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SAMPLE PATH PROPERTIES OF STABLE PROCESSES

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CONTENTS

CHAPTER 1

INTRODUCTION

1.1. SOME TYPICAL PROBLEMS

Many books about probability and statistics mention the weak and strong laws of large numbers for samples from distributions with finite expectation only. However, both laws also hold for distributions with infinite expectation and then the sample average tends to infinity with increasing sample size. One would expect a gradual increase of the average with the size of the sample. In general, however, this proves to be wrong as can be seen in plots of averages of simulated samples. See MIJNHEER (1968). These plots show that the average takes a large jump upwards from time to time and decreases between the jumps. These jumps are due to large observations. This surprising behavior of the sample average constituted a starting point of the present study.

The first problem that arises in this connection is that, in general, there is no simple expression for the distribution function of the sum of two independent random variables. Only for *stable* random variables do we know the distribution of the sum of an arbitrary number of independent and identically distributed random variables. Therefore we shall mainly consider stable random variables. In some cases we also consider independent and identically distributed random variables with the property that suitably normalized sums of these variables have a limiting distribution, which is then necessarily stable.

In the remainder of this section we describe a few typical results concerning the behavior of the sample average. Though these results hold for a wide class of distributions, including certain stable distributions, we assume here, for the sake of simplicity, that $X_1, X_2, \ldots$ are independent and identically distributed random variables with common distribution function $F(x) = 1 - x^{-\alpha}$ for $x > 1$ and $0 < \alpha < 2$. The moments of these random variables satisfy

$$EX_i^p < \infty \qquad \text{for } p < \alpha$$

and

$$EX_i^p = \infty \qquad \text{for } p \geq \alpha.$$

We distinguish three cases, viz. $0 < \alpha < 1$, $\alpha = 1$ and $1 < \alpha < 2$. *Here we give only a rough description of the behavior of the sample average. The exact formulation will be given in the theorems and remarks in the following chapters. There one can also find the values of the constants* $c_1(\alpha)$ *and* $c_2(\alpha)$.

<u>The case $0 < \alpha < 1$</u>

Theorem 6.2.1 implies that

$$\liminf_{n\to\infty} \frac{X_1+\ldots+X_n}{n^{1/\alpha}} (2\log\log n)^{(1-\alpha)/\alpha} = c_1(\alpha) \qquad \text{a.s.}\ .$$

Roughly speaking this means that the sample average tends to infinity at least as fast as

$$c_1(\alpha)n^{(1-\alpha)/\alpha}(2\log\log n)^{-(1-\alpha)/\alpha},$$

while it approaches this lower bound infinitely often. The results in the theorems 8.1.1 and 10.2.1 show that, with probability 1,

$$\limsup_{n\to\infty} \frac{X_1+\ldots+X_n}{n^{1/\alpha}(\log n)^{(1+\varepsilon)/\alpha}} = \begin{cases} 0 & \text{for } \varepsilon > 0 \\ \infty & \text{for } \varepsilon = 0 \end{cases},$$

which implies that the sample average will exceed $n^{(1-\alpha)/\alpha}(\log n)^{1/\alpha}$ infinitely often. The influence of $\max(X_1,\ldots,X_n)$ on the partial sums $X_1+\ldots+X_n$ is studied by DARLING (1952). It appears that the maximal term is the dominating one in the partial sum. See also theorem 10.2.1.

<u>The case $\alpha = 1$</u>

For this case we find (cf. theorems 6.3.1, 8.1.1 and 10.2.1)

$$\liminf_{n\to\infty} \frac{X_1+\ldots+X_n}{(2/\pi)n\log n} = 1$$

and

$$\limsup_{n\to\infty} \frac{X_1+\ldots+X_n}{(2/\pi)n(\log n)^{1+\varepsilon}} = \begin{cases} 0 & \text{for } \varepsilon > 0 \\ \infty & \text{for } \varepsilon = 0 \end{cases}$$

with probability 1.

The case $1 < \alpha < 2$

Now the random variables have finite expectation. By the law of large numbers, the sample average converges with probability 1 to EX_1. Because the variance is infinite the classical law of the iterated logarithm does not hold. However, as a consequence of theorem 6.4.1 we have

$$\liminf_{n\to\infty} \frac{X_1+\ldots+X_n - nEX_1}{n^{1/\alpha}(2\log\log n)^{(\alpha-1)/\alpha}} = c_2(\alpha) \qquad \text{a.s.}$$

and by theorems 8.1.1 and 10.2.1 it follows that, with probability 1,

$$\limsup_{n\to\infty} \frac{X_1+\ldots+X_n - nEX_1}{n^{1/\alpha}(\log n)^{(1+\varepsilon)/\alpha}} = \begin{cases} 0 & \text{for } \varepsilon > 0 \\ \infty & \text{for } \varepsilon = 0 \end{cases} .$$

1.2. ORGANIZATION

As explained in section 1.1 stable distributions play an important role in solving our original problem. The definition and basic properties of these distributions will be given in chapter 2. The general theory of stable distributions was initiated by LÉVY. For examples and applications we refer to FELLER (1971). In other cases too, we shall refer to this or other recent books, rather than to the original literature. For example, for the proof of theorem 2.1.2 we refer to BILLINGSLEY (1968), even though this theorem was already well-known long before 1968.

The explicit form of a stable distribution function is known only in a few special cases. However, expansions for the tails are known in general. These expansions are given in theorem 2.1.7. Sometimes, we can give an asymptotic expression for one tail of the distribution function of a (non-normal) stable random variable in terms of the tail of a standard normal distribution function. The corresponding random variables are called *completely asymmetric*. This relation between the tails of the distribution functions will be applied in many places.

There exists an extensive literature on stable random variables and stable processes. Many authors consider only special cases. Thus, there are papers where the stable random variables are assumed to be either symmetrically distributed or positive. Other authors exclude the case $\alpha = 1$, because in this case we have to take a shift into consideration. In chapters 4,5,6 and 8 we shall extend some theorems which are known only for such special cases. In chapter 4 for example, the theorems in the first two sections are known. In view of the techniques applied, it has been conjectured that the theorems in the last two sections would also hold. We prove that this is indeed the case.

As can be seen in the table of contents, many chapters are divided in four sections, viz. called: the case $\alpha = 2$, the case $0 < \alpha < 1$, the case $\alpha = 1$ and the case $1 < \alpha < 2$. Here α denotes the so-called characteristic exponent of the stable distribution. The reason why we have to consider these cases separately is that the left tail of the distribution function of a completely asymmetric stable random variable differs in these four cases.

In chapter 3 we shall discuss some properties of the Wiener process and other stable processes. In sections 3.3 - 3.6 we prove some technical lemmas for the previously mentioned four cases. Section 3.3 deals with the Wiener process. The lemmas for this case were known before. However, in the proofs quantities were used which are not defined for other stable processes. Here we prove these lemmas in such a way that the proofs for the other stable processes follow the same pattern.

In chapters 4 and 5 we establish generalized laws of the iterated logarithm for completely asymmetric stable processes. The cases $\alpha = 2$ and $0 < \alpha < 1$ are already proved in the literature. The case $1 \leq \alpha < 2$ for small times can be proved by using the lemmas of sections 3.5 and 3.6 For a Wiener process the theorem for large times easily follows from the theorem for small times. For other stable processes separate proofs are necessary for small times and for large times. These proofs are very similar however. The lemmas in chapter 3 are formulated in such a way that they can be applied directly in the proofs for small times. For that reason the theorems for small times are considered first.

In chapter 6 we prove similar theorems for partial sums of independent and identically distributed completely asymmetric stable random variables. The proofs are partly derived from the theorems of chapter 5.

The law of the iterated logarithm describes the local behavior of the sample paths near a fixed point. In chapter 7 we establish Hölder-type theorems for the Wiener process and completely asymmetric stable processes.

Up to this point, we essentially considered only completely asymmetric stable processes or completely asymmetric stable random variables. In chapter 8 we prove laws of the iterated logarithm for arbitrary stable processes and random variables. In the special case of completely asymmetric distributions these theorems supplement the results obtained in previous chapters.

Chapter 9 deals with functional laws of the iterated logarithm. In section 9.1 we summarize some results for the Wiener process. In the other sections we derive similar theorems for completely asymmetric stable processes.

As explained in the introduction our starting point was the behavior of the sample average for random variables with infinite moments. Up to chapter 9 we mainly considered stable processes or stable random variables. In section 10.1 we quote results for non-stable random variables with asymptotically normal partial sums. Finally, in section 10.2 we discuss non-stable random variables with asymptotically stable partial sums.

1.3. ABBREVIATIONS AND CONVENTIONS

Here we explain some conventions and notation which are used throughout this monograph.

Asymptotics

$f(t) = \mathcal{O}(g(t))$ for $t \to t_0$, if $|f(t)g^{-1}(t)|$ is bounded in some neighborhood of t_0;

$f(t) = o(g(t))$ for $t \to t_0$, if $\lim_{t\to t_0} f(t)g^{-1}(t) = 0$;

$f(t) \sim g(t)$ for $t \to t_0$, if $\lim_{t\to t_0} f(t)g^{-1}(t) = 1$.

Probability

$(\Omega,\mathcal{F},\mathcal{P})$ denotes a probability triple. Ω is the sample space, $\mathcal{F}$ is a σ-field of subsets of Ω and $\mathcal{P}$ is a probability measure on $\mathcal{F}$. An element of Ω is denoted by ω. Random variables are denoted by capitals. $X \overset{d}{=} Y$ means: X and Y have the same distribution.

Functions

Let f be a function on the real line, then

$$f(t_0+) = \lim_{t\downarrow t_0} f(t) \quad \text{and} \quad f(t_0-) = \lim_{t\uparrow t_0} f(t).$$

The functions f^+ and f^- are defined by

$$f^+(t) = \max(0,f(t)) \qquad \text{for all real } t$$

$$f^-(t) = \max(0,-f(t)) \qquad \text{for all real } t.$$

Abbreviations

a.s.	almost surely
iff	if and only if
i.i.d.	independent and identically distributed
i.o.	infinitely often
L.I.L.	law of iterated logarithm
r.v.	random variable
w.p.1	with probability 1
□	end of proof.

1.4. SOME PROBABILITY THEORY

Many theorems in the following chapters are of the type $P[A_n \text{ i.o.}] = 0$ or 1 according as some conditions are fullfilled or not. The usual way to prove theorems of this type is to apply the Borel-Cantelli lemma (cf. BREIMAN (1968a)).

LEMMA 1.4.1. *Let* $A_1, A_2, \ldots$ *be a sequence of events.*

a. If $\sum P[A_k] < \infty$ *then* $P[A_k \text{ i.o.}] = 0$
b. If the events $\{A_k\}$ *are independent and if* $\sum P[A_k] = \infty$ *then* $P[A_k \text{ i.o.}] = 1$.

Application of this lemma is made difficult by the assumption of independence in part b. One usually constructs a new sequence of independent events out of the given sequence and applies part b to this new sequence. We shall proceed in a different way and use the following extension of part b. The proof of this extension can be found in SPITZER (1964).

LEMMA 1.4.2. *If* $\sum P[A_k] = \infty$ *and*

$$\liminf_{n\to\infty} \frac{\sum_{j=1}^{n} \sum_{k=1}^{n} P[A_j \cap A_k]}{\left[\sum_{k=1}^{n} P[A_k]\right]^2} \leq c,$$

then $P[A_k \text{ i.o.}] \geq c^{-1}$.

The following result is well-known (cf. BREIMAN (1968a)).

LEMMA 1.4.3. *Let* $S_1, S_2, \ldots$ *be successive sums of i.i.d. random variables, such that* $\max_{1 \leq j < n} P[S_n - S_j < 0] = c < 1$. *Then*

$$P[\max_{1 \leq j \leq n} S_j > x] \leq (1-c)^{-1} P[S_n > x].$$

In sections 3.3 through 3.6 we prove similar lemmas for stable processes.

Let (Ω, F, P) be a probability triple and let $\{X_t\}$ be a collection of r.v.'s indexed by a parameter t in some interval $I \subset \mathbb{R}$. We call this collection a *stochastic process* and write $\{X(t) : t \in I\}$. $F(X(s), 0 \leq s \leq t)$ is the σ-field spanned by $X(s)$ for $0 \leq s \leq t$. A process $\{X(t) : 0 \leq t < \infty\}$ has *independent increments* if for any $t > 0$, $F(X(t+s)-X(t), s > 0)$ is independent of $F(X(s), 0 \leq s \leq t)$. We say that the process has *stationary increments* if the distribution of $X(t+s)-X(t)$, $s \geq 0$, does not depend on t. In the study of sample path properties of stochastic processes (that is the study of $X(\cdot, \omega)$) we need the following concepts.

DEFINITION 1.4.1. A non-negative random variable T will be called a *stopping time* if for every $t \geq 0$,

$$\{T \leq t\} \in F(X(s), 0 \leq s \leq t).$$

Intuitively, we can say that a stopping time T only depends on the stochastic process up to time T. A process satisfies the *strong Markov property* if the process starts afresh at any stopping time T. To be more precise, let $F(X(s), s \leq T)$ be the σ-field of events $B \in F$ such that $B \cap \{T \leq t\} \in F(X(s), s \leq t)$ for all $t \geq 0$. Then the strong Markov property holds if, for any stopping time T, the process $\{X_1(t) : 0 \leq t < \infty\}$, defined by

$$X_1(t) = X(T+t) - X(T),$$

has the same distribution as $\{X(t) : 0 \leq t < \infty\}$ and is independent of $F(X(s), s \leq T)$.

1.5. SOME REAL ANALYSIS

A positive function L, defined on $[x_0,\infty)$ (where x_0 is some positive real number), is said to be *slowly varying* at infinity if, for all $t > 0$,

$$\lim_{x\to\infty} L(tx)\, L^{-1}(x) = 1.$$

An exposition of the theory of slowly varying function can be found in FELLER (1971). The next theorem gives a representation of slowly varying functions. See for proof FELLER (1971).

THEOREM 1.5.1. *A function* L *varies slowly at infinity iff it is of the form*

$$L(x) = a(x)\, \exp\left(\int_1^x y^{-1}\, \varepsilon(y)\, dy\right),$$

where $\varepsilon(x) \to 0$ *and* $a(x) \to a \in (0,\infty)$ *as* $x \to \infty$.

EXAMPLES

a. $L(x) = (\log x)^p$ for $x > 1$ and $p > 0$;

b. $L(x) = e^{(\log x)^p}$ for $x > 1$ and $0 < p < 1$;

c. $L(x) = e^{(\log\log x)^{-1}\log x}$ for $x > e$.

Let f be an arbitrary finite real-valued function on some interval $[a,b]$. Let $P = \{a = x_0 < x_1 < \ldots < x_n = b\}$ be a partition of $[a,b]$. We define

$$S_P^+ f = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))^+$$

and

$$S_P^- f = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))^-.$$

Then

$$S_P^+ f - S_P^- f = f(b) - f(a).$$

We define the *positive variation* of f over $[a,b]$ (resp. *negative variation*) by $V^+f(b) = \sup_P S_P^+f$ (resp. $V^-f(b) = \sup_P S_P^-(f)$). Similarly, for any $t \in [a,b]$, $V^+f(t)$ (resp. $V^-f(t)$) will denote the positive (resp. negative) variation of f over $[a,t]$. If f is of bounded variation over $[a,b]$ we have

$$f(t) - f(a) = V^+f(t) - V^-(t) \qquad \text{for all } t \in [a,b].$$

To conclude we present a short survey of the main properties of non-decreasing functions. For the proofs we refer to the book of SAKS (1964). Let λ be the Lebesgue measure on $[0,1]$.

THEOREM 1.5.2. *Let* f *be a finite non-decreasing function on* $[0,1]$. *Then*

a. f *may be represented uniquely as* f_a+f_s, *where* f_a *is absolutely continuous and* f_s *is singular.*
b. *the pointwise derivative* $\mathring{f}$ *of* f *exists almost everywhere and is a version of the Radon-Nikodym derivative of* f_a *with respect to* λ.

Note that this implies $\mathring{f} = \mathring{f}_a$.

REMARK 1.5.1. A finite singular non-decreasing function f on $[0,1]$ has the following property. For all $\varepsilon > 0$, there exist a finite number of disjoint intervals $(x_i,y_i]$, $i=1,\ldots,n$, such that

1. $\sum_{i=1}^n \lambda(x_i,y_i) < \varepsilon$
2. f increases on $\bigcup_{i=1}^n (x_i,y_i]$ by less than ε.

Moreover, we can find, for all $\varepsilon > 0$, a number $m = m_f$ and a set B_f, which is a union of intervals of the form $(jm^{-1},(j+1)m^{-1})$, such that $\lambda(\overline{B_f}) < \varepsilon$ and f increases less than ε on B_f.

Let f be an arbitrary finite real-valued function on $[0,1]$. For every positive number m we define the function $\pi_m f$ by

$$\pi_m f(jm^{-1}) = f(jm^{-1}) \qquad \text{for } j=0,\ldots,m$$
$$\text{and linear on } [jm^{-1},(j+1)m^{-1}] \qquad \text{for } j=0,\ldots,m-1.$$

The following lemma is an immediate consequence of theorem 1.5.2.b (cf. SAKS (1964)). It may be found in WICHURA (1973), for $m = 2^n$ and $n \to \infty$, with a proof based on the martingale convergence theorem.

LEMMA 1.5.1. *Let* f *be a finite non-decreasing function on* $[0,1]$. *Then the pointwise derivative of* $\pi_m f$ *converges almost everywhere to* $\mathring{f} = \mathring{f}_a$ *for* $m \to \infty$.

CHAPTER 2

STABLE DISTRIBUTIONS

2.1. GENERAL THEORY

In this chapter we summarize the well-known theory of stable distributions. The complete theory of stable distributions has first been given in GNEDENKO-KOLMOGOROV (1954). Most results can also be found in general books on probability, for example BREIMAN (1968a), LUKACS (1970) and especially FELLER (1971) and IBRAGIMOV-LINNIK (1971). For further details we refer to these books.

DEFINITION 2.1.1. The distribution function F is called *stable* if for each n, and i.i.d. random variables $X_1,\dots,X_n$ with common distribution function F, there exist constants $a_n > 0$ and b_n such that the random variable

$$(2.1.1) \qquad a_n^{-1}(X_1+\dots+X_n - b_n)$$

has distribution function F.

THEOREM 2.1.1. *For every stable distribution there exists a unique constant* $\alpha \in (0,2]$ *such that* $a_n = n^{1/\alpha}$.

PROOF. See FELLER (1971). □

The constant α is called the *characteristic exponent* or *index* of the stable distribution. If (2.1.1) holds with $b_n = 0$ the distribution is called *strictly stable.*

THEOREM 2.1.2. *In order that a distribution function* F *be stable, it is necessary and sufficient that its characteristic function is given by*

$$(2.1.2) \qquad \log f(t) = \begin{cases} i\gamma t - c|t|^{\alpha}\{1 - i\beta \operatorname{sign}(t) \tan(\pi\alpha/2)\} & \text{if } \alpha \neq 1 \\ i\gamma t - c|t| - i\beta(2/\pi)ct \log |t| & \text{if } \alpha = 1, \end{cases}$$

where α, β, γ *and* c *are real constants with* $c \geq 0$, $0 < \alpha \leq 2$ *and* $|\beta| \leq 1$.

PROOF. See GNEDENKO-KOLMOGOROV (1954). □

Here α is the characteristic exponent.

Because γ and c merely determine location and scale we shall consider only stable distributions with $\gamma = 0$ and $c = 1$. Note that by doing so we are excluding the degenerate case $c = 0$. From the representation of the characteristic function in theorem 2.1.2 it follows that the distribution function F may be differentiated an arbitrary number of times. Especially it follows that each stable distribution has a continuous density. We shall write $F(.;\alpha,\beta)$ resp. $p(.;\alpha,\beta)$ for the distribution function resp. density of a stable law with parameters α, β, $\gamma = 0$ and $c = 1$. Moreover, the choice $\gamma = 0$ implies that we consider stable random variables with expectation equal to zero (when it is finite). In case $\beta = 0$ the distributions are symmetric. Distributions with $|\beta| = 1$ are commonly called *completely asymmetric stable distributions*. In case $0 < \alpha < 1$ the stable laws with $|\beta| = 1$ are one-sided, i.e. their support is $[0,\infty)$ in case $\beta = 1$ and $(-\infty,0]$ in case $\beta = -1$. Using theorem 2.1.2 one easily proves the following theorems.

THEOREM 2.1.3. *Let* $X_1,\ldots,X_n$ *be i.i.d. with common distribution function* $F(.;\alpha,\beta)$. *Then*

$$X_1+\ldots+X_n \overset{d}{=} n^{1/\alpha}X_1 \qquad \text{if } \alpha \neq 1$$

and

$$X_1+\ldots+X_n-(2/\pi)\beta n\log n \overset{d}{=} n\, X_1 \qquad \text{if } \alpha = 1.$$

Theorem 2.1.3 implies that the norming constant b_n is equal to 0 for $\alpha \neq 1$ and $(2/\pi)\beta n \log n$ for $\alpha = 1$. Because b_n may be unequal to zero for $\alpha = 1$ the proofs of many theorems for this case are more delicate than for $\alpha \neq 1$. For that reason many authors do not give a detailed investigation of the case $\alpha = 1$.

THEOREM 2.1.4. *Let* X_1 *and* X_2 *be i.i.d. with common distribution function* $F(.;\alpha,\beta)$. *Then for arbitrary positive* s *and* t

$$s^{1/\alpha}X_1 + t^{1/\alpha}X_2 \overset{d}{=} (s+t)^{1/\alpha}X_1 \qquad \text{if } \alpha \neq 1$$

and

$$s\,X_1 + t\,X_2 \overset{d}{=} (s+t)X_1 + (2/\pi)\beta\{(s+t)\log(s+t)-s\log s - t\log t\} \quad \text{if } \alpha = 1.$$

THEOREM 2.1.5. *Let* X *be a random variable with distribution function* $F(.;\alpha,\beta)$. *Then there exist i.i.d. random variables* Y_1 *and* Y_2 *with common distribution function* $F(.;\alpha,1)$ *such that:*
in case $\alpha \neq 1$

$$X \overset{d}{=} pY_1 - qY_2,$$

where $p,q > 0$, $p^\alpha+q^\alpha = 1$ and $p^\alpha-q^\alpha = \beta$;

and in case $\alpha = 1$

$$X \overset{d}{=} pY_1+(2/\pi)p\log p - qY_2-(2/\pi)q\log q,$$

where $p,q > 0$, $p+q = 1$ and $p-q = \beta$.

EXAMPLES. There are three cases where $p(.;\alpha,\beta)$ is known explicitly.

1. Normal distribution $f(t) = e^{-t^2}$ $p(x;2,0) = 2\pi^{-\frac{1}{2}}e^{-x^2}$
2. Cauchy distribution $f(t) = e^{-|t|}$ $p(x;1,0) = \pi^{-1}(x^2+1)^{-1}$.
3. $f(t) = e^{-\sqrt{2it}}$ $p(x;\frac{1}{2},1) = (2\pi x^3)^{-\frac{1}{2}}e^{-\frac{1}{2x}}$ for $x > 0$.

Let X be a r.v. with distribution function $F(.;\frac{1}{2},1)$ and let U be a r.v. with the standard normal distribution. Then there exists the following relation between these random variables.

(2.1.3) $$X \overset{d}{=} U^{-2}.$$

ZOLOTAREV (1966) has given integral representations of distribution functions of stable laws. In the following theorem we give the expansions of the densities in the tails of the stable distributions. Because $1-F(-x;\alpha,\beta) = F(x;\alpha,-\beta)$ or $p(-x;\alpha,\beta) = p(x;\alpha,-\beta)$ it is no restriction to assume $\beta \geq 0$. A complete summary of the asymptotic formulas for stable densities has first been given by SKOROHOD (1961). The proofs are also given in the book of IBRAGIMOV and LINNIK (1971). Table 1 and theorem 2.1.6 give the expansions for both tails.

TABLE 1

	$\beta = 1$		$0 \le \beta < 1$	
$0 < \alpha < 1$	$x \downarrow 0$ IV	$x \to \infty$ I	$x \to -\infty$ I^*	$x \to \infty$ I
$\alpha = 1$	$x \to -\infty$ V	$x \to \infty$ II	$x \to -\infty$ II^*	$x \to \infty$ II
$1 < \alpha < 2$	$x \to -\infty$ VI	$x \to \infty$ III	$x \to -\infty$ III^*	$x \to \infty$ III
$\alpha = 2$	$x \to -\infty$	VII		$x \to \infty$

THEOREM 2.1.6.

I. $$p(x;\alpha,\beta) = \frac{1}{\pi x}\sum_{n=1}^{\infty} A_n\, x^{-\alpha n} \qquad \textit{for } x > 0,$$

where

$$(2.1.4) \qquad A_n = \frac{(-1)^{n+1}\Gamma(n\alpha+1)}{n!}\,(1+\beta^2\tan^2(\pi\alpha/2))^{n/2}\sin n[(\pi\alpha/2) + \arctan(\beta\tan(\pi\alpha/2))].$$

II. $$p(x+(2/\pi)\beta\log x;1,\beta) = \frac{1}{\pi x}\sum_{n=1}^{N} A_n\, x^{-n} + O(x^{-N-2}) \qquad \textit{for } x \to \infty,$$

where

$$(2.1.5) \qquad A_n = \frac{1}{n!}\,\mathrm{Im}\int_0^{\infty} e^{-t}t^n(i+i\beta-(2/\pi)\beta\log t)^n dt.$$

III. $$p(x;\alpha,\beta) = \frac{1}{\pi x}\sum_{n=1}^{N} A_n\, x^{-\alpha n} + O(x^{-(N+1)\alpha-1}) \qquad \textit{for } x \to \infty,$$

where A_n *is given by (2.1.4).*

IV. $$p(x;\alpha,1) = (2/\alpha)^{\frac{1}{2}}(2\pi)^{-\frac{1}{2}}(2B(\alpha))^{\frac{1}{2}}(\lambda(\alpha)/2)^{\frac{1}{2}}x^{-1-\lambda(\alpha)/2}\cdot e^{-B(\alpha)x^{-\lambda(\alpha)}}[1 + O(x^{\frac{\lambda(\alpha)}{2}-\varepsilon})] \qquad \textit{for } x \downarrow 0,$$

where

$$(2.1.6) \qquad \lambda(\alpha) = \alpha(1-\alpha)^{-1}$$

and

$$(2.1.7) \qquad B(\alpha) = (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}(\cos(\pi\alpha/2))^{-\frac{1}{1-\alpha}}.$$

V. $p(-x;1,1) = 2^{\frac{1}{2}}(\pi/4)(\frac{2}{\sqrt{\pi e}})\exp\{(\pi x/4)-(2/\pi e)\, e^{\frac{\pi}{2}x}\}\cdot$

$\cdot\{1 + O(e^{-\frac{\pi}{4}x(1-\varepsilon)})\}$ *for* $x \to \infty$.

VI. $p(-x;\alpha,1) = (2\alpha)^{\frac{1}{2}}(2\pi)^{-\frac{1}{2}}(2B(\alpha))^{\frac{1}{2}}(\lambda(\alpha)/2)^{\frac{1}{2}}x^{-1-\lambda(\alpha)/2}\cdot$

$\cdot e^{-B(\alpha)x^{-\lambda(\alpha)}}[1 + O(x^{\frac{\lambda(\alpha)}{2}+\varepsilon})]$ *for* $x \to \infty$,

where $\lambda(\alpha)$ *is defined by (2.1.6) and*

(2.1.8) $B(\alpha) = (\alpha-1)\alpha^{\frac{\alpha}{\alpha-1}}\,|\cos(\pi\alpha/2)|^{\frac{1}{\alpha-1}}$.

VII. $p(x;2,0) = \frac{1}{2\sqrt{\pi}}\, e^{-x^2/4}$ *for all* x.

The formulas in the cases marked by asterisks can be derived from the corresponding formulas without asterisks by the substitution of $-x$ *for* x *and* $-\beta$ *for* β.

REMARK 2.1.1. By using Stirling's formula one easily finds a convergent majorant of the series in theorem 2.1.6 part I. Note that the series $\frac{1}{\pi x}\sum_{n=1}^{\infty} A_n\, x^{-\alpha n}$ in part III of theorem 2.1.6 is divergent. FELLER (1971) and BERGSTRÖM (1952) have given a convergent series expansion for this part of theorem 2.1.6. See for example FELLER (1971). IBRAGIMOV and LINNIK (1971) give more terms in the asymptotic expansions of the left tail of the completely asymmetric stable distributions (the cases IV, V and VI).

From the expansions in theorem 2.1.6 we can deduce the following estimates for the tails of the distribution function. Table 1 and theorem 2.1.7 give a summary of the expansions of the tails of $F(.;\alpha,\beta)$.

THEOREM 2.1.7. *Let* U *be the standard normal random variable.*

I, II *and* III.

$$1 - F(x;\alpha,\beta) \sim \frac{A_1}{\pi\alpha}\, x^{-\alpha} \qquad \textit{for } x \to \infty,$$

where A_1 *is given by (2.1.4) if* $\alpha \neq 1$ *and by (2.1.5) if* $\alpha = 1$.

IV. $F(x;\alpha,1) \sim (2/\alpha)^{\frac{1}{2}} P[U \geq (2B(\alpha))^{\frac{1}{2}} x^{-\frac{\alpha}{2(1-\alpha)}}]$ *for* $x \downarrow 0$,

where $B(\alpha)$ *is given by (2.1.7).*

V. $F(x;1,1) \sim 2^{\frac{1}{2}}\ P[U \geq 2(\pi e)^{-\frac{1}{2}}\ e^{-\pi x/4}]$ *for* $x \to -\infty$.

VI. $F(x;\alpha,1) \sim (2\alpha)^{\frac{1}{2}}\ P[U \geq (2B(\alpha))^{\frac{1}{2}}(-x)^{\frac{\alpha}{2(\alpha-1)}}]$ *for* $x \to -\infty$.

VII. $F(x;2,0) = P[2^{\frac{1}{2}}U \geq -x] \sim \pi^{-\frac{1}{2}}(-x)^{-1}e^{-x^2/4}$ *for* $x \to -\infty$.

The formulas in the cases marked by asterisks can be derived as in theorem 2.1.6.

PROOF. The parts I up to VI easily follow from theorem 2.1.6 by straightforward integration. Part VII is the well-known estimate for the tail of the standard normal distribution function. A proof of this estimate is given in FELLER (1957). □

With these estimates the following lemma is easily proved.

LEMMA 2.1.1. *Let* X *be a random variable with distribution function* $F(.;\alpha,\beta)$. *Then*

$$E|X|^a < \infty \qquad \textit{for all } a < \alpha$$

and

$$E|X|^a = \infty \qquad \textit{for all } a \geq \alpha.$$

We shall make frequent use of the following property of the tail of the standard normal distribution function.

LEMMA 2.1.2. *Let* U *be a standard normal random variable. Then for all* a

$$P[U \geq x + a/x] \sim e^{-a}\ P[U \geq x] \qquad \textit{for } x \to \infty.$$

2.2. DOMAINS OF ATTRACTION

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with common distribution function F.

DEFINITION 2.2.1. The distribution function F belongs to the *domain of attraction* of a non-degenerate distribution function G if there exist norming constants $a_n > 0$, b_n such that the distribution of $a_n^{-1}(X_1+\ldots+X_n - b_n)$ converges weakly to G.

We say a random variable belongs to the domain of attraction of a non-degenerate distribution G if its distribution function does.

THEOREM 2.2.1. *Only stable distribution functions have non-empty domains of attraction.*

PROOF. See IBRAGIMOV and LINNIK (1971). □

NOTATION. By appropriate choice of the norming constants a_n and b_n we may consider the stable distributions with $\gamma = 0$ and $c = 1$ only. We write F (or X) $\in \mathcal{D}(\alpha,\beta)$.

The following criterion can be used for determining whether a distribution function F is in the domain of attraction of a stable law.

THEOREM 2.2.2. F $\in \mathcal{D}(\alpha,\beta)$ *iff*
either $\alpha = 2$ *and*

$$\int_{|y|\le x} y^2 dF(y) \qquad \textit{is slowly varying at infinity}$$

or $0 < \alpha < 2$ *and both*

(i) $x^\alpha\,[1 - F(x)+F(-x)] = L(x)$ *with* $L(x)$ *slowly varying at infinity*

(ii) $\dfrac{F(-x)}{1 - F(x)+F(-x)} \to \dfrac{1-\beta}{2}$ *as* $x \to \infty$.

PROOF. See IBRAGIMOV and LINNIK (1971). □

Let $F \in \mathcal{D}(\alpha,\beta)$. Then a_n must satisfy one of the following conditions. In case $\alpha = 2$

$$(2.2.1) \qquad \frac{n \int_{|x|\le a_n} x^2 dF(x)}{a_n^2} \longrightarrow \frac{1}{2},$$

in case $0 < \alpha < 1$

$$(2.2.2) \qquad \frac{n\ L(a_n)}{a_n^\alpha} \longrightarrow \Gamma(1-\alpha)\ \cos(\pi\alpha/2),$$

in case $\alpha = 1$

$$(2.2.3) \qquad \frac{n\ L(a_n)}{a_n} \longrightarrow \frac{2}{\pi}$$

and in case $1 < \alpha < 2$

$$(2.2.4) \qquad \frac{n\ L(a_n)}{a_n^\alpha} \longrightarrow \frac{\Gamma(2-\alpha)}{\alpha-1}\ |\cos(\tfrac{\pi\alpha}{2})|.$$

The other norming constant b_n may be chosen as follows:

$$b_n = \begin{cases} 0 & \text{for } 0 < \alpha < 1 \\ n\ a_n \int_{-\infty}^{+\infty} \sin(x/a_n)\ dF(x) & \text{for } \alpha = 1 \\ n \int_{-\infty}^{+\infty} x\ dF(x) & \text{for } 1 < \alpha < 2. \end{cases}$$

In all cases it follows that $a_n = n^{1/\alpha}h(n)$, where h is slowly varying at infinity.

DEFINITION 2.2.2. A distribution function F (or a r.v. X with distribution function F) belongs to the *domain of normal attraction* of a stable law with characteristic exponent α $(0 < \alpha \le 2)$ if it belongs to its domain of attraction with $a_n = n^{1/\alpha}h(n)$ where h has a non-zero finite limit for $n \to \infty$.

NOTATION. F (or X) $\in \mathcal{D}_N(\alpha,\beta)$.

REMARK 2.2.1. One can give necessary and sufficient conditions in terms of characteristic functions in order that a random variable belongs to the domain of attraction of a stable law. See BELKIN (1968) and IBRAGIMOV and LINNIK (1971).

REMARK 2.2.2. Let $X_1, X_2, \ldots$ be i.i.d. $\in \mathcal{D}_N(\alpha,\beta)$ with $\alpha \neq 1,2$. CRAMÉR (1963) has shown that, under some restrictions on the tails of the distribution function of X_1,

$$|P[a_n^{-1}(X_1+\ldots+X_n-b_n) \le x] - F(x;\alpha,\beta)| = \mathcal{O}(n^{-1/\alpha}) \qquad \text{for } n \to \infty$$

uniformly in x.

REMARK 2.2.3. Let $X_1, X_2, \ldots$ be positive i.i.d. random variables with

$$(2.2.5) \qquad P[X_1 \ge x] = L(x)x^{-\alpha} \qquad \text{for } x \ge x_0 > 0 \text{ and } \alpha \neq 1,2,$$

where L is a continuous slowly varying function. By theorem 2.2.2 it follows that $X_1 \in \mathcal{D}(\alpha,1)$. By (2.2.2) and (2.2.4) we can take a_n such that

$$(2.2.6) \qquad a_n^{\alpha}\,\Gamma(1-\alpha)\,\cos(\pi\alpha/2) = n\,L(a_n) \qquad \text{for } 0 < \alpha < 1$$

and

$$(2.2.7) \qquad a_n^{\alpha}\,\Gamma(2-\alpha)\,|\cos(\pi\alpha/2)| = (\alpha-1)\,n\,L(a_n) \qquad \text{for } 1 < \alpha < 2.$$

LIPSCHUTZ (1956a) proved the following large deviations result. Let r(n) tend to infinity with n and

$$(2.2.8) \qquad (\log n)^{1-\delta} = \mathcal{O}(r(n)) \qquad \text{for any } \delta > 0.$$

Assume that the function L(x) in (2.2.5) satisfies the following relation

$$(2.2.9) \qquad \frac{L(nx)}{L(n)} = 1 + \frac{l_1(x,n)}{r(n)} + \frac{l_2(x,n)}{r(n)^2} + o\left(\frac{l_2(x,n)}{r(n)^2}\right) \qquad \text{for } n \to \infty$$

for

$$(2.2.10) \qquad r(n)^{-2} \le x < r(n)^{3/\alpha},$$

where $l_1(x,n)$ and $l_2(x,n)$ are $o(r(n)^{\varepsilon})$ for any $\varepsilon > 0$. Take any $\varepsilon \in (0,2)$ and let for $n \to \infty$

$$x_n \downarrow 0, \quad x_n > (B(\alpha)/(2-\varepsilon)\log r(a_n))^{\frac{1-\alpha}{\alpha}} \qquad \text{for } 0 < \alpha < 1$$

and

$$x_n \to -\infty, \quad |x_n| < ((2-\varepsilon)\log r(a_n)/B(\alpha))^{\frac{\alpha-1}{\alpha}} \qquad \text{for } 1 < \alpha < 2.$$

Then in case $0 < \alpha < 1$

$$(2.2.11) \quad P[a_n^{-1}(X_1+\ldots+X_n) \le x_n] \sim (2/\alpha)^{\frac{1}{2}} P[U \ge (2B(\alpha))^{\frac{1}{2}} x_n^{-\frac{\alpha}{2(1-\alpha)}}],$$

where $B(\alpha)$ is defined by (2.1.7)

and in case $1 < \alpha < 2$

$$(2.2.12) \quad P[a_n^{-1}(X_1+\ldots+X_n-EX_1-\ldots-EX_n) \le x_n] \sim$$

$$\sim (2\alpha)^{\frac{1}{2}} P[U \ge (2B(\alpha))^{\frac{1}{2}} (-x_n)^{\frac{\alpha}{2(\alpha-1)}}],$$

where $B(\alpha)$ is defined by (2.1.8).

In chapter 10 we shall discuss LIPSCHUTZ's result and give an interpretation of the assumption (2.2.9) for the function L.

CHAPTER 3

STABLE PROCESSES

In the first two sections of this chapter we shall give the definition and some properties of the Wiener process and other stable processes. In the next sections we prove some technical lemmas. Because we make use of the expansions given in theorem 2.1.7 we have to distinguish the four cases $\alpha = 2$, $0 < \alpha < 1$, $\alpha = 1$ and $1 < \alpha < 2$.

3.1. THE WIENER PROCESS

There exist several constructions of the Wiener process. In this section we give two of these. See ITO and McKEAN (1965) for other constructions.

DEFINITION 1.3.1. $\{W(t) : 0 \le t < \infty\}$ is called a *Wiener process* or *Brownian motion* on a probability triple $(\Omega, \mathcal{F}, P)$ if

(a) $W : [0,\infty) \times \Omega \to \mathbb{R}$;

(b) $W(0,\omega) = 0$ for each ω;

(c) $W(t,.)$ is $\mathcal{F}$-measurable for each t;

(d) for $0 < t_1 < t_2 < \ldots < t_n$, the increments $W(t_1)$, $W(t_2)-W(t_1), \ldots, W(t_n)-W(t_{n-1})$ are independent and normally distributed, with means 0 and variances t_1, $t_2-t_1, \ldots, t_n-t_{n-1}$.

According to Kolmogorov's consistency theorem, there is such a process on a suitably chosen probability triple. We shall always take $\{W(t) : 0 \le t < \infty\}$ to be a separable version. This implies the existence of a set Ω_0 with $P[\Omega_0] = 1$ such that $W(.,\omega)$ is continuous on Ω_0. (See FREEDMAN (1971) or BREIMAN (1968a).)

Let $C[0,\infty)$ be the set of real-valued continuous functions on $[0,\infty)$. We endow $C[0,\infty)$ with the metrizable topology of local uniform convergence. $\mathcal{C}[0,\infty)$ denotes the smallest σ-field containing all open sets in $C[0,\infty)$. Consider the following mapping

$$h : \Omega_0 \to C[0,\infty)$$

defined by $h(\omega) = W(.,\omega)$. This mapping is measurable and defines a probability measure Ph^{-1} on $(C[0,\infty),\mathcal{C}[0,\infty))$. This probability measure is called the *Wiener measure* $P_{2,0}$.

Let $C[0,1]$ be the set of all real-valued continuous functions on the interval $[0,1]$. The natural topology for $C[0,1]$ is the sup-norm topology. $\mathcal{C}[0,1]$ denotes the smallest σ-field containing all open sets in $C[0,1]$. BILLINGSLEY (1968) gives another construction of the Wiener measure on $(C[0,1],\mathcal{C}[0,1])$. Let $U_1,U_2,\ldots$ be i.i.d. with a standard normal distribution (on some $(\Omega,\mathcal{F},\mathcal{P})$). Define the random function

$$(3.1.1) \qquad U_n(t,\omega) = n^{-\frac{1}{2}}(U_1(\omega)+\ldots+U_{[nt]}(\omega)) + n^{-\frac{1}{2}}(nt-[nt])\ U_{[nt]+1}(\omega).$$

Let P_n be the distribution of the random function U_n on C. Then BILLINGSLEY (1968, theorem 9.1) proves that the sequence $\{P_n\}$ converges weakly to a limit and that this limit coincides with the Wiener measure $P_{2,0}$ on $(C[0,1],\mathcal{C}[0,1])$. Let W be a measurable mapping from some $(\Omega,\mathcal{F},\mathcal{P})$ to $(C[0,1],\mathcal{C}[0,1])$ with the property

$$\mathcal{P}[\{\omega : W(\omega) \in A\}] = P_{2,0}[A] \qquad \text{for } A \in \mathcal{C}[0,1].$$

Denote the value at t of $W(\omega)$ by $W(t,\omega)$. Then $\{W(t) : 0 \le t \le 1\}$ is a Wiener process on $[0,1]$ with continuous paths.

In a similar fashion WHITT (1970a) proves the existence of the Wiener measure on $(C[0,\infty),\mathcal{C}[0,\infty))$.

PROPERTIES. *Let* $\{W(t) : 0 \le t < \infty\}$ *be a Wiener process, then so are*

1. $\{-W(t) : 0 \le t < \infty\}$
2. $\{W(t+\tau)-W(\tau) : 0 \le t < \infty\ ,\ \tau\ (\textit{fixed}) > 0\}$
3. $\{tW(t^{-1}) : 0 \le t < \infty\}$
4. $\{c^{-\frac{1}{2}}W(ct) : 0 \le t < \infty\ ,\ c\ (\textit{fixed}) > 0\}$
5. $\{W(t_1)-W(t_1-t) : 0 \le t \le t_1\ (\textit{fixed})\ \}$

The proofs of these properties are easy.

THEOREM 3.1.1. *Let* $\{W(t) : 0 \le t < \infty\}$ *be a Wiener process. Then*

a. *The strong Markov property holds*
b. *For almost all* ω *the function* $W(.,\omega)$ *is nowhere differentiable and of unbounded variation on every interval.*

PROOF. BREIMAN (1968a). □

THEOREM 3.1.2. *Let* $\{W(t) : 0 \le t \le 1\}$ *be a Wiener process. Then for* $x \ge 0$

$$P[\max_{0 \le t \le 1} W(t) \ge x] = 2\, P[W(1) \ge x].$$

PROOF. BILLINGSLEY (1968). □

There exists an extensive literature on the Wiener process. See for example FREEDMAN (1971) and ITO and McKEAN (1965). We shall give other properties of the Wiener process in the following chapters. In these chapters we consider the local behavior of the *sample path* $W(t,\omega)$ for small and large values of t (L.I.L.), a Hölder-type theorem and Strassen's theorem.

3.2. STABLE PROCESSES

One may give constructions of stable processes analogous to the ones for the Wiener process. Let X be a random variable with distribution function $F(.;\alpha,\beta)$, $0 < \alpha < 2$ and $|\beta| \le 1$.

DEFINITION 3.2.1. $\{X(t) : 0 \le t < \infty\}$ is called a *stable process* on a probability space (Ω,F,P) if

(a) $X : [0,\infty) \times \Omega \to \mathbb{R}$;

(b) $X(0,\omega) = 0$ for each ω;

(c) $X(t,.)$ is F-measurable for each t;

(d) for $0 < t_1 < t_2 < \ldots < t_n$, the increments
$X(t_1),\ X(t_2)-X(t_1),\ldots,X(t_n)-X(t_{n-1})$
are independent and
in case $\alpha \ne 1,2$ they are distributed like
$$t_1^{1/\alpha}X,\ (t_2-t_1)^{1/\alpha}X,\ldots,(t_n-t_{n-1})^{1/\alpha}X$$
in case $\alpha = 1$ they are distributed like
$$t_1X + (2/\pi)\beta t_1 \log t_1,\ldots,(t_n-t_{n-1})X + (2/\pi)\beta(t_n-t_{n-1})\log(t_n-t_{n-1}).$$

REMARK 3.2.1. Condition (d) in definition 3.2.1 may be replaced by the conditions

(d_1) $\{X(t) : 0 \le t < \infty\}$ has stationary and independent increments

and

$$(d_2)\ X(t) \overset{d}{=} t^{1/\alpha}X \qquad \text{for } \alpha \neq 1,2$$
$$\overset{d}{=} t\,X + (2/\pi)\beta t\log t \qquad \text{for } \alpha = 1.$$

Let $D[0,\infty)$ be the set of real-valued functions on $[0,\infty)$ which are right-continuous and have finite left-hand limits. Then there exists a version of $\{X(t) : 0 \le t < \infty\}$ with all sample paths in $D[0,\infty)$ (cf. BREIMAN (1968a)). One may construct the stable measure $P_{\alpha,\beta}$ (with $0<\alpha<2$ and $|\beta|\le 1$) on $D[0,\infty)$ just as $P_{2,0}$ is constructed on $C[0,\infty)$ in section 3.1.

We now give a construction of the stable measure similar to Billingsley's construction of the Wiener measure. Let $D[0,1]$ be the set of real-valued functions on $[0,1]$ which are right-continuous and have finite left-hand limits. SKOROHOD (1956) has defined several topologies on $D[0,1]$. In appendix 1 we shall give the definitions of two topologies, viz. the J_1- and M_1-topology. Let $\mathcal{D}[0,1]$ be the σ-field of Borel-sets for the J_1-topology.

Let $X_1, X_2, \ldots$ be i.i.d. with common distribution $F(.;\alpha,\beta)$. Define the sequence of random elements $X_n(t)$ of $D[0,1]$ by

$$X_n(t) = n^{-1/\alpha}(X_1+\ldots+X_{[nt]}) \qquad \text{if } \alpha \neq 1$$
$$n^{-1}(X_1+\ldots+X_{[nt]} - (2/\pi)\beta[nt]\log n) \qquad \text{if } \alpha = 1.$$

By SKOROHOD (1957, theorem 2.7) the distribution of X_n converges weakly under the J_1-topology to a limit and this limit coincides with the stable measure $P_{\alpha,\beta}$.

Both the J_1- and M_1-topologies can be generalized to topologies on $D[0,\infty)$. See for example STONE (1963) and WHITT (1970b). Then we can prove the existence of the stable measure on $D[0,\infty)$ in a similar way.

For $0 < \alpha < 1$ and $\beta = 1$ we shall give another construction for the stable process. Let $X_1, X_2, \ldots$ be i.i.d. random variables with common distribution function F, and let $\{Y(t) : 0 \le t < \infty\}$ be a Poisson process with parameter $\lambda > 0$ and independent of the random variables X_k, $k = 1,2,\ldots$. Define the process $\{\tilde{X}(t) : 0 \le t < \infty\}$ by

$$\tilde{X}(t) = X_1+\ldots+X_{Y(t)}.$$

In other words: denote the jump points of the Poisson process by $T_1, T_2, \ldots,$

let the process $\tilde{X}(t)$ have a jump of height X_1 at time T_1, height X_2 at time T_2 etc. and be constant between two successive jump points. The process $\{\tilde{X}(t) : 0 \le t < \infty\}$ is called a *compound Poisson process*. Then

$$E\, e^{iu\,\tilde{X}(t)} = \exp\{\lambda t \int_0^\infty [\exp(iux) - 1]\, dF(x).$$

The stable process $\{X(t) : 0 \le t < \infty\}$ with $\alpha \in (0,1)$ and $\beta = 1$ satisfies

$$E\, e^{iu\,X(t)} = \exp\{mt \int_0^\infty [\exp(iux) - 1]\, \frac{dx}{x^{1+\alpha}},$$

with $m = \alpha\{\Gamma(1-\alpha)\sin(\pi\alpha/2)\}^{-1}$, corresponding to the choice $\lambda dF(x) = mx^{-1-\alpha}dx$. For more details of this construction we refer to the book of BREIMAN (1968a). We see from this construction that the sample paths of $X(t)$ are non-decreasing pure jump functions. Thus $X(t)$ has only upward jumps and between two successive jumps the sample paths are constant.

THEOREM 3.2.1. *Let* $\{X(t) : 0 \le t < \infty\}$ *be a stable process* ($0<\alpha<2$ *and* $|\beta|\le 1$). *Then*
a. The strong Markov property holds
b. There are no fixed discontinuities.

PROOF. BREIMAN (1968a). □

Stable processes with $|\beta| = 1$ are called *completely asymmetric*. Processes with $\beta = 1$ (resp. $\beta = -1$) have only positive (resp. negative) jumps. In case $\beta = 0$ the stable processes are symmetric. (See also property 1 below.) The completely asymmetric stable process with $\alpha = \frac{1}{2}$ and $\beta = 1$ can be obtained from the Wiener process in the following way.

THEOREM 3.2.2. *Let* $\{W(t) : 0 \le t < \infty\}$ *be a Wiener process. Define* $X(t) = \min\{v : W(v) = t\}$. *Then* $\{X(t) : 0 \le t < \infty\}$ *is a completely asymmetric stable process with* $\alpha = \frac{1}{2}$ *and* $\beta = 1$.

PROOF. See ITO and McKEAN (1965). □

PROPERTIES. *Let* $\{X(t) : 0 \le t < \infty\}$ *be a stable process with parameters* α *and* β. *Then*

1. $\{-X(t) : 0 \le t < \infty\}$ *is a stable process with parameters* α *and* $-\beta$.
2. $\{X(t+\tau)-X(\tau) : 0 \le t < \infty\ ,\ \tau$ *(fixed)* $> 0\}$ *is a stable process with parameters* α *and* β.

3. *In case* $\alpha \neq 1$

$$t^{1/\alpha}X(t^{-1}) \overset{d}{=} t^{-1/\alpha}X(t)$$

for $t > 0$.

In case $\alpha = 1$

$$\frac{X(t^{-1})-(2/\pi)\beta t^{-1}\log t^{-1}}{t^{-1}} \overset{d}{=} \frac{X(t)-(2/\pi)\beta t\log t}{t}$$

for $t > 0$.

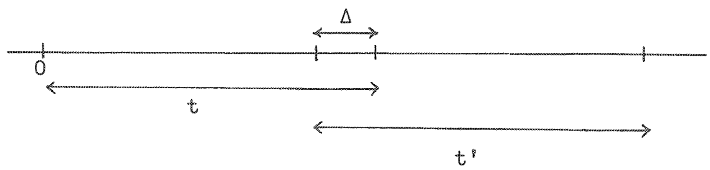
4. *For* $\alpha \neq 1$
$\{c^{-1/\alpha}X(ct) : 0 \leq t < \infty, c > 0\}$ *is a stable process with parameters* α *and* β,

for $\alpha = 1$
$\{c^{-1}X(ct)-(2/\pi)\beta t\log c : 0 \leq t < \infty, c > 0\}$ *is a stable process with parameters* $\alpha = 1$ *and* β.

5. $\{X(t_1)-X((t_1-t)-) : 0 \leq t \leq t_1(\textit{fixed})\}$ *is a stable process with parameters* α *and* β. *(We define* $X(0-) = 0$.)

3.3. SOME LEMMAS FOR THE CASE $\alpha = 2$

In this section we consider the Wiener process $\{W(t) : 0 \leq t < \infty\}$. We shall prove some lemmas, which are the tools in the proofs of the theorems in the following chapters (in the case $\alpha = 2$). Consider two intervals of length t and t' as below.



Let $t' \leq t$, denote the length of their intersection by Δ and suppose that $0 < \Delta < t$. Then we have

$$0 < \Delta \leq t' \leq t \tag{3.3.1}$$

and

$$\Delta < t. \tag{3.3.2}$$

Let ϕ be a non-negative, continuous and non-increasing function on $(0,\infty)$. We shall give bounds for the probability

$$P_I = P[W(t) \le -t^{\frac{1}{2}}\phi(t) \wedge W(t-\Delta+t')-W(t-\Delta) \le -(t')^{\frac{1}{2}}\phi(t')].$$

We distinguish three cases, which are - roughly speaking - characterized by

1. Δ/t near 0,
2. Δ/t bounded away from 0 and 1,
3. Δ/t near 1.

Define the function ψ by

$$\psi(t^{-1}) = \phi(t) \tag{3.3.3}$$

U is a standard normal random variable.

LEMMA 3.3.1. *Let* $\phi(s) \to \infty$ *for* $s \to 0$. *For all positive* ε *there exist positive constants* t_0 *and* δ *such that*

$$P_I \le (1+\varepsilon)\, P[U \le -\phi(t)]\, P[U \le -\phi(t')]$$

for all Δ, t', t *satisfying (3.3.1), (3.3.2),* $t \le t_0$ *and* $\Delta t^{-1}\psi^2(1/t)\psi^2(1/t') \le \delta$, *where* ψ *is defined by (3.3.3).*

PROOF. Take $\varepsilon > 0$ and c a positive number smaller than $\log(1+\varepsilon)$.

$$\begin{aligned} P_I &= P[W(t) \le -t^{\frac{1}{2}}\phi(t) \wedge W(t-\Delta+t')-W(t-\Delta) \le -(t')^{\frac{1}{2}}\phi(t') \wedge \\ &\qquad \wedge W(t)-W(t-\Delta) \le -t^{\frac{1}{2}}\phi(t)-(t-\Delta)^{\frac{1}{2}}(-\phi(t)+c/\phi(t))] + \\ &\quad + P[W(t) \le -t^{\frac{1}{2}}\phi(t) \wedge W(t-\Delta+t')-W(t-\Delta) \le -(t')^{\frac{1}{2}}\phi(t') \wedge \\ &\qquad \wedge W(t)-W(t-\Delta) > -t^{\frac{1}{2}}\phi(t)-(t-\Delta)^{\frac{1}{2}}(-\phi(t)+c/\phi(t))] \le \\ &\le P[U \le -(t/\Delta)^{\frac{1}{2}}\phi(t)-(t-\Delta)^{\frac{1}{2}}\Delta^{-\frac{1}{2}}(-\phi(t)+c/\phi(t))] + \qquad (3.3.4) \\ &\quad + P[W(t-\Delta) \le (t-\Delta)^{\frac{1}{2}}(-\phi(t)+c/\phi(t)]\, P[W(t-\Delta+t')-W(t-\Delta) \le \\ &\qquad \le -(t')^{\frac{1}{2}}\phi(t')]. \end{aligned}$$

The first probability in (3.3.4) can be bounded by

$$(3.3.5) \qquad P[U \le -(1-\Delta/t)^{\frac{1}{2}} c(t/\Delta)^{\frac{1}{2}}/\phi(t)].$$

Choose $c\delta^{-1} > 2+\varepsilon$. Then for $\Delta t^{-1}\psi^2(1/t)\psi^2(1/t') < \delta$ the probability in (3.3.5) is less than

$$P[U \le -(1-\Delta/t)^{\frac{1}{2}}(2+\varepsilon)\phi(t')] =$$

$$= o(P[U \le -\phi(t)]\; P[U \le -\phi(t')]) \qquad \text{for } t \text{ (and } t') \to 0.$$

The first factor in the second term of (3.3.4) is equal to

$$P[U \le -\phi(t)+c/\phi(t)].$$

The desired result follows from lemma 2.1.2. □

LEMMA 3.3.2. *Let* $\phi(s) \to \infty$ *for* $s \to 0$. *For every constant* $c \in (0,1)$ *there exist two positive constants* C_1 *and* C_2 *(independent of* Δ, t' *and* t*) such that*

$$P_I \le C_1\, e^{-C_2\psi^2(t^{-1})}\, P[U \le -\phi(t')]$$

for all Δ, t', t *satisfying (3.3.1), (3.3.2) and* $\Delta t^{-1}\psi^2(1/t')/\psi^2(1/t) \le c$, *where* ψ *is defined by (3.3.3).*

PROOF. Choose a number $a \in (0,1)$ such that $(1-a)^2-(a^2+1)c > 0$.

$$\begin{aligned} P_I = {} & P[W(t) \le -t^{\frac{1}{2}}\phi(t) \wedge W(t-\Delta+t')-W(t-\Delta) \le -(t')^{\frac{1}{2}}\phi(t') \wedge \\ & \wedge W(t)-W(t-\Delta) \le -t^{\frac{1}{2}}\phi(t)(1-a(1-\Delta/t)^{\frac{1}{2}})] + \\ & + P[W(t) \le -t^{\frac{1}{2}}\phi(t) \wedge W(t-\Delta+t')-W(t-\Delta) \le -(t')^{\frac{1}{2}}\phi(t') \wedge \\ & \wedge W(t)-W(t-\Delta) > -t^{\frac{1}{2}}\phi(t)(1-a(1-\Delta/t)^{\frac{1}{2}})] \le \end{aligned}$$

$$(3.3.6) \qquad \begin{aligned} & \le P[U \le -(t/\Delta)^{\frac{1}{2}}\phi(t)(1-a(1-\Delta/t)^{\frac{1}{2}})] + \\ & + P[W(t-\Delta) \le -a(t-\Delta)^{\frac{1}{2}}\phi(t)]\; P[W(t-\Delta+t')-W(t-\Delta) \le -(t')^{\frac{1}{2}}\phi(t')]. \end{aligned}$$

The last term in (3.3.6) can easily be bounded by using theorem 2.1.7 VII. By means of the same theorem we can show that for $t' \to 0$

$$P[U \le -(t/\Delta)^{\frac{1}{2}}\phi(t)(1-a(1-\Delta/t)^{\frac{1}{2}})] \le P[U \le -(t/\Delta)^{\frac{1}{2}}\phi(t)(1-a)] =$$
$$= o(P[U \le -a\phi(t)]\, P[U \le -\phi(t')])$$

because $t' \to 0$ implies $(t/\Delta)^{\frac{1}{2}}\phi(t) \to \infty$. If t' (and hence t) is bounded away from zero the result of the lemma is trivial. □

LEMMA 3.3.3. *Let* $\phi(s) \to \infty$ *for* $s \to 0$. *Let* $c \in (0,1)$ *and* $C > 0$ *be two constants. Then there exist two positive constants* C_3 *and* C_4 *such that*

$$P_I \le C_3\, e^{-C_4((t-\Delta)/t)\psi^2(t^{-1})}\, P[U \le -\phi(t)]$$

for all Δ, t', t *satisfying (3.3.1), (3.3.2),* $\Delta t^{-1} \in (c,1)$ *and* $(1-\Delta t^{-1})^{\frac{1}{2}}\psi(t^{-1}) > C$, *where* ψ *is defined by (3.3.3).*

PROOF. Just as in the proofs of the lemmas 3.3.1 and 3.3.2 we have for any constant A

$$\begin{aligned} P_I \le\ & P[U \le -\phi(t)(1+(t-\Delta)A/t)] + \\ & + P[U \le -(t/(t-\Delta))^{\frac{1}{2}}\phi(t)+(\Delta/(t-\Delta))^{\frac{1}{2}}\phi(t)(1+(t-\Delta)A/t)]\cdot \\ & \cdot P[U \le -\phi(t')]. \end{aligned}$$

We take $0 < A < \frac{1}{2}$. By theorem 2.1.7 VII we know that there exist two positive constants A_1 and A_2 such that

$$P[U \le -\phi(t)(1+(t-\Delta)A/t)] \le A_1\, e^{-A_2((t-\Delta)/t)\psi^2(t^{-1})}\, P[U \le -\phi(t)]. \tag{3.3.7}$$

There exists a positive constant c_1 (independent of Δ and t) such that for all Δ and t with $\Delta/t \in (c,1)$

$$-(t/(t-\Delta))^{\frac{1}{2}}\phi(t)+(\Delta/(t-\Delta))^{\frac{1}{2}}\phi(t)(1+(t-\Delta)A/t) \le -c_1(t-\Delta)^{\frac{1}{2}}t^{-\frac{1}{2}}\phi(t).$$

Then by theorem 2.1.7 VII it follows that there are two positive constants B_1 and B_2 such that

$$P[U \le -c_1(t-\Delta)^{\frac{1}{2}}t^{-\frac{1}{2}}\phi(t)] \le B_1 e^{-B_2((t-\Delta)/t)\psi^2(t^{-1})}. \tag{3.3.8}$$

From the estimates (3.3.7), (3.3.8) and the monotonicity of ϕ the desired result easily follows if we take $C_3 = A_1+B_1$ and $C_4 = \min(A_2,B_2)$. □

REMARK 3.3.1. Results similar to the lemmas 3.3.1, 3.3.2 and 3.3.3 are proved in the paper of CHUNG, ERDÖS and SIRAO (1959). They make use of the magnitude of the correlationcoefficient of the random variables $W(t)$ and $W(t-\Delta+t')-W(t-\Delta)$, which is equal to $\Delta(tt')^{-\frac{1}{2}}$. Our formulation in terms of the ratio of the length Δ of the intersection and the length t of the largest interval can also be used in case we are considering stable processes.

REMARK 3.3.2. Let I_1 and I_2 be two arbitrary intervals of $[0,\infty)$ with length t and t' and length of the intersection $\Delta > 0$. We write $x(I)$ for $x(r)-x(s)$ for any real function x; s and r are the endpoints of an interval I. One easily sees that we can deduce similar bounds as in lemmas 3.3.1, 3.3.2 and 3.3.3 for the probability

$$P[W(I_1) \le -t^{\frac{1}{2}}\phi(t) \wedge W(I_2) \le -(t')^{\frac{1}{2}}\phi(t')].$$

We conclude this section by stating the following result of KIEFER (1969).

LEMMA 3.3.4. *Let* T, L, δ *and* x *be positive numbers with* $T < L$. *Then*

a. $P[\sup_{0\le t_1<t_2\le T} |W(t_1)-W(t_2)| \ge x] \le 4\, P[|W(T)| \ge x]$

and

b. $P[\sup_{\substack{0\le t_1<t_2\le L \\ |t_2-t_1|\le T}} |W(t_1)-W(t_2)| \ge x] \le 4(L-T+\delta)\delta^{-1}P[|W(T+2\delta)| \ge x].$

3.4. THE CASE $0 < \alpha < 1$

In this section we give the analogous lemmas for the case $0<\alpha<1$. Let first $\{X(t) : 0 \le t < \infty\}$ be the completely asymmetric stable process ($\beta=1$) with characteristic exponent $\alpha \in (0,1)$ and ϕ a positive continuous non-decreasing function on $(0,\infty)$ with the property $\phi(s) \to 0$ for $s \to 0$. We define the function ψ by

$$(3.4.1) \qquad \psi(t^{-1}) = \{2B(\alpha)\}^{\frac{1}{2}}\{\phi(t)\}^{-\frac{\alpha}{2(1-\alpha)}},$$

where $B(\alpha)$ is defined by (2.1.7). Let U be a standard normal random variable and X a r.v. with the same distribution as X(1). By theorem 2.1.7 IV we have

$$P[X \le \phi(t)] \sim (2/\alpha)^{\frac{1}{2}} P[U \ge \psi(t^{-1})] \qquad \text{for } t \downarrow 0. \tag{3.4.2}$$

Let I_1 and I_2 be two arbitrary intervals of $[0,\infty)$ with length t and t' and length of the intersection $\Delta > 0$. In the first three lemmas we give bounds for the probability

$$P_I = P[X(I_1) \le t^{1/\alpha}\phi(t) \wedge X(I_2) \le (t')^{1/\alpha}\phi(t')].$$

Again it is no restriction to suppose that the intervals are situated as in section 3.3 and satisfy (3.3.1) and (3.3.2). In that case the proof of the first three lemmas can be found in MIJNHEER (1973).

LEMMA 3.4.1. *For all positive* ε *there exist positive constants* t_0 *and* δ *such that*

$$P_I \le (1+\varepsilon)\, P[X \le \phi(t)]\, P[X \le \phi(t')]$$

for all Δ, t', t *satisfying (3.3.1), (3.3.2),* $t \le t_0$ *and* $\Delta t^{-1}\psi^2(t^{-1}) < \delta$, *where* ψ *is defined by (3.4.1).*

LEMMA 3.4.2. *For every constant* $c \in (0,1)$ *there exist positive constants* C_1 *and* C_2 *(independent of* Δ, t' *and* t*) such that*

$$P_I \le C_1\, e^{-C_2\psi^2(t^{-1})}\, P[X \le \phi(t')]$$

for all Δ, t', t *satisfying (3.3.1), (3.3.2) and* $\Delta t^{-1} < c$, *where* ψ *is defined by (3.4.1).*

LEMMA 3.4.3. *Let* $c \in (0,1)$ *and* $C > 0$ *be two constants. Then there exist two constants* C_3 *and* C_4 *such that*

$$P_I \le C_3\, e^{-C_4((t-\Delta)/t)\psi^2(t^{-1})}\, P[X \le \phi(t)]$$

for all Δ, t', t *satisfying (3.3.1), (3.3.2),* $\Delta t^{-1} \in (c,1)$ *and* $(1-\Delta t^{-1})^{\frac{1}{2}}\psi(t^{-1}) > C$, *where* ψ *is defined by (3.4.1).*

In the following lemma we consider the process $\{X(t) : 0 \le t < \infty\}$ with $0 < \alpha < 1$ and $|\beta| \le 1$. This lemma is the analogue of lemma 1.4.3 for stable processes.

LEMMA 3.4.4. *Let* $\{X(t) : 0 \le t < \infty\}$ *be a stable process with* $0 < \alpha < 1$, $|\beta| \le 1$ *and let* $k(\alpha,\beta) = P[X(1) \le 0]$. *Then for all positive* t *and* x

$$P[\sup_{0\le s\le t} X(s) \ge x] \le (1-k(\alpha,\beta))^{-1} P[X(t) \ge x].$$

PROOF. Similar to the proof of lemma 2.2 in MIJNHEER (1973). □

REMARK 3.4.3. In case $\beta = 1$ the sample paths are non-decreasing. Then we have

$$P[\sup_{0\le s\le t} X(s) \ge x] = P[X(t) \ge x].$$

REMARK 3.4.4. BREIMAN (1965) has shown

$$k(\alpha,\beta) = P[X(1) \le 0] = \frac{1}{2} - \frac{1}{\pi\alpha} \arctan(\beta \tan(\pi\alpha/2))$$

for $0 < \alpha < 1$.

3.5. THE CASE $\alpha = 1$

In this section we give similar lemmas as in sections 3.3 and 3.4. Let first $\{X(t) : 0 \le t < \infty\}$ be the completely asymmetric stable process with $\alpha = \beta = 1$, let ϕ be a non-negative, non-increasing function on $(0,\infty)$ and $\phi(s) \to \infty$ for $s \to 0$. Define the function ψ by

$$(3.5.1) \qquad \psi(t^{-1}) = 2(\pi e)^{-\frac{1}{2}} \exp(\pi\phi(t)/4).$$

Let U be the standard normal r.v. and X a r.v. with distribution function $F(.;1,1)$. Then we have by theorem 2.1.7 V

$$(3.5.2) \qquad P[X \le -\phi(t)] \sim 2^{\frac{1}{2}} P[U \ge \psi(t^{-1})] \qquad \text{for } t \downarrow 0.$$

Let I_1 and I_2 be two intervals of $[0,\infty)$ with lengths t and t', and length of the intersection $\Delta > 0$. We shall give bounds for the probability

$$P_I = P[X(I_1)-(2/\pi)\, t \log t \le -t\phi(t) \wedge$$
$$\wedge\; X(I_2)-(2/\pi)\, t' \log t' \le -t'\phi(t')].$$

The proofs of the first four lemmas do not differ appreciably from those of the lemmas in section 3.3. Again we may restrict ourselves to intervals situated as in section 3.3 and satisfying (3.3.1) and (3.3.2). We shall only work out the points of difference between the proofs of the first two lemmas and the corresponding ones in MIJNHEER (1973). In that paper the proofs of lemmas 3.5.3 and 3.5.4 are given.

LEMMA 3.5.1. *For all* $\varepsilon > 0$ *there exist positive constants* t_0 *and* δ *such that*

$$P_I \le (1+\varepsilon)\, P[X \le -\phi(t)]\, P[X \le -\phi(t')]$$

for all Δ, t', t *satisfying (3.3.1), (3.3.2),* $t \le t_0$ *and* $\Delta t^{-1}\psi^2(1/t)\psi^2(1/t') < \delta$, *where* ψ *is defined by (3.5.1).*

PROOF. Take $\varepsilon > 0$ and c a positive number smaller than $\log(1+\varepsilon)$. As in the proof of lemma 3.3.1 we obtain

$$(3.5.3)\qquad P_I \le P[X(t)-X(t-\Delta) \le (2/\pi)(t \log t - (t-\Delta)\log(t-\Delta)) - \Delta\phi(t) + (4/\pi)(t-\Delta)\log(1-c\psi^{-2}(1/t))] + P[X \le -\phi(t')]\;\; P[X \le -\phi(t)-(4/\pi)\log(1-c\psi^{-2}(1/t))].$$

The first probability on the right in (3.5.3) is equal to

$$(3.5.4)\qquad P[X \le \frac{2}{\pi}\,\frac{t\log t-(t-\Delta)\log(t-\Delta)-\Delta\log\Delta}{\Delta} - \phi(t) + \frac{2}{\pi}\,(\frac{t}{\Delta}-1)\log(1-c\psi^{-2}(t^{-1}))].$$

We now use the assumption $\Delta t^{-1}\psi^2(1/t)\psi^2(1/t') < \delta$ with $\delta < c(2+\varepsilon)^{-1}\, e^{-\frac{1}{2}}$ – implying that Δ/t is small for small t – and apply theorem 2.1.7 V. In this way we show, as in the proof of lemma 3.3.1 that the probability in (3.5.4) is

$$o(P[X \le -\phi(t)]\; P[X \le -\phi(t')] \qquad \text{for } t \downarrow 0.$$

The second term on the right in (3.5.3) easily gives the desired result by using theorem 2.1.7 V and lemma 2.1.2. □

LEMMA 3.5.2. *For every constant* $c \in (0,1)$ *there exist two positive constants* C_1 *and* C_2 *(independent of* Δ, t' *and* t*) such that*

$$P_I \leq C_1 \, e^{-C_2\psi^2(t^{-1})} \, P[X \leq -\phi(t')]$$

for all Δ, t', t *satisfying (3.3.1), (3.3.2) and* $\Delta t^{-1}\psi^2(1/t')/\psi^2(1/t) < c$, *where* ψ *is defined by (3.5.1).*

PROOF. Define the positive number a_0 by $(\frac{1}{2a_0^2})^{1/c} = 1 + \frac{1}{2a_0^2}$. Choose $a \in (0,a_0)$. Just as in the proof of lemma 3.5.1 we have

$$(3.5.5) \qquad P_I \leq P[X \leq -\phi(t) + \frac{4}{\pi}\frac{t-\Delta}{\Delta}\log a + \frac{2}{\pi}\frac{t\log t-\Delta\log\Delta-(t-\Delta)\log(t-\Delta)}{\Delta}] +$$

$$+ P[X \leq -\phi(t')] \;\; P[X \leq -\phi(t)-(4/\pi)\log a]$$

After some algebra one finds that the first term on the right in (3.5.5) is

$$o(P[X \leq -\phi(t')] \;\; P[U \leq -a\psi(t^{-1})]) \qquad \text{for } t' \downarrow 0. \;\; □$$

LEMMA 3.5.3. *Let* $c \in (0,1)$ *and* $C > 0$ *be two constants. Then there exist two constants* C_3 *and* C_4 *such that*

$$P_I \leq C_3 \, e^{-C_4((t-\Delta)/t)\psi^2(t^{-1})} \, P[X \leq -\phi(t)]$$

for all Δ, t', t *satisfying (3.3.1), (3.3.2),* $\Delta t^{-1} \in (c,1)$ *and* $(1-\Delta t^{-1})^{\frac{1}{2}}\psi(t^{-1}) > C$, *where* ψ *is defined by (3.5.3).*

LEMMA 3.5.4. *Let the function* x *be such that for some constants* c_1 *and* c_2

$$c_1 < -x(p) + (2/\pi)\log p < c_2$$

for all positive p. *Define the constant* k_1 *by*

$$k_1^{-1} = P[X(1) \leq c_1-1].$$

Then for all positive t *and for sufficiently large* p

a. $$P\left[\inf_{t-tp^{-1}\le s\le t} \frac{X(s)-(2/\pi)s\log s}{s} \le -x(p)\right] \le k_1\, P\left[\frac{X(t)-(2/\pi)t\log t}{t} \le -x(p)\right] = k_1\, P[X(1) \le -x(p)]$$

b. $$P\left[\inf_{\substack{0\le r\le tp^{-1}\\ t-tp^{-1}\le s\le t}} \frac{X(s)-X(r)-(2/\pi)(s-r)\log(s-r)}{s-r} \le -x(p)\right] \le k_1^2\, P[X(1) \le -x(p)].$$

In the following lemma we not only consider the completely asymmetric stable process with $\alpha = \beta = 1$, but all stable processes with $\alpha = 1$ and $|\beta| \le 1$.

LEMMA 3.5.5. *Let* $\{X(t) : 0 \le t < \infty\}$ *be a stable process with* $\alpha = 1$ *and* $-1 < \beta \le 1$. *Then for any pair of positive numbers* b_1 *and* b_2 *we have*

$$P\left[\sup_{b_1\le s\le b_2} \frac{X(s)-(2/\pi)\beta s\log s}{s} \ge x\right] = O(x^{-1}) \qquad \textit{for } x \to \infty.$$

PROOF. We distinguish the cases $\beta \ge 0$ and $\beta < 0$. Let $x > 0$, $B_1 = (2/\pi) \min_{b_1\le s\le b_2} s\log s$ and $B_2 = (2/\pi) \max_{b_1\le s\le b_2} s\log s$. The event

$$\{\omega : \sup_{b_1\le s\le b_2} \frac{X(s)-(2/\pi)\beta s\log s}{s} \ge x\}$$

is contained in

$$\{\omega : \sup_{b_1\le s\le b_2} X(s) \ge b_1x + \beta B_1\} \qquad \text{for } \beta \ge 0$$

and in

$$\{\omega : \sup_{b_1\le s\le b_2} X(s) \ge b_1x + \beta B_2\} \qquad \text{for } \beta < 0.$$

In both cases the proof of the lemma follows a similar pattern as the proof of lemma 2.2 in MIJNHEER (1973). We sketch the proof only for $\beta < 0$. Let Γ be the event that there exists some $s \in [b_1,b_2]$ with $X(s) > b_1x + \beta B_2$. The r.v. S is defined on Γ to be the infimum of these numbers s. By the right-continuity

$$X(S) \geq b_1x + \beta B_2.$$

By the strong Markov property we have for $s \in [b_1, b_2)$ and $B_3 = \max(0,(b_2-b_1)\log(b_2-b_1))$

$$P[X(b_2)-X(s) \geq (2/\pi)\beta\ B_3 \mid \Gamma \wedge S = s] =$$
$$= P[(b_2-s)\ X(1) \geq (2/\pi)\beta\ B_3-(2/\pi)\beta(b_2-s)\log(b_2-s)] \geq P[X(1) \geq 0].$$

Denote this last probability by p. Then

$$P[\Gamma] \leq p^{-1}\ P[X(1) \geq b_2^{-1}(b_1x+\beta B_2+(2/\pi)\beta\ B_3-(2/\pi)\beta b_2\log b_2)].$$

By the estimate in theorem 2.1.7 II this part of the lemma easily follows. □

3.6. THE CASE $1 < \alpha < 2$

In this section we give lemmas corresponding to the lemmas in section 3.3 for the case $1 < \alpha < 2$. $\{X(t) : 0 \leq t < \infty\}$ is the completely asymmetric stable process with $1 < \alpha < 2$ and $\beta = 1$. Let ϕ be a non-negative, continuous, non-increasing function on $(0,\infty)$ with $\phi(t) \to \infty$ for $t \to 0$. Define the function ψ by

$$\psi(t^{-1}) = \{2B(\alpha)\}^{\frac{1}{2}}\{\phi(t)\}^{\frac{\alpha}{2(\alpha-1)}}. \tag{3.6.1}$$

The r.v. X has the same distribution as X(1). Then by theorem 2.1.7 VI

$$P[X \leq -\phi(t)] \sim (2\alpha)^{\frac{1}{2}}\ P[U \geq \psi(t^{-1})] \qquad \text{for } t \to 0. \tag{3.6.2}$$

Let I_1 and I_2 be two intervals of $[0,\infty)$ with length t and t', and length of the intersection $\Delta > 0$. We give bounds for the probability

$$P_I = P[X(I_1) \leq -t^{1/\alpha}\phi(t) \wedge X(I_2) \leq -(t')^{1/\alpha}\phi(t')].$$

We may restrict ourselves to intervals situated as in section 3.3 and satisfying (3.3.1) and (3.3.2). The proofs of lemmas 3.6.1 and 3.6.2 follow the exact lines of the corresponding lemmas for the case $\alpha = 2$. The proofs of lemmas 3.6.3 and 3.6.4 are given in MIJNHEER (1973).

LEMMA 3.6.1. *For all positive* ε *there exist positive constants* t_0 *and* δ *such that*

$$P_I \le (1+\varepsilon)\; P[X \le -\phi(t)]\; P[X \le -\phi(t')]$$

for all Δ, t', t *satisfying (3.3.1), (3.3.2),* $t \le t_0$ *and* $\Delta t^{-1}\psi^2(1/t)\psi^2(1/t') < \delta$, *where* ψ *is defined by (3.6.1).*

LEMMA 3.6.2. *For every constant* $c \in (0,1)$ *there exist two positive constants* C_1 *and* C_2 *(independent of* Δ, t' *and* t*) such that*

$$P_I \le C_1\, e^{-C_2\psi^2(t^{-1})}\; P[U \le -\phi(t')]$$

for all Δ, t', t *satisfying (3.3.1), (3.3.2) and* $\Delta t^{-1}\psi^2(1/t')/\psi^2(1/t) < c$, *where* ψ *is defined by (3.6.1).*

LEMMA 3.6.3. *Let* $c \in (0,1)$ *and* $C > 0$ *be two constants. Then there exist two positive constants* C_3 *and* C_4 *such that*

$$P_I \le C_3\, e^{-C_4((t-\Delta)/t)\psi^2(t^{-1})}\; P[X \le -\phi(t)]$$

for all Δ, t', t *satisfying (3.3.1), (3.3.2),* $\Delta t^{-1} \in (c,1)$ *and* $(1-\Delta t^{-1})^{\frac{1}{2}}\psi(t^{-1}) > C$, *where* ψ *is defined by (3.6.1).*

In the following lemma we not only consider the completely asymmetric stable processes with $1 < \alpha < 2$ and $\beta = 1$, but all stable processes with $1 < \alpha < 2$ and $|\beta| \le 1$.

LEMMA 3.6.4. *Let* $\{X(t) : 0 \le t < \infty\}$ *be a stable process with* $1 < \alpha < 2$ *and* $|\beta| \le 1$. *Define the constant* $k(\alpha,\beta)$ *by*

$$k(\alpha,\beta) = P[X(1) \le 0].$$

Then for all positive t *and all negative* x

a. $P[\inf_{0\le s\le t} X(s) \le x] \le k^{-1}(\alpha,\beta)\; P[X(t) \le x]$

b. $P[\inf_{0\le r\le s\le t} \{X(s)-X(t)\} \le x] \le k^{-2}(\alpha,\beta)\; P[X(t) \le x]$.

CHAPTER 4

GENERALIZED LAWS OF THE ITERATED LOGARITHM FOR SMALL TIMES

In this chapter we are interested in the local behavior near $t = 0$ of the sample paths of the Wiener process and the completely asymmetric stable processes. In the case of completely asymmetric stable processes ($\beta = 1$) we obtain sharp lower asymptotic results by using the relation between the left tail of the distribution of the completely asymmetric stable laws ($\beta = 1$) and the tail of the standard normal distribution as given in theorem 2.1.7 parts IV, V and VI. We shall prove the result only for the case $\alpha = 1$. The proofs in the other cases are similar and can be found in the literature.

4.1. THE CASE $\alpha = 2$

In this section we formulate *Kolmogorov's integral test.*

THEOREM 4.1.1. *Let* $\{W(t) : 0 \le t < \infty\}$ *be a Wiener process. Let* $\phi(t)$ *be positive, continuous and non-increasing for sufficiently small* t *and define* $\psi(t^{-1}) = \phi(t)$. *Then*

$$P[\{\omega: \textit{there exists some } t_0(\omega) > 0 \textit{ such that } W(t,\omega) \le t^{\frac{1}{2}}\phi(t) \textit{ for all } t \le t_0(\omega)\}] = 0 \textit{ or } 1$$

according as the integral

$$I(\psi) = \int^{\infty} \psi(t)t^{-1}e^{-\frac{1}{2}\psi^2(t)}dt \tag{4.1.1}$$

diverges or converges.

PROOF. By property 3 of section 3.1 theorem 4.1.1 is equivalent to the generalized L.I.L. for large times. The proof for that case is given by MOTOO (1959). One can also give a proof by making use of the Borel-Cantelli lemma. This proof is similar to the proof in section 4.3 and rests on the lemmas of section 3.3. □

As a consequence of this theorem we have

$$\limsup_{t \downarrow 0} \frac{W(t)}{(2t \log\log t^{-1})^{\frac{1}{2}}} = 1 \qquad \text{a.s.}$$

and by symmetry

$$\liminf_{t\downarrow 0} \frac{W(t)}{(2t \log\log t^{-1})^{\frac{1}{2}}} = -1 \qquad \text{a.s.}\ .$$

4.2. THE CASE $0 < \alpha < 1$

THEOREM 4.2.1. *Let* $\{X(t) : 0 \le t < \infty\}$ *be a completely asymmetric stable process with* $0 < \alpha < 1$ *and* $\beta = 1$. *Let* $\phi(t)$ *be positive, continuous and non-decreasing for sufficiently small* t *and define the function* ψ *by*

$$\psi(t^{-1}) = \{2B(\alpha)\}^{\frac{1}{2}}\{\phi(t)\}^{-\frac{\alpha}{2(1-\alpha)}},$$

where the constant $B(\alpha)$ *is given in (2.1.7). Then*

$$P[\{\omega: \text{there exists some } t_0(\omega) > 0 \text{ such that } X(t,\omega) \ge t^{1/\alpha}\phi(t) \text{ for all } t \le t_0(\omega)\}] = 0 \text{ or } 1$$

according as the integral (4.1.1) diverges or converges.

PROOF. BREIMAN (1968b) has given a proof following Motoo's proof in the case $\alpha = 2$. □

As a consequence of this theorem we have

$$\liminf_{t\downarrow 0} \frac{X(t)}{t^{1/\alpha}(2\log\log t^{-1})^{-(1-\alpha)/\alpha}} = \{2B(\alpha)\}^{(1-\alpha)/\alpha} \qquad \text{a.s.}\ .$$

This result was first proved by FRISTEDT (1964). Similar results were obtained for increasing processes with stationary independent increments (these processes are also called *subordinators* and are not necessarily stable) by FRISTEDT and PRUITT (1971).

4.3. THE CASE $\alpha = 1$

THEOREM 4.3.1. *Let* $\{X(t) : 0 \le t < \infty\}$ *be a completely asymmetric stable* *process with* $\alpha = \beta = 1$. *Let* $\phi(t)$ *be positive, continuous and non-increasing for sufficiently small* t *and define the function* ψ *by*

$$(4.3.1) \qquad \psi(t^{-1}) = 2(\pi e)^{-\frac{1}{2}}\exp(\pi\phi(t)/4).$$

Then

$$(4.3.2) \qquad P[\{\omega: \textit{there exists some } t_0(\omega) > 0 \textit{ such that } X(t,\omega)-(2/\pi)t\log t \geq$$
$$\geq -t\phi(t) \textit{ for all } t \leq t_0(\omega)\}] = 0 \textit{ or } 1$$

according as the integral (4.1.1) diverges or converges.

Below we give a proof of theorem 4.3.1 based on the Borel-Cantelli lemma. We need the following lemmas. Similar results are to be found in LIPSCHUTZ (1956b) and FELLER (1943).

Define the sequence $\{t_k\}$ by

$$(4.3.3) \qquad t_k = e^{k/\log k} \qquad\qquad k=1,2,\ldots$$

and for $\delta > 0$ the functions

$$(4.3.4) \qquad \psi_1(t) = \{2(1-\delta)\log\log t\}^{\frac{1}{2}}$$

and

$$(4.3.5) \qquad \psi_2(t) = \{2(1+\delta)\log\log t\}^{\frac{1}{2}}.$$

LEMMA 4.3.1. *Let* $\delta > 0$ *and let* ψ_1 *and* ψ_2 *be defined by (4.3.4) and (4.3.5). If theorem 4.3.1 holds for all functions* ϕ *satisfying*

$$(4.3.6) \qquad \psi_1(t) \leq \psi(t) \leq \psi_2(t),$$

where ψ *is defined by (4.3.1), then it holds in general.*

PROOF. *i.* We first prove the following assertion. Let $I(\psi) < \infty$ then $\psi(t) > \psi_1(t)$ for sufficiently large t. Assume that the set $\{t:\psi(t) \leq \psi_1(t)\}$ is not bounded, then there exists an increasing sequence $\{v_n\}$ with $\psi(v_n) \leq$ $\leq \psi_1(v_n)$. Then for sufficiently large m and $n \to \infty$

$$I(\psi) > \int_{v_m}^{v_n} \psi(t)t^{-1}e^{-\frac{1}{2}\psi^2(t)}dt > \psi_1(v_n)e^{-\frac{1}{2}\psi_1^2(v_n)} \int_{v_m}^{v_n} t^{-1}dt \to \infty,$$

which contradicts $I(\psi) < \infty$.

ii. Let ϕ be an arbitrary function satisfying the conditions of theorem 4.3.1 and $I(\psi) < \infty$. Define the function $\hat{\psi}$ by

$$(4.3.7) \qquad \hat{\psi}(t) = \min(\max(\psi_1(t),\psi(t)),\psi_2(t)).$$

Let $\hat{\phi}$ correspond to $\hat{\psi}$ as ϕ does to ψ by (4.3.1). From the assertion in part *i.* of the proof we have $\psi(t) > \psi_1(t)$ for sufficiently large t. This implies $\hat{\psi}(t) = \min(\psi(t),\psi_2(t))$ for large t and $I(\hat{\psi}) < I(\psi)+I(\psi_2) < \infty$. The assumption that theorem 4.3.1 is proved for $\hat{\phi}$ gives for almost all ω

$$X(t,\omega)-(2/\pi)t \log t \geq -t\hat{\phi}(t) \qquad \text{for } t \leq t_0(\omega)$$

and hence certainly, since $\hat{\phi}(t) \leq \phi(t)$, we have for almost all ω

$$X(t,\omega)-(2/\pi)t \log t \geq -t\phi(t) \qquad \text{for } t \leq t_0(\omega).$$

Thus the lemma is proved in the convergent case.

iii. Let ϕ be an arbitrary function satisfying the conditions of theorem 4.3.1 and $I(\psi) = \infty$. Define the function $\hat{\psi}$ by (4.3.7). If the set $\{t:\psi(t) \leq \psi_1(t)\}$ is bounded we have for sufficiently large t $\psi_1(t) < \psi(t)$ implying $\hat{\psi}(t) = \min(\psi(t),\psi_2(t))$. This implies $I(\hat{\psi}) = \infty$. If, on the contrary, the set $\{t:\psi(t) \leq \psi_1(t)\}$ is not bounded, we obtain $I(\hat{\psi}) = \infty$, by an argument similar as in part *i.* Hence, by the assumption of the lemma, for almost all ω there exists a decreasing sequence $\{t_n'\}$ (which depends on ω) such that

$$X(t_n',\omega)-(2/\pi)t_n' \log t_n' < -t_n'\hat{\phi}(t_n').$$

Because $I(\psi_2) < \infty$ we have for almost all ω

$$X(t_n',\omega)-(2/\pi)t_n' \log t_n' \geq -t_n'\phi_2(t_n')$$

for sufficiently large n. Then we have for sufficiently large n

$\hat{\phi}(t_n') < \phi_2(t_n')$ implying $\phi(t_n') \le \hat{\phi}(t_n')$. This yields for almost all ω

$$X(t_n',\omega)-(2/\pi)t_n' \log t_n' < -t_n'\phi(t_n'). \quad \square$$

LEMMA 4.3.2. *Let* ψ *be a positive, continuous and non-decreasing function satisfying* $\psi(t) \le \psi_2(t)$. *Let the sequence* $\{t_k\}$ *be defined by (4.3.3). Then*

$$I(\psi) < \infty \quad \text{iff} \quad \sum_k \frac{1}{\psi(t_k)}\, e^{-\frac{1}{2}\psi^2(t_k)} < \infty.$$

PROOF. From (4.3.3) it follows that

$$(4.3.8) \qquad (t_k-t_{k-1})t_k^{-1} \sim (\log k)^{-1} \qquad \text{for } k \to \infty.$$

From the proof of lemma 4.3.1 part *i* we know that $I(\psi) < \infty$ implies $\psi(t) > \psi_1(t)$ for sufficiently large t. Because $\psi(t)t^{-1}e^{-\frac{1}{2}\psi^2(t)}$ is decreasing for large t, we have for sufficiently large k

$$(4.3.9) \qquad \left(\frac{t_k-t_{k-1}}{t_k}\right) \psi(t_k)e^{-\frac{1}{2}\psi^2(t_k)} \le \int_{t_{k-1}}^{t_k} \frac{\psi(t)}{t}\, e^{-\frac{1}{2}\psi^2(t)}dt \le$$

$$\le \left(\frac{t_k-t_{k-1}}{t_{k-1}}\right) \psi(t_{k-1})e^{-\frac{1}{2}\psi^2(t_{k-1})}.$$

In case $I(\psi) < \infty$ the function ψ satisfies for large t

$$1-\delta \le (2 \log\log t)^{-1}\psi^2(t) \le 1+\delta.$$

Then by (4.3.3) and (4.3.8) we have for some positive constant a_1

$$\left(\frac{t_k-t_{k-1}}{t_k}\right) \psi(t_k) \ge \frac{a_1}{\psi(t_k)}\ .$$

Then one part of the assertion in the lemma now follows easily.

In case $\sum_k \frac{1}{\psi(t_k)}\, e^{-\frac{1}{2}\psi^2(t_k)} < \infty$, the assumption $\psi(t) \le \psi_2(t)$ guarantees the existence of a constant a_2 such that

$$\left(\frac{t_k-t_{k-1}}{t_{k-1}}\right)\psi(t_{k-1}) \le \frac{a_2}{\psi(t_{k-1})}\ .$$

This implies the other part of the assertion in the lemma. □

PROOF of theorem 4.3.1. By the preceding lemmas we may restrict ourselves to the case where $\psi_1 \le \psi \le \psi_2$. This implies

$$(4.3.10)\qquad (4/\pi)\log\left[2^{-1}(\pi e)^{\frac{1}{2}}\{2(1-\delta)\log\log t^{-1}\}^{\frac{1}{2}}\right] \le \phi(t) \le$$
$$\le (4/\pi)\log\left[2^{-1}(\pi e)^{\frac{1}{2}}\{2(1+\delta)\log\log t^{-1}\}^{\frac{1}{2}}\right].$$

Hence

$$(4.3.11)\qquad \phi(t) \sim (2/\pi)\log\log\log t^{-1} \qquad \text{for } t \downarrow 0.$$

Suppose the integral (4.1.1) converges and let the sequence t_k be defined by (4.3.3). Consider the events

$$A_k:\ \inf_{t_{k+1}^{-1}<t\le t_k^{-1}} \frac{X(t)-(2/\pi)t\log t}{t} < -\phi(t_k^{-1}) \qquad \text{for } k=1,2,\dots\ .$$

Then

$$P[A_k] \le k_1\ P[X(1) \le -\phi(t_k^{-1})]$$
by lemma 3.5.4.a
$$\sim k_1\ 2^{\frac{1}{2}}\ P[U \ge \psi(t_k)] \qquad \text{for } k \to \infty$$
by theorem 2.1.7 V
$$\sim k_1\pi^{-\frac{1}{2}}\ \{\psi(t_k)\}^{-1}e^{-\frac{1}{2}\psi^2(t_k)} \qquad \text{for } k \to \infty$$
by theorem 2.1.7 VII.

By lemma 4.3.2 $I(\psi) < \infty$ implies $\sum_k P[A_k] < \infty$. Hence from the Borel-Cantelli lemma it follows that $P[A_k \text{ i.o.}] = 0$. Thus for almost all ω there exists a number $k_0(\omega)$ such that

$$X(t,\omega)-(2/\pi)t\log t \geq -t\phi(t_k^{-1}) \geq -t\phi(t)$$

for $t \in [t_{k+1}^{-1}, t_k^{-1}]$ and $k \geq k_0$.

Suppose the integral (4.1.1) diverges. With the same sequence $\{t_k\}$ we define the events

$$B_k: X(t_k^{-1})-(2/\pi)t_k^{-1}\log t_k^{-1} < -t_k^{-1}\phi(t_k^{-1}).$$

By theorem 2.1.7 part V and part VII

$$P[B_k] \sim \pi^{-\frac{1}{2}}\{\psi(t_k)\}^{-1}e^{-\frac{1}{2}\psi^2(t_k)} \qquad \text{for } k \to \infty.$$

By lemma 4.3.2 divergence of the integral (4.1.1) implies $\sum_k P[B_k] = \infty$. In order to apply the extension of the Borel-Cantelli lemma we have to compute

$$\liminf_{n\to\infty} \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} P[B_i \wedge B_j]}{\{\sum_{j=1}^{n} P[B_j]\}^2} .$$

Consider for fixed i and $j(> i)$ the term $P[B_i \wedge B_j] =$
$= P[X(t_i^{-1}) \leq (2/\pi)t_i^{-1}\log t_i^{-1}-t_i^{-1}\phi(t_i^{-1}) \wedge X(t_j^{-1}) \leq (2/\pi)t_j^{-1}\log(t_j^{-1})-t_j^{-1}\phi(t_j^{-1})]$.
By making use of the lemmas in section 3.5 we can obtain the following results.

a. For each $\varepsilon > 0$ and $\delta > 0$ there exists a number i_0 such that for all $i \geq i_0$ and $j \geq i+(\log i)^{2+\delta}$ we have by lemma 3.5.1

$$(4.3.12) \qquad P[B_i \wedge B_j] \leq (1+\varepsilon)\, P[B_i]\, P[B_j].$$

b. Let M be an arbitrary positive (large) number. We now consider events with

$$(4.3.13) \qquad M^{-1}\log i \leq j-i < (\log i)^{2+\delta}.$$

By lemma 3.5.2 it follows that there exist constants i_1, C_1 and C_2 such that for $i \geq i_1$

$$(4.3.14)\qquad P[B_i \wedge B_j] \le C_1 e^{-C_2\psi^2(t_i)}\ P[B_j].$$

Let, for fixed j, R_j be the number of values i satisfying (4.3.13). Then $R_j = O((\log j)^{2+\delta})$ for $j \to \infty$. By (4.3.10) and (4.3.14) we have that there exists a constant a_1 such that for every j

$$(4.3.15)\qquad \sum_i^* P[B_i \wedge B_j] \le a_1\ P[B_j],$$

where $\sum_i^*$ denotes the summation, for fixed j, over all events B_i satisfying (4.3.13) and $i \ge i_1$.

c. For indices satisfying

$$(4.3.16)\qquad i < j < i+M^{-1}\log i$$

there exist, by lemma 3.5.3, constants i_2, C_3 and C_4 such that for $i \ge i_2$

$$P[B_i \wedge B_j] \le C_3 e^{-C_4((t_j-t_i)/t_j)\psi^2(t_i)}\ P[B_i].$$

By (4.3.3) and (4.3.6) we have for $i \ge i_2$

$$P[B_i \wedge B_j] \le C_5 e^{-C_6(j-i)}\ P[B_i],$$

where C_5 and C_6 are positive constants. Hence for $i \ge i_2$ there exists a constant a_2 such that

$$(4.3.17)\qquad \sum_j^{**} P[B_i \wedge B_j] \le a_2\ P[B_i],$$

where $\sum_j^{**}$ restricts the summation to all values of j satisfying (4.3.16).

Let $i_3 = \max(i_0,i_1,i_2)$. For $n > i_3$ we have, by (4.3.12), (4.3.15) and (4.3.17),

$$\sum_{i=1}^n \sum_{j=1}^n P[B_i \wedge B_j] = \sum_{j=1}^n P[B_j] + 2\sum_{i<j}\sum P[B_i \wedge B_j] \le$$

$$\le (1+2i_3+2a_1+2a_2)\sum_{j=1}^n P[B_j] + (1+\varepsilon)\sum_{i=1}^n\sum_{j=1}^n P[B_i]\,P[B_j].$$

Hence

$$\liminf_{n\to\infty} \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} P[B_i \wedge B_j]}{\{\sum_{j=1}^{n} P[B_j]\}^2} = 1$$

and the divergence part of the theorem follows from lemma 1.4.2. □

Two consequences of theorem 4.3.1 are

(4.3.18) $$\liminf_{t\downarrow 0} \{\frac{X(t)-(2/\pi)t\log t}{t} + \frac{2}{\pi}\log(\pi e\log\log t^{-1})\} = $$
$$= \frac{2}{\pi}\log 2 \qquad \text{a.s.}$$

and hence

(4.3.19) $$\liminf_{t\downarrow 0} \frac{X(t)}{(2/\pi)t\log t} = 1 \qquad \text{a.s. .}$$

4.4. THE CASE $1 < \alpha < 2$

THEOREM 4.4.1. *Let* $\{X(t) : 0 \le t < \infty\}$ *be a completely asymmetric stable process with* $1 < \alpha < 2$ *and* $\beta = 1$, *let* ϕ *be a positive, continuous and non-increasing function and define*

$$\psi(t^{-1}) = \{2B(\alpha)\}^{\frac{1}{2}}\{\phi(t)\}^{\overline{2(\alpha-1)}\,\alpha} ,$$

where $B(\alpha)$ *is defined by (2.1.8). Then*

$$P[\{\omega: \textit{there exists some } t_0(\omega) > 0 \textit{ such that } X(t,\omega) \ge -t^{1/\alpha}\phi(t)$$
$$\textit{for all } t \le t_0(\omega)\}] = 0 \textit{ or } 1$$

according as the integral (4.1.1) diverges or converges.

PROOF. The proof is similar to the proof for the case $\alpha = 1$ and to those of generalized L.I.L. theorems for $t \to \infty$ and partial sums. (See chapters 5 and 6.) □

This theorem implies

$$\liminf_{t\downarrow 0} \frac{X(t)}{t^{1/\alpha}(2\log\log t^{-1})^{(\alpha-1)/\alpha}} = -\{2B(\alpha)\}^{-(\alpha-1)/\alpha} \quad \text{a.s.}\ .$$

CHAPTER 5

GENERALIZED LAWS OF THE ITERATED LOGARITHM FOR LARGE TIMES

The duality between small and large times is given in property 3 of section 3.1 for the Wiener process and in property 3 of section 3.2 for other stable processes. By using this duality we obtain the generalized laws of the iterated logarithm for large times. In this chapter we shall not give proofs of the theorems but only formulate the results, make some remarks and give references (if they exist). Theorem 5.1.1 for the Wiener process follows immediately from property 3 of section 3.1 and theorem 4.1.1. The assertion in property 3 of section 3.2 is weaker than that of property 3 of section 3.1. Therefore, the theorems for stable processes with $\alpha \neq 2$ do not follow in this way. They can be proved by making use of exactly the same methods as in chapter 4.

5.1. THE CASE $\alpha = 2$

THEOREM 5.1.1. *Let* $\{W(t) : 0 \le t < \infty\}$ *be a Wiener process,* ϕ *a positive, continuous and non-decreasing function and take* $\psi = \phi$. *Then*

$$P[\{\omega: \text{there exists some } t_0(\omega) > 0 \text{ such that } W(t,\omega) \le t^{\frac{1}{2}}\phi(t) \text{ for all } t \ge t_0(\omega)\}] = 0 \text{ or } 1$$

according as the integral (4.1.1) diverges or converges.

An elegant proof of this theorem is given by MOTOO (1959).

As a consequence of this theorem we have *Khintchine's classical law of the iterated logarithm*

$$\limsup_{t\to\infty} \frac{W(t)}{(2t \log\log t)^{\frac{1}{2}}} = 1 \qquad \text{a.s.}$$

and by symmetry

$$\liminf_{t\to\infty} \frac{W(t)}{(2t \log\log t)^{\frac{1}{2}}} = -1 \qquad \text{a.s. .}$$

5.2. THE CASE $0 < \alpha < 1$

THEOREM 5.2.1. *Let* $\{X(t) : 0 \le t < \infty\}$ *be a completely asymmetric stable process with* $0 < \alpha < 1$ *and* $\beta = 1$. *Let* ϕ *be a positive, continuous and non-increasing function and take*

$$\psi(t) = \{2B(\alpha)\}^{\frac{1}{2}}\{\phi(t)\}^{-\frac{\alpha}{2(1-\alpha)}}.$$

Then

$$P[\{\omega: \textit{there exists some } t_0(\omega) > 0 \textit{ such that } X(t,\omega) \ge t^{1/\alpha}\phi(t) \textit{ for all } t \ge t_0(\omega)\}] = 0 \textit{ or } 1$$

according as the integral (4.1.1) diverges or converges.

A proof is given by BREIMAN (1968b) following MOTOO's proof for the Wiener process.

As a consequence we have

$$\liminf_{t\to\infty} \frac{X(t)}{t^{1/\alpha}(2\log\log t)^{-(1-\alpha)/\alpha}} = \{2B(\alpha)\}^{(1-\alpha)/\alpha} \qquad \text{a.s. .}$$

This last result was first proved by FRISTEDT (1964). For general increasing processes with stationary independent increments similar results are obtained by FRISTEDT and PRUITT (1971).

5.3. THE CASE $\alpha = 1$

THEOREM 5.3.1. *Let* $\{X(t) : 0 \le t < \infty\}$ *be a completely asymmetric stable process with* $\alpha = \beta = 1$. *Let* ϕ *be a positive, continuous and non-decreasing function and take*

$$\psi(t) = 2(\pi e)^{-\frac{1}{2}} \exp(\pi\phi(t)/4).$$

Then

$$P[\{\omega: \textit{there exists some } t_0(\omega) > 0 \textit{ such that } X(t,\omega)-(2/\pi)t\log t \geq$$
$$\geq -t\phi(t) \textit{ for all } t \geq t_0(\omega)\}] = 0 \textit{ or } 1$$

according as the integral (4.1.1) diverges or converges.

As a consequence we have

$$(5.3.1) \qquad \liminf_{t\to\infty} \left\{\frac{X(t)-(2/\pi)t\log t}{t} + (2/\pi)\log(\pi e \log\log t)\right\} = (2/\pi)\log 2 \qquad \text{a.s.}$$

and

$$(5.3.2) \qquad \liminf_{t\to\infty} \frac{X(t)}{(2/\pi)t\log t} = 1 \qquad \text{a.s.}.$$

This last consequence is also proved by MILLAR (1972). See also section 6.3.

5.4. THE CASE $1 < \alpha < 2$

THEOREM 5.4.1. *Let* $\{X(t) : 0 \leq t < \infty\}$ *be a completely asymmetric stable process with* $1 < \alpha < 2$ *and* $\beta = 1$. *Let* ϕ *be a positive, continuous and non-decreasing function and take*

$$\psi(t) = \{2B(\alpha)\}^{\frac{1}{2}}\{\phi(t)\}^{\frac{\alpha}{2(\alpha-1)}}.$$

Then

$$P[\{\omega: \textit{there exists some } t_0(\omega) > 0 \textit{ such that } X(t,\omega) \geq -t^{1/\alpha}\phi(t)$$
$$\textit{for all } t \geq t_0(\omega)\}] = 0 \textit{ or } 1$$

according as the integral (4.1.1) diverges or converges.

As a consequence we have

$$\liminf_{t\to\infty} \frac{X(t)}{t^{1/\alpha}(2\log\log t)^{(\alpha-1)/\alpha}} = -\{2B(\alpha)\}^{-(\alpha-1)/\alpha} \qquad \text{a.s.}.$$

CHAPTER 6

GENERALIZED LAWS OF THE ITERATED LOGARITHM FOR PARTIAL SUMS

Throughout this chapter $X_1, X_2, \ldots$ will be i.i.d. random variables. Write $S_n = X_1 + \ldots + X_n$. The theorems will be formulated for the standard normal r.v. and completely asymmetric stable random variables. Partially the theorems follow from the results in chapter 5, because we now consider the processes at discrete points $t=1,2,\ldots$. As we saw in section 4.3 the proofs of generalized L.I.L. theorems rest on the Borel-Cantelli lemma. It is therefore obvious that these theorems are also true for those random variables in the domain of attraction for which the distribution function of the normalized sum converges sufficiently fast to the corresponding stable distribution. This is discussed further in chapter 10.

6.1. THE CASE $\alpha = 2$

Following FELLER (1943) we first sketch the historic development of the L.I.L.. Let Y be a randomly selected point of the interval (0,1) and let its binary expansion be given by

$$Y = \sum_{n=1}^{\infty} Y_n \, 2^{-n}.$$

We define $X_n = 2Y_n - 1$. Then the random variables $X_1, X_2, \ldots$ are i.i.d. with common distribution $P[X_1=1] = P[X_1=-1] = \frac{1}{2}$. The sum S_n is the difference of the frequencies of occurence of the digits 1 and 0 among the first n places in the expansion of Y.

1. HAUSDORFF (1913):

$$S_n = o(n^{\frac{1}{2}+\varepsilon}) \qquad \text{a.s.} \qquad \text{for every } \varepsilon > 0.$$

2. HARDY-LITTLEWOOD (1914):

$$S_n = O((n \log n)^{\frac{1}{2}}) \qquad \text{a.s. .}$$

3. STEINHAUS (1922):

$$\limsup_{n\to\infty} S_n/(2n \log n)^{\frac{1}{2}} \le 1 \qquad \text{a.s.} \; .$$

4. KHINTCHINE (1923):

$$S_n = O((n \log \log n)^{\frac{1}{2}}) \qquad \text{a.s.} \; .$$

5. KHINTCHINE (1924):

$$\limsup_{n\to\infty} S_n/(2n \log \log n)^{\frac{1}{2}} = 1 \qquad \text{a.s.} \; .$$

6. LÉVY (1933):

$$P[S_n > n^{\frac{1}{2}}(2 \log \log n + a \log \log \log n)^{\frac{1}{2}} \text{i.o.}] = \begin{cases} 0 & \text{if } a > 3 \\ 1 & \text{if } a \le 1. \end{cases}$$

7. KOLMOGOROV-ERDÖS (1942):

If ϕ is non-decreasing, then

$$P[S_n > n^{\frac{1}{2}}\phi(n) \text{ i.o.}] = 0 \text{ or } 1$$

according as the integral $I(\phi)$, defined by (4.1.1), converges or diverges.

The last result gives a complete solution for i.i.d. Bernoulli trials. The above results have been extended in various directions. For example to other random variables with finite or infinite variance, not identically distributed r.v.'s or dependent r.v.'s.

HARTMAN and WINTNER (1941) show that

$$(6.1.1) \qquad \limsup_{n\to\infty} \frac{S_n}{(2n \log \log n)^{\frac{1}{2}}} = 1 \qquad \text{a.s.}$$

for i.i.d. $X_1, X_2, \ldots$ with $E\, X_i = 0$ and $\sigma^2(X_i) = 1$, i.e. $X_i \in \mathcal{D}_N(2,0)$. The case $X_i \in \mathcal{D}(2,0)$ and $\sigma^2(X_i) = \infty$ is studied by FELLER (1968). STRASSEN (1964) proves a beautiful generalization of Hartman-Wintner's result, that we shall discuss in chapters 9 and 10. In most proofs of L.I.L. type theorems the rate of convergence in the central limit theorem plays an important role. STRASSEN, however, obtains his result by embedding the r.v.'s X_i in the

Wiener process.

FELLER (1943) generalizes the Kolmogorov-Erdös result to general random variables X_k subject to some conditions. For example to i.i.d. random variables X_k satisfying $E\, X_k = 0$ and $E|X_k|^{2+\varepsilon} < \infty$ for some positive ε. In this section we formulate his theorem for i.i.d. random variables with a standard normal distribution.

THEOREM 6.1.1. *Let* $X_1, X_2, \ldots$ *be i.i.d. random variables with a standard normal distribution and* ϕ *a positive, continuous and non-decreasing function on* $(0,\infty)$. *Then*

$$P[S_n \geq n^{\frac{1}{2}}\phi(n) \text{ i.o.}] = 0 \text{ or } 1$$

according as $I(\phi)$, *defined in (4.1.1), converges or diverges.*

REMARK 6.1.1. For almost all ω there exists, for all $v \in [-1,1]$, a sequence $\{n_k(v,\omega)\}$ such that

$$\lim_{k\to\infty} \frac{S_{n_k(v,\omega)}(\omega)}{\{n_k(v,\omega)\}^{\frac{1}{2}}\{2 \log\log n_k(v,\omega)\}^{\frac{1}{2}}} = v.$$

6.2. THE CASE $0 < \alpha < 1$

THEOREM 6.2.1. *Let* $X_1, X_2, \ldots$ *be i.i.d. random variables with common distribution function* $F(.;\alpha,1)$ *with* $0 < \alpha < 1$. *Let* ϕ *be a positive, continuous and non-increasing function on* $(0,\infty)$ *and take*

$$\psi(t) = \{2B(\alpha)\}^{\frac{1}{2}}\{\phi(t)\}^{-\frac{\alpha}{2(1-\alpha)}}. \tag{6.2.1}$$

Then

$$P[S_n \leq n^{1/\alpha}\phi(n) \text{ i.o.}] = 0 \text{ or } 1$$

according as the integral (4.1.1) converges or diverges.

PROOF. The proof of this theorem, and extended to the case of positive, continuous r.v.'s in the domain of attraction of a completely asymmetric law (with some restrictions on the right-hand tail) is given by LIPSCHUTZ (1956b) and KALINAUSKAĬTE (1971).

This theorem implies

$$(6.2.2)\qquad \liminf_{n\to\infty} \frac{S_n}{n^{1/\alpha}(2\log\log n)^{-(1-\alpha)/\alpha}} = \{2B(\alpha)\}^{(1-\alpha)/\alpha} \qquad \text{a.s.} \ .$$

REMARK 6.2.1. For almost all ω there exists, for all $v \geq 1$, a sequence $\{n_k(v,\omega)\}$ such that

$$\lim_{k\to\infty} \frac{S_{n_k(v,\omega)}(\omega)}{\{n_k(v,\omega)\}^{1/\alpha}\{2\log\log n_k(v,\omega)\}^{-(1-\alpha)/\alpha}} = v\ \{2B(\alpha)\}^{(1-\alpha)/\alpha}.$$

6.3. THE CASE $\alpha = 1$

THEOREM 6.3.1. *Let* $X_1, X_2, \ldots$ *be i.i.d. random variables with common distribution function* $F(.;1,1)$. *Let* ϕ *be a positive, continuous and non-decreasing function on* $(0,\infty)$ *and take*

$$(6.3.1)\qquad \psi(t) = 2(\pi e)^{-\frac{1}{2}}\exp(\pi\phi(t)/4).$$

Then

$$P[S_n-(2/\pi)n\log n \leq -n\phi(n) \text{ i.o.}] = 0 \text{ or } 1$$

according as the integral (4.1.1) converges or diverges.

PROOF. MIJNHEER (1972). □

REMARK 6.3.1. Take

$$\phi(t) = (2/\pi)\log(\pi e\log\log t)-(2/\pi)\log 2+(2/\pi)\log\lambda.$$

By (6.3.1) this is equivalent with

$$\psi(t) = (2\lambda \log \log t)^{\frac{1}{2}}.$$

Then

$$P[S_n-(2/\pi)n\log n \le -n\phi(n) \text{ i.o.}] = 0 \text{ or } 1$$

according as $\lambda > 1$ or ≤ 1.

This implies

$$(6.3.2) \qquad \liminf_{n\to\infty} \left\{\frac{S_n-(2/\pi)n\log n}{n} + \frac{2}{\pi}\log(\pi e \log\log n)\right\} = \frac{2}{\pi}\log 2 \qquad \text{a.s.}\ .$$

REMARK 6.3.2. As a consequence we have

$$(6.3.3) \qquad \liminf_{n\to\infty} \frac{S_n}{(2/\pi)n\log n} = 1 \qquad \text{a.s.}\ .$$

The result (6.3.3) was proved by MILLER (1967) in case $X_i \in \mathcal{D}(\alpha,1)$ with some restrictions on the right tail.

From the expansions for the tails of the distribution function (theorem 2.1.7 part II and part V) one easily proves

$$(6.3.4) \qquad S_n/\{(2/\pi)n\log n\} \xrightarrow{P} 1$$

and

$$(6.3.5) \qquad E\, X_1 = \infty.$$

The latter implies

$$S_n/n \longrightarrow \infty \qquad \text{a.s.}\ .$$

From a paper of CHOW and ROBBINS (1961) we know that

$$(6.3.6) \qquad \limsup_{n\to\infty} \frac{S_n}{(2/\pi)n\log n} = \infty \qquad \text{a.s.}\ .$$

The result (6.3.6) is also a consequence of theorem 8.1.1.

Let us now consider the results (6.3.2) up to (6.3.6). Roughly speaking we can say that the average S_n/n tends to ∞ like $(2/\pi)\log n$. Moreover, from (6.3.3) and (6.3.6) we have the surprising result: for almost all ω there exist (infinite) sequences $\{n_k(\omega)\}$ and $\{m_k(\omega)\}$ such that

$$\lim_{k\to\infty} \frac{S_{n_k(\omega)}(\omega)}{(2/\pi)n_k(\omega)\log(n_k(\omega))} = 1$$

and

$$\lim_{k\to\infty} \frac{S_{m_k(\omega)}(\omega)}{(2/\pi)m_k(\omega)\log(m_k(\omega))} = \infty .$$

6.4. THE CASE $1 < \alpha < 2$

THEOREM 6.4.1. *Let* $X_1, X_2, \ldots$ *be i.i.d. random variables with common distribution function* $F(.;\alpha,1)$ *with* $1 < \alpha < 2$. *Let* ϕ *be a positive, continuous and non-decreasing function on* $(0,\infty)$ *and take*

$$(6.4.1) \qquad \psi(t) = \{2B(\alpha)\}^{\frac{1}{2}}\{\phi(t)\}^{\frac{\alpha}{2(\alpha-1)}} .$$

Then

$$P[S_n \le -n^{1/\alpha}\phi(n) \text{ i.o.}] = 0 \text{ or } 1$$

according as the integral (4.1.1) converges or diverges.

PROOF. The convergence part of this theorem follows immediately from the convergence part of theorem 5.4.1.

The r.v.'s S_n, $n=1,2,\ldots$, have the same distribution as a completely asymmetric stable process ($\alpha \in (1,2)$; $\beta = 1$) $\{X(t) : 0 \le t < \infty\}$ at the points $t=1,2,\ldots$. The divergence part of theorem 5.4.1 implies that for almost all ω there exist a sequence $t_k = t_k(\omega)$ such that

$$(6.4.2) \qquad X(t_k) \le -t_k^{1/\alpha}\phi(t_k).$$

We shall show that the inequality (6.4.2) is also true for infinitely many integer values of t. Let n_k be defined, for each k, as the nearest integer to $e^{k/\log k}$. Define the events

$$B_k: S_{n_k} \le -n_k^{1/\alpha}\phi(n_k) .$$

As in the proof of theorem 4.3.1 we have $\sum P[B_k] = \infty$ and by making use of the lemmas in section 3.6 and by lemma 1.4.2 it follows that

$$P[B_k \text{ i.o.}] = 1. \qquad \square$$

As a consequence we have

$$\liminf_{n\to\infty} \frac{S_n}{n^{1/\alpha}(2\log\log n)^{(\alpha-1)/\alpha}} = -\{2B(\alpha)\}^{(\alpha-1)/\alpha} \qquad \text{a.s. .}$$

REMARK 6.4.1. Note that the distribution function $F(.;\alpha,1)$, with $1 < \alpha < 2$, has support $(-\infty,\infty)$. LIPSCHUTZ (1956b) has established an integral-test for partial sums of positive, continuous random variables $\in \mathcal{D}(\alpha,1)$ with some assumptions on the right tail. See also KALINAUSKAĬTE (1971).

CHAPTER 7

HÖLDER-TYPE THEOREMS

The generalized L.I.L. theorems in chapter 4 give the local behavior of the sample paths near $t = 0$. By the properties 2 of sections 3.1 and 3.2 we obviously have the same behavior in the neighbourhood of every fixed point $\tau(>0)$. In this chapter we consider processes on $[0,1]$ and we study a modulus of continuity result for the Wiener process and completely asymmetric stable processes.

7.1. THE CASE $\alpha = 2$

Consider the Wiener process $\{W(t) : 0 \le t \le 1\}$. We saw in theorem 3.1.1.b that almost all sample paths are continuous functions on $[0,1]$. Let ϕ be a positive, continuous and non-increasing function. Consider the probability

$$(7.1.1) \qquad P[\{\omega\text{: there exists some } \Delta_0(\omega) > 0 \text{ such that } |W(t+\Delta,\omega)-W(t,\omega)| \le \le \Delta^{\frac{1}{2}}\phi(\Delta) \text{ for all } 0 \le t \le 1-\Delta \text{ and } 0 < \Delta \le \Delta_0(\omega)\}].$$

In this section we establish an integral test, comparable to the criterion in the generalized L.I.L. for $W(t)$ at time $t = 0$, for deciding whether the probability in (7.1.1) has the value zero or one. Concerning this problem of the modulus of continuity of $W(t)$ we have the following historic development. Let the function ψ be defined by

$$\psi(t^{-1}) = \phi(t).$$

1. LÉVY (1937):

$$\psi(t) = c(2 \log t)^{\frac{1}{2}}.$$

The probability in (7.1.1) is zero for $c < 1$ and one for $c > 1$. As a consequence of this result we have

$$\lim_{\varepsilon\downarrow 0} \sup_{\substack{0\le t\le 1-\Delta \\ 0<\Delta<\varepsilon}} \frac{|W(t+\Delta)-W(t)|}{(2\Delta \log(\Delta^{-1}))^{\frac{1}{2}}} = 1 \qquad \text{a.s.} .$$

2. SIRAO (1954):

$$\psi(t) = (2 \log t + c \log\log t)^{\frac{1}{2}}.$$

The probability in (7.1.1) is zero for $c < -1$ and one for $c > 5$.

3. CHUNG, ERDÖS and SIRAO (1959):

The probability in (7.1.1) is zero or one according as the integral

$$(7.1.2) \qquad J(\psi) = \int^{\infty} \psi^3(t) e^{-\frac{1}{2}\psi^2(t)} dt$$

diverges or converges.

REMARK 7.1.1. Let the function ψ be defined by

$$(7.1.3) \qquad \psi(t) = [2 \log t + 5 \log_{(2)}(t) + 2\sum_{k=3}^{n-1} \log_{(k)}(t) + c \log_{(n)}(t)]^{\frac{1}{2}},$$

where $\log_{(k)}(t) = \log(\log_{(k-1)}(t))$ and $n \geq 3$. Then integral (7.1.2) converges for $c > 2$ and diverges for $c \leq 2$.

7.2. THE CASE $0 < \alpha < 1$

In this section we shall establish a similar integral test for the completely asymmetric stable process $\{X(t) : 0 \leq t \leq 1\}$ with characteristic exponent $0 < \alpha < 1$ and $\beta = 1$. Let ϕ be a positive, continuous and non-decreasing function. We define the function ψ by

$$(7.2.1) \qquad \psi(t^{-1}) = \{2B(\alpha)\}^{\frac{1}{2}} \{\phi(t)\}^{-\frac{\alpha}{2(1-\alpha)}}.$$

THEOREM 7.2.1. *Let ϕ and ψ be defined as above and let $\{X(t) : 0 \leq t \leq 1\}$ be the completely asymmetric stable process with $0 < \alpha < 1$ and $\beta = 1$. Then*

$$(7.2.2) \qquad P[\{\omega: \text{there exists some } \Delta_0(\omega) > 0 \text{ such that } X(t+\Delta,\omega) - X(t,\omega) \geq$$
$$\geq \Delta^{1/\alpha}\phi(\Delta) \text{ for all } 0 \leq t \leq 1-\Delta \text{ and } 0 < \Delta \leq \Delta_0(\omega)\}] =$$
$$= 0 \text{ or } 1$$

according as the integral (7.1.2) diverges or converges.

As in the proof of the L.I.L. type theorems we may restrict ourselves to special choices for ψ. We define the functions ψ_1 and ψ_2 by

$$(7.2.3) \qquad \psi_1(t) = (2 \log t - 10 \log \log t)^{\frac{1}{2}}$$

and

$$(7.2.4) \qquad \psi_2(t) = (2 \log t + 10 \log \log t)^{\frac{1}{2}}.$$

LEMMA 7.2.1. *Let ψ_1 and ψ_2 be defined by (7.2.3) and (7.2.4). If theorem 7.2.1 holds for all functions ϕ such that*

$$(7.2.5) \qquad \psi_1(t) \le \psi(t) \le \psi_2(t),$$

where ψ is defined in (7.2.1), then it holds in general.

PROOF. The proof of this lemma has the same pattern as the one of lemma 4.3.1. We follow the proof of lemma 1 in the paper of CHUNG, ERDÖS and SIRAO (1959). Define the function $\hat{\psi}$ by

$$\hat{\psi}(t) = \min(\max(\psi(t), \psi_1(t)), \psi_2(t)).$$

Let $\hat{\phi}$ correspond to $\hat{\psi}$ as ϕ does to ψ by (7.2.1). From the proof of CHUNG, ERDÖS and SIRAO we borrow the following results:

$$(7.2.6) \qquad \text{If } J(\psi) < \infty \text{ then } \hat{\psi}(t) \le \psi(t) \text{ for large } t.$$

$$(7.2.7) \qquad J(\psi) < \infty \text{ iff } J(\hat{\psi}) < \infty.$$

Suppose $J(\psi) < \infty$ and hence that $J(\hat{\psi}) < \infty$ and

$$(7.2.8) \qquad \hat{\phi}(h) \ge \phi(h) \text{ for sufficiently small } h$$

by (7.2.7) and (7.2.6). Then it follows from the assumption of the lemma that for $\hat{\phi}$ the probability in (7.2.2) is equal to 1. Then for almost all ω we have

$$X(t+\Delta,\omega)-X(t,\omega) \ge \Delta^{1/\alpha}\hat{\phi}(\Delta) \ge \Delta^{1/\alpha}\phi(\Delta)$$

for all $t \in [0,1-\Delta]$ and Δ sufficiently small. Thus the lemma is proved in the convergence case.

Suppose $J(\psi) = \infty$ and hence $J(\hat{\psi}) = \infty$. By the assumption of the lemma the probability in (7.2.2) is equal to 0 for $\hat{\phi}$. Hence, for almost all ω there exist sequences $\{t_n\}, \{t_n'\}$, $t_n' > t_n$ with the properties

$$(7.2.9) \qquad X(t_n',\omega)-X(t_n,\omega) < (t_n'-t_n)^{1/\alpha}\hat{\phi}(t_n'-t_n)$$

and

$$(7.2.10) \qquad t_n'-t_n \downarrow 0 \qquad\qquad \text{for } n \to \infty.$$

Because $J(\psi_2) < \infty$ we have by the assumption of the lemma that there exists, for almost all ω, a number Δ_0 such that

$$X(t+\Delta,\omega)-X(t,\omega) > \Delta^{1/\alpha}\phi_2(\Delta)$$

for all t and $\Delta \le \Delta_0$. Together with (7.2.9) this implies, for sufficiently large n,

$$(7.2.11) \qquad \hat{\phi}(t_n'-t_n) \le \phi(t_n'-t_n).$$

Now (7.2.9) and (7.2.11) imply for almost all ω

$$X(t_n',\omega)-X(t_n,\omega) < (t_n'-t_n)^{1/\alpha}\phi(t_n'-t_n).$$

This proves the lemma in the divergence case. □

PROOF of theorem 7.2.1. By the lemma 7.2.1 we may restrict our attention to the case where (7.2.5) holds. This is equivalent to

$$(7.2.12) \qquad \{2B(\alpha)\}^{\frac{1-\alpha}{\alpha}}\{2 \log t^{-1} + 10 \log \log t^{-1}\}^{-\frac{1-\alpha}{\alpha}} \le \phi(t) \le$$
$$\le \{2B(\alpha)\}^{\frac{1-\alpha}{\alpha}}\{2 \log t^{-1} - 10 \log \log t^{-1}\}^{-\frac{1-\alpha}{\alpha}}$$

and yields

$$\phi(t) \sim \{B(\alpha)\}^{\frac{1-\alpha}{\alpha}}\{\log t^{-1}\}^{-\frac{1-\alpha}{\alpha}} \qquad\qquad \text{for } t \downarrow 0.$$

Thus the restriction (7.2.5) implies that $\phi(t) \to 0$ for $t \to 0$.

Suppose the integral (7.1.2) converges. For $p=1,2,\dots$, $k=0,1,\dots,2^p$,

$j=[p/3],\dots,p$ and $j+k \le 2^p$ we define the event $D^p_{j,k}$ by

$$X(\frac{j+k}{2^p})-X(\frac{k}{2^p}) < (\frac{j+2}{2^p})^{1/\alpha}\phi(\frac{j+2}{2^p}).$$

By theorem 2.1.7 IV we have uniformly in j and k

$$P[D^p_{j,k}] = P[X(1) < (\frac{j+2}{j})^{1/\alpha}\phi(\frac{j+2}{2^p})]$$

$$\sim (2/\alpha)^{\frac{1}{2}} P[U > (\frac{j+2}{j})^{-\frac{1}{2(1-\alpha)}}\psi(\frac{2^p}{j+2})] =$$

$$= O(1)\ P[U > \psi(\frac{2^p}{j+2})] \qquad \text{for } p \to \infty,$$

since

$$(\frac{j+2}{j})^{-\frac{1}{2(1-\alpha)}}\psi(\frac{2^p}{j+2}) = \psi(\frac{2^p}{j+2}) + O(1/\psi(\frac{2^p}{j+2})) \qquad \text{for } p \to \infty.$$

This and convergence of the integral (7.1.2) imply (see the proof of CHUNG, ERDÖS and SIRAO in the case $\alpha = 2$)

$$\sum_{p=1}^{\infty}\sum_{k=0}^{2p}\sum_{j=[p/3]}^{p} P[D^p_{j,k}] < \infty$$

and hence $P[D^p_{j,k} \text{ i.o.}] = 0$.

For arbitrary fixed $t, t+\Delta \in [0,1]$ and $\Delta < \frac{1}{2}$ we define integers p, j and k by

$$(7.2.13) \qquad (p+1)2^{-p-1} < \Delta \le p2^{-p}$$

and

$$(7.2.14) \qquad (k-1)2^{-p} < t \le k2^{-p} < (j+k)2^{-p} \le t+\Delta < (j+k+1)2^{-p}.$$

This implies $[p/3] \le j \le p$ for $p \ge 9$ and

$$X(t+\Delta,\omega)-X(t,\omega) \ge X(\frac{j+k}{2^p},\omega)-X(\frac{k}{2^p},\omega).$$

Hence, for almost all ω, we have for sufficiently small Δ (i.e. sufficiently large p and all $t \in [0,1-\Delta]$)

$$X(t+\Delta,\omega)-X(t,\omega) > \left(\frac{j+2}{2^p}\right)^{1/\alpha} \phi\left(\frac{j+2}{2^p}\right).$$

Because of the monotonicity of ϕ the right-hand member is larger than $\Delta^{1/\alpha}\phi(\Delta)$. Thus the theorem is proved for the case of convergence.

In the divergence case we define the event $E^p_{j,k}$ by

$$X\left(\frac{j+k}{2^p}\right)-X\left(\frac{k}{2^p}\right) < \left(\frac{j}{2^p}\right)^{1/\alpha} \phi\left(\frac{j}{2^p}\right)$$

for $p=1,2,\ldots$, $k=0,1,\ldots,2^p$, $j=[p/2]+1,\ldots,p$ and $j+k \le 2^p$. It is sufficient to prove $P[E^p_{j,k} \text{ i.o.}] = 1$. To prove this assertion we apply lemma 1.4.2. We order the events $E^p_{j,k}$. If $E_n = E^p_{j,k}$ and $E_{n'} = E^{p'}_{j',k'}$ then $n < n'$ iff one of the following conditions holds:

1. $p < p'$
2. $p = p'$ and $j > j'$
3. $p = p'$, $j = j'$ and $k < k'$.

This arrangement implies $j2^{-p} \ge j'2^{-p'}$ for $n < n'$. Divergence of the integral (7.1.2) implies $\sum P[E_n] = \infty$. (See the proof for $\alpha = 2$.) Consider two events $E_n = E^p_{j,k}$ and $E_{n'} = E^{p'}_{j',k'}$ with $n < n'$ and let $\Delta_{n,n'} > 0$ denote the length of the intersection of $[k2^{-p},(j+k)2^{-p}]$ and $[k'2^{-p'},(j'+k')2^{-p'}]$. We arrive at the following three conclusions.

1. By lemma 3.4.1 there exist, for any positive ε, a number p_0 and a positive constant δ such that

$$(7.2.15) \qquad P[E_n \wedge E_{n'}] \le (1+\varepsilon)\, P[E_n]\, P[E_{n'}]$$

for all events E_n and $E_{n'}$, with $n < n'$, $p \ge p_0$ and

$$(7.2.16) \qquad \Delta_{n,n'} j^{-1}2^p\psi^2(j^{-1}2^p) < \delta.$$

2. Let $0 < c < 1$. Computations similar to those in the paper of CHUNG, ERDÖS and SIRAO (1959) yield for fixed n'

(7.2.17) $\sum^{*} P[E_n \wedge E_{n'}] \le M_1\, P[E_{n'}]$,

where $\sum^{*}$ denotes the summation over all events E_n with $n < n'$ such that $\Delta_{n,n'} j^{-1} 2^{p} \le c$ and for which (7.2.16) does not hold. M_1 is a constant independent of n'.

3. In case

(7.2.18) $\frac{1}{2} \le c \le \Delta_{n,n'} j^{-1} 2^{p} < 1$

(the choice $c \ge \frac{1}{2}$ in (7.2.18) restricts the values of p' to $p' = p$, $p+1$ or $p+2$) the conditions of lemma 3.4.3 are fulfilled for large p. Following the computations in the proof for $\alpha = 2$ we obtain for every fixed n

(7.2.19) $\sum^{**} P[E_n \wedge E_{n'}] \le M_2\, P[E_n]$,

where $\sum^{**}$ restricts the summation to all $n' > n$ for which (7.2.18) holds and where M_2 is a constant.

From the estimates (7.2.15), (7.2.17) and (7.2.19) it follows that

$$\liminf_{N\to\infty} \Big(\sum_{n=1}^{N} P[E_n]\Big)^{-2} \sum_{n=1}^{N} \sum_{n=1}^{N} P[E_n \wedge E_{n'}] =$$

$$= \liminf_{N\to\infty} \Big(\sum_{n=1}^{N} P[E_n]\Big)^{-2} \cdot 2 \cdot \sum \sum_{n<n'}^{N} P[E_n \wedge E_{n'}] \le 1+\varepsilon.$$

Letting $\varepsilon \downarrow 0$ we obtain lim inf ≤ 1. Now we can apply lemma 1.4.2 in order to conclude $P[E_n \text{ i.o.}] = 1$. □

REMARK 7.2.1. Taking

$$\phi(\Delta) = \{2B(\alpha)\}^{\frac{1-\alpha}{\alpha}} \{2(1+\delta)\log(\Delta^{-1})\}^{-\frac{1-\alpha}{\alpha}}$$

we find that the probability in (7.2.2) is zero or one according as $\delta \le 0$ or $\delta > 0$. Hence one obtains

$$\lim_{\varepsilon\downarrow 0} \inf_{\substack{0\le t\le 1-\Delta \\ 0<\Delta<\varepsilon}} \frac{X(t+\Delta)-X(t)}{\Delta^{1/\alpha}\{2\log(\Delta^{-1})\}^{-(1-\alpha)/\alpha}} = \{2B(\alpha)\}^{(1-\alpha)/\alpha} \quad \text{a.s.} .$$

This result was first proved by HAWKES (1971).

7.3. THE CASE $\alpha = 1$

THEOREM 7.3.1. *Let ϕ be a non-negative, continuous and non-increasing function and $\{X(t) : 0 \leq t \leq 1\}$ the completely asymmetric stable process with $\alpha = \beta = 1$. Define the function ψ by*

$$\psi(t^{-1}) = 2(\pi e)^{-\frac{1}{2}}\exp(\pi\phi(t)/4).$$

Then

$$\text{(7.3.1)} \qquad P[\{\omega: \textit{there exists some } \Delta_0(\omega) > 0 \textit{ such that } X(t+\Delta,\omega)-X(t,\omega)+ -(2/\pi)\Delta\log\Delta \geq -\Delta\phi(\Delta) \textit{ for all } 0 \leq t \leq 1-\Delta \textit{ and } 0 < \Delta \leq \Delta_0(\omega)\}] = 0 \textit{ or } 1$$

according as the integral (7.1.2) diverges or converges.

PROOF. Again we may restrict ourselves to functions ψ satisfying (7.2.5). Hence

$$\phi(t) \sim (2/\pi)\log\log t^{-1} \qquad \text{for } t \downarrow 0$$

and this implies $\phi(t) \to \infty$ for $t \downarrow 0$.

Assume (7.1.2) converges. For $p=1,2,\ldots$, $k=0,1,\ldots,2^p$, $j=[p/3],\ldots,p$ and $j+k+1 \leq 2^p$ we define the event $D^p_{j,k}$ by

$$\inf_{0\leq r,s\leq 2^{-p}} \left\{\frac{X((j+k)2^{-p}+s)-X(k2^{-p}-r)-(2/\pi)(j2^{-p}+r+s)\log(j2^{-p}+r+s)}{j2^{-p}+r+s}\right\} < -\phi\left(\frac{j+2}{2^p}\right).$$

The restriction (7.2.5) implies that the conditions in lemma 3.5.4b are fulfilled uniformly in j. Thus

$$P[D^p_{j,k}] \leq k_1^2\; P[X(1) \leq -\phi(\tfrac{j+2}{2^p})].$$

By theorem 2.1.7 V this implies that, uniformly in j and k,

$$P[D^p_{j,k}] = O(1)\; P[U \geq \psi(\tfrac{2^p}{j+2})] \qquad \text{for } p \to \infty.$$

Convergence of (7.1.2) gives, as in the proof of CHUNG, ERDÖS and SIRAO for the case $\alpha = 2$, $P[D^p_{j,k}$ i.o.$] = 0$. For arbitrary $t, t+\Delta \in [0,1]$ we define p, j and k by (7.2.13) and (7.2.14). For almost all ω, we have for sufficiently large p

$$\frac{X(t+\Delta)-X(t)-(2/\pi)\Delta\log\Delta}{\Delta} \geq$$

$$\inf_{0\leq r,s\leq 2^{-p}} \left\{\frac{X((j+k)2^{-p}+s)-X(k2^{-p}-r)-(2/\pi)(j2^{-p}+r+s)\log(j2^{-p}+r+s)}{j2^{-p}+r+s}\right\} \geq$$

$$\geq -\phi\left(\frac{j+2}{2^p}\right) \geq -\phi(\Delta).$$

In the divergence case we define $E^p_{j,k}$ by

$$X((j+k)2^{-p})-X(k2^{-p})-(2/\pi)j2^{-p}\log(j2^{-p}) < -j2^{-p}\phi(j2^{-p})$$

for $p=1,2,\ldots$, $k=0,1,\ldots,2^p$, $j=[p/2]+1,\ldots,p$ and $j+k \leq 2^p$. The remainder of the proof closely resembles the proof of theorem 7.2.1. However, the necessary estimation of the lim inf occurring in lemma 1.4.2 differs on one point. This difference arises in connection with lemma 3.5.2. We want to use this lemma for the case $0 < \Delta t^{-1} < c < 1$. In that case $\Delta t^{-1}\psi^2(1/t')/\psi^2(1/t)$ is not necessarily less than one. However, one only has to invoke lemma 3.5.2 in case that $p'-5\log p' < p < p'$. Then by the restriction (7.2.5) we know that for any pair of constants (c, c_1) with $0 < c < c_1 < 1$, the restriction $\Delta t^{-1} < c$ implies $\Delta t^{-1}\psi^2(1/t')/\psi^2(1/t) < c_1$ for sufficiently large p' (or p).

Then, as in the proof of theorem 7.2.1 we can show $P[E_n$ i.o.$] = 1$ and hence the theorem is proved. □

REMARK 7.3.1. Taking

$$\phi(\Delta) = (2/\pi)\log(\pi e \log(\Delta^{-1}))-(2/\pi)\log 2 + (2/\pi)\log(1+\delta)$$

the probability in (7.3.2) is zero or one according as $\delta \leq 0$ or $\delta > 0$. Hence

$$\lim_{\varepsilon\downarrow 0}\left[\inf_{\substack{0\leq t\leq 1-\Delta\\ 0<\Delta<\varepsilon}}\left\{\frac{X(t+\Delta)-X(t)-(2/\pi)\Delta\log\Delta}{\Delta} + \frac{2}{\pi}\log(\pi e\log(\Delta^{-1}))\right\}\right] = \frac{2}{\pi}\log 2 \qquad \text{a.s.}$$

and

$$\lim_{\substack{\varepsilon\downarrow 0}} \inf_{\substack{0\le t\le 1-\Delta \\ 0<\Delta<\varepsilon}} \left\{\frac{X(t+\Delta)-X(t)-(2/\pi)\Delta\log\Delta}{(2/\pi)\Delta\log\log(\Delta^{-1})}\right\} = -1 \qquad \text{a.s.}\ .$$

7.4. THE CASE $1 < \alpha < 2$

THEOREM 7.4.1. *Let* ϕ *be a non-negative, continuous and non-increasing function and* $\{X(t) : 0 \le t \le 1\}$ *the completely asymmetric stable process with* $1 < \alpha < 2$ *and* $\beta = 1$. *Define the function* ψ *by*

$$(7.4.1) \qquad \psi(t^{-1}) = \{2B(\alpha)\}^{\frac{1}{2}}\{\phi(t)\}^{\frac{\alpha}{2(\alpha-1)}} .$$

Then

$$(7.4.2) \qquad P[\{\omega: \textit{there exists some } \Delta_0(\omega) > 0 \textit{ such that } X(t+\Delta,\omega)-X(t,\omega) \ge$$
$$\ge -\Delta^{1/\alpha}\phi(\Delta) \textit{ for all } 0 \le t \le 1-\Delta \textit{ and } 0 < \Delta \le \Delta_0(\omega)\}] =$$
$$= 0 \textit{ or } 1$$

according as the integral (7.1.2) diverges or converges.

PROOF. Again we may restrict ourselves to functions ψ satisfying (7.2.5). Hence it follows by (7.4.1)

$$\phi(t) \sim \{B(\alpha)\}^{\frac{\alpha-1}{\alpha}}\{\log t^{-1}\}^{\frac{\alpha-1}{\alpha}} \qquad \text{for } t \downarrow 0$$

and this implies $\phi(t) \to \infty$ for $t \downarrow 0$.

Suppose the integral (7.1.2) converges. For $p=1,2,\ldots$, $k=0,1,\ldots,2^p$, $j=[p/3],\ldots,p$ and $j+k+1 \le 2^p$ we define the event $D^p_{j,k}$ by

$$\inf_{0\le r,s\le 2^{-p}} \{X(\tfrac{j+k}{2^p}+s)-X(\tfrac{k}{2^p}-r)\} < -(\tfrac{j}{2^p})^{1/\alpha}\phi(\tfrac{j+2}{2^p}).$$

By lemma 3.6.4b we have

$$P[D^p_{j,k}] \le k^{-2}(\alpha,1)\ P[X(\tfrac{j+2}{2^p}) \le -(\tfrac{j}{2^p})^{1/\alpha}\phi(\tfrac{j+2}{2^p})]$$
$$= k^{-2}(\alpha,1)\ P[X(1) \le -(\tfrac{j}{j+2})^{1/\alpha}\phi(\tfrac{j+2}{2^p})].$$

Consequently, by theorem 2.1.7 VI we have, uniformly in j and k,

$$P[D^p_{j,k}] = O(1)\ P[U \geq \psi(\frac{2^p}{j+2})] \qquad \text{for } p \to \infty.$$

Hence it follows that $P[D^p_{j,k}$ i.o.$] = 0$. For any t and Δ we define integers p, j and k by (7.2.13) and (7.2.14). For almost all ω, we have for sufficiently large p

$$X(t+\Delta,\omega)-X(t,\omega) \geq \inf_{0\leq r,s\leq 2^{-p}} \{X(\frac{j+k}{2^p}+s,\omega)-X(\frac{k}{2^p}-r,\omega)\} \geq$$

$$\geq -(\frac{j}{2^p})^{1/\alpha}\phi(\frac{j+2}{2^p}) \geq -\Delta^{1/\alpha}\phi(\Delta).$$

In the divergence case we define the events $E^p_{j,k}$ by

$$X(\frac{j+k}{2^p})-X(\frac{k}{2^p}) < -(\frac{j}{2^p})^{1/\alpha}\phi(\frac{j}{2^p})$$

for $p=1,2,\ldots$, $k=0,1,\ldots,2^p$, $j=[p/2]+1,\ldots,p$ and $j+k \leq 2^p$. The proof that $P[E^p_{j,k}$ i.o.$] = 1$ is the same as for the case $\alpha = 1$. □

REMARK 7.4.1. Taking

$$\phi(\Delta) = \{2B(\alpha)\}^{-\frac{\alpha-1}{\alpha}}\{2(1+\delta)\log(\Delta^{-1})\}^{\frac{\alpha-1}{\alpha}}$$

we have that the probability in (7.4.2) is zero or one according as $\delta \leq 0$ or $\delta > 0$. Hence

$$\lim_{\varepsilon\downarrow 0}\ \inf_{\substack{0\leq t\leq 1-\Delta\\ 0<\Delta<\varepsilon}} \frac{X(t+\Delta)-X(t)}{\Delta^{1/\alpha}\{2\log(\Delta^{-1})\}^{(\alpha-1)/\alpha}} = -\{2B(\alpha)\}^{-(\alpha-1)/\alpha} \quad \text{a.s.}\ .$$

CHAPTER 8

L.I.L.-TYPE THEOREMS FOR THE HEAVY TAILS

In the chapters 4,5 and 6 we have proved generalized laws of the iterated logarithm for completely asymmetric stable processes ($\beta=1$) for $t \downarrow 0$, $t \to \infty$ and for partial sums. In that way we obtain lower limits for the rate of growth of the sample paths of these stable processes for small and for large times. In the proofs we made use of the relation between the left tail of the distribution function $F(.;\alpha,1)$ and the tail of the standard normal distribution function, as given in theorem 2.1.7 parts IV, V and VI. In this chapter we shall obtain upper limits for the rate of growth of the sample paths of completely asymmetric stable processes ($\beta=1$). We apply the expansions of the right tail of the corresponding distribution functions $F(.;\alpha,1)$ given in theorem 2.1.7 parts I, II and III. For the other stable distributions ($|\beta|\neq 1$) we have the same expansions for *both* tails of the distribution function. By using these expansions we also obtain upper- and lowerbounds for the rate of growth for stable processes with $|\beta| \neq 1$. In this chapter we establish integral tests similar to the criteria in the chapters 4,5 and 6. We distinguish three cases: partial sums, stable processes for $t \to \infty$ and stable processes for $t \downarrow 0$.

8.1. PARTIAL SUMS

We first give some early results.

1. LÉVY (1931) - MARCINKIEWICZ (1939):

Let $X_1,X_2,\ldots$ be independent random variables with distribution function $F_1,F_2,\ldots$. Suppose that, uniformly for large x and all k,

$$cx^{-\alpha} < 1 - F_k(x)+F_k(-x) < Cx^{-\alpha},$$

where α,c and C are positive constants with $\alpha \in (0,2)$. In case $1 \le \alpha < 2$ we assume

$$\lim_{t\to\infty} \int_{-t}^{t} x\, dF_k(x) = 0.$$

Let λ be a positive increasing function such that $\lambda(2t)/\lambda(t) \to 1$ for $t \to \infty$.

Define the sequence $\{a_n\}$ by

$$a_n = \{n(\log n)\lambda(\log n)\}^{1/\alpha}.$$

Then

$$P[|X_1+\ldots+X_n| > a_n \text{ i.o.}] = 0 \text{ or } 1$$

according as $\sum \frac{1}{n\lambda(n)}$ converges or diverges.

LÉVY proved this result in case $0 < \alpha < 1$. In chapter 10 we formulate an extension, proved by FELLER (1946), without conditions on $\{a_n\}$. Other authors also proved similar results using the methods used for the L.I.L. for the case $\alpha = 2$. We mention the following ones.

2. LIPSCHUTZ (1956b):

Let $X_1, X_2, \ldots$ be positive i.i.d. random variables with common distribution function $F \in \mathcal{D}(\alpha, 1)$ with $\alpha \neq 1,2$ and let ψ be a positive continuous non-decreasing function. Then

in case $0 < \alpha < 1$

$$P[X_1+\ldots+X_n > a_n\psi(n) \text{ i.o.}] = 0 \text{ or } 1;$$

in case $1 < \alpha < 2$

$$P[X_1+\ldots+X_n-nEX_1 > a_n\psi(n) \text{ i.o.}] = 0 \text{ or } 1$$

according as

$$K(\psi) = \int^{\infty} \frac{1}{t\psi^{\alpha}(t)}\, dt \tag{8.1.1}$$

converges or diverges. The constants a_n are defined by (2.2.2) and (2.2.4).

3. CHOVER (1966):

Let $X_1, X_2, \ldots$ be i.i.d. with common distribution function $F(.;\alpha,0)$ with $0 < \alpha < 2$. Then

$$P[|X_1+\ldots+X_n| > n^{1/\alpha}(\log n)^{(1+\varepsilon)/\alpha} \text{ i.o.}] = 0 \text{ or } 1$$

according as $\varepsilon > 0$ or $\varepsilon < 0$.

4. HEYDE (1969):

Let $X_1, X_2, \ldots$ be i.i.d. random variables with common distribution function $F \in \mathcal{D}_N(\alpha,\beta)$ with $\alpha \neq 1,2$ and $|\beta| \neq 1$. Then

$$P[|X_1+\ldots+X_n| > n^{1/\alpha}(\log n)^{(1+\varepsilon)/\alpha} \text{ i.o.}] = 0 \text{ or } 1$$

according as $\varepsilon > 0$ or $\varepsilon < 0$.

In the sections 2 and 3 of this chapter we shall refer to similar theorems for stable processes for large and small times.

Let $X_1, X_2, \ldots$ be i.i.d. random variables with common distribution function $F(.;\alpha,\beta)$. We define the sequence $\{T(n;\alpha,\beta)\}$ by

$$(8.1.2) \qquad T(n;\alpha,\beta) = (X_1+\ldots+X_n)n^{-1/\alpha} \qquad \text{for } \alpha \neq 1$$

$$= (X_1+\ldots+X_n - (2/\pi)\beta n \log n)n^{-1} \qquad \text{for } \alpha = 1.$$

By theorem 2.1.3 it follows that for every α and β

$$(8.1.3) \qquad T(n;\alpha,\beta) \overset{d}{=} X_1 \qquad \text{for all } n.$$

In this section we shall prove the following theorem.

THEOREM 8.1.1. *Let the sequence* $\{T(n;\alpha,\beta)\}$ *be defined by (8.1.2) and let* ψ *be a positive, continuous and non-decreasing function. Then*

a. for $\alpha \in (0,2)$ *and* $\beta \in (-1,1]$

$$P[T(n;\alpha,\beta) \geq \psi(n) \text{ i.o.}] = 0 \text{ or } 1$$

according as the integral $K(\psi)$, *defined in (8.1.1), converges or diverges*

b. for $\alpha \in (0,2)$ *and* $\beta \in [-1,1)$

$$P[T(n;\alpha,\beta) \leq -\psi(n) \text{ i.o.}] = 0 \text{ or } 1$$

according as the integral $K(\psi)$, *defined in (8.1.1), converges or diverges.*

Let $\varepsilon > 0$. We define the function ψ_1 and ψ_2 by

$$(8.1.4) \qquad \psi_1(t) = (\log t)^{\frac{1-\varepsilon}{\alpha}}$$

and

$$(8.1.5) \qquad \psi_2(t) = (\log t)^{\frac{1+\varepsilon}{\alpha}} .$$

In the proof of theorem 8.1.1 we apply the following lemma.

LEMMA 8.1.1. *Let* $\varepsilon > 0$ *and let* ψ_1 *and* ψ_2 *be defined by (8.1.4) and (8.1.5). If theorem 8.1.1 holds for all functions* ψ *satisfying*

$$(8.1.6) \qquad \psi_1(t) \le \psi(t) \le \psi_2(t)$$

then it holds in general.

PROOF. The proof has a similar pattern as the proof of lemma 4.3.1.

i. In the same way as in the proof of part *i* of lemma 4.3.1 we show that convergence of $K(\psi)$ implies $\psi(t) > \psi_1(t)$ for sufficiently large t.

ii. Let ψ be an arbitrary function satisfying the conditions of theorem 8.1.1 and $K(\psi) < \infty$. Define the function $\hat{\psi}$ by

$$(8.1.7) \qquad \hat{\psi}(t) = \min(\max(\psi_1(t),\psi(t)),\psi_2(t)).$$

Then, for sufficiently large t, we have $\hat{\psi}(t) = \min(\psi(t),\psi_2(t))$, implying $K(\hat{\psi}) < \infty$. The function $\hat{\psi}$ clearly satisfies (8.1.6). By the assumption that theorem 8.1.1 holds for all functions satisfying (8.1.6) we have

$$P[T(n;\alpha,\beta) \ge \hat{\psi}(n) \text{ i.o.}] = 0$$

and obviously

$$P[T(n;\alpha,\beta) \ge \psi(n) \text{ i.o.}] = 0.$$

iii. Let ψ be an arbitary function satisfying the conditions of theorem 8.1.1 and $K(\psi) = \infty$. Define $\hat{\psi}$ by (8.1.6). In the same way as in the proof of part *iii* of lemma 4.3.1 we obtain $K(\hat{\psi}) = \infty$. By the assumption that theorem 8.1.1 holds for all functions satisfying (8.1.6) we have

$$P[T(n;\alpha,\beta) \geq \hat{\psi}(n) \text{ i.o.}] = 1$$

and

$$P[T(n;\alpha,\beta) > \psi_2(n) \text{ i.o.}] = 0.$$

Thus there exists a sequence $\{n_k\}$ such that $\hat{\psi}(n_k) < \psi_2(n_k)$ and

$$P[T(n_k;\alpha,\beta) \geq \hat{\psi}(n_k) \text{ i.o.}] = 1. \tag{8.1.8}$$

The inequality $\hat{\psi}(n_k) < \psi_2(n_k)$ implies $\hat{\psi}(n_k) \geq \psi(n_k)$. This yields, in view of (8.1.8),

$$P[T(n;\alpha,\beta) \geq \psi(n) \text{ i.o.}] = 1.$$

Thus the restriction (8.1.6) can also be made in case the integral (8.1.1) diverges. □

PROOF of theorem 8.1.1. Because $F(x;\alpha,\beta) = 1-F(-x;\alpha,-\beta)$ we have only to prove part a of the theorem. By lemma 8.1.1 we may restrict ourselves to functions ψ satisfying $\psi_1 \leq \psi \leq \psi_2$.

Suppose $K(\psi) < \infty$. Let $c > 1$ and let n_r denote the largest integer smaller than c^r. We define the following events

$$A_n: \quad T(n;\alpha,\beta) \geq \psi(n);$$

$$B_r: \quad \max_{n_r<n\leq n_{r+1}} S_n \geq n_r^{1/\alpha}\psi(n_r) \qquad \text{for } \alpha \neq 1,$$

$$\max_{n_r<n\leq n_{r+1}} T(n;\alpha,\beta) \geq \psi(n_r) \qquad \text{for } \alpha = 1$$

and for $\alpha \neq 1$

$$C_r: \quad S_{n_{r+1}} \geq n_r^{1/\alpha}\psi(n_r).$$

Then

$$\limsup A_n \subset \limsup B_r. \tag{8.1.10}$$

By lemma 1.4.3 and remark 1.4.1 there exists, for $\alpha \neq 1$, a constant $k(\alpha,\beta)$ such that $P[B_r] \leq k^{-1}(\alpha,\beta)\, P[C_r]$. The expansion of the tail of $F(.;\alpha,\beta)$ in theorem 2.1.7 parts I and III implies

$$P[C_r] = P[X_1 \geq (n_r/n_{r+1})^{1/\alpha}\psi(n_r)] = O(\frac{1}{\psi^{\alpha}(n_r)}) \qquad \text{for } r \to \infty.$$

Let $\{X(t) : 0 \leq t < \infty\}$ be a stable process for which the r.v. $X(1)$ has distribution function $F(.;1,\beta)$. Then

$$P[\max_{n_r<n\leq n_{r+1}} \frac{X(n)-(2/\pi)\beta n\log n}{n} \geq \psi(n_r)] \leq$$

$$\leq P[\sup_{1\leq t\leq n_r^{-1}n_{r+1}} \frac{X(n_r t)-(2/\pi)\beta n_r t\log(n_r t)}{n_r t} \geq \psi(n_r)] =$$

$$= P[\sup_{1<t\leq n_r^{-1}n_{r+1}} \frac{X(t)-(2/\pi)\beta t\log t}{t} \geq \psi(n_r)]$$

by property 4 of section 3.2. By lemma 3.5.5 it follows that for $\alpha = 1$

$$P[B_r] = O(\frac{1}{\psi(n_r)}) \qquad \text{for } r \to \infty.$$

Thus we have for all $\alpha \in (0,2)$ that there exists some positive constant k such that

$$\sum P[B_r] \leq k\cdot K(\psi) < \infty .$$

This yields $P[\limsup A_n] = P[\limsup B_r] = 0$.

Suppose $K(\psi) = \infty$. Because, for every positive λ,

$$\{\limsup \frac{T(n;\alpha,\beta)}{\psi(n)} > \lambda \}$$

is a tail event we have

$$P[\limsup \frac{T(n;\alpha,\beta)}{\psi(n)} > \lambda] = 0 \text{ or } 1.$$

Thus, in order to prove the divergence part of the theorem we only have to show

$$P[T(n;\alpha,\beta) \geq \psi(n) \text{ i.o.}] > 0.$$

Define the sequence n_r as in the convergence part with c so large that $1-2(c-1)^{-1/\alpha} \geq \frac{1}{2}$. Define the events

$$D_r\colon\ \psi(n_r) \leq T(n_r;\alpha,\beta) \leq 2\psi(n_r).$$

Then there exists a positive constant k such that $P[D_r] \geq k \dfrac{1}{\psi^{\alpha}(n_r)}$ for $r \to \infty$ and this yields $\sum_r P[D_r] = \infty$. In case $\alpha \neq 1$ we have for $r < s$

$$P[D_r \wedge D_s] = P[n_r^{1/\alpha}\psi(n_r) \leq S_{n_r} \leq 2n_r^{1/\alpha}\psi(n_r) \wedge n_s^{1/\alpha}\psi(n_s) \leq S_{n_s} \leq 2n_s^{1/\alpha}\psi(n_s)] \leq$$

$$\leq P[D_r]\ P[S_{n_s}-S_{n_r} \geq n_s^{1/\alpha}\psi(n_s)-2n_r^{1/\alpha}\psi(n_r)] \leq$$

$$\leq P[D_r]\ P[X_1 \geq \left(\frac{n_s}{n_s-n_r}\right)^{1/\alpha}\psi(n_s)-2\left(\frac{n_r}{n_s-n_r}\right)^{1/\alpha}\psi(n_r)] \leq k_1 P[D_r]\ P[D_s],$$

where the constant k_1 can be chosen independent of r and s.

In case $\alpha = 1$ we have for $r < s$

$$P[D_r \wedge D_s] = P[n_r\psi(n_r) \leq S_{n_r} -(2/\pi)\beta n_r \log n_r \leq 2n_r\psi(n_r) \wedge$$
$$\wedge\ n_s\psi(n_s) \leq S_{n_s} -(2/\pi)\beta n_s \log n_s \leq 2n_s\psi(n_s)]$$

$$\leq P[D_r]\ P[S_{n_s}-S_{n_r} \geq n_s\psi(n_s)+(2/\pi)\beta n_s \log n_s - 2n_r\psi(n_r)-(2/\pi)\beta n_r \log n_r]$$

$$\leq P[D_r]\ P[X_1 \geq (n_s-n_r)^{-1}(n_s\psi(n_s) + (2/\pi)\beta n_s \log n_s - 2n_r\psi(n_r) +$$
$$- (2/\pi)\beta n_r \log n_r - (2/\pi)\beta(n_s-n_r)\log(n_s-n_r)]$$

$$\leq P[D_r]\ P[X_1 \geq \tfrac{1}{2}\psi(n_s) + \beta A(r,s)],$$

where $A(r,s) = \dfrac{2}{\pi}\,\dfrac{n_s \log n_s - n_r \log n_r - (n_s-n_r)\log(n_s-n_r)}{n_s-n_r}$ is uniformly bounded by a constant which depends only on c. By the expansion in theorem 2.1.7 part II we have

$$P[D_r \wedge D_s] \le k_1 P[D_r]\, P[D_s],$$

where k_1 may be chosen independent of r and s. Hence for all $\alpha \in (0,2)$ lemma 1.4.2 yields $P[D_r \text{ i.o.}] > 0$. □

REMARK 8.1.1. Divergence of the integral $K(\psi)$ implies $K(\lambda\psi) = \infty$ for all positive λ. Consequently

$$\limsup \frac{T(n;\alpha,\beta)}{\psi(n)} \ge \lambda \qquad \text{a.s.}$$

and this yields

$$\limsup \frac{T(n;\alpha,\beta)}{\psi(n)} = \infty \qquad \text{a.s.}$$

in case $K(\psi) = \infty$.

CHOVER (1966) makes use of his version of theorem 8.1.1 to prove that

$$\limsup_{n\to\infty} |T(n;\alpha,0)|^{\frac{1}{\log\log n}} = e^{1/\alpha} \qquad \text{a.s..}$$

This result is extended to the cases $|\beta| \ne 1$ (and $\alpha\ne1$) by HEYDE (1969). CHOVER (1966) has also given some other limit points of the sequence $\{|T(n;\alpha,0)|^{\frac{1}{\log\log n}}\}$. We define the sequence $\{\tilde{T}(n;\alpha,\beta)\}$, for $0 < \alpha < 2$ and $|\beta| \le 1$, by

$$(8.1.11) \qquad \tilde{T}(n;\alpha,\beta) = \operatorname{sign}(T(n;\alpha,\beta))\, |T(n;\alpha,\beta)|^{\frac{1}{\log\log n}}.$$

The following theorem gives some limit points of this sequence $\{\tilde{T}(n;\alpha,\beta)\}$.

THEOREM 8.1.2. *Let the sequence* $\{\tilde{T}(n;\alpha,\beta)\}$ *be defined by (8.1.11) for* $0 < \alpha < 2$ *and* $|\beta| \le 1$. *Then, with probability* 1, *all points of the following intervals are limit points of* $\{\tilde{T}(n;\alpha,\beta)\}$.

$[-e^{1/\alpha}, e^{1/\alpha}]$	for $1 \le \alpha < 2$	and $\lvert\beta\rvert \ne 1$
$[-e^{1/\alpha}, 1]$	$1 \le \alpha < 2$	$\beta = -1$
$[-1, e^{1/\alpha}]$	$1 \le \alpha < 2$	$\beta = 1$
$[-e^{1/\alpha}, -e^{-\frac{1}{1-\alpha}}] \cup [e^{-\frac{1}{1-\alpha}}, e^{1/\alpha}]$	$0 < \alpha < 1$	$\lvert\beta\rvert \ne 1$
$[-e^{1/\alpha}, -1]$	$0 < \alpha < 1$	$\beta = -1$
$[1, e^{1/\alpha}]$	$0 < \alpha < 1$	$\beta = 1.$

REMARK 8.1.2. In case $1 \le \alpha < 2$ and all β and in case $0 < \alpha < 1$ and $|\beta| = 1$ theorem 8.1.2 gives all a.s. limit points. I do not know about the points in the interval $(-e^{-\frac{1}{1-\alpha}}, e^{-\frac{1}{1-\alpha}})$ in case $0 < \alpha < 1$ and $|\beta| \ne 1$.

In the proof of theorem 8.1.1 we need the following lemma, which is a simple extension of a lemma proved by SPITZER (1956).

LEMMA 8.1.2. *Let the sequence* $\{T(n;\alpha,\beta)\}$ *be defined by (8.1.2) and let* $\{a_n\}$ *be a non-increasing sequence of positive real numbers. Then for all* $\alpha \ge 1$

$$P[0 \le T(n;\alpha,\beta) \le a_n \text{ i.o.}] = 0 \textit{ or } 1$$

according as the series $\sum a_n$ *converges or diverges.*

PROOF. Denote the event

$$(8.1.12) \qquad 0 \le T(n;\alpha,\beta) \le a_n$$

by D_n. Then $P[D_n] = P[0 \le X_1 \le a_n]$. Because each stable random variable has a bounded density it follows that

$$P[D_n] = \mathcal{O}(a_n) \qquad \text{for } n \to \infty.$$

Then the convergence part easily follows.

In the divergence part we may suppose, without loss of generality, that $a_n \le \frac{1}{n \log n}$. We compute for $m > n$

$$P[D_n \wedge D_m] = P[0 \le T(n;\alpha,\beta) \le a_n \wedge 0 \le T(m;\alpha,\beta) \le a_m]$$

$$= \begin{cases} P[0 \le S_n \le n^{1/\alpha}a_n \wedge 0 \le S_m \le m^{1/\alpha}a_m] & \text{if } \alpha \ne 1 \\ \\ P[0 \le S_n-(2/\pi)\beta n\log n \le na_n \wedge 0 \le S_m-(2/\pi)\beta m\log m \le ma_m] & \text{if } \alpha = 1. \end{cases}$$

We first consider the case $\alpha \ne 1$.

$$P[D_n \wedge D_m] = \int_0^{n^{1/\alpha}a_n} \{\int_{-y}^{m^{1/\alpha}a_m-y} f_{m-n}(x)dx\}\ f_n(y)dy,$$

where f_j is the density of $S_j = X_1+\ldots+X_j$ for $j=1,2,\ldots$. Because the random variables X_j, $j=1,2,\ldots$, have a stable distribution f_n satisfies

$$(8.1.13)\qquad f_n(.) = n^{-1/\alpha}f_1(n^{-1/\alpha}.) \qquad\qquad \text{for } \alpha \ne 1$$

and f_1 is bounded. Thus

$$(8.1.14)\qquad P[D_n \wedge D_m] \le M^2(\frac{m}{m-n})^{1/\alpha} a_n\ a_m.$$

In case $\alpha = 1$ we obtain the same upperbound. Now the density f_n satisfies

$$(8.1.15)\qquad f_n(. + (2/\pi)\beta n\log n) = n^{-1}f_1(n^{-1}.).$$

If $m > 2n$ we have $m(m-n)^{-1} < 2$ and because f_1 is bounded away from zero near the origine, (8.1.14) implies the existence of a constant k_0 such that

$$(8.1.16)\qquad P[D_n \wedge D_m] \le k_0 P[D_n]\ P[D_m].$$

By (8.1.14) we have for fixed n

$$\sum_{m=n+1}^{2n} P[D_n \wedge D_m] \le M^2 a_n \sum_{m=n+1}^{2n} (\frac{m}{m-n})^{1/\alpha} a_m.$$

Some algebra shows that for $n \to \infty$

$$\sum_{m=n+1}^{2n} (\frac{m}{m-n})^{1/\alpha} a_m = \begin{cases} O((\log n)^{-1}) & \text{for } 1 < \alpha \le 2 \\ \\ O(1) & \text{for } \alpha = 1. \end{cases}$$

By lemma 1.4.2 the divergence part of the lemma follows immediately. □

PROOF of theorem 8.1.2. Let the subsequence n_k be defined by $n_k = [\gamma^{k^\delta}]$ with $\gamma > 1$ and $\delta > 1$. One can show that for $0 < \alpha < 2$ and $-1 < \beta \le 1$

$$P[T(n_k;\alpha,\beta) \ge \psi(k) \text{ i.o.}] = 0 \text{ or } 1$$

according as $K(\psi)$ converges or diverges. The proof of this assertion is similar to that of theorem 8.1.1 and is therefore omitted. It easily follows that $e^{\frac{1}{\alpha\delta}}$ is, with probability 1, a limit point of $\tilde{T}(n;\alpha,\beta)$. Thus, w.p.1, every point of $[1,e^{1/\alpha}]$ is limit point of $\tilde{T}(n;\alpha,\beta)$ for all α and $-1 < \beta \le 1$. Theorem 8.1.1 part a. implies that, in case $\alpha \in (0,2)$ and $-1 < \beta \le 1$,

$$\limsup \tilde{T}(n;\alpha,\beta) = e^{1/\alpha} \quad \text{a.s. .} \tag{8.1.17}$$

Theorem 6.2.1 implies that, in case $\alpha \in (0,1)$ and $\beta = 1$,

$$\liminf \tilde{T}(n;\alpha,\beta) = 1 \quad \text{a.s. .} \tag{8.1.18}$$

For $\alpha \in (0,1)$, it follows that the set of all limit points of $\{\tilde{T}(n;\alpha,1)\}$ coincides almost surely with $[1,e^{1/\alpha}]$. Because

$$F(-x;\alpha,-\beta) = 1-F(x;\alpha,\beta) \tag{8.1.19}$$

we have also proved theorem 8.1.2 in case $0 < \alpha < 1$ and $\beta = -1$.

In case $1 \le \alpha < 2$ and $|\beta| \le 1$ we define the subsequence n_k by $n_k = \max(k,[\gamma^{k^\delta}])$ for fixed $\gamma > 1$ and $\delta > 0$. Repeating the argument of lemma 8.1.2 we can show

$$P[0 \le T(n_k;\alpha,\beta) \le a_k \text{ i.o.}] = 0 \text{ or } 1$$

according as $\sum a_k < \infty$ or $= \infty$. Then we easily obtain that all points of $[-1,1]$ are limit points of $\tilde{T}(n;\alpha,\beta)$. In case $1 < \alpha < 2$ theorem 6.4.1 implies

$$\liminf \tilde{T}(n;\alpha,1) = -1 \quad \text{a.s. .} \tag{8.1.20}$$

In case $\alpha = 1$ the result (8.1.20) follows from theorem 6.3.1. This completes the proof for the case $1 \le \alpha < 2$ and $\beta = 1$ and because of (8.1.19) also for the other cases with $1 \le \alpha < 2$.

In case $0 < \alpha < 1$ and $|\beta| \neq 1$ we define the subsequence n_k by $n_k = \max(k,[\gamma^{k^\delta}])$ for $\gamma > 1$ and $\delta \ge 1-\alpha$. Now we can show that

$$P[0 \le T(n_k;\alpha,\beta) \le a_k \text{ i.o.}] = 0 \text{ or } 1$$

according as $\sum a_k$ converges or diverges. Remark that, also in case $0 < \alpha < 1$, we can define the events D_n by (8.1.12) and give an upperbound for $P[D_n \wedge D_m]$ as in (8.1.14). Then we obtain the limit points in $[-1,-e^{-\frac{1}{1-\alpha}}] \cup [e^{-\frac{1}{1-\alpha}},1]$. □

8.2. LARGE TIMES

In this section we prove the analogue of theorem 8.1.1 for stable processes. Let $\{X(t) : 0 \le t < \infty\}$ be a stable process. KHINTCHINE (1937) has given an integral test in order to determine whether the event

$$(8.2.1) \qquad \{\omega\text{: there exists some } t_0(\omega) > 0 \text{ such that } |X(t,\omega)| \le t^{1/\alpha}\psi(t) \text{ for all } t \ge t_0(\omega)\}$$

has probability zero or one. FRISTEDT (1967) has given a similar result for subordinators. Symmetric processes with stationary, independent increments (not necessarily stable) are studied by FRISTEDT (1971).

As in section 8.1 we define the process $\{T(t;\alpha,\beta) : 0 \le t < \infty\}$, with $\alpha \in (0,2)$ and $\beta \in [-1,1]$ by

$$(8.2.2) \qquad T(t;\alpha,\beta) = t^{-1/\alpha}X(t) \qquad \text{for } \alpha \neq 1$$
$$= t^{-1}\{X(t)-(2/\pi)\beta t\log t\} \qquad \text{for } \alpha = 1.$$

It follows from the definition of a stable process that

$$T(t;\alpha,\beta) \overset{d}{=} X(1) \qquad \text{for } t > 0,\ \alpha \in (0,2) \text{ and } \beta \in [-1,1].$$

THEOREM 8.2.1. *Let the process* $\{T(t;\alpha,\beta) : 0 \le t < \infty\}$ *be defined by (8.2.2) and let* ψ *be a positive, continuous and non-decreasing function. Then* *a. For* $\alpha \in (0,2)$ *and* $\beta \in (-1,1]$

$$P[\{\omega: \text{there exists some } t_0(\omega) > 0 \text{ such that } T(t;\alpha,\beta) \le \psi(t) \text{ for all } t \ge t_0(\omega)\}] = 0 \text{ or } 1$$

according as the integral (8.1.1) diverges or converges

b. For $\alpha \in (0,2)$ *and* $\beta \in [-1,1)$

$$P[\{\omega: \text{there exists some } t_0(\omega) > 0 \text{ such that } T(t;\alpha,\beta) \ge -\psi(t) \text{ for all } t \ge t_0(\omega)\}] = 0 \text{ or } 1$$

according as the integral (8.1.1) diverges or converges.

PROOF. Again it is sufficient only to prove part a and we may suppose that $(\log t)^{\frac{1-\varepsilon}{\alpha}} \le \psi(t) \le (\log t)^{\frac{1+\varepsilon}{\alpha}}$. We have only to prove the convergence part because theorem 8.1.1 implies the divergence part.

Define the events C_r, $r=1,2,\ldots$, by

$$\sup_{2^{r-1} \le t \le 2^r} T(t;\alpha,\beta) > \psi(2^{r-1}).$$

By lemmas 3.4.4, 3.5.5 and 3.6.4b it follows that

$$P[C_r] = O((\psi(2^{r-1}))^{-\alpha}) \qquad \text{for } r \to \infty.$$

It follows, as in the proof of theorem 8.1.1, that for all $\alpha \in (0,2)$ $\sum P[C_r] < \infty$, implying $P[C_r \text{ i.o.}] = 0$. Therefore, for almost all ω, there exists a number $r_0(\omega)$ such that

$$\sup_{2^{r-1} \le t \le 2^r} T(t;\alpha,\beta) \le \psi(2^{r-1})$$

for all $r \ge r_0(\omega)$. Then the theorem follows by making use of the monotonicity of ψ. □

8.3. SMALL TIMES

The duality between small and large times for stable processes, given in property 3 of section 3.2, indicates that we may establish a similar theorem for small times (cf. the references given in section 8.2). The proof of the following theorem follows the same pattern as the proof of theorem 8.1.1 and is omitted.

THEOREM 8.3.1. *Let the process* $\{T(t;\alpha,\beta) : 0 \le t < \infty\}$ *be defined by (8.2.1) and let* ψ *be a positive, continuous and non-decreasing function. Then*

a. For $\alpha \in (0,2)$ *and* $\beta \in (-1,1]$

$$P[\{\omega: \textit{there exists some } t_0(\omega) > 0 \textit{ such that } T(t;\alpha,\beta) \le \psi(t^{-1}) \textit{ for all } t \le t_0(\omega)\}] = 0 \textit{ or } 1$$

according as the integral (8.1.1) diverges or converges

b. For $\alpha \in (0,2)$ *and* $\beta \in [-1,1)$

$$P[\{\omega: \textit{there exists some } t_0(\omega) > 0 \textit{ such that } T(t;\alpha,\beta) \ge -\psi(t^{-1}) \textit{ for all } t \le t_0(\omega)\}] = 0 \textit{ or } 1$$

according as the integral (8.1.1) diverges or converges.

CHAPTER 9

FUNCTIONAL LAW OF THE ITERATED LOGARITHM

To state the theorems in this chapter we remember that $C[0,1]$ is the Banach space of all real-valued continuous functions on $[0,1]$ with sup-norm $\|\cdot\|_c$ and metric d_c. The set $D[0,1]$ will be the set of real-valued functions on $[0,1]$ which are right-continuous and have finite left-hand limits. In appendix 1 we define two topologies on $D[0,1]$. In section 1.5 we have defined the mapping

$$\pi_m : D[0,1] \rightarrow C[0,1]$$

as the following piecewise linear approximation

(9.0.1) $\quad \pi_m x(j/m) = x(j/m) \qquad$ for $j=0,1,\ldots,m$

and linear on the sub-intervals $[j/m,(j+1)/m]$ for $j=0,\ldots,m-1$.

DEFINITION 9.0.1. Let K be the subset of absolutely continuous functions $x \in C[0,1]$, such that $x(0) = 0$ and

$$\int_0^1 (\dot{x}(t))^2 dt \le 1.$$

The set K is compact. (See for example FREEDMAN (1971), lemma 78(d).) Let C_0^+ (resp. C^+) be the subclasses of $C[0,1]$ of increasing (resp. non-decreasing) functions. Then we have

$$C_0^+ \subset C^+ \subset C[0,1].$$

The increasing and non-decreasing functions of K constitute the subclasses K_0^+ and K^+ with

$$K_0^+ \subset K^+ \subset K.$$

Every finite non-decreasing function x is almost everywhere differentiable and we denote his derivative by $\dot{x}$. From theorem 1.5.2 we know that $\dot{x}$ is a version of the Radon-Nikodym derivative of the absolutely continuous part x_a of x with respect to Lebesgue measure. We define the mapping I

$$I : C^{+} \to \mathbb{R}$$

by

$$(9.0.2) \qquad I\,x = \int_0^1 (\dot{x}(t))^2 dt.$$

9.1. THE CASE $\alpha = 2$

STRASSEN (1964) proved the following functional law of the iterated logarithm for the Wiener process $\{W(t) : 0 \le t < \infty\}$. Let the sequence $\{f_n : n \ge 3\}$

$$f_n : [0,1] \times \Omega \to \mathbb{R}$$

be defined by

$$f_n(t,\omega) = W(nt,\omega)/(2n \log\log n)^{\frac{1}{2}}$$

for $n=3,4,\ldots$.

THEOREM 9.1.1. *For almost all* ω*, the indexed subset*

$$\{f_n(\cdot,\omega) : n \ge 3\}$$

of $C[0,1]$ *is relatively compact, with limit set* K.

In fact STRASSEN proved the theorem for the Wiener process in $\mathbb{R}^k$. By using the Skorohod representation (see chapter 10) of a random variable $Y \in \mathcal{D}_N(2,0)$ he proved the so called *strong invariance principle*. This strong invariance principle will be stated in chapter 10. VERVAAT (1972) has obtained similar results in $C[0,\infty)$ instead of $C[0,1]$.

9.2. THE CASE $0 < \alpha < 1$

Let $\{X(t) : 0 \le t < \infty\}$ be a completely asymmetric stable process with $0 < \alpha < 1$ and $\beta = 1$. We introduce the following mapping

$$D_\alpha : C_0^+ \to C_0^+$$

defined by

$$(9.2.1) \qquad D_\alpha x(t) = \int_0^t [\dot{x}(y)]^{-\frac{\alpha}{2(1-\alpha)}} dy.$$

Define the sequences of functions $\{f_n : n \geq 3\}$ and $\{g(.,n,m,.) : m \in \mathbb{N},\ n \geq 3\}$

$$f_n : [0,1] \times \Omega \to \mathbb{R}$$

and

$$g(.,n,m,.) : [0,1] \times \mathbb{N} \times \mathbb{N} \times \Omega \to \mathbb{R}$$

by

$$(9.2.2) \qquad f_n(t,\omega) = (2 \log \log n)^{\frac{1-\alpha}{\alpha}} \{2B(\alpha)\}^{-\frac{1-\alpha}{\alpha}} n^{-\frac{1}{\alpha}} X(nt,\omega)$$

and

$$(9.2.3) \qquad g(t,n,m,\omega) = D_\alpha \pi_m f_n(t,\omega).$$

THEOREM 9.2.1. *Let* $\varepsilon > 0$.

a. For almost all ω *and all* m *there exists a number* $n_0 = n_0(\varepsilon,m,\omega)$ *such that*

$$d_c(g(.,n,m,\omega),K^+) < \varepsilon$$

for all $n \geq n_0$.

b. For all $h \in K^+$ *there exists a number* $m_0(\varepsilon,h)$ *such that*

$$P[\{\omega: d_c(g(.,n,m,\omega),h) < \varepsilon \text{ for infinitely many } n\}] = 1$$

for all $m \geq m_0(\varepsilon,h)$.

For $r=2,3,\ldots,$ let n_r be the nearest integer to

$$(9.2.4) \qquad e^{r/(\log r)^2}$$

For a positive integer m, there exists an integer $r(m)$ such that for $r \geq r(m)$

$$(9.2.5) \qquad \frac{n_{r+1}}{n_r} < \frac{m}{m-1} .$$

Obviously this implies that, for $r \geq r(m)$, we have $jn_{r+1} < (j+1)n_r$ for all $j=0,\ldots,m-1$. For fixed m and all $r \geq r(m)$ we define the random variables $A_{j,r}$ $(j=0,\ldots,m-1)$ by

$$(9.2.6) \qquad A_{j,r} = [X((j+1)n_r m^{-1})-X(jn_{r+1}m^{-1})]^{-\frac{\alpha}{1-\alpha}} n_{r+1}^{\frac{1}{1-\alpha}} 2B(\alpha)\, m^{-\frac{1}{1-\alpha}}$$

In the proof of theorem 9.2.1 we need the following lemma. The proof of this lemma will be given in appendix 2.

LEMMA 9.2.1. *Let the sequence* $\{n_r\}$ *be defined by (9.2.4) and the random variables* $A_{j,r}$ *by (9.2.6) for* $r \geq r(m)$. *Then, for* $\varepsilon > 0$ *there exists a number* k *(depending on* m *and* ε, *but not on* r) *such that for all* $r \geq r(m)$,

$$P[A_{0,r}+\ldots+A_{m-1,r} > (1+\varepsilon)^2\, 2\log\log n_r] < kr^{-1-\varepsilon}.$$

PROOF of theorem 9.2.1.

Part a. By the definition of $g(.,n,m,\omega)$

$$I(g) = I(D_\alpha \pi_m f_n) = \sum_{j=0}^{m-1} [m(f_n((j+1)/m)-f_n(j/m))]^{-\frac{\alpha}{1-\alpha}} m^{-1}$$

$$= m^{-\frac{1}{1-\alpha}} 2B(\alpha)(2\log\log n)^{-1} \sum_{j=0}^{m-1} [n^{-\frac{1}{\alpha}}(X(n(j+1)/m)-X(nj/m))]^{-\frac{\alpha}{1-\alpha}}.$$

Taking n_r as in (9.2.4) and $r \geq r(m)$ so that (9.2.5) is fulfilled, we define the events B_r by

$$(9.2.7) \qquad B_r = \{\omega: \max_{n_r \leq n \leq n_{r+1}} I(D_\alpha \pi_m f_n(.,\omega)) > (1+\varepsilon)^2\}.$$

Because the paths of the completely asymmetric stable processes for $0 < \alpha < 1$ are increasing functions we have

$$P[B_r] \le P[\{\omega\colon m^{-\frac{1}{1-\alpha}}\, 2B(\alpha)(2\log\log n_r)^{-1}\cdot$$

$$\cdot \sum_{j=0}^{m-1} [n_{r+1}^{-\frac{1}{\alpha}}(X(n_r(j+1)/m,\omega)-X(n_{r+1}j/m,\omega))]^{-\frac{\alpha}{1-\alpha}} > (1+\varepsilon)^2\}]$$

$$= P[A_{0,r}+\ldots+A_{m-1,r} > (1+\varepsilon)^2\; 2\log\log n_r],$$

where $A_{j,r}$ is defined in (9.2.6). By lemma 9.2.1

$$P[B_r] \le kr^{-1-\varepsilon}$$

and hence $\sum P[B_r] < \infty$. By the Borel-Cantelli lemma it follows that for almost all ω there exists a number $n_0(\varepsilon,m,\omega)$ such that

$$I(D_\alpha \pi_m f_n(\cdot,\omega)) \le (1+\varepsilon)^2$$

for $n \ge n_0(\varepsilon,m,\omega)$. Since $g(\cdot,n,m,\omega)$ is obviously increasing, it follows that $(1+\varepsilon)^{-1}\, g(\cdot,n,m,\omega) \in K_0^+$ for almost all ω and $n \ge n_0(\varepsilon,m,\omega)$ and because any function in K_0^+ is bounded by 1,

$$d_c(g(\cdot,n,m,\omega),(1+\varepsilon)^{-1}\, g(\cdot,n,m,\omega)) < \varepsilon$$

for almost all ω and $n \ge n_0(\varepsilon,m,\omega)$.

Part b. Fix $h \in K^+$ and $\varepsilon > 0$. Because $I((1+\varepsilon)^{-1}h) < 1$ and $d_c(h,(1+\varepsilon)^{-1}h) < \varepsilon$ we shall further assume $I(h) < 1$. By uniform continuity, there exists an integer $m_0(\varepsilon,h)$ such that for all $m \ge m_0(\varepsilon,h)$

(9.2.8) $$a_j = h((j+1)/m)-h(j/m) < \varepsilon/4 \qquad \text{for } j=0,\ldots,m-1.$$

Choose $m > \max(m_0(\varepsilon,h),16\varepsilon^{-2})$ and define the sequence $\{n_r\}$ by $n_r = m^r$ and $\delta = m\varepsilon^2/16-1$. Then $\delta > 0$ and by theorem 6.2.1 we have

$$P[X(n) \le n^{\frac{1}{\alpha}}\, (2B(\alpha))^{\frac{1-\alpha}{\alpha}}\, (2(1+\delta)\log\log n)^{-\frac{1-\alpha}{\alpha}} \text{ i.o.}] = 0.$$

This implies: for almost all ω, there exists an integer $r_0(\varepsilon,\omega,m)$ such that

$$(9.2.9) \qquad g(m^{-1},n_r,m,\omega) =$$

$$= [(2\log\log n_r)^{\frac{1-\alpha}{\alpha}}\, n_r^{-\frac{1}{\alpha}}\, (2B(\alpha))^{-\frac{1-\alpha}{\alpha}}\, m\, X(n_r/m,\omega)]^{-\frac{\alpha}{2(1-\alpha)}}\, m^{-1}$$

$$\leq \varepsilon/4 \qquad\qquad \text{for } r \geq r_0(\varepsilon,\omega,m).$$

Choose positive ε_j for $j=1,\ldots,m-1$ such that

$$(9.2.10) \qquad \begin{cases} \varepsilon_j < a_j/m & \text{if } a_j \neq 0, \\ \varepsilon_j < \varepsilon(4m)^{-1} & \text{otherwise,} \end{cases}$$

and such that

$$(9.2.11) \qquad m \sum_{j=1}^{m-1} (a_j + \varepsilon_j)^2 < 1;$$

this is possible because $m \sum_{j=1}^{m-1} a_j^2 \leq I(h) < 1$. (By lemma 80 in chapter 1 of FREEDMAN (1971).)

Define for $j=1,\ldots,m-1$ the events

$$(9.2.12) \qquad C_r^{(j)} = \{\omega\colon (a_j-\varepsilon_j)^+ \leq g((j+1)/m,n_r,m,\omega)-g(j/m,n_r,m,\omega) \leq a_j + \varepsilon_j\}$$

and

$$D_r = \bigcap_{j=1}^{m-1} C_r^{(j)}.$$

The events $C_r^{(j)}$, $j=1,\ldots,m-1$ and $r=1,\ldots,$ are independent. Therefore the events D_r are also independent and $P[D_r] = \Pi_{j=1}^{m-1} P[C_r^{(j)}]$. By (9.2.2) and (9.2.3)

$$P[C_r^{(j)}] = P[(a_j-\varepsilon_j)^+\, m^{\frac{1}{2}}(2\log\log n_r)^{\frac{1}{2}} \leq$$

$$\leq \{2B(\alpha)\}^{\frac{1}{2}}\, X(1)^{-\frac{\alpha}{2(1-\alpha)}} \leq (a_j+\varepsilon_j)\, m^{\frac{1}{2}}\, (2\log\log n_r)^{\frac{1}{2}}].$$

Because of theorem 2.1.6 IV there is a number k such that

$$P[C_r^{(j)}] \geq k\, e^{-m(a_j+\varepsilon_j)^2 \log\log n_r},$$

and hence there exists a number k_1 such that

$$P[D_r] \geq k_1 \, e^{-m \log\log n_r \sum_{j=1}^{m-1} (a_j+\varepsilon_j)^2} \geq \frac{k_1}{r \log m}$$

by (9.2.11). This implies $\sum P[D_r] = \infty$. The Borel-Cantelli lemma gives $P[D_r \text{ i.o.}] = 1$. Consequently, there exists for almost all ω a sub-sequence $\hat{n}_r(\omega)$ such that

$$d_c(g(\cdot,\hat{n}_r,m,\omega),h) \leq d_c(g(\cdot,\hat{n}_r,m,\omega),\pi_m h) + d_c(\pi_m h,h)$$
$$\leq \varepsilon/4 + \sum_{j=1}^{m-1} \varepsilon_j + \varepsilon/4 < \varepsilon. \quad \square$$

Using compactness of K^+, we obtain from theorem 9.2.1 part a that, for almost all ω and fixed m, the limit points (which in general may depend on ω) of $g(\cdot,n,m,\omega)$ are in K^+. Note that K^+ is a non-random set. Theorem 9.2.1 part b ensures that a.s. there are $g(\cdot,n,m,\omega)$ arbitrarily close to any (fixed) point in K^+. Note that the exceptional nullset in part b depends on h. The existence of a countable dense subset of K^+ (see FREEDMAN (1971) lemma 100) implies that for almost all ω there exist, for every $h \in K^+$, increasing sequences $n_k(\omega)$ and m_k such that

$$\lim_{k\to\infty} d_c(g(\cdot,n_k(\omega),m_k,\omega),h) = 0.$$

We now transform theorem 9.2.1 into one for the sequence $\{f_n\}$. For every non-decreasing function x on [0,1] we define $J_x = \{t: x(t) < \infty\}$.

DEFINITION 9.2.1. Let $K(\alpha)$ be the set of non-decreasing functions on [0,1] with the properties

1. $x(0) = 0$;
2. x is strictly increasing on J_x;
3. $\int_{J_x} [\dot{x}(t)]^{-\frac{\alpha}{1-\alpha}} \, dt \leq 1$.

Note that a function in $K(\alpha)$ may have positive jumps. With regard to finiteness of a function $x \in K(\alpha)$ we distinguish the following cases:

i. $x(1) < \infty$. Then $J_x = [0,1]$ and x satisfies $I(D_\alpha x_a) \le 1$.

ii. x tends continuously to infinity in some point $t_0 \in (0,1]$. Then $J_x = [0,t_0)$.

iii. x jumps to infinity at t_0 (i.e. $x(t_0-) < \infty$ and $x(t_0+) = \infty$). We redefine $x(t_0) = x(t_0-)$ and have $J_x = [0,t_0]$.

THEOREM 9.2.2. *Let the sequence* $\{f_n\}$ *be defined by (9.2.2) and let the set* $K(\alpha)$ *as in definition 9.2.1. Then*

a. *Let f be an arbitrary non-decreasing function on* $[0,1]$ *and assume that there is a set A, with* $P[A] > 0$, *and such that for all* $\omega \in A$ *there exists a sequence* $n_k = n_k(\omega)$ *for which* $f_{n_k}(.,\omega) \to f(.)$ *in all continuity points of f in* J_f. *Then* $f \in K(\alpha)$.

b. *Let* $f \in K(\alpha)$. *Then, for almost all* ω, *there exists a sequence* $n_k = n_k(\omega)$ *such that* f_{n_k} *converges to f in all continuity points of f in* J_f.

PROOF. WICHURA found an error in the original proof. We now give a corrected proof, following the original idea but making use of lemma 1.5.1.

<u>Part a</u>. Let $f(1) < \infty$. We first prove that f has to be strictly increasing on $[0,1]$. Suppose f is constant on some subinterval J of J_f. Then there exists, for large m, a number j such that $[jm^{-1},(j+1)m^{-1}]$ is contained in J. Because f_{n_k} converges to f in every point of J, we have for all $\omega \in A$ that

$$g(\tfrac{j+1}{m},n_k,m,\omega)-g(\tfrac{j}{m},n_k,m,\omega) = m^{-1}[m(f_{n_k}(\tfrac{j+1}{m},\omega)-f_{n_k}(\tfrac{j}{m},\omega))]^{-\frac{\alpha}{2(1-\alpha)}}$$

tends to infinity for $k \to \infty$. Since every function in K^+ is bounded by 1, this contradicts the fact that $g(.,n_k,m,.)$ approaches K^+ as guaranteed by theorem 9.2.1 part a.

The restriction to $[0,1]$ in (9.2.2) is arbitrary. Every (finite) increasing function f on an interval has at most countably many discontinuities. We can choose $t_0 > 1$ such that f is continuous in every point $jm^{-1}t_0$, with $j=0,\dots,m$, for all m, and prove the theorem on $[0,1]$ by way of the corresponding theorem on $[0,t_0]$. Thus we may, without loss of generality, suppose that f is continuous in every point jm^{-1}, with $j=0,\dots,m$, for all m.

Let $\varepsilon > 0$. The above remarks imply that, for sufficiently large k,

$$d_c(\pi_m f_{n_k},\pi_m f) < \varepsilon \qquad \text{on } A \tag{9.2.13}$$

and also

$$(9.2.14) \qquad d_c(D_\alpha \pi_m f_{n_k}, D_\alpha \pi_m f) < \varepsilon \qquad \text{on } A.$$

Thus, on A, the sequence $D_\alpha \pi_m f_{n_k}$ tends to the limit $D_\alpha \pi_m f$. Theorem 9.2.1 part a implies that $D_\alpha \pi_m f \in K^+$. Now it remains to show that $D_\alpha f_a \in K^+$. We shall prove that $D_\alpha \pi_m f$ converges to $D_\alpha f_a$ as m tends to infinity. As K^+ is closed, this will imply that $D_\alpha f_a \in K^+$. Consider the sequence $\{D_\alpha \pi_m f\}_{m=0}^\infty$. Lemma 1.5.1 and Fatou's lemma give

$$(9.2.15) \qquad \liminf_{m\to\infty} D_\alpha \pi_m f \geq D_\alpha f_a$$

and from Jensen's inequality we obtain

$$(9.2.16) \qquad D_\alpha \pi_m f_a \leq D_\alpha f_a.$$

Because

$$\frac{d}{dt} \pi_m f_a(t) \leq \frac{d}{dt} \pi_m f(t) \qquad \text{for all } t \in [0,1]$$

we have

$$(9.2.17) \qquad D_\alpha \pi_m f_a \geq D_\alpha \pi_m f.$$

Together with (9.2.15) and (9.2.16) the result in (9.2.17) implies

$$\lim_{m\to\infty} D_\alpha \pi_m f = D_\alpha f_a.$$

This completes the proof of part a for the case where $f(1) < \infty$.

In case $f(1) = \infty$ and $J_f = [0,t_0]$ for some $t_0 \in (0,1)$ the proof is similar to the case $J_f = [0,1]$ if we divide J_f into m disjoint intervals of equal length. In case $J_f = [0,t_0)$ for some $t_0 \in (0,1]$ we can repeat the proof for every interval $[0,t_1]$ with $t_1 < t_0$.

Part b. Take $\varepsilon > 0$, $f \in K(\alpha)$ and suppose $J_f = [0,1]$. From definition 9.2.1 it follows that f is strictly increasing and $I(D_\alpha f_a) \leq 1$. An argument similar to (9.2.16) and (9.2.17) yields $I(D_\alpha \pi_m f) \leq 1$, implying $D_\alpha \pi_m f \in K_0^+$. Define for $j=0,\dots,m-1$

$$(9.2.18) \qquad a_j = a_j^{(m)} = D_\alpha \pi_m f((j+1)m^{-1}) - D_\alpha \pi_m f(jm^{-1}).$$

Then $a_j > 0$ for $j=0,\dots,m-1$. Choose $m > 1$ so large that $a_j < \varepsilon$ for all j.

Now we basically repeat the proof of part b of theorem 9.2.1, but we apply lemma 1.4.2. Again we may suppose $I(D_\alpha \pi_m f) < 1$. Choose for $j=0,\dots,m-1$ positive numbers ε_j as in (9.2.10) and such that

$$(9.2.19) \qquad m \sum_{j=0}^{m-1} (a_j + \varepsilon_j)^2 < 1.$$

Then we have

$$(9.2.20) \qquad m \sum_{j=0}^{m-1} (a_j - \varepsilon_j)^2 < 1.$$

Define for $j=0,\dots,m-1$ the events $C_r^{(j)}$ by (9.2.12) and the events E_r by

$$(9.2.21) \qquad E_r = \bigcap_{j=0}^{m-1} C_r^{(j)}.$$

The events E_r are not independent because E_r is not independent of $C_s^{(0)}$ for $s > r$. For fixed r the events $C_r^{(0)},\dots,C_r^{(m-1)}$ are independent and as in the proof of theorem 9.2.1.b it follows that $\sum P[E_r] = \infty$.

Consider $P[E_r \wedge E_s]$ for $r < s$. By the independence of the increments of stable processes we have

$$P[E_r \wedge E_s] = P[E_r \wedge C_s^{(0)}] \prod_{j=1}^{m-1} P[C_s^{(j)}].$$

By theorem 2.1.7 part IV and calculations similar to those in section 7.2 we have: there exists a constant k (independent of r and s) and a number r_0 such that

$$(9.2.22) \qquad P[E_r \wedge E_s] \le k\, P[E_r]\, P[E_s]$$

for $r \ge r_0$ and $s \ge r + 2\log\log r$. In the case $r < s \le r + 2\log\log r$ we obtain

$$(9.2.23)\quad P[E_r \wedge E_s] \le P[E_r] \prod_{j=1}^{m-1} P[C_s^{(j)}] \le$$

$$\le k_1\, P[E_r]\, e^{-m \sum_{j=1}^{m-1} (a_j-\varepsilon_j)^2 \log\log n_s}$$

where k_1 is independent of r and s.

Then, using (9.2.20), (9.2.22) and (9.2.23), we find

$$\liminf \{\sum_{r=1}^{n} P[E_r]\}^{-2} \sum_{r=1}^{n} \sum_{s=1}^{n} P[E_r \wedge E_s] \le k.$$

Lemma 1.4.2 implies $P[E_r \text{ i.o.}] = 1$ and hence, in particular, $P[\bigcup_{r=1}^{\infty} E_r] = 1$. Therefore, for almost all ω, the sequence n_r contains an index $\hat{n}(\omega)$ for which

$$d_c(D_\alpha \pi_m f_{\hat{n}}, D_\alpha \pi_m f) < \sum_{j=0}^{m-1} \varepsilon_j < \varepsilon/4.$$

More precisely, for j=1,...,m,

$$|D_\alpha \pi_m f_{\hat{n}}(jm^{-1}) - \sum_{i=0}^{j-1} a_i| < \sum_{i=0}^{j-1} \varepsilon_i \qquad \text{a.s. .}$$

By the definition of D_α and since $a_j \neq 0$, for all j, we have for all j

$$|f_{\hat{n}}(jm^{-1}) - f(jm^{-1})| < c\,\varepsilon \qquad \text{a.s. ,}$$

where $c = m\{(1-m^{-1})^{-\frac{2(1-\alpha)}{\alpha}} - 1\}$. Note that this constant can be bounded by another constant c_1, which depends on α, but not on m. Thus, independently of ε (and m), we have for all j

$$(9.2.24)\quad |f_{\hat{n}}(jm^{-1}) - f(jm^{-1})| < c_1\varepsilon \qquad \text{a.s. .}$$

Let $\eta_1, \eta_2, \ldots$ be a decreasing sequence of real numbers tending to zero. For each η_i there exists a number $m_i > 1$ such that $a_j^{(m_i)} < \eta_i$ for $j=0,\ldots$ $\ldots,m_i-1$, where $a_j^{(m)}$ is defined by (9.2.18). The result (9.2.24) yields the existence of a set A_i, with $P[A_i] = 1$, such that for each $\omega \in A_i$ there is a $\hat{n}_i(\omega)$ with the property

$$(9.2.25) \qquad |f_{\hat{n}_i}(jm_i^{-1},\omega) - f(jm_i^{-1})| < c_1\ \eta_i$$

for $j=0,\ldots,m_i$. Obviously we have $P[\cap A_i] = 1$. Fix some $\omega \in \cap A_i$. Because $f(1) < \infty$, the sequence $f_{\hat{n}_i}$ is uniformly bounded. Helly's first theorem yields the existence of a subsequence $f_{\tilde{n}_i}$ which converges weakly to some non-decreasing bounded function $\tilde{f}$. By (9.2.25) we have, for any $\omega \in \cap A_i$, that $f = \tilde{f}$ in all continuity points of f.

In case $J_f = [0,t_0]$ (resp. $[0,t_0)$) with $0 < t_0 < 1$ (resp. $0 < t_0 \le 1$) we proceed as in part a. □

REMARK 9.2.1. WICHURA (1973) has independently proved results of a similar nature for partial sum processes. He extends the space $D[0,\infty)$ with functions that may have the value ∞ and he also extends the M_1-topology to this (new) space. Define the sequence of functions f_n

$$f_n : [0,\infty) \times \Omega \to \mathbb{R}$$

analogous to f_n in (9.2.2) using the partial sum process. Then he proved relative compactness of $\{f_n\}$ in the (extended) M_1-topology.

REMARK 9.2.2. As a consequence of theorem 9.2.1 part a we have for all integers m

$$\limsup_{n\to\infty} \quad D_\alpha \pi_m f_n(1) \le 1 \qquad \text{a.s.}$$

or equivalently

$$\limsup_{n\to\infty} \frac{(2B(\alpha))^{\frac{1}{2}}}{(2\log\log n)^{\frac{1}{2}}} \sum_{j=0}^{m-1} \left[\frac{X((j+1)n/m)-X(jn/m)}{(n/m)^{1/\alpha}}\right]^{-\frac{\alpha}{2(1-\alpha)}} \le 1 \qquad \text{a.s.}\ .$$

9.3. THE CASE $\alpha = 1$

In this section we consider the completely asymmetric stable process $\{X(t) : 0 \le t < \infty\}$ with characteristic exponent $\alpha = \beta = 1$. The mappings π_m and I are defined by (9.0.1) and (9.0.2). Let C be the subclass of $C[0,1]$ of almost everywhere differentiable functions. D_1 is defined by

$$D_1 : C \to C^+$$

and

$$(9.3.1) \qquad D_1x(t) = 2(\pi e)^{-\frac{1}{2}} \int_0^t e^{-\frac{\pi}{4}\dot{x}(y)} dy.$$

Define the sequences of functions $\{f_n(t,\omega), n \geq 3\}$ and $\{g(t,n,m,\omega);$ $n \geq 3,\ m \in N\}$ by

$$(9.3.2) \qquad f_n(t,\omega) = n^{-1}\ \{X(nt,\omega)-(2/\pi)nt \log n\} + (2/\pi)t \log(2\log \log n)$$

and

$$(9.3.3) \qquad g(t,n,m,\omega) = D_1\pi_m f_n(t,\omega).$$

Let n_r be defined by (9.2.4) and let $r(m)$ be as in (9.2.5). Define the random variables $A_{j,r}$ for $j=0,\ldots,m-1$ and $r \geq r(m)$ by

$$(9.3.4) \qquad A_{j,r} = ((j+1)n_r/m-jn_{r+1}/m)^{-1}\ \{X((j+1)n_r/m)-X(jn_{r+1}/m) +$$
$$- (2/\pi)((j+1)n_r/m-jn_{r+1}/m) \log ((j+1)n_r/m)-jn_{r+1}/m)\}.$$

In the proof of the first theorem in this section we need the following lemma. The proof of this lemma will be given in appendix 2.

LEMMA 9.3.1. *Fix* m. *Let* n_r, $r(m)$ *and* $A_{j,r}$ *be as above, let* $\varepsilon > 0$ *and define*

$$\bar{A}_r = \sum_{j=0}^{m-1} 4(\pi e)^{-1} \exp(-(\pi/2)\ A_{j,r}\ (1-\varepsilon_{j,r})),$$

where $\varepsilon_{j,r} = O((\log r)^{-2})$ *for* $r \to \infty$. *Then*

$$P[\bar{A}_r > 2(1+\varepsilon)\log \log n_r \text{ for infinitely many } r] = 0.$$

THEOREM 9.3.1. *Let* $\{g(.,n,m,\omega) : n \geq 3,\ m \in N\}$ *be defined by (9.3.3). Then theorem 9.2.1 is true for this sequence.*

PROOF.

Part a. We shall only give the points of difference with the proof of theorem 9.2.1. Remember that the sample paths of the process $X(t)$ are in $D[0,1]$ and may decrease (continuously) but that all jumps have to be positive. We make use of the results in section 7.3.

Define the random variables $B_{j,n}$ and $C_{j,n}$ for $j=0,\dots,m-1$ and $n \in \mathbb{N}$ by

$$B_{j,n} = (n/m)^{-1}[X(n(j+1)/m)-X(nj/m)-(2/\pi)(n/m)\log(n/m)] \tag{9.3.5}$$

and

$$C_{j,n} = 2(\pi e)^{-\frac{1}{2}} \exp(-\pi B_{j,n}/4). \tag{9.3.6}$$

Then

$$I(g) = \sum_{j=0}^{m-1} 4(\pi e)^{-1}(2\log\log n)^{-1} e^{-\pi B_{j,n}/2} =$$

$$= (2\log\log n)^{-1} \sum_{j=0}^{m-1} C_{j,n}^2.$$

Let n_r be defined by (9.2.4). Then we can find for every n a number r such that $n_r \le n < n_{r+1}$. If $r \ge r(m)$ so that (9.2.5) is fulfilled, we can write

$$B_{j,n} = Q_1 + Q_2 + Q_3 + Q_4,$$

where

$$Q_1 = A_{j,r}(n_r(j+1)/n-n_{r+1}j/n),$$

$$Q_2 = [\{(n-n_r)(j+1)/m\}^{-1}\{X(n(j+1)/m)-X(n_r(j+1)/m) +$$
$$- (2/\pi)(n-n_r)(j+1)m^{-1}\log((n-n_r)(j+1)/m)\}] \cdot (n-n_r)(j+1)n^{-1},$$

$$Q_3 = [\{(n_{r+1}-n)j/m\}^{-1}\{X(n_{r+1}j/m)-X(nj/m)-(2/\pi)(n_{r+1}-n)jm^{-1} \cdot$$
$$\cdot \log((n_{r+1}-n)j/m)\}] \cdot (n_{r+1}-n)jn^{-1}$$

and

$$Q_4 = (2/\pi)(n/m)^{-1}\{-(n/m)\log(n/m)+(n-n_r)(j+1)m^{-1}\log((n-n_r)(j+1)/m) +$$
$$+ (n_{r+1}-n)jm^{-1}\log((n_{r+1}-n)j/m) +$$
$$+ (n_r(j+1)m^{-1}-n_{r+1}jm^{-1})\log(n_r(j+1)m^{-1}-n_{r+1}jm^{-1})\}.$$

If $n = n_r$, we define $Q_2 = 0$.

First we consider Q_1. By the definition of n_r we have $(n_r(j+1)-jn_{r+1})/n = 1-O((\log r)^{-2})$ for $r \to \infty$. Hence by lemma 9.3.1

$$P[\sum_{j=0}^{m-1} 4(\pi e)^{-1} \exp(-(\pi/2)A_{j,r}((j+1)n_r-jn_{r+1})n^{-1}) > (1+\varepsilon)\, 2\log\log n_r \text{ for infinitely many } r] = 0.$$

Consequently, for almost all ω, there exists a number $r_1 = r_1(\varepsilon,m,\omega)$ such that

$$(2\log\log n_r)^{-1} \sum_{j=0}^{m-1} 4(\pi e)^{-1}\, e^{-\frac{\pi}{2} A_{j,r}((j+1)n_r n^{-1}-jn_{r+1}n^{-1})} \le 1+\varepsilon \text{ for all } r \ge r_1. \tag{9.3.7}$$

Next we turn to Q_2 and assume $n > n_r$. Define the process $\{\tilde{X}_r(t) : 0 \le t \le 1\}$ by

$$\tilde{X}_r(t) = n_{r+1}^{-1}\{X(n_{r+1}t)-(2/\pi)n_{r+1}t\log n_{r+1}\}. \tag{9.3.8}$$

The expression in square brackets in Q_2 is distributed as $X(1)$ and equals

$$mn_{r+1}(j+1)^{-1}(n-n_r)^{-1}\{\tilde{X}_r((j+1)nm^{-1}n_{r+1}^{-1})-\tilde{X}_r((j+1)n_r m^{-1}n_{r+1}^{-1}) + - (2/\pi)(j+1)(n-n_r)m^{-1}n_{r+1}^{-1}\log((j+1)(n-n_r)m^{-1}n_{r+1}^{-1})\} \tag{9.3.9}$$

By property 4 in section 3.2, the process $\{\tilde{X}_r(t) : 0 \le t \le 1\}$ is a stable process with $\alpha = \beta = 1$. Now we have

$$n_{r+1}^{-1} \le n_{r+1}^{-1}(n-n_r) \le 1-n_r n_{r+1}^{-1} = O((\log r)^{-2}) \qquad \text{for } r \to \infty. \tag{9.3.10}$$

Hence, by theorem 7.3.1, for almost all ω there exists a number $r_2(\varepsilon,m,\omega)$ such that (9.3.9) is larger than or equal to

$$-(2/\pi)\log(\pi e/2)-(2/\pi)\log\log(m(j+1)^{-1}n_{r+1}(n-n_r)^{-1})-(2/\pi)\log(1+\varepsilon) \tag{9.3.11}$$

for $r \ge r_2$. Consequently Q_2 is bounded from below by a function of r, say $-\phi(r)$, and by (9.3.10) and (9.3.11) $\phi(r) = O((\log r)^{-1})$ for $r \to \infty$. A similar lower bound can be given for Q_3.

Q_4 can be expanded for large r. This term is $O(\log\log r(\log r)^{-2})$ for $r \to \infty$ for every j. Using all these estimates we have for almost all ω: there exists a number $r_3(\varepsilon,m,\omega)$ such that

$$(9.3.12) \quad I(g(.,n,m,\omega)) \le (1+\varepsilon)^2$$

for $n \ge n_{r_3(\varepsilon,m,\omega)}$.

<u>Part b</u>. Again we may suppose $I(h) < 1$. It is possible to give a proof similar to that of theorem 9.2.1.b by using theorem 6.3.1. We shall not do so. We shall follow the proof of theorem 9.2.2.b instead.

Fix m such that (9.2.8) holds. We first consider the case where h is strictly increasing, so that $a_j > 0$ for all $j=0,\dots,m-1$. Choose ε_j for $j=0,\dots,m-1$ such that (9.2.10), (9.2.19) and (9.2.20) are fulfilled. Define the events $C_r^{(j)}$ and E_r as in (9.2.12) and (9.2.21), where $g(.,n,m,\omega)$ is defined by (9.3.3). Take $n_r = m^r$. Using the definition of $g(.,n,m,\omega)$ and properties of the completely asymmetric stable process we find that

$$\begin{aligned} P[C_r^{(j)}] &= P[(a_j-\varepsilon_j) \le 2(\pi e)^{-\frac{1}{2}}m^{-1}\exp\{-(\pi/4)(m/n_r)X(n_r(j+1)/m) + \\ &\quad - X(n_r j/m)-(2/\pi)(n_r/m)\log n_r\}\exp\{-\tfrac{1}{2}\log(2\log\log n_r) \le \\ &\quad \le (a_j+\varepsilon_j)] \\ &= P[(a_j-\varepsilon_j)m^{\frac{1}{2}}(2\log\log n_r)^{\frac{1}{2}} \le 2(\pi e)^{-\frac{1}{2}}\exp(-\pi X(1)/4) \le \\ &\quad \le (a_j+\varepsilon_j)m^{\frac{1}{2}}(2\log\log n_r)^{\frac{1}{2}}]. \end{aligned}$$

Now we use theorem 2.1.7 V to prove $\sum_r P[E_r] = \infty$ (c.f. section 9.2). In order to apply lemma 1.4.2 we have to bound

$$\{P[E_r]\,P[E_s]\}^{-1}\,P[E_r \wedge E_s] = \{P[E_r]\,P[C_s^{(0)}]\}^{-1}\,P[E_r \wedge C_s^{(0)}]$$

for $r < s$. As in the case $0 < \alpha < 1$ we have

$$P[C_s^{(0)}] \sim \sqrt{2}\,P[U \ge (a_0-\varepsilon_0)m^{\frac{1}{2}}(2\log\log n_s)^{\frac{1}{2}}] \qquad \text{for } s \to \infty.$$

For $s > r+1$ we can bound $P[E_r \wedge C_s^{(0)}]$ by

$$P[E_r]\, P[A_- \le 2(\pi e)^{-\frac{1}{2}} \exp\{-(\pi/4)m(n_s - mn_r)^{-1}(X(n_s/m) - X(n_r) + $$
$$- (2/\pi)(n_s/m - n_r)\log(n_s/m - n_r))\} \le A_+],$$

where

$$A_\pm = \{(a_0 \pm \varepsilon_0)m^{\frac{1}{2}}(2\log\log n_s)^{\frac{1}{2}}\}^{\frac{n_s}{n_s - mn_r}} \cdot \left\{e^{O(m^{r-s+1}\log(m^{r-s}))}\right\}^{\frac{n_s}{n_s - mn_r}} \cdot$$
$$\cdot \{\prod_{j=0}^{m-1} (a_j \mp \varepsilon_j)m^{\frac{1}{2}}(2\log\log n_r)^{\frac{1}{2}}\}^{-\frac{n_r}{n_s - mn_r}} .$$

After some algebra (9.2.22) follows for $r \ge r_0$ and $s \ge r + 2\log\log r$. In the other cases the estimate (9.2.23) can be derived. Thus it follows that $P[E_r \text{ i.o.}] = 1$. Therefore, for almost all ω there exists a subsequence $\hat{n}_r(\omega)$ such that

$$d_c(g(\cdot, \hat{n}_r, m, \omega), h) < \varepsilon.$$

If h is not strictly increasing, then some of the a_j will vanish for large m. We distinguish two cases. In case $a_0 = 0$ we have $P[C_s^{(0)}] > 1-\delta$ for $s \ge s_0(\delta)$ and it follows that

$$\{P[E_r]\, P[E_s]\}^{-1}\, P[E_r \wedge E_s] \le (1-\delta)^{-1} \qquad \text{for } s \ge s_0(\delta)$$

and all $r < s$. If $a_0 \ne 0$ and $a_j = 0$ for some $j \in \{1,\dots,m-1\}$, then we replace the product in A_+ by

$$\Pi^*\{(a_j - \varepsilon_j)m^{\frac{1}{2}}(2\log\log n_r)^{\frac{1}{2}}\}^{-\frac{n_r}{n_s - mn_r}},$$

where Π^* means the product of all those factors with $a_j \ne 0$. □

We recall that in the case where $0 < \alpha < 1$ the f_n are non-decreasing and that we have characterized their limit points within the class of non-decreasing functions that may be infinite from some point on. Now the f_n belong to $D[0,1]$ and we shall consider limit points in the class $\bar{D}[0,1]$ of functions f on $[0,1]$ which have no discontinuities but jumps and are such that $-\infty < f(t) \leq \infty$ for all t and that $f = \infty$ on $[t,1]$ whenever $f(t) = \infty$. For $f \in \bar{D}[0,1]$ we shall denote the interval where f is finite by J_f. We define the following subclass of $\bar{D}[0,1]$.

DEFINITION 9.3.1. Let K(1) be the set of functions x on [0,1] with the properties

1. $x(0) = 0$ and x is bounded below;
2. $x = x_s + x_a$, where x_s is non-decreasing and singular with respect to Lebesgue measure and x_a is absolutely continuous;
3. $4(\pi e)^{-1} \int_{J_x} \exp(-\pi\dot{x}(t)/2)\,dt \leq 1$.

Note that a function in K(1) cannot have negative jumps. Moreover, 2. implies that x is differentiable almost everywhere on J_x so that the integral in 3. is well defined.

THEOREM 9.3.2. *Let the sequence* $\{f_n\}$ *be defined by (9.3.2) and the set* K(1) *as in definition 9.3.1. Then*

a. *Let* f *be in* $\bar{D}[0,1]$ *and assume that there is a set* A, *with* $P[A] > 0$, *and such that, for all* $\omega \in A$, *there exists a sequence* $n_k = n_k(\omega)$ *for which* $f_{n_k}(.,\omega) \to f(.)$ *in all continuity points of* f *in* J_f. *Then* $f \in K(1)$.

b. *Let* $f \in K(1)$. *Then, for almost all* ω *there exists a sequence* $n_k = n_k(\omega)$ *such that* f_{n_k} *converges to* f *in all continuity points of* f *in* J_f.

PROOF.

Part a. We follow the proof of theorem 9.2.2 part a. Again we suppose that $J_f = [0,1]$. Also, because $f \in \bar{D}[0,1]$, it has at most countably many discontinuities. By the argument in the proof of theorem 9.2.2 part a we may assume that, for all m, the points jm^{-1}, $j=0,\ldots,m$, are continuity points of f.

By theorem 9.3.1 part a the set of limit points of $g(.,n,m,\omega) = D_1\pi_m f_n(.,\omega)$ is, for every fixed m, w.p. 1 contained in K^+. From Schwarz's inequality we have (see FREEDMAN (1971) lemma 78a) for $g \in K^+$ and all positive integers m

$$0 \leq g((j+1)/m)-g(j/m) \leq m^{-\frac{1}{2}}$$

for $j=1,\ldots,m$. It easily follows that f cannot have negative jumps. Take

$\varepsilon > 0$. We have for sufficiently large k

$$(9.3.13) \qquad d_c(\pi_m f_{n_k}, \pi_m f) < \varepsilon \qquad \text{on } A.$$

and

$$(9.3.14) \qquad d_c(D_1\pi_m f_{n_k}, D_1\pi_m f) < \varepsilon \qquad \text{on } A.$$

Thus on A, the sequence $D_1\pi_m f_{n_k}$ tends to the limit $D_1\pi_m f$. It follows that $D_1\pi_m f \in K^+$ for all m. Consequently

$$4(\pi e)^{-1} \sum_{j=0}^{m-1} (m^{-1}+(\pi/2)(f((j+1)/m)-f(j/m))^-) \le$$

$$\le 4(\pi m e)^{-1} \sum_{j=0}^{m-1} \exp\{(\pi/2)m^{-1}(f((j+1)/m)-f(j/m))^-\} \le$$

$$\le I(D_1\pi_m f) + 4(\pi e)^{-1} \le 1 + 4(\pi e)^{-1}.$$

This yields

$$\sum_{j=0}^{m-1} (f((j+1)/m)-f(j/m))^- \le e/2 \qquad \text{for all m.}$$

Thus, the negative variation V^-f is a finite continuous function on $[0,1]$. Now, the assumption $J_f = [0,1]$ implies that f is of bounded variation on J_f and thus we can write $f = V^+f - V^-f$, where V^+f and V^-f are both non-decreasing functions.

Applying martingale theory WICHURA (1973) shows V^-f is absolutely continuous and $\lim_{n\to\infty} D_1\pi_{2^n} f = D_1 f_a$. Because K^+ is closed we have $D_1 f_a \in K^+$ and this implies $f \in K(1)$.

Part b. Let $f \in K(1)$ and $J_f = [0,1]$. As in the proof of theorem 9.2.2 we have, by Jensen's inequality and properties of functions in K(1),

$$D_1\pi_m f \le D_1\pi_m f_a \le D_1 f_a.$$

This yields $\pi_m f \in K(1)$. In the proof of theorem 9.3.1.b we have already proved $P[E_r \text{ i.o.}] = 1$. From the definition of D_1 we easily obtain that, for sufficiently large m, there exist, for almost all ω, infinitely many numbers n such that

$$(9.3.15) \qquad -(4/\pi)\log(1+m^{-1}) \le f_n(jm^{-1},\omega)-f(jm^{-1}) \le -(4/\pi)\log(1-m^{-1})$$

for $j=0,\ldots,m$. It follows that, for almost all ω, there exists $\{n_k(\omega)\}$ such that f_{n_k} converges to f on a non-random countable dense subset of [0,1].

In order to conclude $f_{n_k} \to f$ in the continuity points of f we first prove the following assertion. For all $\varepsilon > 0$ and $0 \le t < 1$, there exists a real constant $\Delta > 0$ (independent of t) such that for almost all ω there is a number $n_0 = n_0(\omega)$ such that

$$\inf_{0<\delta\le\Delta} \{f_n(t+\delta)-f_n(t)\} > -\varepsilon \qquad \text{for } n \ge n_0.$$

Define the event C_n by

$$\inf_{0<\delta\le\Delta} \{f_n(t+\delta)-f_n(t)\} \le -\varepsilon.$$

As in the proof of lemmas 3.5.4 and 3.5.5 we can show that there exists a constant k such that

$$P[C_n] \le k\, P[f_n(t+\Delta)-f_n(t) \le -(2/\pi)\log(\pi e(1+\varepsilon)/4)]$$

for sufficiently small Δ. In a similar fashion one shows that

$$P[D_r] \le k\, P[\min_{n_r\le n\le n_{r+1}} \{f_n(t+\Delta)-f_n(t)\} \le -(2/\pi)\log(\pi e(1+\varepsilon)/4)],$$

where n_r is defined by (9.2.4) and $D_r = \bigcup_{n=n_r}^{n_{r+1}} C_n$. Hence by lemma 3.5.4

$$P[D_r] \le k_1\, P[X(1) \le -(2/\pi)\log(\tfrac{1}{2}\pi e(1+\varepsilon)\log\log n_r)].$$

Theorem 2.1.7 part V yields $\sum P[D_r] < \infty$. Then the assertion follows from the Borel-Cantelli lemma.

For every continuity point of f we choose points t_1, t_2 in the countable dense subset where f_{n_k} converges to f and such that $t_1 \le t \le t_2$, $t_2-t_1 \le \Delta$

and $|f(t_2)-f(t_1)| < \varepsilon$. Then, for almost all ω,

$$\limsup_{k\to\infty} \quad |f_{n_k}(t)-f(t)| \le 2\varepsilon$$

for all t. □

REMARK 9.3.1. As a consequence of theorem 9.3.1.a we have for all integers m

$$\limsup_{n\to\infty} \quad D_1\pi_m f_n(1) \le 1 \qquad \text{a.s.}$$

or equivalently

$$\limsup_{n\to\infty} \left\{\frac{2}{\sqrt{\pi e}}\frac{1}{\sqrt{2\log\log n}} \sum_{j=0}^{m-1} e^{-\frac{\pi}{4}[X(\frac{n(j+1)}{m})-X(\frac{nj}{m}) - \frac{2}{\pi}\frac{n}{m}\log\frac{n}{m}](n/m)^{-1}}\right\}$$

$$\le 1 \qquad \text{a.s.} \; .$$

9.4. THE CASE $1 < \alpha < 2$

Let $\{X(t) : 0 \le t < \infty\}$ be the completely asymmetric stable process with $1 < \alpha < 2$. Define the sequences $\{f_n(.,\omega) : n \ge 3\}$, $\{g(.,n,m,\omega) : n \ge 3, m \in \mathbb{N}\}$ and the function D_α by

$$f_n : [0,1] \times \Omega \to \mathbb{R} \qquad \text{for } n \ge 3$$

and

$$(9.4.1) \qquad f_n(t,\omega) = X(nt,\omega)\, n^{-\frac{1}{\alpha}}\{2B(\alpha)\}^{\frac{\alpha-1}{\alpha}}(2\log\log n)^{-\frac{\alpha-1}{\alpha}} ;$$

$$D_\alpha : C \to C^+$$

$$(9.4.2) \qquad D_\alpha x(t) = \int_0^t \{[\dot{x}(v)]^-\}^{\frac{\alpha}{2(\alpha-1)}}\, dv$$

and

$$(9.4.3) \qquad g(t,n,m,\omega) = D_\alpha \pi_m f_n(t,\omega).$$

Let n_r be defined by (9.2.4) and let $r(m)$ be as in (9.2.5). Define the random variables $A_{j,r}$ for $j=0,\ldots,m-1$ and $r \geq r(m)$ by

$$(9.4.4) \qquad A_{j,r} = 2B(\alpha)m^{\frac{1}{\alpha-1}}((j+1)n_r-jn_{r+1})^{-\frac{1}{\alpha-1}}\{[X((j+1)n_r/m)-X(jn_{r+1}/m)]^-\}^{\frac{\alpha}{\alpha-1}}.$$

In the proof of the first theorem in this section we need the following lemma. The proof of this lemma will be given in appendix 2.

LEMMA 9.4.1. *Fix* m. *Let* n_r, $r(m)$ *and* $A_{j,r}$ *be as above, let* $\varepsilon > 0$ *and* $\varepsilon_{j,r} = O((\log r)^{-2})$ *for* $r \to \infty$. *Then*

$$P[\sum_{j=0}^{m-1} A_{j,r}(1-\varepsilon_{j,r}) > 2(1+\varepsilon)\log\log n_r \textit{ for infinitely many } r] = 0.$$

THEOREM 9.4.1. *Let* $\{g(.,n,m,\omega) : n \geq 3,\ m \in N\}$ *be defined by (9.4.3). Then theorem 9.2.1 is true for this sequence.*

PROOF.

Part a. It is sufficient to prove that for almost all ω there exists a number $n_0 = n_0(\varepsilon,m,\omega)$ such that $I(g) = I(D_\alpha \pi_m f_n(t,\omega)) < (1+\varepsilon)^2$ for $n \geq n_0$.

$$(9.4.5) \qquad I(g) = \sum_{j=0}^{m-1} \{[X((j+1)n/m,\omega)-X(jn/m,\omega)]^- n^{-\frac{1}{\alpha}} m\}^{\frac{\alpha}{\alpha-1}} m^{-1} 2B(\alpha).$$

$$\cdot\ (2\log\log n)^{-1} =$$

$$= (2\log\log n)^{-1} \sum_{j=0}^{m-1} C_{j,n}^2,$$

where the random variable $C_{j,n}$ is defined by

$$C_{j,n} = \{[X((j+1)n/m)-X(jn/m)]^-\}^{\frac{\alpha}{2(\alpha-1)}} n^{-\frac{1}{2(\alpha-1)}} m^{\frac{1}{2(\alpha-1)}} \{2B(\alpha)\}^{\frac{1}{2}}.$$

Suppose $n \geq n_{r(m)}$. For every such n we can find an integer $r \geq r(m)$ such that $n_r \leq n < n_{r+1}$. Then we can write

(9.4.6) $$n^{-1/\alpha}\{X((j+1)n/m)-X(jn/m)\} = Q_1 + Q_2 + Q_3,$$

where

$$Q_1 = n^{-1/\alpha}\{X((j+1)n/m)-X((j+1)n_r/m)\},$$

$$Q_2 = n^{-1/\alpha}\{X((j+1)n_r/m)-X(jn_{r+1}/m)\}$$

and

$$Q_3 = n^{-1/\alpha}\{X(jn_{r+1}/m)-X(jn_r/m)\}.$$

Define the process $\{\tilde{X}_r(t) : 0 \le t \le 1\}$ by

$$\tilde{X}_r(t) = X(n_{r+1}t)n_{r+1}^{-1/\alpha}.$$

This process is a completely asymmetric stable process with characteristic exponent α. Then

(9.4.7) $$\{(j+1)(n-n_r)/m\}^{-1/\alpha}\{X((j+1)n/m)-X((j+1)n_r/m\} = \\ = m^{1/\alpha}n_{r+1}^{1/\alpha}\{(j+1)(n-n_r)\}^{-1/\alpha}\{\tilde{X}_r((j+1)nm^{-1}n_{r+1}^{-1})-\tilde{X}_r((j+1)n_rm^{-1}n_{r+1}^{-1})\}.$$

By theorem 7.4.1 Q_1 is for almost all ω bounded from below by

$$-(2B(\alpha))^{-\frac{\alpha-1}{\alpha}}\{2(1+\varepsilon)\log(mn_{r+1}(j+1)^{-1}(n-n_r)^{-1})\}^{\frac{\alpha-1}{\alpha}}\{(j+1)(1-n_r/n)/m\}^{\frac{1}{\alpha}}$$

for $r > r_0(\varepsilon,m,\omega)$. Using (9.3.10) we have that $Q_1 > -\varepsilon$ for $r > r_1(\varepsilon,m,\omega)$.

In the same way one shows $Q_3 > -\varepsilon$ for $r > r_2(\varepsilon,m,\omega)$. In view of the lower bounds for Q_1 and Q_3 we find

$$C_{j,n}^2 = [(Q_1+Q_2+Q_3)^-]^{\frac{\alpha}{\alpha-1}} m^{-\frac{1}{\alpha-1}} \{2B(\alpha)\} \le [(Q_2-2\varepsilon)^-]^{\frac{\alpha}{\alpha-1}} m^{-\frac{1}{\alpha-1}} \{2B(\alpha)\}$$

$$\le [Q_2^-+2\varepsilon]^{\frac{\alpha}{\alpha-1}} m^{-\frac{1}{\alpha-1}} \{2B(\alpha)\} \le A_{j,r}\{(j+1)n_r/n-jn_{r+1}/n\}^{\frac{1}{\alpha-1}} + 2c\varepsilon$$

for $r \ge r_3(\varepsilon,m,\omega)$ and some constant c. Note that $(j+1)n_r/n-jn_{r+1}/n =$

$= 1-O((\log r)^{-2})$ for $r \to \infty$. By lemma 9.4.1 we have for almost all ω and $r \geq r_4(\varepsilon,m,\omega)$

$$I(g) < (1+\varepsilon) + (2\log\log n)^{-1}\, 2c\varepsilon.$$

This implies that for almost all ω there exists a number $n_0(\varepsilon,m,\omega)$ such that $I(g) < (1+\varepsilon)^2$ for all $n \geq n_0(\varepsilon,m,\omega)$.

Part b. Again we may suppose $I(h) < 1$. Define n_r, a_j, ε_j, $C_r^{(j)}$ and E_r as in the proof of theorem 9.2.1.b. Then

$$P[E_r] = \prod_{j=0}^{m-1} P[C_r^{(j)}] = \prod_{j=0}^{m-1} P[(a_j-\varepsilon_j)^+ m^{\frac{1}{2}}(2\log\log n_r)^{\frac{1}{2}} \leq$$

$$\leq (2B(\alpha))^{\frac{1}{2}}\{[X(1)]^-\}^{\frac{\alpha}{2(\alpha-1)}} \leq (a_j+\varepsilon_j)m^{\frac{1}{2}}(2\log\log n_r)^{\frac{1}{2}}].$$

Using theorem 2.1.7 part VI we have $\sum P[E_r] = \infty$.

Consider $P[E_r \wedge C_s^{(0)}]$ for $r < s$. This probability can be bounded by

$$P[E_r]\, P[-A_+ \leq (n_s/m-n_r)^{-1/\alpha}\{X(n_s/m)-X(n_r)\} \leq -A_-]$$

where

$$A_\pm = (1-mn_r/n_s)^{-\frac{1}{\alpha}}\{[(a_0\pm\varepsilon_0)^+ m^{\frac{1}{2}}(2\log\log n_s)^{\frac{1}{2}}]^{\frac{2(\alpha-1)}{\alpha}}(2B(\alpha))^{-\frac{\alpha-1}{\alpha}} +$$

$$-\sum_{j=0}^{m-1}[(a_j\mp\varepsilon_j)^+ m^{\frac{1}{2}}(2\log\log n_r)^{\frac{1}{2}}]^{\frac{2(\alpha-1)}{\alpha}}(2B(\alpha))^{\frac{\alpha-1}{\alpha}}(n_r/n_s)^{\frac{1}{\alpha}}\}.$$

In the case $a_0 \neq 0$ the quantity A_- tends to ∞ if $s-r \to \infty$. Then

$$P[-A_+ \leq (n_s/m-n_r)^{-\frac{1}{\alpha}}\{X(n_s/m)-X(n_r)\} \leq -A_-] \sim$$

$$\sim (2\alpha)^{\frac{1}{2}}\, P[B_- \leq U \leq B_+], \qquad \text{for } s-r \to \infty$$

where $B_\pm = -\{2B(\alpha)\}^{\frac{1}{2}}\, A_\pm^{\frac{\alpha}{2(\alpha-1)}}$.

Then (9.2.22) follows for $r \geq r_0$ and $s > r+2\log\log r$.

In the case $a_0 = 0$ it follows that

$$\{P[E_r]\, P[E_s]\}^{-1}\, P[E_r \wedge E_s] \leq \{P[C_s^{(0)}]\}^{-1} < (1-\varepsilon)^{-1}$$

for $s \geq s_0(\varepsilon)$.

The proof can be finished as in the cases $0 < \alpha < 1$ and $\alpha = 1$. □

DEFINITION 9.4.1. Let $K(\alpha)$ be the set of non-decreasing functions with the properties

1. $x(0) = 0$;
2. x is absolutely continuous on $[0,1]$;
3. $\int_0^1 [\dot{x}(t)]^{\frac{\alpha}{\alpha-1}}\, dt \leq 1$.

The following theorem deals with non-decreasing limit points only.

THEOREM 9.4.2. *Let the sequence* $\{f_n\}$ *be defined by (9.4.1) and let the set* $K(\alpha)$ *as in definition 9.4.1. Let* $\varepsilon > 0$.

a. *Let* f *be an arbitrary non-decreasing function and assume that there is a set* A *with* $P[A] > 0$ *and such that, for almost all* ω, *there exists a sequence* $n_k = n_k(\omega)$ *for which the sequence* $f_{n_k}(.,\omega)$ *converges to* $-f$. *Then* $f \in K(\alpha)$ *and for almost all* $\omega \in A$ $d_c(f_{n_k},-f) \to 0$ *for* $k \to \infty$.

b. *For all* $f \in K(\alpha)$ *there exists a number* $m_0(\varepsilon,f)$ *such that*

$$P[\{\omega:\ d_c([\pi_m f_n(.,\omega)]^-, f) < \varepsilon \text{ for infinitely many } n\}] = 1$$

for all $m \geq m_0(\varepsilon,f)$.

PROOF.

<u>Part a</u>. As in the proofs of theorems 9.2.2 and 9.3.2, every (non-increasing) limit point f of $\{f_n\}$ satisfies $g_m = D_\alpha \pi_m f \in K^+$ for every m. It follows that $g = \lim g_m \in K^+$. Computations similar to those in the proof of theorem 9.2.2 part a give $g = D_\alpha f_a$.

It remains to prove $f \equiv f_a$. Let $\varepsilon > 0$. Suppose $f_s(1) = f(1)-f_a(1) > 0$. Construct a set B_f as in remark 1.5.1. This set consists of intervals $(jm^{-1},(j+1)m^{-1})$ for a certain m. Let n_0 be the number of interval contained in $\bar{B}_f$. For $n_0 > 0$, there exists an integer j with $(jm^{-1},(j+1)m^{-1}) \in \bar{B}_f$ and

$$m\cdot(f((j+1)m^{-1})-f(jm^{-1})) \geq m(f_s(1)-\varepsilon)n_0^{-1}.$$

This implies that the function $D_\alpha \pi_m f$ increases on $\bar{B}_f$ more than

$$(f_s(1)-\varepsilon)^{\frac{\alpha}{2(\alpha-1)}} \cdot (n_0 m^{-1})^{\frac{\alpha-2}{2(\alpha-1)}}.$$

This contradicts $D_\alpha \pi_m f \in K^+$.

Part b. This part of the theorem follows directly from theorem 9.4.1 part b. □

REMARK 9.4.1. Theorem 9.4.2 part b suggests that the negative variation of any limit point is absolutely continuous. This is indeed the case, as follows from WICHURA (1973), who also describes all other limit points.

REMARK 9.4.2. As a consequence of part a of theorem 9.4.1 we have, for all integers m,

$$\limsup_{n\to\infty} \quad D_\alpha \pi_m f_n(1) \leq 1 \qquad \text{a.s.}$$

or, equivalently,

$$\limsup_{n\to\infty} \frac{\{2B(\alpha)\}^{\frac{1}{2}}}{(2\log\log n)^{\frac{1}{2}}} \sum_{j=0}^{m-1} \left\{\frac{m}{n}\left[X\left(\frac{j+1}{m}\right)-X\left(\frac{j}{m}\right)\right]^-\right\}^{\frac{\alpha}{2(\alpha-1)}} \leq 1 \qquad \text{a.s. .}$$

CHAPTER 10

DOMAINS OF ATTRACTION

In the preceding chapters we considered stable processes or partial sums of i.i.d. stable random variables. Occasionally we have already noted that the theorems also hold for more general processes or for partial sums of random variables in the domain of attraction. Here we consider some of these generalizations in more detail. In section 10.1 we give some results for the case $\alpha = 2$ (normal distribution and the Wiener process). In section 10.2 we study the case $0 < \alpha < 2$ and $|\beta| \leq 1$.

10.1. THE CASE $\alpha = 2$

Let $X_1, X_2, \ldots$ be i.i.d. random variables with $X_i \in \mathcal{D}_N(2,0)$. Suppose $EX_i = 0$ and $\sigma^2(X_i) = 1$. The next theorem shows that we can embed these i.i.d. random variables in a Wiener process $\{W(t) : 0 \leq t < \infty\}$ on some probability triple (Ω, F, P). The proofs of the theorems in this section can be found in FREEDMAN (1971).

THEOREM 10.1.1. *There exist non-negative random variables* $T_1 \leq T_2 \ldots$ *on* (Ω, F, P) *such that*

a. T_1, T_2-T_1, $T_3-T_2, \ldots$ *are i.i.d. random variables*
b. $ET_1 = EX_1^2$
c. $W(T_1)$, $W(T_2)-W(T_1)$, $W(T_3)-W(T_2), \ldots$ *are i.i.d. random variables*
d. $W(T_1) \stackrel{d}{=} X_1$.

The representation of random variables $\in \mathcal{D}_N(2,0)$, given in theorem 10.1.1 is called the *Skorohod representation*. The Skorohod representation permits us to generalize theorems for a Wiener process or partial sums of independent r.v.'s with distribution function $F(.;2,0)$ to theorems for partial sums of random variables in $\mathcal{D}_N(2,0)$. We formulate, for example, the *strong invariance principle* proved by STRASSEN (1964). Define, for each integer $n \geq 3$ and all ω, the function $f_n(.,\omega) \in C[0,1]$ by

$$(10.1.1) \qquad f_n(i/n,\omega) = \begin{cases} (2n \log\log n)^{-\frac{1}{2}}(X_1(\omega)+\ldots+X_i(\omega)) & \text{for } i=0,\ldots,n; \\ \\ \text{and linear on } [i/n,(i+1)/n] & \text{for } i=0,\ldots,n-1. \end{cases}$$

THEOREM 10.1.2. *Let* $X_1,X_2,\ldots$ *be i.i.d. random variables with* $EX_1 = 0$ *and* $\sigma^2(X_1) = 1$ *and let* $\{f_n\}$ *be defined by (10.1.1). For almost all* ω, *the indexed subset* $\{f_n(.,\omega) : n \geq 3\}$ *of* $C[0,1]$ *is relatively compact, with limit set* K, *which is given in definition 9.0.1.*

As a consequence we have the law of the iterated logarithm of HARTMAN and WINTNER (1941).

THEOREM 10.1.3. *Let* $X_1,X_2,\ldots$ *be i.i.d. random variables with* $EX_1 = 0$ *and* $\sigma^2(X_1) = 1$. *Then*

$$\limsup_{n\to\infty} \frac{X_1+\ldots+X_n}{(2n \log\log n)^{\frac{1}{2}}} = 1 \qquad \text{a.s.}\ .$$

In case X_i, $i=1,2,\ldots$, are i.i.d. r.v.'s with $X_i \in \mathcal{D}(2,0)$ and $\sigma^2(X_i) = \infty$ we have by theorem 2.2.2 that

$$H(x) = \int_{|y|\leq x} y^2 d\,F(y)$$

is slowly varying at infinity. Let a_n be defined by (2.2.1) then $a_n^{-1}(X_1+\ldots$ $\ldots+X_n)$ converges weakly to a standard normal r.v.. FELLER (1968) has studied the question under which restrictions on H

$$\limsup_{n\to\infty} \frac{X_1+\ldots+X_n}{a_n(2\log\log n)^{\frac{1}{2}}} = 1 \qquad \text{a.s.}\ .$$

10.2. THE CASE $\alpha \neq 2$

In this section we consider i.i.d. random variables X_i, $i=1,2,\ldots$, in the domain of attraction of stable distributions. The definitions, criteria for attraction and norming constants are given in section 2.2.

We begin with a quite general result of FELLER (1946) that has important implications for the problem at hand. The proof rests on Kronecker's lemma and three-series criterion.

THEOREM 10.2.1. *Let* $Y_1, Y_2, \dots$ *be i.i.d. random variables with* $E|Y_i| = \infty$. *Then, for any sequence* y_n*, for which* $y_n n^{-1}$ *increases, we have*

$$P[|Y_1+\dots+Y_n| > y_n \text{ i.o.}] = 0 \text{ or } 1$$

according as

$$\sum P[|Y_n| > y_n] \text{ converges or diverges.}$$

Obviously theorem 10.2.1 implies that

$$P[|Y_1+\dots+Y_n| > y_n \text{ i.o.}] = P[|Y_n| > y_n \text{ i.o.}].$$

Suppose that X_i, $i=1,2,\dots$, are positive i.i.d. random variables with

$$(10.2.1) \qquad P[X_1 \geq x] = L(x)x^{-\alpha} \qquad \text{for } x \geq x_0 > 0 \text{ and } 0 < \alpha < 1,$$

where L is slowly varying at infinity. This implies $X_1 \in \mathcal{D}(\alpha,1)$. Application of theorem 10.2.1 yields

$$P[X_1+\dots+X_n > y_n \text{ i.o.}] = 0 \text{ or } 1$$

according as

$$\sum L(y_n)y_n^{-\alpha} \text{ converges or diverges,}$$

provided $y_n n^{-1}$ increases. A similar result could be obtained by generalizing theorem 8.1.1 but then one would have to impose restrictions on L.

Next we consider analogues of the results in section 6.2. As a consequence of theorem 6.2.1 we obtained

$$(6.2.2) \qquad \liminf_{n\to\infty} \frac{X_1+\dots+X_n}{n^{1/\alpha}(2\log\log n)^{-(1-\alpha)/\alpha}} = \{2B(\alpha)\}^{(1-\alpha)/\alpha} \qquad \text{a.s.}$$

for i.i.d. random variables X_i, $i=1,2,\dots$, with a completely asymmetric stable distribution with $0 < \alpha < 1$ and $\beta = 1$. Now we consider random variables with distribution function given by (10.2.1). Define the norming constants a_n by (2.2.2). We can ask under what conditions on the slowly varying function L

$$(10.2.2) \qquad \liminf_{n\to\infty} \frac{X_1+\ldots+X_n}{a_n(2\log\log n)^{-(1-\alpha)/\alpha}} = \{2B(\alpha)\}^{(1-\alpha)/\alpha} \qquad \text{a.s.} .$$

An extension of theorem 6.2.1 given by LIPSCHUTZ (1956b) yields that (10.2.2) holds under the restrictions given in remark 2.2.3. In particular, for the case $X_i \in \mathcal{D}_N(\alpha,1)$, i.e. if $L(x)$ tends to a finite constant for $x \to \infty$, (10.2.2) is always true. Under a slightly weaker condition than given in remark 2.2.3 we shall prove the following theorem. Let $\varepsilon > 0$ and define the sequences b_n and c_n, for $n > 1$ by

$$(10.2.3) \qquad b_n = (\log n)^{-\frac{1+\varepsilon}{2-\alpha}}$$

and

$$(10.2.4) \qquad c_n = (\log n)^{\frac{1+\varepsilon}{\alpha}} .$$

THEOREM 10.2.2. *Let* $\varepsilon > 0$ *and let* $\{b_n\}$ *and* $\{c_n\}$ *be defined by (10.2.3) and (10.2.4). Let* $X_1, X_2, \ldots$ *be i.i.d. random variables with distribution function given by (10.2.1). Assume that*

$$(10.2.5) \qquad L(a_n x)/L(a_n) \longrightarrow 1 \qquad \text{for } n \to \infty$$

uniformly in $x \in [b_n, c_n]$. *Then (10.2.2) is true.*

PROOF. Using (10.2.1), (10.2.4) and (2.2.2) we find

$$\sum P[X_n > a_n c_n] = \sum a_n^{-\alpha} c_n^{-\alpha} L(a_n c_n) < \infty.$$

The Borel-Cantelli lemma implies that, w.p. 1, $X_n \le a_n c_n$ for sufficiently large n.

Define the truncated random variables

$$X'_n = X_n \qquad \text{if } X_n < a_n b_n$$
$$0 \qquad \text{otherwise.}$$

By properties of slowly varying functions (FELLER (1971) theorem 2 of section VIII.9) we obtain

$$E(X'_n)^2 \sim \alpha(2-\alpha)^{-1}(a_n b_n)^{2-\alpha} L(a_n b_n) \qquad \text{for } n \to \infty.$$

This yields, by (2.2.2) and (10.2.5)

$$\sum a_n^{-2}(2\log\log n)^{2(1-\alpha)/\alpha} E(X'_n)^2 < \infty.$$

It follows from theorem 3.27 in BREIMAN (1968a) that

$$\frac{X'_1+\ldots+X'_n}{a_n(2\log\log n)^{-(1-\alpha)/\alpha}} \longrightarrow 0 \qquad \text{a.s. .}$$

Thus, only the random variables $X''_n = X_n \cdot 1_{[a_n b_n, a_n c_n]}$, obtained by truncation at $a_n b_n$ and $a_n c_n$, contribute to the lim inf in (10.2.2).

Let $\Delta > 0$. There exists a number $n(\Delta)$ such that for all $n \geq n(\Delta)$ we have

$$(10.2.6) \qquad |L(a_n x)/L(a_n)-1| < \Delta$$

for all $x \in [b_n, c_n]$. This implies, by theorem 2.1.7 part I and (2.2.2), the existence of a number Δ_1 such that

$$(10.2.7) \qquad 1-F((1-\Delta_1)^{-1/\alpha}x,\alpha,1) \leq P[X_n > a_n n^{-1/\alpha}x] \leq 1-F((1+\Delta_1)^{-1/\alpha}x,\alpha,1)$$

for $x \in [n^{1/\alpha}b_n, n^{1/\alpha}c_n]$ and n sufficiently large.

Define random variables $\tilde{X}_n$ with distribution functions $\tilde{F}_n$ such that

$$(10.2.8) \qquad 1-F((1-\Delta_1)^{-1/\alpha}x,\alpha,1) \leq 1-\tilde{F}_n(x) \leq 1-F((1+\Delta_1)^{-1/\alpha}x,\alpha,1).$$

Then we can deduce upper- and lower bounds for the distribution function of $\tilde{S}_n = \tilde{X}_1+\ldots+\tilde{X}_n$. Just as only the truncated r.v.'s X''_n contribute to the lim inf in (10.2.2), we can now prove a similar assertion for the r.v.'s $\tilde{X}''_n = \tilde{X}_n \cdot 1_{[n^{1/\alpha}b_n, n^{1/\alpha}c_n]}$. Therefore, we may restrict our attention to the random variables $\tilde{X}_n$. Because we can take Δ (and Δ_1) arbitrarily small we can show that (10.2.2) is true for the random variables $\tilde{X}_n$. Therefore, (10.2.2) holds for i.i.d. r.v.'s X_i, i=1,2,..., with distribution function given by (10.2.1) and satisfying (10.2.5). This completes the proof. □

REMARK 10.2.1. The assumption (10.2.5) is comparable with (2.2.9). Comparing the interval $[b_n, c_n]$ with the interval in (2.2.10), we see that $[b_n, c_n]$ is shorter.

REMARK 10.2.2. The above results show that we may consider the random variables $\tilde{X}_n$ for solving our problem. The property (10.2.8) of their distribution functions suggests that $\tilde{X}_n$ can be embedded in a completely asymmetric stable process, by using a stopping time which degenerates at the value 1, for n tending to infinity. Such an embedding technique might also enable us to work under a weaker condition than (10.2.5) which does not imply that the stopping times degenerate. So far, however, I have not been able to prove the existence of such stopping times even under condition (10.2.5).

REMARK 10.2.3. In view of the techniques applied in this section, it must be possible, by similar reasoning, to prove results in case $\alpha \geq 1$.

APPENDIX 1

TOPOLOGIES ON D[0,1]

In section 3.2 we defined D[0,1] as the set of all real-valued functions on [0,1], which are right-continuous and have finite left-hand limits. In SKOROHOD (1956) five topologies on D[0,1] are studied. We shall define two of these below.

Let Λ denote the class of strictly increasing, continuous mappings of [0,1] onto itself. If $\lambda \in \Lambda$, then $\lambda(0) = 0$ and $\lambda(1) = 1$. For $x,y \in D[0,1]$ we define the metric

$$d_{J_1}(x,y) = \inf_{\lambda\in\Lambda}\{\sup_t |x(t)-y(\lambda(t))| + \sup_t |t-\lambda(t)|\}.$$

This metric defines the J_1-*topology*. The sequence $x_n \in D[0,1]$ is J_1-convergent (or converges in the J_1-topology) to a function $x \in D[0,1]$ if

$$\lim_{n\to\infty} d_{J_1}(x_n,x) = 0.$$

The *graph* Γ_x of $x \in D[0,1]$ is the closed set of pairs (t,z), such that z lying between $x(t-)$ and $x(t)$. A *parametric representation* of the graph Γ_x is a pair of functions (τ,ζ) such that

$$\tau : [0,1] \to [0,1]$$

is continuous and non-decreasing,

$$\zeta : [0,1] \to \mathbb{R}$$

is continuous, and such that $(t,z) \in \Gamma_x$ iff a number $u \in [0,1]$ can be found with $t = \tau(u)$ and $z = \zeta(u)$. Note that if (τ_1,ζ_1) and (τ_2,ζ_2) are parametric representations of Γ_x, there exists a non-decreasing function λ such that $\tau_1 = \tau_2\circ\lambda$ and $\zeta_1 = \zeta_2\circ\lambda$. Define a metric R in $\mathbb{R}^2$ by

$$R((t_1,z_1),(t_2,z_2)) = |t_1-t_2| + |z_1-z_2|.$$

Let $x,y \in D[0,1]$ and let (τ_x,ζ_x) and (τ_y,ζ_y) be parametric representations of their graphs. We define

$$d_{M_1}(x,y) = \inf \sup_u \; R((\tau_x(u),\tau_y(u)),(\zeta_x(u),\zeta_y(u))),$$

where the infimum is taken over all parametric representations of Γ_x and Γ_y. This metric defines the M_1-*topology*.

Convergence in the J_1-topology implies convergence in the M_1-topology. The converse is not true. For the proof of this assertion and necessary and sufficient conditions for convergence in both topologies we refer to SKOROHOD (1956).

APPENDIX 2

In this appendix we shall give the proofs of the lemmas 9.2.1, 9.3.1 and 9.4.1. Throughout h_i denotes the density of the chi-square distribution with i degrees of freedom.

PROOF of lemma 9.2.1. It is sufficient to prove the lemma only for sufficiently large r. The intervals $(jn_{r+1}m^{-1},(j+1)n_r m^{-1})$, $j=0,\ldots,m-1$, are disjoint. This implies that the random variables $A_{j,r}$, defined in (9.2.6), are independent.

$A_{j,r}$ has the same distribution as $b_{j,r}\cdot 2B(\alpha)[X(1)]^{-\alpha/(1-\alpha)}$, where

$$b_{j,r} = \{(j+1)n_r n_{r+1}^{-1}-j\}^{-1/(1-\alpha)}.$$

By theorem 2.1.6 part IV we have the following expansion for the right tail of the density $f_{j,r}$ of $b_{j,r}^{-1}A_{j,r}$

$$f_{j,r}(x) = (4\pi\alpha x)^{-\frac{1}{2}}e^{-x/2}\{1 + O(x^{-\frac{1}{2}+\varepsilon})\} \qquad \text{for } x \to \infty.$$

Note that the density $f_{j,r}$ is independent of j and r. If $0 < \alpha \le \frac{1}{2}$ it follows, from theorem 2.1.6 part I, that there exists a constant $c = c(\alpha)$ such that $f_{j,r} \le ch_1$. On the other hand, if $\frac{1}{2} < \alpha < 1$ there is, for every $x_0 > 0$, a constant $c = c(\alpha,x_0)$ such that $f_{j,r}(x) \le ch_1(x)$ for $x \ge x_0$.

Choose $0 < \delta < \varepsilon$. For sufficiently large r we have

$$1 < b_{j,r} < 1+\delta \qquad \text{for } j=0,\ldots,m-1.$$

Then we have in case $\alpha \in (0,\frac{1}{2}]$

$$\begin{aligned} P[A_{0,r}+\ldots+A_{m-1,r} > 2(1+\varepsilon)^2\log\log n_r] &\le \\ &\le P[b_{0,r}^{-1}A_{0,r}+\ldots+b_{m-1,r}^{-1}A_{m-1,r} > (1+\delta)^{-1}(1+\varepsilon)^2\, 2\log\log n_r] \\ &\le c^m\, P[\chi^2_m \ge (1+\delta)^{-1}(1+\varepsilon)^2\, 2\log\log n_r] \\ &\le kr^{-1-\varepsilon}. \end{aligned}$$

For the case $\alpha \in (\frac{1}{2},1)$ we only give the proof for $m = 2$. For $m > 2$ the proof is similar. Consider

$$I(x) = h_2^{-1}(x) \int_0^x f_{0,r}(t)f_{1,r}(x-t)dt$$

for $x > 2x_0$. Then

$$I(x) = h_2^{-1}(x) \{\int_0^{x_0} + \int_{x_0}^{x-x_0} + \int_{x-x_0}^{x}\}$$

$$\leq c\, h_2^{-1}(x) \int_0^{x_0} f_{0,r}(t)h_1(x-t)dt + c^2 + c\, h_2^{-1}(x) \int_{x-x_0}^{x} h_1(t)f_{1,r}(x-t)dt.$$

The first and last terms on the right are $O((x-x_0)^{-\frac{1}{2}})$ for $x \to \infty$. This implies the existence of a constant c_2 such that

$$I(x) \leq c_2 \qquad\qquad \text{for } x \geq 2x_0.$$

Consequently the density of $b_{0,r}^{-1}A_{0,r} + b_{1,r}^{-1}A_{1,r}$ is bounded by $c_2h_2(x)$ for $x \geq 2x_0$ and the lemma follows. □

PROOF of lemma 9.3.1. Using a similar argument as in the proof of lemma 9.2.1 we can prove the following assertion. For all $x_0,C > 0$ there exist a number $r_0 = r_0(x_0)$ and a constant $k_0 = k_0(x_0,r_0,C)$ (both independent of j,r and x) such that the density of the random variable

$$4(\pi e)^{-1} \exp(-(\pi/2)A_{j,r}(1-\varepsilon_{j,r}))$$

can be bounded by $k_0h_1(x)$ for all $x \in [x_0,C \log r]$ and $r \geq r_0$.

Choose a constant C such that $C > 2m(1+\varepsilon/2)$. For fixed r the random variables $A_{0,r},\dots,A_{m-1,r}$, defined in (9.3.4), are independent. Denote the density of $\bar{A}_r$ by $\bar{g}_r$. In a similar way as in the proof of lemma 9.2.1 we can show $\bar{g}_r(x) < kh_m(x)$ for $x \in [mx_0,C \log r]$ and some constant k (independent of r). Thus, for sufficiently large r,

$$r^{1+\varepsilon/2}\, P[\bar{A}_r > 2(1+\varepsilon)\log\log n_r]$$

$$= r^{1+\varepsilon/2}\, \{P[\bar{A}_r \geq C \log r] + P[2(1+\varepsilon)\log\log n_r < \bar{A}_r < C \log r]\}$$

$$\leq r^{1+\varepsilon/2}\, P[\bar{A}_r \geq C \log r] + kr^{1+\varepsilon/2}\, P[\chi_m^2 > 2(1+\varepsilon)\log\log n_r].$$

Just as in the proof of lemma 9.2.1 the last term on the right is bounded. Because $\bar{A}_r$ is a sum of i.i.d. random variables we have

$$P[\bar{A}_r \geq C \log r] \leq m\, P[4(\pi e)^{-1}\exp(-(\pi/2)X(1)(1-\varepsilon_{j,r})) \geq Cm^{-1}\log r].$$

By theorem 2.1.7 parts V and VII it follows that

$$r^{1+\varepsilon/2}\, P[\bar{A}_r \geq C \log r] = o(1) \qquad \text{for } r \to \infty.$$

Now apply the Borel-Cantelli lemma. □

PROOF of lemma 9.4.1. Let $f_{j,r}$ be the density of $A_{j,r}$. From theorem 2.1.6 part VI it follows that

$$\lim_{x\to\infty} h_1^{-1}(x) f_{j,r}(x) = (\alpha/2)^{\frac{1}{2}}.$$

By the continuity of $f_{j,r}$ we have for all $x_0 > 0$ that there exist numbers $r_0 = r_0(x_0)$ and $k_0 = k_0(r_0, x_0)$ such that this density is bounded by $k_0 h_1(x)$ for all $x \geq x_0$. Then we prove as in case $0 < \alpha < 1$, for sufficiently large r,

$$P[\sum_{j=0}^{m-1} A_{j,r}(1-\varepsilon_{j,r}) > 2(1+\varepsilon)\log\log n_r]$$

$$\leq P[\sum_{j=0}^{m-1} A_{j,r} > 2(1+\varepsilon/2)\log\log n_r] \leq kr^{-1-\varepsilon/4}.$$

Now the Borel-Cantelli lemma yields the desired result. □

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AUTHOR INDEX

SUMMARY

$\alpha = 2$	$0 < \alpha < 1$	
$U_1 + \ldots + U_n \overset{d}{=} n^{\frac{1}{2}} U_1$	$X_1 + \ldots + X_n \overset{d}{=} n^{1/\alpha} X_1$	
	From	
$P[U \geq x] \sim (2\pi)^{-\frac{1}{2}} x^{-1} e^{-\frac{1}{2}x^2}$ for $x \to \infty$.	$P[X \leq x] \sim (2/\alpha)^{\frac{1}{2}}\, P[U \geq (2B(\alpha))^{\frac{1}{2}} x^{-\frac{\alpha}{2(\alpha-1)}}]$ for $x \downarrow 0$.	$P[X \leq -x$
$\liminf_{t \downarrow 0} \frac{W(t)}{(2t \log\log(t^{-1}))^{\frac{1}{2}}} = -1$ a.s.	$\liminf_{t \downarrow 0} \frac{X(t)}{t^{1/\alpha}(2\log\log(t^{-1}))^{-(1-\alpha)/\alpha}} = (2B(\alpha))^{\frac{1-\alpha}{\alpha}}$ a.s.	$\liminf_{t \downarrow 0}$ $\liminf_{t \downarrow 0}$
$\liminf_{t \to \infty} \frac{W(t)}{(2t \log\log t)^{\frac{1}{2}}} = -1$ a.s.	$\liminf_{t \to \infty} \frac{X(t)}{t^{1/\alpha}(2\log\log t)^{-(1-\alpha)/\alpha}} = (2B(\alpha))^{\frac{1-\alpha}{\alpha}}$ a.s.	$\liminf_{t \to \infty}$ $\liminf_{t \to \infty}$
$\liminf_{n \to \infty} \frac{U_1 + \ldots + U_n}{(2n \log\log n)^{\frac{1}{2}}} = -1$ a.s.	$\liminf_{n \to \infty} \frac{X_1 + \ldots + X_n}{n^{1/\alpha}(2\log\log n)^{-(1-\alpha)/\alpha}} = (2B(\alpha))^{\frac{1-\alpha}{\alpha}}$ a.s.	$\liminf_{n \to \infty}$ $\liminf_{n \to \infty}$
$\limsup_{\varepsilon \downarrow 0} \sup_{\substack{0 \leq t \leq 1-\Delta \\ 0 < \Delta \leq \varepsilon}} \frac{\lvert W(t+\Delta) - W(t) \rvert}{(2\Delta \log(\Delta^{-1}))^{\frac{1}{2}}} = 1$ a.s.	$\liminf_{\varepsilon \downarrow 0} \inf_{\substack{0 \leq t \leq 1-\Delta \\ 0 < \Delta \leq \varepsilon}} \frac{X(t+\Delta) - X(t)}{\Delta^{1/\alpha}(2\log(\Delta^{-1}))^{-(1-\alpha)/\alpha}} = (2B(\alpha))^{\frac{1-\alpha}{\alpha}}$ a.s.	lim in $\varepsilon \downarrow 0$ $0 \leq t$ $0 < \Delta$ lim in $\varepsilon \downarrow 0$ $0 \leq t$ $0 < \Delta$
limit points of $(2n \log\log n)^{-\frac{1}{2}} W(nt)$ in $K = \{x : x(0) = 0,\ x$ absolutely continuous $\int_0^1 (\dot{x}(t))^2 dt \leq 1\}$	limit points of $(2\log\log n)^{(1-\alpha)/\alpha} (2B(\alpha))^{-(1-\alpha)/\alpha} n^{-1/\alpha} X(nt)$ in $K(\alpha) = \{x : x(0) = 0,\ x$ strictly increasing on J_x $\int_{J_x} [\dot{x}(t)]^{-\alpha/(1-\alpha)} dt \leq 1\}$	

)F RESULTS

$\alpha = 1$	$1 < \alpha < 2$
$X_1+\ldots+X_n-(2/\pi)\beta n\log n \overset{d}{=} nX_1$	$X_1+\ldots+X_n \overset{d}{=} n^{1/\alpha}X_1$
here on $\beta = 1$	
$\cdot\ 2^{\frac{1}{2}}\ P[U \geq 2(\pi e)^{-\frac{1}{2}}e^{\pi x/4}]$ for $x \to \infty$.	$P[X \leq -x] \sim (2\alpha)^{\frac{1}{2}}\ P[U \geq (2B(\alpha))^{\frac{1}{2}}x^{\frac{\alpha}{2(\alpha-1)}}]$ for $x \to \infty$.
$\frac{t)-(2/\pi)t\log t}{t} + \frac{2}{\pi}\log(\pi e \log\log(t^{-1}))\Big\} = \frac{2}{\pi}\log 2$ a.s. $\frac{X(t)}{\pi)t\log t} = 1$ a.s.	$\liminf\limits_{t\downarrow 0} \frac{X(t)}{t^{1/\alpha}(2\log\log(t^{-1}))^{(\alpha-1)/\alpha}} = -(2B(\alpha))^{-\frac{\alpha-1}{\alpha}}$ a.s.
$\frac{t)-(2/\pi)t\log t}{t} + \frac{2}{\pi}\log(\pi e \log\log t)\Big\} = \frac{2}{\pi}\log 2$ a.s. $\frac{X(t)}{\pi)t\log t} = 1$ a.s.	$\liminf\limits_{t\to\infty} \frac{X(t)}{t^{1/\alpha}(2\log\log t)^{(\alpha-1)/\alpha}} = -(2B(\alpha))^{-\frac{\alpha-1}{\alpha}}$ a.s.
$\frac{+\ldots+X_n-(2/\pi)n\log n}{n} + \frac{2}{\pi}\log(\pi e\log\log n)\Big\} = \frac{2}{\pi}\log 2$ a.s. $\frac{+\ldots+X_n}{\pi)n\log n} = 1$ a.s.	$\liminf\limits_{n\to\infty} \frac{X_1+\ldots+X_n}{n^{1/\alpha}(2\log\log n)^{(\alpha-1)/\alpha}} = -(2B(\alpha))^{-\frac{\alpha-1}{\alpha}}$ a.s.
$\Big\{\frac{X(t+\Delta)-X(t)-(2/\pi)\Delta\log\Delta}{\Delta} + \frac{2}{\pi}\log(\pi e\log(\Delta^{-1}))\Big\} = \frac{2}{\pi}\log 2$ a.s. $\frac{X(t+\Delta)-X(t)-(2/\pi)\Delta\log\Delta}{(2/\pi)\Delta\log\log(\Delta^{-1})} = -1$ a.s.	$\liminf\limits_{\substack{\varepsilon\downarrow 0}}\ \inf\limits_{\substack{0\leq t\leq 1-\Delta \\ 0<\Delta\leq\varepsilon}} \frac{X(t+\Delta)-X(t)}{\Delta^{1/\alpha}(2\log(\Delta^{-1}))^{(\alpha-1)/\alpha}} = -(2B(\alpha))^{-\frac{\alpha-1}{\alpha}}$ a.s.
ıit points of $\{X(nt)-(2/\pi)nt\log n\} + (2/\pi)t(\log\log n)$ in $) = \{x : x(0) = 0,\ x = x_s+x_a,\ x_s$ non-decreasing $4(\pi e)^{-1}\int_{J_x}\exp(-\pi\dot{x}(t)/2)dt \leq 1\}$	non-increasing limit points of $(2\log\log n)^{-(\alpha-1)/\alpha}(2B(\alpha))^{(\alpha-1)/\alpha}n^{-1/\alpha}X(nt)$ in $K(\alpha) = \{x : x(0) = 0,$ x absolutely continuous $\int_0^1[-\dot{x}(t)]^{\alpha/(\alpha-1)}dt \leq 1\}$.