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Approximation of the Solution to the Moment Problem in a Hilbert Space

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In this paper we obtain a construction of the solution to a moment problem. We use our results to derive a truncation error for *sinc*-interpolation, which generalizes the error bounds in the literature to the case of nonuniform sampling.

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0. Introduction.

In this paper we approximate the solution f to a moment problem, by means of truncation. The moment problem consists of finding an element f of a Hilbert space $\mathcal{H}$ which satisfies

$$\langle f, \varphi_i \rangle_{\mathcal{H}} = g_i, \quad \forall i \in \mathbb{Z} \tag{0.1}$$

where $\{g_i\} \in \ell^2(\mathbb{Z})$ and the system of vectors $\{\varphi_i\}_{i\in \mathbb{Z}}$ lies in $\mathcal{H}$, which has inner product $\langle\ ,\ \rangle_{\mathcal{H}}$. The space $\ell^2(\mathbb{Z})$ is the set of sequences of complex numbers $\{g_i\}_{i\in \mathbb{Z}}$ such that $\sum_{i\in \mathbb{Z}} |g_i|^2 < \infty$. Without further conditions on the system $\{\varphi_i\}_{i\in \mathbb{Z}}$, (0.1) need not to have a solution. It turns out that a sufficient condition for (0.1) to have a solution is that $\{\varphi_i\}_{i\in \mathbb{Z}}$ is a Riesz basis, cf. Young [12] . The computation of f involves the inversion of an infinite matrix. For practical reasons, we want to work with finite matrices. This problem can be circumvented by first solving the truncated problem,

$$\langle f_n, \varphi_i \rangle_{\mathcal{H}} = g_i, \quad \forall i \in \{-n, ..., n\}. \tag{0.2}$$

Repeating this procedure for each $n \in I\!N$, we obtain a sequence f_n. These functions f_n are given in closed form, involving only finite sums and inverses of finite matrices. In section 2 we prove that f can be approximated by f_n,

$$\lim_{n\to\infty} \|f - f_n\| = 0.$$

In section 3 we introduce the space of bandlimited functions, i.e. functions whose Fourier transforms have compact support. It turns out that for bandlimited functions f, the inner product $\langle f, \varphi_i \rangle$ is a point evaluation of f at, say, t_i. If $t_i = i$ for all $i \in \mathbb{Z}$, then we say the function f is sampled uniformly, otherwise f is said to be sampled nonuniformly. The main application is to derive a bound for the truncation error in the case of nonuniform sampling, which is an extension of an estimate of Butzer [1] . In the literature Butzer [1] , Butzer and Splettstösser [3] , Butzer, Splettstösser and Stens [4] , and Papoulis [9] , estimates for the truncation error are given only for uniform sampling. In section 4 we make some remarks on the estimates from literature.

1. Preliminaries

In this section we introduce notions wich we use in later sections. A sequence of vectors $\{\varphi_i\}_{i\in \mathbb{Z}}$ is a Riesz basis (see Young [12] p. 31) if there exists a bounded linear invertible operator T on $\mathcal{H}$ such that

$$T\varphi_i = h_i, \quad \forall i \in \mathbb{Z}, \tag{1.1}$$

where $\{h_i\}_{i\in \mathbb{Z}}$ is an orthonormal basis for $\mathcal{H}$. An operator T is invertible if its inverse, denoted by T^{-1}, exists and is bounded.

The next theorem (cf. Young [12] Theorem 9, p. 32) characterizes Riesz bases, in terms of its Gram matrix and of completeness of a system of vectors. A sequence $\{\varphi_i\} \subset \mathcal{H}$ is complete if its linear span, denoted by $\text{span}\{\varphi_i\}_{i\in \mathbb{Z}}$, lies dense in $\mathcal{H}$. The Gram matrix of $\{\varphi_i\}$ is defined by

$$G_{ij} := \langle \varphi_j, \varphi_i \rangle_{\mathcal{H}}, \quad \forall i,j \in \mathbb{Z}.$$

In the case of a Riesz basis G is the matrix representation of the operator $(TT^*)^{-1}$, with respect to the basis $\{h_i\}$. So,

$$\|T^{-1}\| = \|G\|^{1/2}, \quad \text{and} \quad \|T\| = \|G^{-1}\|^{1/2}. \tag{1.2}$$

Theorem 1.1 . *The following statements are equivalent.*

(i) $\{\varphi_i\}_{i\in \mathbb{Z}}$ *is a Riesz basis.*
(ii) $\{\varphi_i\}$ *is complete and there exist positive real numbers* A, B *such that for each* $n \in \mathbb{N}$ *and for each finite sequence* $\{c_i\}_{-n,\dots,n}$

$$A \sum_{i=-n}^{n} |c_i|^2 \le \| \sum_{i=-n}^{n} c_i\varphi_i\|^2 \le B \sum_{i=-n}^{n} |c_i|^2.$$

(iii) $\{\varphi_i\}$ *is complete and the Gram matrix* G *of* $\{\varphi_i\}$ *generates a bounded linear invertible operator on* $\ell^2(\mathbb{Z})$.

Throughout the rest of this paper the system $\{\varphi_i\}_{i\in \mathbb{Z}}$ denotes a Riesz basis. By Theorem 1.1 it follows that the definition of Riesz basis is independent of the choice of the orthonormal system $\{h_i\}$.

Two systems $\{\psi_i\}, \{\varphi_i\}$ are called biorthogonal if

$$\langle \varphi_i, \psi_j \rangle_{\mathcal{H}} = \delta_{ij}, \forall i,j \in \mathbb{Z}.$$

A Riesz basis $\{\varphi_i\}$ has a unique biorthogonal system $\{\psi_i\}$, given by $\psi_i = T^*h_i$, for $i \in \mathbb{Z}$. The biorthogonal sequence also is a Riesz basis. Any $f \in \mathcal{H}$ can uniquely be written as (cf. Higgins [6])

$$f = \sum_{i\in \mathbb{Z}} \langle f, \varphi_i \rangle_{\mathcal{H}} \psi_i. \tag{1.3}$$

From this it follows that the moment problem (0.1) has the unique solution

$$f = \sum_{i\in \mathbb{Z}} g_i \psi_i. \tag{1.4}$$

If we want to compute the system $\{\psi_i\}$, we need a formula for the operator T, which may be hard to find. An alternative formula for $\{\psi_i\}$ is obtained by Zwaan [13]

$$\psi_i = \sum_{j\in \mathbb{Z}} \overline{(G^{-1})_{ij}} \varphi_j, \quad \forall i \in \mathbb{Z}.$$

The problem in this formula is the inversion of the infinite matrix G. In section 2 we circumvent this inconvenience by inverting the truncated matrix. We thus obtain an approximation of the system $\{\psi_i\}$ and of the solution f.

We construct an orthonormal basis $\{h_i\}_{i\in \mathbb{Z}}$ for $\mathcal{H}$ in such a way that $\{h_i\}_{i=-n,\dots,n}$ is an orthonormal basis for

$$\mathcal{H}_n := \text{span}\{\varphi_{-n},\dots,\varphi_n\},$$

e.g. by Gram-Schmidt orthogonalization. In this case the operator T given by (1.1) leaves all the subspaces $\mathcal{H}_n$ invariant, and

$$T\varphi_i = h_i, \quad \forall i \in \mathbb{Z}.$$

Note that the adjoint of T need not to leave the subspaces $\mathcal{H}_n$ invariant. Define the restriction of T to $\mathcal{H}_n$ by $T_n := T_{|\mathcal{H}_n}$. Denoting the adjoint of T_n in $\mathcal{H}_n$ by T_n^*, the system $\{\psi_i^n\}_{-n,\dots,n} \subset \mathcal{H}_n$ can be defined as

$$\psi_i^n := T_n^* h_i, \quad \forall i \in \{-n,\dots,n\}, \tag{1.5}$$

which is the unique biorthogonal system of $\{\varphi_i\}_{-n,\dots,n}$ in $\mathcal{H}_n$. An alternative formula for ψ_i^n is

$$\psi_i^n = \sum_{j=-n}^{n} \overline{(G_n^{-1})_{ij}}\varphi_j.$$

Here G_n is the truncated Gram matrix,

$$(G_n)_{ij} := G_{ij}, \quad \forall i,j \in \{-n,\dots,n\}.$$

A (not necessarily unique) solution to (0.2) can now be given as

$$f_n = \sum_{i=-n}^{n} g_i \psi_i^n. \tag{1.6}$$

(1.6) is not unique, because other solutions can be obtained by adding elements to f_n, which are orthogonal to span$\{\varphi_{-n},\dots,\varphi_n\}$. The following result (cf. Young [12] Proposition 1, p.147) characterizes solutions to an arbitrary moment problem.

Proposition 1.2 . *Let $\mathbb{I} \subset \mathbb{Z}$ be an arbitrary index set and let $\{g_i\} \in \ell^2(\mathbb{I})$. If the problem*

$$\langle f,\varphi_i\rangle_{\mathcal{H}} = g_i, \quad \forall i \in \mathbb{I}, \tag{1.7}$$

has a solution, then there exists a unique minimum norm solution which lies in the subspace $\overline{\text{span}}\{\varphi_i\}_{i\in\mathbb{I}} \subset \mathcal{H}$.

It follows that $f_n \in \mathcal{H}_n$, given by formula (1.6), is the unique minimum norm solution to (0.2) in $\mathcal{H}$.

2. Construction of the solution to the moment problem.

The aim of this section is to prove that $\|f - f_n\| \to 0$, (for $n \to \infty$) where $f \in \mathcal{H}$ and $f_n \in \mathcal{H}_n$ are the unique and the unique minimum norm solution to (0.1) and (0.2), respectively.

Introduce the projection operator $P_n : \mathcal{H} \to \mathcal{H}_n$, by

$$P_n f = \sum_{i=-n}^{n} \langle f, \varphi_i \rangle_{\mathcal{H}} \psi_i^n. \tag{2.1}$$

P_n is a normal operator ($P_n^* P_n = P_n P_n^*$) from $\mathcal{H}$ onto $\mathcal{H}_n$ and it reduces to the identity operator on $\mathcal{H}_n$, i.e. $P_n g = g$ for $g \in \mathcal{H}_n$. If $f \in \mathcal{H}$ is the solution to (0.1), then the minimum norm solution f_n to (0.2) can be written as $f_n = P_n f$. For any $g \in \mathcal{H}_n$ we have

$$(Id - P_n)f = (Id - P_n)(f - g).$$

Hence

$$\|(Id - P_n)f\| \leq \|Id - P_n\| \text{dist}(f, \mathcal{H}_n), \tag{2.2}$$

where

$$\text{dist}(f, \mathcal{H}_n) = \inf_{h \in \mathcal{H}_n} \|f - h\|_{\mathcal{H}}.$$

We know that for all $f \in \mathcal{H}$

$$\lim_{n \to \infty} \text{dist}(f, \mathcal{H}_n) = 0. \tag{2.3}$$

Note that $\|(Id - P_n)f\|$ is the error due to truncation of the moment problem (0.1). The next theorem proves that $\|Id - P_n\| \leq c$, where c is a constant independent of n.

Theorem 2.1 . *Let $\{\varphi_i\}_{i \in \mathbb{Z}}$ be a Riesz basis for $\mathcal{H}$, and let P_n be given by (2.1). Then*

$$\|Id - P_n\| \leq 1 + (\|G^{-1}\| \; \|G\|)^{1/2}, \quad \forall n \in \mathbb{N}. \tag{2.4}$$

Proof:

Using $\psi_i = T^* h_i$, and (1.5), we obtain

$$\|P_n f\| = \| \sum_{i=-n}^{n} \langle f, \varphi_i \rangle \psi_i^n \| = \| \sum_{i=-n}^{n} \langle f, \varphi_i \rangle_{\mathcal{H}} T_n^* h_i \| \leq$$

$$\|T_n\| \| \sum_{i=-n}^{n} \langle f, \varphi_i \rangle_{\mathcal{H}} h_i \| \leq \|T\| \| \sum_{i \in \mathbb{Z}} \langle f, \varphi_i \rangle_{\mathcal{H}} T^{*^{-1}} \psi_i \| \leq \|T\| \|T^{-1}\| \|f\|.$$

Hence, by (1.2) and Theorem 1.1. (iii),

$$\|Id - P_n\| \leq 1 + (\|G\| \|G^{-1}\|)^{1/2} < \infty, \quad \forall n \in \mathbb{N}.$$

This proves the estimate. □

By (2.2) and (2.4) it follows that,

$$\|(Id - P_n)f\| \leq (1 + (\|G^{-1}\| \; \|G\|)^{1/2}) \text{ dist}(f, \mathcal{H}_n). \tag{2.5}$$

Hence, by (2.3), for all $f \in \mathcal{H}$,

$$\lim_{n \to \infty} \|(Id - P_n)f\| = 0. \tag{2.6}$$

We have proved the following result.

Corollary 2.2 . *If $\{\varphi_i\}_{i\in\mathbb{Z}}$ is a Riesz basis and if f_n (formula (1.6)) is the minimum norm solution of the truncated problem (0.2), then $\{f_n\}_{n\in\mathbb{N}}$ converges to the solution of problem (0.1).*

This applies in particular to biorthogonal sequences $\{\psi_i\}_{i\in\mathbb{Z}}$ of a Riesz basis. It follows by definition of P_n that $\psi_i^n = P_n\psi_i$, for $i \in \{-n, ..., n\}$. Hence by (2.6)

$$\lim_{n\to\infty} \|\psi_i^n - \psi_i\| = 0, \tag{2.7}$$

for $i \in \mathbb{Z}$. This procedure of solving the truncated problem (0.2), instead of (0.1), is an application of a projection method of Natterer [8] .

3. Truncation error for nonuniform sampling

In this section we derive a formula for the truncation error in the case of nonuniform sampling of a bandlimited function. We apply the results of the previous section in the case that $\mathcal{H}$ is the space of bandlimited functions and $\varphi_i :=$ sinc$_T(. - t_i\pi/r)$, where $\{t_i\}$ is a sequence of real numbers. Here the *sinc*-function is given for $t \in \mathbb{R}$, by

$$\text{sinc}_r(t) := \begin{cases} \dfrac{\sin(rt)}{rt}, & t \neq 0 \\ 1, & t = 0 \end{cases}.$$

The space of bandlimited functions, also refered to as the Paley-Wiener space, consists of all $L^2(\mathbb{R})$-functions f such that the Fourier transform of f, denoted by $\hat{f}$, is zero outside the interval $[-r, r]$.

Definition 3.1. $\mathbb{P}_r := \{f \in L^2(\mathbb{R}) | \text{supp } \hat{f} \subset [-r, r]\}$

If we define the inner product $\langle\ ,\ \rangle_{P_r}$ on $\mathbb{P}_r$ by,

$$\langle f, g\rangle_{P_r} := \int_R f(x)\overline{g(x)}dx,$$

then $\mathbb{P}_r$ is a Hilbert space. By the theorem of Paley-Wiener (see Young [12] Theorem 18, p. 101) any $f \in \mathbb{P}_r$ can be extended to an entire function $f : \mathbb{C} \to \mathbb{C}$ which satisfies $|f(z)| \leq \|f\|_{P_r}\, e^{r|Imz|}$, for all $z \in \mathbb{C}$. Hence any element of $\mathbb{P}_r$ satisfies the inequality,

$$\|f\|_\infty \leq \|f\|_{P_r}, \quad \forall f \in \mathbb{P}_r. \tag{3.1}$$

Here the ∞-norm is defined by $\|f\|_\infty := \sup_{t\in R}|f(t)|$.

The system $\{\varphi_i\}_{i\in\mathbb{Z}}$ is a Riesz basis for $\mathbb{P}_r$ if the sequence of t_i's satisfies

$$|t_i - i| \leq \alpha < 1/4, \quad \forall i \in \mathbb{Z}. \tag{3.2}$$

If $t_i = i$ for all $i \in \mathbb{Z}$, then $\{\varphi_i\}$ is an orthonormal basis for $\mathbb{P}_r$.

The point evaluation can be written in terms of the φ_i's,

$$\langle f, \varphi_i\rangle_{P_r} = (\sqrt{\pi/r})f(t_i\pi/r), \quad \forall f \in \mathbb{P}_r. \tag{3.3}$$

With (3.3) and (1.3) we write an arbitrary element f lying in $\mathcal{H} = \mathbb{P}_r$ as

$$f = \sum_{i\in \mathbb{Z}} (\sqrt{\pi/r}) f(t_i\pi/r)\psi_i,$$

and the projection from f onto $\mathcal{H}_n$ as (cf. definitions (2.1) and (1.5))

$$f_n := P_n f = \sum_{i=-n}^{n} (\sqrt{\pi/r}) f(t_i\pi/r)\psi_i^n.$$

The distance from f to $\mathcal{H}_n$ can be expressed in terms of the system $\{h_i\}$ (cf. section 2),

$$\text{dist}(f,\mathcal{H}_n) = \Big(\sum_{|i|>n} |\langle f, h_i\rangle_{P_r}|^2\Big)^{1/2}.$$

Because $\{\varphi_i\}$ is a Riesz basis, we obtain by (1.1),

$$\text{dist}(f,\mathcal{H}_n) = \Big(\sum_{|i|>n} |\langle T^* f, \varphi_i\rangle_{P_r}|^2\Big)^{1/2} = \Big(\sum_{|i|>n} (\pi/r)|(T^*f)(t_i\pi/r)|^2\Big)^{1/2}.$$

Generalizing Butzer [1] , we assume T^*f to satisfy,

$$|(T^*f)(t)| \le M_{T^*f}\; 1/|t|^{\gamma}, \tag{3.4}$$

for $t \in \mathbb{R} \setminus \{0\}$ and $\gamma > 1/2$. Here M_{T^*f} is a constant which depends on T^*f. It follows by a straightforward computation (for $n > 0$), and by (3.4) that

$$\text{dist}\;(f,\mathcal{H}_n) \le \sqrt{2}\; M_{T^*f}\; (r/\pi)^{(\gamma-1/2)}\; \frac{(n-1/4)^{(1-2\gamma)/2}}{\sqrt{2\gamma-1}}. \tag{3.5}$$

Define the truncation error as $e_{\text{tr}} := \|f - f_n\|_\infty$. By Theorem 2.1, and (3.5) we have

$$e_{\text{tr}} \le \Big(1 + (\|G^{-1}\|\|G\|)^{1/2}\Big)\;\Big(\sqrt{2}\; M_{T^*f}\; (r/\pi)^{(\gamma-1/2)}\; \frac{(n-1/4)^{(1-2\gamma)/2}}{\sqrt{2\gamma-1}}\Big). \tag{3.6}$$

A remark is in order. In the case of uniform sampling (i.e. $\alpha = 0$ or, equivalently, $t_i = i$ for $i \in \mathbb{Z}$) T^* is the identity on $\mathbb{P}_r$, and $G = G^{-1} = Id$. In the case of nonuniform sampling (i.e. $\alpha \ne 0$) the norms of G and G^{-1} are estimated in Zwaan [14] ,

$$\|G\|^{1/2} \le 1 + \lambda \text{ and } \|G^{-1}\|^{1/2} \le \frac{1}{1-\lambda},$$

where $\lambda := 1 - \cos\pi\alpha + \sin\pi\alpha$.

4. Conclusions and Remarks

From (3.6) it follows that the moment problem (0.1) is stable for truncation (i.e. $\lim f_n = f$, for $n \to \infty$), if the t_i's satisfy (3.2). The rate of convergence is governed by the norms of the matrices G and G^{-1}. In the case of uniform sampling (i.e. $\alpha = 0$ or, alternatively $t_i = i$, for $i \in \mathbb{Z}$) the number $\|G\|\|G^{-1}\|$ is equal to one, but if we sample nonuniformly, especially when α is close to $1/4$, this term may become large. So, in the case of uniform sampling, the truncated solution f_n may converge faster to the solution f than in the case of nonuniform sampling.

Next we make some remarks on estimates of the truncation error which are given in the literature. The estimates given by Butzer [1] , Butzer and Splettstösser [3] , Butzer, Splettstösser and Stens [4] are valid for functions f which are sampled uniformly. Furthermore f is assumed to lie in the Lipschitz class of order α, given by

$$\{f \in C(\mathbb{R}) | \sup_{|h|<\delta} \|f(.+h) - f(.)\| \leq L\delta^{\alpha}\}.$$

The estimate from Butzer [1] , Lemma 2,

$$\| \sum_{|i|>n} \sqrt{\pi/r} f(i\pi/r) \| \leq \sqrt{2} M_f (r/\pi)^{\gamma-1/2}\ n^{(1-2\gamma)/2}, \tag{4.1}$$

holds for functions f that satisfy the additional estimate

$$|f(t)| \leq M_f\ 1/|t|^{\gamma}, \tag{4.2}$$

for $t \in \mathbb{R} \setminus \{0\}$. This can be proved by straightforward computation. Note that for uniform sampling T^* is the identity operator on $\mathbb{P}_r$ and $G = G^{-1} = Id$, hence condition (3.4) reduces to (4.2) and (3.6) reduces to an error bound which is similar to (4.1). By using *de la Valleé Poussin* kernels, Theorem 6.1. of Butzer and Splettstösser [3] provides the error bound, (if f satisfies (4.2) and if s is such that $t \to t^s f(t)$ belongs to the Lipschitz class of order α)

$$e_{\mathrm{tr}} := \|f(.) - \sum_{i=-n}^{n} f(i\pi/r) \mathrm{sinc}_{\pi}(. - i\pi/r)\| \leq c\ n^{-s-\alpha} \ln n.$$

In Butzer [1] and Butzer, Splettstösser and Stens [4] a similar error is stated for functions f in a special subspace of $L^1(\mathbb{R})$,

$$e_{\mathrm{tr}} = O(n^{(-s-\alpha)}).$$

The truncation error is expressed in terms of its own energy, by Papoulis [9] , p. 142, in the following manner. Define, for $f \in \mathbb{P}_r$,

$$e_{\mathrm{tr}}(t) := f(t) - \sum_{i=-n}^{n} \sqrt{\pi/r} f(i\pi/r) \mathrm{sinc}_{\pi}(t - i\pi/r).$$

Since $e_{\mathrm{tr}} \in \mathbb{P}_r$, it follows by (3.1) that $|e_{\mathrm{tr}}(t)| \leq \|e_{\mathrm{tr}}(.)\|_{L^2}$.

In this paper we obtained a new bound for the truncation error in the case of nonuniform sampling, for functions $f \in \mathbb{P}_r$. We approximated the solution to the moment problem (0.1) and used this procedure to derive the error bound (3.6).

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