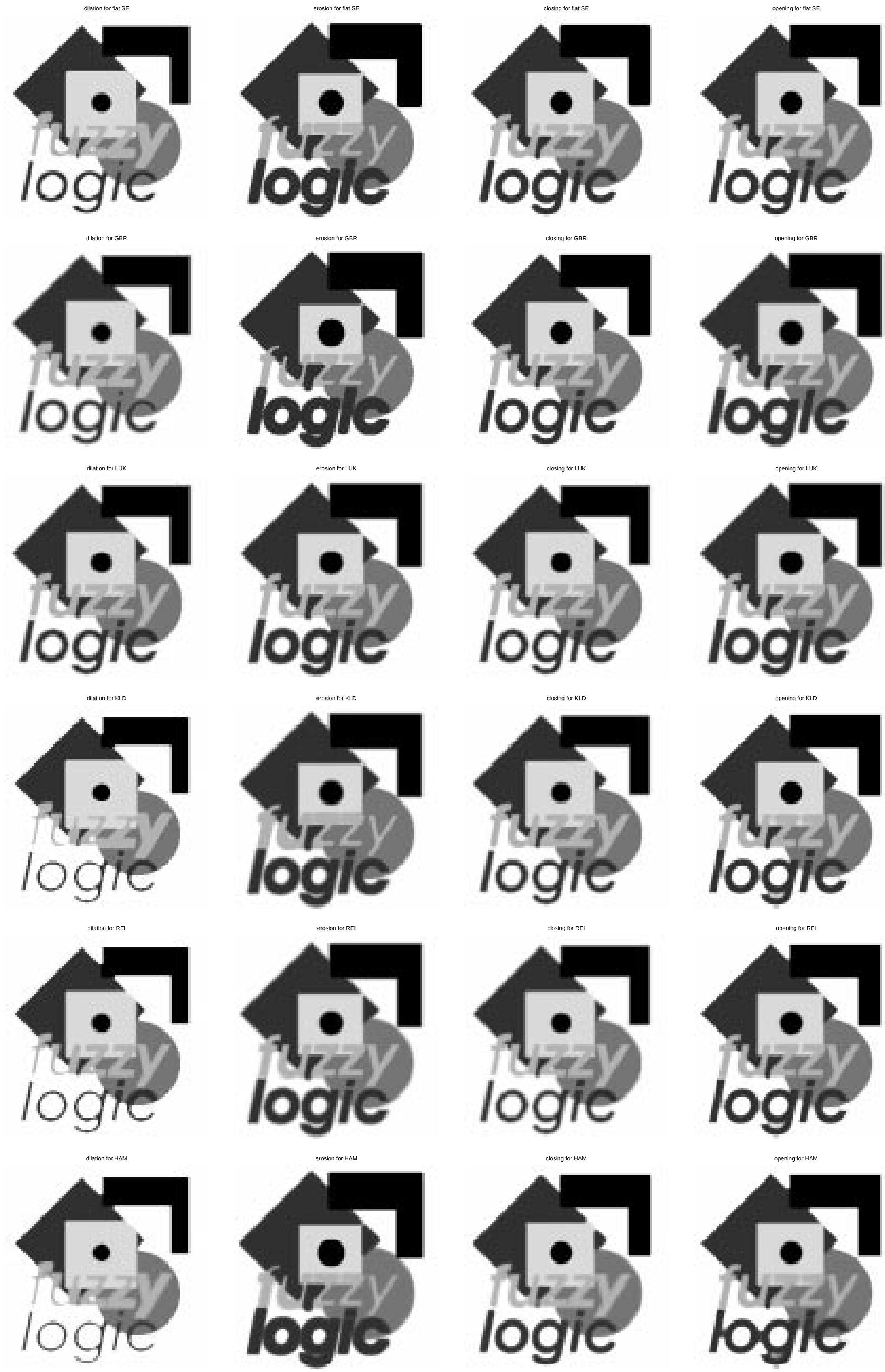

Figure 2: Left to right: dilation, erosion, closing and opening. Top to bottom: flat operator, fuzzy operator using Gödel-Brouwer conjunction, fuzzy operator using Lukasiewicz conjunction, fuzzy operator using Kleene-Dienes conjunction, fuzzy operator using Reichenbach conjunction, and fuzzy operator using Hamacher conjunction.

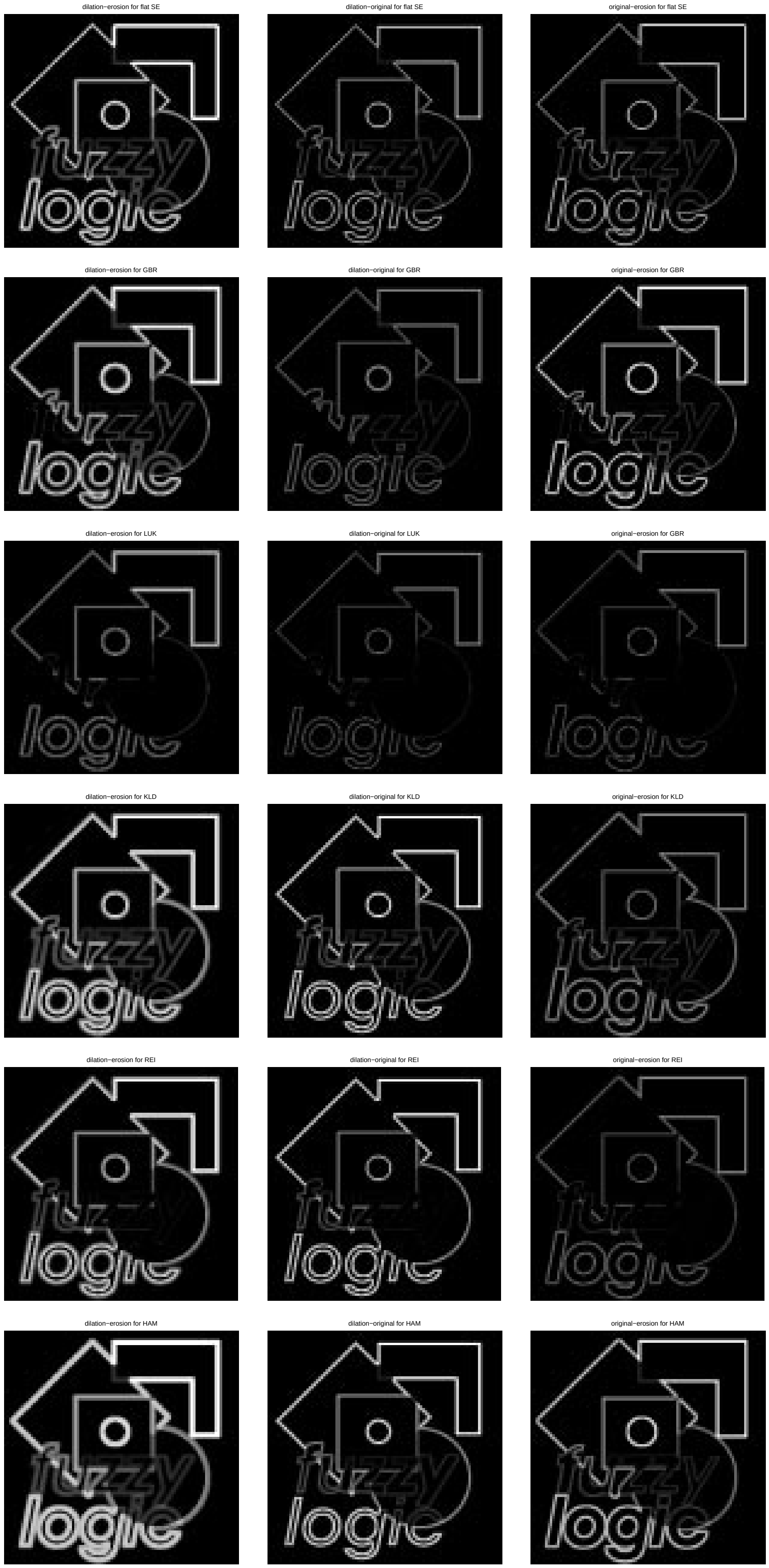

Figure 3: Left to right: dilation minus erosion, dilation minus original, and original minus erosion. Top to bottom: Same as in Fig. 1.

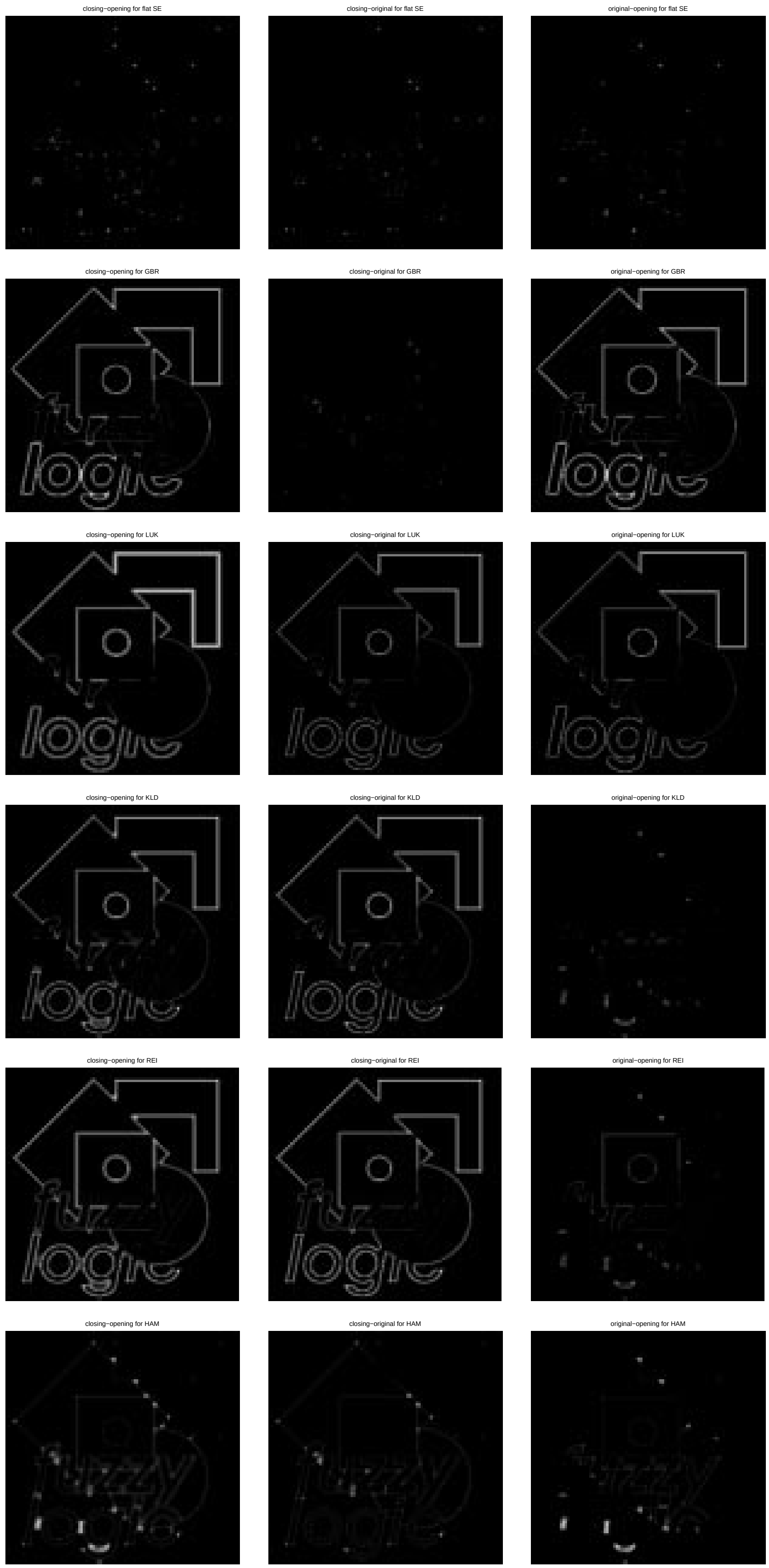

Figure 4: Left to right: Closing minus opening, closing minus original, and original minus opening. Top to bottom: Same as in Fig. 1.

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