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Exceptional Presentations of Three Generalized Hexagons of Order 2

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The generalized hexagons associated with $L_3(2)$, $U_3(3)$, ${}^3D_4(2)$, respectively, are presented as subconfigurations of the projective plane over the complex, quaternionic, octonionic field, respectively. This leads to an embedding of Aut ${}^3D_4(2)$ in the Lie group of type $F_4(\mathbb{R})$.

INTRODUCTION

Exceptional presentations of the generalized hexagon of order (2, 1) on 21 points in the complex projective plane, of the dual of the classical generalized hexagon of order (2, 2) on 63 points in the quaternionic projective plane and of the generalized hexagon of order (2, 8) on 819 points in the octonionic projective plane are known to exist (by J. H. Conway, A. Neumaier, and S. Norton). No construction of the generalized hexagon on 819 points has appeared in the literature so far (to the best of the author's knowledge). The intention of this article is to change that situation. It is shown how to obtain presentations for the generalized hexagons on 21 points and on 63 points from the one on 819 points. The construction of the generalized hexagon on 819 points is such that an embedding of the automorphism group of ${}^3D_4(2)$ in the Lie group of type $F_4(\mathbb{R})$ results.

1. PECULIARITIES OF THE OCTONIONS

Let $\mathbb{O}$ be the real division algebra of the *octonions* (also called *octaves* or *Cayley numbers*). Choose an $\mathbb{R}$ basis $e_0 = 1, e_1, \dots, e_7$ such that multiplication in $\mathbb{O}$ is determined by the rules

$$e_i^2 = -1 \qquad (i = 1, 2, \dots, 7) \tag{1}$$

and

$$e_i e_j = e_k \tag{2}$$

whenever (ijk) is one of the 3-cycles $(1+r, 2+r, 4+r)$, where i, j, k, r run through the integers modulo 7 and take their values in $\{1, 2, \ldots, 7\}$. The antiautomorphism $x \to \bar{x}$ of order 2 defined by

$$\bar{x} = \xi_0 - \sum_{i=1}^{7} \xi_i e_i \qquad \text{whenever} \quad x = \sum_{i=0}^{7} \xi_i e_i \in \mathbb{O} \tag{3}$$

is called *conjugation*. The real part $\mathrm{Re}(x)$ of an element x of $\mathbb{O}$, is given by

$$\mathrm{Re}(x) = \tfrac{1}{2}(x + \bar{x}). \tag{4}$$

We recall that $\mathbb{O}$ is nonassociative and satisfies the following equations for $x, y, z \in \mathbb{O}$:

$$x(yx) = (xy)x \qquad \text{(hence also denoted by } xyx\text{)}, \tag{5}$$

$$x(y\bar{x}) = (xy)\bar{x} \qquad \text{(hence also denoted by } xy\bar{x}\text{)}, \tag{6}$$

$$(zxz)y = z(x(zy)) \qquad \text{and} \qquad y(zxz) = ((yz)x)z, \tag{7}$$

$$(zx)(yz) = z(xy)z, \tag{8}$$

$$x(xy) = x^2 y \qquad \text{and} \qquad \bar{x}(xy) = (\bar{x}x)y. \tag{9}$$

For more details and an excellent introduction, the reader is referred to [7] or [8].

We shall need a particular element of $\mathbb{O}$,

$$a = \tfrac{1}{2} \sum_{i=0}^{7} e_i \tag{10}$$

and a particular subset of $\mathbb{O}$,

$$Q = \{-e_i, e_i \mid i = 0, 1, \ldots, 7\}. \tag{11}$$

The following relations hold for $d \in Q$:

$$\begin{aligned} &ada + a = -2d\bar{a}d, \qquad ada - a = -2\bar{d}, \qquad \text{if} \quad \mathrm{Re}\, ad = \tfrac{1}{2}, \\ &ada + a = -2\bar{d}, \qquad ada - a = 2d\bar{a}d, \qquad \text{if} \quad \mathrm{Re}\, ad = -\tfrac{1}{2}. \end{aligned} \tag{12}$$

The stabilizer in $\mathrm{Aut}\, \mathbb{O}$ of Q is denoted by $\mathscr{C}$. This group is known as (see [4, 7]):

$$\mathscr{C} = \langle (1234567), (124)(365), \delta^{-}_{\{1,2,4,7\}}(12)(46) \rangle, \tag{13}$$

where a permutation π stands for the $\mathbb{R}$-linear transformation induced on $\mathbb{O}$ by the permutation $e_i \to e_{\pi(i)}$ $(i = 0, 1, ..., 7)$ and δ_K^- for $K \subseteq \{0, 1, ..., 7\}$ stands for the $\mathbb{R}$-linear map sending e_i to $-e_i$ whenever $i \in K$ and fixing e_i if $i \notin K$ $(i = 0, 1, ..., 7)$.

The group $\mathscr{C}$ is a nonsplit extension of a (diagonal) group of order 2^3 by $PSL_2(7)$. Thus, $\mathscr{C}$ has order $2^6 \cdot 3 \cdot 7$. The stabilizer in $\mathscr{C}$ of α, denoted by $\mathscr{C}_0$, is a non-Abelian group of order 21,

$$\mathscr{C}_0 = \langle (1234567), (124)(365) \rangle. \tag{14}$$

The set $(Q\alpha)Q$ will be used frequently in the sequel; we record some of its properties here,

$$(Q\alpha)Q = \mathscr{C}(\alpha) \cup \mathscr{C}(-\alpha) = \left\{ \frac{1}{2} \sum_{i=0}^{7} \varepsilon_i e_i \;\middle|\; \varepsilon_i \in \{-1, 1\}; \prod_{i=0}^{7} \varepsilon_i = 1 \right\}. \tag{15}$$

Let $c, d, e, f \in Q$. Then

$$(c\alpha)d = (e\alpha)f \Rightarrow c = \pm e \quad \text{and} \quad d = \pm f, \tag{16}$$

$$c(d\alpha) \in Q\alpha, \tag{17}$$

$$(\alpha c)(cd) \in (Q\alpha)d, \tag{18}$$

$$(Q\alpha)d = Q(\alpha d). \tag{19}$$

By use of $\mathscr{C}_0$ and $\mathscr{C}$, the verification of these equalities may be reduced to a slight amount of work. Details are omitted here.

2. The Exceptional Jordan Algebra $\mathbb{J}_3(\mathbb{F})$

Let $\mathbb{F}$ be a division subalgebra of $\mathbb{O}$. The *exceptional Jordan algebra* $\mathbb{J}_3(\mathbb{F})$ is defined on the set of 3×3 Hermitian matrices (with respect to conjugation) over $\mathbb{F}$. Its multiplication is given by

$$A \circ B = \tfrac{1}{2}(AB + BA) \qquad (A, B \in \mathbb{J}_3(\mathbb{F})), \tag{20}$$

where AB stands for the usual matrix product of A and B. Note that $A^2 = A \circ A$. Let I be the 3×3 identity matrix. Multiplication is commutative, but nonassociative (see [8]). The Jordan algebra $\mathbb{J}_3(\mathbb{F})$ has a natural inner product $(\cdot, \cdot)$ given by

$$(A, B) = \operatorname{Re} \operatorname{Trace}(AB) \qquad (A, B \in \mathbb{J}_3(\mathbb{F})). \tag{21}$$

Here, $\operatorname{Aut} \mathbb{J}_3(\mathbb{F})$, the automorphism group of $\mathbb{J}_3(\mathbb{F})$, preserves this inner product.

Let $P(\mathbb{F})$ be the set of idempotents in $\mathbb{J}_3(\mathbb{F})$ having trace 1. Then, $P(\mathbb{F})$ together with $\{\{A \in P(\mathbb{F}) \mid A \circ B = 0\} \mid B \in P(\mathbb{F})\}$ for the collection of lines, is a projective plane over $\mathbb{F}$. This plane is stabilized by Aut $\mathbb{J}_3(\mathbb{F})$ and the restriction of Aut $\mathbb{J}_3(\mathbb{F})$ to $P(\mathbb{F})$ is faithful. Moreover, Aut $\mathbb{J}_3(\mathbb{F})$ preserves the polarity on $P(\mathbb{F})$ determined by the map $A \mapsto \{B \in P(\mathbb{F}) \mid A \circ B = 0\}$ on $P(\mathbb{F})$. For $p \in P(\mathbb{F}')$, where $\mathbb{F}'$ is a commutative subfield of $\mathbb{F}$, let $\sigma_p \colon \mathbb{J}_3(\mathbb{F}) \to \mathbb{J}_3(\mathbb{F})$ be the map given by

$$\sigma_p(A) = ((1-2p)A)(1-2p) \qquad (A \in \mathbb{J}_3(\mathbb{F})). \tag{22}$$

Then,

$$\sigma_p \in \text{Aut } \mathbb{J}_3(\mathbb{F}) \text{ and } \sigma_p^2 = 1 \text{ if } p \in P(\mathbb{F}') \text{ for } \mathbb{F}' \text{ a commutative subfield of } \mathbb{F}. \tag{23}$$

This observation is easily verified since by [8] any two maximal commutative subfields of $\mathbb{O}$ are in the same Aut $\mathbb{O}$ orbit. Note that σ_p, when defined, is on $P(\mathbb{F})$ the unique involutory homology with center p contained in Aut $\mathbb{J}_3(\mathbb{F})$. This explains the importance of $\phi\sigma_p\phi^{-1}$; it is the unique involutory homology with center ϕp contained in Aut $\mathbb{J}_3(\mathbb{F})$. As Aut $\mathbb{J}_3(\mathbb{F})$ is transitive on $P(\mathbb{F})$, this yields that for any $q \in P(\mathbb{F})$ there is a canonical involutory homology with center q which is an automorphism of $\mathbb{J}_3(\mathbb{F})$. We denote this automorphism by σ_q; this is in accordance with the notation of (22). Moreover, for $p \in P(\mathbb{F})$ and $\phi \in$ Aut $\mathbb{J}_3(\mathbb{F})$, we have

$$\phi\sigma_p\phi^{-1} = \sigma_{\phi p}. \tag{24}$$

For $\tau \in \text{Aut}(\mathbb{F})$, let $\hat{\tau}$ denote the automorphism of $\mathbb{J}_3(\mathbb{F})$ given by

$$\hat{\tau}(A) = (\tau a_{ij})_{1 \leqslant i,j \leqslant 3} \qquad \text{if} \quad A = (a_{ij})_{1 \leqslant i,j \leqslant 3} \in \mathbb{J}_3(\mathbb{F}). \tag{25}$$

For π a permutation of 1, 2, 3, denote by $\tilde{\pi}$, the automorphism of $\mathbb{J}_3(\mathbb{F})$ given by

$$\tilde{\pi}(A) = (a_{\pi(i)\pi(j)})_{1 \leqslant i,j \leqslant 3} \qquad \text{if} \quad A = (a_{ij})_{1 \leqslant i,j \leqslant 3} \in \mathbb{J}_3(\mathbb{F}). \tag{26}$$

3. The Generalized Hexagon of Order (2, 8)

For $\pi = (123)$; $i \in \{1, 2, 3\}$; $x \in (Q\alpha)Q$; $c, d, e \in Q$ (see (11)) define the following elements of $\mathbb{J}_3(\mathbb{O})$ in $P(\mathbb{O})$:

$$p(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{27}$$

$$p(i) = \tilde{\pi}^{i-1} p(1), \tag{28}$$

$$p(1, e) = \tfrac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & e \\ 0 & \bar{e} & 1 \end{pmatrix}, \tag{29}$$

$$p(i, e) = \tilde{\pi}^{i-1} p(1, e), \tag{30}$$

$$p(1, x, e) = \tfrac{1}{4} \begin{pmatrix} 2 & \overline{ex} & \bar{x} \\ ex & 1 & e \\ x & \bar{e} & 1 \end{pmatrix}, \tag{31}$$

$$p(i, x, e) = \tilde{\pi}^{i-1} p(1, x, e). \tag{32}$$

Recall from (10) that $\alpha = \frac{1}{2} \sum_{i=0}^{7} e_i$. Consider the following subsets of $P(\mathbb{O})$:

$$L_0 = \{ p(i) \mid i = 1, 2, 3 \}, \tag{33}$$

$$L_1 = \{ p(i, e) \mid i = 1, 2, 3; e \in Q \}, \tag{34}$$

$$L_2 = \{ p(i, x, e) \mid i = 1, 2, 3; e \in Q; x \in (Q\alpha) e \}. \tag{35}$$

The set $H = L_0 \cup L_1 \cup L_2$ has $3(1 + 16 + 16^2) = 819$ elements. This set is turned into a graph $(H, \sim)$ by requiring,

$$p \sim q \Leftrightarrow (p, q) = 0 \qquad (p, q \in H). \tag{36}$$

Note that L_0 is a maximal clique of $(H, \sim)$. Let $\underline{\underline{h}}$ be the set of all maximal cliques of H; its members are called *lines*. Denote by D, the subgroup

$$D = \langle \sigma_p, \hat{\tau}, \tilde{\pi} \mid p \in \{ p(1, \alpha, 1) \} \cup L_0 \cup L_1 ; \tau \in \mathscr{C}_0 ; \pi \in \mathrm{Sym}(3) \rangle \tag{37}$$

of Aut $\mathbb{J}_3(\mathbb{O})$. The following result comprises the presentation we seek:

THEOREM. *Let $(H, \sim)$, $\underline{\underline{h}}$, D, and $(\cdot, \cdot)$ be as described. Then*:

(i) *D stabilizes H and its restriction to H is faithful. Its image is a transitive group of automorphisms of $(H, \sim)$.*

(ii) *For any two distinct points p, q of H the inner product (p, q) is one of* $0, \frac{1}{4}, \frac{1}{2}$.

(iii) *The tuple $(H, \underline{\underline{h}})$ is the classical generalized hexagon of order* $(2, 8)$ *and* $D \cong \mathrm{Aut}({}^3D_4(2))$, *an extension of* ${}^3D_4(2)$ *by an element of order* 3.

Proof. (i) First, we establish the hardest part of the proof, namely, the verification that D stabilizes H. Frequent use is made of the formulae (5)–(19).

It is immediate from the construction of H that H is invariant under $\hat{\tau}$ for $\tau \in \mathscr{C}_0$ and under $(\widetilde{123})$. Let $\pi = (23)$. Then for any $e \in Q$ and $x \in (Q\alpha)Q$,

$$\tilde{\pi} p(1, x, e) = p(1, ex, \bar{e}), \tag{38}$$

while $x \in (Q\alpha)e$ implies $(ex)\bar{e} \in Q\alpha$, so that $p(1, x, e) \in H$ leads to $p(1, ex, \bar{e}) \in H$. Moreover,

$$\tilde{\pi} p(2, x, e) = p(3, ex, \bar{e}). \tag{39}$$

As above, $p(2, x, e) \in H$ implies $(ex)\bar{e} \in Q\alpha$ so that $\pi p(2, x, e) \in H$. It readily follows that $\tilde{\pi} L_2$ is contained in H, and $\tilde{\pi} H = H$. Since (123) and (23) generate Sym(3), the result is that $\tilde{\pi} H = H$ for any $\pi \in \mathrm{Sym}(3)$.

Let $q = p(1)$. For $e \in Q$ and $x \in (Q\alpha)e$, we have

$$\begin{aligned}
\sigma_q p(i) &= p(i) \qquad (i = 1, 2, 3),\\
\sigma_q p(1, e) &= p(1, e),\\
\sigma_q p(i, e) &= p(i, -e) \qquad (i = 2, 3)\\
\sigma_q p(1, x, e) &= p(1, -x, e),\\
\sigma_q p(2, x, e) &= p(2, -x, -e),\\
\sigma_q p(3, x, e) &= p(3, x, -e).
\end{aligned} \tag{40}$$

Thus, $\sigma_{p(1)} H = H$. In view of (24), it follows that $\sigma_q H = H$ for any $q \in L_0$.

Next, let $q = p(1, e)$ for some $e \in Q$. For any $d \in Q$ and $x \in (Q\alpha)d$, we have

$$\begin{aligned}
\sigma_q p(1) &= p(1),\\
\sigma_q p(2) &= p(3),\\
\sigma_q p(1, d) &= p(1, e\bar{d}e),\\
\sigma_q p(2, d) &= p(3, \overline{ed}),\\
\sigma_q p(1, x, d) &= p(1, -\bar{e}(dx), e\bar{d}e),\\
\sigma_q p(2, x, d) &= p(3, -e(dx)e, -\overline{ed}).
\end{aligned} \tag{41}$$

From Eqs. (5)–(9) and (17)–(19), it is readily deduced that $\sigma_{p(1,e)} H = H$. Applying $(\widetilde{123})$, we get $\sigma_q H = H$ for any $q \in L_1$. Finally, let $q = p(1, \alpha, 1)$. For $d \in Q$, we have

$$
\begin{aligned}
\sigma_q p(1) &= p(1, 1), \\
\sigma_q p(2) &= p(1, \alpha, -1), \\
\sigma_q p(3) &= p(1, -\alpha, -1), \\
\sigma_q p(1, d) &= p(1), && \text{if} \quad d = 1, \\
&= p(1, -1), && \text{if} \quad d = -1, \\
&= p(1, d\alpha, -1), && \text{if} \quad d \in Q \backslash \{\pm 1\}, \\
\sigma_q p(2, d) &= p(2, -d\alpha d, -d), && \text{if} \quad \operatorname{Re} \alpha \bar{d} = \tfrac{1}{2}, \\
&= p(3, -\alpha\bar{d}, -\bar{d}), && \text{if} \quad \operatorname{Re} \alpha \bar{d} = -\tfrac{1}{2}.
\end{aligned}
\tag{42}
$$

This shows that $\sigma_q(L_0 \cup L_1) \subseteq H$.

In verifying that $\sigma_q L_2 \subseteq H$, we may restrict considerations to $\sigma_q p$ for $p = p(i, y, d)$ with:

(a) $i \in \{1, 2\}$ (as $(\widetilde{23})\,\sigma_q = \sigma_q(\widetilde{23})$).

(b) $d = 1, -1, e_1, -e_1$ (as $\mathscr{C}_0$ acts on Q with these octonions as orbit representatives and $\hat{\tau}\sigma_q = \sigma_q \hat{\tau}$ for $\tau \in \mathscr{C}_0$).

(c) $y = (cx)d$ with $c = 1, -1, e_1, -e_1, e_3, -e_3$ (as $\tau = (124)(365)$ acts on Q and stabilizes $\{\pm e_0, \pm e_1, \alpha\}$ pointwise).

(d) Moreover, if $d = \pm 1$, we may take $c \in \{1, -1, e_1, -e_1\}$ (for then all of $\mathscr{C}_0$ stabilizes d).

Thus, we need to check whether $\sigma_q p \in H$ for 40 particular points p. This is done in Table I. It should be remarked that (still) many equalities listed are superfluous. For instance, $\sigma_q p(2, -e_1 \alpha e_1, -e_1) = p(2, e_1)$, by (42).

Thus, $\sigma_{p(1,\alpha,1)}$ stabilizes H. The conclusion is that the generators of D, and hence D itself stabilize H. The restriction of D to H induces a permutation group of H. Since H contains an $\mathbb{R}$ basis of $\mathbb{J}_3(\mathbb{O})$, this restriction is faithful and D may be viewed as a group of permutations on H. Since D consists of automorphisms of $\mathbb{J}_3(\mathbb{O})$, it preserves the inner product $(\cdot, \cdot)$ and therefore adjacency $\sim$ in H. We obtain that D is a subgroup of $\operatorname{Aut}(H, \sim)$. From (38)–(42) and Table I it is readily seen that D is transitive on H. This proves (i).

(ii) Since D is transitive the claim need only be checked for pairs p, q, where $p = p(1)$ and $q \in H \backslash \{p\}$. Thus, the proof amounts to the observation that the 1, 1 coefficient of any matrix $q \in H \backslash \{p(1)\}$ is one of $0, \frac{1}{4}, \frac{1}{2}$.

(iii) To establish that $(H, \underline{\underline{h}})$ is a generalized hexagon of order (2, 8) one only need show (see [5]) that $(H, \sim)$ is a distance-regular graph with intersection array (18, 16, 16; 1, 1, 9) in the terminology of [1]. From the preceding formulae, it is easily obtained that the stabilizer in D of $p(1)$ has

TABLE I

Images of σ_q for $q = p(1, \alpha, 1)$ on 40 Points

p_i	$\sigma_q p_1$	$\sigma_q p_2$
$p(i, \alpha, 1)$	$p(1, \alpha, 1)$	$p(3, -1)$
$p(i, -\alpha, -1)$	$p(3)$	$p(2, 1)$
$p(i, \alpha e_1, e_1)$	$p(3, -e_1 \alpha, 1)$	$p(2, e_1 \alpha, 1)$
$p(i, -\alpha e_1, -e_1)$	$p(3, -e_1 \alpha e_1, -e_1)$	$p(3, e_1 \alpha, -1)$
$p(i, -\alpha, 1)$	$p(1, -\alpha, 1)$	$p(3, -\alpha, 1)$
$p(i, \alpha, -1)$	$p(2)$	$p(2, \alpha, -1)$
$p(i, -\alpha e_1, e_1)$	$p(2, e_1 \alpha, -1)$	$p(1, e_1 \alpha e_1, e_1)$
$p(i, \alpha e_1, -e_1)$	$p(2, \alpha e_1, -e_1)$	$p(1, \alpha e_1, -e_1)$
$p(i, e_1 \alpha, 1)$	$p(1, -e_1 \alpha, 1)$	$p(2, \alpha e_1, e_1)$
$p(i, -e_1 \alpha, -1)$	$p(1, -e_1)$	$p(3, e_1 \alpha e_1, e_1)$
$p(i, e_1 \alpha e_1, e_1)$	$p(2, -\alpha e_1, e_1)$	$p(3, e_1)$
$p(i, -e_1 \alpha e_1, -e_1)$	$p(3, -e_1 \alpha, -1)$	$p(2, e_1)$
$p(i, -e_1 \alpha, 1)$	$p(1, e_1 \alpha, 1)$	$p(1, e_1 \alpha e_1, e_1)$
$p(i, e_1 \alpha, -1)$	$p(1, e_1)$	$p(1, -\alpha e_1, e_1)$
$p(i, -e_1 \alpha e_1, e_1)$	$p(3, -e_1 \alpha e_1, e_1)$	$p(3, \alpha e_1, -e_1)$
$p(i, e_1 \alpha e_1, -e_1)$	$p(2, -e_1 \alpha, 1)$	$p(2, e_1 \alpha e_1, -e_1)$
$p(i, (e_3 \alpha) e_1, e_1)$	$p(3, -(e_1 \alpha) e_6, e_6)$	$p(1, -(e_6 \alpha) e_4, -e_4)$
$p(i, -(e_3 \alpha) e_1, -e_1)$	$p(2, (e_7 \alpha) e_5, e_5)$	$p(1, -(e_7 \alpha) e_4, e_4)$
$p(i, -(e_3 \alpha) e_1, e_1)$	$p(2, (e_2 \alpha) e_6, e_6)$	$p(2, -(e_5 \alpha) e_2, e_2)$
$p(i, (e_3 \alpha) e_1, -e_1)$	$p(3, (e_1 \alpha) e_5, e_5)$	$p(3, (e_4 \alpha) e_2, e_2)$

orbits $\{p(2), p(3)\} \cup \{p(1, e) \mid e \in Q\}$, $\{p(i, e), p(1, (c\alpha)e, e) \mid e \in Q\}$, and $\{p(i, (c\alpha)e, e) \mid i = 2, 3; c, e \in Q\}$. Details are omitted, but it is noted that $\sigma_{p(1,\alpha,1)}\sigma_{p(1,d)}\sigma_{p(1,\alpha,1)}\sigma_{p(1,e)}\sigma_{p(1,\alpha,1)}$ fixes $p(1)$ for all $d, e \in Q \backslash \{\pm 1\}$. This implies that D is distance transitive on H and that $p, q \in H$ are of distance 1 (2, 3, resp.) iff $(p, q) = 0$ ($\frac{1}{2}, \frac{1}{4}$, resp.). The distance transitivity of D accounts for the distance regularity of $(H, \sim)$. It is now straightforward to compute the actual intersection array.

By [6], the generalized hexagon $(H, \underline{\underline{h}})$ is the unique one of order $(2, 8)$, i.e., the classical one associated with the group ${}^3D_4(2)$.

As for D, so far we have that D and ${}^3D_4(2)$ are subgroups of $\mathrm{Aut}(H, \sim)$ which are distance transitive. But $\sigma_{p(1)}$ fixes all vertices of H adjacent to $p(1)$, and leaves invariant all lines containing a point adjacent to $p(1)$ (cf. (40)), so it corresponds to the unique central involution of ${}^3D_4(2)$ associated with $p(1)$ (cf. [6, 9]). Therefore, D contains $\{\phi\sigma_{p(1)}\phi^{-1} \mid \phi \in D\}$ which by the transitivity of D is the set of all central involutions in ${}^3D_4(2)$. By simplicity of ${}^3D_4(2)$, these involutions generate all of ${}^3D_4(2)$, so that ${}^3D_4(2)$ is contained in D.

Standard permutation representation theoretic arguments yield that $\mathrm{Aut}(H, \sim)$ is a subgroup of $\mathrm{Aut}({}^3D_4(2))$. On the other hand, $(H, \sim)$ is isomorphic to the graph whose vertex set is the conjugacy class of central involutions and in which two vertices are adjacent whenever they commute. Therefore $\mathrm{Aut}(H, \sim) = \mathrm{Aut}({}^3D_4(2))$, up to isomorphism, and ${}^3D_4(2)$ is a normal subgroup of D.

Now $\tau = (235)(476)$, regarded as an element of $\mathscr{C}$ (cf. [13]), induces an automorphism $\hat{\tau}$ of $(H, \sim)$ that fixes $p(1)$ and three of the nine lines through $p(1)$. Since the stabilizer of $p(1)$ in ${}^3D_4(2)$ induces $PSL_2(8)$ in its natural action on these 9 lines, $\hat{\tau}$ is not contained in ${}^3D_4(2)$. This yields that $|D|$ is a multiple of $|{}^3D_4(2)| \cdot 3$. But Aut ${}^3D_4(2)$ is well known to be of order $3 \cdot |{}^3D_4(2)|$. Hence, $D = \mathrm{Aut}(H, \sim) = \langle {}^3D_4(2), \hat{\tau} \rangle$ of order $2^{12} \cdot 3^5 \cdot 7^2 \cdot 13$. This ends the proof of the theorem. ∎

4. Two More Generalized Hexagons of Order 2

Let $Q_1 = \{1, -1, e_1, -e_1\}$ and $Q_2 = \{1, -1\}$. For $j = 1, 2$, write

$$L_1^j = \{p(i, e) \mid i = 1, 2, 3; e \in Q_j\},$$

$$L_2^j = \{p(i, x, e) \mid i = 1, 2, 3; e \in Q_j; x \in (Q_j \alpha) e\},$$

and define $H_j = L_0 \cup L_1^j \cup L_2^j$. Thus H_1 is the set of fixed points of $\hat{\tau}$ in H (where τ is as in the previous section) and H_2 is the set of fixed points of $\hat{\mathscr{C}}_0$ in H.

It is straightforward to see that the subgraph $(H_j, \sim)$ of $(H, \sim)$ is the point graph of a generalized hexagon of order $(2, 3 \dot{-} j)$. In fact, $(H_1, \sim)$ is the dual of the classical generalized hexagon associated with $U_3(3)$ (see [4]) and $(H_2, \sim)$ is the unique generalized hexagon associated with $PSL_2(7)$ (whose line graph is the Heawood graph on 14 points).

As $H_1 \subseteq \mathbb{J}_3(\mathbb{R}(\alpha, e_1))$, the orthogonality preserving map $v \to vv^*$ (where v^* is the usual conjugate transpose of v) from $(\mathbb{R}(\alpha, e_1))^3$ to $\mathbb{J}_3(\mathbb{R}(\alpha, e_1))$ exhibits this presentation of the dual classical generalized hexagon as a well known one on the "root system" of the quaternionic reflection group $W(Q)$ studied in [3] (note that $\mathbb{R}(\alpha, e_1)$ is indeed a quaternion division algebra). Similarly, $H_2 \subseteq \mathbb{J}_3(\mathbb{R}(\alpha))$ corresponds to the root system of the complex reflection group $W(J_3(4))$ studied in [2].

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