Centrum voor Wiskunde en Informatica
Centre for Mathematics and Computer Science

A.B. Olde Daalhuis

Asymptotic expansions of an integral containing a phase function with three saddle points

Department of Analysis, Algebra and Geometry Report AM-R8922 December

The Centre for Mathematics and Computer Science is a research institute of the Stichting Mathematisch Centrum, which was founded on February 11, 1946, as a nonprofit institution aiming at the promotion of mathematics, computer science, and their applications. It is sponsored by the Dutch Government through the Netherlands Organization for the Advancement of Research (N.W.O.).

Asymptotic Expansions of an Integral containing a Phase Function with three Saddle Points

A.B. Olde Daalhuis

Centre for Mathematics and Computer Science
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

New asymptotic expansions are given for a special function of two complex variables. Our approach is based on an integral representation. The phase function of the integral has three saddle points, and for certain combinations of the variables two of these saddle points coalesce.

1980 Mathematics Subject Classification: 41A60, 30E15, 33A40.
Keywords & phrases: uniform asymptotic expansions of integrals, saddle point method, Airy approximation.

1. Introduction

The integral

$$Q(h,\mu)=\frac{1}{\sqrt{\pi}}\int_{-\infty-i\beta}^{\infty-i\beta} e^{-(u-\mu)^2+h/u}\,du \qquad \beta>0 \tag{1.1}$$

plays a role in electrodynamics, radio physics etc. The function $Q(h,\mu)$ cannot be expressed, in a simple way, in terms of the well-known special functions of mathematical physics. In [2] it is shown that $Q(h,\mu)$ can be viewed as a generating function of parabolic cylinder functions:

$$Q\left(\frac{s}{i\sqrt{2}},\frac{z}{i\sqrt{2}}\right)=e^{z^2/4}\sum_{n=0}^{\infty}\frac{s^n}{n!}D_{-n}(z).$$

In [2] several analytical properties are derived, and numerical aspects are considered, especially for complex values of the parameters with μ large and h a constant. In this paper we further investigate the function $Q(h,\mu)$, and we derive new asymptotic expansions for large μ and $h=\gamma\mu^3$, where γ is an uniformity parameter and μ, $h\in\mathbb{R}$.
Substituting $u=\mu t$, we obtain

$$Q(h,\mu)=\frac{\mu}{\sqrt{\pi}}\int_{-\infty-i\beta}^{\infty-i\beta} e^{-\mu^2((t-1)^2-\gamma/t)}\,dt. \tag{1.2}$$

Report AM-R8922
Centre for Mathematics and Computer Science
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

We consider the following cases:

Case 1 : $h > 0$, $\mu > 0$, thus $\gamma > 0$.

Case 2 : $h > 0$, $\mu < 0$, or $h < 0$, $\mu > 0$, thus $\gamma < 0$.

Case 3 : $h < 0$, $\mu < 0$, thus $\gamma > 0$.

Because of $Q(-h,-\mu) = \overline{Q(h,\mu)}$, we only give expansions in case 1 and case 2.
In case 1 we use the saddle point method. It will appear that we have to consider two subcases.
But we also give an expansion in terms of Airy functions which is valid for $\gamma > 0$.
In case 2 we also use the saddle point method, and again we have two subcases.

2. Case 1 : Expansions when both parameters are positive

2.1. Saddle points

We write

$$f(t) = (t-1)^2 - \frac{\gamma}{t}, \qquad \gamma > 0. \tag{2.1.1}$$

This function has three saddle points β_1, β_2 and β_3. By writing

$$g(t) := \tfrac{1}{2}t^2 f'(t) = t^2(t-1) + \tfrac{1}{2}\gamma = (t-\beta_1)(t-\beta_2)(t-\beta_3), \tag{2.1.2}$$

we have

$$\begin{aligned} &\beta_1 + \beta_2 + \beta_3 = 1, \\ &\beta_1\beta_2 + \beta_2\beta_3 + \beta_3\beta_1 = 0, \\ &\beta_1\beta_2\beta_3 = -\tfrac{1}{2}\gamma. \end{aligned} \tag{2.1.3}$$

For the remaining part of the analysis we only use these facts about the saddle points. Because of $g(0) = \frac{1}{2}\gamma > 0$, $\lim_{t\to-\infty} g(t) = -\infty$ and g is real valued and monotonous on $(-\infty, 0]$, it follows that one and only one saddle point is in $(-\infty, 0)$, say

$$\beta_1 \in (-\infty, 0). \tag{2.1.4}$$

Notice that by (2.1.3), it is impossible that $\beta_1 = \beta_2 = \beta_3$, and only when $\gamma = \frac{8}{27}$ two saddle points coalesce: $\beta_2 = \beta_3 = \frac{2}{3}$. Because of this we shall split up this case in two subcases 1) $\gamma \in (0, \frac{8}{27})$ and 2) $\gamma > \frac{8}{27}$.
By considering (2.1.3) it is easy to show that

$$f(\beta_j) = 3\beta_j^2 - 4\beta_j + 1. \tag{2.1.5}$$

2.2. Saddle point method when $\gamma \in (0, \frac{8}{27})$

The condition $\gamma \in (0, \frac{8}{27})$ implies $g(0) > 0$, $g(\frac{2}{3}) < 0$ and $g(1) > 0$. Thus we may conclude

$$\beta_1 < 0 < \beta_2 < \tfrac{2}{3} < \beta_3 < 1. \tag{2.2.1}$$

From (2.1.5) and the graph of f on $\mathbb{R}$ it easily follows that

$$f(\beta_3) < f(\beta_2) < 1 < f(\beta_1). \tag{2.2.2}$$

Thus the main contribution to the integral in (1.2) comes from β_3. The steepest descent curve we consider is given by

$$\rho(t)=\begin{cases} t & t\in(-\infty,\beta_1]\cup[\beta_2,\infty),\\ t-i\sqrt{\frac{g(t)}{1-t}} & t\in[\beta_1,\beta_2].\end{cases} \tag{2.2.3}$$

See Figure 2.2.1.

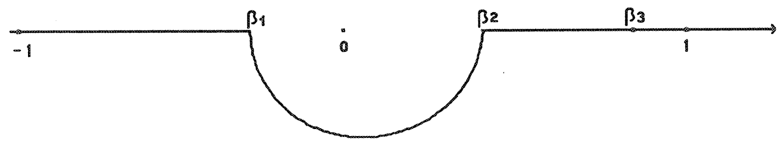

FIGURE 2.2.1. Steepest descent curve for the case $0<\gamma<\frac{8}{27}$.

On this path, f is real valued and f attains its minimum at β_3. From OLVER [3, P. 127] and (2.1.5) we obtain

$$Q(h,\mu)\sim\frac{2}{\sqrt{\pi}}\mu e^{-\mu^2(3\beta_3^2-4\beta_3+1)}\sum_{s=0}^{\infty}\Gamma(s+\tfrac{1}{2})\frac{a_{2s}}{\mu^{2s+1}},\qquad \mu\to\infty. \tag{2.2.4}$$

Formulas for the first three coefficients are

$$\begin{aligned} a_0&=\frac{1}{2}\frac{\beta_3^{\frac{1}{2}}}{(3\beta_3-2)^{\frac{1}{2}}}, \qquad a_2=-\frac{3}{4}\frac{(\beta_3-1)(\beta_3+1)}{\beta_3^{\frac{1}{2}}(3\beta_3-2)^{\frac{7}{2}}},\\ a_4&=\frac{5}{16}\frac{(\beta_3-1)(15\beta_3^3-9\beta_3^2-15\beta_3+1)}{\beta_3^{\frac{3}{2}}(3\beta_3-2)^{\frac{13}{2}}}. \end{aligned} \tag{2.2.5}$$

Notice that $3\beta_3^2-4\beta_3+1<0$ and that the coefficients have a singularity at $\beta_3=\frac{2}{3}$, which is the coalescing position of the saddle points β_2 and β_3 as $\gamma\to\frac{8}{27}$. Also, notice that the coefficients a_{2s} $(s\geq 1)$ all vanish when $\beta_3=1$ (i.e., $\gamma=0$). Expansion (2.2.4) is valid as $\mu\to\infty$, uniformly with respect to $\gamma\in[0,\gamma_0]$, where $0<\gamma_0<\frac{8}{27}$, γ_0 fixed.

2.3. Saddle point method when $\gamma>\frac{8}{27}$

If we take $\gamma>\frac{8}{27}$, then f has only one real saddle point. Thus we have $\beta_2,\beta_3\notin\mathbb{R}$, and by (2.1.3) we also have $\beta_2+\beta_3,\ \beta_2\beta_3\in\mathbb{R}$. Thus

$$\beta_2=\overline{\beta}_3,\qquad \beta_1\in(-\infty,0). \tag{2.3.1}$$

We write $\beta_2=x+iy$ and $\beta_3=x-iy$. Then (2.1.3) gives

$$\begin{aligned}\beta_1+2x&=1,\\ 2x\beta_1+x^2+y^2&=0,\\ \beta_1(x^2+y^2)&=-\tfrac{1}{2}\gamma.\end{aligned} \tag{2.3.2}$$

So we obtain

$$\begin{aligned} \beta_1 &= 1 - 2x, \\ y^2 &= 3x^2 - 2x. \end{aligned} \tag{2.3.3}$$

It follows that $x > \frac{2}{3}$ and $\beta_1 < -\frac{1}{3}$. Moreover, by (2.1.5), we have

$$\begin{aligned} f(\beta_1) &= 12x^2 - 4x, \\ f(\beta_{2,3}) &= -6x^2 + 2x + 1 \pm iy(6x - 4). \end{aligned} \tag{2.3.4}$$

Since $f(\beta_1) > Re(f(\beta_{2,3}))$, the main contribution to the integral in (1.2) comes from β_2 and β_3. A steepest descent path through β_3 is given by

$$\{p + iq \in \mathbb{C} \mid p, q \in \mathbb{R} \mid q(2(p-1) + \frac{\gamma}{p^2 + q^2}) + y(6x - 4) = 0\}. \tag{2.3.5}$$

See Figure 2.3.1.

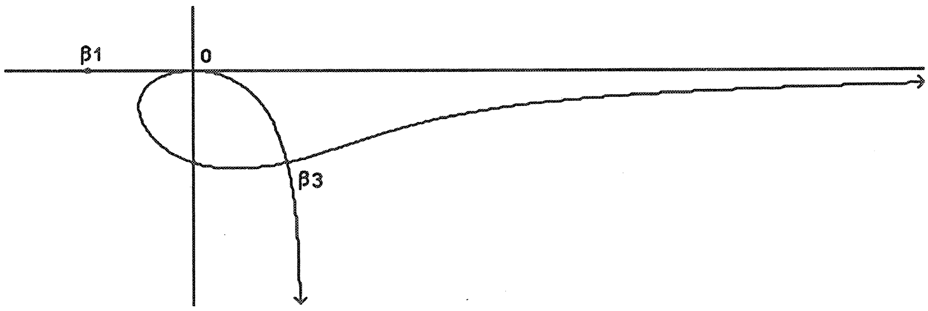

FIGURE 2.3.1. Steepest descent curve through β_3, for the case $\frac{8}{27} < \gamma < \infty$.

A possible integration curve $\tilde{\rho}$ is given in Figure 2.3.2. It exists of a part of the steepest descent path through β_3, and of the negative real axis. The main contribution comes from the steepest descent path through β_3.

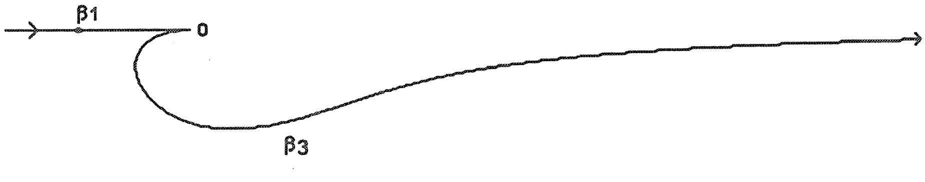

FIGURE 2.3.2. Contour of integration for the case $\frac{8}{27} < \gamma < \infty$.

Thus, from OLVER [3, P. 127] and (2.3.4), we obtain

$$Q(h, \mu) \sim \frac{2}{\sqrt{\pi}} \mu e^{-\mu^2(-6x^2+2x+1)+i\mu^2(y(6x-4))} \sum_{s=0}^{\infty} \Gamma(s + \tfrac{1}{2}) \frac{a_{2s}}{\mu^{2s+1}}, \qquad \mu \to \infty, \tag{2.3.6}$$

with a_{2s} as in (2.2.4). This expansion is valid as $\mu \to \infty$, uniformly with respect to $\gamma \in [\gamma_1, \infty)$, where $\gamma_1 > \frac{8}{27}$, γ_1 fixed.

2.4. Airy function expansion

When γ ranges over $(0, \infty)$, f has two coalescing saddle points at $\gamma = \frac{8}{27}$. We use the following transformation, suggested by CHESTER, FRIEDMAN and URSELL [1]

$$(t-1)^2 - \frac{\gamma}{t} = -\tfrac{1}{3}s^3 + \alpha s + c. \tag{2.4.1}$$

We prescribe that the saddle point $t = \beta_3$ must correspond with $s = -\sqrt{\alpha}$, and $t = \beta_2$ must correspond with $s = \sqrt{\alpha}$. It follows that

$$\begin{aligned} c &= \tfrac{1}{2}(1 + 4\beta_1 - 3\beta_1^2), \\ \alpha^{\frac{3}{2}} &= \tfrac{9}{4}(\beta_2^2 - \beta_3^2) - 3(\beta_2 - \beta_3). \end{aligned} \tag{2.4.2}$$

The quantity α can be viewed as an analytic function of γ, $\gamma > 0$. Mapping (2.4.1) is not conformal at the remaining saddle point $t = \beta_1$, and at $t = 0$. Thus possible s–integration curves must stay away from these singularities. It is easy to prove that this is possible for all $\gamma > 0$. The curves in the s–domain, which correspond with ρ and $\widetilde{\rho}$, begin at $s = -\infty$ and end at $s = e^{-\frac{\pi i}{3}}\infty$. So we have

$$Q(h,\mu) = \frac{\mu}{\sqrt{\pi}} \int_{-\infty}^{\infty e^{-\frac{\pi i}{3}}} e^{\mu^2(\frac{1}{3}s^3 - \alpha s - c)} k(s)ds, \tag{2.4.3}$$

where

$$k(s) = \frac{dt}{ds} = -\tfrac{1}{2}t^2\left(\frac{s-\sqrt{\alpha}}{t-\beta_2}\right)\left(\frac{s+\sqrt{\alpha}}{t-\beta_3}\right)\left(\frac{1}{t-\beta_1}\right). \tag{2.4.4}$$

Starting from (2.4.3), we write

$$k(s) = k_0(s) = a_0 + b_0 s + (s^2 - \alpha)l_0(s), \tag{2.4.5}$$

where a_0, b_0 and l_0 are to be determined. We now substitute (2.4.5) in (2.4.3). An integration by parts yields

$$\begin{aligned} \frac{e^{-\frac{2\pi i}{3}}}{2\pi i} \int_{-\infty}^{\infty e^{-\frac{\pi i}{3}}} e^{\mu^2(\frac{1}{3}s^3 - \alpha s)} k_0(s)ds =& Ai_{-1}(\mu^{\frac{4}{3}}\alpha)a_0\mu^{-\frac{2}{3}} - Ai'_{-1}(\mu^{\frac{4}{3}}\alpha)b_0\mu^{-\frac{4}{3}} \\ & - \mu^{-2}\frac{e^{-\frac{2\pi i}{3}}}{2\pi i} \int_{-\infty}^{\infty e^{-\frac{\pi i}{3}}} e^{\mu^2(\frac{1}{3}s^3 - \alpha s)} k_1(s)ds, \end{aligned} \tag{2.4.6}$$

where $k_1(s) = l_0'(s)$ and $Ai_{-1}(z) = Ai(e^{\frac{2\pi i}{3}}z)$. The above procedure can be repeated, and we obtain

$$\begin{aligned} Q(h,\mu) =& 2\pi i \frac{\mu}{\sqrt{\pi}} e^{\frac{2\pi i}{3} - \mu^2 c} Ai_{-1}(\mu^{\frac{4}{3}}\alpha) \sum_{n=0}^{m-1} (-1)^n a_n \mu^{-2n-\frac{2}{3}} \\ & - 2\pi i \frac{\mu}{\sqrt{\pi}} e^{\frac{2\pi i}{3} - \mu^2 c} Ai'_{-1}(\mu^{\frac{4}{3}}\alpha) \sum_{n=0}^{m-1} (-1)^n b_n \mu^{-2n-\frac{4}{3}} \\ & + (-1)^m \frac{1}{\sqrt{\pi}} \mu^{1-2m} e^{-\mu^2 c} \int_{-\infty}^{\infty e^{-\frac{\pi i}{3}}} e^{\mu^2(\frac{1}{3}s^3 - \alpha s)} k_m(s)ds, \end{aligned} \tag{2.4.7}$$

where

$$k_m(s) = a_m + b_m s + (s^2 - \alpha) l_m(s), \qquad k_{m+1}(s) = l'_m(s), \tag{2.4.8}$$

and

$$\begin{aligned} a_m &= \tfrac{1}{2}(k_m(\sqrt{\alpha}) + k_m(-\sqrt{\alpha})), \\ b_m &= \frac{k_m(\sqrt{\alpha}) - k_m(-\sqrt{\alpha})}{2\sqrt{\alpha}}. \end{aligned} \tag{2.4.9}$$

It can be shown that k_m, l_m, and hence a_m, b_m, are analytic functions of α, and hence of γ, $\gamma > 0$. Formulas for the first two coefficients are

$$\begin{aligned} a_0 &= -\tfrac{1}{2}\Big(\frac{3}{4(1-\beta_1)}\Big)^{\frac{1}{6}}\Big(\frac{\beta_2}{\sqrt{\beta_2-\beta_1}} + \frac{\beta_3}{\sqrt{\beta_3-\beta_1}}\Big), \\ b_0 &= \tfrac{1}{2}\Big(\frac{4(1-\beta_1)}{3}\Big)^{\frac{1}{6}}\frac{1}{\sqrt{\beta_3-\beta_1}}\Big(1 - \frac{\beta_2}{(\beta_2-\beta_1) + \sqrt{\beta_2-\beta_1}\sqrt{\beta_3-\beta_1}}\Big). \end{aligned} \tag{2.4.10}$$

The expansion (2.4.7) is valid as $\mu \to \infty$, uniformly with respect to γ, for γ in compact subsets of $(0, \infty)$, and possibly for $\gamma \geq 0$.

3. Case 2: some remarks

Because of $\gamma < 0$, we have $g(1) < 0$, $\lim_{t\to\infty} g(t) = \infty$ and g is real valued and monotonous on $[1, \infty)$. Thus we may conclude that one and only one saddle point is in $(1, \infty)$, say $\beta_3 \in (1, \infty)$. It is easy to verify that g does not have zeros on $(-\infty, 1]$. However (2.1.3) yields $\beta_1 + \beta_2 < 0$ and $\beta_1\beta_2 > 0$. Consequently

$$\beta_3 \in (1, \infty), \qquad \beta_1 = \overline{\beta}_2, \qquad \beta_1, \beta_2 \notin \mathbb{R}. \tag{3.1}$$

We write again $\beta_1 = x + iy$, $\beta_2 = x - iy$ and obtain

$$\begin{aligned} f(\beta_{1,2}) &= -6x^2 + 2x + 1 \pm iy(6x - 4), \\ f(\beta_3) &= 12x^2 - 4x, \\ x &< 0. \end{aligned} \tag{3.2}$$

For determining the dominant saddle point we consider the equation $Re(f(\beta_{1,2})) = f(\beta_3)$. This gives the changing point $\gamma_0 = \frac{1}{6} - \frac{1}{6}\sqrt{3}$. In subcase 1, that is $\gamma \in (\gamma_0, 0)$, β_3 is dominant, whereas in subcase 2, that is $\gamma \in (-\infty, \gamma_0)$, β_1 and β_2 are dominant.
Subcase 1 goes like $\gamma \in (0, \frac{8}{27})$, and gives expansion (2.2.4), which is valid uniformly with respect to $\gamma \in [\gamma_0, 0]$.
Subcase 2 goes like $\gamma \in (\frac{8}{27}, \infty)$, and gives expansion (2.3.6), which is valid uniformly with respect to $\gamma \in [\gamma_1, \gamma_0]$, where $\gamma_1 < \gamma_0$, γ_1 fixed.

Observe that for small values of γ two saddle points, β_1 and β_2, are close to the origin. For a treatment of that asymptotic phenomenon a Bessel function may be used. However, when $|\gamma|$ is small, the dominant saddle point is β_3. Therefore it is not needed to consider this particular case of coalescing saddle points near a pole of $f(t)$.

ACKNOWLEDGEMENT I gratefully thank Dr. N. M. Temme for the great number of valuable comments and his thorough reading of the manuscript.

REFERENCES

1. C. CHESTER, B. FRIEDMAN, AND F. URSELL, (1957). *An extension of the method of steepest descent,* Proc. Cambridge Philos. Soc., 53, pp. 599-611.

2. A.M. GLUTSYUK, M.K. KERIMOV, AND N.N. REZNIK, (1986). *Computation of a special function of two complex variables,* U.S.S.R. Comput. Maths. Math. Phys., Vol. 26, No. 6, pp. 107-117.

3. F.W.J. OLVER, (1974). *Asymptotics and Special Functions,* Academic Press, New York, 1974.