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Part I. The simply transitive case

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Mathematical Morphology on Homogeneous Spaces

Part I. The simply transitive case

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Mathematical morphology as originally developed by Matheron and Serra is a theory of image transformations invariant under the group of Euclidean translations. Since this framework turns out to be too restricted for many practical applications, various generalizations have recently been proposed. First the translation group may be replaced by an arbitrary commutative group. Secondly, one may consider more general object spaces, such as the set of all convex subsets of the plane or the set of grey-level functions on the plane, requiring a formulation in terms of complete lattices. So far symmetry properties have been incorporated by assuming that the allowed image transformations are invariant under a certain commutative group of automorphisms on the lattice. In this paper we embark upon another generalization of mathematical morphology by dropping the assumption that the invariance group is commutative. To this end we consider an arbitrary homogeneous space, i.e. a set on which a transitive but not necessarily commutative group of invertible transformations is defined. As our object space we then take the Boolean algebra of all subsets of this homogeneous space. In Part I we consider the case that the transformation group is simply transitive, or equivalently, that the basic set is itself a group. The general transitive case is considered in Part II. For clarity of exposition as well as to emphasize the connection with classical Euclidean morphology, we have restricted ourselves to the case of Boolean lattices, which is appropriate for binary image transformations.

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1. Introduction.

Mathematical morphology was originally developed at the Paris School of Mines as a set-theoretical approach to image analysis [12,19]. It has a strong algebraic component, studying image transformations with a simple geometrical interpretation and their decomposition and synthesis in terms of set-theoretical operations. Other aspects are the probabilistic one, modeling (images of) samples of materials by random sets, and the integral geometric one which is concerned with image functionals. Although the main object of our present study is the algebraic approach we emphasize that our main motivation comes from the geometrical side, in the sense that various image transformations used in mathematical morphology today (dilations, erosions, openings, closings) have a straightforward geometrical analogue in a more general context. It is then a natural question to ask whether a corresponding algebraic description can be found. From a practical point of view the importance of such an algebraic decomposition theory no doubt derives from the fact that it enables fast and efficient implementations on digital computers and special image analysis hardware. Since we will not deal with such questions here we refer the reader to [6] for an elementary introduction to Euclidean morphology with emphasis on implementation.

In the original approach of Matheron and Serra [12,19] a two-dimensional image of, let us say, a planar section of a porous material is modeled as a subset X of the plane. In order to reveal the structure of the material, the image is probed by translating small subsets B, called *structuring elements*, of various forms and sizes over the image plane and recording the locations h where certain relations (e.g. 'B_h included in X', 'B_h hits X', etc.) between the image X and the translate B_h of the structuring element B over the vector h are satisfied (see Fig.1a). In this way one can construct a large class of image transformations which are compatible with translations of the image plane, or to put it differently, are invariant under the Euclidean translation group. The underlying idea here is that the form or shape of objects in the image does not depend on the relative location with respect to an arbitrary origin and that therefore the transformations performed on the image should respect this. Notice that the basic object of study, the 'object space', is not the reference space (the plane in our example) itself, but the collection of subsets of this reference space, and the transformations defined on this collection of subsets.

Now in practice one encounters various situations where this framework is too restrictive. One of the earliest examples is mentioned in Serra's book [19, p.17], where a photograph is shown of the trees in a forest, taken by putting the camera at ground level and aiming towards the sky. Such photographs are used to measure the amount of sunshine in the woods. The resulting image shows a clear radial symmetry with an intrinsic origin (the projection point of the zenith). It is clear that in this case we need image transformations which are adapted to the symmetries of this polar structure. It turns out that in fact one obtains a straightforward generalization of Euclidean morphology by replacing the Euclidean translations by an arbitrary *abelian* (*commutative*) group [9,15]. In the case of the example mentioned above, this would be the group generated by rotations and multiplications with respect to the origin. Here the size of the structuring element increases with increasing distance from the origin (Fig.1b). Another example occurs in the analysis of traffic scenes, where the goal is to recognize the shape of automobiles with a camera on a bridge overlooking a highway [3]. In this case the size of the structuring element has to be adapted according to the law of perspective (Fig.1c). It is not difficult to show that in this case there is again invariance under a commutative group. Notice that in the two examples just mentioned we have a variable structuring element as a function of position. This has been taken as the starting point by Serra and others to introduce arbitrary assignments of subsets to each point of the plane and define dilations and erosions accordingly, completely giving up invariance under a symmetry group. However, in the examples just given the situation is different in the sense that there is a definite group connecting structuring elements at different locations, although their sizes differ. Actually, a metrical concept like 'size' does not enter at all into the definition of the classical morphological operations. Only the group property of the Euclidean vector addition is involved, which explains why an extension to arbitrary groups is possible. In fact we will argue in Part II that without a concept of invariance (under a group, or otherwise), one cannot even give a meaningful answer to the question when sets at different locations are 'of the same shape' or not.

Instead of changing the symmetry group of the object space one may generalize the object space itself. For example, instead of all subsets of the plane one may want to study a smaller collection, such as the open or closed sets or the convex sets. In that case the original approach is no longer valid since the union of an arbitrary collection of closed or convex sets is not necessarily closed or convex, the intersection of an arbitrary collection of open sets not necessarily open, etc. These difficulties can be overcome by taking as the object space a so-called *complete lattice*, i.e. an ordered set $\mathcal{L}$ such that any subset of $\mathcal{L}$ has a supremum (smallest upper bound) and infimum (greatest lower bound), generalizing the set operations of union and intersection. This is the approach initiated by Serra and Matheron [20,21], as well as Heijmans [9]. A general study of this topic has recently been made by Heijmans and Ronse, see [10,17]. If one does not assume any invariance property one can only prove generalities. But again invariance under a group of automorphisms of the lattice may be introduced, as in [9,10,17], where so far the assumption made is always that the group is *commutative*. This enables a complete characterization of dilations, erosions, openings, closings, increasing transformations, etc. Another situation where a lattice formulation is in order, arises when one wants to go from binary images with their Boolean image algebra to grey-level images, i.e. *functions*, defined on the basic reference space. Following Sternberg [22] one has introduced the so-called *umbras* to deal with this case [19,21,23]. After introducing an extra dimension for the function values one performs the binary Euclidean operations in this enlarged space and translates the results back to the original space. However, for a mathematically satisfactory approach complete lattices are required, see Ronse [16].

In this paper we want to generalize morphology by dropping the assumption that the invariance group is commutative. To this end we consider an arbitrary *homogeneous space*, i.e. a set $\mathcal{X}$ on which a transitive but not necessarily commutative group Γ of invertible transformations is defined. Here *transitive* means that for any pair of points in the set there is a transformation in the group mapping one point on the other. If this mapping is unique we say that the transformation group is *simply transitive* or *regular*. As the object space of interest from a morphological point of view we take here the Boolean algebra of all subsets of this homogeneous space. We present two examples for basic motivation. First of all one may extend Euclidean morphology in the plane by including rotations. In many situations one does not want to distinguish between rotated versions of the same object. In that case it is appropriate to use the full Euclidean group of motions (the group generated by translations and rotations) as (non-commutative) invariance group. This is for example the basic assumption made in integral geometry to give a complete characterization (Hadwiger's Theorem) of functionals of compact, convex sets in $\mathbb{R}^n$ [8]. As our second basic example we mention the sphere with its symmetry group of three-dimensional rotations, again a non-abelian group. Various motivations can be given here. First of all the earth is spherical to a good approximation and this has to be taken into account when analyzing pictures taken by weather satellites. Secondly, pictures of virus particles show them to be nearly spherical with antibodies attached randomly to the surface, and a morphological description of the particle distribution on the surface is of interest. Thirdly, from a theoretical point of view we observe that integral geometry and geometric probability on the sphere have been well investigated in the past [13,18]. Since there is a clear connection between these fields and mathematical morphology (see Serra [19], Chapters 4, 13), it is of interest to develop morphology for the sphere as well. Here we can do no more than indicate how the sphere fits into our general framework, but clearly this case is important enough to warrant an in-depth study per se. Another area of possible research is the question of how to take the projective geometry of the imaging process into account, since clearly the symmetry of a two-dimensional plane is not the same as the symmetry of the three-dimensional world of which it is a projection. For a nice exposition of symmetry groups in nature, see the book by Weyl [25].

In this paper we develop the theory for simply transitive groups (all abelian transformation groups fall in this category, see Part II). It is easy to see that in this case there is a one-to-one correspondence between elements of $\mathcal{X}$ and those of Γ: let ω (the 'origin') be an arbitrary point in $\mathcal{X}$, and associate to any $x \in \mathcal{X}$ the unique transformation in Γ mapping ω to x. So in the simply transitive case we can assume without loss of generality that $\mathcal{X}$ coincides with the group Γ. This will be taken as the starting point in this paper, where we study the Boolean algebra $\mathcal{P}(\Gamma)$ of subsets of an arbitrary group Γ. Of course this is precisely the situation in Euclidean morphology, where the group is that of the Euclidean translations. In Part II we will consider the general transitive case, enabling us to analyze the examples mentioned above (the Euclidean plane with the translation-rotation group, the sphere with the rotation group) as particular cases. It turns out that the general case can be handled by embedding the object space of interest (the set $\mathcal{P}(\mathcal{X})$ of subsets of $\mathcal{X}$) into another one (the set $\mathcal{P}(\Gamma)$ of subsets of Γ), which has a simply transitive transformation group. So the results for the latter case, although rather technical, have to be developed first in depth. In Part II we will then be able to tackle the geometrically more interesting case and also illustrate the theory by various examples. The possibility of an extension to complete lattices will be considered in future work.

Now some remarks about the organization of this paper. In section 2 we first review Euclidean morphology and present a number of lattice-theoretical concepts. In section 3 we then generalize Euclidean morphology to the Boolean lattice of all subsets of an arbitrary group, ordered by set-inclusion. In particular we generalize the classical Minkowski set operations, as well as dilations, erosions, openings and closings which are 'translation-invariant' in a generalized sense, i.e. invariant under certain automorphisms induced by the transformation group. We point out the connection to the theory of residuated lattices and ordered semigroups [4,5]. A complete characterization of these operations is given in section 4, where we also prove a general representation theorem for translation-invariant mappings, generalizing earlier results of Matheron [12] and Banon and Barrera [1].

2. Review of morphological concepts.

In this section we first outline some elementary concepts and results from classical Euclidean morphology (section 2.1), followed by a few general lattice-theoretical concepts which are needed below (section 2.2).

2.1. Euclidean morphology.

Let E be the Euclidean space $\mathbb{R}^n$ or the discrete grid $\mathbb{Z}^n$. By $\mathcal{P}(E)$ we denote the set of all subsets of E ordered by set-inclusion, henceforth called the 'object space'. A binary image can be represented as a subset X of E. Now E is a commutative group under vector addition: we write $x+y$ for the sum of two vectors x and y, and $-x$ for the inverse of x. Then we can define the following elementary algebraic operations:

$$\textit{Minkowski addition}: \quad X \oplus A = \{x + a : x \in X, a \in A\} = \bigcup_{a \in A} X_a \tag{2.1}$$

$$\textit{Minkowski subtraction}: \quad X \ominus A = \bigcap_{a \in A} X_{-a}, \tag{2.2}$$

where X_a is the translate of the set X along the vector a:

$$X_a = \{x + a : x \in X\}. \tag{2.3}$$

Here we have followed the original definitions of Hadwiger [8], which is also the convention in [10,17,22,23]. Matheron [12] and Serra [20] use a slightly different definition for the Minkowski subtraction, see remark 2.1 below.

We collect some standard algebraic properties of Minkowski addition and subtraction [8]. Here E is the Euclidean space, o is the origin of E, $\emptyset$ the empty set, X an arbitrary subset of E.

$$\begin{aligned}
&X \oplus \{o\} = X; \quad X \ominus \{o\} = X \\
&X \oplus \emptyset = \emptyset; \quad X \oplus E = E \\
&X \ominus \emptyset = E; \quad \emptyset \ominus X = \emptyset; \quad E \ominus X = E \\
&X \oplus A = A \oplus X \\
&(X \oplus A) \oplus B = X \oplus (A \oplus B) \\
&(X \ominus A) \ominus B = X \ominus (A \oplus B) \\
&(X \cup Y) \oplus A = (X \oplus A) \cup (Y \oplus A) \\
&(X \cap Y) \ominus A = (X \ominus A) \cap (Y \ominus A) \\
&X \oplus (A \cup B) = (X \oplus A) \cup (X \oplus B) \\
&X \ominus (A \cup B) = (X \ominus A) \cap (X \ominus B)
\end{aligned} \tag{2.4}$$

The transformations $\delta_A : X \mapsto X \oplus A$ and $\varepsilon_A : X \mapsto X \ominus A$ are called a *dilation* and *erosion* by the structuring element A, respectively. There is a simple geometrical interpretation of these operations:

$$\textit{Dilation}: \quad X \oplus A = \{h \in E : (\check{A})_h \cap X \neq \emptyset\} \tag{2.5}$$

$$\textit{Erosion}: \quad X \ominus A = \{h \in E : A_h \subseteq X\}, \tag{2.6}$$

where the *reflected* or *symmetric* set $\check{A}$ of A is defined by

$$\check{A} = \{-a : a \in A\}. \tag{2.7}$$

There exists a *duality relation* with respect to set-complementation (X' denotes the complement of the set X):

$$X \oplus A = (X' \ominus \check{A})'; \quad \delta_A(X) = (\varepsilon_{\check{A}}(X'))', \tag{2.8}$$

i.e. dilating an image by A gives the same result as eroding the background by $\check{A}$. To any mapping $\psi : \mathcal{P}(E) \to \mathcal{P}(E)$ we can associate the *dual* mapping $\psi' : \mathcal{P}(E) \to \mathcal{P}(E)$ defined by $\psi'(X) = \{\psi(X')\}'$. To avoid confusion with other forms of duality to be discussed below, we will refer to ψ' as the *Boolean dual* of ψ.

Remark 2.1. Matheron and Serra define the Minkowski subtraction of X by A as follows: $X \ominus A = \bigcap_{a \in A} X_a$. Then one has to write $X \ominus \check{A}$ in Eq.(2.6). The advantage of this definition is that the duality relation (2.8) does not involve a reflection of the structuring element. But it complicates the expression of adjunctions (see below), which is a notion persisting in lattices without complementation.

Two characteristic properties of dilation are:

(*i*) *Distributivity w.r.t. union* : $$(\bigcup_{i \in I} X_i) \oplus A = \bigcup_{i \in I}(X_i \oplus A) \tag{2.9}$$

(*ii*) *Translation invariance* : $$(X \oplus A)_h = X_h \oplus A. \tag{2.10}$$

Similar properties hold for the erosion with intersection instead of union. A consequence of the distributivity property is that dilation and erosion are *increasing* mappings, i.e. mappings such that for all $X, Y \in \mathcal{P}(E)$, $X \subseteq Y$ implies that $\psi(X) \subseteq \psi(Y)$.

Other important increasing transformations are the opening and closing by a structuring element A (the closing is defined slightly differently in [12,19]):

Opening : $$X \circ A := (X \ominus A) \oplus A = \bigcup_{h \in E} \{A_h : A_h \subseteq X\} \tag{2.11}$$

Closing : $$X \bullet A := (X \oplus A) \ominus A = \bigcap_{h \in E} \{(\check{A}')_h : ((\check{A}')_h \supseteq X\}. \tag{2.12}$$

The opening is the union of all the translates of the structuring element which are included in the set X. Opening and closing are related by Boolean duality: $(X' \circ A)' = X \bullet \check{A}$. A more general definition of dilations, erosions, openings and closings will be given in the next subsection in the framework of complete lattices.

We end this review of Euclidean morphology by presenting a theorem by Matheron [12], which gives a characterization in the Euclidean case of translation-invariant increasing mappings.

Theorem 2.2. *A mapping $\psi : \mathcal{P}(E) \to \mathcal{P}(E)$ is increasing and translation-invariant if and only if ψ can be decomposed as a union of erosions, or alternatively, as an intersection of dilations:*

$$\psi(X) = \bigcup_{A \in \mathcal{V}(\psi)} X \ominus A$$

$$= \bigcap_{A \in \mathcal{V}(\psi')} X \oplus \check{A}$$

where $\mathcal{V}(\psi) = \{A \in \mathcal{P}(E) : o \in \psi(A)\}$ is the kernel of ψ, and ψ' is the Boolean dual of ψ.

2.2. Lattice-theoretical concepts.

The object spaces of interest in mathematical morphology are not restricted to Boolean algebras. For example, if one is interested in convex subsets of the plane or grey-level images one has to introduce the notion of complete lattices. This approach has recently been initiated by Serra [20], Serra et al. [21] and Heijmans and Ronse [10,17]. Although the present generalization of mathematical morphology is confined to Boolean lattices, it is nevertheless advantageous to summarize a few lattice-theoretical concepts which will be needed below. The reader may want to skip this subsection at first reading and refer back to it later. For a full discussion, see [10,17]. A general introduction to lattice theory is Birkhoff [4].

A complete lattice $(\mathcal{L}, \leq)$ is a partially ordered set $\mathcal{L}$ with order relation $\leq$, a supremum or join operation written $\bigvee$ and an infimum or meet operation written $\bigwedge$, such that every (finite or infinite) subset of $\mathcal{L}$ has a supremum (smallest upper bound) and an infimum (greatest lower bound). In particular there exist two universal bounds, the least element written O and the greatest element I. In the case of the power lattice $\mathcal{P}(E)$ of all subsets of a set E, the order relation is set-inclusion $\subseteq$, the supremum is the union $\bigcup$ of sets, the infimum is the intersection $\bigcap$ of sets, the smallest element is the empty set $\emptyset$ and the largest element is the set E itself. If $(\mathcal{L}, \leq)$ is a complete lattice, a reverse ordering $\geq$ is defined by $X \geq Y \iff Y \leq X$. The principle of *lattice duality* states that to every statement on the lattice $(\mathcal{L}, \leq)$ corresponds a dual one on $(\mathcal{L}, \geq)$, obtained by interchanging $\leq$ and $\geq$, $\bigvee$ and $\bigwedge$, O and I. An *atom* is an element X such that for any $Y \in \mathcal{L}$, $O \leq Y \leq X$ implies that $Y = O$ or $Y = X$. A complete lattice $\mathcal{L}$ is called *atomic* if every element of $\mathcal{L}$ is the supremum of the atoms less than or equal to it. It is called *Boolean* if (i) suprema are distributive over infima and vice versa, and (ii) every element X has a unique complement X', defined by $X \vee X' = X$, $X \wedge X' = O$. The power lattice $\mathcal{P}(E)$ is an atomic complete Boolean lattice.

Since we are interested in image transformations, a main object of study is the set $\mathcal{O} := \mathcal{L}^{\mathcal{L}}$ of all maps (operators) $\psi : \mathcal{L} \to \mathcal{L}$. Operators are generally written in greek letters, with α, ϕ, δ, ε being reserved for openings, closings, dilations and erosions. The identity operator $X \mapsto X$ is written $id_{\mathcal{L}}$. The composition of two operators ψ_1 and ψ_2 is defined by $\psi_1\psi_2(X) = \psi_1(\psi_2(X)), X \in \mathcal{L}$. Instead of $\psi\psi$ we write ψ^2.

The power lattice $\mathcal{O} := \mathcal{L}^{\mathcal{L}}$ inherits the complete lattice structure of $\mathcal{L}$. The ordering, supremum and infimum in $\mathcal{O}$ are denoted by $\leq, \bigvee, \bigwedge$ as well, and for any subset $Q \subseteq \mathcal{O}$ they are defined by

$$\psi_1 \leq \psi_2 \iff \psi_1(X) \leq \psi_2(X), \quad \forall X \in \mathcal{L}, \tag{2.13}$$

$$(\bigvee Q)(X) = \bigvee_{\eta \in Q} \eta(X), \quad \forall X \in \mathcal{L}, \tag{2.14}$$

$$(\bigwedge Q)(X) = \bigwedge_{\eta \in Q} \eta(X), \quad \forall X \in \mathcal{L}. \tag{2.15}$$

In the case that $\mathcal{L}$ is itself a power lattice $\mathcal{P}(E)$ with ordering $\subseteq$, we will write $\bigcup Q$ and $\bigcap Q$ instead of $\bigvee Q$ and $\bigwedge Q$.

A mapping $\psi : \mathcal{L} \to \mathcal{L}$ is called *increasing (isotone, order-preserving)* when $X \leq Y \implies \psi(X) \leq \psi(Y)$ for all $X, Y \in \mathcal{L}$. An *automorphism* $\psi : \mathcal{L} \to \mathcal{L}$ is a bijection such that for any $X, Y \in \mathcal{L}$, $X \leq Y$ if and only if $\psi(X) \leq \psi(Y)$. Given a group $\mathbb{T}$ of automorphisms of $\mathcal{L}$, a mapping $\psi : \mathcal{L} \to \mathcal{L}$ is called T-*invariant* or a T-*mapping* if it commutes with all $\tau \in \mathrm{T}$, i.e. if $\psi(\tau(X)) = \tau(\psi(X))$ for all $X \in \mathcal{L}$, $\tau \in \mathrm{T}$ (we will refer to $\tau(X)$ as the *translation of* X *by* τ). Accordingly, we will speak below of T-dilations, T-erosions, etc. If no invariance under a group is required, one may set $\mathrm{T} = \{id_{\mathcal{L}}\}$.

Next we give a general definition of dilations and erosions, which are examples of increasing mappings.

Definition 2.3. Let $\mathcal{L}$ be a complete lattice. A dilation $\delta : \mathcal{L} \to \mathcal{L}$ is a mapping commuting with suprema. An erosion $\varepsilon : \mathcal{L} \to \mathcal{L}$ is a mapping commuting with infima. In other words, for any subset $\{X_i : i \in I\}$ of $\mathcal{L}$ it is true that

$$\delta(\bigvee_{i\in I} X_i) = \bigvee_{i\in I} \delta(X_i), \tag{2.16}$$

$$\varepsilon(\bigwedge_{i\in I} X_i) = \bigwedge_{i\in I} \varepsilon(X_i). \tag{2.17}$$

In particular, $\delta(O) = O$ and $\varepsilon(I) = I$.

The following definition generalizes the notion of Euclidean openings and closings.

Definition 2.4. A mapping $\psi : \mathcal{L} \to \mathcal{L}$ is called:
(*a*) *idempotent*, if $\psi^2 = \psi$;
(*b*) *extensive*, if for every $X \in \mathcal{L}$, $\psi(X) \geq X$;
(*c*) *anti-extensive*, if for every $X \in \mathcal{L}$, $\psi(X) \leq X$;
(*d*) a *closing*, if it is increasing, extensive and idempotent;
(*e*) an *opening*, if it is increasing, anti-extensive and idempotent;
(*f*) an *involution*, if $\psi^2 = id_{\mathcal{L}}$.

Of fundamental importance is the concept of *adjunction*.

Definition 2.5. Let $\mathcal{L}$ be a complete lattice. A pair of mappings (ε, δ) on $\mathcal{L}$ is called an *adjunction* if for every $X, Y \in \mathcal{L}$, the following equivalence holds:

$$\delta(X) \leq Y \iff X \leq \varepsilon(Y).$$

We list a number of properties of adjunctions, see [7,10,17].

Lemma 2.6. *The following holds:*

(*a*) *In an adjunction* (ε, δ), ε *is an erosion and* δ *a dilation.*
(*b*) *For every dilation* δ *there is a unique erosion* ε *such that* (ε, δ) *is an adjunction;* ε *is given by* $\varepsilon(Y) = \bigvee\{X \in \mathcal{L} : \delta(X) \leq Y\}$, *and is called the upper adjoint of* δ.
(*c*) *For every erosion* ε *there is a unique dilation* δ *such that* (ε, δ) *is an adjunction;* δ *is given by* $\delta(X) = \bigwedge\{Y \in \mathcal{L} : X \leq \varepsilon(Y)\}$, *and is called the lower adjoint of* ε.
(*d*) δ *is* T-*invariant if and only if* ε *is* T-*invariant: if so, we call* (ε, δ) *a* T-*adjunction.*
(*e*) *For any adjunction, we have* $\delta\varepsilon \leq id_{\mathcal{L}}$, $\varepsilon\delta \geq id_{\mathcal{L}}$, $\delta\varepsilon\delta = \delta$ *and* $\varepsilon\delta\varepsilon = \varepsilon$. *In particular* $\delta\varepsilon$ *is an opening and* $\varepsilon\delta$ *is a closing.*
(*f*) *Given two* T-*adjunctions* (ε, δ) *and* (ε', δ'), $(\varepsilon'\varepsilon, \delta\delta')$ *is a* T-*adjunction.*
(*g*) *If* $(\varepsilon_j, \delta_j)$ *is a* T-*adjunction for every* $j \in J$, $(\bigwedge_{j\in J} \varepsilon_j, \bigvee_{j\in J} \delta_j)$ *is a* T-*adjunction.*

Definition 2.7. Let $\varepsilon : \mathcal{L} \to \mathcal{L}$ be an erosion with adjoint dilation $\delta : \mathcal{L} \to \mathcal{L}$. A *morphological opening (closing)* is an opening (closing) of the form $\delta\varepsilon$ ($\varepsilon\delta$).

Next we recall some general properties of openings and closings. The supremum of a collection of openings is again an opening. The greatest opening on $\mathcal{L}$ is $id_{\mathcal{L}}$, where the ordering of mappings is defined by (2.13).

Definition 2.8. The *domain of invariance* of a mapping $\psi : \mathcal{L} \to \mathcal{L}$ is the set

$$\mathsf{Inv}(\psi) := \{X \in \mathcal{L} : \psi(X) = X\}.$$

Openings are completely characterized by their domain of invariance: $\alpha_1 = \alpha_2 \iff \mathsf{Inv}(\alpha_1) = \mathsf{Inv}(\alpha_2)$.

Definition 2.9. Let B be an element of $\mathcal{L}$. The *structural* $\mathbb{T}$*-opening* by the structuring element B is the mapping

$$\alpha_B^{\mathbb{T}}(X) = \bigvee\{\tau(B) : \tau \in \mathbb{T},\ \tau(B) \le X\}, \quad X \in \mathcal{L}. \tag{2.18}$$

Similarly the *structural closing* $\phi_B^{\mathbb{T}}$ by the structuring element B is defined by the formula

$$\phi_B^{\mathbb{T}}(X) = \bigwedge\{\tau(B) : \tau \in \mathbb{T},\ \tau(B) \ge X\}, \quad X \in \mathcal{L}. \tag{2.19}$$

As the name suggests, structural openings and closings are defined in terms of a single structuring element. Notice that (2.11) is a structural opening by the structuring element A and (2.12) is a structural closing by the structuring element $\check{A}'$.

An important result is the following characterization of $\mathbb{T}$-openings:

Proposition 2.10. *Let α be a $\mathbb{T}$-opening on $\mathcal{L}$. Then α is a supremum of structural $\mathbb{T}$-openings, i.e.*

$$\alpha(X) = \bigvee\{\alpha_B^{\mathbb{T}}(X) : B \in \mathcal{B}\}, \quad X \in \mathcal{L}, \tag{2.20}$$

where $\mathcal{B}$ is the domain of invariance of α. The subset $\mathcal{B}$ in this formula may be replaced by any subset $\mathcal{B}'$ which generates $\mathcal{B}$ under group translations and infinite suprema.

In the Euclidean case, a structural opening by B is also a morphological opening: $\alpha_B^{\mathbb{T}}(X) = \delta_B\varepsilon_B(X) = (X \ominus B) \oplus B$. The corresponding representation (2.20) of Euclidean openings on $\mathcal{P}(E)$ as a union of morphological openings was originally proved by Matheron [12].

3. Mathematical morphology on non-commutative groups.

As explained in the introduction, our aim in this paper is to generalize Euclidean mathematical morphology as reviewed in section 2.1 to the power lattice $\mathcal{P}(\Gamma)$, i.e. the set of subsets of Γ ordered by set-inclusion, where Γ is an arbitrary group. The classical Euclidean case corresponds to the case where Γ is the abelian group of vector additions (translations). Our first step will be to find a generalization of the Minkowski operations, the main problem being how to overcome the non-commutativity of the group Γ. Subsequently we define generalized dilations and erosions invariant under Γ, followed by a discussion of adjunctions as well as openings and closings. But first we will look at a pair of automorphism groups of $\mathcal{P}(\Gamma)$ which are essential in what follows.

3.1. Left and right translations on $\mathcal{P}(\Gamma)$.

Let Γ be an arbitrary group. To be consistent with the notation in Part II, we denote elements of Γ by g, h, k, etc., and subsets of Γ by the corresponding capitals G, H, K. The product of two group elements g and h is written gh. For a fixed $g \in \Gamma$, one can define the mappings $h \mapsto gh$ and $h \mapsto hg$ for any $h \in \Gamma$. These mappings are called *left translation by* g and *right translation by* g, respectively [2,14,24]. This definition can be trivially extended to subsets of the group as follows:

$$\textit{left translation}: \quad \lambda_g : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma), \quad \lambda_g(H) = \{gh : h \in H\}, \tag{3.1a}$$

$$\textit{right translation}: \quad \rho_g : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma), \quad \rho_g(H) = \{hg : h \in H\}. \tag{3.1b}$$

Instead of $\lambda_g(H)$ and $\rho_g(H)$ we will usually write gH and Hg. It is straightforward to check that the left and right translations on the lattice $\mathcal{P}(\Gamma)$ preserve unions, intersections and complements:

$$g(G \cup H) = (gG) \cup (gH), \; g(G \cap H) = (gG) \cap (gH), \; (gG)' = gG',$$

and similarly for right translations. So the sets $\Gamma^\lambda := \{\lambda_g : g \in \Gamma\}$ and $\Gamma^\rho := \{\rho_g : g \in \Gamma\}$ are both automorphism groups of $\mathcal{P}(\Gamma)$.

Remark 3.1. Notice that $\lambda_g \lambda_h = \lambda_{gh}$, $\rho_g \rho_h = \rho_{hg}$, so Γ^λ is isomorphic to Γ under the correspondence $g \leftrightarrow \lambda_g$, and Γ^ρ is isomorphic to Γ under the correspondence $g \leftrightarrow \rho_g^{-1}$. This is related to the concept of the *dual* Γ^* of a group Γ, which is obtained by defining a dual product $*$ in Γ by $g * h = hg$. It is easy to see that the groups Γ^λ and Γ^ρ are dual. So we only need to give proofs for invariance with respect to left translations, say. The right-invariant counterparts follow then by group duality. For easy reference we nevertheless give most results in left- and right-invariant form.

A simple yet fundamental observation is that left and right translations commute. Summarizing:

Lemma 3.2. *Let Γ be a group. Then the groups Γ^λ and Γ^ρ of left and right translations are: (i) automorphism groups of the lattice $\mathcal{P}(\Gamma)$; (ii) isomorphic to Γ. Moreover, left and right translations commute: $\lambda_g \rho_h = \rho_h \lambda_g$.*

Finally, we define left and right translation-invariant mappings.

Definition 3.3. A mapping $\psi : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$ is called *left translation-invariant* when $\lambda_g \psi = \psi \lambda_g$ for all $g \in \Gamma$. Similarly, a mapping $\psi : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$ is called *right translation-invariant* when $\rho_g \psi = \psi \rho_g$ for all $g \in \Gamma$.

For brevity we will speak of *left-invariant* or *λ-mappings* and *right-invariant* or *ρ-mappings.*

3.2. Generalization of the Minkowski operations.

Since Γ is a group we can use the group operation to define a multiplication on subsets of Γ, which leads to the generalization of the Minkowski addition.

Definition 3.4. Let G, H be subsets of the group Γ. The *product of G by H*, denoted by $G \otimes H$, is the subset of Γ defined by

$$G \otimes H = \{gh : g \in G, h \in H\}, \tag{3.2a}$$

$$G \otimes \emptyset = \emptyset \otimes G = \emptyset. \tag{3.2b}$$

It is immediate that, with e the unit element of Γ, $G \otimes \{e\} = \{e\} \otimes G = G$.

Remark 3.5. We notice in passing that $\mathcal{P}(\Gamma)$ is a *monoid* under the multiplication $\otimes$, i.e. a semigroup with unit element $\{e\}$. Since $\mathcal{P}(\Gamma)$ is a complete lattice as well, and the multiplication $\otimes$ is distributive over unions (see Proposition 3.8(a) below), we have an example here of a so-called *complete lattice-ordered monoid* or *cl-monoid*, see Birkhoff [4] or Blyth and Janowitz [5].

We can write (3.2a) in the alternative forms

$$G \otimes H = \bigcup_{g \in G} gH = \bigcup_{h \in H} Gh. \tag{3.3}$$

The similarity with the Minkowski addition (2.1) is clear. Next we generalize the Minkowski subtraction.

Definition 3.6. Let G, H be subsets of the group Γ. The *left residual* of G by H, denoted by $G \obslash H$, is the subset of Γ defined by

$$G \obslash H = \{g \in \Gamma : gH \subseteq G\}. \tag{3.4a}$$

The *right residual* of G by H, denoted by $G \oslash H$, is the subset of Γ defined by

$$G \oslash H = \{g \in \Gamma : Hg \subseteq G\}. \tag{3.4b}$$

Remark 3.7. The above definition of residuals is standard in the theory of *residuated semigroups.* The left residual of G by H is characterized by the property that it is the largest subset K of Γ such that when multiplied on the right by H it is included in G:

$$\begin{aligned} &(i) \quad (G \obslash H) \otimes H \subseteq G \\ &(ii) \quad K \otimes H \subseteq G \implies K \subseteq G \obslash H, \end{aligned}$$

with a similar statement for right residuals, see Birkhoff [4] or Blyth and Janowitz [5]. The definition 3.6 also applies if Γ is just a semigroup instead of a group. Of course the fact that we assume Γ to be a group enables us to derive more specific results. As far as notation is concerned, in residuation theory one usually writes GH, $G \mathbin{\cdot\,.} H$, $G \mathbin{.\,\cdot} H$ instead of $G \otimes H$, $G \obslash H$ and $G \oslash H$, respectively. With our choice of notation we maintain some resemblance to the symbols $\oplus$, $\ominus$ which are used in Euclidean morphology.

Using the group nature of Γ we easily derive the following equivalences:

$$gH \subseteq G \iff gh \in G,\ \forall h \in H \iff g \in Gh^{-1},\ \forall h \in H \iff g \in \bigcap_{h \in H} Gh^{-1}.$$

Hence,

$$G \obslash H = \bigcap_{h \in H} Gh^{-1}, \qquad G \oslash H = \bigcap_{h \in H} h^{-1}G\,, \tag{3.5}$$

where the result for the right residual can be shown similarly. Both formulas reduce to the Euclidean Minkowski subtraction $G \ominus H$ if the group Γ is commutative, as a glance at Eq.(2.2) makes clear. Note that

$$\{g\} \otimes G = gG,\ G \otimes \{g\} = Gg, \tag{3.6a}$$

$$G \obslash \{g\} = Gg^{-1},\ G \oslash \{g\} = g^{-1}G. \tag{3.6b}$$

Next we prove a number of algebraic properties of the set product and the residuals, generalizing the formulas (2.4). For a proof of (a)-(f) in an abstract lattice-theoretical context, see [4,5].

Proposition 3.8. *Let $G, H, K \subseteq \Gamma$ and $g, h, k \in \Gamma$. Then the following holds:*

(a) $G \otimes (H \cup K) = (G \otimes H) \cup (G \otimes K)$ — *$\cup$-distributivity*
$(G \cup H) \otimes K = (G \otimes K) \cup (H \otimes K)$

(b) $(G \otimes H) \otimes K = G \otimes (H \otimes K)$ — *associativity*

(c) $(G \cap H) \obslash K = (G \obslash K) \cap (H \obslash K)$ — *$\cap$-distributivity*
$(G \cap H) \oslash K = (G \oslash K) \cap (H \oslash K)$

(d) $G \obslash (H \cup K) = (G \obslash H) \cap (G \obslash K)$
$G \oslash (H \cup K) = (G \oslash H) \cap (G \oslash K)$

(e) $G \otimes H \subseteq K \iff G \subseteq K \obslash H \iff H \subseteq K \oslash G$

(f) $(G \obslash H) \obslash K = G \obslash (K \otimes H)$ — *iteration*
$(G \oslash H) \oslash K = G \oslash (H \otimes K)$
$(G \oslash H) \obslash K = (G \obslash K) \oslash H$

(g) $(gH) \otimes K = g(H \otimes K); \quad H \otimes (Kg) = (H \otimes K)g$ — *Γ-invariance*
$(gH) \obslash K = g(H \obslash K); \quad (Hg) \oslash K = (H \oslash K)g$

(h) $H \obslash (gK) = (H \obslash K)g^{-1}; \quad H \obslash (Kg) = (Hg^{-1}) \obslash K$
$H \oslash (gK) = (g^{-1}H) \oslash K; \quad H \oslash (Kg) = g^{-1}(H \oslash K)$

PROOF. In cases where pairs of statements occur which differ only by left-right symmetry, we prove only one of them. In all proofs we use without comment that translations commute with unions and intersections.

(a) $G \otimes (H \cup K) = \bigcup_{g \in G} g(H \cup K) = \bigcup_{g \in G}(gH \cup gK) = (\bigcup_{g \in G} gH) \bigcup (\bigcup_{g \in G} gK)$ $= (G \otimes H) \cup (G \otimes K)$, which proves the left distributivity of the set product.

(b) Using that multiplication in a group is associative, we find $(G \otimes H) \otimes K = \bigcup_{g \in G, h \in H, k \in K} (gh)k = \bigcup_{g \in G, h \in H, k \in K} g(hk) = G \otimes (H \otimes K)$.

(c) $(G \cap H) \obslash K = \bigcap_{k \in K} (G \cap H)k^{-1} = \bigcap_{k \in K} (Gk^{-1}) \cap (Hk^{-1}) =$ $(\bigcap_{k \in K} Gk^{-1}) \bigcap (\bigcap_{k \in K} Hk^{-1}) = (G \obslash K) \cap (H \obslash K)$.

(d) $G \obslash (H \cup K) = \bigcap_{m \in H \cup K} Gm^{-1} = (\bigcap_{m \in H} Gm^{-1}) \bigcap (\bigcap_{m \in K} Gm^{-1})$ $= (G \obslash H) \cap (G \obslash K)$.

(e) $G \otimes H \subseteq K \iff \forall h \in H : Gh \subseteq K \iff \forall h \in H : G \subseteq Kh^{-1} \iff$ $G \subseteq \bigcap_{h \in H} Kh^{-1} = K \obslash H$. Similarly, $G \otimes H \subseteq K \iff \forall g \in G : gH \subseteq K \iff$ $\forall g \in G : H \subseteq g^{-1}K \iff H \subseteq \bigcap_{g \in G} g^{-1}K = K \oslash G$.

(f) $(G \obslash H) \obslash K = \bigcap_{k \in K} (G \obslash H)k^{-1} = \bigcap_{k \in K} (\bigcap_{h \in H} Gh^{-1})k^{-1} = \bigcap_{h \in H, k \in K} G(kh)^{-1}$ $= \bigcap_{m \in K \otimes H} Gm^{-1} = G \obslash (K \otimes H)$. In a similar way one proves that $(G \oslash H) \oslash K =$ $G \oslash (H \otimes K)$. Finally, $(G \oslash H) \obslash K = \bigcap_{h \in H, k \in K} h^{-1}Gk^{-1} = (G \obslash K) \oslash H$.

(g) Follows from (b) and the identities (3.6).

(h) $H \obslash (gK) = \bigcap_{m \in gK} Hm^{-1} = \bigcap_{k \in K} H(gk)^{-1} = \bigcap_{k \in K} (Hk^{-1})g^{-1} =$ $(\bigcap_{k \in K} Hk^{-1})g^{-1} = (H \obslash K)g^{-1}$. The other results are proved similarly. □

As in the Euclidean case there exixts a duality by complementation. First we need some definitions.

Definition 3.9. Let G be a subset of Γ. The *reflected set* of G is the set $\check{G} = \{g^{-1} : g \in G\}$. The *complement* of G is the set $G' = \{g \in \Gamma : g \notin G\}$. The complement of the reflected set is denoted by $\hat{G} := (\check{G})'$.

Lemma 3.10. *Let G, H be subsets of Γ. Then,*

(*a*) $(G')' = (\check{G})^{\vee} = (\hat{G})^{\wedge} = G$
(*b*) $(\check{G})' = (G')^{\vee}$
(*c*) $(gG)' = gG'$; $(Gg)' = G'g$
(*d*) $(gG)^{\vee} = \check{G}g^{-1}$; $(Gg)^{\vee} = g^{-1}\check{G}$
(*e*) $(G \otimes H)^{\vee} = \check{H} \otimes \check{G}$
(*f*) $(G \cup H)' = G' \cap H'$, $(G \cup H)^{\vee} = \check{G} \cup \check{H}$,
$(G \cap H)^{\vee} = \check{G} \cap \check{H}$, $(G \cup H)^{\wedge} = \hat{G} \cap \hat{H}$
(*g*) $(G \otimes H)' = G' \odot \check{H} = H' \odot \check{G}$
(*h*) $(G \odot H)^{\vee} = \check{G} \odot \check{H}$
(*i*) $(G \otimes H)^{\wedge} = \hat{G} \odot H = \hat{H} \odot G$

PROOF. We only prove (*g*)-(*i*). The other items are obvious.
(*g*) $(G \otimes H)' = (\bigcup_{h \in H} Gh)' = \bigcap_{h \in H} G'h = G' \odot \check{H}$. Also, $(G \otimes H)' = (\bigcup_{g \in G} gH)' = \bigcap_{g \in G} gH' = H' \odot \check{G}$.
(*h*) $(G \odot H)^{\vee} = (\bigcap_{h \in H} Gh^{-1})^{\vee} = \bigcap_{h \in H} h\check{G} = \check{G} \odot \check{H}$.
(*i*) Follows from (*e*) and (*g*). □

By making use of duality by complementation one may derive pairs of equivalent results, e.g. consider Prop.3.8(*g*). Start with $(gH) \otimes K = g(H \otimes K)$. Take complements of both sides and use Lemma 3.10(*g*) to find $(gH') \odot \check{K} = g(H' \odot \check{K})$. Since K and H are arbitrary, we get $(gH) \odot K = g(H \odot K)$, which is the third item of Prop. 3.8(*g*). All this is completely analogous to the Euclidean case.

3.3. Dilations, erosions, openings and closings.

Now that we have generalized the Minkowski operations we are in a position to define various morphological transformations which are invariant under the group Γ. We start with a discussion of dilations and erosions.

Because of the non-commutativity of the set product (3.2) there are two possibilities to generalize the dilation (2.5). We may consider, for a fixed $H \in \mathcal{P}(\Gamma)$, the mapping $G \mapsto G \otimes H$, as well as the mapping $G \mapsto H \otimes G$. This leads us to the following definition.

Definition 3.11. Let $H \in \mathcal{P}(\Gamma)$. The *left dilation* δ_H^{λ} and *right dilation* δ_H^{ρ} by the structuring element H are the mappings: $\mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$ defined by

$$\delta_H^{\lambda}(G) = G \otimes H \ , \qquad \delta_H^{\rho}(G) = H \otimes G \ . \tag{3.7}$$

That these mappings are dilations (i.e. commute with arbitrary unions, see section 2.2), is readily proved by extending Prop.3.8(*a*) to distributivity with respect to infinite unions. The reason for the terminology is that left (right) dilations are left (right) translation-invariant, see Prop.3.8(*g*).

Next we show that left and right dilations can be decomposed in terms of the automorphisms of the lattice $\mathcal{P}(\Gamma)$. From (3.3) it is immediate that

$$\delta_H^\lambda(G) = \bigcup_{h\in H} Gh = \bigcup_{g\in G} gH, \tag{3.8a}$$

$$\delta_H^\rho(G) = \bigcup_{h\in H} hG = \bigcup_{g\in G} Hg. \tag{3.8b}$$

Defining the union and intersection of left and right translations pointwise (i.e. by the ordering inherited from $\mathcal{P}(\Gamma)$, see section 2.2), (3.8) can be written in operator form as

$$\delta_H^\lambda = \bigcup_{h\in H} \rho_h, \qquad \delta_H^\rho = \bigcup_{h\in H} \lambda_h. \tag{3.9}$$

Since left and right translations commute, we see that δ_H^λ commutes with left translations and δ_H^ρ commutes with right translations. Below we will show that all left- and right-invariant dilations have this form. In a similar way we define left- and right-invariant erosions.

Definition 3.12. Let $H \in \mathcal{P}(\Gamma)$. The *left erosion* ε_H^λ and *right erosion* ε_H^ρ by the structuring element H are the mappings: $\mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$ defined by

$$\varepsilon_H^\lambda(G) = G \odot H, \qquad \varepsilon_H^\rho(G) = G \odot H. \tag{3.10}$$

We will also write λ-*dilation*/ λ-*erosion* instead of left dilation/erosion, with a similar convention for the right-invariant counterparts.

Again we decompose left and right erosions in terms of left and right translations. Just as there are two equivalent forms for the left and right dilation (3.8), one can derive two forms for the erosions. To see this take the complement of (3.8a), which by (3.7) equals the complement of $G \otimes H$:

$$\bigcap_{h\in H} G'h = \bigcap_{g\in G} gH' = (G \otimes H)' = G' \odot \check{H},$$

where we have used Lemma 3.10(g). Since this formula holds for arbitrary $G, H \in \mathcal{P}(\Gamma)$ we find (the proof for the right erosion is analogous),

$$\varepsilon_H^\lambda(G) = \bigcap_{h\in H} Gh^{-1} = \bigcap_{g\in G'} g\hat{H}, \tag{3.11a}$$

$$\varepsilon_H^\rho(G) = \bigcap_{h\in H} h^{-1}G = \bigcap_{g\in G'} \hat{H}g, \tag{3.11b}$$

where, as before, $\hat{H} = \check{H}'$. In operator form,

$$\varepsilon_H^\lambda = \bigcap_{h\in H} \rho_h^{-1}, \qquad \varepsilon_H^\rho = \bigcap_{h\in H} \lambda_h^{-1}. \tag{3.12}$$

The following lemma shows that as soon as we have proved a result for left-invariant dilations, there is a corresponding result for right-invariant dilations, as well as for left- or right-invariant erosions. First we need a definition.

Definition 3.13. Let $\psi : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$ be an arbitrary mapping. The *Boolean dual* ψ' of ψ is the mapping defined by $\psi'(G) = (\psi(G'))'$. The *reflection* $\check{\psi}$ of ψ is the mapping defined by $\check{\psi}(G) = (\psi(\check{G}))^\vee$. The *dual reflection* of ψ is the mapping $\hat{\psi}$ defined by $\hat{\psi}(G) = (\psi(\hat{G}))^\wedge$.

Lemma 3.14. *Let $\psi : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$ be an arbitrary mapping. Then,*

(a) $(\psi')' = (\check{\psi})^\vee = (\hat{\psi})^\wedge = \psi$

(b) $(\check{\psi})' = (\psi')^\vee$

(c) *ψ is an increasing λ-mapping $\iff$ ψ' is an increasing λ-mapping;*
ψ is a dilation $\iff$ ψ' is an erosion. In particular, $(\delta^\lambda_H)' = \varepsilon^\lambda_{\check{H}}$.

(d) *ψ is right-invariant $\iff$ $\check{\psi}$ is left-invariant.*
In particular, $(\lambda_h)^\vee = \rho_h^{-1}, (\delta^\lambda_H)^\vee = \delta^\rho_{\check{H}}, (\varepsilon^\lambda_H)^\vee = \varepsilon^\rho_{\check{H}}$.

(e) $(\delta^\lambda_H)^\wedge = \varepsilon^\rho_H$.

PROOF. Items (a) and (b) follow from Lemma 3.10(a,b).

(c) Let ψ be increasing. Then if $G \subseteq H$, $G' \supseteq H'$, so $\psi(G') \supseteq \psi(H')$. Therefore if $G \subseteq H$, then $\psi'(G) = (\psi(G'))' \subseteq (\psi(H'))' = \psi'(H)$, hence ψ' is increasing. The converse is proved similarly. Also, let ψ be left-invariant, $g \in \Gamma, H \subseteq \Gamma$. Then $\psi'(gH) = \{\psi((gH)')\}' = \{\psi(gH')\}' = \{g\psi(H')\}' = g\{\psi(H')\}' = g\psi'(H)$, hence ψ' is left-invariant. Next, let ψ be a dilation. Then $\psi'(\cap X_i) = \{\psi((\cap X_i)')\}' = \{\psi(\cup X_i')\}' = \{\cup\psi(X_i')\}' = \cap\{\psi(X_i')\}' = \cap\psi'(X_i)$, hence ψ' is an erosion; the reverse implication is proved similarly. Finally, $(\delta^\lambda_H)'(G) = (\delta^\lambda_H(G'))' = (G' \otimes H)' = G \odot \check{H} = \varepsilon^\lambda_{\check{H}}(G)$, where we have used Lemma 3.10(g).

(d) Let ψ be right-invariant, $g \in \Gamma, H \subseteq \Gamma$. Then $\check{\psi}(gH) = \{\psi((gH)^\vee)\}^\vee = \{\psi(\check{H}g^{-1})\}^\vee = \{\psi(\check{H})g^{-1}\}^\vee = g\{\psi(\check{H})\}^\vee = g\check{\psi}(H)$, where we used Lemma 3.10(d). So we have shown that if ψ is right-invariant, $\check{\psi}$ is left-invariant. The reverse statement is proved similarly. Also, $(\lambda_g)^\vee(H) = \{\lambda_g(\check{H})\}^\vee = \{g\check{H}\}^\vee = Hg^{-1} = \rho_g^{-1}(H)$; and $(\delta^\lambda_H)^\vee = (\bigcup_{h\in H}\rho_h)^\vee = \bigcup_{h\in H}(\rho_h)^\vee = \bigcup_{h\in H}\lambda_h^{-1} = \delta^\rho_{\check{H}}$. The result for the erosion follows in the same way.

(e) Follows from (c) and (d). □

Remark 3.15. Here is an example of how this lemma can be used. Suppose the following statement has been proved: ψ increasing $\Longrightarrow$ ψ' increasing. To show the converse, apply this statement to ψ'. Then we find: ψ' increasing $\Longrightarrow$ ψ'' increasing, but since the complementation operator is an involution ($\psi'' = \psi$) the proof is complete. In a similar way we can use results for left-invariant dilations to derive counterparts for right-invariant dilations (using $(\check{\psi})^\vee = \psi$) or for right-invariant erosions (using $(\hat{\psi})^\wedge = \psi$).

Next we make a few remarks about adjunctions. By Prop. 3.7(e) we have the equivalences:

$$\delta^\lambda_H(G) \subseteq K \iff G \subseteq \varepsilon^\lambda_H(K)$$
$$\delta^\rho_H(G) \subseteq K \iff G \subseteq \varepsilon^\rho_H(K)$$

We call $(\varepsilon^\lambda_H, \delta^\lambda_H)$ a *left-invariant adjunction* (λ-*adjunction*) and similarly we call $(\varepsilon^\rho_H, \delta^\rho_H)$ a *right-invariant adjunction* (ρ-*adjunction*). In particular all the properties of adjunctions as summarized in Lemma 2.6 hold for these adjunctions. So ε^λ_H is the upper adjoint of δ^λ_H, δ^λ_H is the lower adjoint of ε^λ_H, etc. Lemma 3.14(c-e) expresses the relation between the duality by complementation, reflection and adjoint pairs.

From the properties of adjunctions (see section 2.2) we know that we can build so-called *morphological* openings and closings from dilations and erosions. In particular, the mappings $\delta_H^\lambda \varepsilon_H^\lambda$ and $\delta_H^\rho \varepsilon_H^\rho$ are left- and right-invariant morphological openings; and the mappings $\varepsilon_H^\lambda \delta_H^\lambda$ and $\varepsilon_H^\rho \delta_H^\rho$ are left- and right-invariant morphological closings. As in the Euclidean case, these mappings are also so-called *structural* openings and closings (see Def. 2.9). Explicitly:

Proposition 3.16. *For all $G, H \in \mathcal{P}(\Gamma)$,*

$$\begin{aligned}
\alpha_H^\lambda(G) &:= \bigcup_{g\in\Gamma}\{gH : gH \subseteq G\} = (G \odot H) \otimes H = \delta_H^\lambda \varepsilon_H^\lambda(G) \\
\alpha_H^\rho(G) &:= \bigcup_{g\in\Gamma}\{Hg : Hg \subseteq G\} = H \otimes (G \odot H) = \delta_H^\rho \varepsilon_H^\rho(G) \\
\phi_H^\lambda(G) &:= \bigcap_{g\in\Gamma}\{gH : gH \supseteq G\} = (G \otimes \hat{H}) \odot \hat{H} = \varepsilon_{\hat{H}}^\lambda \delta_{\hat{H}}^\lambda(G) \\
\phi_H^\rho(G) &:= \bigcap_{g\in\Gamma}\{Hg : Hg \supseteq G\} = (\hat{H} \otimes G) \odot \overline{\hat{H}} = \varepsilon_{\hat{H}}^\rho \delta_{\hat{H}}^\rho(G)
\end{aligned}$$

where $\hat{H} = \check{H}'$.

PROOF. We only prove the first and third formula. From the definition Eq.(3.4a) of the left residual we have $(G \odot H) \otimes H = (\bigcup_{g\in\Gamma}\{g : gH \subseteq G\}) \otimes H = \bigcup_{g\in\Gamma}\{gH : gH \subseteq G\}$, which proves the result for the left-invariant opening. Using Boolean duality, we have $\bigcap_{g\in\Gamma}\{gH : gH \supseteq G\} = \bigcap_{g\in\Gamma}\{gH : gH' \subseteq G'\} = (\bigcup_{g\in\Gamma}\{gH' : gH' \subseteq G'\})' = ((G' \odot H') \otimes H')' = (G \otimes \check{H}') \odot \check{H}' = (G \otimes \hat{H}) \odot \hat{H}$, proving the third line. □

This proposition contains the geometrical interpretation of the morphological openings and closings. For example, the left-invariant opening $\delta_H^\lambda \varepsilon_H^\lambda(G)$ is the union of all left translates of H which are contained in G, etc. All this is completely analogous to the situation in Euclidean morphology. The following properties related to behaviour under translations are immediate:

$$\alpha_H^\lambda(gG) = g\alpha_H^\lambda(G), \quad \alpha_{gH}^\lambda(G) = \alpha_H^\lambda(G). \tag{3.13}$$

Similar properties can be proved for closings and the right-invariant counterparts of both by using the identities

$$(\phi_H^\lambda)' = \alpha_{H'}^\lambda, \quad (\alpha_H^\lambda)^\vee = \alpha_{\check{H}}^\rho, \tag{3.14}$$

which follow from Lemma 3.14.

Summarizing the results of this section, we have generalized the Minkowski operations and the associated dilations and erosions, forming adjoint pairs invariant under either left or right translations. Finally we have constructed the morphological openings and closings which correspond to these adjunctions and provided a simple geometrical interpretation for them. The question which we take up in the final section is whether all adjunctions, openings and closings are of the form found above. Also, the representation theorem of Matheron for increasing translation-invariant mappings will be generalized.

4. Characterization Theorems.

This section treats the representation theorems for adjunctions, openings and closings, as well as general translation-invariant mappings. We start with the characterization of adjunctions. Then follows a discussion of kernels of mappings $\psi : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$. Subsequently we extend the results of [1] concerning a representation theorem for arbitrary translation-invariant mappings, obtaining decompositions of increasing or decreasing translation-invariant mappings as special cases. We end with a discussion of openings and closings.

4.1. General form of adjunctions on $\mathcal{P}(\Gamma)$.

The question whether all left- and right-invariant dilations and erosions have the form (3.9) and (3.12), respectively, is answered by the following proposition.

Proposition 4.1. *A pair (ε, δ) of mappings from $\mathcal{P}(\Gamma)$ to itself is a left-invariant adjunction if and only if*

$$\delta = \delta_H^\lambda = \bigcup_{h \in H} \rho_h \,, \quad \varepsilon = \varepsilon_H^\lambda = \bigcap_{h \in H} \rho_h^{-1}, \tag{4.1}$$

for some $H \in \mathcal{P}(\Gamma)$. A corresponding statement holds for right-invariant adjunctions.

PROOF. We have seen above that (4.1) is a λ-adjunction. Therefore it remains to prove the 'only if' part. So assume that (ε, δ) is a λ-adjunction. Let $H = \delta(\{e\})$, where e is the unit element of Γ. Then, for each $g \in \Gamma$,

$$\delta(\{g\}) = \delta(\lambda_g\{e\}) = \lambda_g \delta(\{e\}) = \lambda_g(H).$$

Hence, for each $G \in \mathcal{P}(\Gamma)$,

$$\delta(G) = \delta(\bigcup_{g \in G} \{g\}) = \bigcup_{g \in G} \delta(\{g\}) = \bigcup_{g \in G} \lambda_g(H) = G \otimes H = \delta_H^\lambda(G),$$

proving that each left-invariant dilation has the form as in (4.1). To complete the proof, observe that if ε is a λ-erosion, then its lower adjoint δ is a dilation, so $\delta = \delta_H^\lambda$ for some $H \in \mathcal{P}(\Gamma)$, whose unique upper adjoint is ε_H^λ. Hence $\varepsilon = \varepsilon_H^\lambda$. □

4.2. Kernels of mappings $\psi : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$.

Definition 4.2. The *kernel* of a mapping $\psi : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$, denoted by $\mathcal{V}(\psi)$, is the family of subsets of Γ defined by

$$\mathcal{V}(\psi) = \{G \in \mathcal{P}(\Gamma) : e \in \psi(G)\},$$

where e denotes the unit element of the group Γ.

Proposition 4.3. *There is a 1-1 correspondence between subsets of the lattice $\mathcal{P}(\Gamma)$ and λ-mappings (ρ-mappings) $\psi : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$. More precisely, to any λ-mapping (ρ-mapping) ψ corresponds a family $\mathcal{B}$ of subsets of Γ, where $\mathcal{B}$ is the kernel of ψ. Conversely, to any subset $\mathcal{B} \subseteq \mathcal{P}(\Gamma)$ corresponds one λ-mapping ψ^λ defined by $\psi^\lambda(G) = \{h \in \Gamma : G \in h\mathcal{B}\}$ and one ρ-mapping ψ^ρ defined by $\psi^\rho(G) = \{h \in \Gamma : G \in \mathcal{B}h\}$, both with kernel $\mathcal{B}$.*

Here we have used the notation $h\mathcal{B} = \{hB : B \in \mathcal{B}\}$, $\mathcal{B}h = \{Bh : B \in \mathcal{B}\}$. The proof is completely analogous to the Euclidean case, see e.g. Matheron [12, chapter 8], and is omitted here.

The following lemma shows the relation between the kernel of a mapping and that of its dual, reflection and dual reflection, respectively.

Lemma 4.4. *Let $\psi : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$ be a mapping with kernel $\mathcal{V}(\psi)$. Then the kernels of the dual ψ', the reflection $\check{\psi}$ and the dual reflection $\hat{\psi}$ are given by:*

(a) $\mathcal{V}(\psi') = \{G \in \mathcal{P}(\Gamma) : G' \notin \mathcal{V}(\psi)\}$
(b) $\mathcal{V}(\check{\psi}) = \{G \in \mathcal{P}(\Gamma) : \check{G} \in \mathcal{V}(\psi)\}$
(c) $\mathcal{V}(\hat{\psi}) = \{G \in \mathcal{P}(\Gamma) : \hat{G} \notin \mathcal{V}(\psi)\}$

Proof.

(a) $\mathcal{V}(\psi') = \{G \in \mathcal{P}(\Gamma) : e \in (\psi(G'))'\} = \{G \in \mathcal{P}(\Gamma) : e \notin \psi(G')\} = \{G \in \mathcal{P}(\Gamma) : G' \notin \mathcal{V}(\psi)\}$.
(b) $\mathcal{V}(\check{\psi}) = \{G \in \mathcal{P}(\Gamma) : e \in (\psi(\check{G}))^{\vee}\} = \{G \in \mathcal{P}(\Gamma) : e \in \psi(\check{G})\} = \{G \in \mathcal{P}(\Gamma) : \check{G} \in \mathcal{V}(\psi)\}$.
(c) $\mathcal{V}(\hat{\psi}) = \{G \in \mathcal{P}(\Gamma) : G' \notin \mathcal{V}(\check{\psi})\} = \{G \in \mathcal{P}(\Gamma) : \hat{G} \notin \mathcal{V}(\psi)\}$. □

4.3. Decomposition of translation-invariant mappings.

In a recent paper, Banon and Barrera [1] generalized Matheron's theorem 2.2 to arbitrary translation-invariant mappings (not necessarily increasing) on $\mathcal{P}(E)$, where E denotes Euclidean space. Following the simplified proof in [11], we extend this result here to the case $\mathcal{P}(\Gamma)$ with Γ a non-commutative group, getting as a by-product a generalization of Matheron's theorem. We only formulate the left translation-invariant case. The right translation-invariant case is obtained by left-right symmetry.

Define, for $F, G, H \in \mathcal{P}(\Gamma)$, the *left wedge transform* of G by the pair (F, H) by

$$\begin{aligned} G \circledwedge (F,H) &:= \{g \in \Gamma : gF \subseteq G \subseteq gH\} \\ &= (G \odot F) \cap (G' \odot H'), \end{aligned}$$

where the second line follows from the definition Eq.(3.4a) of the left residual. In the Euclidean case, this operation is a slight modification of the hit-or-miss transform [19]. Clearly the mapping $G \mapsto G \circledwedge (F, H)$ is left translation-invariant. Two cases are of special interest:

(a) $G \circledwedge (F, \Gamma) = G \odot F$,
(b) $G \circledwedge (\emptyset, H) = G' \odot H'$.

Define also the 'interval' between sets as

$$[F, H] = \{G \in \mathcal{P}(\Gamma) : F \subseteq G \subseteq H\}.$$

Clearly, $[F, H]$ and $G \circledwedge (F, H)$ are both empty if $F \not\subseteq H$.

Definition 4.5. Let ψ be a mapping on $\mathcal{P}(\Gamma)$, with kernel $\mathcal{V}(\psi)$ given by Def.4.2. The *bi-kernel* of ψ is defined by

$$\mathcal{W}(\psi) = \{(F, H) \in \mathcal{P}(\Gamma) \times \mathcal{P}(\Gamma) : \; [F, H] \subseteq \mathcal{V}(\psi)\}.$$

If ψ is increasing and F is an element of $\mathcal{V}(\psi)$, then the whole interval $[F,H]$ is included in $\mathcal{V}(\psi)$ if $H \supseteq F$. Similarly, if ψ is decreasing and $H \in \mathcal{V}(\psi)$, then $[F,H]$ is included in $\mathcal{V}(\psi)$ if $F \subseteq H$. Hence

$$\psi \textit{ increasing}, \; F \in \mathcal{V}(\psi) \Longrightarrow (F,\Gamma) \in \mathcal{W}(\psi) \tag{4.2a}$$
$$\psi \textit{ decreasing}, \; H \in \mathcal{V}(\psi) \Longrightarrow (\emptyset,H) \in \mathcal{W}(\psi). \tag{4.2b}$$

Now we can prove:

Theorem 4.6. Representation of translation-invariant mappings
The mapping $\psi : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$ is left translation-invariant if and only if

$$\psi(G) = \bigcup_{(F,H)\in\mathcal{W}(\psi)} G \circledA (F,H). \tag{4.3}$$

PROOF. It is clear that ψ as given by (4.3) is a left-invariant mapping, since it is a union of such mappings. Conversely, let ψ be a left-invariant mapping. We show that ψ has the form (4.3). Given $G \in \mathcal{P}(\Gamma)$, let $Z = \bigcup_{(F,H)\in\mathcal{W}(\psi)} G \circledA (F,H)$. We show that $\psi(G) = Z$.

(*a*) $\psi(G) \supseteq Z$: Let $g \in G \circledA (F,H)$ for some $(F,H) \in \mathcal{W}(\psi)$. Then $gF \subseteq G \subseteq gH$, hence $F \subseteq g^{-1}G \subseteq H$ and so $g^{-1}G \in [F,H] \subseteq \mathcal{V}(\psi)$ by assumption on (F,H). It follows that $e \in \psi(g^{-1}G) = g^{-1}\psi(G)$, where e is the identity of Γ and we used left invariance of ψ. Therefore $g \in \psi(G)$, hence $\psi(G) \supseteq Z$.

(*b*) $\psi(G) \subseteq Z$: Let $g \in \psi(G)$. Then, using left invariance, $e \in g^{-1}\psi(G) = \psi(g^{-1}G)$, hence $g^{-1}G \in \mathcal{V}(\psi)$ and therefore $(g^{-1}G, g^{-1}G) \in \mathcal{W}(\psi)$. Combining this with the obvious fact that $G \circledA (g^{-1}G, g^{-1}G) \supseteq \{g\}$, we conclude that $g \in Z$ and so $\psi(G) \subseteq Z$. □

Corollary 4.7. *If $\psi : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$ is an increasing λ-mapping it can be decomposed as a union of λ-erosions, or an intersection of λ-dilations:*

$$\psi(G) = \bigcup_{F\in\mathcal{V}(\psi)} G \odot F = \bigcap_{F\in\mathcal{V}(\psi')} G \otimes \check{F}. \tag{4.4a),}$$

where ψ' is the Boolean dual of ψ. If $\psi : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$ is a decreasing λ-mapping, it can be similarly decomposed:

$$\psi(G) = \bigcup_{H\in\mathcal{V}(\psi)} G' \odot H' = \bigcap_{H\in\mathcal{V}(\psi')} G' \otimes \check{H}. \tag{4.4b}$$

PROOF. By application of the above theorem to an increasing λ-mapping, and using (4.2a) combined with the obvious fact that $G \circledA (F,H)$ is increasing in H, we have

$$\psi(G) = \bigcup_{F\in\mathcal{V}(\psi)} G \circledA (F,\Gamma) = \bigcup_{F\in\mathcal{V}(\psi)} G \odot F.$$

To prove the representation as an intersection of dilations, observe that the dual mapping ψ' of ψ is itself left-invariant and increasing, see Lemma 3.14. So, applying the decomposition just proved to ψ', we get

$$\psi'(G) = \bigcup_{F\in\mathcal{V}(\psi')} G \odot F.$$

Now we take again the Boolean dual of ψ', using Lemma 3.10(g) and the fact that $\psi'' = \psi$ to find,

$$\psi(G) = \Big(\bigcup_{F \in \mathcal{V}(\psi')} G' \odot F \Big)' = \bigcap_{F \in \mathcal{V}(\psi')} G \otimes \check{F}.$$

This completes the proof for increasing λ-mappings. The proof for decreasing λ-mappings is analogous. □

4.4. Decomposition of openings and closings.

Recall from section 2.2 that the domain of invariance of a mapping $\psi : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$ is the subset of $\mathcal{P}(\Gamma)$ defined by $\mathsf{Inv}(\psi) = \{G \in \mathcal{P}(\Gamma) : \psi(G) = G\}$.

Theorem 4.8. *A mapping $\psi : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$ is a left-invariant opening if and only if ψ has the representation,*

$$\psi(G) = \bigcup_{H \in \mathcal{B}} \alpha_H^{\lambda}(G) , \tag{4.5}$$

for some subset $\mathcal{B}$ of the lattice $\mathcal{P}(\Gamma)$, with $\alpha_H^{\lambda}(G) = (G \odot H) \otimes H$. Moreover, $\mathsf{Inv}(\psi)$ is the class of sets generated by $\mathcal{B}$ under left translations and infinite unions and any subset $\mathcal{B}$ which generates $\mathsf{Inv}(\psi)$ in this way defines the same opening ψ.

Proof. We only have to prove the 'only if' part, since a union of λ-openings is a λ-opening (see section 2.2). So assume that ψ is a λ-opening. Applying Prop.2.10 of section 2.2 with $\mathbb{T} = \Gamma^{\lambda}$, one finds that ψ has the form (4.5) with α_H^{λ} the structural λ-opening by the structuring element H. Since from Prop.3.16, $\alpha_H^{\lambda} = \delta_H^{\lambda} \varepsilon_H^{\lambda}$, the proof is complete. □

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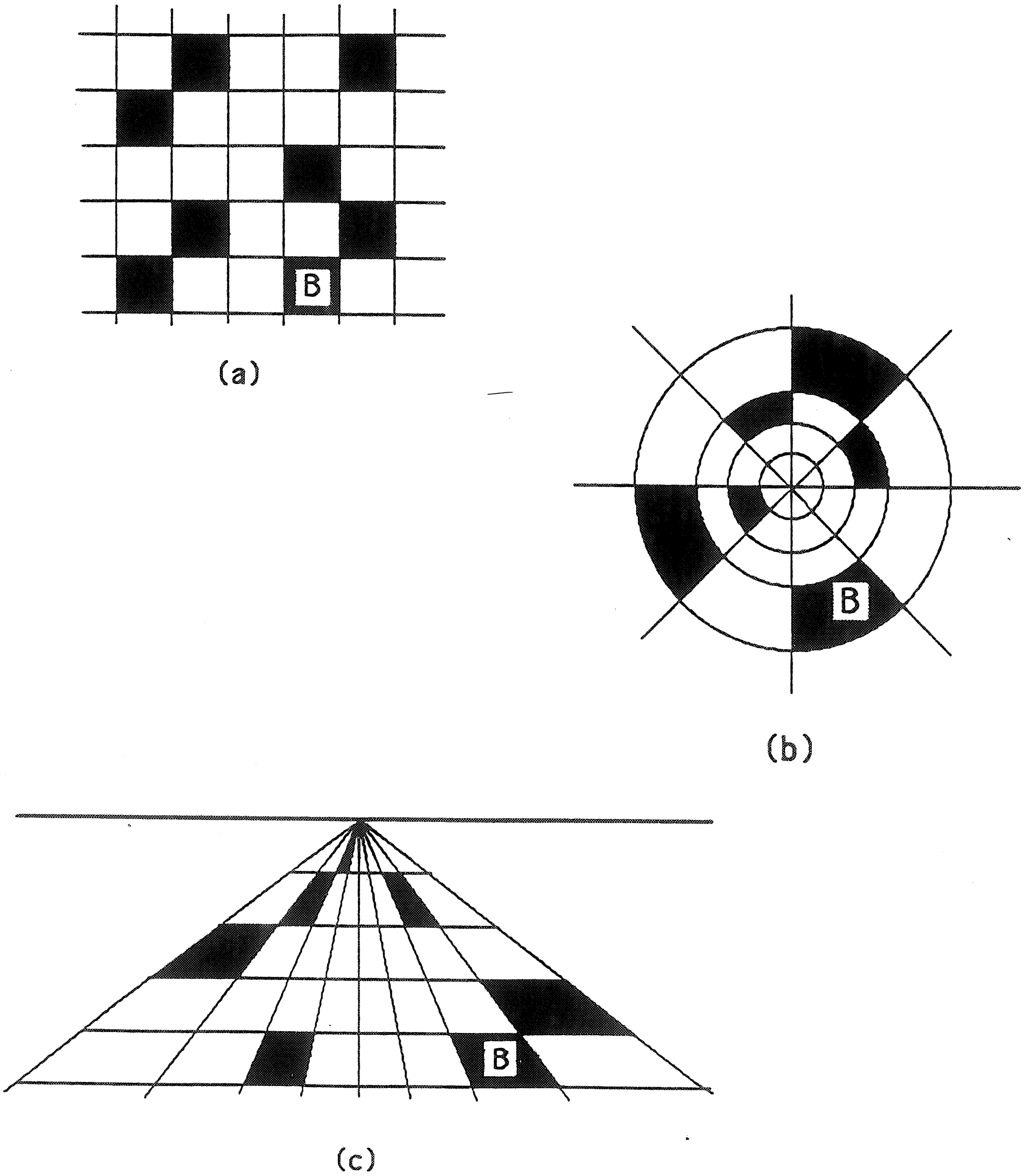

Figure 1. *Copies (dark) of a structuring element B under:*

(a) Euclidean translation
(b) rotations and scalar multiplication
(c) perspective transformation