CHAPTER 5

EASY RECIPES FOR MORPHOLOGICAL FILTERS[1]

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5.1 INTRODUCTION

A *(morphological) filter* is an increasing operator ψ on a complete lattice $\mathcal{L}$ which is idempotent:

$$\psi^2 = \psi.$$

For some basic literature on morphological filters the reader may refer to the second book edited by Serra [23] (in particular Chap. 6 by Matheron and Chap. 8 by Serra), the tutorial paper by Serra and Vincent [24], to our book [5, Chaps. 12–13], and to [2, 7, 17].

This paper discusses various classes of morphological filters such as openings and closings, annular filters, alternating sequential filters (or AS-filters), and self-dual filters. A large part of the mathematical theory for morphological filters holds on arbitrary complete lattices. However, when we give concrete examples, we often restrict ourselves to the case of subsets of a Euclidean or discrete space.

This paper aims to be mathematically rigorous in the sense that it gives precise definitions and propositions. As a general rule, proofs will be included when they provide additional insight. However, if a proof is rather technical, or when it requires substantial preparations, it will be omitted; in such cases appropriate references will be given.

In the remainder of this section we summarize the contents of this paper. Section 5.2 contains a brief discussion of the complete lattice framework for morphology. In Section 5.3 we discuss two elementary classes of filters, namely, openings and closings. A simple and general (but not the only) way to get openings and closings is by composing dilations and erosions that form adjunctions; a generalization

[1] This paper is a revised and extended version of our lecture "Morphological Filters," which was presented at the summer school, Morphological Image and Signal Processing, September 27–30, 1995, Zakopane, Poland.

of this idea leads to a class of filters which is relatively unknown, the *adjunctional filters*. In Section 5.4, annular filters will be treated. Such filters are given by very simple mathematical expressions. By composing openings and closings, one obtains a class of filters which has proved its importance in practice: the *alternating sequential filters*. These filters are discussed in Section 5.5. In Section 5.6, we introduce operators which are closely related to filters, namely *overfilters, underfilters, inf-overfilters*, and *sup-underfilters*, and describe several methods for their construction. In this section we also treat the *rank-max opening* and the *rank-min closing*, and introduce a new AS-filter obtained by composition of such openings and closings. The class of AS-filters introduced in Section 5.5 can be generalized by composing overfilters and underfilters instead of openings and closings. This generalization is the topic of Section 5.7; we also present some interesting examples there. A general way to construct idempotent operators is by iteration of operators which are not idempotent. In Section 5.8 it is explained that pointwise convergence of the iterates ψ^n of an operator ψ to a limit operator ψ^∞ (along with the continuity of ψ) guarantees that ψ^∞ is idempotent. An important class of operators which satisfy this pointwise convergence criterion are the operators which are *activity-extensive*. These operators are introduced in Section 5.9. There we also discuss the *center operator*. Section 5.10 deals with *self-dual filters*. Such filters treat foreground and background identically (unlike openings, closings, and AS-filters), and as such they are of great importance. The simplest self-dual filter is a special case of the annular filter treated in Section 5.4. For this filter we can give an explicit expression. All other (nontrivial) self-dual filters we know of cannot be expressed by an explicit formula, but their action is described in terms of an iteration procedure. Such a procedure uses a self-dual operator which is activity-extensive. We give detailed results (using the center operator) on the construction of such operators, and some explicit examples.

The filters discussed in this paper are all applied to the same binary input image, the right image in Fig. 5–1. It is obtained from the left image by "adding" salt-and

Figure 5–1. The undistorted image (left) and our test image (right). The size of these images is 128×128 pixels. The black pixels represent the foreground, the white pixels the background.

pepper noise; approximately 15% of the pixels have been affected by this noise. In all images shown in this paper, the black pixels represent the foreground and the white pixels the background.

Although many of our examples concern binary images (our test image is also binary), we can apply them to gray-scale images, too. For that goal we have to consider the usual flat operator extension discussed in [5, 22].

We conclude with some remarks about notation. If E is a set, then we denote by $\mathcal{P}(E)$ the power set of E comprising all subsets of E. When we write $\mathbb{E}^d$, we mean the d-dimensional product of $\mathbb{E}$, where $\mathbb{E}$ is a group; in practical cases $\mathbb{E} = \mathbb{Z}$ or $\mathbb{R}$ with the additive group structure. Using the notation $\mathbb{E}^d$ enables us to treat the discrete case $\mathbb{Z}^d$ and the continuous case $\mathbb{R}^d$ simultaneously.

5.2 MORPHOLOGY ON COMPLETE LATTICES

By its very nature, mathematical morphology is set-oriented, and as such directed toward binary images. However, from the early days of morphology onward, there has been a need for a more general theory covering different object spaces, in particular gray-scale images. Matheron and Serra [23] were the first to observe that a general framework for morphology can be achieved if one starts from the assumption that the object space is a complete lattice. This idea has been carried further by various people, in particular Heijmans and Ronse [9, 19] and Roerdink [13, 14]. A comprehensive account of the complete lattice framework can be found in [5].

5.2.1 BASIC THEORY

In this subsection we give some basic results. In the next section some of them will be applied to gray-scale images.

DEFINITION 5–1. A *complete lattice* is a set $\mathcal{L}$ with a partial ordering '$\leqslant$' such that every subset $\mathcal{H}$ of $\mathcal{L}$ has an infimum (greatest lower bound) and a supremum (least upper bound). The least element of $\mathcal{L}$ is denoted by O, the greatest element by I.

See [1] or [5] for further details. Throughout this section we assume that $\mathcal{L}$ is a complete lattice. A simple example is the family $\mathcal{P}(E)$ ordered by inclusion.

A well-known principle in the theory of partially ordered sets is the *duality principle*. This principle originates from the (trivial) fact that if $\mathcal{L}$ is a partial ordered set, then $\mathcal{L}$ with the dual partial ordering $\leqslant'$ defined by "$X \leqslant' Y$ if and only if $Y \leqslant X$" is a partial ordered set, too. As a result, to every definition or statement referring to $\leqslant$ there corresponds a dual one referring to $\leqslant'$.

Let $\mathcal{L}, \mathcal{M}$ be complete lattices and let $\psi : \mathcal{L} \to \mathcal{M}$; by this notation we mean that ψ is an operator from $\mathcal{L}$ to $\mathcal{M}$. We say that ψ is *increasing* if $X \leqslant X'$ implies that $\psi(X) \leqslant \psi(X')$. It is *decreasing* if $X \leqslant X'$ implies that $\psi(X) \geqslant \psi(X')$. On the collection of operators from $\mathcal{L}$ to $\mathcal{M}$ one can define a partial ordering as follows: $\psi \leqslant \psi'$ if $\psi(X) \leqslant \psi'(X)$, for every $X \in \mathcal{L}$. The set of operators, as well as the set of increasing operators, constitutes a complete lattice under this partial ordering.

An operator $\psi : \mathcal{L} \to \mathcal{L}$ is called an *automorphism* if it is increasing and bijective. One can easily show that every automorphism satisfies $\psi(\bigvee_{i \in I} X_i) = \bigvee_{i \in I} \psi(X_i)$ and $\psi(\bigwedge_{i \in I} X_i) = \bigwedge_{i \in I} \psi(X_i)$, for every family of sets X_i. An operator ψ is called a *negation* if it is decreasing, bijective, and satisfies $\psi^2 = \mathsf{id}$. Here id, or $\mathsf{id}_{\mathcal{L}}$, denotes the *identity operator* given by $\mathsf{id}(X) = X$, for every $X \in \mathcal{L}$. A negation satisfies $\psi(\bigvee_{i \in I} X_i) = \bigwedge_{i \in I} \psi(X_i)$ and $\psi(\bigwedge_{i \in I} X_i) = \bigvee_{i \in I} \psi(X_i)$. If ψ is a negation, then we call $X^* = \psi(X)$ the *negative* of X. (Although our notation may suggest otherwise, negations are not unique in general.) On the Boolean lattice $\mathcal{P}(E)$, the complement operator $X \mapsto X^c$ defines a negation.

Let $\psi : \mathcal{L} \to \mathcal{M}$ and suppose that both complete lattices possess a negation, then the *negative operator* $\psi^* : \mathcal{L} \to \mathcal{M}$ is defined by

$$\psi^*(X) = \left[\psi(X^*)\right]^*.$$

An operator $\psi : \mathcal{L} \to \mathcal{L}$, where $\mathcal{L}$ possesses a negation, is called *self-dual* if

$$\psi^* = \psi.$$

The key notion in mathematical morphology is that of an adjunction.

DEFINITION 5–2. Let $\mathcal{L}, \mathcal{M}$ be complete lattices, let $\varepsilon : \mathcal{L} \to \mathcal{M}$ and $\delta : \mathcal{M} \to \mathcal{L}$. The pair (ε, δ) is called an *adjunction* between $\mathcal{L}$ and $\mathcal{M}$ if

$$\delta(Y) \leqslant X \iff Y \leqslant \varepsilon(X), \tag{5–1}$$

for $X \in \mathcal{L}$ and $Y \in \mathcal{M}$.

PROPOSITION 5–1. *If (ε, δ) is an adjunction between $\mathcal{L}$ and $\mathcal{M}$, then*

(a) $\varepsilon(\bigwedge_{i \in I} X_i) = \bigwedge_{i \in I} \varepsilon(X_i)$ *for every collection X_i in $\mathcal{L}$; in particular, ε is an increasing operator.*

(b) $\delta(\bigvee_{i \in I} Y_i) = \bigvee_{i \in I} \delta(Y_i)$ *for every collection Y_i in $\mathcal{M}$; in particular, δ is an increasing operator.*

(c) $\varepsilon\delta \geqslant \mathsf{id}_{\mathcal{M}}$ *and* $\delta\varepsilon \leqslant \mathsf{id}_{\mathcal{L}}$.

(d) $\varepsilon\delta\varepsilon = \varepsilon$ *and* $\delta\varepsilon\delta = \delta$.

PROOF. (a) Suppose that (ε, δ) is an adjunction between $\mathcal{L}$ and $\mathcal{M}$; we show that ε is an erosion. Suppose $X_i \in \mathcal{L}$ for $i \in I$; given $Y \in \mathcal{M}$, it holds that $\delta(Y) \leqslant \bigwedge_{i \in I} X_i$ if and only if $\delta(Y) \leqslant X_i$ for every $i \in I$. This, however, is equivalent to $Y \leqslant \varepsilon(X_i)$ for every $i \in I$; that is, $Y \leqslant \bigwedge_{i \in I} \varepsilon(X_i)$. On the other hand, by the adjunction relation, $\delta(Y) \leqslant \bigwedge_{i \in I} X_i$ if and only if $Y \leqslant \varepsilon(\bigwedge_{i \in I} X_i)$. But this implies $\varepsilon(\bigwedge_{i \in I} X_i) = \bigwedge_{i \in I} \varepsilon(X_i)$.

(b) Dual statement of (a).

(c) Choosing $X = \delta(Y)$ in Eq. 5–1, we get $Y \leqslant \varepsilon\delta(Y)$, which proves the first relation. The second follows by duality.

(d) From (c) and the increasingness of ε and δ we get that $\varepsilon\delta\varepsilon \geqslant \varepsilon$ and $\varepsilon\delta\varepsilon \leqslant \varepsilon$; therefore, equality holds. Similarly, it follows that $\delta\varepsilon\delta = \delta$. ■

An operator ε which satisfies the relation given under (a) is called *erosion*. An operator satisfying relation (b) is called *dilation*. Thus an adjunction is formed by a dilation and an erosion satisfying the adjunction relation 5–1.

PROPOSITION 5–2. *With every erosion $\varepsilon : \mathcal{L} \to \mathcal{M}$ there corresponds a unique dilation $\delta : \mathcal{M} \to \mathcal{L}$ such that (ε, δ) is an adjunction. This dilation is given by*

$$\delta(Y) = \bigwedge \{X \in \mathcal{L} \mid Y \leqslant \varepsilon(X)\}.$$

Dually, with every dilation $\delta : \mathcal{M} \to \mathcal{L}$ there corresponds a unique erosion $\varepsilon : \mathcal{L} \to \mathcal{M}$ such that (ε, δ) is an adjunction. This erosion is given by

$$\varepsilon(X) = \bigvee \{Y \in \mathcal{M} \mid \delta(Y) \leqslant X\}.$$

PROOF. Suppose that ε is an erosion, and let δ be given by the expression above. We show that Eq. 5–1 holds. First, if $\delta(Y) \leqslant X$, then by applying ε to both sides and using the fact that it distributes over infima, we get

$$\varepsilon\delta(Y) = \bigwedge \{\varepsilon(X') \mid Y \leqslant \varepsilon(X')\} \leqslant \varepsilon(X),$$

and therefore $Y \leqslant \varepsilon(X)$. On the other hand, if $Y \leqslant \varepsilon(X)$, then by definition $\delta(Y) \leqslant X$. This proves Eq. 5–1.

It remains to us to prove uniqueness of δ. Suppose δ' is another operator such that ε, δ' satisfy Eq. 5–1. Then

$$\delta'(Y) \leqslant X \iff Y \leqslant \varepsilon(X) \iff \delta(Y) \leqslant X.$$

But this yields immediately that $\delta(Y) = \delta'(Y)$. The second part of the statement follows by the duality principle. ■

The adjunction relation, though mathematically very simple, provides pairs of operators ε, δ with special properties as illustrated by Propositions 5–1 and 5–2. Also, the proof of our next result can be easily established by using this adjunction relation.

PROPOSITION 5–3.

(a) *Let* (ε, δ) *and* (ε', δ') *be adjunctions between* $\mathcal{L}$ *and* $\mathcal{M}$. *Then* $\varepsilon' \leqslant \varepsilon$ *if and only if* $\delta' \geqslant \delta$.

(b) *Let* $(\varepsilon_i, \delta_i)$ *be an adjunction between* $\mathcal{L}$ *and* $\mathcal{M}$ *for every* $i \in I$. *Then* $(\bigwedge_{i \in I} \varepsilon_i, \bigvee_{i \in I} \delta_i)$ *is an adjunction between* $\mathcal{L}$ *and* $\mathcal{M}$ *as well.*

(c) *Let* (ε, δ) *be an adjunction between* $\mathcal{L}$ *and* $\mathcal{M}$, *and let* (ε', δ') *be an adjunction between* $\mathcal{M}$ *and* $\mathcal{N}$. *Then* $(\varepsilon'\varepsilon, \delta\delta')$ *is an adjunction between* $\mathcal{L}$ *and* $\mathcal{N}$.

If both $\mathcal{L}$ and $\mathcal{M}$ possess a negation and (ε, δ) is an adjunction between $\mathcal{L}$ and $\mathcal{M}$, then $(\delta^*, \varepsilon^*)$ is an adjunction between $\mathcal{M}$ and $\mathcal{L}$.

The operators ε_A, δ_A on $\mathcal{P}(\mathbb{E}^d)$ given by

$$\varepsilon_A(X) = X \ominus A = \{h \in \mathbb{E}^d \mid A_h \subseteq X\} = \bigcap_{a \in A} X_{-a}$$

$$\delta_A(X) = X \oplus A = \bigcup_{a \in A} X_a$$

define an adjunction. Here X_h denotes the translate of X over a vector h, i.e., $X_h = \{x + h \mid x \in X\}$. Furthermore, A is a given subset of $\mathbb{E}^d$, called a *structuring element*.

5.2.2 APPLICATION TO GRAY-SCALE FUNCTIONS

We start with some notation and terminology. We represent gray-scale images mathematically as functions $F : \mathbb{E}^d \to \mathcal{T}$, where $\mathcal{T}$ is the gray-value set. Depending on the application at hand we may choose for $\mathcal{T}$ the set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$, $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, +\infty\}$, $\overline{\mathbb{Z}}_+ = \mathbb{Z}_+ \cup \{+\infty\}$, the bounded interval $[0, 1]$, or the finite set $\{0, 1, \ldots, N\}$. All these examples have in common a complete lattice structure.

By $\mathrm{Fun}(\mathbb{E}^d, \mathcal{T})$ we represent the set of all functions $F : \mathbb{E}^d \to \mathcal{T}$. If $\mathcal{T} = \overline{\mathbb{R}}$, we shall simply write $\mathrm{Fun}(\mathbb{E}^d)$; in all other cases we will include the gray-value set in our notation.

We assume throughout this subsection that $\mathcal{T} = \overline{\mathbb{R}}$. Most of its content, however, carries over to the case $\mathcal{T} = \overline{\mathbb{Z}}$.

For $h \in \mathbb{E}^d$ and $F \in \mathrm{Fun}(\mathbb{E}^d)$, the *horizontal translate* F_h is defined by

$$F_h(x) = F(x-h), \quad x \in \mathbb{E}^d.$$

The *vertical translate* $F + v$, where $v \in \mathbb{R}$, is defined by

$$(F+v)(x) = F(x) + v, \quad x \in \mathbb{E}^d.$$

Given a function $G \in \mathrm{Fun}(\mathbb{E}^d)$, we define the operators

$$\Delta_G(F) = F \oplus G, \quad \mathcal{E}_G(F) = F \ominus G,$$

respectively given by

$$(F \oplus G)(x) = \bigvee_{h \in \mathbf{E}^d} \big[F(x-h) + G(h)\big], \tag{5–2}$$

$$(F \ominus G)(x) = \bigwedge_{h \in \mathbf{E}^d} \big[F(x+h) - G(h)\big]. \tag{5–3}$$

We call G an *additive structuring function*. In the case of ambiguous expressions we use the convention that $s + t = -\infty$ if $s = -\infty$ or $t = -\infty$, and $s - t = +\infty$ if $s = +\infty$ or $t = -\infty$. The following result is easily proved.

PROPOSITION 5–4. *The pair* $(\mathcal{E}_G, \Delta_G)$ *defines an adjunction on* $\mathrm{Fun}(\mathbb{E}^d)$.

Both Δ_G and $\mathcal{E}_G$ are translation invariant with respect to horizontal as well as vertical translations, i.e., both operators have the following property:

$$\Psi(F_h + v) = \big[\Psi(F)_h\big] + v,$$

for $h \in \mathbb{E}^d$ and $v \in \mathbb{R}$. An operator with this property is called a *T-operator*. If Ψ is only invariant under horizontal translations, i.e.,

$$\Psi(F_h) = \big[\Psi(F)_h\big],$$

then it is called an *H-operator*.

The mapping $F \to -F$, where $(-F)(x) = -F(x)$ defines a negation on $\mathrm{Fun}(\mathbb{E}^d)$. Writing $F^* = -F$, we have the following duality relations:

$$(F \oplus G)^* = F^* \ominus \check{G}, \quad (F \ominus G)^* = F^* \oplus \check{G},$$

where $\check{G}$ is the reflection of G with respect to the origin, that is, $\check{G}(x) = G(-x)$.

A general way to construct T-operators is by using (extensions of) Boolean functions. If b is an increasing Boolean function of n variables, then we extend b to a function mapping $\overline{\mathbb{R}}^n$ into $\overline{\mathbb{R}}$ as follows: products are replaced by infima, sums by suprema, 0 by $-\infty$ and 1 by $+\infty$. For example, if $b(u_1, u_2, u_3) = u_1 + u_2u_3$, then $b(t_1, t_2, t_3) = t_1 \vee (t_2 \wedge t_3)$.

Let $A = \{a_1, \dots, a_n\}$ be a finite structuring element and b an increasing Boolean function of n variables, define the increasing T-operator Ψ_b on $\mathrm{Fun}(\mathbb{E}^d)$ by

$$\Psi_b(F)(x) = b\big(F(x+a_1), \dots, F(x+a_n)\big).$$

In fact, gray-scale operators obtained using Boolean functions all belong to the class of so-called *flat operators*.

Evidently, if $\mathcal{T}$ is a complete lattice, then $\mathrm{Fun}(E, \mathcal{T})$ is also a complete lattice with partial ordering

$$F \leqslant F' \text{ iff } F(x) \leqslant F'(x), \text{ for } x \in E.$$

Heijmans and Ronse [9] (see also [5, Sect. 5.1]) have given a complete description of adjunctions on $\mathrm{Fun}(E, \mathcal{T})$, where E is an arbitrary set and $\mathcal{T}$ a complete lattice.

PROPOSITION 5–5. *The pair $(\mathcal{E}, \Delta)$ is an adjunction on* $\mathrm{Fun}(E, \mathcal{T})$ *if and only if for every $x, y \in E$ there exists an adjunction $(e_{y,x}, d_{x,y})$ on $\mathcal{T}$ such that*

$$\Delta(F)(y) = \bigvee_{x \in E} d_{x,y}\big(F(x)\big)$$
$$\mathcal{E}(F)(x) = \bigwedge_{y \in E} e_{y,x}\big(F(y)\big).$$

We take $E = \mathbb{E}^d$ and focus on adjunctions in which both the dilation and the erosion are H-operators; such adjunctions are called *H-adjunctions*. The next proposition follows easily from the previous one.

PROPOSITION 5–6. *The pair $(\mathcal{E}, \Delta)$ is an H-adjunction on* $\mathrm{Fun}(\mathbb{E}^d, \mathcal{T})$ *if and only if for every $h \in \mathbb{E}^d$ there exists an adjunction (e_h, d_h) on $\mathcal{T}$ such that*

$$\Delta(F)(y) = \bigvee_{h \in \mathbb{E}^d} d_h\big(F(y-h)\big)$$
$$\mathcal{E}(F)(x) = \bigwedge_{h \in \mathbb{E}^d} e_h\big(F(x+h)\big).$$

If $\mathcal{T} = \overline{\mathbb{R}}$, we can obtain the adjunction $(\mathcal{E}_G, \Delta_G)$ using the additive structuring function G if we take $d_h(t) = t + G(h)$ and $e_h(t) = t - G(h)$. In [5, Sect. 11–5] we discuss some other H-adjunctions. Below we apply Proposition 5–6 to the case where the gray-value set $\mathcal{T}$ is finite.

If $\mathcal{T} = \{0, 1, \dots, N\}$, the adjunction given by Eqs. 5–2 and 5–3 becomes meaningless, since $\mathcal{T}$ is not closed under addition and subtraction. If one tries to overcome this problem by truncating values below 0 and above N, one does not get adjunctions: see [4] or [5, Sect. 11–9]. It turns out that we can use the characterization of H-adjunctions given in Proposition 5–6. This characterization utilizes adjunctions on $\mathcal{T}$, in the present case $\{0, 1, \dots, N\}$.

Define, for $v \in \mathbb{Z}$, the operation $t \mapsto t \dot{+} v$ on $\{0, 1, \dots, N\}$ given by

$$\begin{cases} 0 \dot{+} v = 0, & \\ t \dot{+} v = 0, & \text{if } t > 0 \text{ and } t + v \leqslant 0, \\ t \dot{+} v = t + v, & \text{if } t > 0 \text{ and } 0 \leqslant t + v \leqslant N, \\ t \dot{+} v = N, & \text{if } t > 0 \text{ and } t + v > N, \end{cases}$$

and the operation $t \mapsto t \dot{-} v$ by

$$\begin{cases} t \dot{-} v = 0, & \text{if } t < N \text{ and } t - v \leqslant 0, \\ t \dot{-} v = t - v, & \text{if } t < N \text{ and } 0 \leqslant t - v \leqslant N, \\ t \dot{-} v = N, & \text{if } t < N \text{ and } t - v > N, \\ N \dot{-} v = N. & \end{cases}$$

Let, for example, $N = 10$. Then $(6 \dot{+} 5) \dot{-} 4 = 10$ and $6 \dot{+} (5 \dot{-} 4) = 7$. The operation $\dot{+}$ is neither commutative ($0 \dot{+} 1 \neq 1 \dot{+} 0$) nor associative ($(3 \dot{+} 0) \dot{+} 5 = 3 \dot{+} 5 = 8 \neq 3 = 3 \dot{+} 0 = 3 \dot{+} (0 \dot{+} 5)$).

A simple computation shows that the pair $e(t) = t \dot{-} v$, $d(t) = t \dot{+} v$ defines an adjunction on $\{0, 1, \dots, N\}$ for every $v \in \mathbb{Z}$. For an illustration, see Fig. 5–2.

In combination with Proposition 5–6, this yields an interesting class of H-adjunctions with dilation and erosion, respectively, given by

$$(F \dot{\oplus} G)(x) = \bigvee_{h \in \operatorname{dom} G} \big(F(x-h) \dot{+} G(h)\big),$$
$$(F \dot{\ominus} G)(x) = \bigwedge_{h \in \operatorname{dom} G} \big(F(x+h) \dot{-} G(h)\big).$$

Here G is a function with domain $\operatorname{dom}(G)$ and values in $\mathbb{Z}$. In fact, one takes $d_h(t) = t \dot{+} G(h)$, $e_h(t) = t \dot{-} G(h)$, for $h \in \operatorname{dom}(G)$ and $d_h \equiv 0$, $e_h \equiv N$ for $h \notin \operatorname{dom}(G)$.

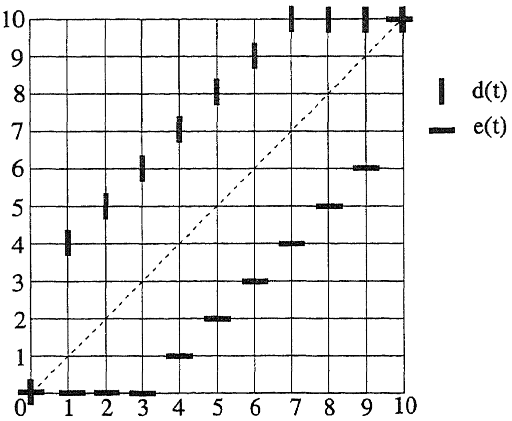


Figure 5–2. The pair $e(t) = t \dot{-} 3,\ d(t) = t \dot{+} 3$ forms an adjunction on $\mathcal{T} = \{0, 1, \ldots, 10\}$.

It is easy to verify that

$$
\begin{aligned}
(F \dot{+} v) \dot{\oplus} G &= (F \dot{\oplus} G) \dot{+} v, \\
(F \dot{-} v) \dot{\ominus} G &= (F \dot{\ominus} G) \dot{-} v,
\end{aligned}
$$

if $v \geqslant 0$. More results can be found in [5, Sect. 11–9].

5.3 Openings and Closings

This section contains a brief description of some basic properties of openings and closings, and introduces adjunctional filters.

5.3.1 Basic Facts

DEFINITION 5–3. An *opening* α is an operator on a complete lattice $\mathcal{L}$ which is increasing, idempotent, and anti-extensive ($\alpha(X) \leqslant X$ for every $X \in \mathcal{L}$). Dually, a *closing* β is an operator which is increasing, idempotent and extensive ($\beta(X) \geqslant X$ for every X).

The results presented in this section are mostly concerned with openings; analogous statements for closings follow from the duality principle [5].

If ψ is an operator on the complete lattice $\mathcal{L}$, then the *invariance domain* of ψ is

$$\mathrm{Inv}(\psi) = \{X \in \mathcal{L} \mid \psi(X) = X\}.$$

Elements of $\mathrm{Inv}(\psi)$ are sometimes called *fixpoints* or *roots*. From (c)–(d) in Proposition 5–1 the following result is clear:

PROPOSITION 5–7. *If* (ε, δ) *is an adjunction between* $\mathcal{L}$ *and* $\mathcal{M}$, *then* $\delta\varepsilon$ *is an opening on* $\mathcal{L}$ *and* $\varepsilon\delta$ *is a closing on* $\mathcal{M}$.

The opening resulting from Minkowski subtraction followed by Minkowski addition, i.e.,

$$X \circ A = (X \ominus A) \oplus A = \bigcup \{A_h \mid h \in \mathbb{E}^d \text{ and } A_h \subseteq X\}$$

is called a *structural opening*. The invariance domain of the opening $\delta\varepsilon$ is $\mathrm{Ran}(\delta)$. The previous result can be extended as follows.

PROPOSITION 5–8. *Let* α *be an opening on the complete lattice* $\mathcal{M}$ *and let* (ε, δ) *be an adjunction between* $\mathcal{L}$ *and* $\mathcal{M}$, *then* $\delta\alpha\varepsilon$ *is an opening on* $\mathcal{L}$ *with invariance domain* $\{\delta(Y) \mid Y \in \mathrm{Inv}(\alpha)\}$.

PROOF. It is evident that $\alpha' = \delta\alpha\varepsilon$ is increasing. Furthermore, $\delta\alpha\varepsilon \leqslant \delta\mathrm{id}\varepsilon = \delta\varepsilon \leqslant \mathrm{id}$, hence α' is anti-extensive. It remains to prove that $\alpha'^2 \geqslant \alpha'$ (note that the reverse inequality is trivial by the anti-extensivity of α'). Now

$$\alpha'^2 = \delta\alpha\varepsilon\delta\alpha\varepsilon \geqslant \delta\alpha^2\varepsilon = \delta\alpha\varepsilon = \alpha',$$

where we have used that $\varepsilon\delta \geqslant \mathrm{id}$.

Every fixpoint of α' is of the form $\delta(Y)$, where $Y \in \mathrm{Inv}(\alpha)$. To prove the converse, take $Y \in \mathrm{Inv}(\alpha)$ and consider $\alpha'\delta(Y)$. Since α' is anti-extensive, we have $\alpha'\delta(Y) \leqslant \delta(Y)$. On the other hand, since $\varepsilon\delta \geqslant \mathrm{id}$,

$$\alpha'\delta(Y) = \delta\alpha\varepsilon\delta(Y) \geqslant \delta\alpha(Y) = \delta(Y),$$

where we used that $\alpha(Y) = Y$. This concludes the proof. ■

PROPOSITION 5–9. *Let* α_1, α_2 *be openings on the complete lattice* $\mathcal{L}$. *The following assertions are equivalent*:

(i) $\alpha_1 \leqslant \alpha_2$;

(ii) $\alpha_1\alpha_2 = \alpha_2\alpha_1 = \alpha_1$;

(iii) $\mathrm{Inv}(\alpha_1) \subseteq \mathrm{Inv}(\alpha_2)$.

In particular, $\alpha_1 = \alpha_2$ *if and only if* $\mathrm{Inv}(\alpha_1) = \mathrm{Inv}(\alpha_2)$.

PROOF. Let α_1, α_2 be openings.

(i) $\Rightarrow$ (ii): If $\alpha_1 \leqslant \alpha_2$, then $\alpha_1\alpha_2 \geqslant \alpha_1\alpha_1 = \alpha_1$. Since the reverse inequality is trivially satisfied, one gets $\alpha_1\alpha_2 = \alpha_1$. The identity $\alpha_2\alpha_1 = \alpha_1$ is proved in a similar way.

(ii) $\Rightarrow$ (iii): Let $X \in \text{Inv}(\alpha_1)$, that is, $\alpha_1(X) = X$. Then $\alpha_2(X) = \alpha_2\alpha_1(X) = \alpha_1(X) = X$, and therefore $X \in \text{Inv}(\alpha_2)$.

(iii) $\Rightarrow$ (i): As $\alpha_1(X) \in \text{Inv}(\alpha_1) \subseteq \text{Inv}(\alpha_2)$, one gets $\alpha_1(X) = \alpha_2\alpha_1(X) \leqslant \alpha_2(X)$. ∎

If $\alpha_1(X) = X \circ A$ and $\alpha_2(X) = X \circ B$, then these equivalent conditions hold if A is B-open, i.e., $A \circ B = A$. We recall the following result.

PROPOSITION 5–10. *If α_i, $i \in I$, are openings, then $\bigvee_{i \in I} \alpha_i$ is an opening, too.*

PROOF. Let α_i, $i \in I$, be openings, and put $\alpha = \bigvee_{i \in I} \alpha_i$. It is evident that α is increasing and anti-extensive. We show that it is idempotent. By the anti-extensivity, it follows immediately that $\alpha^2 \leqslant \alpha$. The converse inequality also holds, since

$$\alpha^2 = \bigvee_{i \in I} \alpha_i \alpha \geqslant \bigvee_{i \in I} \alpha_i \alpha_i = \bigvee_{i \in I} \alpha_i = \alpha.$$

This proves the result. ∎

However, it is easy to construct examples which show that neither the infimum nor the composition of two openings is an opening in general.

5.3.2 ANNULAR OPENING

An opening on $\mathcal{P}(\mathbb{E}^d)$ which is given by a simple expression, but which is not of structural type, is the *annular opening* given by $X \mapsto X \cap X \oplus \check{A}$, where $A \subseteq \mathbb{E}^d$ is a structuring element which is symmetric, i.e., $A = \check{A}$. Here $\check{A}$ is the reflected structuring element: $\check{A} = \{-a \mid a \in A\}$. The proof that this operator defines an opening indeed, is not very difficult; see e.g., [23] or [5, Prop. 4–27]. An illustration of the effect of the annular opening can be found in Fig. 5–3.

In [5] we discuss various manifestations of the annular opening; see also [19]. In the next section we discuss a generalization of the annular opening, called an *annular filter*.

5.3.3 ADJUNCTIONAL FILTERS

If (ε, δ) is an adjunction on the complete lattice $\mathcal{L}$, then $\varepsilon\delta$ is a closing and $\delta\varepsilon$ an opening; see Proposition 5–7. More generally, if $k \geqslant 1$ then $\varepsilon^k\delta^k$ is a closing and $\delta^k\varepsilon^k$ is an opening. This follows easily from the observation that $(\varepsilon^k, \delta^k)$ is an adjunction, too. The composition $\varepsilon\delta^2\varepsilon$ is a filter, for

$$\varepsilon\delta^2\varepsilon\varepsilon\delta^2\varepsilon = \varepsilon(\delta^2\varepsilon^2\delta^2)\varepsilon = \varepsilon\delta^2\varepsilon,$$

where we have used that $\delta^2\varepsilon^2\delta^2 = \delta^2$ since $(\varepsilon^2, \delta^2)$ is an adjunction [5].

In [3] we have established the following general result:

PROPOSITION 5–11. *Let* (ε, δ) *be an adjunction on* $\mathcal{L}$ *and let* ψ *be of the form*

$$\psi = \varepsilon^{e_n}\delta^{d_n} \cdots \varepsilon^{e_2}\delta^{d_2}\varepsilon^{e_1}\delta^{d_1},$$

where $e_i, d_i \geqslant 0$ *are integers and* $\sum_{i=1}^{n} e_i = \sum_{i=1}^{n} d_i$. *Then* ψ *is a filter.*

For example, $\varepsilon^3\delta^2\varepsilon\delta^2$ is a filter. The filters given by Proposition 5–11 are called *adjunctional filters*; refer to [3] for additional results.

5.4 ANNULAR FILTERS

In what follows, δ_A and ε_A denote *dilation* and *erosion* on $\mathcal{P}(\mathbb{E}^d)$ by the structuring element A, respectively; that is

$$\delta_A(X) = X \oplus A \text{ and } \varepsilon_A(X) = X \ominus A.$$

In Section 5.3.2 we introduced annular openings. Such openings have the form $\alpha = \mathrm{id} \wedge \delta_A$. Dually, annular closings are given by $\beta = \mathrm{id} \vee \varepsilon_B$. Here A, B are symmetric structuring elements which do not contain the origin. Annular filters, which were introduced by the author in [8] and investigated in great detail in [10, 20], are a combination of both operators.

5.4.1 ANNULAR FILTERS FOR BINARY IMAGES

Let A, B be structuring elements in $\mathbb{E}^d$ which are symmetric and which do not contain the origin. Consider the operator

$$\omega = (\mathrm{id} \wedge \delta_A) \vee \varepsilon_B. \tag{5–4}$$

Throughout this subsection we assume that

$$A \cap B \neq \emptyset. \tag{5–5}$$

As a result we have

$$\varepsilon_B \leqslant \delta_A,$$

and thus we find that ω can alternatively be written as $\omega = (\mathrm{id} \vee \varepsilon_B) \wedge \delta_A$. In the sequel we write

$$\omega = \delta_A \wedge \mathrm{id} \vee \varepsilon_B,$$

showing that the expression is independent of the order in which the infimum and supremum are computed. In [10] the following result has been established:

PROPOSITION 5–12. *Let A, B be as before; the operator $\omega = \delta_A \wedge \mathrm{id} \vee \varepsilon_B$ is a filter if and only if*

$$A \cap B \cap (A \oplus B) \neq \emptyset. \tag{5–6}$$

It is self-dual if and only if $A = B$.

We mention four examples for $\mathbb{Z}^2$ where Eqs. 5–5 and 5–6 hold. Observe that in all examples one may interchange A and B.

$$A = B = \boxed{\begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \underline{\cdot} & \bullet \\ \bullet & \bullet & \bullet \end{array}}$$

$$A = \boxed{\begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \underline{\cdot} & \bullet \\ \bullet & \bullet & \bullet \end{array}} \quad \text{and} \quad B = \boxed{\begin{array}{ccc} & \bullet & \\ \bullet & \underline{\cdot} & \bullet \\ & \bullet & \end{array}}$$

$$A = \boxed{\begin{array}{ccccc} & & \bullet & & \\ \bullet & \bullet & \underline{\cdot} & \bullet & \bullet \\ & & \bullet & & \end{array}} \quad \text{and} \quad B = \boxed{\begin{array}{ccc} & \bullet & \\ \bullet & \underline{\cdot} & \bullet \\ & \bullet & \end{array}}$$

$$A = \boxed{\begin{array}{ccc} \bullet & \cdot & \bullet \\ \bullet & \underline{\cdot} & \bullet \\ \bullet & \cdot & \bullet \end{array}} \quad \text{and} \quad B = \boxed{\begin{array}{ccc} \bullet & \bullet & \bullet \\ \cdot & \underline{\cdot} & \cdot \\ \bullet & \bullet & \bullet \end{array}}$$

Considering A and B as sets which determine foreground and background adjacency, respectively, one obtains an interesting geometric interpretation of the annular filter. Thus, saying that two points $x, y \in \mathbb{Z}^2$ are foreground adjacent if $x - y \in A$ and background adjacent if $x - y \in B$, the operator ω given by Eq. 5–4 removes points in X which have no foreground neighbors in X, and it adds points from X^c which have no background neighbors in X^c.

The first example, with $A = B$, yields a self-dual annular filter. For this structuring element we apply the annular opening and the annular filter to our test image; see

Figure 5–3. Annular opening (left) and annular filter (right).

Fig. 5–3. Observe that the annular opening removes only isolated noise pixels from the foreground (black pixels), whereas the annular filter removes all isolated noise pixels, that is, from the foreground as well as from the background.

5.4.2 ANNULAR FILTERS FOR GRAY-SCALE IMAGES

Consider the complete lattice of gray-scale images modeled by $\operatorname{Fun}(\mathbb{E}^d)$. For a structuring function A with domain $\operatorname{dom}(A) \subseteq \mathbb{E}^d$ and range in $\overline{\mathbb{R}}$, the gray-scale dilation Δ_A and gray-scale erosion $\mathcal{E}_A$ are given by (see Eqs. 5–2, 5–3):

$$\begin{aligned}\Delta_A(F)(x) &= \bigvee_{h \in \operatorname{dom}(A)} \big(F(x-h) + A(h)\big)\\ \mathcal{E}_A(F)(x) &= \bigwedge_{h \in \operatorname{dom}(A)} \big(F(x+h) - A(h)\big).\end{aligned}$$

In [19] it was shown that the operator $\mathrm{id} \wedge \Delta_A$ is an opening, the annular opening for gray-scale images, if

(i) $\operatorname{dom}(A)$ is symmetric
(ii) $A(x) + A(-x) \geqslant 0$ for $x \in \operatorname{dom}(A)$.

In order that $\mathrm{id} \wedge \Delta_A$ is not the identity mapping, one has to assume that

$$0 \notin \operatorname{dom}(A).$$

Given two symmetric structuring functions A and B, we define the structuring function $A \sqcap B$ as follows:

$$\begin{aligned}\operatorname{dom}(A \sqcap B) = \big\{x \in \operatorname{dom}(A) \cap \operatorname{dom}(B) \mid &\min(A(x), B(x))\\ &+ \min(A(-x), B(-x)) \geqslant 0\big\},\end{aligned}$$

and

$$(A \sqcap B)(x) = \min(A(x), B(x)) \quad \text{if } x \in \operatorname{dom}(A \sqcap B).$$

The next result can be found in [20].

PROPOSITION 5–13. *Let A, B be two structuring functions which satisfy* (i)–(ii) *above, as well as*

$$\big[A \oplus B \oplus (A \sqcap B)\big](0) \geqslant 0;$$

then $\Omega = \Delta_A \wedge \mathrm{id} \vee \mathcal{E}_B$ is a morphological filter.

In [20] some examples are given.

5.5 AS-Filters

Perhaps the most interesting class of morphological filters is obtained by composing openings and closings.

PROPOSITION 5–14. *Let $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_N$ be openings and let $\beta_1 \leqslant \beta_2 \leqslant \cdots \leqslant \beta_N$ be closings. Every composition of operators of these two sequences is a filter.*

The result in this form was first stated by Schonfeld and Goutsias [21]. We introduce the following notation: if $\psi_1, \psi_2, \ldots$ are operators, then

$$(\psi)_n = \psi_n \psi_{n-1} \cdots \psi_1, \quad n \geqslant 1.$$

More generally, if $\phi_1, \phi_2, \ldots$ is another sequence of operators, then

$$\begin{aligned}(\psi\phi)_n &= \psi_n\phi_n\psi_{n-1}\phi_{n-1}\cdots\psi_1\phi_1,\\ (\psi\phi\psi)_n &= \psi_n\phi_n\psi_n\psi_{n-1}\phi_{n-1}\psi_{n-1}\cdots\psi_1\phi_1\psi_1.\end{aligned}$$

In what follows we fix a sequence of openings

$$\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_N,$$

and a sequence of closings

$$\beta_1 \leqslant \beta_2 \leqslant \cdots \leqslant \beta_N.$$

From Proposition 5–14 we get that the compositions $(\alpha\beta)_n, (\beta\alpha)_n, (\alpha\beta\alpha)_n, (\beta\alpha\beta)_n$ are filters. These filters are called *alternating sequential filters* or *AS-filters* [5, 23, 24]. Furthermore, the sequences $(\alpha\beta)_n, (\beta\alpha)_n, (\alpha\beta\alpha)_n, (\beta\alpha\beta)_n$ are absorbing in the following sense: a sequence of operators $\psi_1, \psi_2, \ldots$ is said to be *absorbing* if

$$\psi_n\psi_m = \psi_n, \quad n \geqslant m.$$

In fact, one can easily show that for any family of increasing operators ψ_n for which $(\psi)_n$ is a filter, one automatically gets that the sequence $(\psi)_n$ is absorbing.

The following inequalities are easily established:

$$(\alpha\beta\alpha)_n \leqslant \left\{ \begin{matrix} (\alpha\beta)_n \\ (\beta\alpha)_n \end{matrix} \right\} \leqslant (\beta\alpha\beta)_n.$$

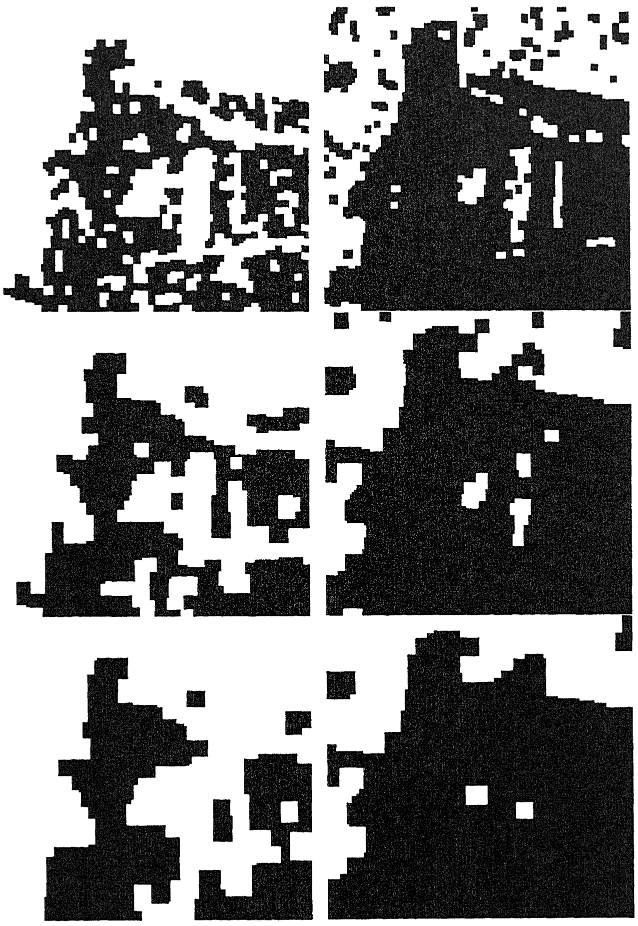


Figure 5–4. $(\beta\alpha)_n(X)$ (left) and $(\alpha\beta)_n(X)$ (right). $n = 1, 2, 3$ in the first, second, and third row, respectively.

In practice (discrete case) one usually constructs AS-filters starting from a structuring element A, and defining $\alpha_n(X) = X \circ nA$ and $\beta_n(X) = X \bullet nA$, where $nA = A \oplus \cdots \oplus A$ (n terms). In Fig. 5–4 we apply $(\alpha\beta)_n$ and $(\beta\alpha)_n$ to our test image of Fig. 5–1; we choose for A the 3×3 square.

5.6 OVERFILTERS AND INF-OVERFILTERS

This section discusses overfilters and inf-overfilters. From the duality principle [5] we know that analogous results hold for the dual concepts, underfilters and sup-underfilters.

5.6.1 DEFINITIONS AND BASIC PROPERTIES

DEFINITION 5–4. An increasing operator ψ is called

(a) an *overfilter* if $\psi^2 \geqslant \psi$;

(b) an *inf-overfilter* if $\psi(\mathrm{id} \wedge \psi) = \psi$;

(c) an *underfilter* if $\psi^2 \leqslant \psi$;

(d) a *sup-underfilter* if $\psi(\mathrm{id} \vee \psi) = \psi$.

We make some simple observations. It is clear that overfilters and underfilters are dual in the sense of the duality principle. The same is true for inf-overfilters and sup-underfilters. In this section we mostly restrict ourselves to (inf-) overfilters. Since $\psi(\mathrm{id} \wedge \psi) \leqslant \psi^2$, every inf-overfilter is also an overfilter. To prove that an operator ψ is an inf-overfilter, we only have to show that $\psi(\mathrm{id} \wedge \psi) \geqslant \psi$, as the reverse inequality is trivial for increasing operators.

PROPOSITION 5–15. *The family of (inf-) overfilters is closed under suprema.*

PROOF. Assume that ψ_i, $i \in I$, are inf-overfilters, and put $\psi = \bigvee_{i \in I} \psi_i$. Thus $\psi(\mathrm{id} \wedge \psi) \geqslant \psi_i(\mathrm{id} \wedge \psi_i) = \psi_i$, which yields immediately that $\psi(\mathrm{id} \wedge \psi) \geqslant \bigvee_{i \in I} \psi_i = \psi$. Therefore, ψ is an inf-overfilter. ■

In [5] one can find detailed results concerning the lattice structure of the class of filters and (inf-) overfilters; see also [17, 18, 23].

PROPOSITION 5–16. *If ψ is an overfilter (inf-overfilter, underfilter, sup-underfilter), then ψ^n is such as well, for every $n \geqslant 1$.*

PROOF. For overfilters, the result is obvious. Now assume that ψ is an inf-overfilter. Then

$$\psi^n(\mathrm{id} \wedge \psi^n) \geqslant \psi^n(\mathrm{id} \wedge \psi) = \psi^{n-1}\psi(\mathrm{id} \wedge \psi) \geqslant \psi^{n-1}\psi = \psi^n.$$

Here we have used that $\psi^n \geqslant \psi$, if ψ is an (inf-) overfilter. ■

The next result shows that inf-overfilters provide a useful tool for the construction of openings.

PROPOSITION 5–17. *If ψ is an inf-overfilter, then* $\mathrm{id} \wedge \psi$ *is an opening.*

PROOF. Let ψ be an inf-overfilter and $\alpha = \mathrm{id} \wedge \psi$. It is evident that α is increasing and anti-extensive. It is also idempotent, for

$$\alpha^2 = (\mathrm{id} \wedge \psi)(\mathrm{id} \wedge \psi) = \mathrm{id} \wedge \psi \wedge \psi(\mathrm{id} \wedge \psi) = \mathrm{id} \wedge \psi = \alpha.$$

This proves the result. ■

The dual statement says that $\mathrm{id} \vee \psi$ is a closing when ψ is a sup-underfilter. Combining the latter two propositions gives that $\mathrm{id} \wedge \psi^n$ is an opening for every $n \geqslant 1$ if ψ is an inf-overfilter. Since $\psi \leqslant \psi^2 \leqslant \psi^3 \leqslant \cdots$, we find that

$$\mathrm{id} \wedge \psi \leqslant \mathrm{id} \wedge \psi^2 \leqslant \mathrm{id} \wedge \psi^3 \leqslant \cdots .$$

The next result is obvious.

PROPOSITION 5–18. *Suppose that $\mathcal{L}$ possesses a negation. If ψ is an (inf-) overfilter, then ψ^* is a (sup-) underfilter.*

In the following proposition we sum up various ways to construct overfilters and inf-overfilters.

PROPOSITION 5–19.

(a) *Let (ε, δ) be an adjunction between $\mathcal{L}$ and $\mathcal{M}$ and let $\psi : \mathcal{M} \to \mathcal{L}$ be an increasing operator such that $\psi \geqslant \delta$, then $\psi\varepsilon$ is an inf-overfilter.*

(b) *Let (ε, δ) and (ε', δ') be adjunctions between $\mathcal{L}$ and $\mathcal{M}$ such that $\varepsilon' \leqslant \varepsilon$ and $\delta' \geqslant \delta$. If ψ is an (inf-) overfilter on $\mathcal{M}$, then $\delta'\psi\varepsilon$ is an (inf-) overfilter on $\mathcal{L}$.*

(c) *Let α be an opening and $\alpha \leqslant \psi$, then $\alpha\psi$ and $\psi\alpha\psi$ are overfilters, whereas $\psi\alpha$ and $\alpha\psi\alpha$ are inf-overfilters.*

(d) *Let ψ be an (inf-) overfilter and $\phi \geqslant \mathrm{id}$ then $\phi\psi$ is an (inf-) overfilter.*

(e) *If ψ is an overfilter and $\phi \geqslant \psi$, then $\phi\psi$ and $\psi\phi$ are overfilters.*

(f) *If ψ is an inf-overfilter and $\phi \geqslant \mathrm{id} \wedge \psi$, then $\phi\psi$ is an inf-overfilter.*

(g) *If ψ is an overfilter and β a closing, then $\beta\psi$, $\psi\beta\psi$, $\psi\beta$, $\beta\psi\beta$ are overfilters.*

(h) *If ψ is an inf-overfilter and β a closing, then $\beta\psi$, $\psi\beta\psi$ are inf-overfilters.*

PROOF. For a full proof we refer to [3]. Here we only prove (a) and (f).

(a) Let $\varepsilon, \delta, \psi$ be as stated. Then

$$\psi\varepsilon(\mathrm{id} \wedge \psi\varepsilon) \geqslant \psi\varepsilon(\mathrm{id} \wedge \delta\varepsilon) = \psi\varepsilon\delta\varepsilon = \psi\varepsilon,$$

where we have used that $\delta\varepsilon \leqslant \mathrm{id}$ and that $\varepsilon\delta\varepsilon = \varepsilon$.

(f) In this case,

$$\begin{aligned}\phi\psi(\mathrm{id} \wedge \phi\psi) &\geqslant \phi\psi(\mathrm{id} \wedge (\mathrm{id} \wedge \psi)\psi) \\ &= \phi\psi(\mathrm{id} \wedge \psi \wedge \psi^2) \\ &= \phi\psi(\mathrm{id} \wedge \psi) = \phi\psi,\end{aligned}$$

where we have used that $\psi^2 \geqslant \psi$, since ψ is an overfilter. From (a) [and also from (b)] we get that $\delta'\varepsilon$ is an inf-overfilter, if $\delta' \geqslant \delta$. ■

5.6.2 RANK-MAX OPENINGS

Ronse and Heijmans [19] (see also [5, Sect. 6–6]) have shown that every translation-invariant inf-overfilter on $\mathcal{P}(\mathbb{E}^d)$ is of the form

$$\psi(X) = \bigcap_{k \in K} \bigcup_{j \in J} (X \ominus B_j) \oplus A_{kj}, \tag{5–7}$$

where A_{kj}, B_j are structuring elements such that $B_j \subseteq A_{kj}$ for $k \in K$ and $j \in J$. As an illustration of this result we discuss the rank-max opening, first discussed by Ronse [15]; see also [5, 19].

Let A be a finite structuring element containing n points, and let $\mathcal{B}_k$ contain all subsets of A which contain k points (where $k \leqslant n$). It is evident that

$$\bigcup_{B \in \mathcal{B}_k} X \ominus B = \rho_{A,k}(X),$$

where $\rho_{A,k}$ is the kth rank operator which is defined by: $h \in \rho_{A,k}(X)$ if and only if $X \cap A_h$ contains at least k points [5]. The composition $\delta_A \varepsilon_B$ is an inf-overfilter, for $B \in \mathcal{B}_k$, hence

$$\bigvee_{B \in \mathcal{B}_k} \delta_A \varepsilon_B = \delta_A \Big(\bigvee_{B \in \mathcal{B}_k} \varepsilon_B \Big) = \delta_A \rho_{A,k}$$

is an inf-overfilter, too; here we have used Proposition 5–15. Note that the previous expression is a special case of Eq. 5–7.

The opening $\alpha_{A,k} = \mathrm{id} \wedge \delta_A \rho_{A,k}$ is called a *rank-max opening*. For $k = n$ it coincides with the structural opening $X \mapsto X \circ A$, whereas for $k = 1$ it yields the identity operator.

The rank-max openings are "more flexible" than the structural opening: the latter one preserves translates of A which fit entirely inside X. The rank-max opening $\rho_{A,k}$ preserves those portions $X \cap A_h$ which contain at least k points. It is evident that

$$\alpha_{A,n} \leqslant \alpha_{A,n-1} \leqslant \cdots \leqslant \alpha_{A,1} = \mathrm{id}.$$

Let us denote the dual closing, the *rank-min closing* by $\beta_{A,k}$:

$$\beta_{A,k} = \mathrm{id} \vee \varepsilon_{\check{A}} \rho_{A,n+1-k}.$$

It follows easily that $(\alpha_{A,k})^* = \beta_{A,k}$. We have

$$\beta_{A,n} \geqslant \beta_{A,n-1} \geqslant \cdots \geqslant \beta_{A,1} = \mathrm{id}.$$

We can use the rank-max openings and rank-min closings to construct AS-filters. Define e.g.,

$$(\beta_A\alpha_A)_m = \beta_{A,m}\alpha_{A,m}\beta_{A,m-1}\alpha_{A,m-1}\cdots\beta_{A,2}\alpha_{A,2},$$

where $m \leqslant n$. The AS-filter $(\alpha_A\beta_A)_m$ is defined analogously.

In Fig. 5–5 we give an illustration of these two filters for the case that A is the 3×3 square (hence $n = 9$).

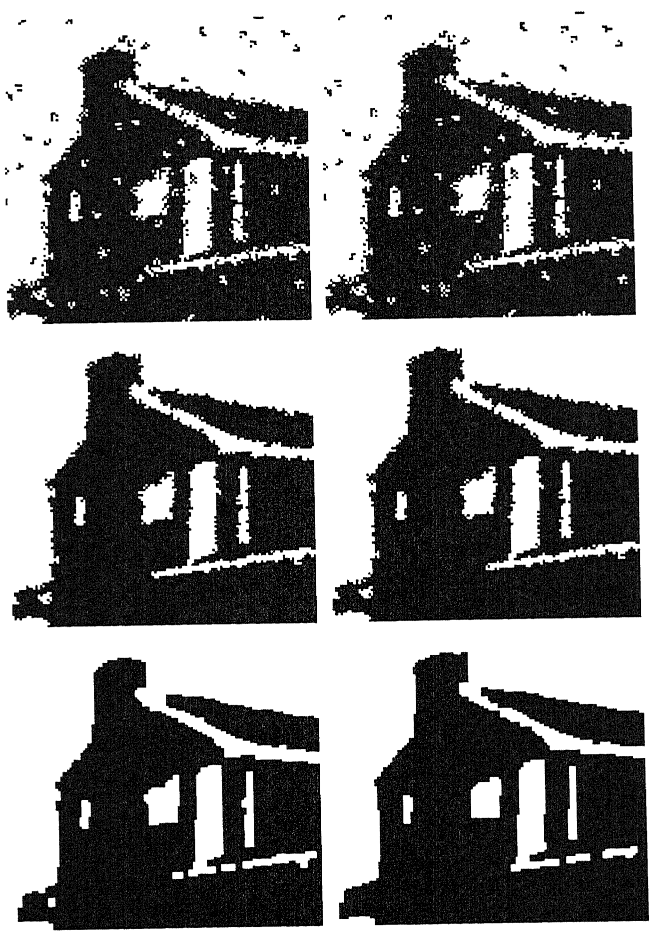


Figure 5–5. $(\beta_A\alpha_A)_n(X)$ (left) and $(\alpha_A\beta_A)_n(X)$ (right); $n = 5$ in the first row, $n = 7$ in the second row, and $n = 9$ in the third row.

5.7 Generalized AS-Filters

In this section we extend the class of AS-filters obtained by composing openings and closings, as discussed in Section 5.5. The basic idea is to use overfilters instead of openings and underfilters instead of closings. The exposition in this section is extracted from [3]; see also [6, 7].

PROPOSITION 5–20. *Assume that ϕ is an overfilter, that ψ is an underfilter, and that $\phi \leqslant \psi$. The compositions $\phi\psi$, $\psi\phi$, $\phi\psi\phi$, $\psi\phi\psi$ are filters, and*

$$\phi \leqslant \phi\psi\phi \leqslant \left\{ \begin{matrix} \psi\phi \\ \phi\psi \end{matrix} \right\} \leqslant \psi\phi\psi \leqslant \psi.$$

PROOF. That, e.g., $\psi\phi$ is a filter follows from $\psi\phi\psi\phi \leqslant \psi^3\phi \leqslant \psi\phi$ and $\psi\phi\psi\phi \geqslant \psi\phi^3 \geqslant \psi\phi$. In the same fashion, one shows that the other compositions are filters, too. Furthermore,

$$\phi \leqslant \phi^3 \leqslant \phi\psi\phi \leqslant \phi\psi^2 \leqslant \phi\psi \leqslant \phi^2\psi \leqslant \psi\phi\psi \leqslant \psi^3 \leqslant \psi.$$

The inequalities with $\psi\phi$ instead of $\phi\psi$ follow analogously. ■

We consider translation-invariant operators on $\mathcal{P}(\mathbb{E}^d)$. Let $A \subseteq A'$; then $\phi(X) = (X \ominus A) \oplus A'$ defines an inf-overfilter. Dually, if $B \subseteq B'$, then $\psi(X) = (X \oplus B) \ominus B'$ defines a sup-underfilter. Now $\phi \leqslant \psi$ iff

$$(X \ominus A) \oplus A' \subseteq (X \oplus B) \ominus B',$$

for every $X \subseteq \mathbb{E}^d$. It is easy to see that this condition holds iff $A' \oplus B' \subseteq A \oplus B$. Since the reverse inclusion is trivially satisfied, we arrive at the following set of conditions:

$$A \subseteq A', \;\; B \subseteq B', \;\; A \oplus B = A' \oplus B'. \tag{5–8}$$

PROPOSITION 5–21. *Suppose that (A_i, A_i'), $i \in I$, and (B_j, B_j'), $j \in J$, are pairs of structuring elements such that*

$$A_i \subseteq A_i', \;\; B_j \subseteq B_j', \;\; A_i \oplus B_j = A_i' \oplus B_j',$$

for every $i \in I$ and $j \in J$. Then

$$\phi(X) = \bigcup_{i \in I} (X \ominus A_i) \oplus A_i'$$

is an inf-overfilter,

$$\psi(X) = \bigcap_{j \in J} (X \oplus B_j) \ominus B'_j$$

is a sup-underfilter, and $\phi \leqslant \psi$.

We present two examples.

EXAMPLE 5–1. Let $A' = B'$ be the 3×3 square and

$$A = B = \boxed{\begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \underline{\cdot} & \bullet \\ \bullet & \bullet & \bullet \end{array}}$$

Let $\phi(X) = (X \ominus A) \oplus A'$ and $\psi(X) = (X \oplus B) \ominus B'$.

In the first row of Fig. 5–6 we depict $(\psi\phi)(X)$ and $(\phi\psi)(X)$. Compare these images with $(\beta\alpha)_1(X)$ and $(\alpha\beta)_1(X)$, respectively, in the first row of Fig. 5–4.

EXAMPLE 5–2. Let

$$A_1 = \boxed{\begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \underline{\cdot} & \cdot \\ \bullet & \bullet & \bullet \end{array}}, \quad A_2 = \boxed{\begin{array}{ccc} \bullet & \cdot & \bullet \\ \bullet & \underline{\cdot} & \bullet \\ \bullet & \bullet & \bullet \end{array}},$$

$$B_1 = \boxed{\begin{array}{ccc} \bullet & \bullet & \bullet \\ \cdot & \underline{\cdot} & \bullet \\ \bullet & \bullet & \bullet \end{array}}, \quad B_2 = \boxed{\begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \underline{\cdot} & \bullet \\ \bullet & \cdot & \bullet \end{array}},$$

and let $A' = B'$ be the 3×3 square. Define $\phi(X) = ((X \ominus A_1) \cup (X \ominus A_2)) \oplus A'$ and $\psi(X) = ((X \oplus B_1) \cap (X \oplus B_2)) \ominus B'$. The images $(\psi\phi)(X)$ and $(\phi\psi)(X)$ are depicted in the second row of Fig. 5–6. In [7] we present a variant of this latter example where we have rotation invariance.

Now we consider generalized AS-filters that use more than one overfilter and one underfilter.

PROPOSITION 5–22. *Assume that* $\phi_1, \phi_2, \ldots$ *are overfilters and that* $\psi_1, \psi_2, \ldots$ *are underfilters and that the following conditions are satisfied*:

$$\begin{aligned} &\phi_n \leqslant \psi_n, \\ &\phi_n \phi_{n-1} \geqslant \phi_n, \\ &\psi_n \psi_{n-1} \leqslant \psi_n. \end{aligned} \tag{5–9}$$

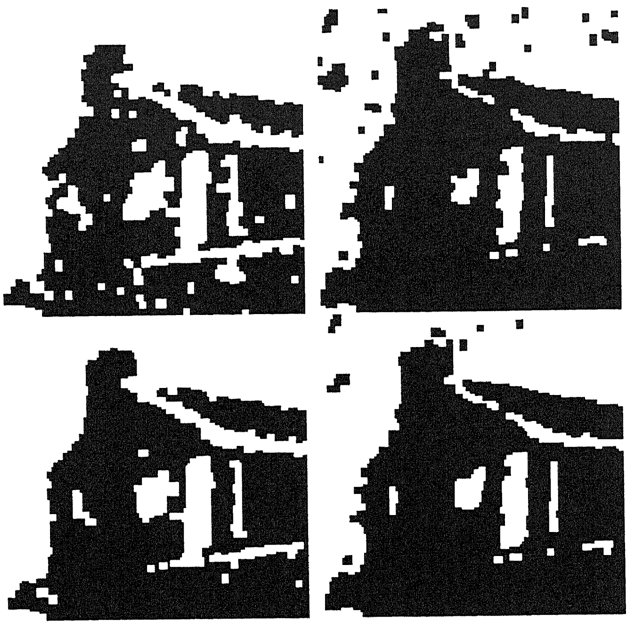


Figure 5–6. $(\psi\phi)(X)$ and $(\phi\psi)(X)$ of Example 5–1 (top row) and Example 5–2 (bottom row).

Then $(\phi\psi)_n, (\psi\phi)_n, (\phi\psi\phi)_n, (\psi\phi\psi)_n$ *are absorbing sequences of filters and*

$$(\phi\psi\phi)_n \leqslant \left\{ \begin{matrix} (\phi\psi)_n \\ (\psi\phi)_n \end{matrix} \right\} \leqslant (\psi\phi\psi)_n. \tag{5–10}$$

PROOF. To show that $(\phi\psi)_n$ is a filter, we must show that it is an underfilter and an overfilter at the same time. Note first that

$$(\psi)_n = \psi_n \cdots \psi_3\psi_2\psi_1 \leqslant \psi_n \cdots \psi_3\psi_2 \leqslant \psi_n \cdots \psi_3 \leqslant \cdots \leqslant \psi_n.$$

We find that

$$\begin{aligned}
(\phi\psi)_n(\phi\psi)_n &= \phi_n\big(\psi_n(\phi\psi)_n\phi_n\psi_n\big)(\phi\psi)_{n-1} \\
&\leqslant \phi_n\big(\psi_n(\psi\psi)_n\psi_n^2\big)(\phi\psi)_{n-1} \\
&\leqslant \phi_n\big(\psi_n(\psi)_n\psi_n\big)(\phi\psi)_{n-1} \\
&\leqslant \phi_n(\psi_n^3)(\phi\psi)_{n-1} \\
&\leqslant \phi_n\psi_n(\phi\psi)_{n-1} \\
&= (\phi\psi)_n.
\end{aligned}$$

This proves that the composition$(\phi\psi)_n$ is an underfilter. To show that it is an overfilter, we use that

$$(\phi)_n = \phi_n \phi_{n-1} \cdots \phi_1 \geqslant \phi_n.$$

Therefore,

$$\begin{aligned}(\phi\psi)_n(\phi\psi)_n &\geqslant (\phi\phi)_n(\phi\psi)_n \\ &\geqslant (\phi)_n(\phi\psi)_n \\ &\geqslant \phi_n(\phi\psi)_n \\ &= \phi_n^2\psi_n(\phi\psi)_{n-1} \\ &\geqslant \phi_n\psi_n(\phi\psi)_{n-1} \\ &= (\phi\psi)_n.\end{aligned}$$

Thus $(\phi\psi)_n$ is a filter.

To show that the other three compositions define filters, one can use similar arguments. Furthermore, e.g.,

$$(\phi\psi\phi)_n \leqslant (\phi\psi\psi)_n \leqslant (\phi\psi)_n,$$

since every ψ_n is an underfilter. ■

The conditions in Eq. 5–9 hold if we make the (stronger) assumption that

$$\cdots \leqslant \phi_3 \leqslant \phi_2 \leqslant \phi_1 \leqslant \psi_1 \leqslant \psi_2 \leqslant \psi_3 \leqslant \cdots. \tag{5–11}$$

We present some examples.

EXAMPLE 5–3. Let ϕ be an overfilter and ψ an underfilter such that $\phi \leqslant \psi$. Fix $N \geqslant 1$ and define, for $n = 1, 2, \ldots, N$:

$$\phi_n = \phi^{N+1-n}, \quad \psi_n = \psi^{N+1-n}.$$

Then Eq. 5–11 holds.

For the next example we need some preparation. Let ψ be an increasing translation-invariant operator on $\mathcal{P}(\mathbb{E}^d)$, and let (ε, δ) be a translation-invariant adjunction on $\mathcal{P}(\mathbb{E}^d)$. Thus δ is of the form $\delta(X) = X \oplus A$, for some structuring element $A \subseteq \mathbb{E}^d$. It follows that

$$\begin{aligned}\psi\delta(X) &= \psi\Big(\bigcup_{a\in A} X_a\Big) \supseteq \bigcup_{a\in A} \psi(X_a) \\ &= \bigcup_{a\in A} [\psi(X)]_a = \psi(X) \oplus A = (\delta\psi)(X).\end{aligned}$$

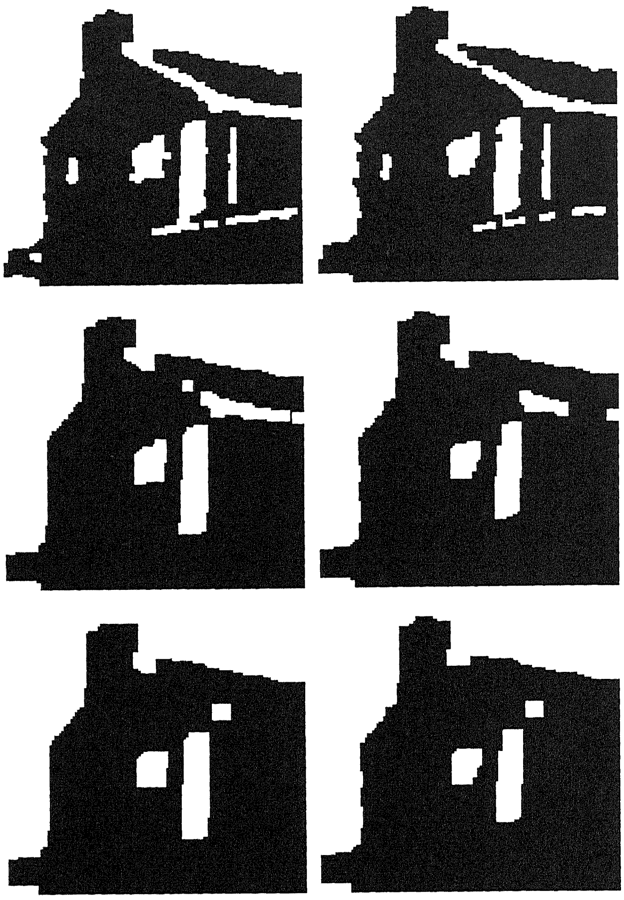


Figure 5–7. The images $(\psi\phi)_n(X)$ and $(\psi\phi)_n(X)$ of Example 5–5: $n = 1, 2, 3$ in the first, second, and third row, respectively.

Thus we find that $\psi\delta \geqslant \delta\psi$. Similarly, it follows that $\psi\varepsilon \leqslant \varepsilon\psi$. These relations yield that

$$\delta\psi\varepsilon \leqslant \psi \quad \text{and} \quad \varepsilon\psi\delta \geqslant \psi. \tag{5–12}$$

EXAMPLE 5–4. Consider the following translation-invariant operators on $\mathcal{P}(\mathbb{E}^d)$: an adjunction (ε, δ), an overfilter ϕ, and an underfilter ψ. Assume, moreover, that $\phi \leqslant \psi$. By Proposition 5–19(b) we know that $\phi_n = \delta^n\phi\varepsilon^n$ are overfilters, and that $\psi_n = \varepsilon^n\psi\delta^n$ are underfilters. Furthermore, by Eq. 5–12, $\phi_n = \delta\phi_{n-1}\varepsilon \leqslant \phi_{n-1}$, and dually, $\psi_n \geqslant \psi_{n-1}$. Therefore, the conditions in Eq. 5–11 hold.

EXAMPLE 5–5. Let α_n, β_n be openings and closings, respectively, and let ξ be an increasing operator such that

$$\cdots \leqslant \alpha_2 \leqslant \alpha_1 \leqslant \xi \leqslant \beta_1 \leqslant \beta_2 \leqslant \cdots . \tag{5–13}$$

Define $\phi_n = \alpha_n \xi$ and $\psi_n = \beta_n \xi$. From Proposition 5–19(c) we derive that ϕ_n are overfilters; dually, ψ_n are underfilters. It is obvious that Eq. 5–11 holds.

Suppose, for example, that ξ is the median operator on $\mathcal{P}(\mathbb{Z}^2)$ using the rhombus as structuring element (origin and four horizontal and vertical neighbors). Let α_n, β_n be the opening and closing, respectively, with the $(2n+1) \times (2n+1)$ square, and define $\phi_n = \alpha_n \xi$ and $\psi_n = \beta_n \xi$. It is easy to see that the conditions in Eq. 5–13 are satisfied. In Fig. 5–7 we depict the corresponding AS-filters $(\psi\phi)_n$ and $(\phi\psi)_n$ for $n = 1, 2, 3$. Comparing these images with those in Fig. 5–4, we see that the new AS-filters introduced here perform substantially better than the classical ones described in Section 5.5; see [6].

5.8 ITERATION

In this section we explain how to construct morphological filters by iteration of increasing operators which are not idempotent. Though we restrict attention to operators on $\mathcal{P}(E)$, most of the results can be extended to complete lattices; refer to [11] and [5, Chap. 13].

5.8.1 CONVERGENCE

Let $X_n \subseteq E$, $n \geqslant 1$, and $X \subseteq E$: we say that $X_n \to X$ (X_n converges to X) if $X_n(h) \to X(h)$ as $n \to \infty$, for every $h \in E$. Here $X(\cdot)$ is the characteristic function associated with the set X. It is easy to see that the following assertions are equivalent:

(i) $X_n \to X$,

(ii) $h \in X$ iff $h \in X_n$ for n large enough.

DEFINITION 5–5. An operator ψ on $\mathcal{P}(E)$ is said to be *continuous* if $X_n \to X$ implies that $\psi(X_n) \to \psi(X)$.

Let ψ, ψ_n be operators on $\mathcal{P}(E)$, $n \geqslant 1$: we say that $\psi_n \to \psi$ (ψ_n converges to ψ) if $\psi_n(X) \to \psi(X)$ for every $X \in \mathcal{P}(E)$. In this section we are concerned mostly with sequences ψ^n consisting of iterates of a given operator ψ.

5.8.2 FINITE WINDOW OPERATORS

DEFINITION 5–6. Let ψ be an increasing operator on $\mathcal{P}(E)$, and assume that $W(h) \subseteq E$ is a finite set for every $h \in E$. We say that ψ is a finite window operator with window W if

$$h \in \psi(X) \iff h \in \psi\big(X \cap W(h)\big),$$

for $h \in E$ and $X \subseteq E$.

Note that if $E = \mathbb{E}^d$ and ψ is translation invariant, we can take $W(h) = W_h$, where $W \subseteq \mathbb{E}^d$ is a finite set. If ψ is a finite window operator, then its dual ψ^* is such as well. Furthermore, compositions, finite suprema, and finite infima of finite window operators are finite window operators.

For our purposes, the main property of a finite window operator is given by the following result; a proof can be found in [11].

PROPOSITION 5–23. *Every finite window operator is continuous.*

5.8.3 ITERATION AND IDEMPOTENCE

Assume that ψ is a continuous operator on $\mathcal{P}(E)$ and that $\psi^n \to \psi^\infty$, where ψ^∞ is another operator on $\mathcal{P}(E)$. Note that we do not assume that ψ or ψ^∞ are increasing, nor that ψ^∞ is continuous. Then $\psi^n = \psi\psi^{n-1} \to \psi\psi^\infty$, as ψ is continuous. This yields that $\psi\psi^\infty = \psi^\infty$; hence $\psi^n\psi^\infty = \psi^\infty$ for every $n \geqslant 1$. Letting $n \to \infty$, we get $\psi^\infty\psi^\infty = \psi^\infty$. Thus we arrive at the following result:

PROPOSITION 5–24. *If ψ is a continuous operator on $\mathcal{P}(E)$ and $\psi^n \to \psi^\infty$, then ψ^∞ is idempotent. In particular, if ψ is also increasing, then ψ^∞ is a filter.*

PROOF. On the one hand $\psi^{n+1} \to \psi^\infty$ as $n \to \infty$, but on the other hand

$$\psi^{n+1} = \psi\psi^n \to \psi\psi^\infty \text{ as } n \to \infty,$$

by the continuity of ψ. This yields that $\psi\psi^\infty = \psi^\infty$, and more generally, $\psi^n\psi^\infty = \psi^\infty$, for every $n \geqslant 1$. Letting $n \to \infty$, this yields

$$\psi^\infty\psi^\infty = \psi^\infty,$$

meaning that ψ^∞ is idempotent. It is obvious that ψ^∞ is increasing, and we conclude that ψ^∞ is a filter. ■

For example, if $\psi \geqslant \mathrm{id}$ (i.e., ψ is extensive), then $\psi^2 \geqslant \psi$, hence $\psi^3 \geqslant \psi^2$, etc. This yields immediately that $\psi^n \to \psi^\infty$, where ψ^∞ is given by $\psi^\infty(X) = \bigcup_{n\geqslant 1} \psi^n(X)$. If ψ is also increasing, then ψ^∞ is a closing.

We present a simple example of an increasing, extensive operator ψ for which ψ^∞ is not idempotent; it is easy to see that this operator is not continuous. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\overline{\mathbb{N}} = \mathbb{N} \cup \{+\infty\}$. For $X \subseteq \overline{\mathbb{N}}$, we define $X + 1 = \{x + 1 \mid x \in X\}$, where $+\infty + 1 = +\infty$. Let the operator ψ on $\mathcal{P}(\overline{\mathbb{N}})$ be given by

$$\psi(X) = \begin{cases} X \cup (X + 1), & \text{if } X \neq \mathbb{N} \\ \overline{\mathbb{N}}, & \text{if } X = \mathbb{N}. \end{cases}$$

Obviously, ψ is increasing and extensive. For $k = 1, 2, \ldots$ we have $\psi^k(\{0\}) = \{0, 1, \ldots, k\}$, hence $\psi^\infty(\{0\}) = \mathbb{N}$. However, $\psi\psi^\infty(\{0\}) = \psi(\mathbb{N}) = \overline{\mathbb{N}}$. This implies that $(\psi^\infty)^2(\{0\}) = \overline{\mathbb{N}} \neq \psi^\infty(\{0\})$.

Most of the previous results can be extended to gray-scale functions. We recall the following definition.

DEFINITION 5–7. A partially ordered set $\mathcal{T}$ is called a *chain* if for every two elements $s, t \in \mathcal{T}$ we have $s \leqslant t$ or $t \leqslant s$. It is called a *complete chain* if it is both a chain and a complete lattice.

For example, $\overline{\mathbb{Z}}$ and $\overline{\mathbb{R}}$ with the natural ordering are complete chains.

Consider the space $\mathrm{Fun}(E, \mathcal{T})$, where $\mathcal{T}$ is a subset of $\overline{\mathbb{R}}$ which is a complete chain. (In [5, Chap. 13] we consider also the case where $\mathcal{T}$ is an arbitrary complete lattice.) We say that the sequence of functions F_n converges to F, $F_n \to F$, if $F_n(x) \to F(x)$ as $n \to \infty$, for every $x \in E$.

Definition 5–5 generalizes easily to function operators: an operator ψ on $\mathrm{Fun}(E, \mathcal{T})$ is *continuous* if $F_n \to F$ implies that $\psi(F_n) \to \psi(F)$. Proposition 5–24 remains valid in this case.

PROPOSITION 5–25. *If ψ is a continuous operator on* $\mathrm{Fun}(E, \mathcal{T})$ *and $\psi^n \to \psi^\infty$, then ψ^∞ is idempotent. If, furthermore, ψ is increasing, then ψ^∞ is a filter.*

In Section 5.9.3 we present a class of operators, the so-called activity-extensive operators, for which the sequence ψ^n converges.

5.9 ACTIVITY ORDERING AND CENTER OPERATOR

5.9.1 ACTIVITY ORDERING

Activity ordering is a partial ordering on $\mathcal{O}(\mathcal{L})$, the complete lattice of operators on $\mathcal{L}$ (where $\mathcal{L}$ is a complete lattice), which provides a tool to compare the effect of two different operators. The notion of "activity ordering" is due to Serra [23]; see also [12]. A comprehensive discussion can also be found in [5].

DEFINITION 5–8. Given two operators ϕ, ψ on the complete lattice $\mathcal{L}$, we say that ψ is *more active than* ϕ, denoted by $\phi \preccurlyeq \psi$, if

$$\mathrm{id} \wedge \psi \leqslant \mathrm{id} \wedge \phi \quad \text{and} \quad \mathrm{id} \vee \psi \geqslant \mathrm{id} \vee \phi.$$

For example, if ϕ and ψ are both extensive, then $\phi \preccurlyeq \psi$ if and only if $\phi \leqslant \psi$. However, if both operators are anti-extensive, then $\phi \preccurlyeq \psi$ iff $\phi \geqslant \psi$. It is evident that every operator is more active than id, the identity operator. On the other hand, if $\mathcal{L} = \mathcal{P}(E)$, then the complement operator $X \mapsto X^c$ is more active than any other operator. However, this last observation cannot be generalized to arbitrary negations.

PROPOSITION 5–26. *If* $\mathcal{L} = \mathcal{P}(E)$ *or* $\mathrm{Fun}(E, \mathcal{T})$, *where* $\mathcal{T}$ *is a complete chain, then* '$\preccurlyeq$' *defines a partial ordering on* $\mathcal{O}(\mathcal{L})$.

PROOF. We consider the case $\mathcal{L} = \mathcal{P}(E)$. The proof for $\mathcal{L} = \mathrm{Fun}(E, \mathcal{T})$ is quite similar. It is obvious that $\preccurlyeq$ is reflexive and transitive. We show that it is anti-symmetric. Let ψ, ϕ be two operators such that $\phi \preccurlyeq \psi$ and $\psi \preccurlyeq \phi$, and take $X \subseteq E$. Then $X \cap \phi(X) = X \cap \psi(X)$ and $X \cup \phi(X) = X \cup \psi(X)$, hence $\phi(X) = \psi(X)$. ■

Suppose that $\mathcal{L}$ possesses a negation ν. If $\phi \preccurlyeq \psi$ then $\mathrm{id} \wedge \psi \leqslant \mathrm{id} \wedge \phi$, hence $(\mathrm{id} \wedge \psi)\nu \leqslant (\mathrm{id} \wedge \phi)\nu$. This gives us $\nu \wedge \psi\nu \leqslant \nu \wedge \phi\nu$. Applying ν at both sides yields $\nu^2 \vee \nu\psi\nu \geqslant \nu^2 \vee \nu\phi\nu$. Using that $\nu^2 = \mathrm{id}$ and $\nu\psi\nu = \psi^*$, we get that $\mathrm{id} \vee \psi^* \geqslant \mathrm{id} \vee \phi^*$. Similarly, we find that $\mathrm{id} \vee \psi \geqslant \mathrm{id} \vee \phi$ implies that $\mathrm{id} \wedge \psi^* \leqslant \mathrm{id} \wedge \phi^*$. Thus we arrive at the following result.

PROPOSITION 5–27. *Let* $\mathcal{L}$ *be a complete lattice which possesses a negation. Then* $\phi \preccurlyeq \psi$ *if and only if* $\phi^* \preccurlyeq \psi^*$.

5.9.2 CENTER OPERATOR

In Proposition 5–26 we have seen that the relation '$\preccurlyeq$' defines a partial ordering if $\mathcal{L} = \mathcal{P}(E)$ or $\mathrm{Fun}(E, \mathcal{T})$, with $\mathcal{T}$ a complete chain. In the first case, a much stronger result holds.

PROPOSITION 5–28. *The family of operators on* $\mathcal{P}(E)$ *endowed with the activity ordering* '$\preccurlyeq$' *is a complete lattice. Given a collection of operators* ψ_i, $i \in I$, *on* $\mathcal{P}(E)$, *the activity infimum and supremum are given, respectively, by*

$$\curlywedge_{i \in I} \psi_i = \Big[\mathrm{id} \wedge \Big(\bigvee_{i \in I} \psi_i\Big)\Big] \vee \Big(\bigwedge_{i \in I} \psi_i\Big), \tag{5–14}$$

$$\curlyvee_{i \in I} \psi_i = \Big[\nu \wedge \Big(\bigvee_{i \in I} \psi_i\Big)\Big] \vee \Big(\bigwedge_{i \in I} \psi_i\Big). \tag{5–15}$$

The operator $\curlywedge_{i \in I} \psi_i$ is called the *center* of the operators ψ_i; the operator $\curlyvee_{i \in I} \psi_i$ is called the *anti-center* [5, 12, 23]. It is obvious that the center is an increasing operator, given that every ψ_i is increasing. For the anti-center this is not true in general. In this paper, our interest only concerns the center. Observe that the annular filter ω discussed in Section 5.4 is the center operator of ε_B and δ_A.

It is straightforward to show that (cf. Proposition 5–27):

$$\curlywedge_{i\in I} \psi_i^* = \big(\curlywedge_{i\in I} \psi_i\big)^*, \qquad \curlyvee_{i\in I} \psi_i^* = \big(\curlyvee_{i\in I} \psi_i\big)^*.$$

The center of the family ψ_i has the form

$$\gamma = (\mathrm{id} \wedge \psi) \vee \phi = (\mathrm{id} \vee \phi) \wedge \psi, \tag{5–16}$$

where $\phi = \bigwedge_{i\in I} \psi_i$ and $\psi = \bigvee_{i\in I} \psi_i$. Note that $\phi \leqslant \psi$.

If $\mathcal{L} = \mathrm{Fun}(E, \mathcal{T})$, where $\mathcal{T}$ is a complete chain, then the center γ given by Eq. 5–16 also has the interpretation of the activity infimum. Because $\mathrm{Fun}(E, \mathcal{T})$ possesses no complement operator (though it may possess a negation), the activity supremum does not exist in general.

The center operator on $\mathrm{Fun}(E, \mathcal{T})$ has an interesting geometric interpretation that explains why this operator is called center. Let ψ_1, ψ_2 be operators on $\mathrm{Fun}(E, \mathcal{T})$; put $\phi = \psi_1 \wedge \psi_2$ and $\psi = \psi_1 \vee \psi_2$. Define γ by Eq. 5–16; thus γ is the center of ψ_1 and ψ_2. Define the mapping $m : \mathcal{T}^3 \to \mathcal{T}$ as follows: given t_1, t_2, t_3, let $m(t_1, t_2, t_3)$ be the value t_i which lies between the other two. In fact, m is given by the formula

$$m(t_1, t_2, t_3) = (t_1 \wedge t_2) \vee (t_1 \wedge t_3) \vee (t_2 \wedge t_3).$$

It is not difficult to show that

$$\gamma(F)(x) = m\big(F(x), \psi_1(F)(x), \psi_2(F)(x)\big);$$

see [5, Sect. 3–6]. Refer to Fig. 5–8 for an illustration.

DEFINITION 5–9. A lattice $\mathcal{L}$ is said to be *modular* if the condition

$$(X \vee Y) \wedge Y' = \big(X \wedge Y'\big) \vee Y \quad \text{if} \quad Y \leqslant Y'$$

holds, for $X, Y, Y' \in \mathcal{L}$.

It is obvious that on a modular lattice the identity $(\mathrm{id} \wedge \psi) \vee \phi = (\mathrm{id} \vee \phi) \wedge \psi$ holds for any two operators ϕ, ψ with $\phi \leqslant \psi$. Thus, we can extend the definition of the center γ given in Eq. 5–16 to arbitrary modular lattices. We point out that distributivity of a lattice implies modularity. In particular, $\mathcal{P}(E)$ and $\mathrm{Fun}(E, \mathcal{T})$, where $\mathcal{T}$ is a chain, are modular.

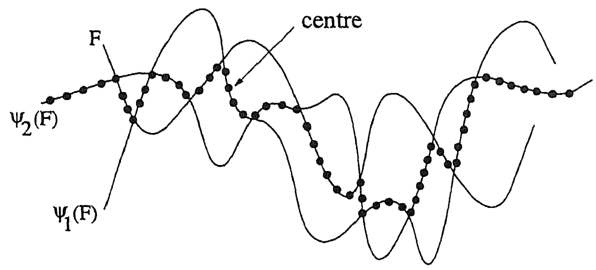


Figure 5–8. The center of the operators ψ_1 and ψ_2 is indicated by the black dots.

5.9.3 ACTIVITY-EXTENSIVE OPERATORS

We start with a definition.

DEFINITION 5–10. An operator ψ is called activity-extensive if $\psi^n \preccurlyeq \psi^{n+1}$, for every $n \geqslant 1$.

It is evident that every increasing operator which is extensive or anti-extensive is activity-extensive, but the converse is not true.

On $\mathrm{Fun}(E, \mathcal{T})$ there exists an interesting characterization of activity-extensive operators.

DEFINITION 5–11. Assume that $\mathcal{T}$ is a complete lattice. The sequence F_n, $n \geqslant 1$, in $\mathrm{Fun}(E, \mathcal{T})$ is called *pointwise monotone* if the sequence $F_n(x)$ is either increasing or decreasing for every $x \in E$.

If $\mathcal{T} = \{0, 1\}$, then $\mathrm{Fun}(E, \mathcal{T})$ is isomorphic to $\mathcal{P}(E)$. In this case, a sequence X_n is pointwise monotone if, for fixed $h \in E$, the sequence $X_n(h)$, where $X_n(\cdot)$ is the characteristic function of the set X_n, is of the form $0, 0, \ldots, 0, 1, 1, \ldots$ or $1, 1, \ldots, 1, 0, 0, \ldots$.

PROPOSITION 5–29. *The operator ψ on* $\mathrm{Fun}(E, \mathcal{T})$, *where $\mathcal{T}$ is a complete lattice, is activity-extensive if and only if the sequence $\psi^n(F)$ is pointwise monotone, for every function F.*

PROOF. We prove only the only if-statement. Let $F \in \mathrm{Fun}(E, \mathcal{T})$ and put $F_n = \psi^n(F)$ for $n \geqslant 0$. Given $x \in E$, assume that $n \geqslant 1$ is such that $F_0(x) = \cdots = F_{n-1}(x) < F_n(x)$. From $F_{n+1}(x) \vee F(x) \geqslant F_n(x) \vee F(x)$ it follows that $F_{n+1}(x) \geqslant F_n(x)$. Repeating this argument, we find that $F_n(x) \leqslant F_{n+1}(x) \leqslant F_{n+2}(x) \leqslant \cdots$. This proves the assertion. The case that $F_{n-1}(x) > F_n(x)$ is treated analogously. ■

An operator ψ on $\mathcal{P}(E)$ is activity-extensive if, for every X, the sequence $X \cap \psi^n(X)$ is decreasing, and the sequence $X^c \cap \psi^n(X)$ is increasing. It is well known [5, 16, 22] that an important class of operators on $\mathrm{Fun}(E)$ is the *flat operators*, i.e., operators generated by a set operator.

PROPOSITION 5–30. *An increasing operator on $\mathcal{P}(E)$ is activity-extensive if and only if its flat extension to* $\mathrm{Fun}(E)$ *is activity-extensive.*

The proof of this result can be found in [5, Prop. 13–44].

If ψ is an activity-extensive operator on $\mathrm{Fun}(E, \mathcal{T})$, where $\mathcal{T}$ is an arbitrary complete lattice, then $\psi^n(F)$ is pointwise monotone, for every function F. Fix $x \in E$; define $\psi^\infty(F)$ to be $\bigvee_{n \geqslant 1} \psi^n(F)(x)$ if the sequence $\psi^n(F)(x)$ is increasing, and $\bigwedge_{n \geqslant 1} \psi^n(F)(x)$ if it is decreasing. It follows immediately that $\psi^n \to \psi^\infty$ as $n \to \infty$.

PROPOSITION 5–31. *Let $\mathcal{T}$ be an arbitrary complete lattice and ψ an activity-extensive operator on* $\mathrm{Fun}(E, \mathcal{T})$; *then the sequence of iterates ψ^n converges (pointwise).*

We exploit this fact in Section 5.10.2. The next result, which is the main result of this section, and which is due to Serra [23], shows a simple, yet general, way to construct nontrivial activity-extensive operators.

PROPOSITION 5–32. *Let $\mathcal{L}$ be a complete modular lattice, ϕ an overfilter and ψ an underfilter such that $\phi \leqslant \psi$. The center operator $\gamma = (\mathrm{id} \wedge \psi) \vee \phi$ is activity-extensive.*

PROOF. To prove that $\gamma^{n-1} \preccurlyeq \gamma^n$, we show that

$$\gamma^n = \left(\mathrm{id} \wedge \psi\gamma^{n-1}\right) \vee \phi\gamma^{n-1} = \left(\mathrm{id} \vee \phi\gamma^{n-1}\right) \wedge \psi\gamma^{n-1}, \tag{5–17}$$

for every $n \geqslant 1$. Using the modularity of $\mathcal{L}$, we infer that

$$\begin{aligned} \mathrm{id} \wedge \gamma^n &= \mathrm{id} \wedge \psi\gamma^{n-1}, \\ \mathrm{id} \vee \gamma^n &= \mathrm{id} \vee \phi\gamma^{n-1}. \end{aligned}$$

In particular, using that $\gamma \leqslant \psi$ and that ψ is an underfilter, we get

$$\mathrm{id} \wedge \gamma^n \leqslant \mathrm{id} \wedge \psi^2\gamma^{n-2} \leqslant \mathrm{id} \wedge \psi\gamma^{n-2} \leqslant \mathrm{id} \wedge \gamma^{n-1}.$$

Analogously,

$$\mathrm{id} \vee \gamma^n \geqslant \mathrm{id} \vee \gamma^{n-1}.$$

Thus we have demonstrated that $\gamma^{n-1} \preccurlyeq \gamma^n$.

We prove Eq. 5–17 by induction. For $n = 1$ the result is obvious. Assume that it holds for n, then

$$\begin{aligned}\gamma^{n+1} &= \big((\mathrm{id} \wedge \psi) \vee \phi\big)\gamma^n \\ &= \big(\gamma^n \wedge \psi\gamma^n\big) \vee \phi\gamma^n \\ &= \big[\big((\mathrm{id} \vee \phi\gamma^{n-1}) \wedge \psi\gamma^{n-1}\big) \wedge \psi\gamma^n\big] \vee \phi\gamma^n.\end{aligned}$$

Since $\psi\gamma^{n-1} \geqslant \psi^2\gamma^{n-1} \geqslant \psi\gamma^n$, we get

$$\begin{aligned}\gamma^{n+1} &= \big[(\mathrm{id} \vee \phi\gamma^{n-1}) \wedge \psi\gamma^n\big] \vee \phi\gamma^n \\ &= \big(\mathrm{id} \vee \phi\gamma^{n-1} \vee \phi\gamma^n\big) \wedge \psi\gamma^n \\ &= \big(\mathrm{id} \vee \phi\gamma^n\big) \wedge \psi\gamma^n.\end{aligned}$$

Here we used the modularity of $\mathcal{L}$ and the fact that $\phi\gamma^{n-1} \leqslant \phi^2\gamma^{n-1} \leqslant \phi\gamma^n$. ■

Let ψ be an increasing operator, $\alpha \leqslant \psi$ an opening and $\beta \geqslant \psi$ a closing. From Proposition 5–19(c) and its dual we get that $\alpha\psi$ is an overfilter and that $\beta\psi$ is an underfilter. It is obvious that

$$\alpha\psi \leqslant \psi \leqslant \beta\psi.$$

Thus, Proposition 5–32 applies, and we arrive at the following result [8]:

PROPOSITION 5–33. *Let $\mathcal{L}$ be a modular lattice, ψ an increasing operator, α an opening, β a closing, and assume that $\alpha \leqslant \psi \leqslant \beta$. Then*

$$\pi = (\mathrm{id} \wedge \beta\psi) \vee \alpha\psi$$

is activity-extensive and $\pi \preccurlyeq \psi$. Furthermore, if α' is an opening and β' a closing such that $\alpha' \leqslant \alpha$ and $\beta' \geqslant \beta$, and if $\pi' = (\mathrm{id} \wedge \beta'\psi) \vee \alpha'\psi$, then $\pi' \preccurlyeq \pi$.

We point out that $(\mathrm{id} \wedge \psi\beta) \vee \psi\alpha$ is activity-extensive as well. However, this modification of ψ turns out to be less interesting than π; see [8]. Proposition 5–33 forms the basis for the construction of self-dual filters, as discussed in the following section.

5.10 SELF-DUAL FILTERS

This section discusses self-dual filters, that is, morphological filters which satisfy $\psi^* = \psi$. We have seen one instance of a self-dual filter in Section 5–4, namely, the annular filter for which the structuring elements governing foreground and background adjacency coincide. This particular filter is given by a simple explicit expression. As we observed earlier, it is the center of a dilation δ_A and its negative erosion $\varepsilon_A = \delta_A^*$, where A is a symmetric structuring element which does not contain the origin. The self-dual filters considered in this section are not as simple as the annular filter; they are all obtained by iteration of an increasing operator which is self-dual.

5.10.1 SELF-DUAL OPERATORS

Before we discuss the construction of self-dual filters by iteration, we explain how to build self-dual operators, as these form the main ingredient for this iteration procedure. The center operator is the most important tool for the construction of self-dual operators. This is the content of our first proposition.

PROPOSITION 5–34. *If ψ_i, $i \in I$, is a family of operators on $\mathcal{P}(E)$ such that with every ψ_i the negative operator ψ_i^* is also a member of the family, then the center $\gamma = \curlywedge_{i\in I} \psi_i$ is a self-dual operator.*

The proof is easy if one uses the explicit expressions in Eq. 5–14. Throughout the remainder of this section we restrict ourselves to translation-invariant operators on $\mathcal{P}(\mathbb{E}^d)$.

The median operator is the best-known example of a self-dual operator. More generally, if ψ_b is the morphological operator derived from a structuring element $A = \{a_1, a_2, \ldots, a_n\}$ and a Boolean function b of n variables, i.e.,

$$h \in \psi_b(X) \quad \text{if} \quad b\big(X(a_1 + h), \ldots, X(a_n + h)\big) = 1,$$

then ψ_b is self-dual if and only if $b^* = b$.

EXAMPLE 5–6. Let A be the 3×3 square, and let $\rho_{A,s}$ be the corresponding rank operators (see Sect. 5.6.2). It is easy to see that

$$\rho_{A,s}^* = \rho_{A,10-s}, \qquad s = 1, 2, \ldots, 9.$$

Furthermore, $\rho_{A,10-s} \leqslant \rho_{A,s}$ if $s \leqslant 5$. The center of $\rho_{A,s}$ and $\rho_{A,10-s}$, written as η_s, is given by

$$\eta_s = (\mathrm{id} \wedge \rho_{A,s}) \vee \rho_{A,10-s}, \qquad s = 1, 2, 3, 4, 5.$$

Evidently, η_s is a self-dual operator. Furthermore, one can easily show that

$$\mathrm{id} = \eta_1 \preccurlyeq \eta_2 \preccurlyeq \eta_3 \preccurlyeq \eta_4 \preccurlyeq \eta_5.$$

Figure 5–9. From left to right and top to bottom: $\eta_2(X), \eta_3(X), \eta_4(X), \eta_5(X)$. This figure shows clearly that η_{s+1} is more active than η_s.

The operator η_5 is the median operator. In Fig. 5–9 we apply $\eta_2, \eta_3, \eta_4, \eta_5$ to our test image.

In [5, Chap. 13] and [8] we have presented a comprehensive treatment of the construction of self-dual operators based on the concept of *switch operator*. Here we only summarize some of the main results.

PROPOSITION 5–35. *An increasing, translation-invariant operator on $\mathcal{P}(\mathbb{E}^d)$ is self-dual if and only if it is of the form $\psi = \psi_{\mathcal{A}}$, where*

$$\psi_{\mathcal{A}}(X) = \Big(X \cap \bigcap_{A \in \mathcal{A}} X \oplus \check{A}\Big) \cup \bigcup_{A \in \mathcal{A}} X \ominus A, \tag{5–18}$$

where $\mathcal{A} \subseteq \mathcal{P}(\mathbb{E}^d)$ is a collection of structuring elements which satisfy

$$0 \notin A \quad \textit{and} \quad A \cap B \neq \emptyset,$$

for $A, B \in \mathcal{A}$.

If $A \cap B \neq \emptyset$, then $X \ominus B \subseteq X \oplus \check{A}$. This yields that the operator given by Eq. (5–18) is the center of $X \mapsto \bigcup_{A \in \mathcal{A}} X \ominus A$ and its negative $X \mapsto \bigcap_{A \in \mathcal{A}} X \oplus \check{A}$.

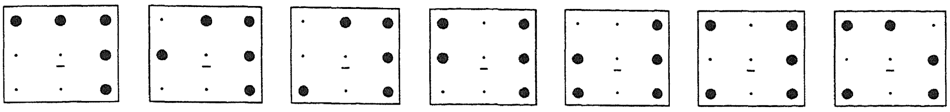
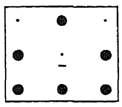


Figure 5–10. Collection $\mathcal{A}$ of structuring elements associated with the median operator.

There exists the following interpretation of Eq. 5–18. If a point h lies not in X, then h lies in the transformed image $\psi_{\mathcal{A}}(X)$ if and only if $A_h \subseteq X$ for some $A \in \mathcal{A}$. Dually, if $h \in X$ then $h \notin \psi_{\mathcal{A}}(X)$ if $A_h \subseteq X^c$ for some $A \in \mathcal{A}$.

For the median operator associated with the 3×3 square, the collection $\mathcal{A}$ contains the structuring elements depicted in Fig. 5–10.

The collection $\mathcal{A}$ in Eq. 5–18 is not uniquely determined by ψ. For example, if



is added to the collection in Fig. 5–10, one still obtains the median operator. But we have the following result:

PROPOSITION 5–36. *Let $\mathcal{A}, \mathcal{B}$ be two collections of structuring elements in $\mathcal{P}(\mathbb{E}^d)$. The following two assertions are equivalent:*

(i) *for every $A \in \mathcal{A}$ there exists a $B \in \mathcal{B}$ such that $B \subseteq A$;*

(ii) $\psi_{\mathcal{A}} \preccurlyeq \psi_{\mathcal{B}}$.

In particular, this result implies that with any subcollection $\mathcal{B}$ of the structuring elements depicted in Fig. 5–10, there corresponds a self-dual operator $\psi_{\mathcal{B}}$ which is less active than the median operator. In the next subsection we discuss an alternative way to diminish the activity of a self-dual operator.

EXAMPLE 5–7. The self-dual operators η_k can be represented as in Eq. 5–18 using a collection of structuring elements $\mathcal{A}_k$ consisting of all subsets A of the 3×3 square with $0 \notin A$ and containing $10 - k$ points. For example, $\mathcal{A}_4$ consists of 28 structuring elements, namely:

and rotations.

5.10.2 Construction of Self-dual Filters

Suppose we have an increasing, translation-invariant operator ψ which is self-dual. The next result gives an easy criterion for the activity-extensivity of ψ; see [8, Prop. 6–3].

PROPOSITION 5–37. *The operator $\psi_{\mathcal{A}}$ given by Eq. 5–18 is activity-extensive if and only if $0 \in \psi^n(A)$, for every $A \in \mathcal{A}$ and $n \geqslant 1$.*

PROOF. "only if": if $A \in \mathcal{A}$, then $0 \notin A$ and $0 \in \psi(A)$. If there exists an integer $n > 1$ such that $0 \notin \psi^n(A)$, then the sequence $\psi^k(A)$ is not pointwise monotone, hence ψ is not activity-extensive.

"if": suppose that $0 \in \psi^n(A)$ for $A \in \mathcal{A}$ and $n \geqslant 1$, and that ψ is not activity-extensive. Then there is a set X such that $\psi^n(X)$ is not pointwise monotone. Without loss of generality, we can assume that $0 \notin X$, $0 \in \psi(X)$ and $0 \notin \psi^m(X)$, for some $m > 1$. Using Eq. 5–18, we derive that $0 \in \bigcup_{A \in \mathcal{A}} X \ominus A$, which means that $A \subseteq X$ for some $A \in \mathcal{A}$. By assumption, $0 \in \psi^n(A) \subseteq \psi^n(X)$ for $n \geqslant 1$, a contradiction. This yields the result. ■

To obtain self-dual operators which are activity-extensive, we combine Proposition 5–33 and Proposition 5–34. (The continuity of these operators will be guaranteed by the fact that we restrict ourselves to finite window operators.)

Let ψ be a self-dual operator and let α be an opening with $\alpha \leqslant \psi$. The negative closing $\beta = \alpha^*$ satisfies $\beta \geqslant \psi$. Now Proposition 5–33 gives us that the center

$$\pi = (\mathrm{id} \wedge \beta\psi) \vee \alpha\psi \tag{5–19}$$

is activity-extensive, whereas Proposition 5–34 guarantees that π is self-dual (for $(\beta\psi)^* = \alpha\psi$).

REMARK 5–1. Alternatively, one can start with an increasing operator ψ such that $\psi \leqslant \psi^*$ and an opening $\alpha \leqslant \psi$. The negative closing $\beta = \alpha^*$ satisfies $\beta \geqslant \psi^*$, and the center of $\alpha\psi$ and $\beta\psi^*$ is self-dual and activity-extensive.

Before we present some examples on $\mathcal{P}(\mathbb{Z}^2)$, we give a criterion which guarantees that $\alpha \leqslant \psi$. The structural opening $\alpha_B(X) = X \circ B$ satisfies $\alpha_B \leqslant \psi$ if $B \subseteq \psi(B)$. We say that B is *persistent with respect to* ψ if $B \subseteq \psi(B)$. The following result is obvious.

PROPOSITION 5–38. *Let ψ be an increasing, translation-invariant operator on $\mathcal{P}(\mathbb{E}^d)$, and let B_i, $i \in I$, be persistent with respect to ψ. Then the opening $\alpha(X) = \bigcup_{i \in I} X \circ B_i$ satisfies $\alpha \leqslant \psi$.*

EXAMPLE 5–8. Let μ be the median operator with the 3×3 square as structuring element. The effect of μ on our test image can be seen in Fig. 5–9; recall that $\mu = \eta_5$. The set B given by

$$B = \begin{array}{cccc} & \bullet & \bullet & \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ & \bullet & \bullet & \end{array}$$

is persistent with respect to μ. Let $\alpha(X) = X \circ B$ and $\beta(X) = X \bullet B$. The modification π given by Eq. 5–19 is self-dual and activity-extensive. In the first row of Fig. 5–11 we depict $\pi(X)$ and $\pi^\infty(X)$.

EXAMPLE 5–9. The median operator discussed in the previous example is of the form $\mu = \psi_{\mathcal{A}}$, where $\mathcal{A}$ are the structuring elements depicted in Fig. 5–10. Consider the subcollection $\mathcal{B}$ of $\mathcal{A}$ which lacks the first structuring element and its three 90°-rotations. Now Proposition 5–36 yields that $\psi_{\mathcal{B}} \preccurlyeq \psi_{\mathcal{A}}$. The operator $\psi_{\mathcal{B}}$ is not activity-extensive; for example, the following pattern oscillates with period 2.

$$\begin{array}{|cccccccc|} \hline \bullet & \bullet & \circ & \circ & \bullet & \bullet & \circ & \circ \\ \circ & \circ & \bullet & \bullet & \circ & \circ & \bullet & \bullet \\ \bullet & \bullet & \circ & \circ & \bullet & \bullet & \circ & \circ \\ \circ & \circ & \bullet & \bullet & \circ & \circ & \bullet & \bullet \\ \hline \end{array}$$

It is evident that the structuring element B is persistent with respect to $\psi_{\mathcal{B}}$, but we can find a smaller structuring element with this property, namely

$$B' = \begin{array}{cc} \bullet & \bullet \\ \bullet & \bullet \end{array}$$

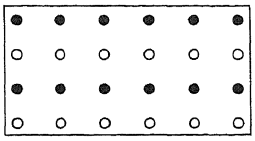
Let α be the structural opening $\alpha(X) = X \circ B'$ and let β be the negative closing. Again, the modification π of $\psi_{\mathcal{B}}$ given by Eq. 5–19 is self-dual and activity-extensive. The images $\pi(X)$ and $\pi^\infty(X)$ are depicted in the second row of Fig. 5–11.

EXAMPLE 5–10. Consider the operator η_4 introduced in Example 5–6. This operator can be represented in the form Eq. 5–18 where $\mathcal{A}$ is given by

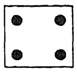
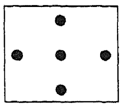
$$\begin{array}{|ccc|} \hline \bullet & \bullet & \bullet \\ \bullet & \underline{\cdot} & \cdot \\ \bullet & \bullet & \cdot \\ \hline \end{array} \quad \begin{array}{|ccc|} \hline \bullet & \bullet & \bullet \\ \bullet & \underline{\cdot} & \cdot \\ \bullet & \cdot & \bullet \\ \hline \end{array} \quad \begin{array}{|ccc|} \hline \bullet & \bullet & \bullet \\ \bullet & \underline{\cdot} & \cdot \\ \cdot & \bullet & \bullet \\ \hline \end{array} \quad \begin{array}{|ccc|} \hline \bullet & \bullet & \bullet \\ \cdot & \underline{\cdot} & \cdot \\ \bullet & \bullet & \bullet \\ \hline \end{array}$$

Figure 5–11. First three rows: $\pi(X)$ and $\pi^{\infty}(X)$ of Examples 5–8, 5–9, and 5–10. Bottom row: $\psi_{\mathcal{B}}(X)$ and $\psi_{\mathcal{B}}^{\infty}(X)$ of Example 5–11.

and their 45°-rotations. The operator η_4 is not activity-extensive since the pattern

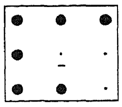
 etc.

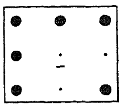
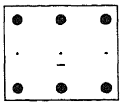
is 2-periodic. The structuring elements

 and 

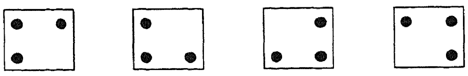
are persistent. Let α be the union of the structural openings associated with these two structuring elements; let $\beta = \alpha^*$ and let π be the modification of η_4 obtained from Eq. 5–19. The action of π and π^∞ can be seen from the third row in Fig. 5–11.

EXAMPLE 5–11. Consider the subcollection $\mathcal{B}$ of of the collection $\mathcal{A}$ of the previous example which is obtained by deleting the elements

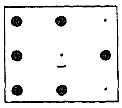
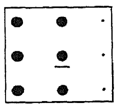
 + 45°-rotations,

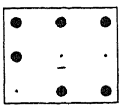
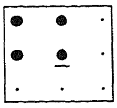
 and  + 90°-rotations.

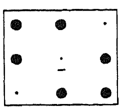
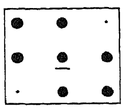
We use Proposition 5–37 to show that $\psi_{\mathcal{B}}$ is activity-extensive. Therefore we must consider the sequences $\psi^n_{\mathcal{B}}(B)$ for $B \in \mathcal{B}$. First we note that the triangles



are invariant under $\psi_{\mathcal{B}}$. Now

 $\xrightarrow{\psi_{\mathcal{B}}}$  $\xrightarrow{\psi_{\mathcal{B}}}$ idem

 $\xrightarrow{\psi_{\mathcal{B}}}$  $\xrightarrow{\psi_{\mathcal{B}}}$ idem

 $\xrightarrow{\psi_{\mathcal{B}}}$  $\xrightarrow{\psi_{\mathcal{B}}}$ idem

This implies that $\psi_{\mathcal{B}}$ is activity-extensive. The bottom row in Fig. 5–11 depicts the transformed sets $\psi_{\mathcal{B}}(X)$ and $\psi_{\mathcal{B}}^{\infty}(X)$. These figures clearly show the invariance of the triangles depicted above.

REFERENCES

[1] Birkhoff, G., *Lattice Theory*, 3rd Ed., Vol. 25, American Mathematical Society, Providence, RI, 1967.

[2] Charif-Chefchaouni, M. and D. Schonfeld, "Morphological Representation of Nonlinear Filters," *J. Math. Imaging and Vision* **4**, 215–232, 1994.

[3] Heijmans, H. J. A. M., "Composing Morphological Filters," *IEEE Trans. Image Process.* **6** (5), 713–723, 1997.

[4] Heijmans, H. J. A. M., "Theoretical Aspects of Gray-Level Morphology," *IEEE Trans. Patt. Anal. Mach. Intell.* **13**, 568–582, 1991.

[5] Heijmans, H. J. A. M., *Morphological Image Operators*, Academic Press, Boston, 1994.

[6] Heijmans, H. J. A. M., "A New Class of Alternating Sequential Filters," in *1995 IEEE Workshop on Nonlinear Signal and Image Processing*, I. Pitas, ed., Vol. I, *Proc. IEEE*, 30–33, 1995.

[7] Heijmans, H. J. A. M., "Morphological Filters for Dummies," in *Mathematical Morphology and its Application to Image and Signal Processing*, P. Maragos, R. W. Shafer, and M. A. Butt, eds., Kluwer Academic Publishers, Boston, 127–137, 1996.

[8] Heijmans, H. J. A. M., "Self-Dual Morphological Operators and Filters," *J. Math. Imaging and Vision* **6**(1), 15–36, 1996.

[9] Heijmans, H. J. A. M., and C. Ronse, "The Algebraic Basis of Mathematical Morphology – Part I: Dilations and Erosions," *Comp. Vis. Graph. Image Process.* **50**, 245–295, 1990.

[10] Heijmans, H. J. A. M., and C. Ronse, "Annular Filters for Binary Images," *IEEE Trans. Image Process.*, to appear.

[11] Heijmans, H. J. A. M., and J. Serra, "Convergence, Continuity and Iteration in Mathematical Morphology," *J. Vis. Comm. Image Rep.* **3**, 84–102, 1992.

[12] Meyer, F., and J. Serra, "Contrasts and Activity Lattice," *Sign. Proc.* **16**, 303–317, 1989.

[13] Roerdink, J. B. T. M., "Mathematical Morphology with Non-Commutative Symmetry Groups," in *Mathematical Morphology in Image Processing*, E. R. Dougherty, ed., Chap. 7, pp. 205–254, Marcel Dekker, New York, 1993.

[14] Roerdink, J. B. T. M., "On the Construction of Translation and Rotation Invariant Morphological Operators," in *Mathematical Morphology: Theory and Hardware*, R. M. Haralick, ed., Oxford University Press, in press.

[15] Ronse, C., "Extraction of Narrow Peaks and Ridges in Images by Combination of Local Low Rank and Max Filters: Implementation and Applications to Clinical Angiography," Working Document WD47, Philips Research Laboratory, Brussels, Belgium, 1988.

[16] Ronse, C., "Why Mathematical Morphology Needs Complete Lattices," *Sign. Proc.* **21**, 129–154, 1990.

[17] Ronse, C., "Lattice-Theoretical Fixpoint Theorems in Morphological Image Filtering," *J. Math. Imaging and Vision* **4**, 19–41, 1994.

[18] Ronse, C., "Openings: Main Properties, and How to Construct Them," in *Mathematical Morphology: Theory and Hardware*, R. M. Haralick, ed., Oxford University Press, in press.

[19] Ronse, C., and H. J. A. M. Heijmans, "The Algebraic Basis of Mathematical Morphology – Part II: Openings and Closings," *CVGIP: Image Understanding* **54**, 74–97, 1991.

[20] Ronse, C., and H. J. A. M. Heijmans, "A Lattice-Theoretical Framework for Annular Filters in Morphological Image Processing," *Applicable Algebra in Engineering, Communication, and Computing* **9**, 45–89, 1998.

[21] Schonfeld, D., and J. Goutsias, "Optimal Morphological Pattern Restoration from Noisy Binary Images," *IEEE Trans. Patt. Anal. Mach. Intell.* **13**, 14–29, 1991.

[22] Serra, J., *Image Analysis and Mathematical Morphology*, Academic Press, London, 1982.

[23] Serra, J., ed., *Image Analysis and Mathematical Morphology*, Vol. II: *Theoretical Advances*, Academic Press, London, 1988.

[24] Serra, J., and L. Vincent, "On Overview of Morphological Filtering," *IEEE Trans. Circ. Sys. Sign. Proc.* **11**, 47–108, 1992.