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Graph Morphology

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This paper presents a systematic theory for the construction of morphological operators on graphs. Graph morphology extracts structural information from graphs using predefined test probes called structuring graphs. Structuring graphs have a simple structure and are relatively small compared to the graph that is to be transformed. A neighbourhood function on the set of vertices of a graph is constructed by relating individual vertices to each other whenever they belong to a local instantiation of the structuring graph. This function is used to construct dilations and erosions. The concept of structuring graph is also used to define openings and closings. The resulting morphological operators are invariant under symmetries of the graph. Graph morphology resembles classical morphology (which uses structuring elements to obtain translation-invariant operators) to a large extent. However, not all results from classical morphology have analogues in graph morphology because the local graph structure may be different at different vertices.

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1. Introduction

In many fields of research (e.g. geography, histology, robotics) the objects of interest and their interrelations are represented by a graph. In a graph the vertices correspond to objects and the edges

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represent the relations between objects. Objects may be characterized by a finite number of (numeric or symbolic) parameters such as the average grey-value, area, colour, temperature, shape etc. The relation between objects generally corresponds to some kind of distance or (dis-) similarity measure. The objects themselves may also have a graph representation describing their primitive components and their spatial layout together with their local and global (i.e. intrinsic) topological properties of invariance with respect to a set of transformations. The structure of a complex object can be defined as the "*group of all interrelations among its subparts*" (see Matheron [9]). None of these interrelations is intrinsically more interesting than the others. The interrelations between objects (and/or their component parts) that are of interest depend on the property that is being studied. Thus, the choice of the type of information that is to be extracted from the image constitutes or frames the objects and their interrelations. Note that images can also be considered as graphs when the topology (connectivity) of the support grid is taken into account.

Mathematical morphology is a set-theoretic approach to image processing and analysis. It considers images to be sets in the underlying support space and manipulates them using set-based operators. Consistent with its literal meaning, the purpose of mathematical morphology is the quantitative description of geometrical structures. Unlike purely statistical approaches to image processing which ignore spatial relationships between picture elements and which do not provide information about shape, mathematical morphology uses small probes with well defined shapes, called structuring elements, to extract specific shape information from images. Individual elements of the support grid are related to each other spatially whenever they belong to a local instantiation of the structuring element. This approach is based on the *logical* relations between image features rather than arithmetic ones. Different types of information can be extracted from the signal by varying the shape and size of the structuring element.

Mathematical morphology was originally developed for binary imagery (for which set operations are the most common). However, it has recently been extended to arbitrary complete lattices [14,8,12]. The space of (binary or multilevel) functions on a graph whose vertices are given numerical (or symbolic) attributes can be related to a special kind of complete lattice. Hence graphs are also amenable to morphological transformations. In this case the structuring element is a graph that is relatively small and simple in comparison to the graph that is to be transformed. This is in contrast with Euclidean mathematical morphology where the structuring element is merely a set of points that has no additional (internal) structure.

In graph morphology individual vertices of a graph are related to each other whenever they belong to a local instantiation of the structuring graph. Thus the neighbourhood relations between the vertices of a graph induce a large collection of morphological transformations. These transformations have a straightforward geometrical interpretation (e.g. erosions, dilations, openings, closings, skeletons etc.). Recently, Vincent [21,18,20,19] initiated the extension of morphology to graphs. By using simple neighbourhood functions he restricted his study to a subclass of morphological transformations. The collection of morphological transformations on graphs is considerably extended by the introduction of more general neighbourhood functions. This extension is essential if specific structures (e.g. objects) are to be detected within a graph (subgraph isomorphisms). The aim of the present study is to initiate the development of a general theory of morphology on graphs.

Object recognition in graphs generally involves the detection of local instantiations of the object's graph representation. This can be done by placing the (relatively small and simple) graph representation of the object at every vertex of the graph and monitoring the (partial) matches. Procedures like these are intricate and computationally expensive. In graph morphology objects can be defined as the invariants of morphological filters. Thus, it seems interesting to use graph morphological transformations for object recognition.

Just like images, graphs come in several different types, depending on the range of values that the attributes (labels) of the vertices (pixels) can adopt. In this study we will consider *multilevel graphs* (also called labelled graphs in the literature). In these graphs each vertex is labeled with one of n (usually 2^k) values which are strictly ordered. A *binary graph* is a special case of a multilevel graph where $n = 2$. Thus, each vertex in a binary graph can only have one of two values (usually 0 and 1).

This paper is organized as follows. Section 2 briefly reviews the theory of mathematical morphology on complete lattices. The extension of mathematical morphology to graphs is presented in Section 3. Section 4 introduces the concept of structuring graph and shows how this object defines a neighbourhood for every vertex in a graph. Dilation and erosion of a graph with a structuring graph are defined in Section 5. Here we also examine some basic properties of these transformations. Section 6 shows how one can build openings and closings using the concept of a structuring graph. Some notes about the implementation of morphological operators on graphs can be found in Section 7. Finally, in Section 8, some concluding remarks have been made.

2. Mathematical morphology on complete lattices

Mathematical morphology was originally developed for binary images or, equivalently, for sets. The basic principles of this framework are set inclusion, intersection, union and translation. These concepts can be formalized by the notion of a complete lattice and invariance under an automorphism group of the lattice. To a certain extent classical morphology can be generalized to this abstract framework. Such a generalization was initiated by Matheron and Serra [14] and further developed by Heijmans and Ronse [8,12; see also 15]. Here we recall elements of this abstract theory which we will use in the sequel. For a comprehensive overview of the theory of complete lattices we refer to the monograph of Birkhoff [4]

Definition 2.1. A complete lattice consists of a set $\mathcal{L}$ and a partial order relation $\leq$ on $\mathcal{L}$ with the following properties. For each collection $X_i \in \mathcal{L}$, $i \in I$ there exist two elements $\overline{X}, \underline{X} \in \mathcal{L}$ such that:

(i) for each $i \in I$ we have $\underline{X} \leq X_i \leq \overline{X}$.

(ii) $X_i \leq Y$ for each $i \in I$ implies $\overline{X} \leq Y$, for all $Y \in \mathcal{L}$.

(iii) $Z \leq X_i$ for each $i \in I$ implies $Z \leq \underline{X}$, for all $Z \in \mathcal{L}$.

$\overline{X}$ and $\underline{X}$ are respectively called the *supremum* and the *infimum* of the family X_i, and are denoted as $\overline{X} = \bigvee_{i\in I} X_i$ and $\underline{X} = \bigwedge_{i\in I} X_i$

As mentioned before, an important example of a complete lattice is the power set $\mathcal{P}(V)$ of some set V, with the set inclusion as the order relation, and union and intersection as the supremum and infimum respectively. A second example which plays a prominent role in mathematical morphology is the space of functions mapping some underlying space V into a set $\mathcal{G}$ of grey-levels. The only requirement is that $\mathcal{G}$ is a complete lattice. This function space is denoted by $\mathcal{G}^V$ and its partial order relation is given by $f \le g$ if $f(x) \le g(x)$ for every $x \in V$. The supremum and infimum of a collection of functions are then obtained by taking the pointwise supremum and infimum.

The building blocks of mathematical morphology are dilation and erosion. These two notions are closely related and are defined on a complete lattice $\mathcal{L}$ as follows:

Definition 2.2. A mapping $\delta : \mathcal{L} \to \mathcal{L}$ is called a *dilation* if $\delta(\bigvee_{i\in I} X_i) = \bigvee_{i\in I} \delta(X_i)$ for any collection $X_i \in \mathcal{L}$, $(i \in I)$. A mapping $\varepsilon : \mathcal{L} \to \mathcal{L}$ is called an *erosion* if $\varepsilon(\bigwedge_{i\in I} X_i) = \bigwedge_{i\in I} \varepsilon(X_i)$.

Dilations and erosions are increasing mappings. Recall that a mapping ψ on $\mathcal{L}$ is called *increasing* if $X \le Y$ implies $\psi(X) \le \psi(Y)$ for any pair $X, Y \in \mathcal{L}$. Note the symmetry of these two definitions. We define the dual or opposite of a complete lattice $(\mathcal{L}, \le)$ as the lattice $(\mathcal{L}', \le')$ with the opposite ordering, i.e. $X \le' Y$ if $X \ge Y$. A mapping $\psi : \mathcal{L} \to \mathcal{L}$ which induces a dilation on $\mathcal{L}$ similarly induces an erosion on $\mathcal{L}'$. Dilation and erosion are therefore dual notions.

Another important relation between dilations and erosions is the following. Let δ and ε be two mappings on a complete lattice $\mathcal{L}$. The pair (ε, δ) is called an *adjunction* if

$$\delta(X) \le Y \iff X \le \varepsilon(Y)$$

for all $X, Y \in \mathcal{L}$. If (ε, δ) is an adjunction it can be shown that δ is a dilation and ε is an erosion. Furthermore, to every dilation δ there corresponds a unique erosion ε such that (ε, δ) is an adjunction and vice versa. In this case ε and and δ are called *adjoints*. Let $\mathcal{L}$ be the Boolean lattice $\mathcal{P}(V)$ for some set V. A dilation δ is uniquely characterized by a mapping $N : V \to \mathcal{P}(V)$ in the sense that δ can be represented by $\delta(X) = \bigcup_{x\in V} N(x)$. Notice that one can obtain N from δ by defining $N(x) = \delta(\{x\})$. N is called a *neighbourhood function*. To a large extent this paper will be concerned with the construction of neighbourhood functions on the set of vertices of a graph.

A mapping $\psi : \mathcal{L} \to \mathcal{L}$ is called a *filter* if ψ is increasing and idempotent, i.e., $\psi = \psi \circ \psi$. Two important types of filters are the anti-extensive ones ($\psi(X) \le X$ for all $X \in \mathcal{L}$) which are called *openings* and the extensive ones ($X \le \psi(X)$ for all $X \in \mathcal{L}$) which are called *closings*. Just like dilations and erosions, openings and closings are dual notions. Futhermore, if (ε, δ) is an adjunction, then $\varepsilon\delta$ is a closing and $\delta\varepsilon$ is an opening.

3. Morphology on graphs: a summary of Vincent's work

Hereafter the term "graph" will indicate an undirected graph without loops and multiple edges (also called 1-graphs), unless it is stated otherwise. For a comprehensive exposition on graphs we refer to the monograph of Berge [2]. We denote the *vertices* of a graph G by $V = V(G)$ and its *edges* by $E = E(G)$ and we write $G = (V, E)$. An edge between two vertices v, w is denoted by (v, w). The assumption that G is undirected is made explicit by the equality $(v, w) = (w, v)$ for every edge (v, w) in E. We call w a *1-neighbour* of v if (v, w) is an edge. Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. G will be called a *subgraph* of G' if $V \subseteq V'$ and $E \subseteq E'$. Note that the word 'subgraph' is generally used in a more restricted sense [2]. A mapping $\theta : V \to V'$ will be called a *homomorphism* from G to G' if θ is one-to-one and $(v, w) \in E$ implies that $(\theta(v), \theta(w)) \in E'$. This will be denoted as $G \xrightarrow{\theta} G'$. We often write θv instead of $\theta(v)$. G and G' will be called *homomorphic* if there exists a homomorphism from G to G'. This will be denoted by $G \mathrel{\tilde{\subset}} G'$. A homomorphism from G to G' which is onto (and is therefore a bijection) is called an *isomorphism*. The graphs G and G' are called *isomorphic* if they are related by an isomorphism. This will be denoted by $G \simeq G'$. An isomorphism from the graph G to itself is called an *automorphism* or a *symmetry* of G. The collection of all symmetries of a graph G, denoted by $\mathrm{Sym}(G)$, forms a group called the *symmetry group of* G. The identity mapping id which maps every vertex onto itself is always contained in $\mathrm{Sym}(G)$ and is called the *trivial symmetry of* G. The trivial symmetry is often the only element of $\mathrm{Sym}(G)$ (see Figure 1).

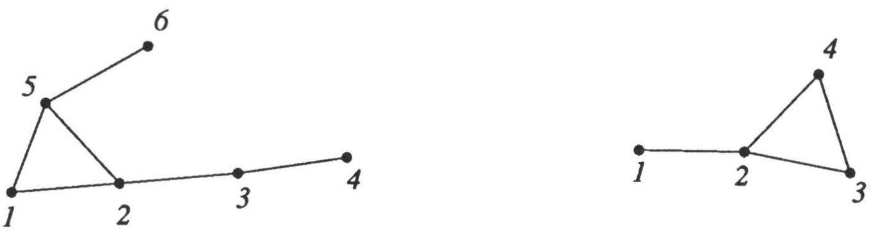

FIGURE 1. *The left graph only has the trivial symmetry. The symmetry group of the right graph contains (besides* id*) the mapping which interchanges the vertices* 3 *and* 4.

A graph is a mathematical structure that has a multitude of applications, e.g., cluster analysis and route planning. In this paper we study binary and multilevel (= grey-level) functions defined on the set of vertices of this type of graphs. Thus we assume that a graph representation of an image is available and we will not address any aspects of image modelling. For reasons of simplicity we assume that there are only finitely many grey-levels, say $0, 1, \ldots, n-1$. A graph is called binary when $n = 2$. For a graph $G = (V, E)$, $\mathcal{L}_n(G)$ denotes the space of all functions mapping V into $\{0, 1, \ldots, n-1\}$. The elements of $\mathcal{L}_n(G)$ are called *multilevel graphs* when $n > 1$. They are denoted by $(f \mid G)$, where G denotes the underlying graph. The underlying graph will not be indicated when its identity is unambiguous. The elements of $\mathcal{L}_2(G)$ are called *binary graphs*. They can be represented by subsets of the vertex set V. We write $\mathcal{L}_2(G)$ as $\mathcal{L}(G)$. Thus we have $\mathcal{L}(G) = \mathcal{P}(V)$. The elements of this space will be denoted by $(X \mid G)$. When there is no ambiguity we will omit the argument G. Let, for any graph G, $\psi(\cdot \mid G)$ represent a mapping from $\mathcal{L}_n(G)$ into itself. Then ψ is called a *multilevel graph operator* if $n > 2$ and a *binary graph operator* if $n = 2$. The multilevel graph operator ψ is called increasing if $\psi(f \mid G) \leq \psi(f' \mid G)$ for any graph G and any two multilevel graphs $f, f' \in \mathcal{L}_n(G)$

for which $f \leq f'$. Increasingness for binary graph operators is defined analogously[1]. In classical morphology (on the continuous Euclidean space $\mathbb{R}^d$ or the discrete space $\mathbb{Z}^d$) it is well-known that any increasing operator on binary images can be extended to grey-level images by thresholding the original image at all grey-levels. The resulting operator on grey-level images is also increasing and is called a *flat operator* [7,13,15]. The same method can be used to extend increasing operators on binary graphs to multilevel graphs. We only give a brief sketch of the main ideas and refer to [7] for an extensive account. Every multilevel graph $(f \mid G) \in \mathcal{L}_n(G)$ corresponds in a unique manner to a sequence $(X_1(f) \mid G), (X_2(f) \mid G), \ldots, (X_{n-1}(f) \mid G)$ of binary graphs such that $X_1(f) \supseteq X_2(f) \supseteq \ldots \supseteq X_{n-1}(f)$. Here $X_i(f) = \{v \in V : f(v) \geq i\}$. From this sequence f can be recovered by means of the identity

$$f(v) = \max\{i = 1, \ldots, n-1 : v \in X_i(f)\},$$

where the maximum of the empty set is defined to be zero. Let $\psi(\cdot \mid G)$ represent an increasing binary graph operator and let $(f \mid G)$ be a multilevel graph. We define $\psi(f \mid G)$ as the function corresponding with the sequence

$$\psi(X_1(f) \mid G) \supseteq \psi(X_2(f) \mid G) \supseteq \ldots \supseteq \psi(X_{n-1}(f) \mid G).$$

Note that the nonincreasingness of this sequence follows from the increasingness of ψ. Alternatively we may define

$$\psi(f \mid G)(v) = \max\{i = 1, \ldots, n-1 : v \in \psi(X_i(f) \mid G)\}.$$

In this paper we only consider increasing binary graph operators and their extension to multilevel graphs.

A graph operator ψ will be called *G-increasing* if ψ increases in G, that is, $\psi(X \mid G) \subseteq \psi(X \mid G')$ for $G \subseteq G'$ and $X \subseteq V(G)$. G-decreasingness of ψ is defined analogously.

For a binary graph operator ψ the dual operator ψ^* is defined by

$$\psi^*(X \mid G) = \big(\psi(X^* \mid G)\big)^*$$

where $X^* = V \setminus X$ represents the complement of X with respect to the vertex set V. For $(X^* \mid G)$ we also write $(X \mid G)^*$, thereby indicating that complementation does not affect the graph G.

Let $G = (V, E)$ represent an arbitrary graph and let $N_1(v)$ be the neighbourhood of vertex v defined as

$$N_1(v) = \{w \in V : (v, w) \in E\} \cup \{v\}.$$

We define the dilation δ and the erosion ε on $\mathcal{L} = \mathcal{P}(V)$ respectively by

$$\begin{aligned} \delta_1(X \mid G) &= \bigcup_{v \in X} N_1(v) \\ \varepsilon_1(X \mid G) &= \{v \in V : N_1(v) \subseteq X\}. \end{aligned} \tag{3.1}$$

[1] Throughout this paper we write $\psi(X \mid G)$ instead of $\psi((X \mid G) \mid G)$.

One may easily verify that $(\varepsilon_1, \delta_1)$ defines an adjunction on $\mathcal{L}$. An alternative expression for δ_1 is

$$\delta_1(X \mid G) = \{v \in V : N_1(v) \cap X \neq \emptyset\}. \tag{3.2}$$

This relation is valid because the 1-neighbourhood is symmetric in the sense that $v \in N_1(w)$ iff $w \in N_1(v)$. In Figure 2 we have depicted an example. The black and white vertices represent respectively X and its complement.

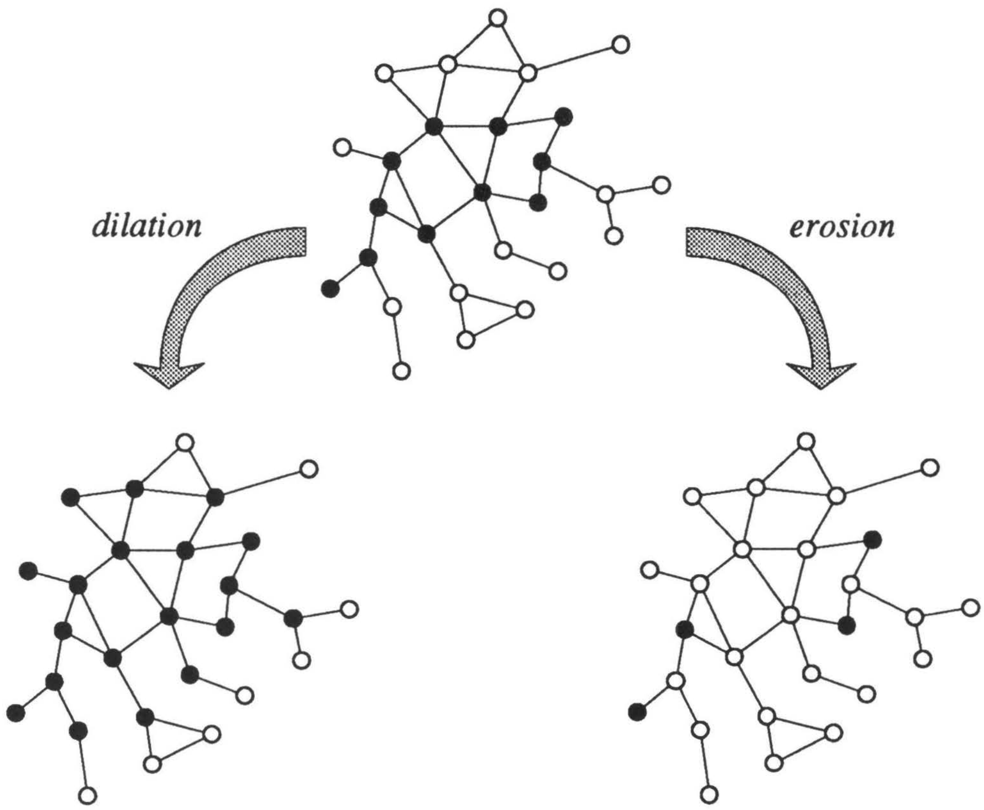

FIGURE 2. *The original binary graph X, its dilation $\delta_1(X)$ and its erosion $\varepsilon_1(X)$.*

Because ε and δ are adjoints the composed mappings $\alpha = \delta\varepsilon$ and $\phi = \varepsilon\delta$ are respectively an opening and a closing. Their action is depicted in Figure 3. Again α and ϕ are dual notions in the sense that opening the foreground X leads to the same result as closing the background X^* and vice versa. It is easy to show that $v \in \alpha(X)$ if v has a neighbour w all of whose neighbours lie in X, that is, $N(w) \subseteq X$.

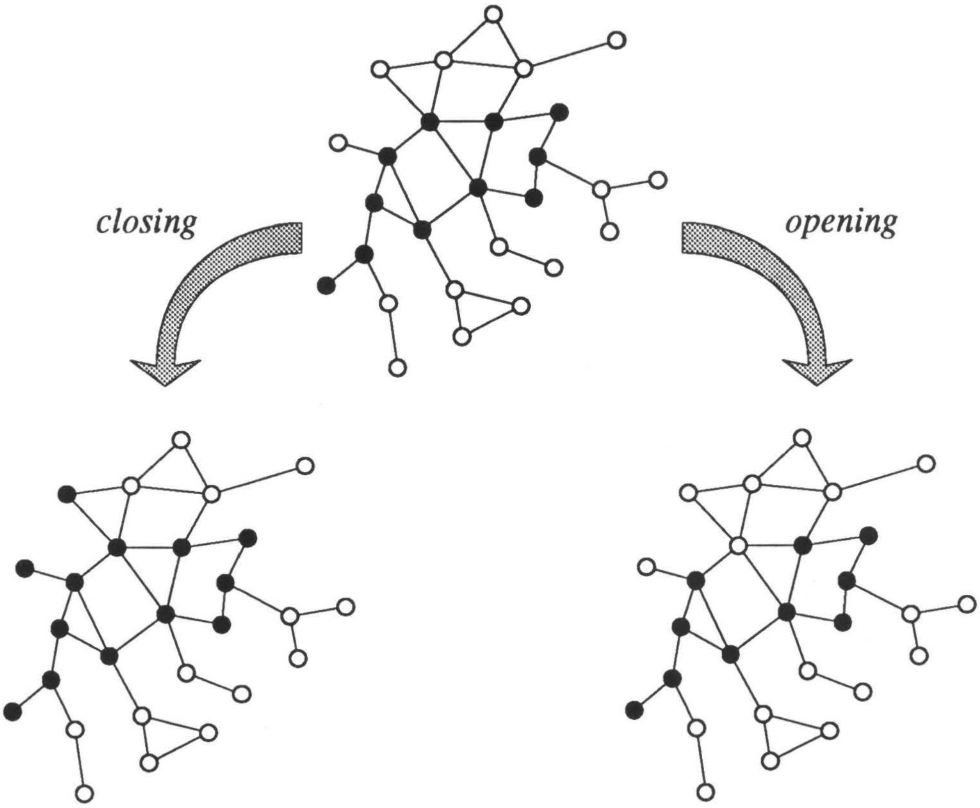

FIGURE 3. *From left to right: the original binary graph X, its closing $\phi(X)$ and its opening $\alpha(X)$.*

It is obvious that δ is G-increasing and that ε is G-decreasing. However, the example in Figure 4 clearly illustrates that α and ϕ are neither G-increasing nor G-decreasing. In this case G is a subgraph of both G' and G''. However, $\alpha(X \mid G')$ is smaller than $\alpha(X \mid G)$ whereas $\alpha(X \mid G'')$ is larger than $\alpha(X \mid G)$. In Section 6 we will discuss structural openings which have the property that they are

G-increasing.

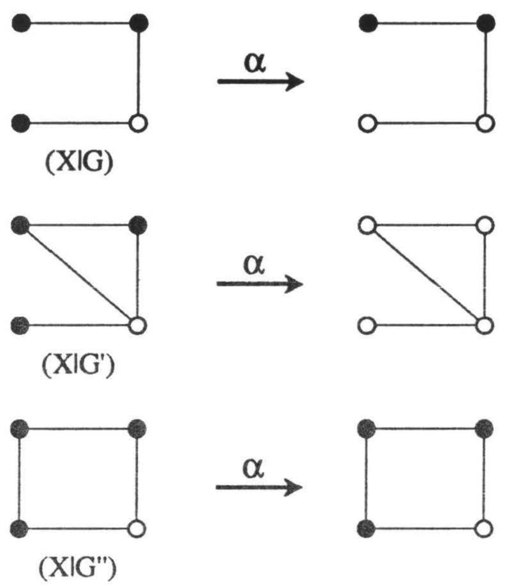

FIGURE 4. *The opening α is neither G-increasing nor G-decreasing: see text.*

Vincent [18,20,21] has shown that many of the classical morphological transformations (distance function, geodesic operators, skeleton, watersheds, etc.) can be extended to binary or multilevel graphs.

4. Neighbourhoods and structuring graphs

The operators discussed in the previous section are not restricted to a particular graph but can be applied to any binary graph. Furthermore these operators are invariant under the symmetries of the graph, i.e.,

$$\tau\delta = \delta\tau.$$

Here the symmetry τ is considered as an operator on $\mathcal{P}(V)$ defined by $\tau X = \{\tau v : v \in X\}$. In this paper we explain how to construct systematically a large family of dilations, erosions, openings and closings with the symmetry-preserving property. The underlying idea is borrowed from classical morphology where one probes the image with "simple" geometrical shapes called *structuring elements.* The resulting operators can be obtained by translation of the original image in combination with logical operands such as **AND** and **OR**. Here we introduce the concept of a *structuring graph* or S-graph, which is a graph with only a small number of vertices and edges and some additional structure. The idea is to construct morphological operators by 'matching' the S-graph at different positions with the binary graph that is to be processed. (N.B. In graph theory the word 'matching' is generally used in a different sense.) The implementation of the resulting operators is is an intricate task that differs considerably from the classical case. Section 7 discusses some aspects of the implementation process. In this section we present a formal definition of an S-graph and show that this object determines a neighbourhood for every vertex in a graph.

A structuring graph (or briefly, S-graph) $\mathcal{A}$ is a graph $G_{\mathcal{A}} = (V_{\mathcal{A}}, E_{\mathcal{A}})$ together with two non-empty subsets $B_{\mathcal{A}}, R_{\mathcal{A}} \subseteq V_{\mathcal{A}}$ called the *buds* and *roots* respectively. An example of a structuring graph is given in Figure 5.

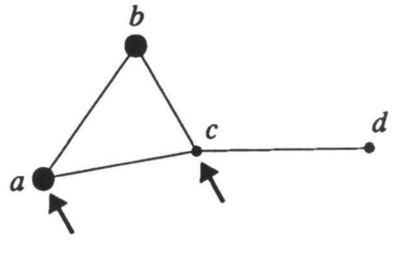

FIGURE 5. *Example of a structuring graph with four vertices $\{a,b,c,d\}$ and four edges. The buds a and b are denoted by black dots and the roots a and c by arrows.*

Note that buds and roots may coincide and that the graph $G_{\mathcal{A}}$ need not be connected. In the sequel $\mathcal{A}$ indicates an S-graph, unless it is stated otherwise.

Let $\mathcal{A}$ be an S-graph and let $G = (V, E)$ be an arbitrary graph. Let θ be a homomorphism from $G_{\mathcal{A}}$ to G. Finally, let $v \in V$. We call θ an embedding of $\mathcal{A}$ into G at v if $v \in \theta(R_{\mathcal{A}})$ (see Figure 6).

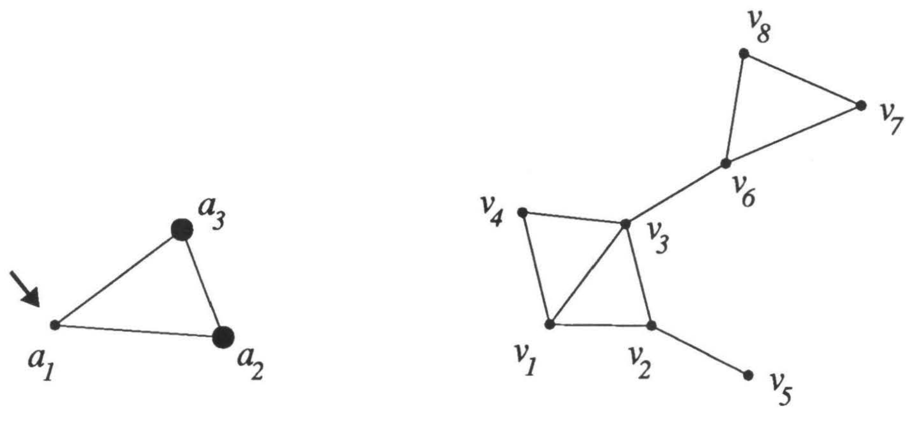

FIGURE 6. *The homomorphism $\theta : G_{\mathcal{A}} \to G$ given by $\theta(a_i) = v_i$ $(i = 1,2,3)$ is an embedding of $\mathcal{A}$ into G at v_1. Furthermore, $N_{\mathcal{A}}(v_1 \mid G) = \{v_2, v_3, v_4\}$.*

With the S-graph $\mathcal{A}$ we can now define the neighbourhood $N_{\mathcal{A}}(v \mid G)$ of v as

$$N_{\mathcal{A}}(v \mid G) = \bigcup \{\theta(B_{\mathcal{A}}) : \theta \text{ is an embedding of } \mathcal{A} \text{ into } G \text{ at } v\}. \tag{4.1}$$

Note that the reciprocal neighbourhood function of N_1 defined in Section 3 is N_1 itself. In the sequel we will show that the reciprocal mapping of a neighbourhood function $N_{\mathcal{A}}$ determined by some S-graph $\mathcal{A}$ corresponds again with an S-graph. To obtain this result we first have to define the reciprocal S-graph. Let $\mathcal{A}$ be an S-graph. We define the reciprocal $\check{\mathcal{A}}$ of an S-graph $\mathcal{A}$ by $G_{\check{\mathcal{A}}} = G_{\mathcal{A}}$, $B_{\check{\mathcal{A}}} = R_{\mathcal{A}}$ and $R_{\check{\mathcal{A}}} = B_{\mathcal{A}}$ (see Figure 9).

FIGURE 9. *An S-graph $\mathcal{A}$ and its reciprocal $\check{\mathcal{A}}$.*

Proposition 4.4. *Let $\mathcal{A}$ be an S-graph and let $\check{\mathcal{A}}$ represent its reciprocal. Then we have*

$$\check{N}_{\mathcal{A}}(v \mid G) = N_{\check{\mathcal{A}}}(v \mid G)$$

for any graph G and every vertex $v \in V(G)$.

PROOF. We only prove the inclusion $\subseteq$. The other part of the assertion is obtained by taking the reciprocal of both terms. Let $w \in \check{N}_{\mathcal{A}}(v \mid G)$. Then $v \in N_{\mathcal{A}}(w \mid G)$ and there exists a homomorphism θ from $G_{\mathcal{A}}$ to G such that $w \in \theta(R_{\mathcal{A}})$ and $v \in \theta(B_{\mathcal{A}})$. This implies that $w \in \theta(B_{\check{\mathcal{A}}})$ and $v \in \theta(R_{\check{\mathcal{A}}})$. Therefore $w \in N_{\check{\mathcal{A}}}(v \mid G)$. This concludes the proof.

∎

5. Dilations and erosions

Let $\mathcal{A}$ be an S-graph and let $N_{\mathcal{A}}$ be its corresponding neighbourhood function. Consider an arbitrary graph G. The binary graph operators $\delta_{\mathcal{A}}$ and $\varepsilon_{\mathcal{A}}$, given by

$$\begin{aligned} \delta_{\mathcal{A}}(X \mid G) &= \bigcup_{v \in X} N_{\mathcal{A}}(v \mid G) \\ \varepsilon_{\mathcal{A}}(X \mid G) &= \{v \in V : N_{\mathcal{A}}(v \mid G) \subseteq X\}, \end{aligned} \tag{5.1}$$

for $X \subseteq V = V(G)$, define respectively a dilation and an erosion on $\mathcal{L}(G) = P(V)$. We call $\delta_{\mathcal{A}}$ a *graph* *dilation* and $\varepsilon_{\mathcal{A}}$ a *graph erosion*.

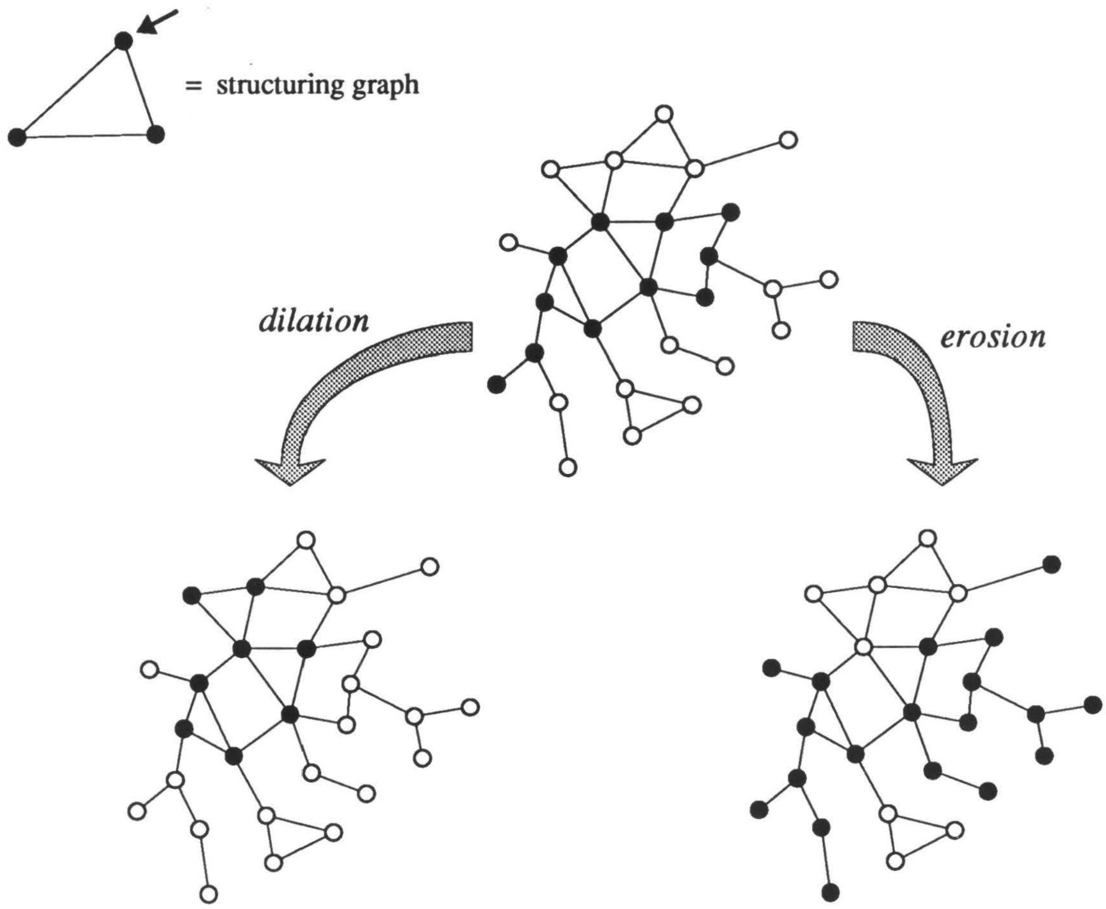

FIGURE 10. *Dilation and erosion of a binary graph by a structuring graph.*

The pair $(\varepsilon_{\mathcal{A}}(\cdot \mid G), \delta_{\mathcal{A}}(\cdot \mid G))$ forms an adjunction on the space $\mathcal{L}(G)$. Furthermore both operators commute with the symmetries of the graph G. Thus, for $X \subseteq V(G)$ and $\tau \in \mathrm{Sym}(G)$,

$$\delta_{\mathcal{A}}(\tau X \mid G) = \tau \delta_{\mathcal{A}}(X \mid G)$$
$$\varepsilon_{\mathcal{A}}(\tau X \mid G) = \tau \varepsilon_{\mathcal{A}}(X \mid G).$$

Recalling that ψ^* denotes the dual operator, we can state the following proposition.

Proposition 5.1. *For any S-graph $\mathcal{A}$ we have*

$$\delta^*_{\mathcal{A}} = \varepsilon_{\check{\mathcal{A}}} \quad \text{and} \quad \varepsilon^*_{\mathcal{A}} = \delta_{\check{\mathcal{A}}}.$$

PROOF. We only prove the first relation. The second follows from duality. Let $(X \mid G)$ represent a binary graph. In the rest of this proof we omit the argument G. Because $y \in N_{\mathcal{A}}(x)$ iff $x \in N_{\check{\mathcal{A}}}(y)$ we

We call this set the $\mathcal{A}$-neighbourhood of v in G. An example is presented in figure 7. One easily checks that for graphs without isolated vertices the neighbourhood function N_1 (as used by Vincent [18,20,21]) can be obtained from the S-graph depicted in Figure 7.

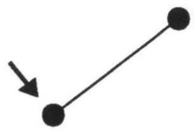

FIGURE 7. *The structuring graph corresponding to the neighbourhood function N_1.*

The following lemma (which we need in the sequel) states that the $\mathcal{A}$-neighbourhood of v in G is composed of subgraphs G' in G which contain v and are isomorphic to $G_{\mathcal{A}}$.

Lemma 4.1. *Let G be a graph and v one of its vertices, then*

$$N_{\mathcal{A}}(v \mid G) = \bigcup \{N_{\mathcal{A}}(v \mid G') : G' \subseteq G,\ G' \simeq G_{\mathcal{A}},\ v \in V(G')\}.$$

PROOF. The inclusion $\supseteq$ is trivial. Therefore we only prove $\subseteq$. Suppose $w \in N_{\mathcal{A}}(v \mid G)$. Then there is an embedding θ of $\mathcal{A}$ in G at v such that $w \in \theta(B_{\mathcal{A}})$. Hence $G' = \theta(G_{\mathcal{A}})$ satisfies $G' \subseteq G$, $G' \simeq G_{\mathcal{A}}$, $v \in V(G')$ and $w \in N_{\mathcal{A}}(v \mid G')$.

∎

From the definition of the neighbourhood $N_{\mathcal{A}}$ it is evident that its cardinality depends upon three factors (which are related to some extent):

(i) the "amount" of structure in $\mathcal{A}$; this depends upon the underlying graph $G_{\mathcal{A}}$,

(ii) the bud set $B_{\mathcal{A}}$; a larger set $B_{\mathcal{A}}$ induces larger neighbourhoods $N_{\mathcal{A}}$ (for fixed $G_{\mathcal{A}}$),

(iii) the root set $R_{\mathcal{A}}$; an S-graph is progressively easier to embed if the number of its roots increases, therefore $N_{\mathcal{A}}$ increases with the number of roots (for fixed $G_{\mathcal{A}}$).

We will now define a partial order $\preceq$ on the set of all S-graphs which corresponds to an ordering on dilations. Let $\mathcal{A}_1, \mathcal{A}_2$ be two S-graphs. We define $\mathcal{A}_1 \preceq \mathcal{A}_2$ if $N_{\mathcal{A}_1}(v \mid G) \subseteq N_{\mathcal{A}_2}(v \mid G)$ for any graph G and $v \in V(G)$. In this case $\mathcal{A}_1$ is called more selective than $\mathcal{A}_2$. If $\mathcal{A}_1 \preceq \mathcal{A}_2$ and $\mathcal{A}_2 \preceq \mathcal{A}_1$ then $\mathcal{A}_1$ and $\mathcal{A}_2$ are equivalent and we write $\mathcal{A}_1 \equiv \mathcal{A}_2$.
N.B. Strictly speaking, $\preceq$ only defines a partial ordering if the equivalence classes associated with the equivalence relation $\equiv$ are considered.

Proposition 4.2. *$\mathcal{A}_1 \preceq \mathcal{A}_2$ if and only if*

(i) $G_{\mathcal{A}_2} \tilde{\subset} G_{\mathcal{A}_1}$,

(ii) $N_{\mathcal{A}_1}(v \mid G_{\mathcal{A}_1}) \subseteq N_{\mathcal{A}_2}(v \mid G_{\mathcal{A}_1})$, *for any* $v \in V_{\mathcal{A}_1}$.

PROOF. "only if": let $\mathcal{A}_1 \preceq \mathcal{A}_2$. We now show that $G_{\mathcal{A}_2} \tilde{\subset} G_{\mathcal{A}_1}$. The second part of the assertion follows from the definition by taking $G = G_{\mathcal{A}_1}$. Let $v \in R_{\mathcal{A}_1}$. Then $N_{\mathcal{A}_1}(v \mid G) \supseteq B_{\mathcal{A}_1}$ since id is an embedding of $\mathcal{A}_1$ in $G_{\mathcal{A}_1}$ at v. Similarly $N_{\mathcal{A}_2}(v \mid G_{\mathcal{A}_1}) \neq \emptyset$. Therefore, there must be an embedding of $\mathcal{A}_2$ in $G_{\mathcal{A}_1}$ (at v), i.e., $G_{\mathcal{A}_2} \tilde{\subset} G_{\mathcal{A}_1}$.
"if": first we note that if $G' \simeq G_{\mathcal{A}_1}$ and $v \in V(G')$ then

$$N_{\mathcal{A}_1}(v \mid G') \subseteq N_{\mathcal{A}_2}(v \mid G').$$

From Lemma 4.1 we get

$$\begin{aligned} N_{\mathcal{A}_1}(v \mid G) &= \bigcup \{N_{\mathcal{A}_1}(v \mid G') : G' \subseteq G,\ G_{\mathcal{A}_1} \simeq G',\ v \in V(G')\} \\ &\subseteq \bigcup \{N_{\mathcal{A}_2}(v \mid G') : G' \subseteq G,\ G_{\mathcal{A}_1} \simeq G',\ v \in V(G')\} \\ &\subseteq \bigcup \{N_{\mathcal{A}_2}(v \mid G') : G' \subseteq G,\ v \in V(G')\} \\ &= N_{\mathcal{A}_2}(v \mid G). \end{aligned}$$

This finishes the proof. ∎

(a)

(b)

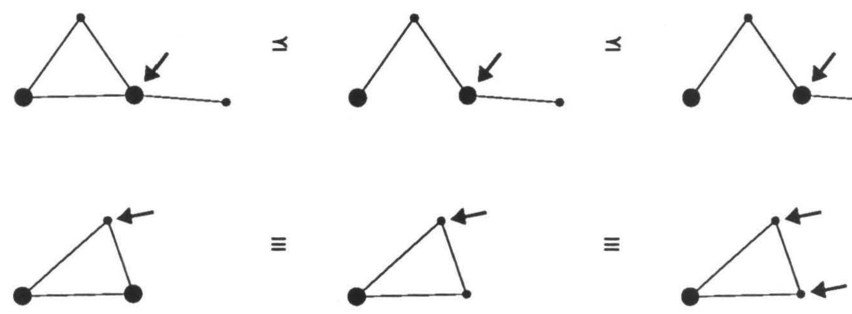

FIGURE 8. *Illustration of (a) the partial order, and (b) equivalence.*

From Proposition 4.2 we directly obtain the following result.

Corollary 4.3. *$\mathcal{A}_1 \equiv \mathcal{A}_2$ if and only if $G_{\mathcal{A}_1} \simeq G_{\mathcal{A}_2}$ and $N_{\mathcal{A}_1}(v \mid G_{\mathcal{A}_1}) = N_{\mathcal{A}_2}(v \mid G_{\mathcal{A}_1})$ for any $v \in V_{\mathcal{A}_1}$.*

The $\mathcal{A}$-neighbourhood of a vertex v depends on the local structure of G near v. Moreover, it is invariant under symmetries of the graph (which is easily deduced from its definition):

$$N_{\mathcal{A}}(\tau v \mid G) = \tau N_{\mathcal{A}}(v \mid G), \tag{4.2}$$

for any $\tau \in \mathrm{Sym}(G)$. We conclude this section with some remarks on the *reciprocal neighbourhood function*. Let $N : V \to \mathcal{P}(V)$ be some neighbourhood function. Then the reciprocal mapping $\check{N} : V \to \mathcal{P}(V)$ is defined by

$$\check{N}(v) = \{w \in V : v \in N(w)\}. \tag{4.3}$$

Proposition 6.1. *For every S-graph $\mathcal{A}$, the graph operator $\alpha_{\mathcal{A}}$ is a symmetry-preserving opening which is G-increasing*

The opening $\alpha_{\mathcal{A}}$ is called a *structural opening.*

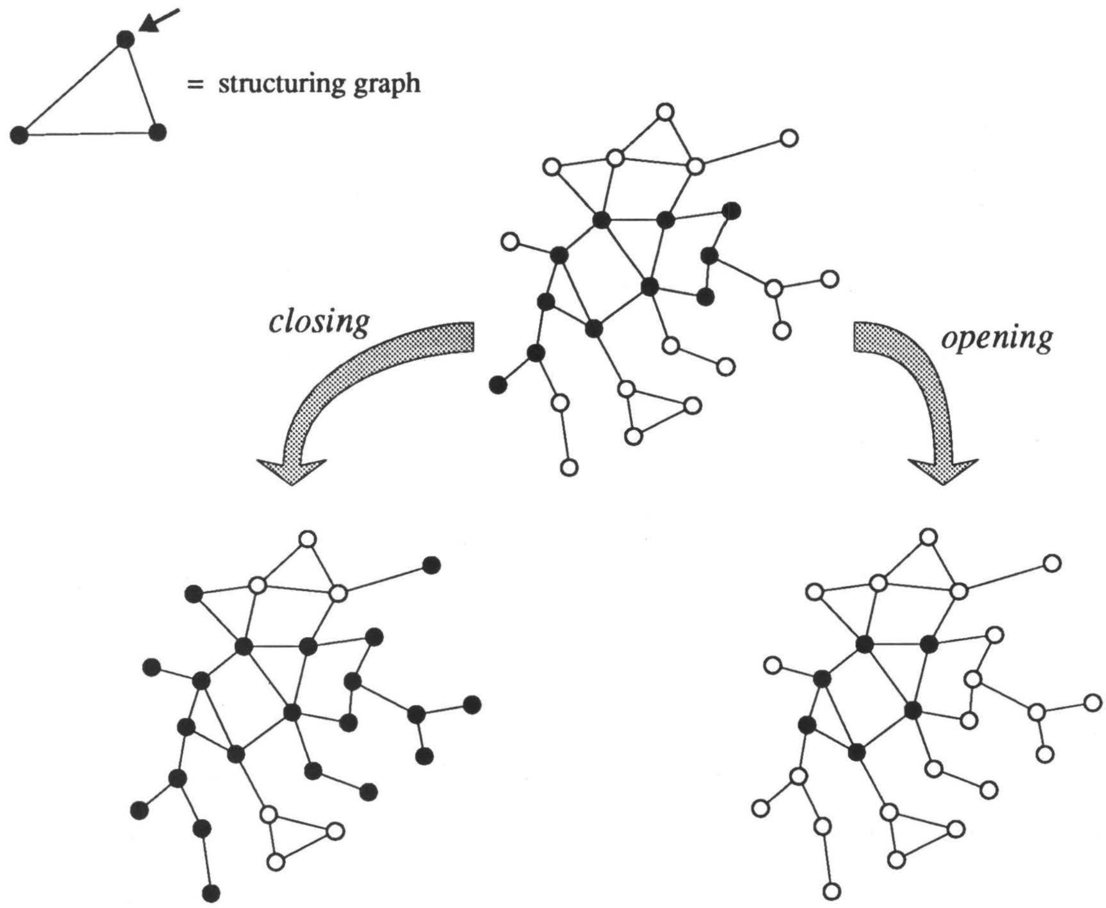

FIGURE 13. *Structural closing and opening of a binary graph.*

The binary graph $(X \mid G)$ is invariant if there exists a homomorphism from $G_{\mathcal{A}}$ into G such that $v \in \theta(B_{\mathcal{A}})$ for every vertex $v \in X$. With every binary graph $(X \mid G)$ we can associate a structuring graph $\mathcal{A} = \mathcal{A}(X \mid G)$ by taking $G_{\mathcal{A}} = G$, $B_{\mathcal{A}} = X$, $R_{\mathcal{A}} = V$. In fact we may choose any (nonempty) subset of $V_{\mathcal{A}} = V$ for $R_{\mathcal{A}}$; for reasons of simplicity we have choosen $R_{\mathcal{A}} = V$.

The class of all invariant graphs is called the invariance domain of $\alpha_{\mathcal{A}}$ and is denoted by $\mathsf{Inv}(\alpha_{\mathcal{A}})$. For convenience we assume that the invariance domain of an operator ψ consists of all S-graphs associated with the binary graphs which are invariant under ψ:

$$\mathsf{Inv}(\psi) = \{\mathcal{A}(X \mid G) : \psi(X \mid G) = (X \mid G)\}.$$

We now prove the following characterization of symmetry-preserving graph operators.

Proposition 6.2. *Let α be a symmetry-preserving graph opening which is G-increasing. Then α can be decomposed as*

$$\alpha = \bigcup_{\mathcal{A}\in \mathsf{Inv}(\alpha)} \alpha_{\mathcal{A}}.$$

PROOF. We define $\alpha' = \bigcup_{\mathcal{A}\in \mathsf{Inv}(\alpha)} \alpha_{\mathcal{A}}$.
$\alpha' \geq \alpha$: let $(X \mid G)$ be a binary graph and define $\mathcal{A} \in \mathsf{Inv}(\alpha)$ as $\mathcal{A} := \mathcal{A}\alpha(X \mid G)$, i.e., $\mathcal{A}$ is the S-graph corresponding with the binary graph $\alpha(X \mid G)$. Then it is easily seen that $\alpha_{\mathcal{A}}(X \mid G) \geq \alpha(X \mid G)$, and therefore $\alpha'(X \mid G) \geq \alpha(X \mid G)$.
$\alpha' \leq \alpha$: let $(X \mid G)$ be a binary graph and $\mathcal{A} \in \mathsf{Inv}(\alpha)$. By definition

$$\alpha_{\mathcal{A}}(X \mid G) = \bigcup\{\theta(B_{\mathcal{A}}) \mid G_{\mathcal{A}} \xrightarrow{\theta} G \text{ and } \theta(B_{\mathcal{A}}) \subseteq X\}.$$

Since $\theta(B_{\mathcal{A}}) \subseteq X$ we get that $\alpha(\theta(B_{\mathcal{A}}) \mid G) \subseteq \alpha(X \mid G)$. By the G-increasingness of α we get that $\alpha(\theta(B_{\mathcal{A}}) \mid \theta(G_{\mathcal{A}})) \subseteq \alpha(X \mid G)$. Since $(B_{\mathcal{A}} \mid G_{\mathcal{A}})$ is invariant, this yields that $\theta(B_{\mathcal{A}}) \subseteq \alpha(X \mid G)$ whence it follows that $\alpha_{\mathcal{A}}(X \mid G) \leq \alpha(X \mid G)$, and hence that $\alpha' \leq \alpha$.

∎

We say that the binary graph $(X \mid G)$ is $\mathcal{A}$-open if $\alpha_{\mathcal{A}}(X \mid G) = (X \mid G)$. Let $\mathcal{A}, \mathcal{B}$ be two S-graphs. We will examine under what conditions a $\mathcal{B}$-open graph is also $\mathcal{A}$-open or, equivalently,

$$\alpha_{\mathcal{A}}\alpha_{\mathcal{B}} = \alpha_{\mathcal{B}}\alpha_{\mathcal{A}} = \alpha_{\mathcal{B}}.$$

It is obvious that this can only be true if the binary graph $(B_{\mathcal{B}} \mid G_{\mathcal{B}})$ corresponding to the S-graph $\mathcal{B}$ is $\mathcal{A}$-open. It turns out that this condition is also sufficient.

Proposition 6.3. *The equalities $\alpha_{\mathcal{A}}\alpha_{\mathcal{B}} = \alpha_{\mathcal{B}}\alpha_{\mathcal{A}} = \alpha_{\mathcal{B}}$ hold if and only if $(B_{\mathcal{B}} \mid G_{\mathcal{B}})$ is $\mathcal{A}$-open.*

PROOF. The "only if"-statement is trivial. Therefore we only prove the "if"-part. Suppose $(B_{\mathcal{B}} \mid G_{\mathcal{B}})$ is $\mathcal{A}$-open. Hence, for every $b \in B_{\mathcal{B}}$ there is an embedding $\theta_b : G_{\mathcal{A}} \to G_{\mathcal{B}}$ and an $a_b \in B_{\mathcal{A}}$ such that

$$\theta_b(a_b) = b \in \theta_b(B_{\mathcal{A}}) \subseteq B_{\mathcal{B}}.$$

We assume that $x \in \alpha_{\mathcal{B}}(X \mid G)$. Then there is an embedding $\theta : G_{\mathcal{B}} \to G$ and a $b \in B_{\mathcal{B}}$ such that

$$\theta(b) = x \in \theta(B_{\mathcal{B}}) \subseteq X.$$

Using the abovementioned θ_b we construct an embedding $\theta \circ \theta_b : G_{\mathcal{A}} \to G$ such that

$$\theta \circ \theta_b(a_b) = x \in \theta \circ \theta_b(B_{\mathcal{B}}) \subseteq X.$$

Therefore we have $x \in \alpha_{\mathcal{A}}(X \mid G)$. Thus we have shown that $\alpha_{\mathcal{B}} \leq \alpha_{\mathcal{A}}$. This implies $\alpha_{\mathcal{B}}\alpha_{\mathcal{B}} \leq \alpha_{\mathcal{A}}\alpha_{\mathcal{B}} \leq \alpha_{\mathcal{B}}$. Because $\alpha_{\mathcal{B}}$ is idempotent this is equivalent to $\alpha_{\mathcal{A}}\alpha_{\mathcal{B}} = \alpha_{\mathcal{B}}$. Since $\alpha_{\mathcal{B}}$ is increasing we also have $\alpha_{\mathcal{B}}\alpha_{\mathcal{B}} \leq \alpha_{\mathcal{B}}\alpha_{\mathcal{A}} \leq \alpha_{\mathcal{B}}$ or $\alpha_{\mathcal{A}}\alpha_{\mathcal{B}} = \alpha_{\mathcal{B}}$.

∎

have

$$\delta^*_{\mathcal{A}}(X \mid G) = [\bigcup_{x \in X^c} N_{\mathcal{A}}(x)]^c = \bigcap_{x \in X^c} \{y \in V : y \notin N_{\mathcal{A}}(x)\}$$
$$= \bigcap_{x \in X^c} \{y \in V : x \notin N_{\check{\mathcal{A}}}(y)\} = \{y \in V : X^c \cap N_{\check{\mathcal{A}}}(y) = \emptyset\}$$
$$= \{y \in V : N_{\check{\mathcal{A}}}(y) \subseteq X\} = \varepsilon_{\check{\mathcal{A}}}(X).$$

∎

Note that these results are analogous to the classical case. However, many other results from classical morphology have no graph theoretical analogues. For example, dilations (or erosions) with two different structuring graphs generally do not commute (see Figure 11).

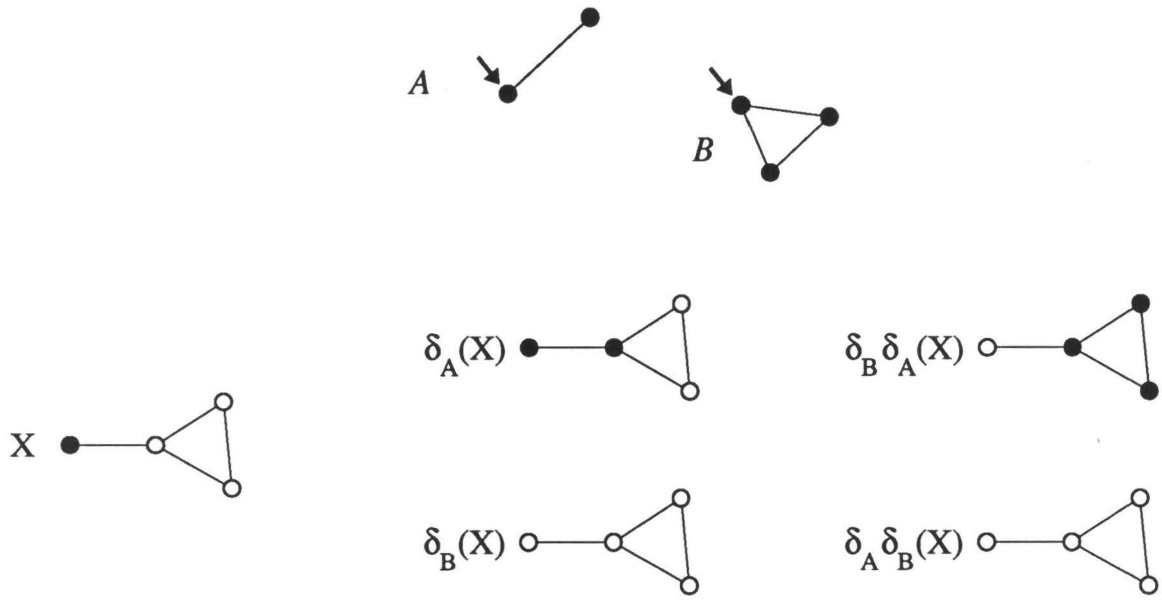

FIGURE 11. *Two dilations which do not commute.*

In classical morphology (on $\mathcal{P}(\mathbb{R}^d)$) the composition of two dilations with structuring elements A and B is again a dilation with a (larger) structuring element: $(X \oplus A) \oplus B = (X \oplus B) \oplus A = X \oplus (A \oplus B) = X \oplus C$, where $C = A \oplus B$. In graph morphology the composition of two dilations with structuring graphs $\mathcal{A}$ and $\mathcal{B}$ is also a dilation. However, there is not always a (larger) S-graph $\mathcal{C}$ such that $\delta_{\mathcal{A}} \circ \delta_{\mathcal{B}} = \delta_{\mathcal{C}}$. In classical morphology Matheron's theorem [9,13] states that any increasing translation invariant operator can be decomposed as a union of erosions, or dually, as an intersection of dilations. It would be useful to have a similar theorem for increasing, symmetry-preserving graph operators in graph morphology. However, the following example shows that there is no such analogue.

Example 5.2. Consider the graph operator $\psi(\cdot \mid G)$ defined as follows. The set $\psi(X \mid G)$ consists of all vertices $v \in V(G)$ which have at least one 1-neighbour in X. Assume that ψ can be written as

$$\psi = \bigcup_{\mathcal{A} \in \Sigma} \varepsilon_{\mathcal{A}},$$

for some collection Σ of S-graphs. Let G be the graph depicted in Figure 12 with vertices u, v, w. In the following we omit the argument G.

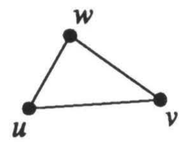

FIGURE 12. *See text.*

For any $A \in \Sigma$ the associated neighbourhood of each of the three vertices is nonempty. Since $\psi(\{u\}) = \{v, w\}$, there is an $A \in \Sigma$ such that $v \in \varepsilon_{\mathcal{A}}(\{u\})$, that is $N_{\mathcal{A}}(v) \subseteq \{u\}$. Since $N_{\mathcal{A}}(v)$ may not be empty, we find that $N_{\mathcal{A}}(v) = \{u\}$. However, by symmetry arguments, $u \in N_{\mathcal{A}}(v)$ iff $w \in N_{\mathcal{A}}(v)$, which is a contradiction. This implies that ψ cannot be written as a union of erosions.

As we already observed in Section 3, any increasing binary graph operator can be extended to multilevel graphs in a unique way. The following explicit expressions can be deduced for the multilevel extensions of the dilation and erosion that were introduced in this section:

$$\delta_{\mathcal{A}}(f \mid G)(v) = \sup\{f(w) : w \in N_{\check{\mathcal{A}}}(v \mid G)\}$$
$$\varepsilon_{\mathcal{A}}(f \mid G)(v) = \inf\{f(w) : w \in N_{\mathcal{A}}(v \mid G)\}.$$

6. Openings and closings

In mathematical morphology openings and closings form dual notions. In this section we will mainly deal with openings. However, in Remark 6.5 we indicate a major difference between structural closings in classical morphology and their analogues in graph morphology.

There are at least two different ways to construct openings on binary (and multilevel) graphs. The first way is to compose an erosion $\varepsilon_{\mathcal{A}}$ and its adjoint dilation $\delta_{\mathcal{A}}$. The resulting operator $\delta_{\mathcal{A}}\varepsilon_{\mathcal{A}}$ is an opening. In classical morphology it is easy to prove that every (translation-invariant) opening can be obtained as a union of these elementary openings. But this fact is not true for all complete lattices as was first noted by Ronse and Heijmans [12]. They introduced the concept of a structural opening and showed that these openings are elementary in the sense that they constitute a base for all openings. In the sequel we will show that the same holds for structural openings in graph morphology. At the end of this section we present an example which shows that there are (structural) openings which can not be constructed from openings of the form $\delta_{\mathcal{A}}\varepsilon_{\mathcal{A}}$.

Let $\mathcal{A}$ be an S-graph. We define the graph operator $\alpha_{\mathcal{A}}$ by

$$\alpha_{\mathcal{A}}(X \mid G) = \bigcup\{\theta(B_{\mathcal{A}}) \mid G_{\mathcal{A}} \xrightarrow{\theta} G \text{ and } \theta(B_{\mathcal{A}}) \subseteq X\}.$$

Note that the roots of $\mathcal{A}$ play no role in this definition. One can easily prove that $\alpha_{\mathcal{A}}$ is indeed an opening.

In classical morphology the identity $\alpha_{\mathcal{A}} = \delta_{\mathcal{A}}\varepsilon_{\mathcal{A}}$ holds. In graph morphology we only have the inequality

$$\delta_{\mathcal{A}}\varepsilon_{\mathcal{A}} \leq \alpha_{\mathcal{A}}. \tag{6.1}$$

To show this let $v \in \delta_{\mathcal{A}}\varepsilon_{\mathcal{A}}(X \mid G)$ for some binary graph $(X \mid G)$. Then $v \in N_{\mathcal{A}}(w \mid G)$ for some $w \in \varepsilon_{\mathcal{A}}(X \mid G)$. But then $v \in N_{\mathcal{A}}(w \mid G) \subseteq X$. This implies that there exists an embedding θ of $G_{\mathcal{A}}$ in G at w such that $v \in \theta(B_{\mathcal{A}}) \subseteq X$. Thus, by definition, $v \in \alpha_{\mathcal{A}}(X \mid G)$.

Below we shall give an example which shows that the class of openings $\delta_{\mathcal{A}}\varepsilon_{\mathcal{A}}$ is too restrictive to build every symmetry-preserving opening.

Example 6.4. Let $\mathcal{B}$ be the S-graph depicted in Figure 14(a) and let $\alpha := \alpha_{\mathcal{B}}$. Consider the graph G depicted in Figure 14(b) with vertices a, b, c, d. There are eight possibilities for the neighbourhood $N_{\mathcal{A}}(a) = N_{\mathcal{A}}(a \mid G)$. Note that the neighbourhoods of b, c, d can be obtained from $N_{\mathcal{A}}(a)$ by symmetry.

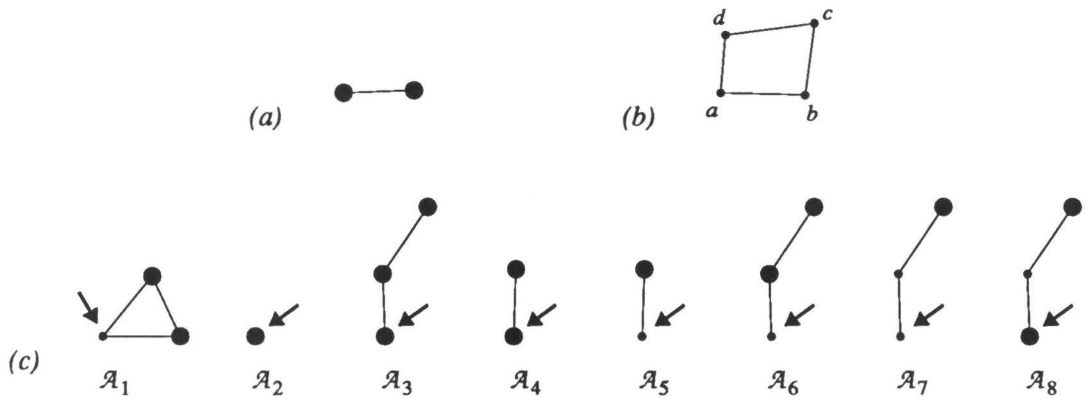

FIGURE 14. *See text.*

The eight possible neighbourhoods are listed as $N_1(a), N_2(a), \dots, N_8(a)$. For each case we give an example of an S-graph $\mathcal{A}_i$ for which $N_{\mathcal{A}_i}(a \mid G) = N_i(a)$, $i = 1, 2, \dots, 8$ (see Figure 14(c)). Note that $\mathcal{A}_i$ is not unique and that our choice is in fact arbitrary. We denote the opening $\delta_{\mathcal{A}_i}\varepsilon_{\mathcal{A}_i}$ by α_i. For the first three neighbourhoods we can give explicit expressions for the corresponding openings α_i. The fourth neighbourhood corresponds to Vincent's construction as described in Section 3.

$N_1(a) = \emptyset$; $\alpha_1(X) = \emptyset$ for any X.

$N_2(a) = \{a\}$; $\alpha_2 = \mathsf{id}$.

$N_3(a) = \{a, b, c, d\}$; $\alpha_3(X) = \emptyset$ if $X \neq V$ and $\alpha_3(V) = V$.

$N_4(a) = \{a, b, d\}$.

$N_5(a) = \{b, d\}$.

$N_6(a) = \{b, c, d\}$.

$N_7(a) = \{c\}$.

$N_8(a) = \{a, c\}$.

Assume that the opening $\alpha = \alpha_B$ can be written as a union of openings of the form $\delta_{\mathcal{A}}\varepsilon_{\mathcal{A}}$. Then there must be a subset $I \subseteq \{1, 2, \ldots, 8\}$ such that

$$\alpha(X) = \bigcup_{i \in I} \alpha_i(X) \tag{6.2}$$

for any $X \subseteq \{a, b, c, d\}$ (the argument G is omitted). First we take $X = \{a, c\}$. Then $\alpha(X) = \emptyset$. Since $\alpha_2(X) = \alpha_5(X) = \alpha_7(X) = \alpha_8(X) = X$ we conclude that $2, 5, 7, 8 \notin I$. Next we take $X = \{a, b\}$. In this case $\alpha(X) = X$ and $\alpha_1(X) = \alpha_3(X) = \alpha_4(X) = \alpha_6(X) = \emptyset$. The combination of these facts yields a contradiction to (6.2). This shows that the structural opening α cannot be written as a union of openings of the form $\delta_{\mathcal{A}}\varepsilon_{\mathcal{A}}$.

Remark 6.5. On the binary image space $\mathcal{P}(\mathbb{R}^d)$ the definition of a translation-invariant structural opening with A as a structuring element is given by:

$$\alpha_A(X) = \bigcup\{A_h : h \in \mathbb{R}^d,\ A_h \subseteq X\}. \tag{6.3}$$

Dually, the structural closing by A is given by

$$\phi_A(X) = \bigcap\{A_h : h \in \mathbb{R}^d,\ A_h \supseteq X\}. \tag{6.4}$$

Between α_A and ϕ_A there is the following duality relation [12]:

$$\phi_A(X) = [\alpha_{A^*}(X^*)]^*, \tag{6.5}$$

where $*$ denotes set complementation. In graph morphology there is no apparent analogue of definition (6.4) of a structural closing. Therefore we resort to formula (6.5). When we ignore the complementation of A in this formula, we can define the structural closing $\phi_{\mathcal{A}}$ by the S-graph $\mathcal{A}$ as

$$\phi_{\mathcal{A}}(X \mid G) = [\alpha_{\mathcal{A}}(X^* \mid G)]^*, \tag{6.6}$$

or alternatively

$$\phi_{\mathcal{A}}(X \mid G) = \bigcap\{V \setminus \theta(B_{\mathcal{A}}) : G_{\mathcal{A}} \xrightarrow{\theta} G,\ \theta(B_{\mathcal{A}}) \cap X = \emptyset\}.$$

An example is presented in Figure 13. Note that a structural closing is G-decreasing. Using Proposition 6.2 it is easy to show that every graph closing ϕ which is symmetry-preserving and G-decreasing can be written as an intersection of structural closings $\phi_{\mathcal{A}}$. More precisely,

$$\phi = \bigcap_{\mathcal{A} \in \mathsf{Inv}(\phi^*)} \phi_{\mathcal{A}}.$$

Using inequality (6.1) and Proposition 5.1 it can be shown that

$$\phi_{\mathcal{A}} \leq \varepsilon_{\check{\mathcal{A}}}\delta_{\check{\mathcal{A}}}.$$

The theory of morphological filters on complete lattices was initiated by Matheron [14, Section 6]. His results can directly be applied to graph morphology. In a companion paper we will present detailed examples of morphological filters on graphs. Here we only give a simple example. An increasing operator ψ is called an *inf-over-filter* if $\psi(\mathsf{id} \wedge \psi) = \psi$ (see [14]). If ψ is an inf-over-filter then $\mathsf{id} \wedge \psi$ is an opening. Let δ_1 and δ_2 be two dilations with $\delta_1 \leq \delta_2$. If ε_1 is the adjoint of δ_1, then $\delta_2\varepsilon_1$ is an inf-over-filter. In this case $\mathsf{id} \wedge \delta_2\varepsilon_1$ is an opening. This general result has the following application in graph morphology. When $\mathcal{A}, \mathcal{B}$ are two structuring graphs with $\mathcal{A} \preceq \mathcal{B}$, then $\mathsf{id} \wedge \delta_{\mathcal{B}}\varepsilon_{\mathcal{A}}$ is an opening.

7. Notes on implementation

The implementation of morphological transformations on graphs involves the choice of appropriate data structures. These data structures should

1. enable the flexible encoding of a vast variety of graphs,
2. allow the *efficient* computation of morphological graph transformations.

An appropriate data structure is derived from the *adjacency matrix* [6, chapter 1]. It belongs to the family of *vertex codings* and is essentially an array **VERT** of vertices. Each of these vertices has some attributes (e.g. its coordinates and its value) as well as a pointer `ptr` towards its neighbours. These neighbours are stored in a second array, called **NEIGH_VERT**, which contains only pointers to elements of **VERT**. The neighbours of the n-th vertex **VERT**[n] can be retrieved by considering the pointers:

$$\texttt{NEIGH\_VERT}[i], \quad \text{for } i = \texttt{ptr}(\texttt{VERT}[n]) \text{ to } \texttt{ptr}(\texttt{VERT}[n+1]) - 1.$$

This data structure, which is described in more details in [20] and [21, pp. 95–98], is illustrated by Figure 15.

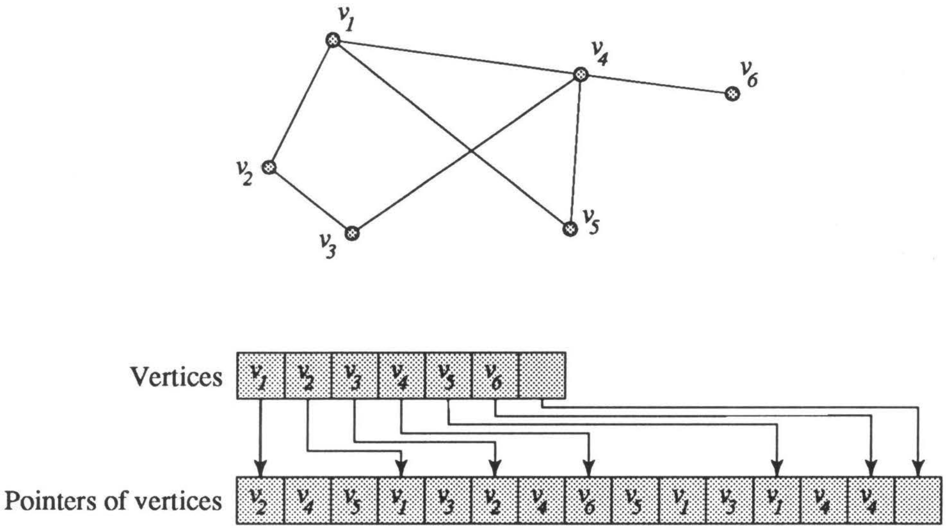

FIGURE 15. *Data structure used for representing graphs and transforming them morphologically.*

It provides *direct* access to the neighbours of a given vertex. This feature is essential in graph morphology. Moreover, this data structure is very general in the sense that it can represent any kind of graph (e.g. non-planar graphs and oriented graphs). Thus, it has all the abovementioned required properties. The complexity of the morphological transformations is directly related to both the structure of the S-graph and the structure of the graph that is to be transformed. We now consider two different cases. First we assume that the graphs that are transformed have no isolated vertices and that the S-graph of Figure 7 is used. In this case the erosions, dilations, openings and closings described in this paper are simply the "classical" ones (see Section 3). Hence, they reduce to simple neighbourhood operations (and so do many other transformations like distance function, geodesic transformations, skeletons, watersheds, etc; [18,19,20]). They can be efficiently implemented by algorithms based on queues of vertices [21, chapters 2 and 5].

For arbitrary structuring graphs the implementation becomes more complex. Let $\mathcal{A}$ be an arbirary structuring graph and let G be the binary or multilevel graph to be transformed. For every morphological operation one has to determine the set $N_{\mathcal{A}}(v|G)$ of associated neighbours of each vertex v. Therefore, the first step of all morphological operations is the construction of an **oriented** graph $G^{\mathcal{A}} = (V, E^{\mathcal{A}})$, such that:

$$E^{\mathcal{A}} = \{(v,w) \in V \times V \mid w \in N_{\mathcal{A}}(v \mid G)\}.$$

Note that the operations described in the previous sections can also be interpreted as morphological transformations of oriented graphs. For example, Figure 16 represents the oriented graph $G^{\mathcal{A}}$ derived from G by means of the S-graph $\mathcal{A}$.

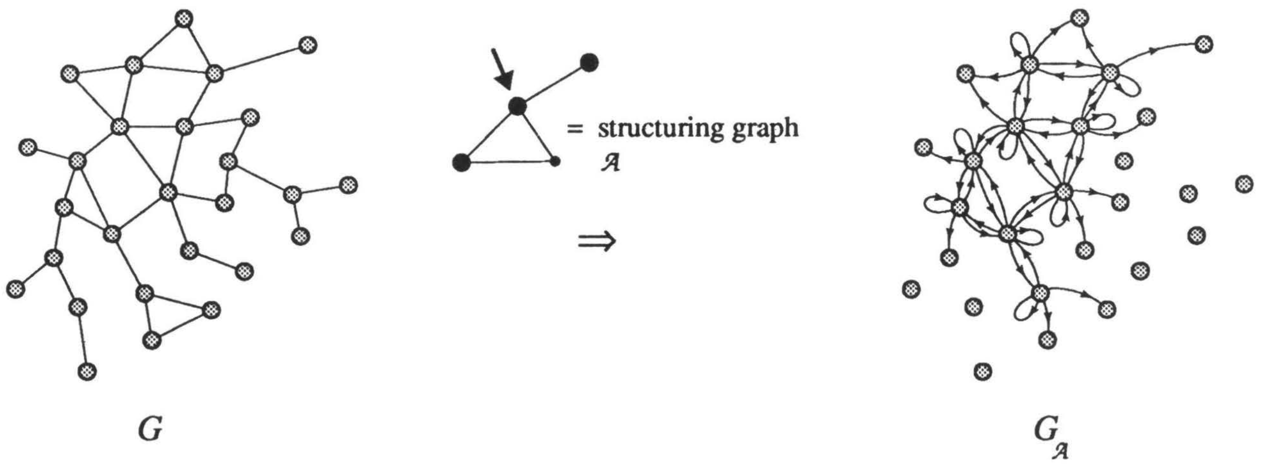

FIGURE 16. *Oriented graph $G^{\mathcal{A}}$ derived from G.*

We now consider the computation of the oriented graph $G^{\mathcal{A}}$. According to the definition of the neighbourhood function $N_{\mathcal{A}}$ in (4.1), we have to determine for each vertex $v \in V$ the union of the $\theta(B_{\mathcal{A}})$ for all embeddings θ of $\mathcal{A}$ into G at v. In other words, we have to determine all the possibilities to "fit" $\mathcal{A}$ in G such that one of the roots of $\mathcal{A}$ is located at v. Here, we will only focus on "small" S-graphs $\mathcal{A}$, since the use of large graph structures is probably not of practical interest. Moreover, the use of large structuring graphs results in extremely complex and time consuming algorithms. Therefore we consider S-graphs $\mathcal{A} = (V_{\mathcal{A}}, E_{\mathcal{A}}, B_{\mathcal{A}}, R_{\mathcal{A}})$ "of size one", i.e.:

$$\forall r \in R_{\mathcal{A}},\ \forall v \in V_{\mathcal{A}} \setminus \{r\},\ \ (r,v) \in E_{\mathcal{A}}.$$

This definition states that an S-graph has size one if every vertex $v \in V_{\mathcal{A}}$ is a neighbour of all the roots. A dilation on a binary or multilevel graph G with an S-graph is equivalent to taking the supremum (union for binary graphs) over $r \in R_{\mathcal{A}}$ of the dilations of G with the S-graphs $\mathcal{A}_r = (V_{\mathcal{A}}, E_{\mathcal{A}}, B_{\mathcal{A}}, \{r\})$. Similar relations can be proved for erosions. In the sequel we will only consider S-graphs of size one with a single root (see Figure 17.).

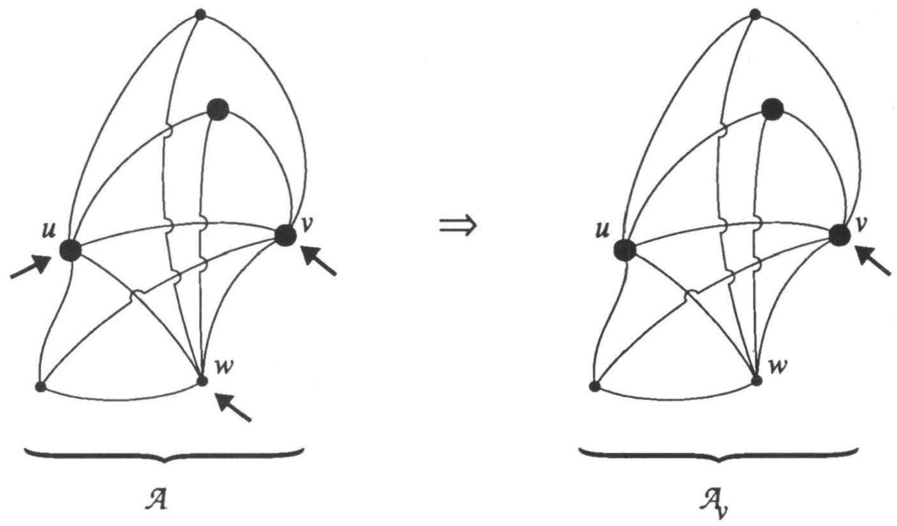

FIGURE 17. *An S-graph $\mathcal{A}$ of size one and one of the S-graphs that can be derived from it by suppressing all its roots but one. Note that $\mathcal{A}$ and $\mathcal{A}_r$ are not equivalent.*

Let $\mathcal{A}$ be a structuring graph of size one such that $R_{\mathcal{A}} = \{r\}$. An edge of $E_{\mathcal{A}}$ with r as one of its extremities is called a *radial* edge. All other edges are called *transversal* edges (see Figure 18).

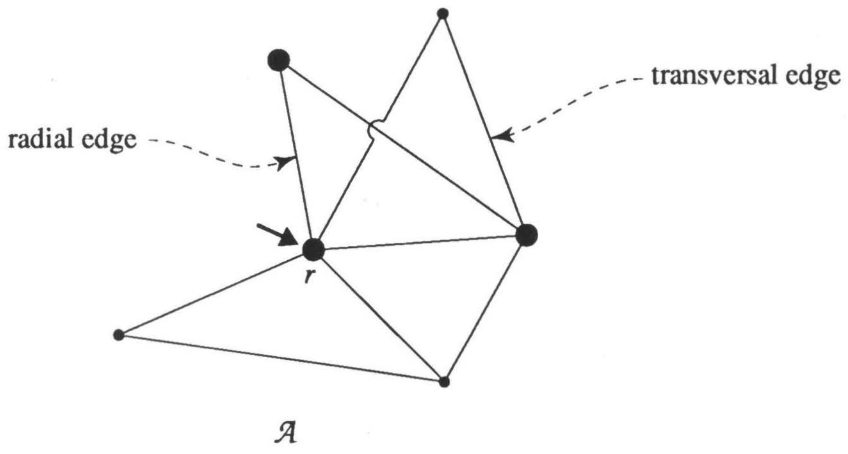

FIGURE 18. *Radial and transversal edges of an S-graph of size one with a unique root.*

We denote by $E^r_{\mathcal{A}}$ the set of the radial edges of $\mathcal{A}$ and by $E^t_{\mathcal{A}}$ the set of its transversal edges. The outline of the algorithm for computing $G^{\mathcal{A}}$ is as follows (the symbol $\leftarrow$ denotes assignment):

- *if* $E^t_{\mathcal{A}} = \emptyset$ *(no transversal edges):*

  for each vertex $v \in V$ *do:*
  - $N_{\mathcal{A}}(v) \leftarrow \emptyset$;
  - *if* $\mathsf{card}(N_1(v)) > \mathsf{card}(E^r_{\mathcal{A}})$ *then*
    - *if* $B_{\mathcal{A}} \setminus \{r\} \neq \emptyset$ *then* $N_{\mathcal{A}}(v) \leftarrow \{w \in V : (v,w) \in E\}$;
    - *if* $r \in B_{\mathcal{A}}$ *then* $N_{\mathcal{A}}(v) \leftarrow N_{\mathcal{A}}(v) \cup \{v\}$;
- *otherwise*

  for each vertex $v \in V$ *do:*
  - $N_{\mathcal{A}}(v) \leftarrow \emptyset$;
  - *if* $\mathsf{card}(N_1(v)) > \mathsf{card}(E^r_{\mathcal{A}})$ *then*

    for every embedding θ *of* $\mathcal{A}$ *into* G *at* v *(exhaustive search):*

    $N_{\mathcal{A}}(v) \leftarrow N_{\mathcal{A}}(v) \cup \theta(B_{\mathcal{A}})$;

In the last case all possible fits have to be computed. This slows down the procedure considerably. However, the computational speed is still acceptable since the number of transversal edges of $\mathcal{A}$ is generally quite small.

8. Final remarks

Note that there is a striking difference between the effects of "classical" transformations (see Section 3) and the effects of transformations with arbitrary structuring graphs. For example, consider the dilation and erosion of the binary graph shown in Figure 10. This example demonstrates that the dilation is not necessarily extensive and, consequently, that the erosion is not necessarily anti-extensive. Furthermore, the number of 1-vertices (i.e. vertices with value 1) of the dilated graph is smaller than that of the eroded graph! (Note that similar phenomena may occur for "classical" dilations and erosions if the origin is not contained in the structuring element.) However, an extensive dilation δ can be constructed from a dilation through an S-graph $\mathcal{A}$ by taking the supremum of $\delta_{\mathcal{A}}$ and the identity mapping:

$$\delta = \mathsf{id} \vee \delta_{\mathcal{A}}.$$

The same procedure can be used to obtain an anti-extensive erosion ε from $\varepsilon_{\mathcal{A}}$:

$$\varepsilon = \mathsf{id} \wedge \varepsilon_{\mathcal{A}}.$$

It can be shown that (ε, δ) is again an adjunction which is symmetry-preserving [7]. The morphological operations with structuring graphs have a *global* effect. For example, the binary erosion may attribute the value 1 to vertices with value 0 that are arbitrarily far away from any original 1-vertex (see Figure 10). As already noted in the introduction, this effect arises because the S-graph has to *match* the underlying graph structure G, which is generally non-periodic (contrary to the digital grid).

Graph morphology was only recently developed. We expect that this theory will be useful for many fields of research. It has already been successfully applied to the study of crack propagation in porous materials [23]. It also provides new tools for the description of object population architectures. Here the relationships between objects are first modelled by a neighbourhood graph such as the Delaunay triangulation [11, Section 5.5], the Gabriel graph [5] or the relative neighbourhood graph [16]. Each object is characterized by a certain number of symbolic or numerical parameters. The resulting binary or multilevel graphs are morphologically transformed to obtain quantitative information about the neighbourhood relationships between the objects. Granulometries on graphs are the most promising tools for this purpose. When special neighbourhood configurations are of interest, these transformations should be combined with operations using S-graphs. At the moment research is being done to investigate how these transformations can be applied to the description of cell population architectures ([20] and [21, Section 5.5]).

Graph morphology can also be used to construct hierarchical image descriptions. It has already been suggested to use classical morphology in the construction of hierarchical image representations [10]. A hierarchical graph representation of an image can be obtained by repeated application of morphological filters to the adjacency graph that represents the neighbourhood relations between the parts of a segmented image. Vincent [21, Chapter 8] introduced a region-merging method based on the watershed transformation on a neighbourhood graph. His results show that the regions that are obtained by this method adapt well to the local image structure [19]. The information that is extracted at each *level of abstraction*, (i.e. at each step in the merging process) depends on the S-graph that is used.

Mathematical morphology is well suited for image segmentation. An existing approach is based on the watershed transformation and uses *markers* to indicate the regions that are to be extracted [22]. However, it is not always possible to determine markers automatically. In such cases it may be useful to transform the image into a neighbourhood graph (representing adjacent regions or adjacent contour elements) and to do all further processing directly on this graph. Beucher [3] already applied this idea without explicitly resorting to morphology on graphs. We expect that the combination of this approach and the S-graph transformations introduced in this paper will result in powerful image segmentation methods. This is presently investigated in a computing environment that was obtained by linking the *Morphograph* package [17] to the *TCL–IMAGE* image processing software [1].

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