Estimating the Intensity of a Cyclic Poisson Process

I Wayan Mangku

Propositions

with the thesis

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by

I Wayan Mangku

1. Lemma 2.1 of chapter 2 of this thesis can easily be generalized to higher dimensions.

2. Consider chapter 5 of the thesis. Suppose that λ is periodic and Lipschitz, and (5.4) holds. Let $A_{n,\delta}$ be as in (5.91). Then, for any δ in a neighborhood of $k\tau$ and any δ' in a neighborhood of $l\tau$, with $\max(k, l) = o(|W_n|)$ as $n \to \infty$, we have that

$$\rho\left(A_{n,\delta}, A_{n,\delta'}\right) \rightarrow 1$$

as $n \to \infty$, $|\delta - k\tau| \to 0$, and $|\delta' - l\tau| \to 0$. Here ρ denotes the correlation coefficient.

3. Let the conditions of Proposition 2 be satisfied and let $B_{n,\delta}$ be as in (5.92). Then, for any δ and δ' as in Proposition 2, we have that

$$\rho\left(B_{n,\delta}, B_{n,\delta'}\right) = \mathcal{O}\left(\sqrt{\min(k,l)/\max(k,l)}\right),$$

as $n \to \infty$, $|\delta - k\tau| \to 0$, and $|\delta' - l\tau| \to 0$.

4. Let $m = m_n = \mathcal{O}(|W_n|^c)$, for some $0 < c < \frac{1}{3}$. Define

$$\bar{\hat{\tau}}^*_{\cdot,n} = \frac{1}{m} \sum_{k=1}^{m} \hat{\tau}^*_{k,n},$$

where $\hat{\tau}^*_{k,n}$ is the estimator defined on page 104 of this thesis. Suppose that the conditions of Theorem 5.4 are satisfied. Then $\hat{\tau}^*_{m,n}$ has smaller asymptotic variance than $\bar{\hat{\tau}}^*_{\cdot,n}$.

5. The second names of Balinese children are periodic with period 4.

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Chapter 1

General introduction

The topic of this thesis is nonparametric estimation of the global and local intensity of a cyclic Poisson process with unknown period. It is supposed that only a single realization of a Poisson process is observed. Since a cyclic Poisson process is a special case of an inhomogeneous Poisson process, we begin with presenting the basic properties of an inhomogeneous Poisson process in section 1.1. The description of a cyclic Poisson process is briefly discussed in section 1.2. In section 1.3, we present an overview of the thesis, while in section 1.4 we discuss briefly some related work in the area of estimating Poisson intensity functions.

1.1 Inhomogeneous Poisson process

Let X denote an inhomogeneous Poisson point process on the real line $\mathbf{R}$ with absolutely continuous σ-finite mean measure μ w.r.t. Lebesgue measure ν and with (unknown) locally integrable intensity function $\lambda : \mathbf{R} \rightarrow \mathbf{R}^+ \cup \{0\}$, i.e., for any bounded Borel set B, we have $\mu(B) = \int_B \lambda(s)ds < \infty$. Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space, and let us suppose that, for some $\omega \in \Omega$, a single realization $X(\omega)$ of the Poisson point process X is observed, though only in a bounded interval (called window) $W \subset \mathbf{R}$. For any set $B \subset \mathbf{R}$, $X(B)$ denotes the number of points of X in B; $\mu(B) = \mathbf{E}X(B)$, for any Borel set B, where $\mathbf{E}$ denotes the expectation. The Poisson process X can be characterized by the following two properties:

(a) $\mathbf{P}\left(X(B) = k\right) = \frac{\mu(B)^k}{k!}e^{-\mu(B)}$, $k = 0, 1, \ldots,$ for each Borel set B with $\mu(B) < \infty$.

(b) For each positive integer m and pairwise disjoint Borel sets B_1, B_2,

$\ldots B_m$ with $\mu(B_j) < \infty$, $j = 1, \ldots, m$ the random variables $X(B_1)$, $X(B_2)$, $\ldots$, $X(B_m)$ are independent.

We refer to Kingman (1993) for an excellent account of the theory of Poisson processes.

This study is concerned with the statistical problem of estimating the 'global' and 'local' intensity, using only a single realization $X(\omega)$ of the Poisson point process X observed only in W. The intensity function λ at a given location $s \in \mathbf{R}$, i.e. the local intensity, can also be expressed as

$$\lambda(s) = \lim_{B \downarrow \{s\}} \mathbf{P}(X(B) = 1)/|B| \tag{1.1}$$

provided λ is continuous at s (Kingman (1993), p. 13). Here $B \downarrow \{s\}$ means that the Borel set B shrinks to $\{s\}$; $|B|$ denotes the Lebesgue measure of a Borel set B and $\{s\}$ is the singleton set, which consists of the point s only.

Since λ is locally integrable, the Poisson point process X always places only a finite number of points in any bounded subset of $\mathbf{R}$. Hence, in order to make consistent estimation possible, one must accumulate the necessary empirical information. One way to achieve this is by adopting a framework called 'increasing domain asymptotics' (see, e.g., Cressie (1993), p. 480). We let the window W depend on $n = 1, 2, \ldots$, in such a way that

$$|W_n| \to \infty, \tag{1.2}$$

as $n \to \infty$, where $|W_n| = \nu(W_n)$ denotes the size (or the Lebesgue measure) of the window W_n. The global intensity θ of the process X, whenever well-defined, can be given by

$$\theta = \lim_{n \to \infty} \frac{\mathbf{E}X(W_n)}{|W_n|}. \tag{1.3}$$

In this set up, a necessary condition for the existence of a consistent estimator is that (cf. Lemma 1.1)

$$\int_{\mathbf{R}} \lambda(s) ds = \infty, \tag{1.4}$$

i.e. $\mu(\mathbf{R}) = \mathbf{E}X(\mathbf{R}) = \infty$, which implies that there are, almost surely, infinite number of points in $\mathbf{R}$ placed there by the point process X. If, on the other hand, $\mathbf{E}X(\mathbf{R}) < \infty$, then there are, almost surely, only a finite number of points in $\mathbf{R}$ placed there by the point process X, and consistent estimation is clearly impossible.

The condition (1.4) also shows up in Rathbun and Cressie (1994), Helmers and Zitikis (1999), and Helmers and Mangku (2000), as a necessary condition for consistency.

Lemma 1.1 *For any Poisson point process X with mean measure μ, if $\mu(\mathbf{R}) = \mathbf{E}X(\mathbf{R}) = \infty$ then for P-almost all ω the point pattern $X(\omega)$ contains infinite many points, i.e. $X(\mathbf{R}) = \infty$. On the other hand, if $\mu(\mathbf{R}) = \mathbf{E}X(\mathbf{R}) < \infty$, then the probability that $X(\omega)$ contains only finitely many points is equal to 1.*

Remark 1.1 Since in (1.2) we only assume that $|W_n| \to \infty$ as $n \to \infty$, we also need that (1.4) holds true if for example we replace $\mathbf{R}$ by $\mathbf{R}^+$ or by any unbounded Borel set. However, the proof of Lemma 1.1 remains valid if $\mathbf{R}$ is replaced by $\mathbf{R}^+$ or by any unbounded Borel set. □

Proof: Suppose $A_1, A_2, \ldots$ are disjoint measurable subsets of $\mathbf{R}$ such that $\bigcup_{i=1}^{\infty} A_i = \mathbf{R}$. Then, we can write

$$X(\mathbf{R}) = \sum_{i=1}^{\infty} X(A_i). \tag{1.5}$$

By Fubini's theorem for nonnegative functions, we have that

$$\mathbf{E}X(\mathbf{R}) = \sum_{i=1}^{\infty} \mathbf{E}X(A_i) = \sum_{i=1}^{\infty} \mu(A_i). \tag{1.6}$$

First, we will show that $\mathbf{E}X(\mathbf{R}) = \infty$ is equivalent to $\sum_{i=1}^{\infty} \mathbf{P}(X(A_i) \geq 1) = \infty$. To prove this, we first write $\mathbf{E}X(\mathbf{R}) = \sum_{i=1}^{\infty} \int_0^{\mu(A_i)} dt$ and

$$\sum_{i=1}^{\infty} \mathbf{P}(X(A_i) \geq 1) = \sum_{i=1}^{\infty} \left(1 - e^{-\mu(A_i)}\right) = \sum_{i=1}^{\infty} \int_0^{\mu(A_i)} e^{-t} dt. \tag{1.7}$$

Let K be a positive real number. We consider two cases, namely, (i) there are infinitely many i such that $\mu(A_i) < K$, and (ii) there are infinitely many i such that $\mu(A_i) \geq K$. Note that, if $\mathbf{E}X(\mathbf{R}) < \infty$ then we are not in case (ii). For case (i), (taking only i such that $\mu(A_i) < K$), we have

$$e^{-K} \sum_{i=1}^{\infty} \int_0^{\mu(A_i)} dt \leq \sum_{i=1}^{\infty} \int_0^{\mu(A_i)} e^{-t} dt \leq \sum_{i=1}^{\infty} \int_0^{\mu(A_i)} dt, \tag{1.8}$$

while for case (ii), (taking only i such that $\mu(A_i) \geq K$), we have

$$\sum_{i=1}^{\infty} \int_0^K e^{-t} dt \leq \sum_{i=1}^{\infty} \int_0^{\mu(A_i)} e^{-t} dt \leq \sum_{i=1}^{\infty} \int_0^{\mu(A_i)} dt. \tag{1.9}$$

Because $\sum_{i=1}^{\infty} \int_0^K e^{-t} dt = \sum_{i=1}^{\infty} \left(1 - e^{-K}\right) = \infty$, we also have that, if $\sum_{i=1}^{\infty} \mathbf{P}(X(A_i) \geq 1) < \infty$ then we are not in case (ii). Hence we have that $\mathbf{E}X(\mathbf{R}) = \infty$ is equivalent to $\sum_{i=1}^{\infty} \mathbf{P}(X(A_i) \geq 1) = \infty$. Because A_i and A_j are disjoint for all $i \neq j$, we have that $X(A_i)$ and $X(A_j)$ are independent, for all $i \neq j$. Therefore, by the Borel-Cantelli lemma, we have that $\mathbf{E}X(\mathbf{R}) = \infty$ is equivalent to $\mathbf{P}(X(\mathbf{R}) = \infty) = 1$. This completes the proof of Lemma 1.1. $\square$

The proof of Lemma 1.1 (generalizing a statement of the author for the case of bounded λ) is due to R. Zitikis.

Though (1.2) and (1.4) are necessary conditions for consistent estimation, one generally requires also some information about the shape of the intensity function in order to be able to estimate θ or $\lambda(s)$, at a given point s, consistently. Parametric models, e.g. the case where $\lambda(s)$ has exponential quadratic and cyclic trends (with known frequency) are particularly convenient (cf. Helmers and Zitikis (1999)), but nonparametric intensity functions can also successfully treated. For example, if we know $\lambda(s)$ to be purely cyclic (with unknown period), one can estimate θ and $\lambda(s)$ consistently, without any further information about the shape of the intensity function. In fact, our aim in this study is to propose and study consistent estimators for θ and $\lambda(s)$ for the important case of cyclic Poisson process (with unknown period τ). To achieve this goal one also requires a consistent estimator for the period τ, while the rate at which one can estimate τ should be sufficiently fast.

1.2 Cyclic Poisson process

In this study we consider the special case of an inhomogeneous Poisson point process, namely *cyclic Poisson point process*, that is a Poisson point process where its intensity function λ is cyclic (periodic) with period $\tau \in \mathbf{R}^+$, i.e. we have

$$\lambda(s + k\tau) = \lambda(s) \tag{1.10}$$

for all $s \in \mathbf{R}$ and $k \in \mathbf{Z}$. If it is not stated otherwise, we consider throughout the case when we do not know the period τ.

The motivation to consider a cyclic Poisson process is twofold. First, in a cyclic Poisson process, the limit in (1.3) is well-defined, and furthermore, we can write the global intensity θ more explicitly as (cf. (2.2))

$$\theta = \frac{1}{\tau} \int_{U_\tau} \lambda(s) ds, \tag{1.11}$$

where U_τ denotes any interval of length τ. Second, and more importantly, periodicity of λ together with (1.2) makes it possible to obtain consistent estimators of λ at a given point s, using only a single realization $X(\omega)$. This is evident, because, since λ is periodic, we not only can use the information in a neighborhood of $s \in W_n$ to estimate λ at s, but also the information in a neighborhood of $s + k\tau$, for any integer k, provided $s + k\tau \in W_n$. Indeed, without imposing a 'structural property' like for instance 'periodicity' on λ, consistent estimation of λ at s using only a single realization $X(\omega)$ is clearly impossible, even though (1.2) holds true.

We also note that, for a cyclic Poisson process, condition (1.4) is automatically satisfied, provided $\theta > 0$. This implication also holds true if $\mathbf{R}$ in (1.4) is replaced by $\mathbf{R}^+$ or by any other unbounded interval. Due to this reason, throughout this thesis, we assume that

$$\theta > 0. \tag{1.12}$$

There are potential applications for the purely cyclic Poisson model (1.10). It has been observed that solar storms are cyclic or, in other words, periodic with peaks occurring about every 11 years; the last peak of storm activity occurred some time between 1989 and 1992. During a peak, a number of large geomagnetic storms can occur. Electrical utilities, especially those located in northern latitudes, are highly susceptible to these storms. It is therefore hoped that the occurrence of the solar storms can be accurately predicted so that operating precautions can be taken to protect the supply of electricity (cf. Molinski (2000)). Other applications can be found in modelling the arrival of patients at an intensive care unit of a hospital (cf. Lewis (1972)). We also refer to page 3 of Kutoyants (1998) for some examples of cyclic intensity functions occurring in applied problems.

1.3 Overview of the thesis

This study is concerned with nonparametric estimation of the *global intensity*, *the intensity function at a given point*, and the *period* of a cyclic

Poisson point process, using only a *single realization* $X(\omega)$ of the cyclic Poisson process X observed in an interval (called window) W_n. An estimator of the global intensity θ is given by

$$\hat{\theta}_n = X(W_n)/|W_n|,$$

where for any interval A, $X(A)$ denotes the number of points in A, and $|A|$ denotes the size (Lebesgue measure) of A. An estimator of the period τ is given by

$$\hat{\tau}_n = \arg\min_{\delta\in\Theta} Q_n(\delta),$$

where Θ denotes the parameter space, $\Theta \subset \mathbf{R}^+$, and for any $\delta \in \Theta$, $Q_n(\delta)$ is given by

$$Q_n(\delta) = \frac{1}{|W_n|}\sum_{i=1}^{N_{n\delta}}\left(X(U_{\delta,i}) - \frac{1}{N_{n\delta}}\sum_{j=1}^{N_{n\delta}} X(U_{\delta,j})\right)^2,$$

with $N_{n\delta} = [|W_n|/\delta]$, which denotes the (maximum) number of adjacent disjoint intervals $U_{\delta,i}$ of length δ in the window W_n. A kernel estimator of the intensity λ at a given point s is given by

$$\hat{\lambda}_{n,K}(s) := \frac{\hat{\tau}_n}{|W_n|}\sum_{k=-\infty}^{\infty}\frac{1}{h_n}\int_{W_n} K\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right) X(dx),$$

where h_n is a sequence of positive real numbers converging to 0, and K is a function $K : \mathbf{R} \to \mathbf{R}$, called the kernel.

Let s_i, $i = 1, \ldots, X(W_n, \omega)$, denote the locations of the points in the realization $X(\omega)$ of the Poisson process X, observed in window W_n. Here $X(W_n, \omega)$ is nothing but the cardinality of the data set $\{s_i\}$. Set $X(W_n, \omega) = m$. Let now $\hat{s}_i$, $i = 1, \ldots, m$, denote the location of the point s_i ($i = 1, \ldots, m$), after translation by a multiple of $\hat{\tau}_n$ such that $\hat{s}_i \in \bar{B}_{\hat{\tau}_n}(s)$, for all $i = 1, \ldots, m$, where $\bar{B}_{\hat{\tau}_n}(s) = [s - \frac{\hat{\tau}_n}{2}, s + \frac{\hat{\tau}_n}{2})$. Let k_n be a sequence of positive integers converging to ∞ as $n \to \infty$. A nearest neighbor estimator of the intensity λ at a given point s is then given by

$$\hat{\lambda}_n(s) = \frac{\hat{\tau}_n k_n}{2|W_n||\hat{s}_{(k_n)} - s|},$$

if $X(W_n) \geq k_n$ and $\hat{\lambda}_n(s) = 0$ otherwise. Here $|\hat{s}_{(k_n)} - s|$ denotes the k_n-th order statistics of $|\hat{s}_1 - s|, \ldots, |\hat{s}_m - s|$, conditionally given $X(W_n) = m$.

In chapter 2 we study asymptotic properties of the estimator $\hat{\theta}_n$ of the global intensity θ. If λ is assumed to be periodic and locally integrable, we have the following results: (i) $\hat{\theta}_n$ is asymptotically unbiased and weakly consistent in estimating θ, (ii) $\hat{\theta}_n$ converges completely to θ, as $n \to \infty$, which also implies strong consistency of $\hat{\theta}_n$ in estimating θ, and (iii) $|W_n|^{1/2}(\hat{\theta}_n - \theta) \stackrel{d}{\to} N(0,\theta)$, as $n \to \infty$, where $N(0,\theta)$ denotes a normal random variable with mean zero and variance θ. Furthermore let, conditionally given $X(W_n)$, $X^*(W_n)$ denote a realization from a Poisson distribution with parameter $X(W_n)$. If $X(W_n)$ happens to be equal to zero, we set $X^*(W_n) = 0$. Define $\hat{\theta}_n^* = X^*(W_n)/|W_n|$. Then we also have (iv) $|W_n|^{1/2}\left(\hat{\theta}_n^* - \hat{\theta}_n\right) \stackrel{d}{\to} N(0,\theta)$, as $n \to \infty$, in P-probability.

In chapter 3 we investigate asymptotic properties of the estimator $\hat{\lambda}_{n,K}(s)$ of λ at a given point s. Suppose that λ is periodic, locally integrable, and s is a Lebesgue point of λ. Then we have the following results: (i) $\hat{\lambda}_{n,K}(s)$ is a consistent estimator of $\lambda(s)$. (ii) $\hat{\lambda}_{n,K}(s)$ converges completely to $\lambda(s)$ as $n \to \infty$, which also implies strong consistency of $\hat{\lambda}_{n,K}(s)$ in estimating $\lambda(s)$. (iii) $Var(\hat{\lambda}_{n,K}(s))$ converges to 0 as $n \to \infty$, provided $|W_n|h_n \to \infty$ as $n \to \infty$. Furthermore, we have an asymptotic approximation to the variance of $\hat{\lambda}_{n,K}$ as follows

$$Var\left(\hat{\lambda}_{n,K}(s)\right) = \frac{\tau\lambda(s)}{|W_n|h_n}\int_{-1}^{1} K^2(x)dx + o\left(\frac{1}{|W_n|h_n}\right),$$

as $n \to \infty$, provided λ is bounded. (iv) $\hat{\lambda}_{n,K}(s)$ is asymptotically unbiased in estimating $\lambda(s)$. If, in addition, λ has finite second derivative λ'' at s, then we have an asymptotic approximation to the bias of $\hat{\lambda}_{n,K}$ as follows

$$\left(\mathbf{E}\hat{\lambda}_{n,K}(s) - \lambda(s)\right) = \frac{\lambda''(s)}{2}h_n^2\int_{-1}^{1} x^2K(x)dx + o(h_n^2) + \mathcal{O}(|W_n|^{-1})$$

as $n \to \infty$. An interesting feature (cf. section 3.2.3) of our results in this chapter is that the kernel K has to be a 'smooth' one, otherwise the effect of estimating τ by $\hat{\tau}_n$ in $\hat{\lambda}_{n,K}(s)$ may destroy the consistency properties of our kernel estimator of $\lambda(s)$.

Parallel to chapter 3, in chapter 4 we investigate the asymptotic properties of the nearest neighbor estimator $\hat{\lambda}_n(s)$ of λ at a given point s. Suppose that λ is periodic, locally integrable, and s is a point at which λ is continuous and positive. Then we have the following results: (i) $\hat{\lambda}_n(s)$ is a consistent estimator of $\lambda(s)$. (ii) $\hat{\lambda}_n(s)$ converges completely to $\lambda(s)$ as $n \to \infty$, which also implies strong consistency of $\hat{\lambda}_n(s)$ in estimating $\lambda(s)$. (iii) $Var(\hat{\lambda}_n(s))$ converges to 0 as $n \to \infty$, and furthermore we have an asymptotic approximation to the variance of $\hat{\lambda}_n$ as follows

$$Var\left(\hat{\lambda}_n(s)\right) = \frac{\lambda^2(s)}{k_n} + o\left(\frac{1}{k_n}\right),$$

as $n \to \infty$. (iv) $\hat{\lambda}_n(s)$ is asymptotically unbiased in estimating $\lambda(s)$. If, in addition, λ has finite second derivative λ'' at s, then we have an asymptotic approximation to the bias of $\hat{\lambda}_n$ as follows

$$\left(\mathbf{E}\hat{\lambda}_n(s) - \lambda(s)\right) = \frac{\tau^2 \lambda''(s) k_n^2}{24\lambda^2(s)|W_n|^2} + o\left(\frac{k_n^2}{|W_n|^2}\right) + \mathcal{O}\left(\frac{1}{|W_n|^{1/2-\epsilon_0}} + \frac{1}{k_n}\right),$$

as $n \to \infty$, where ϵ_0 is a positive real number which can be chosen arbitrarily small.

In a way the only thing we do in chapter 3 is to provide conditions on the rate of convergence of $\hat{\tau}_n$ approaching τ and on the kernel K, such that the asymptotic properties of our kernel estimator of λ we discuss are identical to those that would be obtained if the period τ were known. A similar remark applies to the results on nearest neighbor estimation in chapter 4.

In chapter 5 we focus on estimation of the period τ. In general, τ can be estimated as follows: first estimate $k\tau$, for some positive integer k satisfying $k = k_n = o(|W_n|)$, by $k\hat{\tau}_{n,k}$, which is given by

$$k\hat{\tau}_{n,k} = \arg\min_{\delta \in \Theta_k} Q_n(\delta),$$

where $\hat{\tau}_{n,k}$ denotes the resulting estimator of τ. Here $\Theta_k = (\tau_{k,0}, \tau_{k,1})$ is an open interval, such that no other multiple of τ than $k\tau$ is contained in Θ_k. Suppose that λ is periodic (with period τ) and bounded. Then, for each positive integer k satisfying $k = k_n \sim |W_n|^c$, for some $0 \leq c < \frac{1}{3}$, we have the following results: (i) $\hat{\tau}_{n,k}$ is weakly and strongly consistent in estimating τ. (ii) If $\gamma < \frac{1}{4} + \frac{c}{4}$, we have that $|W_n|^\gamma(\hat{\tau}_{n,k} - \tau)$ converges in probability and completely to zero, as $n \to \infty$. (iii) If, in addition, λ is Lipschitz, then for any $\gamma < \frac{1}{2}$, we have $|W_n|^\gamma \left(\hat{\tau}_{k,n} - \tau\right) \xrightarrow{p} 0$, as $n \to \infty$. In order to obtain asymptotic normality, we need to modify our original estimator $\hat{\tau}_{n,k}$ and obtain a modified estimator $\hat{\tau}^*_{k,n}$ of τ. For each positive integer k satisfying $k = k_n = o(|W_n|)$, define

$$\hat{\tau}^*_{k,n} = \frac{1}{k} \arg\min_{\delta \in \Theta_k} Q^*_n(\delta),$$

where for any $\delta \in \Theta_k$, $Q^*_n(\delta) = Q_n(\delta) + X(W_n \setminus W_{N_{n\delta}})|W_n|^{-1}$. Note that $X(W_n \setminus W_{N_{n\delta}})$ denotes the number of points in a realization of X inside the window W_n, which are not used in the construction of $Q_n(\delta)$. If λ is periodic and Lipschitz, then for each positive integer k satisfying $k = k_n \sim |W_n|^c$, for some $0 \leq c < \frac{1}{3}$, we have

$$|W_n|^{1/2}\left(\hat{\tau}^*_{k,n} - \tau\right) - N\left(0, \sigma_k^2\right) = o_p\left(k^{-1/2}\right),$$

as $n \to \infty$, where

$$\sigma_k^2 = \frac{\tau^3\ \theta}{\int_0^\tau (\lambda(s)-\theta)^2 ds} + \frac{\tau^4\ \theta^3}{4(2\theta k\tau+1)(\int_0^\tau (\lambda(s)-\theta)^2 ds)^2},$$

and $N(0, \sigma_k^2)$ denotes a normal r.v. with mean zero and variance σ_k^2.

1.4 Related work

In this section, we briefly review some related work. To begin with, Rathbun and Cressie (1994) investigated maximum likelihood estimators and Bayes estimators for regular parametric models with unknown finite dimensional parameter. Helmers and Zitikis (1999) consider a uniform kernel type estimator for $\lambda(s)$ in the case where λ is a parametric function of spatial location. These authors focus their attention to the case that X is a Poisson process on $[0, \infty)$ with intensity function

$$\lambda(s) = \exp\left\{\alpha + \beta s + \gamma s^2 + K_1 \sin(\omega_0 s) + K_2 \cos(\omega_0 s)\right\},$$

$s > 0$, where $\alpha, \beta, \gamma, K_1$, and K_2 are unknown parameters, and ω_0 is a known 'frequency'. This model is of importance in diverse fields of applied mathematics. Helmers and Zitikis (1999) obtain L_2-convergence of their estimator, as (1.2) holds. Their estimation method will be especially appropriate when the number of parameters is quite large and maximum likelihood estimation is difficult even numerically to carry out. Dorogovtsev and Kukush (1996) and Kukush and Mishura (1999) investigated consistency properties (including rates of convergence) of a nonparametric MLE of λ, the intensity function of a cyclic Poisson process X, with known period τ. To do this, it is assumed, in addition, that $\lambda|[0, \tau)$, the restriction of λ to $[0, \tau)$, belongs to a Sobolev space of functions on $[0, \tau)$. Also an algorithm for the computation of a nonparametric MLE is given. The paper by Dorogovtsev and Kukush (1996) restricts attention to the case that X is Poisson, while in Kukush and Mishura (1999) X may consists of three components: a drift, a diffusion and a cyclic Poisson process with known period τ.

In all these papers and also in this thesis λ is assumed to be fixed, but the observation window W_n increases, that is (1.2) holds. This approach appears to be a practical one, since the size of W_n is often under control of the researcher. In contrast, the 'infill asymptotic' framework, when the number of points observed in a fixed window W increases ($\lambda = \lambda_n \to \infty$), seems more restrictive for applications. In this setup one typically observes many realizations of a Poisson process X. For instance, Cowling, Hall, and Philips (1996) consider the latter setup and develop bootstrap methods for constructing confidence regions for the intensity function of a nonstationary Poisson process. In section 8.3.2 of Davison and Hinkley (1997) resampling methods for inhomogeneous Poisson processes are discussed within 'infill asymptotic' framework. The same type of asymptotics is employed in section 4.1 of Reiss (1993), where kernel type estimation of smooth Poisson intensity function is considered (cf. also Ellis (1991)). We also refer to chapter 6 of Karr (1986), section 8.5.1 of Cressie (1993), section 13.3.4 of Stoyan and Stoyan (1994), and to Kutoyants (1998) for a recent account of the statistical theory of estimating Poisson intensity functions.

An important paper in the context of the problem of estimating τ, the period of a cyclic Poisson process, is Vere-Jones (1982). He considers the problem of estimating the frequency ω_0 in a cyclic Poisson process X with intensity function

$$\lambda(s) = A \exp \{\rho \cos(\omega_0 s + \phi)\},$$

where $A > 0$, $\rho > 0$, $\omega_0 > 0$, and $\phi (0 < \phi < 2\pi)$ are unknown parameters. Vere-Jones (1982) established consistency of a periodogram estimate, derived asymptotic normality, and showed that the periodogram estimate reduces to a maximum likelihood estimate (of ω_0) in the specific parametric model he considers. A rate of almost sure convergence of order $o(n^{-1})$ is also obtained, where $(0, n)$ denotes the observation interval (cf. also chapter 5 of this thesis).

Chapter 2

Estimation of the global intensity

2.1 Introduction

In this chapter we focus on estimation of the global intensity θ, using only a single realization $X(\omega)$ of the cyclic Poisson process X observed in W_n. This chapter is a revised version of section 2 of Helmers and Mangku (2000). An estimator for this parameter of interest is given by

$$\hat{\theta}_n = X(W_n)/|W_n|. \tag{2.1}$$

One way to obtain the estimator $\hat{\theta}_n$ in (2.1) is as follows. If the Poisson process X is homogeneous, $\mu(B) = \lambda_0 \nu(B) = \lambda_0 |B|$, for some constant $\lambda_0 > 0$ and all Borel sets B, the local intensity is constant, i.e. $\lambda(s) = \lambda_0$ for all $s \in \mathbf{R}$. The global intensity θ is precisely equal to λ_0 in this very special case, and the maximum likelihood method can be applied to estimate θ. Let s_i, $i = 1, \ldots, X(W_n)$ denote the locations of the points in the realization $X(\omega)$ of the Poisson process, observed in W_n. Then, the likelihood of $(s_1, \ldots, s_{X(W_n)})$ is equal to

$$L_n = e^{-\int_{W_n} \lambda(s) ds} \prod_{i=1}^{X(W_n)} \lambda(s_i) = e^{-\lambda_0 |W_n|} \lambda_0^{X(W_n)},$$

where $X(W_n)$ denotes the observed number of points in W_n (cf. Cressie, (1993), p. 655). Maximizing $\ln L_n$ gives us:

$$\frac{d \ln L_n}{d\lambda_0} = \frac{d}{d\lambda_0} (-\lambda_0 |W_n| + X(W_n) \ln \lambda_0) = -|W_n| + \frac{X(W_n)}{\lambda_0} = 0,$$

which directly yields the MLE $\hat{\theta}_n = X(W_n)/|W_n|$ of λ_0 and hence of θ as well.

In Lemma 2.1 we prove that for the cyclic Poisson process θ is well-defined by (1.3) and can now also be written as

$$\theta = \frac{1}{\tau} \int_{U_\tau} \lambda(s) ds, \tag{2.2}$$

where U_τ denote any interval of length τ. In Lemma 2.2 we will show that $\hat{\theta}_n$ is a consistent estimator of the global intensity θ of X. Complete convergence (implying strong consistency) of $\hat{\theta}_n$ is established in Lemma 2.3, while the asymptotic normality $\hat{\theta}_n - \theta$, properly normalized, is derived in Theorem 2.4. A bootstrap CLT for $\hat{\theta}_n - \theta$ is established in Theorem 2.5.

2.2 Consistency

Lemma 2.1 *If λ is periodic (with period τ) and locally integrable, then*

$$\theta_n = \frac{\mathbf{E}X(W_n)}{|W_n|} = \frac{1}{|W_n|} \int_{W_n} \lambda(s) ds \to \theta, \tag{2.3}$$

as $n \to \infty$, with θ as in (2.2). Hence $\hat{\theta}_n$ is asymptotically unbiased in estimating θ.

Proof: Let $N_{n\tau} = [\frac{|W_n|}{\tau}]$. Let $W_{N_{n\tau}}$ denote an interval of length $\tau N_{n\tau}$ contained in W_n, and $R_{n\tau} = W_n \setminus W_{N_{n\tau}}$. Then we can write

$$\theta_n = \frac{|W_{N_{n\tau}}|}{|W_n|} \frac{1}{|W_{N_{n\tau}}|} \int_{W_{N_{n\tau}}} \lambda(s) ds + \frac{1}{|W_n|} \int_{R_{n\tau}} \lambda(s) ds. \tag{2.4}$$

First note that

$$\frac{1}{|W_{N_{n\tau}}|} \int_{W_{N_{n\tau}}} \lambda(s) ds = \theta \tag{2.5}$$

because λ is periodic with period τ. Since $|R_{n\tau}| < \tau$ for all n, we have that

$$\frac{|W_{N_{n\tau}}|}{|W_n|} = \frac{|W_n| - |R_{n\tau}|}{|W_n|} \to 1, \tag{2.6}$$

as $n \to \infty$. Because λ is locally integrable and $|R_{n\tau}| = \mathcal{O}(1)$, as $n \to \infty$, we also know that

$$\int_{R_{n\tau}} \lambda(s) ds = \mathcal{O}(1), \text{ as } n \to \infty.$$

Hence, the first term on the r.h.s. of (2.4) converges to θ, while its second term (by (1.2)) converges to zero, as $n \to \infty$. This completes the proof of Lemma 2.1. □

Lemma 2.2 *If λ is periodic (with period τ) and locally integrable, then*

$$\hat{\theta}_n \xrightarrow{p} \theta, \tag{2.7}$$

as $n \to \infty$.

Proof: To prove (2.7) we must show, for each $\epsilon > 0$,

$$\mathbf{P}(|\hat{\theta}_n - \theta| \geq \epsilon) \to 0, \tag{2.8}$$

as $n \to \infty$. Since $X(W_n)$ has Poisson distribution with parameter $\mu(W_n) = \int_{W_n} \lambda(s)ds$, we know that

$$\mathbf{E}X(W_n) = Var(X(W_n)) = \int_{W_n} \lambda(s)ds.$$

Then we have

$$\mathbf{E}(\hat{\theta}_n) = \frac{1}{|W_n|}\int_{W_n} \lambda(s)ds, \quad \text{and} \quad Var(\hat{\theta}_n) = \frac{1}{|W_n|^2}\int_{W_n} \lambda(s)ds.$$

Now we write

$$\mathbf{P}\left(|\hat{\theta}_n - \theta| \geq \epsilon\right) \leq \mathbf{P}\left(|\hat{\theta}_n - \mathbf{E}\hat{\theta}_n| + |\mathbf{E}\hat{\theta}_n - \theta| \geq \epsilon\right).$$

By Lemma 2.1, for sufficiently large n, we have $|\mathbf{E}\hat{\theta}_n - \theta| \leq \epsilon/2$. Then, for sufficiently large n, we have

$$\mathbf{P}\left(|\hat{\theta}_n - \theta| \geq \epsilon\right) \leq \mathbf{P}\left(|\hat{\theta}_n - \mathbf{E}\hat{\theta}_n| \geq \frac{\epsilon}{2}\right). \tag{2.9}$$

By Chebyshev's inequality and Lemma 2.1, the r.h.s. of (2.9) does not exceed

$$\frac{4Var(\hat{\theta}_n)}{\epsilon^2} = \frac{4}{\epsilon^2|W_n|^2}\int_{W_n} \lambda(s)ds = \frac{4}{\epsilon^2|W_n|}(\theta + o(1)), \tag{2.10}$$

as $n \to \infty$. By (1.2), the r.h.s. of (2.10) is $o(1)$, as $n \to \infty$. This completes the proof of Lemma 2.2. □

Throughout this thesis, for any random variables Y_n and Y on a probability space $(\Omega, \mathcal{A}, \mathbf{P})$, we write $Y_n \xrightarrow{c} Y$ to denote that Y_n *converges completely* to Y, as $n \to \infty$. We say that Y_n converges completely to Y if $\sum_{n=1}^{\infty} \mathbf{P}(|Y_n - Y| > \epsilon) < \infty$, for every $\epsilon > 0$.

Lemma 2.3 *Suppose that λ is periodic (with period τ) and locally integrable. If, in addition, for each $\epsilon > 0$,*

$$\sum_{n=1}^{\infty} \exp\{-\epsilon|W_n|\} < \infty, \tag{2.11}$$

then, as $n \to \infty$,

$$\hat{\theta}_n \xrightarrow{c} \theta. \tag{2.12}$$

Proof: To establish (2.12) we must show

$$\sum_{n=1}^{\infty} \mathbf{P}\left(|\hat{\theta}_n - \theta| > \epsilon\right) < \infty, \tag{2.13}$$

for each $\epsilon > 0$. Now, recall from the proof of Lemma 2.2 that, for sufficiently large n, the probability on the l.h.s. of (2.13) does not exceed that on the r.h.s. of (2.9). Then, to prove (2.13), it suffices to check that the probability on the r.h.s. of (2.9) is summable. By an application of Lemma A.1 (see Appendix), and with θ_n as in (2.3), the probability on the r.h.s. of (2.9) can be bounded above as follows.

$$\begin{aligned}
&\mathbf{P}\left(|\hat{\theta}_n - \mathbf{E}\hat{\theta}_n| \geq \frac{\epsilon}{2}\right) = \mathbf{P}\left(|W_n|^{-1}|X(W_n) - \mathbf{E}X(W_n)| \geq \frac{\epsilon}{2}\right) \\
&= \mathbf{P}\left((\mathbf{E}X(W_n))^{-1/2}|X(W_n) - \mathbf{E}X(W_n)| \geq \frac{\epsilon|W_n|}{2(\mathbf{E}X(W_n))^{1/2}}\right) \\
&\leq 2\exp\left\{-\frac{\epsilon^2 4^{-1}|W_n|^2(\mathbf{E}X(W_n))^{-1}}{2 + \epsilon 2^{-1}|W_n|(\mathbf{E}X(W_n))^{-1}}\right\} = 2\exp\left\{-\frac{\epsilon^2|W_n|}{8\theta_n + 2\epsilon}\right\}.
\end{aligned} \tag{2.14}$$

For sufficiently large n, since by Lemma 2.1 we have $\theta_n = \theta + o(1)$, as $n \to \infty$, the r.h.s. of (2.14) does not exceed $2\exp\{-(\epsilon^2|W_n|)(16\theta+2\epsilon)^{-1}\}$. By assumption (2.11), we can conclude that the quantity on the r.h.s. of (2.14) is summable. This completes the proof of Lemma 2.3. □

In view of Lemma 2.3, we may replace definition (1.3) of the global intensity θ by the following one

$$\theta = \lim_{n\to\infty} \frac{X(W_n)}{|W_n|} \qquad a.s.[\mathbf{P}] \tag{2.15}$$

provided (2.11) holds. Note that (2.15) is similar to the notion of global intensity described in (8.3.22) of Cressie (1993, p. 629).

2.3 Asymptotic normality

Asymptotic normality of $\hat{\theta}_n$, properly normalized, is established in the following theorem.

Theorem 2.4 *If λ is periodic (with period τ) and locally integrable, then*

$$|W_n|^{1/2}\left(\hat{\theta}_n - \theta\right) \xrightarrow{d} N(0,\theta), \tag{2.16}$$

as $n \to \infty$.

Proof: First we write

$$|W_n|^{1/2}\left(\hat{\theta}_n - \theta\right) = |W_n|^{1/2}\left(\hat{\theta}_n - \theta_n\right) + |W_n|^{1/2}(\theta_n - \theta), \tag{2.17}$$

where θ_n is given by the l.h.s. of (2.3). Then, to prove (2.16), it suffices to check

$$|W_n|^{1/2}(\hat{\theta}_n - \theta_n) \xrightarrow{d} N(0,\theta), \tag{2.18}$$

and

$$|W_n|^{1/2}(\theta_n - \theta) \to 0, \tag{2.19}$$

as $n \to \infty$.

First we prove (2.18). The l.h.s. of (2.18) can be written as

$$\begin{aligned} &|W_n|^{1/2}\left(\frac{X(W_n)}{|W_n|} - \frac{\int_{W_n}\lambda(s)ds}{|W_n|}\right) \\ &= \frac{(\int_{W_n}\lambda(s)ds)^{1/2}}{|W_n|^{1/2}}\left(\frac{X(W_n) - \int_{W_n}\lambda(s)ds}{(\int_{W_n}\lambda(s)ds)^{1/2}}\right). \end{aligned} \tag{2.20}$$

By Lemma 2.1, (1.2), and the normal approximation to the Poisson distribution, the r.h.s. of (2.20) can be written as $(\theta^{1/2}+o(1))(N(0,1)+o_p(1))$, which converges in distribution to $N(0,\theta)$ as $n \to \infty$.

Next we prove (2.19). Substituting (2.5) into the r.h.s. of (2.4), and by writing $|W_{N_{n\tau}}|$ as $(|W_n| - |R_{n\tau}|)$, we can simplify the r.h.s. of (2.4) to get

$$\theta_n = \theta - \frac{\theta|R_{n\tau}|}{|W_n|} + \frac{1}{|W_n|}\int_{R_{n\tau}}\lambda(s)ds. \tag{2.21}$$

The l.h.s. of (2.19) now reduces to

$$\begin{aligned} &|W_n|^{1/2}\left(-\frac{\theta|R_{n\tau}|}{|W_n|} + \frac{1}{|W_n|}\int_{R_{n\tau}}\lambda(s)ds\right) \\ &= \left(-\frac{\theta|R_{n\tau}|}{|W_n|^{1/2}} + \frac{\int_{R_{n\tau}}\lambda(s)ds}{|W_n|^{1/2}}\right). \end{aligned} \tag{2.22}$$

Since $|R_{n\tau}| < \tau$ for all n and $\int_{R_{n\tau}} \lambda(s)ds = \mathcal{O}(1)$, as $n \to \infty$, then by (1.2), the r.h.s. of (2.22) is $o(1)$, as $n \to \infty$. This completes the proof Theorem 2.4. □

To conclude this section we derive a bootstrap CLT, parallel to Theorem 2.4. Conditionally given $X(W_n)$, let $X^*(W_n)$ denote a realization from a Poisson distribution with parameter $X(W_n)$. If $X(W_n)$ happens to be equal to zero, we set $X^*(W_n) = 0$. Define

$$\hat{\theta}_n^* = \frac{X^*(W_n)}{|W_n|}. \tag{2.23}$$

To obtain a bootstrap counterpart of (2.16), we replace $\hat{\theta}_n - \theta$ by $\hat{\theta}_n^* - \hat{\theta}_n$, with $\hat{\theta}_n$ as in (2.1), and establish bootstrap consistency, i.e. we shall prove that $|W_n|^{\frac{1}{2}}(\hat{\theta}_n^* - \hat{\theta}_n)$ has - in P-probability - the same limit distribution as $|W_n|^{\frac{1}{2}}(\hat{\theta}_n - \theta)$, that is a normal $(0, \theta)$ distribution. Note that we have employed a 'parametric bootstrap' here. There is no use for Efron's bootstrap, instead our bootstrap is based on a parametric model, namely a Poisson distribution with estimated parameter.

Theorem 2.5 *If λ is periodic (with period τ) and locally integrable, then*

$$|W_n|^{1/2}\left(\hat{\theta}_n^* - \hat{\theta}_n\right) \xrightarrow{d} N(0,\theta), \tag{2.24}$$

as $n \to \infty$, in P-probability. Hence our parametric bootstrap works.

Proof: Since $X^*(W_n)$ has Poisson distribution with parameter $X(W_n)$, it suffices to write the l.h.s. of (2.24) as

$$|W_n|^{1/2}\left(\frac{X^*(W_n)}{|W_n|} - \frac{X(W_n)}{|W_n|}\right) = \left(\frac{X(W_n)}{|W_n|}\right)^{1/2}\left(\frac{X^*(W_n) - X(W_n)}{(X(W_n))^{1/2}}\right). \tag{2.25}$$

By Lemma 2.2, (1.2), and the normal approximation to the Poisson distribution, the r.h.s. of (2.25) can be written as

$$\left(\theta^{1/2} + o_p(1)\right)\left(N(0,1) + o_p(1)\right), \tag{2.26}$$

since $X(W_n) \to \infty$, in P-probability, as $\int_{W_n} \lambda(s)ds \to \infty$, which is implied by $|W_n| \to \infty$ (cf.(1.2)), because $\theta > 0$. Hence, by Slutsky (cf. Serfling (1980), p. 19), the quantity in (2.26) converges in distribution to $N(0, \theta)$, as $n \to \infty$, in P-probability. This completes the proof Theorem 2.5. □

Chapter 3

Kernel estimation of the local intensity

3.1 Introduction

In this chapter we consider kernel type estimation of the intensity function λ at a given point $s \in W_n$, using only a single realization $X(\omega)$ of the cyclic Poisson process X observed in W_n. The requirement $s \in W_n$ can be dropped if we know the period τ. This chapter is a revised version of Helmers, Mangku, and Zitikis (1999) and Helmers, Mangku, and Zitikis (2000).

Let $\hat{\tau}_n$ be any consistent estimator of the period τ, that is, $\hat{\tau}_n \xrightarrow{p} \tau$, as $n \to \infty$. For example, one may use the estimators constructed in chapter 5 or perhaps the estimator investigated by Vere-Jones (1982).

Furthermore, let $K : \mathbf{R} \to \mathbf{R}$ be a function, called a kernel, satisfying assumptions:

(K.1) K is a probability density function,

(K.2) K is bounded,

(K.3) K has support in $[-1, 1]$.

With these notations, we now define the estimator of $\lambda(s)$ as

$$\hat{\lambda}_{n,K}(s) := \frac{\hat{\tau}_n}{|W_n|} \sum_{k=-\infty}^{\infty} \frac{1}{h_n} \int_{W_n} K\left(\frac{x - (s + k\hat{\tau}_n)}{h_n}\right) X(dx), \tag{3.1}$$

where h_n is a sequence of positive real numbers converging to 0, that is,

$$h_n \downarrow 0 \tag{3.2}$$

as $n \to \infty$.

Let us now describe the idea behind the construction of the estimator $\hat{\lambda}_{n,K}(s)$. Note that, since there is only one realization of the Poisson process X available, we have to combine information about the (unknown) value of $\lambda(s)$ from different places of the window W_n. For this reason, the periodicity of λ, that is assumption (1.10), plays a crucial role and leads to the following string of (approximate) equations

$$\begin{aligned} \lambda(s) &= \frac{1}{N_n}\sum_{k=-\infty}^{\infty}\lambda(s+k\tau)\mathbf{I}\{s+k\tau\in W_n\} \\ &\approx \frac{1}{N_n}\sum_{k=-\infty}^{\infty}\frac{1}{|B_{h_n}(s+k\tau)|}\int_{B_{h_n}(s+k\tau)\cap W_n}\lambda(x)dx \\ &\approx \frac{1}{N_n}\sum_{k=-\infty}^{\infty}\frac{1}{2h_n}X(B_{h_n}(s+k\tau)\cap W_n) \\ &\approx \frac{\tau}{|W_n|}\sum_{k=-\infty}^{\infty}\frac{1}{2h_n}X(B_{h_n}(s+k\tau)\cap W_n), \end{aligned} \tag{3.3}$$

where

$$N_n = \#\{k:\ s+k\tau\in W_n\},$$

and $B_h(x)$ denotes the interval $[x-h, x+h]$. We note that, in order to make the first $\approx$ in (3.3) works, we require the assumptions that s is a Lebesgue point of λ and (3.2) holds true. Thus, from (3.3) we conclude that the quantity

$$\lambda_n(s) := \frac{\tau}{|W_n|}\sum_{k=-\infty}^{\infty}\frac{1}{2h_n}X(B_{h_n}(s+k\tau)\cap W_n), \tag{3.4}$$

can be viewed as an estimator of $\lambda(s)$, provided that the period τ is known.

The idea described in (3.3) and (3.4) of constructing an estimator for $\lambda(s)$ resembles that of Helmers and Zitikis (1999) where in a similar fashion a non-parametric estimator for an intensity function which, in addition to the periodic trend, also has a polynomial trend. In Helmers and Zitikis (1999), just like when constructing the estimator $\lambda_n(s)$ in (3.4), the period τ is supposed to be known.

The quantity $\lambda_n(s)$ of (3.4) can be modified in order to get an estimator of intensity functions with unknown periods. To do this, let $\hat{\tau}_n$ be a consistent estimator of τ. For example, $\hat{\tau}_n$ can be the estimator given in Chapter 5, or the one constructed by Vere-Jones (1982), or any other estimator of the period τ. Then, we modify the quantity in (3.4) by replacing the unknown period τ by its estimator $\hat{\tau}_n$ and obtain the following estimator

$$\hat{\lambda}_{n,\bar{K}}(s) = \frac{\hat{\tau}_n}{|W_n|} \sum_{k=-\infty}^{\infty} \frac{1}{2h_n} X\left(B_{h_n}(s + k\hat{\tau}_n) \cap W_n\right) \tag{3.5}$$

of $\lambda(s)$. Note that the estimator $\hat{\lambda}_{n,\bar{K}}(s)$ in (3.5) can be rewritten as

$$\hat{\lambda}_{n,\bar{K}}(s) = \frac{\hat{\tau}_n}{|W_n|} \sum_{k=-\infty}^{\infty} \frac{1}{h_n} \int_{W_n} \frac{1}{2} \mathbf{I}_{[-1,1]}\left(B_{h_n}(s + k\hat{\tau}_n)\right) X(dx). \tag{3.6}$$

By replacing the function $\bar{K} = 2^{-1}\mathbf{I}_{[-1,1]}(\cdot)$ in (3.6) by the general kernel K, we immediately arrive at the estimator introduced in (3.1).

3.2 Consistency

3.2.1 Results

To establish that $\hat{\lambda}_{n,K}(s)$ is a consistent estimator of $\lambda(s)$, we need to impose an additional assumption on the kernel K, that is

(K.4) K has only a finite number of discontinuities.

In section 3.2.3 we will indicate that (K.4) (or (K.4*)) is really needed in order to obtain a consistent kernel type estimator of $\lambda(s)$.

Theorem 3.1 *Let the intensity function λ be periodic and locally integrable, and let the kernel K satisfy assumptions (K.1)–(K.4). Furthermore, let the bandwidth h_n be such that (3.2) holds true, and*

$$h_n|W_n| \to \infty \tag{3.7}$$

as $n \to \infty$. If

$$|W_n||\hat{\tau}_n - \tau|/h_n \xrightarrow{p} 0 \tag{3.8}$$

as $n \to \infty$, then

$$\hat{\lambda}_{n,K}(s) \xrightarrow{p} \lambda(s) \tag{3.9}$$

as $n \to \infty$, provided s is a Lebesgue point of λ. In other words, $\hat{\lambda}_{n,K}(s)$ is a consistent estimator of $\lambda(s)$.

We note that, the assumption (K.4) can be weakened into the following one

(K.4*) For any $\alpha > 0$, there exists a finite collection of disjoint compact intervals $B_1,\ldots, B_{M_\alpha}$ and a continuous function $K_\alpha : \mathbf{R} \to \mathbf{R}$ such that the Lebesgue measure of the set $[-1,1] \setminus \cup_{i=1}^{M_\alpha} B_i$ does not exceed α, and $|K(u) - K_\alpha(u)| \leq \alpha$ for all $u \in \cup_{i=1}^{M_\alpha} B_i$.

Therefore, in the next subsection, we give proof of Theorems 3.1 and 3.2 under the weaker assumption (K.4*). This assumption will allow us to control the fluctuations of the function

$$x \mapsto K\left(\frac{x - (s + k\hat{\tau}_n)}{h_n}\right)$$

depending on the fluctuations of $\hat{\tau}_n$ around τ. In particular, it will exclude functions K like

$$K_0 := \frac{1}{2}\mathbf{I}_{[-1,1]\setminus \mathcal{Q}},$$

where $\mathcal{Q}$ stands for the set of all rational numbers, and $\mathbf{I}_A$ denotes the indicator function of the set A. A more detailed discussion on the necessity of excluding functions like K_0, which satisfies condition (K.1) - (K.3), is given in subsection 3.2.3.

We proceed with our current discussion concerning condition (K.4*), with the note that the measurability of function K (which is implicitly assumed by (K.1)) is only slightly weaker assumption than (K.4*). Indeed, according to the Lusin's theorem (cf., for example, Hewitt and Stromberg (1965), p. 159-160), we have that the measurability of K implies that

(L) For any $\alpha > 0$, there exists a compact set A_α and a continuous function $K_\alpha : \mathbf{R} \to \mathbf{R}$ such that the Lebesgue measure of the set $[-1,1] \setminus A_\alpha$ does not exceed α, and $|K(u) - K_\alpha(u)| \leq \alpha$ for all $u \in A_\alpha$.

Note, that by taking $A_\alpha = \cup_{i=1}^{M_\alpha} B_i$, we immediately obtain that any kernel satisfying assumption (K.4*) also satisfies (L). As it is easy to see, assumption (L) does not exclude the kernel function K_0 which, as it has already been mentioned above, is necessary in order to prove consistency of $\hat{\lambda}_{n,K}(s)$ (cf. subsection 3.2.3).

Though stronger than (L), assumption (K.4*) still covers all the kernel functions K of statistical relevance that we can think of. For example, any kernel K whose set of all discontinuity points can, for any fixed $\alpha > 0$, be covered by a finite collection of open intervals of total size not exceeding α obviously satisfies assumption (K.4*).

Under, naturally, stronger assumptions than those of Theorem 3.1, we also have the complete convergence of the estimator $\hat{\lambda}_{n,K}(s)$ which, in turn, gives a rate of consistency of the estimator $\hat{\lambda}_{n,K}(s)$.

Theorem 3.2 *Let the intensity function λ be periodic and locally integrable, and let the kernel K satisfy assumptions (K.1)–(K.4). Furthermore, let the bandwidth h_n be such that (3.2) holds true, and*

$$\sum_{n=1}^{\infty} \exp\{-\epsilon\sqrt{|W_n|h_n}\} < \infty \tag{3.10}$$

for any $\epsilon > 0$. If

$$|W_n||\hat{\tau}_n - \tau|/h_n \xrightarrow{c} 0, \tag{3.11}$$

then

$$\hat{\lambda}_{n,K}(s) \xrightarrow{c} \lambda(s), \tag{3.12}$$

provided s is a Lebesgue point of λ.

Remark 3.1 One may naturally want to know where the estimator $\hat{\lambda}_{n,K}(s)$ converges when it is not assumed that s is a Lebesgue point. A careful inspection of the proof (given in the next subsection) of Theorem 3.1 shows, for example, that under the assumption

$$\frac{1}{h_n}\int_{-h_n}^{h_n} \lambda(s+x)dx = O(1)$$

as $h_n \downarrow 0$, the estimator $\hat{\lambda}_{n,K}(s)$ estimates

$$\lambda^*(s) := \lim_{h\to 0}\int_{-1}^{1} K(x)\lambda(s+xh)dx, \tag{3.13}$$

provided that the limit in (3.13) exists. For example, if the left- and right-hand limits $\lambda(s-)$ and $\lambda(s-)$ of λ at s exist, then

$$\lambda^*(s) = \lambda(s-)\int_{-1}^{0} K(x)dx + \lambda(s+)\int_{0}^{1} K(x)dx.$$

Consequently, if we assume that the function K is symmetric, then, due to the fact that K is a probability density function by assumption (K.1), we have the following representation

$$\lambda^*(s) = \frac{1}{2}\{\lambda(s-) + \lambda(s+)\}.$$

In turn, if s is a continuity point of λ, then the latter representation implies the following one

$$\lambda^*(s) = \lambda(s), \tag{3.14}$$

as it should be expected. Let us note in passing that if λ is known to be either right- or left-continuous, then we also have equality (3.14), provided that K has "one-sided" supports $[0,1]$ and $[-1,0]$, respectively. □

3.2.2 Proofs

Theorems 3.1 and 3.2 are a consequence of the basic probabilistic tool, which is given in Theorem 3.3.

Theorem 3.3 *Let the intensity function λ be periodic and locally integrable, and let the kernel K satisfy assumptions (K.1)–(K.3) and (K.4*). Furthermore, let the bandwidth h_n be such that (3.2) holds true. Then there exists a constant c such that for every $\epsilon > 0$ there exists a (small) $\beta := \beta(\epsilon) > 0$ and a (large) $n_0(\epsilon)$ such that the bound*

$$\begin{aligned}\mathbf{P}\left(|\hat{\lambda}_{n,K}(s) - \lambda(s)| \geq \epsilon\right) &\leq c\exp\{-\epsilon\sqrt{|W_n|h_n}\} \\ &+ \mathbf{P}\left(|W_n|\,|\hat{\tau}_n - \tau| \geq \beta h_n\right), \end{aligned} \tag{3.15}$$

holds true for all $n \geq n_0(\epsilon)$, provided s is a Lebesgue point of the intensity function λ.

To prove Theorem 3.3, we need the following three lemmas.

Lemma 3.4 *Let the intensity function λ be periodic and locally integrable, and let the kernel K satisfy assumptions (K.1)–(K.3). Furthermore, let the bandwidth h_n be such that (3.2) holds true. Then*

$$\frac{1}{N_n}\sum_{k=-\infty}^{\infty}\frac{1}{h_n}\int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right)\lambda(x)dx \to \lambda(s), \tag{3.16}$$

provided s is a Lebesgue point of λ.

Proof: Obviously,

$$\begin{aligned}&\frac{1}{N_n}\sum_{k=-\infty}^{\infty}\frac{1}{h_n}\int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right)\lambda(x)dx \\ =\ &\frac{1}{N_n}\sum_{k=-\infty}^{\infty}\frac{1}{h_n}\int_{\mathbf{R}} K\left(\frac{x-(s+k\tau)}{h_n}\right)\lambda(x)\mathbf{I}(x\in W_n)dx \\ =\ &\frac{1}{N_n h_n}\int_{\mathbf{R}} K\left(\frac{x}{h_n}\right)\sum_{k=-\infty}^{\infty}\lambda(x+s+k\tau)\mathbf{I}(x+s+k\tau\in W_n)dx.\end{aligned} \tag{3.17}$$

Since λ is periodic with period τ, we have $\lambda(x+s+k\tau) = \lambda(x+s)$. Furthermore, it is obvious that

$$\sum_{k=-\infty}^{\infty}\mathbf{I}(x+s+k\tau\in W_n) \in [N_n - 1, N_n + 1]. \tag{3.18}$$

Consequently, the r.h.s of (3.17) converges to $\lambda(s)$ when $n \to \infty$, provided that

$$\frac{1}{h_n} \int_{\mathbf{R}} K\left(\frac{x}{h_n}\right) \lambda(x+s) dx \to \lambda(s). \tag{3.19}$$

Note that

$$\frac{1}{h_n} \int_{\mathbf{R}} K\left(\frac{x}{h_n}\right) \lambda(s) dx = \lambda(s) \int_{\mathbf{R}} K(x)\, dx = \lambda(s),$$

where we used the assumption that K is a probability density function. Consequently, statement (3.19) follows if

$$\frac{1}{h_n} \int_{\mathbf{R}} K\left(\frac{x}{h_n}\right) \{\lambda(x+s) - \lambda(s)\}\, dx \to 0,$$

when $n \to \infty$. The latter statement obviously follows from the assumptions that K is bounded and with support in $[-1, 1]$, and that s is a Lebesgue point of λ. This completes the proof of Lemma 3.4. □

Denote

$$\begin{aligned} D_n \quad := \quad & \frac{1}{N_n} \sum_{k=-\infty}^{\infty} \frac{1}{h_n} \int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right) X(dx) \\ & -\frac{1}{N_n} \sum_{k=-\infty}^{\infty} \frac{1}{h_n} \int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right) \lambda(x) dx. \end{aligned}$$

Lemma 3.5 *Let the intensity function λ be periodic and locally integrable, and let the kernel K satisfy assumptions (K.1)–(K.3). Furthermore, let the bandwidth h_n be such that (3.2) holds true. Then there is a (large) constant n_1 such that for any constant $c_1 > 0$ there exists another one $c_2 > 0$ such that*

$$\mathbf{P}\left(|D_n| \geq c_1 \epsilon\right) \leq c_2 \exp\left\{-\epsilon\sqrt{|W_n| h_n}\right\}, \tag{3.20}$$

for every $\epsilon > 0$ and all $n \geq n_1$, provided s is a Lebesgue point of λ.

Proof: For every $t > 0$, we have that

$$\mathbf{P}\left(|D_n| \geq c_1 \epsilon\right) \leq \exp\{-c_1 \epsilon t\}\left(\mathbf{E}\exp\{tD_n\} + \mathbf{E}\exp\{-tD_n\}\right). \tag{3.21}$$

To make our further considerations more transparent, we denote

$$Y_k := \int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right) X(dx)$$

and then rewrite D_n as

$$D_n = \frac{1}{N_n} \sum_{k=-\infty}^{\infty} \frac{1}{h_n} \{Y_k - \mathbf{E} Y_k\}. \tag{3.22}$$

Since $h_n \downarrow 0$, the random variables Y_k, $k = 1, 2, \ldots$ are independent for all sufficiently large n (depending on the period τ). Thus, for sufficiently large n, we obtain

$$\mathbf{E} \exp\{\pm t D_n\} = \prod_{k=-\infty}^{\infty} \mathbf{E} \exp\left\{\pm \frac{t}{N_n h_n}(Y_k - \mathbf{E} Y_k)\right\}. \tag{3.23}$$

Using the well known formula for the Laplace transform of the Poisson process, we obtain that

$$\mathbf{E} \exp\left\{\pm \frac{t}{N_n h_n} Y_k\right\} = \exp\left\{\int_{W_n} (e^{K^*(x)} - 1)\lambda(x) dx\right\}, \tag{3.24}$$

where we used the notation

$$K^*(x) := \pm \frac{t}{N_n h_n} K\left(\frac{x - (s + k\tau)}{h_n}\right).$$

Consequently, for every factor on the r.h.s. of (3.23) we have the following formula (cf. Lemma A.3)

$$\begin{aligned} &\mathbf{E} \exp\left\{\pm \frac{t}{N_n h_n} \{Y_k - \mathbf{E} Y_k\}\right\} \\ &= \exp\left\{\int_{W_n} (e^{K^*(x)} - 1 - K^*(x))\lambda(x) dx\right\}. \end{aligned} \tag{3.25}$$

Since $|\exp(x) - 1 - x|$ does not exceed $x^2 \exp(|x|)$, we obtain from (3.25) that

$$\mathbf{E} \exp\left\{\pm \frac{t}{N_n h_n} \{Y_k - \mathbf{E} Y_k\}\right\} \leq \exp\left\{\int_{W_n} |K^*(x)|^2 e^{|K^*(x)|} \lambda(x) dx\right\}. \tag{3.26}$$

We now make the following choice

$$t := \frac{1}{c_1} \sqrt{N_n h_n}. \tag{3.27}$$

Using the assumption that K is bounded and has support in the interval $[-1, 1]$, we obtain from (3.26) with (3.27) that

$$\mathbf{E} \exp\left\{\pm \frac{t}{N_n h_n} \{Y_k - \mathbf{E} Y_k\}\right\} \leq \exp\left\{c \frac{1}{N_n h_n} \mu\left(B_{h_n}(s + k\tau) \cap W_n\right)\right\}, \tag{3.28}$$

for a constant c that does not depend on n. Applying bound (3.28) on the r.h.s. of (3.23), we obtain

$$\mathbf{E}\exp\{\pm tD_n\} \le \exp\left\{c\frac{1}{N_n}\sum_{k=-\infty}^{\infty}\frac{1}{h_n}\mu\left(B_{h_n}(s+k\tau)\cap W_n\right)\right\}. \quad (3.29)$$

Furthermore, we note that the quantity $\mu\left(B_{h_n}(s+k\tau)\cap W_n\right)$ obviously equals to

$$\int_{B_{h_n}(0)} \lambda(s+k\tau+x)\mathbf{I}(s+k\tau+x\in W_n)dx.$$

Consequently, using the periodicity of λ and (3.18) on the r.h.s. of (3.29), we obtain that

$$\mathbf{E}\exp\{\pm tD_n\} \le \exp\left\{c\frac{1}{h_n}\int_{B_{h_n}(0)}\lambda(s+x)dx\right\}.$$

Since s is a Lebesgue point of λ, we have that

$$\frac{1}{2h_n}\int_{B_{h_n}(0)}\lambda(s+x)dx \to \lambda(s),$$

when $n\to\infty$. Thus,

$$\lim_{n\to\infty}\mathbf{E}\exp\{\pm tD_n\} \le c < \infty. \quad (3.30)$$

Bound (3.30), when applied on the r.h.s. of (3.21), implies that

$$\mathbf{P}\left(|D_n|\ge\epsilon\right) \le \exp\left\{-\epsilon\sqrt{N_n h_n}\right\},$$

due to our choice of t as in (3.27). Lemma 3.5 is therefore proved. □

In our next lemma we use the notation

$$\begin{aligned} F_n &= \frac{1}{N_n}\sum_{k=-\infty}^{\infty}\frac{1}{h_n}\int_{W_n}K\left(\frac{x-(s+k\tau)}{h_n}\right)X(dx) \\ &\quad -\frac{1}{N_n}\sum_{k=-\infty}^{\infty}\frac{1}{h_n}\int_{W_n}K\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right)X(dx). \end{aligned} \quad (3.31)$$

Lemma 3.6 *Let the intensity function λ be periodic and locally integrable, and let the kernel K satisfy assumptions $(K.1)-(K.3)$ and $(K.4*)$. Furthermore, let the bandwidth h_n be such that (3.2) holds true. Then there exists a constant c such that for every $\epsilon > 0$, there exists a (small) $\beta := \beta(\epsilon) > 0$ and a (large) $n(\epsilon)\in\mathbf{N}$ such that the bound*

$$\mathbf{P}\left(|F_n|\ge\epsilon\right) \le c\exp\left\{-\epsilon\sqrt{|W_n|h_n}\right\} + \mathbf{P}\left(|W_n||\hat{\tau}_n-\tau|\ge\beta h_n\right), \quad (3.32)$$

holds true for all $n\ge n(\epsilon)$, provided s is a Lebesgue point of λ.

Proof. Fix any $\alpha > 0$ and denote

$$A_\alpha := \bigcup_{i=1}^{M_\alpha} B_i \subset [-1, 1], \tag{3.33}$$

where B_1,..., B_{M_α} are compact disjoint intervals defined in assumption (K.4*). Furthermore, using the (continuous) function K_α of assumption (K.4*) and the Weierstrass's theorem, we get that there exists a Lipschitz function L_α such that $|K(u) - L_\alpha(u)| \leq \alpha$ for all $u \in A_\alpha$. Now, we decompose K on the right-hand side of (3.31) into the following sum of three functions:

$$\begin{aligned} K(u) = \quad & \{K(u) - L_\alpha(u)\}\mathbf{I}_{A_\alpha^c}(u) \\ & +\{K(u) - L_\alpha(u)\}\mathbf{I}_{A_\alpha}(u) \\ & +L_\alpha(u). \end{aligned} \tag{3.34}$$

Using decomposition (3.34), we in turn decompose F_n into the sum of the corresponding three quantities that we are now to define and estimate. Since K and L_α are bounded, we easily see that the first quantity

$$\left| \frac{1}{N_n} \sum_{k=-\infty}^{\infty} \frac{1}{h_n} \int_{W_n} (K - L_\alpha)\left(\frac{x-(s+k\tau)}{h_n}\right) \mathbf{I}_{A_\alpha^c}\left(\frac{x-(s+k\tau)}{h_n}\right) X(dx) \right.$$
$$\left. - \frac{1}{N_n} \sum_{k=-\infty}^{\infty} \frac{1}{h_n} \int_{W_n} (K - L_\alpha)\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right) \mathbf{I}_{A_\alpha^c}\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right) X(dx) \right|$$

does not exceed the sum of the following two quantities

$$F_{n,1} := c(K, L_\alpha) \frac{1}{N_n} \sum_{k=-\infty}^{\infty} \frac{1}{h_n} X\left(\{s + k\tau + h_n A_\alpha^c\} \cap W_n\right),$$

$$F_{n,2} := c(K, L_\alpha) \frac{1}{N_n} \sum_{k=-\infty}^{\infty} \frac{1}{h_n} X\left(\{s + k\hat{\tau}_n + h_n A_\alpha^c\} \cap W_n\right),$$

where $c(K, L_\alpha)$ denotes a constant depending only on $\sup\{|K(u)| : u \in [-1, 1]\}$ and $\sup\{|L_\alpha(u)| : u \in [-1, 1]\}$. The second quantity

$$\left| \frac{1}{N_n} \sum_{k=-\infty}^{\infty} \frac{1}{h_n} \int_{W_n} (K - L_\alpha)\left(\frac{x-(s+k\tau)}{h_n}\right) \mathbf{I}_{A_\alpha}\left(\frac{x-(s+k\tau)}{h_n}\right) X(dx) \right.$$
$$\left. - \frac{1}{N_n} \sum_{k=-\infty}^{\infty} \frac{1}{h_n} \int_{W_n} (K - L_\alpha)\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right) \mathbf{I}_{A_\alpha}\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right) X(dx) \right|$$

does not exceed the sum of the following two quantities

$$F_{n,3} := \alpha \frac{1}{N_n} \sum_{k=-\infty}^{\infty} \frac{1}{h_n} X\left(\{s + k\tau + h_n[-1, 1]\} \cap W_n\right),$$

$$F_{n,4} \quad := \quad \alpha \frac{1}{N_n} \sum_{k=-\infty}^{\infty} \frac{1}{h_n} X\big(\{s + k\hat{\tau}_n + h_n[-1,1]\} \cap W_n\big).$$

Next, without loss of generality we assume that the support of the Lipschitz function L_α is in the interval $[-1,1]$. Using this fact, we obtain that

$$|L_\alpha(u) - L_\alpha(v)| \leq c(L_\alpha)|u - v|\big(\mathbf{I}\{u \in [-1,1]\} + \mathbf{I}\{v \in [-1,1]\}\big) \tag{3.35}$$

for all $u, v \in [-1,1]$. Let $I_0 = (-\infty, -1)$, $I_1 = [-1,1]$, and $I_2 = (1, \infty)$. Consequently, the third quantity

$$\left| \frac{1}{N_n} \sum_{k=-\infty}^{\infty} \frac{1}{h_n} \int_{W_n} L_\alpha\left(\frac{x - (s + k\tau)}{h_n}\right) X(dx) \right.$$
$$\left. - \frac{1}{N_n} \sum_{k=-\infty}^{\infty} \frac{1}{h_n} \int_{W_n} L_\alpha\left(\frac{x - (s + k\hat{\tau}_n)}{h_n}\right) X(dx) \right|$$

does not exceed the sum of the following three quantities

$$F_{n,5} \quad := \quad c(L_\alpha) \frac{1}{N_n} \sum_{k=-\infty}^{\infty} \left|\frac{k(\hat{\tau}_n - \tau)}{h_n}\right| \frac{1}{h_n} X\big(\{s + k\tau + h_n[-1,1]\} \cap W_n\big),$$

$$F_{n,6} \quad := \quad c(L_\alpha) \frac{1}{N_n} \sum_{k=-\infty}^{\infty} \left|\frac{k(\hat{\tau}_n - \tau)}{h_n}\right| \frac{1}{h_n} X\big(\{s + k\hat{\tau}_n + h_n[-1,1]\} \cap W_n\big),$$

$$\bar{F}_{n,7} \quad := \quad c(L_\alpha) \frac{1}{N_n} \sum_{0 \leq i \neq j \leq 2} \sum_{k=-\infty}^{\infty} \frac{1}{h_n}$$
$$X\big(\{s + k\hat{\tau}_n + h_n I_i\} \cap \{s + k\tau + h_n I_j\} \cap W_n\big).$$

Note that the upper bounds $F_{n,5}$ and $F_{n,6}$ correspond to the case where both points $(x - (s + k\hat{\tau}_n))/h_n$ and $(x - (s + k\tau))/h_n$ are in the same interval $[-1,1]$, which is equivalent to the case $x \in \{s + k\hat{\tau}_n + h_n[-1,1]\} \cap \{s + k\tau + h_n[-1,1]\}$, so that we can apply (3.35). The upper bound $\bar{F}_{n,7}$ corresponds to the other cases. Before proceeding further, we estimate $\bar{F}_{n,7}$. We can easily see that if, for example, $i = 0$ and $j = 1$, then

$$\sum_{k=-\infty}^{\infty} X\big(\{s + k\hat{\tau}_n + h_n I_i\} \cap \{s + k\tau + h_n I_j\} \cap W_n\big)$$
$$= \sum_{k=-\infty}^{\infty} X\big(\{s + k\tau + k(\hat{\tau}_n - \tau) + h_n I_0 \cap \{s + k\tau + h_n I_1\} \cap W_n\big)$$
$$\leq \sum_{k=-\infty}^{\infty} X\big([s + k\tau - h_n, s + k\tau - h_n + |k(\hat{\tau}_n - \tau)|) \cap W_n\big).$$

Similar estimates are valid for the other three cases: $i = 1$ and $j = 0$, $i = 1$ and $j = 2$, and $i = 2$ and $j = 1$. These bounds show that $\bar{F}_{n,7}$ does not exceed

$$F_{n,7} := \frac{c(L_\alpha)}{N_n} \sum_{i=1}^{2} \sum_{k=-\infty}^{\infty} \frac{1}{h_n}$$
$$X\left(\left\{s + k\tau + a_i h_n + h_n \left[-\left|\frac{k(\hat{\tau}_n - \tau)}{h_n}\right|, \left|\frac{k(\hat{\tau}_n - \tau)}{h_n}\right|\right]\right\} \cap W_n\right),$$

where $a_1 := -1$ and $a_2 := 1$.

The results obtained above show, in particular, that the probability of the event $F_n \geq \epsilon$ does not exceed the probability of the event $F_{n,1} + \cdots + F_{n,7} \geq \epsilon$. Thus, for any $\beta > 0$,

$$\begin{aligned}\mathbf{P}\left(F_n \geq \epsilon\right) \;&\leq\; \mathbf{P}\left(F_{n,1} + \cdots + F_{n,7} \geq \epsilon,\ |W_n|\,|\hat{\tau}_n - \tau| \leq \beta h_n\right) \\ &+\; \mathbf{P}\left(|W_n|\,|\hat{\tau}_n - \tau| \geq \beta h_n\right).\end{aligned}$$

We shall now estimate $F_{n,1}$,..., $F_{n,7}$ under the restriction $|W_n|\,|\hat{\tau}_n - \tau| \leq \beta h_n$. We start with the observation that even though $F_{n,1}$,..., $F_{n,7}$ are infinite sums they are actually sums of only finite numbers of non-zero summands. Indeed, due to the assumptions $h_n \to 0$ and $|W_n| \to \infty$, we have that, for all sufficiently large n, the summands of $F_{n,1}$,..., $F_{n,7}$ are equal to 0 for all indices k such that

$$|k| > \frac{2}{\tau}|W_n|.$$

Consequently, when estimating $F_{n,1}$,..., $F_{n,7}$ we can restrict ourselves to the summands with indices k such that

$$|k| \leq \frac{2}{\tau}|W_n|.$$

This immediately implies the following bounds

$$\begin{aligned} F_{n,1}, F_{n,2} &\leq c(K, L_\alpha) F^*_{n,1}, \\ F_{n,3}, F_{n,4} &\leq \alpha F^{**}_n, \\ F_{n,5}, F_{n,6} &\leq \tfrac{2\beta}{\tau} c(L_\alpha) F^{**}_n, \\ F_{n,7} &\leq c(L_\alpha) F^*_{n,2}, \end{aligned}$$

where we have denoted

$$F^*_{n,1} \;:=\; \frac{1}{N_n} \sum_{k=-\infty}^{\infty} \frac{1}{h_n} X\left(\left\{s + k\tau + h_n\big[-2\beta/\tau, 2\beta/\tau\big] + h_n A^c_\alpha\right\} \cap W_n\right),$$

$$F_n^{**} := \frac{1}{N_n}\sum_{k=-\infty}^{\infty}\frac{1}{h_n}X\left(\{s+k\tau+h_n[-1-2\beta/\tau, 1+2\beta/\tau]\}\cap W_n\right),$$

$$F_{n,2}^{*} := \frac{1}{N_n}\sum_{i=1}^{2}\sum_{k=-\infty}^{\infty}\frac{1}{h_n}X\left(\{s+k\tau+a_ih_n+h_n[-2\beta/\tau, 2\beta/\tau]\}\cap W_n\right).$$

We now see that if the set A_α^c in $F_{n,1}^*$ is replaced by $\{-1\}\cup\{1\}$, then $F_{n,1}^*$ reduces to $F_{n,2}^*$. To combine these upper bounds, we argue as follows. Define $\bar{A}_\alpha^c = A_\alpha^c \cup \{-1\}\cup\{1\}$, and let F_n^* denote $F_{n,1}^*$ with A_α^c now replaced by $\bar{A}_\alpha^c$. Note that the size of $\bar{A}_\alpha^c$ is the same as that of A_α^c, which does not exceed α. Then we have the bound

$$c(K,L_\alpha)F_{n,1}^* + c(L_\alpha)F_{n,2}^* \le \{c(K,L_\alpha)+c(L_\alpha)\}F_n^*.$$

Consequently, we have proved the following bound

$$\begin{aligned}
&\mathbf{P}\left(|F_n|\ge\epsilon\right)\\
&\le \mathbf{P}\left(\{c(K,L_\alpha)+c(L_\alpha)\}F_n^* + \{\alpha+2\beta\tau^{-1}c(L_\alpha)\}F_n^{**}\ge\epsilon\right)\\
&+\mathbf{P}\left(|W_n|\,|\hat{\tau}_n-\tau|\ge\beta h_n\right).
\end{aligned}$$

The latter bound shows that the proof of Lemma 3.6 is completed if we show that

$$\begin{aligned}
&\mathbf{P}\left(\{c(K,L_\alpha)+c(L_\alpha)\}F_n^* + \{\alpha+\beta c(L_\alpha)\}F_n^{**}\ge\epsilon\right)\\
&\le c\exp\left\{-\epsilon\sqrt{|W_n|h_n}\right\}.
\end{aligned} \tag{3.36}$$

The left-hand side of bound (3.36) does not exceed

$$\mathbf{P}\left(\{c(K,L_\alpha)+c(L_\alpha)\}|F_n^*-\mathbf{E}F_n^*| + \{\alpha+\beta c(L_\alpha)\}|F_n^{**}-\mathbf{E}F_n^{**}|\ge c_\epsilon\right), \tag{3.37}$$

where

$$c_\epsilon := \epsilon - \{c(K,L_\alpha)+c(L_\alpha)\}\mathbf{E}F_n^* - \{\alpha+\beta c(L_\alpha)\}\mathbf{E}F_n^{**}.$$

We now want to show that the parameters α and β can be chosen in such a way that, for example,

$$c_\epsilon \ge \frac{\epsilon}{2} \tag{3.38}$$

when n is sufficiently large. To start with, we note that $\mathbf{E}F_n^{**}$ can be rewritten in the following way

$$2\left(1+\frac{2\beta}{\tau}\right)\frac{1}{N_n}\sum_{k=-\infty}^{\infty}\frac{1}{h_n^*}\int_{W_n}\frac{1}{2}\mathbf{I}_{[-1,1]}\left(\frac{x-(s+k\tau)}{h_n^*}\right)\lambda(x)dx, \tag{3.39}$$

where $h_n^* := (1+2\beta/\tau)h_n$. Using Lemma 3.4 with $K = 2^{-1}\mathbf{I}_{[-1,1]}$, we immediately obtain that the quantity of (3.39) converges to $2(1+2\beta/\tau)\lambda(s)$ when $n \to \infty$, and so does $\mathbf{E}F_n^{**}$. This implies that by choosing $\alpha > 0$ and $\beta > 0$ sufficiently small, we can make the quantity $\{\alpha+\beta c(L_\alpha)\}\mathbf{E}F_n^{**}$ smaller than $\epsilon/4$ for all sufficiently large n. In view of this fact, we obtain the desired bound (3.38), provided that

$$\{c(K, L_\alpha) + c(L_\alpha)\}\mathbf{E}F_n^* \le \frac{\epsilon}{4} \tag{3.40}$$

for all sufficiently large n. We are now to prove (3.40). Denote

$$\mathcal{B} := [-2\beta\tau^{-1}, 2\beta\tau^{-1}] + \bar{A}_\alpha^c$$

for notational simplicity. Then

$$\begin{aligned}
\mathbf{E}F_n^* &= \frac{1}{N_n}\sum_{k=-\infty}^{\infty}\frac{1}{h_n}\mathbf{E}X\left(\{s+k\tau+h_n\mathcal{B}\}\cap W_n\right)\\
&= \frac{1}{N_n}\sum_{k=-\infty}^{\infty}\frac{1}{h_n}\int_{h_n\mathcal{B}}\lambda(x+s+k\tau)\mathbf{I}(x+s+k\tau\in W_n)dx\\
&= \frac{1}{N_n h_n}\int_{h_n\mathcal{B}}\lambda(x+s)\sum_{k=-\infty}^{\infty}\mathbf{I}(x+s+k\tau\in W_n)dx\\
&\le \frac{2}{h_n}\int_{h_n\mathcal{B}}\lambda(x+s)dx\\
&\le \frac{2}{h_n}\left|\int_{h_n\mathcal{B}}\{\lambda(x+s)-\lambda(s)\}dx\right| + 2\lambda(s)|\mathcal{B}|.
\end{aligned}\tag{3.41}$$

Note that the first summand on the right-hand side of (3.41) converges to 0, due to the assumption that s is a Lebesgue point of λ. Thus, in order to achieve the desired bound (3.40) we have to check that by choosing sufficiently small $\alpha > 0$ and $\beta > 0$ we can make the quantity $|\mathcal{B}|$ as small as we want. Here, and only here, we use assumption (K.4*). (We note in passing that if we do not assume (K.4*), then we only have (L). In this case, the set A_α^c can be so scattered over the interval $[-1,1]$ that the set $[-\beta,\beta]+A_\alpha^c$ may fill almost all interval $[-1,1]$ and thus the Lebesgue measure of $[-\beta,\beta]+A_\alpha^c$ may be close, for example, to that of $[-1,1]$ – the case which we want to avoid by assuming (K.4*)).

By choosing $\beta > 0$ sufficiently small, we can achieve the situation when $\mathcal{B}$ is a union of disjoint sets $[-2\beta/\tau, 2\beta/\tau] + B_i$ and $\{-1\}\cup\{1\}$, $i = 1,\dots, M_\alpha$. Consequently,

$$\begin{aligned}
|\mathcal{B}| &= \sum_{i=1}^{M_\alpha}\left|[-2\beta\tau^{-1}, 2\beta\tau^{-1}] + B_i\right| = \sum_{i=1}^{M_\alpha}|B_i| + 4M_\alpha\beta\tau^{-1}\\
&= |A_\alpha^c| + 4M_\alpha\beta\tau^{-1} \le \alpha + 4M_\alpha\beta\tau^{-1}.
\end{aligned}\tag{3.42}$$

Obviously, the right-hand side of (3.42) can be made as small as we want by choosing $\alpha > 0$ and $\beta > 0$ sufficiently small. Thus, the desired bound (3.40) can indeed be achieved for all sufficiently large n. This, in turn, implies that, for all sufficiently large n, the quantity of (3.37) does not exceed

$$\mathbf{P}\left(\{c(K,L_\alpha)+c(L_\alpha)\}|F_n^* - \mathbf{E}F_n^*| + \{\alpha+\beta c(L_\alpha)\}|F_n^{**} - \mathbf{E}F_n^{**}| \geq \frac{\epsilon}{2}\right).$$

The latter quantity does not exceed the sum of $\mathbf{P}(|F_n^* - \mathbf{E}F_n^*| \geq c_1^*\epsilon)$ and $\mathbf{P}(|F_n^{**} - \mathbf{E}F_n^{**}| \geq c_1^{**}\epsilon)$, where $c_1^* > 0$ and $c_1^{**} > 0$ are some constants. Using Lemma 3.5 with the kernel $K := |\mathcal{B}|^{-1}\mathbf{I}_{\mathcal{B}}$ we obtain the bound

$$\mathbf{P}\big(|F_n^* - \mathbf{E}F_n^*| \geq c_1^*\epsilon\big) \leq c_2^* \exp\big\{-\epsilon\sqrt{|W_n|h_n}\big\},$$

Furthermore, an application of Lemma 3.5 with the kernel $K := |\bar{\mathcal{B}}|^{-1}\mathbf{I}_{\bar{\mathcal{B}}}$, where

$$\bar{\mathcal{B}} := \big[-1-2\beta\tau^{-1}, 1+2\beta\tau^{-1}\big]$$

implies

$$\mathbf{P}\left(|F_n^{**} - \mathbf{E}F_n^{**}| \geq c_1^{**}\epsilon\right) \leq c_2^{**} \exp\big\{-\epsilon\sqrt{|W_n|h_n}\big\},$$

Thus, the quantity of (3.37) does not exceed $c\exp\{-\epsilon\sqrt{|W_n|h_n}\}$, which completes the proof of bound (3.36) and, in turn, of Lemma 3.6. □

Proof of Theorem 3.3: Denote

$$\bar{\lambda}_{n,K}(s) := \frac{1}{N_n}\sum_{k=-\infty}^{\infty}\frac{1}{h_n}\int_{W_n} K\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right)X(dx).$$

Elementary algebra shows that

$$\begin{aligned}&\mathbf{P}\left(|\hat{\lambda}_{n,K}(s) - \lambda(s)| \geq \epsilon\right)\\ &\leq \mathbf{P}\left(\left\{\left|\frac{\hat{\tau}_n N_n}{|W_n|}-1\right|+1\right\}|\bar{\lambda}_{n,K}(s)-\lambda(s)| + \left|\frac{\hat{\tau}_n N_n}{|W_n|}-1\right|\lambda(s) \geq \epsilon\right). \quad (3.43)\end{aligned}$$

It is also easy to check that

$$\begin{aligned}\left|\frac{\hat{\tau}_n N_n}{|W_n|}-1\right| &\leq \frac{|\hat{\tau}_n-\tau|}{\tau}\left|\frac{\tau N_n}{|W_n|}-1\right| + \frac{|\hat{\tau}_n-\tau|}{\tau} + \left|\frac{\tau N_n}{|W_n|}-1\right|\\ &\leq \frac{|\hat{\tau}_n-\tau|}{\tau}\left(\frac{\tau}{|W_n|}+1\right) + \frac{\tau}{|W_n|}, \quad (3.44)\end{aligned}$$

where the second bound of (3.44) was obtained using $|\tau N_n - |W_n|| \leq \tau$. Since $|W_n|$ converges to ∞ by assumption, we can make the right-hand side of (3.44) as small as we want provided that we assume $|W_n|\,|\hat{\tau}_n - \tau| \leq \beta h_n$. Consequently, the right-hand side of (3.43) does not exceed

$$\mathbf{P}\left(|\bar{\lambda}_{n,K}(s) - \lambda(s)| \geq \frac{\epsilon}{2}\right) + \mathbf{P}\left(|W_n|\,|\hat{\tau}_n - \tau| \geq \beta h_n\right), \tag{3.45}$$

It is easy to see that Lemmas 3.4 – 3.6 taken together imply that for every $\epsilon > 0$ there exists a (small) $\beta := \beta(\epsilon) > 0$ and a (large) $n(\epsilon) \in \mathbf{N}$ such that the bound

$$\begin{aligned}&\mathbf{P}\left(|\bar{\lambda}_{n,K}(s) - \lambda(s)| \geq \epsilon\right)\\ &\leq c\exp\left\{-\epsilon\sqrt{|W_n|h_n}\right\} + \mathbf{P}\left(|W_n|\,|\hat{\tau}_n - \tau| \geq \beta h_n\right)\end{aligned} \tag{3.46}$$

holds true for all $n \geq n(\epsilon)$. Bounds (3.45) and (3.46) taken together complete the proof of Theorem 3.3. □

3.2.3 The kernel K_0

We will now discuss the role of assumption (K.4*) in our considerations and in Theorem 3.1 in particular, and give an explanation about the necessity to exclude kernel functions like K_0. Let us decompose K_0 as

$$K_0 = K_1 - K_2,$$

where $K_1 := \frac{1}{2}\mathbf{I}_{[-1,1]}$, and $K_2 := \frac{1}{2}\mathbf{I}_{[-1,1]\cap\mathcal{Q}}$, with $\mathcal{Q}$ stands for the set of all rational numbers. Consequently, we have the following decomposition

$$\hat{\lambda}_{n,K_0}(s) = \hat{\lambda}_{n,K_1}(s) - \hat{\lambda}_{n,K_2}(s). \tag{3.47}$$

Note that the kernel K_1 satisfies all four assumptions (K.1)-(K.3), (K.4*). Therefore, by Theorem 3.3, we have the following bound

$$\begin{aligned}\mathbf{P}\left\{|\hat{\lambda}_{n,K_1}(s) - \lambda(s)| \geq \epsilon\right\} \;\leq\; & c\exp\left\{-\epsilon\sqrt{|W_n|h_n}\right\}\\ & +\mathbf{P}\left\{|W_n|\,|\hat{\tau}_n - \tau| \geq \beta h_n\right\},\end{aligned}$$

with the same parameters as in Theorem 3.3. We now easily see that if $h_n|W_n| \to \infty$ and $|W_n||\hat{\tau}_n - \tau|/h_n \xrightarrow{p} 0$, as $n \to \infty$, then $\hat{\lambda}_{n,K_1}(s)$ is a consistent estimator of $\lambda(s)$. In view of this fact and decomposition (3.47), the random variable $\hat{\lambda}_{n,K_0}(s)$ can be a consistent estimator of $\lambda(s)$ if and only if

$$\hat{\lambda}_{n,K_2}(s) \xrightarrow{p} 0, \tag{3.48}$$

as $n \to \infty$. Let us now look at $\hat{\lambda}_{n,K_2}(s)$ more closely. By its definition, $\hat{\lambda}_{n,K_2}(s)$ has the following form

$$\hat{\lambda}_{n,K_2}(s) = \frac{\hat{\tau}_n}{|W_n|}\sum_{k=-\infty}^{\infty}\frac{1}{2h_n}X\left(\{s + k\hat{\tau}_n + h_n\mathcal{Q}\} \cap W_n\right).$$

If $\hat{\tau}_n$ were identically equal to τ, the expectation of the random variable

$$X\left(\{s+k\hat{\tau}_n+h_n\mathcal{Q}\}\cap W_n\right)\ \left[= X\left(\{s+k\tau+h_n\mathcal{Q}\}\cap W_n\right)\right]$$

would obviously be equal to 0, which, in turn, would be a strong evidence that the statement (3.48) holds true (in fact, one can easily verify that it is so under the assumption $\hat{\tau}_n \equiv \tau$). However, if $\hat{\tau}_n$ is a truly random estimator of τ, then the validity of statement (3.48) becomes highly questionable, provided that no additional information about $\hat{\tau}_n$ is available except that $|W_n|\,|\hat{\tau}_n-\tau|/h_n \xrightarrow{p} 0$, for example. To give a more rigorous justification of the latter claim, we note that statement (3.48) can be reduced to showing that, for any $\epsilon>0$ and $\beta>0$,

$$\mathbf{P}\left\{\hat{\lambda}_{n,K_2}(s)\geq\epsilon,\ |W_n|\,|\hat{\tau}_n-\tau|\leq\beta h_n\right\}\to 0, \tag{3.49}$$

as $n\to\infty$. The “restriction” $|W_n|\,|\hat{\tau}_n-\tau|\leq\beta h_n$ in (3.49) actually says that what we really know about the estimator $\hat{\tau}_n$ is only the following confidence interval

$$\hat{\tau}_n\in\tau+\frac{\beta}{|W_n|}h_n[-1,1]. \tag{3.50}$$

With the notation of (3.50), we rewrite (3.49) more explicitly as

$$\mathbf{P}\left\{\frac{\hat{\tau}_n}{|W_n|}\sum_{k=-\infty}^{\infty}\frac{1}{2h_n}X\left(\{s+k\hat{\tau}_n+h_n\mathcal{Q}\}\cap W_n\right)\geq\epsilon,\ \hat{\tau}_n\in\tau+\frac{\beta}{|W_n|}h_n[-1,1]\right\}$$
$$\to 0, \tag{3.51}$$

as $n\to\infty$. If we now use the only available for us information $\hat{\tau}_n\in\tau+\beta|W_n|^{-1}h_n[-1,1]$ to estimate the random variable $X(\{s+k\hat{\tau}_n+h_n\mathcal{Q}\}\cap W_n)$ in (3.51), we shall inevitably end up with the necessity of proving that

$$\mathbf{P}\{\hat{\lambda}^*_{n,K_2}(s)\geq\epsilon\}\to 0, \tag{3.52}$$

as $n\to 0$, where

$$\hat{\lambda}^*_{n,K_2}(s):=\frac{\hat{\tau}_n}{|W_n|}\sum_{k=-\infty}^{\infty}\frac{1}{2h_n}X\left(\{s+k\tau+k\beta|W_n|^{-1}h_n[-1,1]+h_n\mathcal{Q}\}\cap W_n\right).$$

But statement (3.52) appears to be impossible if $\lambda(s)>0$. Indeed, since the interval $\beta|W_n|^{-1}h_n[-1,1]$ has a positive Lebesgue measure (and it does not matter how small it is), we have that the set $k\beta|W_n|^{-1}h_n[-1,1]+h_n\mathcal{Q}$ completely covers the interval $h_n[-1,1]$. This observation immediately implies that

$$\hat{\lambda}^*_{n,K_2}(s)\geq\hat{\lambda}_{n,K_1}(s).$$

But we have already noted above that $\hat{\lambda}_{n,K_1}(s)$ is a consistent estimator of $\lambda(s)$. Thus, $\hat{\lambda}^*_{n,K_2}(s)$ cannot converge in probability to 0, as $n \to \infty$, if $\lambda(s) > 0$.

The above discussion indicates that without additional information about the relationship between X and $\hat{\tau}_n$ in the expression

$$X\left(\{s + k\hat{\tau}_n + h_n \mathcal{Q}\} \cap W_n\right),$$

it may be impossible to prove statements like (3.49) or (3.48). And we emphasize that, by not considering any specific estimator $\hat{\tau}_n$ in the present chapter, we do not have more information about $\hat{\tau}_n$ except that $\hat{\tau}_n$ is a consistent estimator of τ and, possibly, a rate of consistency like $|W_n|\,|\hat{\tau}_n - \tau|/h_n \xrightarrow{p} 0$, as $n \to \infty$. However, it is important to call readers attention that no matter how attractive the problem of including the kernel K_0 into Theorem 3.1 could be from the mathematical point of view, it does not seem relevant from the statistical point of view at all. Indeed, as far as we understand, all the kernels K of statistical relevance satisfy assumptions (K.1)–(K.4), and are thus covered by Theorems 3.1.

3.3 Statistical properties

In this section, we focus on statistical properties of our estimator, i.e. we compute the bias, variance, and mean squared error (MSE) of $\hat{\lambda}_{n,K}$. To obtain our results in this section (cf. also section 4.3) we will need an assumption on the estimator $\hat{\tau}_n$ of τ: there exists constant $C > 0$ and positive integer n_0 such that, for all $n \geq n_0$

$$\mathbf{P}\left(\frac{|W_n|}{a_n}|\hat{\tau}_n - \tau| \leq C\right) = 1$$

for some fixed sequence $a_n \downarrow 0$. Our shorthand notation for this assumption will be :

$$|W_n|\,|\hat{\tau}_n - \tau| = \mathcal{O}(a_n)$$

with probability 1, as $n \to \infty$.

3.3.1 Results

Theorem 3.7 *Suppose that λ is periodic and locally integrable. If, in addition, (3.2) holds, and*

$$|W_n|\,|\hat{\tau}_n - \tau| = \mathcal{O}\left(\delta_n h_n\right) \tag{3.53}$$

with probability 1 as $n \to \infty$, for some fixed sequence $\delta_n \downarrow 0$ as $n \to \infty$, then

$$\mathbf{E}\hat{\lambda}_{n,K}(s) \ \to \ \lambda(s) \tag{3.54}$$

as $n \to \infty$, provided s is a Lebesgue point of λ.

Note that the requirement $|W_n|h_n \to \infty$ as $n \to \infty$, which is needed to obtain weak consistency of $\hat{\lambda}_{n,K}$ (cf. assumption (3.7) in Theorem 3.1), is not needed to establish asymptotic unbiasedness of $\hat{\lambda}_{n,K}$, i.e. (3.54). The validity of condition (3.53) for a suitable estimator $\hat{\tau}_n$ will be discussed in chapter 5. An alternative set of assumptions replacing (3.53) is given in the following remark.

Remark 3.2 Condition (3.53) of Theorem 3.7 can be replaced by the following two assumptions

$$\mathbf{P}\left\{\frac{|W_n|}{\delta_n h_n}|\hat{\tau}_n - \tau| \geq 1\right\}\left\{\frac{|W_n|}{\delta_n h_n}\right\}^{1+\epsilon} \to 0,$$

and

$$\mathbf{E}\left(\mathbf{I}\left\{\frac{|W_n|}{\delta_n h_n}|\hat{\tau}_n - \tau| \geq 1\right\}\left\{\frac{|W_n|}{\delta_n h_n}|\hat{\tau}_n - \tau|\right\}^{1+\epsilon}\right) \to 0$$

as $n \to \infty$, for some $\epsilon > 0$. A proof of this assertion will appear in Helmers, Mangku, and Zitikis (2000). We conjecture that the conditions (3.56) and (3.58) needed in Theorems 3.9 and 3.10 respectively, can also be replaced by assumptions like the one given here.

Theorem 3.8 *Suppose that λ is periodic and locally integrable. If, in addition, (3.2), (3.53), and (3.7) hold true, then*

$$Var\left(\hat{\lambda}_{n,K}(s)\right) = o(1) \tag{3.55}$$

as $n \to \infty$, provided s is a Lebesgue point of λ.

In order to have an explicit asymptotic approximation to the variance of $\hat{\lambda}_{n,K}$ (e.g. Theorem 3.9) and one for the bias of $\hat{\lambda}_{n,K}$ (e.g. Theorem 3.10), we need to impose a stronger assumption than $(K.4)$ on the kernel K, that is $(K.5)$ which is given by

(K.5) K has only finitely many discontinuities and is Lipschitz in between.

Theorem 3.9 *Suppose that λ is periodic and bounded in a neighborhood of s. If, in addition, (3.2) and (3.7) hold, the kernel K satisfies (K.5), and*

$$|W_n|\,|\hat{\tau}_n - \tau| = \mathcal{O}\left(\delta_n h_n^{1/2} |W_n|^{-1/2}\right) \tag{3.56}$$

with probability 1 as $n \to \infty$, for some fixed sequence $\delta_n \downarrow 0$ as $n \to \infty$, then we have

$$Var\left(\hat{\lambda}_{n,K}(s)\right) = \frac{\tau\lambda(s)}{|W_n| h_n} \int_{-1}^{1} K^2(x)dx + o\left(\frac{1}{|W_n| h_n}\right) \tag{3.57}$$

as $n \to \infty$, provided s is a Lebesgue point of λ.

It is clear from the proof of Theorem 3.9 that without the stronger rate assumption (3.56) (cf. also chapter 5), but still assuming (3.53), the $o(|W_n|^{-1}h_n^{-1})$ remainder term on the r.h.s. of (3.57) would only be $o(1)$. As the first term on the r.h.s. of (3.57) is of the exact order $1/(|W_n|h_n)$, this would fail to give us desired asymptotic approximation to the variance of $\hat{\lambda}_{n,K}$. We also need (K.5) to do this.

Theorem 3.10 *Suppose that λ is periodic, locally integrable, (3.2) holds, the kernel K satisfies (K.5), and*

$$|W_n|\,|\hat{\tau}_n - \tau| = \mathcal{O}\left(\delta_n h_n^3\right) \tag{3.58}$$

with probability 1 as $n \to \infty$, for some fixed sequence $\delta_n \downarrow 0$ as $n \to \infty$. If, in addition, K is symmetric and λ has finite second derivative λ'' at s, then

$$\mathbf{E}\hat{\lambda}_{n,K}(s) = \lambda(s) + \frac{\lambda''(s)}{2} h_n^2 \int_{-1}^{1} x^2 K(x)dx + o(h_n^2) + \mathcal{O}(|W_n|^{-1}) \tag{3.59}$$

as $n \to \infty$.

From the proof of Theorem 3.10, it is clear that without the stronger rate assumption (3.58) (cf. also chapter 5), but still assuming (3.53), the $o(h_n^2)$ remainder term on the r.h.s. of (3.59) would only be $o(1)$. As the second term on the r.h.s. of (3.59) is of the exact order h_n^2, this would fail to give us desired asymptotic approximation to the bias of $\hat{\lambda}_{n,K}$. We also need (K.5) to do this.

The r.h.s. of (3.59) yields a familiar expression for the bias of $\hat{\lambda}_{n,K}(s)$ provided $|W_n|^{1/2}h_n \to \infty$ as $n \to \infty$; otherwise the $\mathcal{O}(|W_n|^{-1})$ remainder term will dominate. We have this $\mathcal{O}(|W_n|^{-1})$ remainder term as a consequence of the fact that we approximate the number of k such that $s + k\tau \in W_n$ by $|W_n|/\tau$. This also implies that the absolute value of this $\mathcal{O}(|W_n|^{-1})$ remainder term does not exceed $1/|W_n|$. If τ is known and we replace $\hat{\tau}_n$ in (3.1) by τ and subsequently the factor $|W_n|/\tau$ by the number of k such that $s + k\tau \in W_n$, then the $\mathcal{O}(|W_n|^{-1})$ remainder term on the r.h.s. of (3.59) would disappear.

Corollary 3.11 *Suppose that λ is periodic, locally integrable, (3.2) and (3.7) hold.*

(i) If, in addition, (3.53) holds true, then

$$MSE\left(\hat{\lambda}_{n,K}(s)\right) = Var\left(\hat{\lambda}_{n,K}(s)\right) + Bias^2\left(\hat{\lambda}_{n,K}(s)\right) \to 0 \qquad (3.60)$$

as $n \to \infty$, provided s is a Lebesgue point of λ.

(ii) If (3.56) and (3.58) hold true, the kernel K satisfies (K.5), and λ has finite second derivative λ'' at s, then

$$MSE\left(\hat{\lambda}_{n,K}(s)\right) = \frac{\tau\lambda(s)}{|W_n|h_n}\int_{-1}^{1} K^2(x)dx$$
$$+\frac{1}{4}\left(\lambda''(s)\int_{-1}^{1} x^2K(x)dx\right)^2 h_n^4 + o\left(|W_n|^{-1}h_n^{-1}\right) + o\left(h_n^4\right) \qquad (3.61)$$

as $n \to \infty$, provided K is symmetric.

The first statement of this Corollary is implied by Theorems 3.7 and 3.8, while its second statement is due to Theorems 3.9 and 3.10.

Now, we consider the r.h.s. of (3.61). By minimizing the sum of its first and second terms (the main terms for the variance and the squared bias), we then get the optimal choice of h_n, which is given by

$$h_n = \left[\frac{\tau\lambda(s)\int_{-1}^{1} K^2(x)dx}{\left(\lambda''(s)\int_{-1}^{1} x^2K(x)dx\right)^2}\right]^{\frac{1}{5}} |W_n|^{-\frac{1}{5}}. \qquad (3.62)$$

With this choice of h_n, the optimal rate of decrease of $MSE(\hat{\lambda}_{n,K}(s))$ is of order $\mathcal{O}(|W_n|^{-4/5})$ as $n \to \infty$; and also both (3.56) and (3.58) reduce to the same condition

$$|W_n|\,|\hat{\tau}_n - \tau| = \mathcal{O}\left(\delta_n|W_n|^{-3/5}\right)$$

with probability 1 as $n \to \infty$, for some fixed sequence $\delta_n \downarrow 0$ as $n \to \infty$.

Remark 3.3 The formulas (3.57), (3.59), (3.61), and (3.62) resemble of course closely corresponding ones in the 'classical' kernel density estimation. Let us consider for a moment estimation of a density f, proportional to the intensity function λ and having support in $[0, \tau]$. For simplici-ty, we consider here only the (unrealistic) case where we know θ and τ, where $\theta\tau = \int_0^\tau \lambda(s)ds$ (we assume here that $\theta > 0$). Then we have that $f(s) = \lambda(s)(\theta\tau)^{-1}$, for all $s \in [0, \tau]$. Consequently, the quantity

$$\hat{f}_{n,K}(s) = \hat{\lambda}_{n,K}(s)(\theta\tau)^{-1}$$

can be viewed as an estimate of f at a given point s. Since $\lambda(s) = f(s)\theta\tau$, we also have that $\lambda''(s) = f''(s)\theta\tau$, for all $s \in (0, \tau)$. By (3.57), we then have

$$\begin{aligned} Var\left(\hat{f}_{n,K}(s)\right) &= Var\left(\frac{\hat{\lambda}_{n,K}(s)}{\theta\tau}\right) \\ &= \frac{1}{(\theta\tau)^2}\frac{\tau f(s)(\theta\tau)}{|W_n|h_n}\int_{-1}^{1} K^2(x)dx + o\left(\frac{1}{|W_n|h_n}\right) \\ &= \frac{f(s)}{\theta|W_n|h_n}\int_{-1}^{1} K^2(x)dx + o\left(\frac{1}{|W_n|h_n}\right). \end{aligned} \tag{3.63}$$

Note that, due to our 'increasing domain asymptotic framework', the number of observations $X(W_n)$ in a given window W_n is random. However, since λ is periodic, we know (cf. Lemma 2.1) that $\mathbf{E}X(W_n) \sim \theta|W_n|$. Hence, it seems appropriate to compare $\theta|W_n|$ with the fixed 'sample size n' in the 'classical' kernel density estimation case. If we replace $\theta|W_n|$ on the r.h.s. of (3.63) by n, the r.h.s. of (3.63) indeed reduces to the well-known expression for the variance in the kernel density estimation. From (3.59), we have that

$$\begin{aligned} \mathbf{E}\hat{f}_{n,K}(s) &= \mathbf{E}\frac{\hat{\lambda}_{n,K}(s)}{\theta\tau} \\ &= \frac{\lambda(s)}{\theta\tau} + \frac{f''(s)\theta\tau}{2\theta\tau}h_n^2\int_{-1}^{1} x^2K(x)dx + o(h_n^2) + \mathcal{O}(|W_n|^{-1}) \\ &= f(s) + \frac{f''(s)}{2}h_n^2\int_{-1}^{1} x^2K(x)dx + o(h_n^2) + \mathcal{O}(|W_n|^{-1}). \end{aligned} \tag{3.64}$$

Note that the second term on the r.h.s. of (3.64) is the same as the well-known formula for the asymptotic bias in kernel density estimation. From (3.63) and (3.64), we also can find formulas for $MSE(\hat{f}_{n,K}(s))$ and optimal choice of h_n, when estimating f. These expressions also reduce to the corresponding ones in kernel density estimation, if we replace $\theta|W_n|$ by n. □

In chapter 4 we study the statistical properties of a nearest neighbor estimator of the intensity function of a cyclic Poisson process. It also contains a detailed comparison of the nearest neighbor estimator with the uniform kernel estimator.

3.3.2 Proofs

We begin with two simple lemmas, which will be useful in establishing our results.

Lemma 3.12 *Suppose that λ is periodic and locally integrable. If the kernel K satisfy assumptions (K.1)–(K.3) and the bandwidth h_n be such that (3.2) holds true, then*

$$\mathbf{E}\frac{\tau}{|W_n|}\sum_{k=-\infty}^{\infty}\frac{1}{h_n}\int_{W_n}K\left(\frac{x-(s+k\tau)}{h_n}\right)X(dx)\to\lambda(s) \tag{3.65}$$

as $n\to\infty$, provided s is a Lebesgue point of λ.

Proof: Let N_n denote the number of integers k such that $s+k\tau\in W_n$. Then, the l.h.s. of (3.65) can be written as

$$\frac{N_n\tau}{|W_n|}\mathbf{E}\frac{1}{N_n}\sum_{k=-\infty}^{\infty}\frac{1}{h_n}\int_{W_n}K\left(\frac{x-(s+k\tau)}{h_n}\right)X(dx). \tag{3.66}$$

By Fubini's and Lemma 3.4, the expectation in (3.66) is equal to $\lambda(s)+o(1)$ as $n\to\infty$. Hence, it remains to show that

$$N_n\tau|W_n|^{-1}=1+o(1), \tag{3.67}$$

as $n\to\infty$. It is obvious that $N_n\in\left[\tau|W_n|^{-1}-1,\tau|W_n|^{-1}+1\right]$, which implies $1-\tau|W_n|^{-1}\leq N_n\tau|W_n|^{-1}\leq 1+\tau|W_n|^{-1}$. By (1.2), we then have (3.67). This completes the proof of Lemma 3.12. □

Lemma 3.13 *Suppose that the assumption (3.53) is satisfied. Then, for each positive integer m, we have that*

$$\mathbf{E}\left(\hat{\tau}_n-\tau\right)^{2m}=\mathcal{O}\left(|W_n|^{-2m}\delta_n^{2m}h_n^{2m}\right) \tag{3.68}$$

as $n\to\infty$.

Proof: Easy. □

Proof of Theorem 3.7

Here we give proof of Theorem 3.7 under the weaker assumption $(K.4*)$ which implies $(K.4)$, where $(K.4*)$ is defined as in section 3.2.

We will prove (3.54) by showing

$$\mathbf{E}\hat{\lambda}_{n,K}(s) = \lambda(s) + o(1) \tag{3.69}$$

as $n \to \infty$. First we write $\mathbf{E}\hat{\lambda}_{n,K}(s)$ as

$$\begin{aligned}&\mathbf{E}\left(\hat{\lambda}_{n,K}(s) - \frac{\tau}{|W_n|h_n}\sum_{k=-\infty}^{\infty}\int_{W_n} K\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right)X(dx)\right)\\ &+\frac{\tau}{|W_n|h_n}\mathbf{E}\left(\sum_{k=-\infty}^{\infty}\int_{W_n}\left[K\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right)-K\left(\frac{x-(s+k\tau)}{h_n}\right)\right]X(dx)\right)\\ &+\mathbf{E}\frac{\tau}{|W_n|h_n}\sum_{k=-\infty}^{\infty}\int_{W_n}K\left(\frac{x-(s+k\tau)}{h_n}\right)X(dx).\end{aligned} \tag{3.70}$$

By Lemma 3.12, the third term of (3.70) is equal to $\lambda(s)+o(1)$ as $n \to \infty$. Hence, to prove (3.69), it remains to check that both the first and second terms of (3.70) are $o(1)$ as $n \to \infty$.

First we prove that the first term of (3.70) is $o(1)$ as $n \to \infty$. The absolute value of the quantity in this term can be written as

$$\frac{1}{|W_n|h_n}\mathbf{E}\,|\hat{\tau}_n - \tau|\sum_{k=-\infty}^{\infty}\int_{W_n}K\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right)X(dx). \tag{3.71}$$

For large n, by (3.2) and (3.53), the intervals

$$\{s+k\hat{\tau}_n+h_n[-1,1]\}\cap W_n \quad \text{and} \quad \{s+j\hat{\tau}_n+h_n[-1,1]\}\cap W_n$$

are disjoint with probability 1, provided $k \neq j$. Since K is bounded, there exists fixed constant C_0 such that $K(x) \leq C_0$, for all x in real line. Then we have that

$$\begin{aligned}&\sum_{k=-\infty}^{\infty}\int_{W_n}K\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right)X(dx)\\ &\leq C_0\sum_{k=-\infty}^{\infty}X\left(\{s+k\hat{\tau}_n+h_n[-1,1]\}\cap W_n\right) \leq C_0X(W_n).\end{aligned} \tag{3.72}$$

By (3.72) and Cauchy-Schwarz inequality, the quantity in (3.71) does not exceed

$$\frac{C_0}{|W_n|h_n}\left(\mathbf{E}\left(\hat{\tau}_n-\tau\right)^2\right)^{\frac{1}{2}}\left(\mathbf{E}X^2(W_n)\right)^{\frac{1}{2}}. \tag{3.73}$$

By Lemma 3.13 with $m = 1$ (we take $\delta_n = 1$), we have that the square-root of the first expectation in (3.73) is of order $\mathcal{O}(|W_n|^{-1}h_n)$ as $n \to \infty$. We easily check that $(\mathbf{E}X^2(W_n))^{\frac{1}{2}} = \mathcal{O}(|W_n|)$ as $n \to \infty$, because λ is periodic. Then, the quantity in (3.73) is of order $\mathcal{O}(|W_n|^{-1})$, which is $o(1)$ as $n \to \infty$.

Next we prove that the second term of (3.70) is $o(1)$ as $n \to \infty$. Fix any $\alpha > 0$ and define A_α as given in (3.33). Then we decompose both K in the second term of (3.70) as that in (3.34). Since K and L_α are bounded, we easily see that the quantity

$$\left| \sum_{k=-\infty}^{\infty} \int_{W_n} \left[(K - L_\alpha) \left(\frac{x - (s + k\hat{\tau}_n)}{h_n} \right) \mathbf{I}_{A_\alpha^c} \left(\frac{x - (s + k\hat{\tau}_n)}{h_n} \right) \right.\right.$$
$$\left.\left. -(K - L_\alpha) \left(\frac{x - (s + k\tau)}{h_n} \right) \mathbf{I}_{A_\alpha^c} \left(\frac{x - (s + k\tau)}{h_n} \right) \right] X(dx) \right|$$

does not exceed

$$c(K, L_\alpha) \sum_{k=-\infty}^{\infty} X\left(\{s + k\hat{\tau}_n + h_n A_\alpha^c\} \cap W_n\right)$$
$$+c(K, L_\alpha) \sum_{k=-\infty}^{\infty} X\left(\{s + k\tau + h_n A_\alpha^c\} \cap W_n\right). \tag{3.74}$$

where $c(K, L_\alpha)$ denotes a constant depending only on $\sup\{|K(u)| : u \in [-1, 1]\}$ and $\sup\{|L_\alpha(u)| : u \in [-1, 1]\}$. Note that, by Luzin's and Weierstrass theorems, we may assume without loss of generality that $c(K, L_\alpha) = \mathcal{O}(1)$, as $\alpha \downarrow 0$. The quantity

$$\left| \sum_{k=-\infty}^{\infty} \int_{W_n} \left[(K - L_\alpha) \left(\frac{x - (s + k\hat{\tau}_n)}{h_n} \right) \mathbf{I}_{A_\alpha} \left(\frac{x - (s + k\hat{\tau}_n)}{h_n} \right) \right.\right.$$
$$\left.\left. -(K - L_\alpha) \left(\frac{x - (s + k\tau)}{h_n} \right) \mathbf{I}_{A_\alpha} \left(\frac{x - (s + k\tau)}{h_n} \right) \right] X(dx) \right|$$

does not exceed

$$\alpha \sum_{k=-\infty}^{\infty} X\left(\{s + k\hat{\tau}_n + h_n[-1, 1]\} \cap W_n\right)$$
$$+\alpha \sum_{k=-\infty}^{\infty} X\left(\{s + k\tau + h_n[-1, 1]\} \cap W_n\right). \tag{3.75}$$

Let $I_0 = (-\infty, -1)$, $I_1 = [-1, 1]$, and $I_2 = (1, \infty)$. Then, by (3.35), the quantity

$$\left| \sum_{k=-\infty}^{\infty} \int_{W_n} \left[L_\alpha \left(\frac{x - (s + k\hat{\tau}_n)}{h_n} \right) - L_\alpha \left(\frac{x - (s + k\tau)}{h_n} \right) \right] X(dx) \right|$$

does not exceed

$$c(L_\alpha)\sum_{k=-\infty}^{\infty}\left|\frac{k(\hat{\tau}_n-\tau)}{h_n}\right| X\left(\{s+k\hat{\tau}_n+h_n[-1,1]\}\cap W_n\right)$$
$$+c(L_\alpha)\sum_{k=-\infty}^{\infty}\left|\frac{k(\hat{\tau}_n-\tau)}{h_n}\right| X\left(\{s+k\tau+h_n[-1,1]\}\cap W_n\right)$$
$$+c(L_\alpha)\sum_{0\le i\ne j\le 2}\sum_{k=-\infty}^{\infty} X\left(\{s+k\hat{\tau}_n+h_nI_i\}\cap\{s+k\tau+h_nI_j\}\cap W_n\right). \tag{3.76}$$

Note that the first and second terms of (3.76) correspond to the case where both points $(x-(s+k\hat{\tau}_n))/h_n$ and $(x-(s+k\tau))/h_n$ are in the same interval $[-1,1]$, which is equivalent to the case $x\in\{s+k\hat{\tau}_n+h_n[-1,1]\}\cap\{s+k\tau+h_n[-1,1]\}$, so that we can apply (3.35). The third term of (3.76) corresponds to the other cases. If, e.g., K is a uniform kernel, we may take $L_\alpha = K$ and then the first and second terms of (3.76) can be dropped.

First we consider the quantity in (3.74), (3.75), and the first and second terms of (3.76). Since $s\in W_n$, by condition (3.53), we have with probability 1 that the magnitude any integer k such that $\{s+k\hat{\tau}_n+h_n[-1,1]\}\cap W_n\ne\emptyset$ is at most of order $\mathcal{O}(|W_n|)$ as $n\to\infty$. By assumption (3.53), there exists large positive constant C and positive integer n_0 such that

$$|\hat{\tau}_n-\tau|\le C|W_n|^{-1}\delta_n h_n \tag{3.77}$$

with probability 1, for all $n\ge n_0$. This implies, for sufficiently large n, we have with probability 1

$$k\tau - C\delta_n h_n\le k\hat{\tau}_n\le k\tau + C\delta_n h_n. \tag{3.78}$$

By (3.78), we have with probability 1 that the quantity in (3.74) does not exceed

$$2c(K,L_\alpha)\sum_{k=-\infty}^{\infty} X\left(\{s+k\tau+h_n[-C\delta_n,C\delta_n]+h_nA_\alpha^c\}\cap W_n\right). \tag{3.79}$$

For sufficiently large n, the quantity $C\delta_n$ in (3.78) does not exceed 1. Then for large n, by (3.78) with $C\delta_n$ replaced by 1, we have with probability 1 that the quantity in (3.75) does not exceed

$$2\alpha\sum_{k=-\infty}^{\infty} X\left(\{s+k\tau+h_n[-2,2]\}\cap W_n\right). \tag{3.80}$$

By (3.78), we have with probability 1 that $|k(\hat{\tau}_n - \tau)/h_n| \leq C\delta_n$. Then, a similar argument also shows that sum of the first and second terms of (3.76) does not exceed

$$2c(L_\alpha)C\delta_n \sum_{k=-\infty}^{\infty} X\left(\{s + k\tau + h_n[-2,2]\} \cap W_n\right). \tag{3.81}$$

Next we consider the third term of (3.76). We can easily see that, for example for the case $i = 0$ and $j = 1$, we have

$$\begin{aligned} &\sum_{k=-\infty}^{\infty} X(\{s + k\hat{\tau}_n + h_n I_i\} \cap \{s + k\tau + h_n I_j\} \cap W_n) \\ &= \sum_{k=-\infty}^{\infty} X(\{s + k\tau + k(\hat{\tau}_n - \tau) + h_n(-\infty, -1)\} \\ &\qquad \cap \{s + k\tau + h_n[-1,1]\} \cap W_n) \\ &\leq \sum_{k=-\infty}^{\infty} X([s + k\tau - h_n, s + k\tau - h_n + |k(\hat{\tau}_n - \tau)|) \cap W_n). \end{aligned} \tag{3.82}$$

A similar argument as the one given in (3.82) can be used to treat the other three cases, namely the case $i = 1$ and $j = 0$, $i = 1$ and $j = 2$, and $i = 2$ and $j = 1$. Combining all these results, we have that the third term of (3.76) does not exceed

$$\begin{aligned} c(L_\alpha) \sum_{i=1}^{2} \sum_{k=-\infty}^{\infty} \quad & X([s + k\tau + a_i h_n - |k(\hat{\tau}_n - \tau)|, \\ & s + k\tau + a_i h_n + |k(\hat{\tau}_n - \tau)|] \cap W_n), \end{aligned} \tag{3.83}$$

where $a_1 = -1$ and $a_2 = 1$. Since by (3.78), we have with probability 1 that $|k(\hat{\tau}_n - \tau)| \leq C\delta_n h_n$, the quantity in (3.83) does not exceed

$$\begin{aligned} c(L_\alpha) \sum_{i=1}^{2} \sum_{k=-\infty}^{\infty} X \quad & ([s + k\tau + a_i h_n - C\delta_n h_n, \\ & s + k\tau + a_i h_n + C\delta_n h_n] \cap W_n). \end{aligned} \tag{3.84}$$

Combining all these upper bounds (e.g. (3.79), (3.80), (3.81), and (3.84)), for large n, we have that absolute value of the quantity in the second term of (3.70) does not exceed

$$\begin{aligned} &\frac{2\tau c(K, L_\alpha)}{|W_n| h_n} \mathbf{E} \sum_{k=-\infty}^{\infty} X\left(\{s + k\tau + h_n[-C\delta_n, C\delta_n] + h_n A_\alpha^c\} \cap W_n\right) \\ &+ (2\alpha + 2c(L_\alpha)C\delta_n) \left\{ \frac{\tau}{|W_n| h_n} \mathbf{E} \sum_{k=-\infty}^{\infty} X\left(\{s + k\tau + h_n[-2,2]\} \cap W_n\right) \right\} \end{aligned}$$

$$+\frac{c(L_\alpha)\tau}{|W_n|h_n}\mathbf{E}\sum_{i=1}^{2}\sum_{k=-\infty}^{\infty} X\left(\{s+k\tau+h_n[a_i-C\delta_n,a_i+C\delta_n]\}\cap W_n\right). \quad (3.85)$$

Then, to show that the second term of (3.70) is $o(1)$ as $n\to\infty$, it suffices to check that each term of (3.85) is $o(1)$ as $n\to\infty$.

First we consider the first term of (3.85). By Fubini's, this term is equal to

$$\frac{2\tau c(K,L_\alpha)}{|W_n|h_n}\sum_{k=-\infty}^{\infty}\int_{h_n([-C\delta_n,C\delta_n]+A_\alpha^c)}\lambda(x+s+k\tau)\mathbf{I}(x+s+k\tau\in W_n)dx. \quad (3.86)$$

Since λ is periodic with period τ, we have $\lambda(x+s+k\tau)=\lambda(x+s)$. Obviously we also have that

$$\sum_{k=-\infty}^{\infty}\mathbf{I}(x+s+k\tau\in W_n)\le\frac{2|W_n|}{\tau}. \quad (3.87)$$

Then the quantity in (3.86) does not exceed

$$\begin{aligned}&\frac{4c(K,L_\alpha)}{h_n}\int_{h_n([-C\delta_n,C\delta_n]+A_\alpha^c)}\lambda(x+s)dx\\ &\le 4c(K,L_\alpha)\frac{1}{h_n}\int_{h_n([-C\delta_n,C\delta_n]+A_\alpha^c)}|\lambda(x+s)-\lambda(s)|dx\\ &+4c(K,L_\alpha)\lambda(s)|[-C\delta_n,C\delta_n]+A_\alpha^c|. \end{aligned} \quad (3.88)$$

Since $\delta_n\downarrow 0$ as $n\to\infty$ and $A_\alpha^c\subset[-1,1]$, for sufficiently large n, $[-C\delta_n,C\delta_n]+A_\alpha^c\subset[-2,2]$. This implies the integral on the r.h.s. of (3.88) does not exceed $\int_{-2h_n}^{2h_n}|\lambda(x+s)-\lambda(s)|dx$. Because s is a Lebesgue point of λ, we then have that the first term on the r.h.s. of (3.88) is $o(1)$ as $n\to\infty$. The second term on the r.h.s. of (3.88), by noting that $|A_\alpha^c|\le\alpha$, does not exceed

$$8c(K,L_\alpha)\lambda(s)C\delta_n+4\alpha c(K,L_\alpha)\lambda(s).$$

Since $\delta_n\downarrow 0$ as $n\to\infty$, the second term on the r.h.s. of (3.88) can be made $o(1)$ as $n\to\infty$ by taking $\alpha=\alpha_n\downarrow 0$ as $n\to\infty$ and noting that it is easy to check $c(K,L_\alpha)=\mathcal{O}(1)$ as $\alpha_n\downarrow 0$. Hence the first term of (3.85) is $o(1)$ as $n\to\infty$.

Next we consider the second term of (3.85). To prove this term is $o(1)$ as $n\to\infty$, we first show that

$$\frac{\tau}{|W_n|h_n}\mathbf{E}\sum_{k=-\infty}^{\infty}X\left(\{s+k\tau+h_n[-2,2]\}\cap W_n\right)=\mathcal{O}(1), \quad (3.89)$$

as $n \to \infty$. By Fubini's, the l.h.s. of (3.89) is equal to

$$\begin{aligned} &\frac{\tau}{|W_n|h_n} \sum_{k=-\infty}^{\infty} \int_{-2h_n}^{2h_n} \lambda(x+s+k\tau)\mathbf{I}(x+s+k\tau \in W_n)dx \\ &\leq \frac{2}{h_n} \int_{-2h_n}^{2h_n} \lambda(x+s)dx \\ &\leq 8\alpha \frac{1}{4h_n} \int_{-2h_n}^{2h_n} |\lambda(x+s)-\lambda(s)|dx + 8\lambda(s). \end{aligned} \tag{3.90}$$

Since s is a Lebesgue point of λ, the first term on the r.h.s. of (3.90) is $o(1)$ as $n \to \infty$. Since $\lambda(s)$ is finite, we then have (3.89). By (3.89), the second term of (3.85) is of order $\mathcal{O}(\alpha + \delta_n)$, as $n \to \infty$. By taking $\alpha = \alpha_n \downarrow 0$ as $n \to \infty$, this term is $o(1)$ as $n \to \infty$.

Finally we show the third term of (3.85) is $o(1)$ as $n \to \infty$. By Fubini's, this term can be written as

$$\begin{aligned} &\frac{c(L_\alpha)\tau}{|W_n|h_n} \sum_{i=1}^{2} \sum_{k=-\infty}^{\infty} \int_{h_n(a_i - C\delta_n)}^{h_n(a_i + C\delta_n)} \lambda(x+s+k\tau)\mathbf{I}(x+s+k\tau \in W_n)dx \\ &\leq \frac{2c(L_\alpha)}{h_n} \sum_{i=1}^{2} \int_{h_n(a_i - C\delta_n)}^{h_n(a_i + C\delta_n)} \lambda(x+s)dx \\ &\leq \frac{2c(L_\alpha)}{h_n} \sum_{i=1}^{2} \int_{h_n(a_i - C\delta_n)}^{h_n(a_i + C\delta_n)} |\lambda(x+s)-\lambda(s)|dx + 8c(L_\alpha)C\delta_n\lambda(s). \end{aligned} \tag{3.91}$$

To get the r.h.s. of (3.91) we have used (3.87). Since $\delta_n \downarrow 0$ as $n \to \infty$, the second term on the r.h.s. of (3.91) is $o(1)$ as $n \to \infty$. For sufficiently large n we have that $C\delta_n \leq 1$. Then, the integral in the first term on the r.h.s. of (3.91) does not exceed $\int_{-2h_n}^{2h_n} |\lambda(x+s) - \lambda(s)|dx$. Since s is a Lebesgue point of λ, the first term on the r.h.s. of (3.91) is $o(1)$ as $n \to \infty$. This completes the proof of Theorem 3.7. □

Proof of Theorem 3.8

First we write

$$Var\left(\hat{\lambda}_{n,K}(s)\right) = \mathbf{E}\left(\hat{\lambda}_{n,K}(s)\right)^2 - \left(\mathbf{E}\hat{\lambda}_{n,K}(s)\right)^2. \tag{3.92}$$

Since by Theorem 3.7, the second term on the r.h.s. of (3.92) is equal to $-\lambda^2(s) + o(1)$ as $n \to \infty$, to prove this theorem, it suffices to show that the first term on the r.h.s. of (3.92) is equal to $\lambda^2(s) + o(1)$ as $n \to \infty$.

The first term on the r.h.s. of (3.92) can be written as

$$\frac{1}{|W_n|^2 h_n^2}\mathbf{E}\left(\hat{\tau}_n-\tau\right)^2\left(\sum_{k=-\infty}^{\infty}\int_{W_n}K\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right)X(dx)\right)^2$$
$$+\frac{2\tau}{|W_n|^2 h_n^2}\mathbf{E}\left(\hat{\tau}_n-\tau\right)\left(\sum_{k=-\infty}^{\infty}\int_{W_n}K\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right)X(dx)\right)^2$$
$$+\frac{\tau^2}{|W_n|^2 h_n^2}\mathbf{E}\left(\sum_{k=-\infty}^{\infty}\int_{W_n}K\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right)X(dx)\right)^2. \tag{3.93}$$

First we consider the first term of (3.93). By (3.72) and Cauchy-Schwarz inequality, this term does not exceed

$$C_0^2\left(|W_n|^2 h_n^2\right)^{-1}\left(\mathbf{E}(\hat{\tau}_n-\tau)^4\right)^{\frac{1}{2}}\left(\mathbf{E}X^4(W_n)\right)^{\frac{1}{2}},$$

where C_0 is a positive constant. We know that $(EX^4(W_n))^{\frac{1}{2}}=\mathcal{O}(|W_n|^2)$ as $n\to\infty$. By Lemma 3.13 for $m=2$ (we take $\delta_n=1$), we have that $(\mathbf{E}(\hat{\tau}_n-\tau)^4)^{\frac{1}{2}}=\mathcal{O}(|W_n|^{-2}h_n^2)$ as $n\to\infty$. Hence, the first term of (3.93) is of order $\mathcal{O}(|W_n|^{-2})$, which is $o(1)$ as $n\to\infty$. Using a similar argument, by noting now that $(\mathbf{E}(\hat{\tau}_n-\tau)^2)^{\frac{1}{2}}=\mathcal{O}(|W_n|^{-1}\delta_n h_n)$ as $n\to\infty$, we have the second term of (3.93) is of order $o(|W_n|^{-1}h_n^{-1})$, which (by assumption (3.7)) is $o(1)$ as $n\to\infty$.

Next we consider the third term of (3.93). This term can be written as

$$\frac{\tau^2}{|W_n|^2 h_n^2}\quad\mathbf{E}\quad\left(\sum_{k=-\infty}^{\infty}\int_{W_n}\left[K\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right)-K\left(\frac{x-(s+k\tau)}{h_n}\right)\right]X(dx)\right)^2$$
$$+\frac{2\tau^2}{|W_n|^2 h_n^2}\quad\mathbf{E}\quad\left(\sum_{k=-\infty}^{\infty}\int_{W_n}\left[K\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right)-K\left(\frac{x-(s+k\tau)}{h_n}\right)\right]X(dx)\right.$$
$$\left.\sum_{l=-\infty}^{\infty}\int_{W_n}K\left(\frac{x-(s+l\tau)}{h_n}\right)X(dx)\right)$$
$$+\frac{\tau^2}{|W_n|^2 h_n^2}\quad\mathbf{E}\quad\left(\sum_{k=-\infty}^{\infty}\int_{W_n}K\left(\frac{x-(s+k\tau)}{h_n}\right)X(dx)\right)^2 \tag{3.94}$$

We will show that the third term of (3.94) is equal to $\lambda^2(s)+o(1)$, while the other terms are $o(1)$ as $n\to\infty$.

First we consider the first term of (3.94). From the proof of Theorem 3.7 (cf. the upper bounds in (3.79), (3.80), (3.81), and (3.84)), for sufficiently large n, we have with probability 1 that

$$\left|\sum_{k=-\infty}^{\infty}\int_{W_n}\left[K\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right)-K\left(\frac{x-(s+k\tau)}{h_n}\right)\right]X(dx)\right|$$

$$\leq 2c(K, L_\alpha) \sum_{k=-\infty}^{\infty} X(\{s + k\tau + h_n[-C\delta_n, C\delta_n] + h_n A_\alpha^c\} \cap W_n)$$

$$+2(\alpha + c(L_\alpha)C\delta_n) \sum_{k=-\infty}^{\infty} X(\{s + k\tau + h_n[-2, 2]\} \cap W_n)$$

$$+c(L_\alpha) \sum_{i=1}^{2} \sum_{k=-\infty}^{\infty} X(\{s + k\tau + h_n[a_i - C\delta_n, a_i + C\delta_n]\} \cap W_n),$$

with $a_i = -1$ and $a_2 = 1$. Then, to show that the first term of (3.94) is $o(1)$ as $n \to \infty$, it suffices to check

$$\frac{4\tau^2 c^2(K, L_\alpha)}{|W_n|^2 h_n^2} \mathbf{E} \left(\sum_{k=-\infty}^{\infty} X(\{s + k\tau + h_n[-C\delta_n, C\delta_n] + h_n A_\alpha^c\} \cap W_n) \right)^2 = o(1) \quad (3.95)$$

$$\frac{4\tau^2 (\alpha + c(L_\alpha)C\delta_n)^2}{|W_n|^2 h_n^2} \mathbf{E} \left(\sum_{k=-\infty}^{\infty} X(\{s + k\tau + h_n[-2, 2]\} \cap W_n) \right)^2 = o(1), \quad (3.96)$$

and

$$\frac{\tau^2 c^2(L_\alpha)}{|W_n|^2 h_n^2} \mathbf{E} \left(\sum_{i=1}^{2} \sum_{k=-\infty}^{\infty} X(\{s + k\tau + h_n[a_i - C\delta_n, a_i + C\delta_n]\} \cap W_n) \right)^2 = o(1), \quad (3.97)$$

as $n \to \infty$.

To prove (3.95) we argue as follows. By writing square of a sum as a double sum, we can interchange summations and expectation. Then we distinguish two cases, namely the case where the indexes are the same and the case where the indexes are different. For sufficiently large n, since $h_n \downarrow 0$ as $n \to \infty$,

$$X(\{s + k\tau + h_n[-C\delta_n, C\delta_n] + h_n A_\alpha^c\} \cap W_n)$$

and

$$X(\{s + j\tau + h_n[-C\delta_n, C\delta_n] + h_n A_\alpha^c\} \cap W_n)$$

are independent, provided $k \neq j$. Then, for large n, the expectation on the l.h.s. of (3.95) does not exceed

$$\sum_{k=-\infty}^{\infty} \mathbf{E} X^2(\{s + k\tau + h_n[-C\delta_n, C\delta_n] + h_n A_\alpha^c\} \cap W_n)$$

$$+ \left(\sum_{k=-\infty}^{\infty} \mathbf{E} X(\{s + k\tau + h_n[-C\delta_n, C\delta_n] + h_n A_\alpha^c\} \cap W_n) \right)^2$$

$$\leq 2 \left(\sum_{k=-\infty}^{\infty} \mathbf{E} X(\{s + k\tau + h_n[-C\delta_n, C\delta_n] + h_n A_\alpha^c\} \cap W_n) \right)^2. \quad (3.98)$$

From proof of Theorem 3.7 (recall the quantity in (3.86) does not exceed the r.h.s. of (3.88)), we have that

$$\sum_{k=-\infty}^{\infty} \mathbf{E}X\left(\{s+k\tau+h_n[-C\delta_n, C\delta_n]+h_nA_\alpha^c\}\cap W_n\right)$$
$$= \mathcal{O}\left(\alpha|W_n|h_n+\delta_n|W_n|h_n\right),$$

as $n\to\infty$. This implies that the l.h.s. of (3.95) is of order $\mathcal{O}(\alpha^2+\delta_n^2)$ as $n\to\infty$. By taking now $\alpha=\alpha_n\downarrow 0$, and by noting that it is easy to check $c(K,L_\alpha)=\mathcal{O}(1)$ as $\alpha_n\downarrow 0$, we have that this quantity is $o(1)$ as $n\to\infty$.

Using a similar argument, by noting that

$$\sum_{k=-\infty}^{\infty} \mathbf{E}X\left(\{s+k\tau+h_n[-2,2]\}\cap W_n\right)=\mathcal{O}(|W_n|h_n),$$

as $n\to\infty$ (cf. (3.89)), we also have that

$$\mathbf{E}\left(\sum_{k=-\infty}^{\infty} X\left(\{s+k\tau+h_n[-2,2]\}\cap W_n\right)\right)^2=\mathcal{O}(|W_n|^2h_n^2), \tag{3.99}$$

as $n\to\infty$, which implies the l.h.s. of (3.96) is of order $\mathcal{O}(\alpha^2+\delta_n^2)$, which (by taking $\alpha=\alpha_n\downarrow 0$) is $o(1)$ as $n\to\infty$.

Now we prove (3.97). By a similar argument as the one leads to (3.98), for large n, the expectation on the l.h.s. of (3.97) does not exceed

$$4\left(\sum_{i=1}^{2}\sum_{k=-\infty}^{\infty} \mathbf{E}X\left(\{s+k\tau+h_n[a_i-C\delta_n, a_i+C\delta_n]\}\cap W_n\right)\right)^2. \tag{3.100}$$

Since the third term of (3.85) is $o(1)$ as $n\to\infty$, we conclude that

$$\sum_{i=1}^{2}\sum_{k=-\infty}^{\infty} \mathbf{E}X\left(\{s+k\tau+h_n[a_i-C\delta_n, a_i+C\delta_n]\}\cap W_n\right)$$
$$=o(|W_n|h_n),$$

as $n\to\infty$. This implies the quantity in (3.100) is of order $o(|W_n|^2h_n^2)$, which then implies (3.97). Hence we have proved that the first term of (3.94) is $o(1)$ as $n\to\infty$.

Next we prove that the second term of (3.94) is $o(1)$ as $n\to\infty$. Since K is bounded, i.e. $K(u)\leq C_0$ for all $u\in\mathbf{R}$, the expectation appearing in the third term of (3.94) does not exceed the l.h.s. of (3.99) multiplied by

C_0^2. By (3.99), we know that the third term of (3.94) is $\mathcal{O}(1)$ as $n \to \infty$. We have proved that the first term of (3.94) is $o(1)$ as $n \to \infty$. Then, Cauchy-Schwarz inequality shows that the second term of (3.94) is $o(1)$ as $n \to \infty$.

Finally, we prove the third term of (3.94) is equal to $\lambda^2(s) + o(1)$ as $n \to \infty$. Since $h_n \downarrow 0$ as $n \to \infty$ and the kernel K has support in $[-1, 1]$, for sufficiently large n, we have that

$$\int_{W_n} K\left(\frac{x-(s+j\tau)}{h_n}\right) X(dx) \text{ and } \int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right) X(dx) \quad (3.101)$$

are independent, provided $j \neq k$. Then, for sufficiently large n, the third term of (3.94) can be written as follows

$$\begin{aligned}
&= \frac{\tau^2}{|W_n|^2 h_n^2} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \\
&\quad \mathbf{E} \int_{W_n} K\left(\frac{x-(s+j\tau)}{h_n}\right) X(dx) \int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right) X(dx) \\
&= \frac{\tau^2}{|W_n|^2 h_n^2} \left(\sum_{k=-\infty}^{\infty} \mathbf{E} \int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right) X(dx) \right)^2 \\
&\quad - \frac{\tau^2}{|W_n|^2 h_n^2} \sum_{k=-\infty}^{\infty} \left(\mathbf{E} \int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right) X(dx) \right)^2 \\
&\quad + \frac{\tau^2}{|W_n|^2 h_n^2} \sum_{k=-\infty}^{\infty} \mathbf{E} \left(\int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right) X(dx) \right)^2 . \qquad (3.102)
\end{aligned}$$

Since the kernel K is bounded, the absolute value of the second term on the r.h.s. of (3.102) does not exceed

$$\begin{aligned}
&\frac{\tau^2 C_0^2}{|W_n|^2 h_n^2} \sum_{k=-\infty}^{\infty} (\mathbf{E}X(\{s+k\tau+h_n[-1,1]\})\mathbf{I}(s+k\tau \in W_n))^2 \\
&= \mathcal{O}\left(\frac{1}{|W_n|}\right) = o(1),
\end{aligned}$$

as $n \to \infty$, where C_0 is a positive constant. A similar argument also shows that the third term on the r.h.s. of (3.102) is $o(1)$ as $n \to \infty$. By Lemma 3.12, we can write the first term on the r.h.s. of (3.102) as follows

$$\begin{aligned}
&= \left(\mathbf{E} \frac{\tau}{|W_n|} \sum_{k=-\infty}^{\infty} \frac{1}{h_n} \int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right) X(dx) \right)^2 \\
&= (\lambda(s) + o(1))^2 = \lambda^2(s) + o(1),
\end{aligned}$$

as $n \to \infty$. Hence, the first term on the r.h.s. of (3.92) is equal to $\lambda^2(s) + o(1)$ as $n \to \infty$. This completes the proof of Theorem 3.8. □

Proof of Theorem 3.9

Since we want to prove (3.57) instead of (3.55), it is not enough now to use the result from Theorem 3.7 to simplify the expression for $Var(\hat{\lambda}_{n,K}(s))$. Hence, instead of writing $Var(\hat{\lambda}_{n,K}(s))$ as that in (3.92), here we have to directly compute $Var(\hat{\lambda}_{n,K}(s))$ as follows

$$\begin{aligned}
&Var\left(\hat{\lambda}_{n,K}(s)\right)\\
&= \frac{\tau^2}{|W_n|^2 h_n^2} \quad Var\left(\sum_{k=-\infty}^{\infty}\int_{W_n} K\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right) X(dx)\right)\\
&+\frac{1}{|W_n|^2 h_n^2} \quad Var\left((\hat{\tau}_n-\tau)\sum_{k=-\infty}^{\infty}\int_{W_n} K\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right) X(dx)\right)\\
&+\frac{2\tau}{|W_n|^2 h_n^2} \quad Cov\left((\hat{\tau}_n-\tau)\sum_{k=-\infty}^{\infty}\int_{W_n} K\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right) X(dx),\right.\\
&\qquad\qquad \left.\sum_{k=-\infty}^{\infty}\int_{W_n} K\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right) X(dx)\right). \qquad (3.103)
\end{aligned}$$

The first term of (3.103) can be written as

$$\begin{aligned}
&\frac{\tau^2}{|W_n|^2 h_n^2} \quad Var\left(\sum_{k=-\infty}^{\infty}\int_{W_n} K\left(\tfrac{x-(s+k\tau)}{h_n}\right) X(dx)\right)\\
&+\frac{\tau^2}{|W_n|^2 h_n^2} \quad Var\left(\sum_{k=-\infty}^{\infty}\int_{W_n} \left[K\left(\tfrac{x-(s+k\hat{\tau}_n)}{h_n}\right) - K\left(\tfrac{x-(s+k\tau)}{h_n}\right)\right] X(dx)\right)\\
&+\frac{2\tau^2}{|W_n|^2 h_n^2} \quad Cov\left(\sum_{k=-\infty}^{\infty}\int_{W_n} \left[K\left(\tfrac{x-(s+k\hat{\tau}_n)}{h_n}\right) - K\left(\tfrac{x-(s+k\tau)}{h_n}\right)\right] X(dx),\right.\\
&\qquad\qquad \left.\sum_{l=-\infty}^{\infty}\int_{W_n} K\left(\tfrac{x-(s+l\tau)}{h_n}\right) X(dx)\right). \qquad (3.104)
\end{aligned}$$

We will prove this theorem by showing the first term of (3.104) can be written as the r.h.s. of (3.57), while the second and third terms of (3.104) as well as the second and third terms of (3.103) are of order $o(|W_n|^{-1}h_n^{-1})$ as $n \to \infty$.

First we consider the first term of (3.104). Since for sufficiently large n, the quantity in (3.101) are independent, provided $j \neq k$, this term can be written as follows

$$\frac{\tau^2}{|W_n|^2 h_n^2}\sum_{k=-\infty}^{\infty} Var\left(\int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right) X(dx)\right)$$

$$= \frac{\tau^2}{|W_n|^2 h_n^2} \int_{-h_n}^{h_n} K^2\left(\frac{x}{h_n}\right) \lambda(x+s) \sum_{k=-\infty}^{\infty} \mathbf{I}(x+s+k\tau \in W_n)dx$$

$$= \left(1+\mathcal{O}\left(|W_n|^{-1}\right)\right)\left(\frac{\tau}{|W_n|h_n}\right)\left(\frac{1}{h_n}\int_{-h_n}^{h_n} K^2\left(\frac{x}{h_n}\right)\lambda(x+s)dx\right)$$

$$= \left(1+\mathcal{O}\left(|W_n|^{-1}\right)\right)\left(\frac{\tau}{|W_n|h_n}\right)\left(\frac{1}{h_n}\int_{-h_n}^{h_n} K^2\left(\frac{x}{h_n}\right)\left(\lambda(x+s)-\lambda(s)\right)dx\right)$$

$$+ \left(1+\mathcal{O}\left(|W_n|^{-1}\right)\right)\left(\frac{\tau}{|W_n|h_n}\right)\lambda(s)\int_{-1}^{1} K^2(x)dx. \tag{3.105}$$

Since s is a Lebesgue point of λ and the kernel K is bounded, we have that

$$h_n^{-1}\int_{-h_n}^{h_n} K^2(\frac{x}{h_n})|\lambda(x+s)-\lambda(s)|dx = o(1),$$

as $n \to \infty$. Hence, the first term on the r.h.s. of (3.105) is of order $o(|W_n|^{-1}h_n^{-1})$ as $n \to \infty$. Obviously, the second term on the r.h.s. of (3.105) can be written as the r.h.s. of (3.57). Hence, we have proved that the first term of (3.104) can be written as the r.h.s. of (3.57).

Next we show that the second term of (3.104) is of order $o(|W_n|^{-1}h_n^{-1})$ as $n \to \infty$. Since this term does not exceed the first term of (3.94), it suffices to check that the first term of (3.94) is $o(|W_n|^{-1}h_n^{-1})$ as $n \to \infty$. To do this we argue as follows. Suppose that the kernel K has $(m-1)$ discontinuities. Then, K can be written as a sum of m Lipschitz functions $K_1, \ldots, K_m$ which having disjoint supports $I_1, \ldots, I_m$ respectively. In other words we have that

$$K = \sum_{i=1}^{m} K_i \qquad \text{and} \qquad \bigcup_{i=1}^{m} I_i = [-1,1].$$

Since K_i is a Lipschitz function, for each i $(i = 1, \ldots, m)$ we have that

$$|K_i(u) - K_i(v)| \le c(K_i)|u-v|\left(\mathbf{I}\{u \in I_i\} + \mathbf{I}\{v \in I_i\}\right) \tag{3.106}$$

for all $u, v \in I_i$, where $c(K_i)$ is a positive constant depending on K_i. Let $I_0 = (-\infty, -1)$, $I_{m+1} = (1, \infty)$, and we put $K_i = 0$ when $i = 0$ or $i = m+1$ (by assumption (K.3)). Then, the absolute value of the sum of random variables appearing in the first term of (3.94) can be written as

$$\left|\sum_{k=-\infty}^{\infty}\sum_{i=0}^{m+1}\int_{W_n}\left[K_i\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right) - K_i\left(\frac{x-(s+k\tau)}{h_n}\right)\right]X(dx)\right|$$

$$\le \left|\sum_{i=0}^{m+1}\sum_{k=-\infty}^{\infty}\int_{W_n}\left[K_i\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right) - K_i\left(\frac{x-(s+k\tau)}{h_n}\right)\right]\right.$$

$$\mathbf{I}\left(x \in \{s + k\hat{\tau}_n + h_n I_i\} \cap \{s + k\tau + h_n I_i\}\right) X(dx)|$$
$$+ \left| \sum_{0 \le i \ne j \le m+1} \sum_{k=-\infty}^{\infty} \int_{W_n} \left[K_i \left(\frac{x - (s + k\hat{\tau}_n)}{h_n} \right) - K_j \left(\frac{x - (s + k\tau)}{h_n} \right) \right] \right.$$
$$\mathbf{I}\left(x \in \{s + k\hat{\tau}_n + h_n I_i\} \cap \{s + k\tau + h_n I_j\}\right) X(dx)|. \tag{3.107}$$

Note that, the first term on the r.h.s. of (3.107) correspond to the case where both points $(x-(s+k\hat{\tau}_n))/h_n$ and $(x-(s+k\tau))/h_n$ are in the same interval I_i (for some i, $0 \le i \le m+1$), while its second term corresponds to the case where $(x-(s+k\hat{\tau}_n))/h_n \in I_i$ and $(x-(s+k\tau))/h_n \in I_j$, with $i \ne j$.

First we consider the first term on the r.h.s. of (3.107). Since both points $(x-(s+k\hat{\tau}_n))/h_n$ and $(x-(s+k\tau))/h_n$ are in the same interval I_i and $K_0 = K_{m+1} = 0$, by (3.106), this term does not exceed

$$\sum_{k=-\infty}^{\infty} \sum_{i=1}^{m} c(K_i) \left| \frac{k(\hat{\tau}_n - \tau)}{h_n} \right| X\left(\{s + k\hat{\tau}_n + h_n I_i\} \cap W_n\right)$$
$$+ \sum_{k=-\infty}^{\infty} \sum_{i=1}^{m} c(K_i) \left| \frac{k(\hat{\tau}_n - \tau)}{h_n} \right| X\left(\{s + k\tau + h_n I_i\} \cap W_n\right). \tag{3.108}$$

Since $s \in W_n$, by (3.56), we have with probability 1 that all integer k such that $\{s + k\hat{\tau}_n + h_n I_i\} \cap W_n \ne \emptyset$ satisfy $|k| = \mathcal{O}(|W_n|)$ as $n \to \infty$. Obviously, all integer k such that $\{s + k\tau + h_n I_i\} \cap W_n \ne \emptyset$ also satisfy $|k| = \mathcal{O}(|W_n|)$ as $n \to \infty$. Then by (3.56), we have with probability 1 that

$$|k(\hat{\tau}_n - \tau)/h_n| \le C\delta_n/(|W_n|^{1/2} h_n^{1/2}) \tag{3.109}$$

and

$$k\tau - h_n \left(C\delta_n/(|W_n|^{1/2} h_n^{1/2}) \right) \le k\hat{\tau}_n \le k\tau + h_n \left(C\delta_n/(|W_n|^{1/2} h_n^{1/2}) \right),$$

where C is a positive constant. By (3.7), for large n, we have that $(C\delta_n)/(|W_n|^{1/2} h_n^{1/2}) \le 1$. Note that $|I_i| \le 2$ for all $i = 1, \ldots, m$. Let $c_0 = \sum_{i=1}^{m} c(K_i)$. Then, the sum of the first and second terms of (3.108) does not exceed

$$\frac{2Cc_0\delta_n}{|W_n|^{1/2} h_n^{1/2}} \sum_{k=-\infty}^{\infty} X\left(\{s + k\tau + h_n[-2, 2]\} \cap W_n\right). \tag{3.110}$$

Next we consider the second term on the r.h.s. of (3.107). Since $K(u) \le C_0$ for all $u \in \mathbf{R}$ (because of assumption (K.2)), this term does not exceed

$$2C_0 \sum_{0\le i\ne j\le m+1} \sum_{k=-\infty}^{\infty} X \quad (\{s+k\tau+k(\hat{\tau}_n-\tau)+h_nI_i\}$$
$$\cap\{s+k\tau+h_nI_j\}\cap W_n). \qquad (3.111)$$

For simplicity, we put the indexes $1,\ldots,m$ from left to right in the intervals I_i. By (3.7) and (3.109), we see that with probability 1, $k(\hat{\tau}_n-\tau)\downarrow 0$ faster than $\delta_n h_n^{1/2}|W_n|^{-1/2}\downarrow 0$ and hence also faster than $h_n\downarrow 0$, as $n\to\infty$, for all k such that $\{s+k\tau+k(\hat{\tau}_n-\tau)+h_nI_i\}\cap\{s+k\tau+h_nI_j\}\cap W_n\ne\emptyset$. Then for large n, $|i-j|\ge 2$ implies $\{s+k\tau+k(\hat{\tau}_n-\tau)+h_nI_i\}\cap\{s+k\tau+h_nI_j\}=\emptyset$. Hence for large n, we have with probability 1 that the quantity in (3.111) does not exceed

$$2C_0\sum_{i=0}^{m}\sum_{k=-\infty}^{\infty} X\left(\{s+k\tau+k(\hat{\tau}_n-\tau)+h_nI_i\}\cap\{s+k\tau+h_nI_{i+1}\}\cap W_n\right)$$
$$+2C_0\sum_{i=0}^{m}\sum_{k=-\infty}^{\infty} X\left(\{s+k\tau+h_nI_i\}\cap\{s+k\tau+k(\hat{\tau}_n-\tau)+h_nI_{i+1}\}\cap W_n\right).$$
$$(3.112)$$

From (3.112) we can see that, if $k(\hat{\tau}_n-\tau)\le 0$ then the first term of (3.112) is equal to zero, while if $k(\hat{\tau}_n-\tau)\ge 0$ then its second term is equal to zero. Let a_i $(i=1,\ldots,m-1)$ denote the discontinuity points of K, i.e. the border point of I_i and I_{i+1}, $a_0=-1$, and $a_m=1$. Since, by (3.109) we have $|k(\hat{\tau}_n-\tau)|\le C\delta_n h_n^{1/2}|W_n|^{-1/2}$, the quantity in (3.112) does not exceed

$$2C_0\sum_{i=0}^{m}\sum_{k=-\infty}^{\infty} X \quad (\{s+k\tau+h_na_i+[-C\delta_nh_n^{1/2}|W_n|^{-1/2},$$
$$C\delta_nh_n^{1/2}|W_n|^{-1/2}]\}\cap W_n). \qquad (3.113)$$

Hence, the l.h.s. of (3.107) does not exceed sum of the quantity in (3.110) and (3.113). To prove the first term of (3.94) is $o(|W_n|^{-1}h_n^{-1})$, which implies the second term of (3.104) is $o(|W_n|^{-1}h_n^{-1})$ as $n\to\infty$, it suffices now to check

$$\frac{\delta_n^2}{|W_n|h_n}\mathbf{E}\left(\sum_{k=-\infty}^{\infty} X\left(\{s+k\tau+h_n[-2,2]\}\cap W_n\right)\right)^2=o(|W_n|h_n), \quad (3.114)$$

and

$$\mathbf{E}\ \left(\sum_{i=0}^{m}\sum_{k=-\infty}^{\infty} X(\{s+k\tau+h_na_i+[-C\delta_nh_n^{1/2}|W_n|^{-1/2},\right.$$
$$\left.C\delta_nh_n^{1/2}|W_n|^{-1/2}]\}\cap W_n)\right)^2=o(|W_n|h_n), \qquad (3.115)$$

as $n \to \infty$. By (3.99), we have (3.114). By a similar argument as the one used to show the expectation on the l.h.s. of (3.97) does not exceed the quantity in (3.100), here we also have that the l.h.s. of (3.115) does not exceed

$$4\left(\sum_{i=0}^{m}\sum_{k=-\infty}^{\infty} \mathbf{E}\ X(\{s+k\tau+h_n a_i+[-C\delta_n h_n^{1/2}|W_n|^{-1/2},\right.$$
$$\left. C\delta_n h_n^{1/2}|W_n|^{-1/2}]\}\cap W_n)\right)^2. \tag{3.116}$$

Since λ is assumed to be bounded in a neighborhood of s, sum of expectations in (3.116) can be computed as follows

$$\begin{aligned}
&\sum_{i=0}^{m}\sum_{k=-\infty}^{\infty}\int_{-C\delta_n h_n^{1/2}|W_n|^{-1/2}}^{C\delta_n h_n^{1/2}|W_n|^{-1/2}} \lambda(x+s+k\tau+h_n a_i) \\
&\qquad \mathbf{I}(x+s+k\tau+h_n a_i\in W_n)dx \\
\leq\ & (m+1)(2C\delta_n h_n^{1/2}|W_n|^{-1/2})\lambda_0(2|W_n|/\tau)=\mathcal{O}(\delta_n h_n^{1/2}|W_n|^{1/2}),
\end{aligned} \tag{3.117}$$

as $n \to \infty$. Since the l.h.s. of (3.117) is positive, (3.117) implies that the quantity in (3.116) is of order $\mathcal{O}(\delta_n^2 h_n|W_n|) = o(h_n|W_n|)$ as $n \to \infty$. This implies (3.115). Hence, we have that the second term of (3.104) is $o(|W_n|^{-1}h_n^{-1})$ as $n \to \infty$.

Next, we consider the third term of (3.104). Because the first term of (3.104) can be written as the r.h.s. of (3.57), we know that this term is $\mathcal{O}(|W_n|^{-1}h_n^{-1})$ as $n \to \infty$. Since the second term of (3.104) is $o(|W_n|^{-1}h_n^{-1})$ as $n \to \infty$, an application of Cauchy-Schwarz inequality shows that the third term of (3.104) is $o(|W_n|^{-1}h_n^{-1})$ as $n \to \infty$.

It remains to show the second and third terms of (3.103) are $o(|W_n|^{-1}h_n^{-1})$ as $n \to \infty$. The second term of (3.103) does not exceed the first term of (3.93). From the proof of Theorem 3.8, we know that this term is of order $\mathcal{O}(|W_n|^{-2})$, which is $o(|W_n|^{-1}h_n^{-1})$ as $n \to \infty$. Hence, the second term of (3.103) is of order $o(|W_n|^{-1}h_n^{-1})$ as $n \to \infty$. From the proof above, we know that the first term of (3.103) is $\mathcal{O}(|W_n|^{-1}h_n^{-1})$ as $n \to \infty$. Since the second term of (3.103) is $o(|W_n|^{-1}h_n^{-1})$ as $n \to \infty$, an application of Cauchy-Schwarz inequality shows that the third term of (3.103) is $o(|W_n|^{-1}h_n^{-1})$ as $n \to \infty$. This completes the proof of Theorem 3.9. □

Proof of Theorem 3.10

Recall that $\mathbf{E}\hat{\lambda}_{n,K}(s)$ can be written as the quantity in (3.70). We will prove this theorem by showing that the third term of (3.70) can be written as the r.h.s. of (3.59), while the other terms are of order $o(h_n^2)$ as $n \to \infty$.

First we prove that the first term of (3.70) is $o(h_n^2)$ as $n \to \infty$. From the proof of Theorem 3.7 we know that the absolute value of this term does not exceed the quantity in (3.73). We know that $(EX^2(W_n))^{\frac{1}{2}} = \mathcal{O}(|W_n|)$ as $n \to \infty$. By Lemma 3.13 for $m = 1$ (we take $\delta_n = 1$, provided we replace the condition (3.53) by (3.58), we have that $(E(\hat{\tau}_n - \tau)^2)^{\frac{1}{2}} = \mathcal{O}(|W_n|^{-1}h_n^3)$ as $n \to \infty$. Then, the quantity in (3.73) is of order $\mathcal{O}(|W_n|^{-1}h_n^2)$, which implies that the first term of (3.70) is $o(h_n^2)$ as $n \to \infty$.

Next we prove that the second term of (3.70) is $o(h_n^2)$ as $n \to \infty$. By a similar argument as the one used to prove the upper bounds in (3.110) and (3.113), but with condition (3.56) now replaced by (3.58), we have that

$$
\begin{aligned}
&\left| \sum_{k=-\infty}^{\infty} \int_{W_n} \left[K\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right) - K\left(\frac{x-(s+k\tau)}{h_n}\right) \right] X(dx) \right| \\
\leq\ & 2Cc_0\delta_n h_n^2 \sum_{k=-\infty}^{\infty} X\left(\{s+k\tau+h_n[-2,2]\} \cap W_n\right) \\
&+ 2C_0 \sum_{i=0}^{m} \sum_{k=-\infty}^{\infty} X\left(\{s+k\tau+h_n a_i + [-C\delta_n h_n^3, C\delta_n h_n^3]\} \cap W_n\right).
\end{aligned}
$$

This implies that the absolute value of the second term of (3.70) does not exceed

$$
\begin{aligned}
&2Cc_0\delta_n h_n^2 \left\{ \frac{\tau}{|W_n|h_n} \mathbf{E} \sum_{k=-\infty}^{\infty} X\left(\{s+k\tau+h_n[-2,2]\} \cap W_n\right) \right\} \\
+\ & \frac{2\tau C_0}{|W_n|h_n} \mathbf{E} \sum_{i=0}^{m} \sum_{k=-\infty}^{\infty} X\left(\{s+k\tau+h_n a_i + [-C\delta_n h_n^3, C\delta_n h_n^3]\} \cap W_n\right).
\end{aligned}
\tag{3.118}
$$

By (3.89) we then have that the first term of (3.118) is of order $o(h_n^2)$ as $n \to \infty$. A similar argument as the one in (3.117) also shows that the second term of (3.118) is of order $o(h_n^2)$ as $n \to \infty$. Hence the second term of (3.70) is $o(h_n^2)$ as $n \to \infty$.

Next we prove that the third term of (3.70) can be written as the r.h.s. of (3.59). By Fubini's, this term is equal to

$$\frac{\tau}{|W_n|}\sum_{k=-\infty}^{\infty}\frac{1}{h_n}\int_{\mathbf{R}}K\left(\frac{x-(s+k\tau)}{h_n}\right)\lambda(x)I(x\in W_n)dx$$
$$=\frac{\tau}{|W_n|h_n}\int_{\mathbf{R}}K\left(\frac{x}{h_n}\right)\sum_{k=-\infty}^{\infty}\lambda(x+s+k\tau)\mathbf{I}(x+s+k\tau\in W_n)dx. \tag{3.119}$$

Since λ is periodic with period τ, we have $\lambda(x+s+k\tau)=\lambda(x+s)$. Furthermore, it is obvious that $\sum_{k=-\infty}^{\infty}\mathbf{I}(x+s+k\tau\in W_n)\in[|W_n|/\tau-1,|W_n|/\tau+1]$. Then, the r.h.s of (3.119) can be written as

$$\begin{aligned}&\left(1+\mathcal{O}\left(|W_n|^{-1}\right)\right)\frac{1}{h_n}\int_{\mathbf{R}}K\left(\frac{x}{h_n}\right)\lambda(x+s)dx\\ =\;&\left(1+\mathcal{O}\left(|W_n|^{-1}\right)\right)\frac{1}{h_n}\int_{\mathbf{R}}K\left(\frac{x}{h_n}\right)\{\lambda(x+s)-\lambda(s)\}dx\\ &+\left(1+\mathcal{O}\left(|W_n|^{-1}\right)\right)\frac{1}{h_n}\int_{\mathbf{R}}K\left(\frac{x}{h_n}\right)\lambda(s)dx.\end{aligned} \tag{3.120}$$

By (3.2) and Young's form of Taylor's theorem, we have that

$$\frac{1}{h_n}\int_{-h_n}^{h_n}K\left(\frac{x}{h_n}\right)\lambda(s+x)dx=\int_{-1}^{1}K(x)\lambda(s+xh_n)dx$$
$$=\lambda(s)+\lambda'(s)h_n\int_{-1}^{1}xK(x)dx+\frac{\lambda''(s)}{2}h_n^2\int_{-1}^{1}x^2K(x)dx+o(h_n^2),$$

as $n\to\infty$. Because K is symmetric around zero, we have that $\int_{-1}^{1}xK(x)dx=0$. Then, the l.h.s. of (3.120) can be written as

$$(1+\mathcal{O}(|W_n|^{-1}))(\lambda(s)+\frac{\lambda''(s)}{2}h_n^2\int_{-1}^{1}x^2K(x)dx+o(h_n^2))$$
$$=\lambda(s)+\frac{\lambda''(s)}{2}h_n^2\int_{-1}^{1}x^2K(x)dx+o(h_n^2)+\mathcal{O}(|W_n|^{-1})$$

as $n\to\infty$. Hence, the third term of (3.70) can be written as the r.h.s. of (3.59). This completes the proof of Theorem 3.10. □

Chapter 4

Nearest neighbor estimation of the local intensity

4.1 Introduction

In this chapter we consider nearest neighbor estimation of the intensity function λ at a given point $s \in W_n$, using only a single realization $X(\omega)$ of the cyclic Poisson process X observed in W_n. The requirement $s \in W_n$ can be dropped if we know the period τ. The first part of this chapter is a revised version of Mangku (1999).

As in chapter 3, let $\hat{\tau}$ be any consistent estimator of the period τ, e.g. the one proposed and studied in chapter 5 or perhaps the estimator investigated by Vere-Jones (1982).

Let s_i, $i = 1, \ldots, X(W_n, \omega)$, denote the locations of the points in the realization $X(\omega)$ of the Poisson process X, observed in window W_n. Here $X(W_n, \omega)$ is nothing but the cardinality of the data set $\{s_i\}$.

It is well-known (see, e.g. Cressie (1993), p. 651) that, for any positive integer m, conditionally given $X(W_n) = m$, $(s_1, \ldots, s_m)$ can be viewed as a random sample of size m from a distribution with density f, which is given by

$$f(u) = \frac{\lambda(u)}{\int_{W_n} \lambda(v)dv} \mathbf{I}(u \in W_n), \tag{4.1}$$

while the simultaneous density $f(s_1, \ldots, s_m)$, of $(s_1, \ldots, s_m)$ is given by

$$f(s_1, \ldots, s_m) = \frac{\prod_{i=1}^{m} \lambda(s_i)}{\left(\int_{W_n} \lambda(v)dv\right)^m} \mathbf{I}\left((s_1, \ldots, s_m) \in W_n^m\right). \tag{4.2}$$

Let $\hat{s}_i$, $i = 1, \ldots, m$, denote the location of the point s_i $(i = 1, \ldots, m)$, after translation by a multiple of $\hat{\tau}_n$ such that $\hat{s}_i \in \bar{B}_{\hat{\tau}_n}(s)$, for all $i =$

$1, \ldots, m$, where $\bar{B}_{\hat{\tau}_n}(s) = [s - \frac{\hat{\tau}_n}{2}, s + \frac{\hat{\tau}_n}{2})$. The translation can be described more precisely as follows. We cover the window W_n by $N_{n,\hat{\tau}_n}$ adjacent disjoint intervals $\bar{B}_{\hat{\tau}_n}(s + j\hat{\tau}_n)$, for some integer j, and let $N_{n,\hat{\tau}_n}$ denote the number of such intervals, with $\bar{B}_{\hat{\tau}_n}(s + j\hat{\tau}_n) \cap W_n \neq \emptyset$. Then, for each j, we shift the interval $\bar{B}_{\hat{\tau}_n}(s + j\hat{\tau}_n)$ (together with the data points of $X(\omega)$ contained in this interval) by the amount $j\hat{\tau}_n$ such that after translation the interval coincide with $\bar{B}_{\hat{\tau}_n}(s)$.

Let $k = k_n$ be a sequence of positive integers such that

$$k_n \to \infty, \tag{4.3}$$

and

$$\frac{k_n}{|W_n|} \downarrow 0, \tag{4.4}$$

as $n \to \infty$.

Let now $|\hat{s}_{(k_n)} - s|$ denote the k_n-th order statistics of $|\hat{s}_1 - s|, \ldots, |\hat{s}_m - s|$, given $X(W_n) = m$. A nearest neighbor estimator for λ at the point s, is given by

$$\hat{\lambda}_n(s) = \frac{\hat{\tau}_n k_n}{2|W_n||\hat{s}_{(k_n)} - s|} \tag{4.5}$$

if $X(W_n) \geq k_n$, and $\hat{\lambda}_n(s) = 0$ otherwise.

Let us briefly indicate the relation between the kernel type estimator investigated in chapter 3 and the idea behind the construction of our nearest neighbor estimator. Let, for each ω, $\hat{X}_n(\omega)$ denote the set $\{\hat{s}_i\}$, where for any data point $s_i \in X(\omega)$, $\hat{s}_i$ is obtained from s_i by shifting over a random multiple of $\hat{\tau}_n$ such that $\hat{s}_i \in \bar{B}_{\hat{\tau}_n}(s)$. Here and elsewhere in this chapter let, for any set A, $\hat{X}_n(A)$ denote the number of points $\hat{s}_i$ in A. Then, the 'uniform' kernel estimator in (3.5) can also be written as

$$\hat{\lambda}_{n,\bar{K}}(s) = \frac{\hat{\tau}_n}{|W_n|} \frac{\hat{X}_n(B_{h_n}(s))}{2h_n}. \tag{4.6}$$

To obtain our nearest neighbor estimator (4.5), we replace the (random) number $\hat{X}_n(B_{h_n}(s))$ in (4.6) by a (non-random) positive integer k_n, i.e. $\hat{X}_n(B_{h_n}(s)) = k_n$, which directly yields that we may take $h_n = |\hat{s}_{(k_n)} - s|$, and (4.6) reduces to (4.5). A detailed comparison of (4.5) and (4.6) is given in section 4.4.

We remark that nearest neighbor estimators for estimating an unknown density function have been studied by Loftsgaarden and Quesenberry (1965), Wagner (1973), Moore and Yackel (1977), Ralescu (1995), among others. The condition (4.9) also appears in Wagner (1973). In the construction of our nearest neighbor estimator (4.5) we employ the periodicity of λ (cf. (1.10)) to combine different pieces from our data set, in order to mimic the 'infill asymptotic' framework.

4.2 Consistency

4.2.1 Results

Theorem 4.1 *Suppose that λ is periodic and locally integrable. If, in addition (4.3) and (4.4) hold true, and*

$$\frac{|W_n|^2}{k_n}|\hat{\tau}_n - \tau| \xrightarrow{p} 0, \tag{4.7}$$

as $n \to \infty$, then

$$\hat{\lambda}_n(s) \xrightarrow{p} \lambda(s), \tag{4.8}$$

as $n \to \infty$, for each s at which λ is continuous and positive.

Theorem 4.2 *Suppose that λ is periodic and locally integrable. If, in addition*

$$\sum_{n=1}^{\infty} \exp(-\epsilon k_n) < \infty, \tag{4.9}$$

for each $\epsilon > 0$, (4.4) holds, and

$$\frac{|W_n|^2}{k_n}|\hat{\tau}_n - \tau| \xrightarrow{c} 0, \tag{4.10}$$

then

$$\hat{\lambda}_n(s) \xrightarrow{c} \lambda(s), \tag{4.11}$$

as $n \to \infty$, for each s at which λ is continuous and positive.

Remark 4.1 Since

$$\mathbf{P}(k_n \leq X(W_n)) = \mathbf{P}(k_n/|W_n| \leq X(W_n)/|W_n|) \to 1, \tag{4.12}$$

as $n \to \infty$, (because of (4.4) and by Lemma 2.2 we have $X(W_n)/|W_n| \xrightarrow{p} \theta$, with $\theta > 0$), we can conclude that no matter how we define $\hat{\lambda}_n(s)$ in case $k_n > X(W_n)$, Theorem 4.1 remains valid. To check that the above conclusion also holds for Theorem 4.2, we need to show that

$$\textstyle\sum_{n=1}^{\infty} \mathbf{P}(k_n > X(W_n)) < \infty.$$

But, by (4.4), Lemma A.1 (see Appendix), and (4.9), it is easy to show that $\mathbf{P}(k_n > X(W_n))$ is summable. □

4.2.2 Proofs: the case τ is known

We first consider the situation where we know the period τ. Let $\bar{s}_i$, $i = 1, \ldots, X(W_n, \omega)$, denote the location of the points s_i $(i = 1, \ldots, X(W_n, \omega))$, after translation by a multiple of τ such that $\bar{s}_i \in \bar{B}_\tau(s)$, for all $i = 1, \ldots, X(W_n, \omega)$, where $\bar{B}_\tau(s) = [s - \frac{\tau}{2}, s + \frac{\tau}{2})$. By periodicity of λ, we have that $\lambda(\bar{s}_i) = \lambda(s_i)$, for each $i = 1, \ldots, X(W_n, \omega)$. For any $A \subset \bar{B}_\tau(s)$, let $\bar{X}_n(A)$ denote the number of points $\bar{s}_i$ in A. Then, of course, $\bar{X}_n(\bar{B}_\tau(s)) = X(W_n)$, where $\bar{X}_n$ is a Poisson process with intensity function

$$\lambda_n(u) = \lambda(u) \sum_{j=-\infty}^{\infty} \mathbf{I}(u + j\tau \in W_n)$$

(cf. Kingman (1993), Superposition Theorem and Restriction Theorem, p. 16-17). As a result, (cf. (4.1) and (4.2)), conditionally given $\bar{X}_n(\bar{B}_\tau(s)) = m$, $(\bar{s}_1, \ldots, \bar{s}_m)$ can be viewed as a random sample of size m from a distribution with density $\bar{f}$, which is given by

$$\bar{f}(u) = \frac{\lambda_n(u)}{\int_{W_n} \lambda(v)dv} \mathbf{I}(u \in \bar{B}_\tau(s)) = \frac{\lambda_n(u)}{\int_{\bar{B}_\tau(s)} \lambda_n(v)dv} \mathbf{I}(u \in \bar{B}_\tau(s)),$$

while the simultaneous density $\bar{f}(\bar{s}_1, \ldots, \bar{s}_m)$, of $(\bar{s}_1, \ldots, \bar{s}_m)$ is given by

$$\bar{f}(\bar{s}_1, \ldots, \bar{s}_m) = \frac{\prod_{i=1}^{m} \lambda_n(\bar{s}_i)}{\left(\int_{W_n} \lambda(v)dv\right)^m} \mathbf{I}\left((\bar{s}_1, \ldots, \bar{s}_m) \in \bar{B}_\tau(s)^m\right).$$

For any real number $x \geq 0$, define

$$\begin{aligned} H_n(x) &= \mathbf{P}\left(|\bar{s}_i - s| \leq x \;\; |X(W_n) = m\right) \\ &= \mathbf{P}\left(s - x \leq \bar{s}_i \leq s + x \;\; |X(W_n) = m\right) \\ &= \int_{s-x}^{s+x} \frac{\lambda_n(u)}{\int_{W_n} \lambda(v)dv} \mathbf{I}(u \in \bar{B}_\tau(s))du. \end{aligned} \tag{4.13}$$

Now we consider the order statistics of the random sample

$$|\bar{s}_1 - s|, \ldots, |\bar{s}_m - s|$$

of size m from H_n. Let $|\bar{s}_{(k)} - s|$ denote the k-th order statistic of the sample $|\bar{s}_1 - s|, \ldots, |\bar{s}_m - s|$. Define

$$\bar{\lambda}_n(s) = \frac{\tau k_n}{2|W_n||\bar{s}_{(k_n)} - s|}. \tag{4.14}$$

Note that, if we replace τ and $\bar{s}_{(k_n)}$ in $\bar{\lambda}_n(s)$ by $\hat{\tau}_n$ and $\hat{s}_{(k_n)}$ respectively, then $\bar{\lambda}_n(s)$ reduces to the estimator $\hat{\lambda}_n(s)$ given in (4.5). We will now first prove that our Theorems are true, when $\hat{\lambda}_n(s)$ is replaced by $\bar{\lambda}_n(s)$. In section 3 we will show that our Theorems are valid for $\hat{\lambda}_n(s)$ as well.

Lemma 4.3 *Suppose that λ is periodic (with period τ), and locally integrable. If, in addition (4.3) and (4.4) hold, then*

$$\bar{\lambda}_n(s) \xrightarrow{p} \lambda(s), \tag{4.15}$$

as $n \to \infty$, for each s at which λ is continuous and positive.

Lemma 4.4 *Suppose that λ is periodic (with period τ), and locally integrable. If, in addition (4.9) and (4.4) hold, then*

$$\bar{\lambda}_n(s) \xrightarrow{c} \lambda(s), \tag{4.16}$$

as $n \to \infty$, for each s at which λ is continuous and positive.

Proof of Lemma 4.3 In view of Remark 4.1, we may assume, without loss of generality, that $k_n \leq X(W_n)$.

To prove (4.15), we must show that,

$$\mathbf{P}\left(\left|\frac{\tau k_n}{2|W_n||\bar{s}_{(k_n)} - s|} - \lambda(s)\right| \geq \epsilon\right) \to 0 \tag{4.17}$$

as $n \to \infty$, for each sufficiently small $\epsilon > 0$. Choose $\epsilon < \lambda(s)$. Then, a simple calculation shows that, the probability on the l.h.s. of (4.17) is equal to

$$\begin{aligned} &\mathbf{P}\left(\frac{\tau k_n}{2|W_n|(\lambda(s)-\epsilon)} \leq |\bar{s}_{(k_n)} - s| \text{ or } \frac{\tau k_n}{2|W_n|(\lambda(s)+\epsilon)} \geq |\bar{s}_{(k_n)} - s|\right) \\ &\leq \mathbf{P}\left(|\bar{s}_{(k_n)} - s| \geq \frac{\tau k_n}{2|W_n|(\lambda(s)-\epsilon)}\right) \\ &+\mathbf{P}\left(|\bar{s}_{(k_n)} - s| \leq \frac{\tau k_n}{2|W_n|(\lambda(s)+\epsilon)}\right). \end{aligned} \tag{4.18}$$

Then, to prove (4.17), it suffices to check that

$$\mathbf{P}\left(|\bar{s}_{(k_n)} - s| \geq \frac{\tau k_n}{2|W_n|(\lambda(s)-\epsilon)}\right) \to 0 \tag{4.19}$$

and

$$\mathbf{P}\left(|\bar{s}_{(k_n)} - s| \leq \frac{\tau k_n}{2|W_n|(\lambda(s)+\epsilon)}\right) \to 0 \tag{4.20}$$

as $n \to \infty$, for each $\epsilon > 0$. Here we only give proof of (4.19), because the proof of (4.20) is similar.

Recall that $X(W_n)$ is a Poisson random variable with

$$\mathbf{E}X(W_n) = Var(X(W_n)) = \int_{W_n} \lambda(s)ds.$$

Since λ is cyclic (with period τ), by Lemma 2.1 we have that

$$\int_{W_n} \lambda(s)ds = \theta|W_n| + \mathcal{O}(1),$$

as $n \to \infty$. Let

$$C_{1,n} = [\theta|W_n| - (\theta|W_n|)^{1/2}a_n] \tag{4.21}$$

$$C_{2,n} = [\theta|W_n| + (\theta|W_n|)^{1/2}a_n], \tag{4.22}$$

where a_n is an arbitrary sequence such that $a_n \to \infty$ and $a_n = o(|W_n|^{1/2})$, as $n \to \infty$. Then, we can write the probability on the l.h.s. of (4.19) as

$$\begin{aligned}
&\sum_{m=k_n}^{\infty} \mathbf{P}\left(|\bar{s}_{(k_n)} - s| \geq \frac{\tau k_n}{2|W_n|(\lambda(s)-\epsilon)} \,|X(W_n) = m\right) \mathbf{P}\left(X(W_n) = m\right) \\
\leq\; & \sum_{m=k_n}^{C_{1,n}-1} \mathbf{P}\left(X(W_n) = m\right) + \sum_{m=C_{2,n}+1}^{\infty} \mathbf{P}\left(X(W_n) = m\right) \\
+\; & \max_{C_{1,n} \leq m \leq C_{2,n}} \mathbf{P}\left(X(W_n) = m\right) \cdot \\
& \cdot \sum_{m=C_{1,n}}^{C_{2,n}} \mathbf{P}\left(|\bar{s}_{(k_n)} - s| \geq \frac{\tau k_n}{2|W_n|(\lambda(s)-\epsilon)}|X(W_n) = m\right).
\end{aligned} \tag{4.23}$$

It suffices now to show that each term on the r.h.s. of (4.23) converges to zero, as $n \to \infty$.

First we show that the first term on the r.h.s. of (4.23) is $o(1)$, as $n \to \infty$. Since $|\mathbf{E}X(W_n) - \theta|W_n|| = \mathcal{O}(1)$, as $n \to \infty$, this quantity is equal to

$$\begin{aligned}
&\mathbf{P}\left(X(W_n) \leq C_{1,n} - 1\right) \leq \mathbf{P}\left(X(W_n) \leq \theta|W_n| - (\theta|W_n|)^{1/2}a_n\right) \\
&\leq \mathbf{P}\left(|X(W_n) - \mathbf{E}X(W_n)| \geq (\theta|W_n|)^{1/2}a_n - |\mathbf{E}X(W_n) - \theta|W_n||\right) \\
&= \mathbf{P}\left((\mathbf{E}X(W_n))^{-1/2}|X(W_n) - \mathbf{E}X(W_n)| \geq \mathcal{O}(1)a_n\right) \\
&\leq \mathcal{O}(1)\exp\left(-\frac{a_n^2}{2+o(1)}\right),
\end{aligned} \tag{4.24}$$

which is $o(1)$, since $a_n \to \infty$, as $n \to \infty$. Here we have used Lemma A.1 of the Appendix. A similar argument also shows that the second term on the r.h.s. of (4.23) is $o(1)$, as $n \to \infty$.

Next we prove that the third term on the r.h.s. of (4.23) is $o(1)$, as $n \to \infty$. Let $m = m_n$ be a positive integer, such that $C_{1,n} \le m_n \le C_{2,n}$. Then $m_n \sim \theta|W_n|$, which implies that $k_n/m_n = o(1)$, as $n \to \infty$ (by (4.4)). Recall that $X(W_n)$ has a Poisson distribution with parameter $\mu(W_n) = \int_{W_n} \lambda(s)ds$. A simple calculation, using Stirling's formula, shows that

$$\max_{m_n,\ C_{1,n} \le m_n \le C_{2,n}} \mathbf{P}\left(X(W_n) = m_n\right) = \mathcal{O}(|W_n|^{-1/2}),$$

as $n \to \infty$. It is well-known (see, e.g. Reiss (1989), p. 15) that, conditionally given $\bar{X}_n(\bar{B}_\tau(s)) = X(W_n) = m_n$, $|\bar{s}_{(k_n)} - s|$ has exactly the same distribution as $H_n^{-1}(Z_{k_n:m_n})$, where $Z_{k_n:m_n}$ is the k_n-th order statistic of a sample $Z_1, \ldots, Z_{m_n}$ of size m_n from the uniform $(0,1)$ distribution. (We remark in passing that $k_n \le m_n$ for all n sufficiently large). Note that a similar device was employed by Ralescu (1995) in his analysis of multivariate nearest neighbor density estimators. As a result, the third term on the r.h.s. of (4.23) is equal to

$$\mathcal{O}(|W_n|^{-1/2}) \sum_{m_n=C_{1,n}}^{C_{2,n}} \mathbf{P}\left(H_n^{-1}(Z_{k_n:m_n}) \ge \frac{\tau k_n}{2|W_n|(\lambda(s)-\epsilon)}\right). \tag{4.25}$$

First note that, by choosing $\epsilon < \lambda(s)$, we have

$$\begin{aligned} \frac{\tau k_n}{2|W_n|(\lambda(s)-\epsilon)} &= \frac{\tau k_n}{2\lambda(s)|W_n|\left(1-\frac{\epsilon}{\lambda(s)}\right)} \ge \frac{\tau k_n}{2\lambda(s)|W_n|}\left(1+\frac{\epsilon}{\lambda(s)}\right) \\ &= \frac{\tau k_n}{2\lambda(s)|W_n|} + \frac{\tau\epsilon k_n}{2\lambda^2(s)|W_n|}. \end{aligned} \tag{4.26}$$

We know that, for each m_n,

$$\mathbf{E}Z_{k_n:m_n} = k_n/(m_n+1)$$

and

$$Var(Z_{k_n:m_n}) = \mathcal{O}(k_n/(m_n^2)).$$

We now need a stochastic expansion for $H_n^{-1}(Z_{k_n:m_n})$. First we simplify the r.h.s. of (4.13) to get for any $x \ge 0$

$$\begin{aligned} H_n(x) &= \frac{(|W_n|/\tau + \mathcal{O}(1))}{(\theta|W_n| + \mathcal{O}(1))}\int_{s-x}^{s+x} \lambda(u)\mathbf{I}(u \in \bar{B}_\tau(s))du \\ &= \left(\frac{1}{\theta\tau} + \mathcal{O}(|W_n|^{-1})\right)\int_{s-x}^{s+x} \lambda(u)\mathbf{I}(u \in \bar{B}_\tau(s))du \\ &= \frac{1}{\theta\tau}\int_{s-x}^{s+x} \lambda(u)\mathbf{I}(u \in \bar{B}_\tau(s))du + \mathcal{O}(|W_n|^{-1}), \end{aligned} \tag{4.27}$$

as $n \to \infty$, uniformly in x. This because $\int_{s-x}^{s+x} \lambda(u)\mathbf{I}(u \in \bar{B}_\tau(s))du \le \theta\tau$. Define function $H(x)$, which is equal to the first term on the r.h.s. of

(4.27) for $x \geq 0$, and zero otherwise. The density h of H is given by

$$\begin{aligned} h(x) &= \frac{\lambda(s+x)\mathbf{I}(s+x \in B_\tau(s))}{\theta\tau} \\ &\quad + \frac{\lambda(s-x)\mathbf{I}(s-x \in B_\tau(s))}{\theta\tau}, \end{aligned} \tag{4.28}$$

for any $x > 0$, while $h(0)$ denote the right hand derivative of H at zero. Next note that

$$\begin{aligned} H_n^{-1}(Z_{k_n:m_n}) &= \inf\{x : H_n(x) > Z_{k_n:m_n}\} \\ &= \inf\{x : H(x) > Z_{k_n:m_n} + \mathcal{O}(|W_n|^{-1})\} \\ &= H^{-1}\left(Z_{k_n:m_n} + \mathcal{O}(|W_n|^{-1})\right), \end{aligned} \tag{4.29}$$

as $n \to \infty$. Here and elsewhere in this chapter we define $H^{-1}(t) = \inf\{x : H(x) > t\}$, $0 \leq t < 1$. Now we compute $H^{-1}(0)$. Since $\lambda(s) > 0$ and λ is continuous at s, we see from the first term on the r.h.s. of (4.27) that $H(x) > 0$, while $x > 0$. In other words, the first term on the r.h.s. of (4.27) is equal to zero, if and only if, $x = 0$. Hence $H^{-1}(0) = 0$. Since h is right continuous at 0, the first (right hand) derivative of H^{-1} at 0 can be computed as

$$H^{-1'}(0) = \frac{1}{h\left(H^{-1}(0)\right)} = \frac{1}{h(0)} = \frac{\theta\tau}{2\lambda(s)}. \tag{4.30}$$

Since $H^{-1'}(0)$ is finite, by Young's form for Taylor's theorem (Serfling (1980), p. 45), we can write

$$\begin{aligned} & H^{-1}\left(\frac{k_n}{m_n+1} + \mathcal{O}(|W_n|^{-1})\right) \\ &= H^{-1}(0) + \left(\frac{k_n}{m_n+1} + \mathcal{O}(|W_n|^{-1})\right) H^{-1'}(0)(1+o(1)) \\ &= \frac{\theta\tau k_n}{2\lambda(s)(m_n+1)} + o\left(\frac{k_n}{|W_n|}\right), \end{aligned} \tag{4.31}$$

as $n \to \infty$. Because λ is continuous at s, we have

$$\begin{aligned} H^{-1'}\left(\frac{k_n}{m_n+1} + \mathcal{O}(|W_n|^{-1})\right) &= \frac{1}{h\left(H^{-1}\left(\frac{k_n}{m_n+1} + \mathcal{O}(|W_n|^{-1})\right)\right)} \\ = \frac{1}{h\left(|o(1)|\right)} = \frac{\theta\tau}{2\lambda(s+|o(1)|)} &= \frac{\theta\tau}{2\lambda(s)} + o(1), \end{aligned} \tag{4.32}$$

as $n \to \infty$.

Let $\tilde{Z}_{k_n:m_n} = Z_{k_n:m_n} - \mathbf{E}Z_{k_n:m_n} = Z_{k_n:m_n} - k_n/(m_n+1)$. Let us write

$$\begin{aligned} H_n^{-1}(Z_{k_n:m_n}) &= H_n^{-1}(Z_{k_n:m_n})\mathbf{I}(|\tilde{Z}_{k_n:m_n}| \leq \epsilon_n) \\ &+ H_n^{-1}(Z_{k_n:m_n})\mathbf{I}(|\tilde{Z}_{k_n:m_n}| > \epsilon_n), \end{aligned}$$

where ϵ_n is a sequence of positive real numbers such that $\epsilon_n \downarrow 0$ as $n \to \infty$. Because

$$H^{-1'}(k_n/(m_n+1) + \mathcal{O}(|W_n|^{-1})) = \mathcal{O}(1),$$

as $n \to \infty$, by Young's form for Taylor's theorem, we can write $H_n^{-1}(Z_{k_n:m_n})\mathbf{I}(|\tilde{Z}_{k_n:m_n}| \leq \epsilon_n)$ as (cf. (4.29))

$$\begin{aligned} & H_n^{-1}(Z_{k_n:m_n})\mathbf{I}(|\tilde{Z}_{k_n:m_n}| \leq \epsilon_n) \\ = \;& H^{-1}\left(Z_{k_n:m_n} + \mathcal{O}(|W_n|^{-1})\right)\mathbf{I}(|\tilde{Z}_{k_n:m_n}| \leq \epsilon_n) \\ = \;& \left\{H^{-1}\left(\frac{k_n}{m_n+1} + \mathcal{O}(|W_n|^{-1})\right)\right. \\ & + \left(Z_{k_n:m_n} - \frac{k_n}{m_n+1}\right)H^{-1'}\left(\frac{k_n}{m_n+1} + \mathcal{O}(|W_n|^{-1})\right) \\ & \left. + o\left(Z_{k_n:m_n} - \frac{k_n}{m_n+1}\right)\right\}\mathbf{I}(|\tilde{Z}_{k_n:m_n}| \leq \epsilon_n), \end{aligned} \tag{4.33}$$

as $n \to \infty$. Substituting (4.31) and (4.32) into the r.h.s. of (4.33), we then have

$$\begin{aligned} H_n^{-1}(Z_{k_n:m_n})\mathbf{I}(|\tilde{Z}_{k_n:m_n}| \leq \epsilon_n) = & \left\{\frac{\theta\tau k_n}{2\lambda(s)(m_n+1)} + o\left(\frac{k_n}{|W_n|}\right)\right. \\ + \left(\frac{\theta\tau}{2\lambda(s)}\right)\tilde{Z}_{k_n:m_n} + o\left(\tilde{Z}_{k_n:m_n}\right) & \left.\right\}\mathbf{I}(|\tilde{Z}_{k_n:m_n}| \leq \epsilon_n), \end{aligned} \tag{4.34}$$

as $n \to \infty$. Since $m_n \geq C_{1,n}$, the first term on the r.h.s. of (4.34) does not exceed

$$\begin{aligned} \frac{\theta\tau k_n}{2\lambda(s)\left([\theta|W_n| - (\theta|W_n|)^{1/2}a_n] + 1\right)} & \leq \frac{\theta\tau k_n}{2\lambda(s)\left(\theta|W_n| - (\theta|W_n|)^{1/2}a_n\right)} \\ = \frac{\theta\tau k_n}{2\theta\lambda(s)|W_n|\left(1 - (\theta|W_n|)^{-1/2}a_n\right)} & = \frac{\tau k_n}{2\lambda(s)|W_n|} + o\left(\frac{k_n}{|W_n|}\right), \end{aligned} \tag{4.35}$$

as $n \to \infty$. Combining (4.34), (4.35), and (4.26), and by noting also that the first term on the r.h.s. of (4.35) cancels with the first term on the r.h.s. of (4.26), we find that, for sufficiently large n, the probability appearing in (4.25) does not exceed

$$\begin{aligned} & \mathbf{P}\left(\frac{\theta\tau}{\lambda(s)}|\tilde{Z}_{k_n:m_n}|\mathbf{I}(|\tilde{Z}_{k_n:m_n}| \leq \epsilon_n) + \left|o\left(\tilde{Z}_{k_n:m_n}\right)\right|\mathbf{I}(|\tilde{Z}_{k_n:m_n}| \leq \epsilon_n)\right. \\ & \quad \left. + H_n^{-1}(Z_{k_n:m_n})\mathbf{I}(|\tilde{Z}_{k_n:m_n}| > \epsilon_n) > \frac{\tau\epsilon k_n}{4\lambda^2(s)|W_n|}\right) \\ & \leq \mathbf{P}\left(|\tilde{Z}_{k_n:m_n}| > \frac{\epsilon k_n}{12\theta\lambda(s)|W_n|}\right) + \mathbf{P}\left(|o(\tilde{Z}_{k_n:m_n})| > \frac{\tau\epsilon k_n}{12\lambda^2(s)|W_n|}\right) \\ & \quad + \mathbf{P}\left(H_n^{-1}(Z_{k_n:m_n})\mathbf{I}(|\tilde{Z}_{k_n:m_n}| > \epsilon_n) > \frac{\tau\epsilon k_n}{12\lambda^2(s)|W_n|}\right). \end{aligned} \tag{4.36}$$

First note that, for sufficiently large n, the second term on the r.h.s. of (4.36) does not exceed its first term. Now we notice that $H_n^{-1}(Z_{k_n:m_n}) \leq H_n^{-1}(1) = \frac{\tau}{2}$. Then we find that the third probability on the r.h.s. of (4.36) does not exceed $\mathbf{P}(|\tilde{Z}_{k_n:m_n}| > \epsilon_n)$. For convenience we take $\epsilon_n = (\epsilon k_n)/(12\theta\lambda(s)|W_n|)$. Then, the r.h.s. of (4.36) does not exceed

$$3\mathbf{P}\left(|\tilde{Z}_{k_n:m_n}| > \frac{\epsilon k_n}{12\theta\lambda(s)|W_n|}\right).$$

Therefore, for sufficiently large n, the quantity in (4.25) does not exceed

$$\begin{aligned} &\mathcal{O}(|W_n|^{-1/2})\,(C_{2,n} - C_{1,n} + 1)\,\mathbf{P}\left(|\tilde{Z}_{k_n:m_n}| > \frac{\epsilon k_n}{12\theta\lambda(s)|W_n|}\right) \\ &\leq \mathcal{O}(1)a_n\mathbf{P}\left(\left|Z_{k_n:m_n} - \frac{k_n}{m_n+1}\right| \geq \frac{\epsilon k_n}{12\theta\lambda(s)|W_n|}\right), \end{aligned} \tag{4.37}$$

as $n \to \infty$. By Chebyshev's inequality, we find that the probability on the r.h.s. of (4.37) is of order $\mathcal{O}(k_n^{-1})$, as $n \to \infty$. By (4.3) and choosing now $a_n = o(k_n)$, as $n \to \infty$, we have that the r.h.s. of (4.37) is $o(1)$ as $n \to \infty$. Hence (4.19) is proved. This completes the proof of Lemma 4.3. □

Proof of Lemma 4.4 To establish (4.16), we must show that

$$\sum_{n=1}^{\infty}\mathbf{P}\left(\left|\frac{\tau k_n}{2|W_n||\bar{s}_{(k_n)} - s|} - \lambda(s)\right| \geq \epsilon\right) < \infty, \tag{4.38}$$

for each $\epsilon > 0$. By (4.18), to prove (4.38) it suffices to show, for each $\epsilon > 0$,

$$\sum_{n=1}^{\infty}\mathbf{P}\left(|\bar{s}_{(k_n)} - s| \geq \frac{\tau k_n}{2|W_n|(\lambda(s) - \epsilon)}\right) < \infty, \tag{4.39}$$

and

$$\sum_{n=1}^{\infty}\mathbf{P}\left(|\bar{s}_{(k_n)} - s| \leq \frac{\tau k_n}{2|W_n|(\lambda(s) + \epsilon)}\right) < \infty. \tag{4.40}$$

Here we only give the proof of (4.39), because the proof of (4.40) is similar. To prove (4.39), it suffices clearly to show that, each of the terms on the r.h.s. of (4.23) converges completely to zero, as $n \to \infty$.

Let $C_{1,n}$ and $C_{2,n}$ be as given in (4.21) and (4.22). In order to deal with the first and second term of (4.23), the sequence a_n will now have to satisfy, in addition to the assumption $a_n = o(|W_n|^{1/2})$ which was already needed in the proof of Lemma 4.3, the additional requirement

$$\sum_{n=1}^{\infty} \exp(-a_n^2/3) < \infty.$$

The argument given in (4.24) will then imply that these terms converge completely to zero, as $n \to \infty$.

It remains to show that the third term on the r.h.s. of (4.23) also converges completely to zero, as $n \to \infty$. To do this, it is clear from the proof of Lemma 4.3, that it suffices now to check that the r.h.s. of (4.37) is summable, for each $\epsilon > 0$.

Let us now consider the probability appearing on the r.h.s. of (4.37). For sufficiently large n, by Lemma A.4 (see Appendix), there exists a positive constant C_0 such that the probability on the r.h.s. of (4.37) does not exceed

$$2 \exp\left\{-C_0 t_n^2\right\},$$

where

$$t_n = \left(\frac{m_n}{k_n/(m_n+1)\,(1-k_n/(m_n+1))}\right)^{1/2} \frac{k_n \epsilon}{12\theta\lambda(s)|W_n|}$$

which (for sufficiently large n) can be replaced with impunity by $\epsilon k_n^{\frac{1}{2}}/(24\lambda(s))$. Hence, for sufficiently large n, the r.h.s. of (4.37) does not exceed

$$\begin{aligned}
&\mathcal{O}(1) a_n \exp\left\{-\frac{C_0\epsilon^2}{576(\lambda(s))^2} k_n\right\} \\
&= \mathcal{O}(1) \exp\left\{\log a_n - \frac{C_0\epsilon^2}{1152(\lambda(s))^2} k_n\right\} \exp\left\{-\frac{C_0\epsilon^2}{1152(\lambda(s))^2} k_n\right\} \\
&= \mathcal{O}(1) \exp\left\{-\frac{C_0\epsilon^2}{1152(\lambda(s))^2} k_n\right\}, \qquad (4.41)
\end{aligned}$$

provided we require a_n to satisfy $\log a_n = o(k_n)$, as $n \to \infty$. Note that, e.g. the choice $a_n = (k_n)^{1/2}$ satisfies each of the three conditions imposed on a_n, namely $a_n = o(|W_n|^{1/2})$, $\sum_{n=1}^{\infty} \exp(-a_n^2/3) < \infty$, and $\log a_n = o(k_n)$, provided (4.4) and (4.9). By assumption (4.9), we have that the r.h.s. of (4.41) is summable. Hence (4.39) is proved. This completes the proof of Lemma 4.4. □

4.2.3 Proofs

Proof of Theorem 4.1 To prove (4.8), it suffices to check that

$$\frac{\tau k_n}{2|W_n||\hat{s}_{(k_n)} - s|} \xrightarrow{p} \lambda(s), \tag{4.42}$$

and

$$\left| \frac{\hat{\tau}_n k_n}{2|W_n||\hat{s}_{(k_n)} - s|} - \frac{\tau k_n}{2|W_n||\hat{s}_{(k_n)} - s|} \right| \xrightarrow{p} 0, \tag{4.43}$$

as $n \to \infty$, for each s at which λ is continuous and positive.

First, we prove (4.42). To do this, we must show that

$$\mathbf{P}\left(\left| \frac{\tau k_n}{2|W_n||\hat{s}_{(k_n)} - s|} - \lambda(s) \right| \geq \epsilon \right) \to 0 \tag{4.44}$$

as $n \to \infty$, for each sufficiently small $\epsilon > 0$. Choose $\epsilon < \lambda(s)$. Then, a simple calculation like the one leading from (4.17) to (4.19) and (4.20), shows that it suffices to check

$$\mathbf{P}\left(|\hat{s}_{(k_n)} - s| \geq \frac{\tau k_n}{2|W_n|(\lambda(s) - \epsilon)} \right) \to 0 \tag{4.45}$$

and

$$\mathbf{P}\left(|\hat{s}_{(k_n)} - s| \leq \frac{\tau k_n}{2|W_n|(\lambda(s) + \epsilon)} \right) \to 0 \tag{4.46}$$

as $n \to \infty$, for each $\epsilon > 0$. We only prove (4.45), because the proof of (4.46) is similar.

Recall that s_i, $(i = 1, \ldots, m)$ denotes the location of the points in the realization $X(\omega)$ of the Poisson process X. Let $\hat{\jmath}_i$ denote the random integer, depending on $\hat{\tau}_n$ and s_i, such that $\hat{s}_i = s_i + \hat{\jmath}_i \hat{\tau}_n$. Similarly, let $\bar{\jmath}_i$ denote an integer, depending on τ and s_i, such that $\bar{s}_i = s_i + \bar{\jmath}_i \tau$. If $s_{(k_n)}$ denotes the point corresponding to $\hat{s}_{(k_n)}$ before translation, then obviously $\hat{s}_{(k_n)} = s_{(k_n)} + \hat{\jmath}_{k_n} \hat{\tau}_n$. Furthermore we have that

$$\begin{aligned} |\hat{s}_{(k_n)} - s| &= |s_{(k_n)} + \hat{\jmath}_{k_n} \hat{\tau}_n - s| \\ &\leq |s_{(k_n)} + \bar{\jmath}_{k_n} \tau - s| + |\hat{\jmath}_{k_n} \hat{\tau}_n - \bar{\jmath}_{k_n} \tau| \\ &\leq |\bar{s}_{(k_n)} - s| + |\hat{\jmath}_{k_n}||\hat{\tau}_n - \tau| + \tau |\hat{\jmath}_{k_n} - \bar{\jmath}_{k_n}| \end{aligned} \tag{4.47}$$

To prove (4.45), it suffices now to check, for each $\epsilon > 0$,

$$\mathbf{P}\left(|\bar{s}_{(k_n)} - s| \geq \frac{\tau k_n}{6|W_n|(\lambda(s) - \epsilon)} \right) \to 0, \tag{4.48}$$

$$\mathbf{P}\left(|\hat{\jmath}_{k_n}||\hat{\tau}_n - \tau| \geq \frac{\tau k_n}{6|W_n|(\lambda(s) - \epsilon)} \right) \to 0, \tag{4.49}$$

and

$$\mathbf{P}\left(|\hat{\jmath}_{k_n} - \bar{\jmath}_{k_n}| \geq \frac{k_n}{6|W_n|(\lambda(s) - \epsilon)}\right) \to 0, \tag{4.50}$$

as $n \to \infty$. First note that, the proof of (4.19) also yields (4.48). Since $|\hat{\jmath}_{k_n}| = \mathcal{O}_p(|W_n|)$, as $n \to \infty$, assumption (4.7) yields that

$$|\hat{\jmath}_{k_n}||\hat{\tau}_n - \tau| = o_p(k_n/|W_n|),$$

as $n \to \infty$, which directly implies (4.49). Hence, it remains to check (4.50).

Here we only give the proof of (4.50) for the case $\hat{\tau}_n \geq \tau$ and $\hat{\jmath}_{k_n}$, $\bar{\jmath}_{k_n}$ are both positive; because the proofs of the other seven cases are similar and therefore omitted. Since $\hat{\tau}_n \geq \tau$, we also know that $\hat{\jmath}_{k_n} \leq \bar{\jmath}_{k_n}$. Hence we have that $\hat{\tau}_n = \tau + |\hat{\tau}_n - \tau|$ and $\hat{\jmath}_{k_n} = \bar{\jmath}_{k_n} - |\hat{\jmath}_{k_n} - \bar{\jmath}_{k_n}|$. Then, we can write

$$\begin{aligned}
\hat{s}_{(k_n)} &= s_{k_n} + \hat{\jmath}_{k_n}\hat{\tau}_n \\
&= s_{k_n} + (\bar{\jmath}_{k_n} - |\hat{\jmath}_{k_n} - \bar{\jmath}_{k_n}|)(\tau + |\hat{\tau}_n - \tau|) \\
&= \bar{s}_{k_n} + \bar{\jmath}_{k_n}|\hat{\tau}_n - \tau| - \tau|\hat{\jmath}_{k_n} - \bar{\jmath}_{k_n}| - |\hat{\jmath}_{k_n} - \bar{\jmath}_{k_n}||\hat{\tau}_n - \tau|.
\end{aligned} \tag{4.51}$$

Since $\hat{s}_{(k_n)} \in [s - \frac{\hat{\tau}_n}{2}, s + \frac{\hat{\tau}_n}{2})$, it follows now from (4.51) that

$$\begin{aligned}
& s - \frac{\tau}{2} - \frac{|\hat{\tau}_n - \tau|}{2} \\
& \leq \bar{s}_{k_n} + \bar{\jmath}_{k_n}|\hat{\tau}_n - \tau| - \tau|\hat{\jmath}_{k_n} - \bar{\jmath}_{k_n}| - |\hat{\jmath}_{k_n} - \bar{\jmath}_{k_n}||\hat{\tau}_n - \tau| \\
& < s + \frac{\tau}{2} + \frac{|\hat{\tau}_n - \tau|}{2}.
\end{aligned} \tag{4.52}$$

Since we also know that (4.52) holds true for any value $\bar{s}_{k_n} \in [s - \frac{\tau}{2}, s + \frac{\tau}{2})$, (4.52) directly yields that

$$-\tfrac{|\hat{\tau}_n - \tau|}{2} \leq \bar{\jmath}_{k_n}|\hat{\tau}_n - \tau| - \tau|\hat{\jmath}_{k_n} - \bar{\jmath}_{k_n}| - |\hat{\jmath}_{k_n} - \bar{\jmath}_{k_n}||\hat{\tau}_n - \tau| \leq \tfrac{|\hat{\tau}_n - \tau|}{2},$$

which is equivalent to

$$\left(\bar{\jmath}_{k_n} - \frac{1}{2}\right)|\hat{\tau}_n - \tau| < (\tau + o_p(1))\,|\hat{\jmath}_{k_n} - \bar{\jmath}_{k_n}| \leq \left(\bar{\jmath}_{k_n} + \frac{1}{2}\right)|\hat{\tau}_n - \tau|. \tag{4.53}$$

Since $\bar{\jmath}_{k_n} = \mathcal{O}(|W_n|)$, as $n \to \infty$, together with assumption (4.7), we find that

$$|\hat{\jmath}_{k_n} - \bar{\jmath}_{k_n}| = o_p(k_n|W_n|^{-1}),$$

as $n \to \infty$, which implies (4.50). Hence (4.42) is proved.

Next we prove (4.43). The l.h.s. of (4.43) can be written as

$$\frac{\tau k_n}{2|W_n||\hat{s}_{(k_n)} - s|} \frac{1}{\tau} |\hat{\tau}_n - \tau| = \mathcal{O}_p(1)\ o_p(k_n|W_n|^{-2}) = o_p(1), \tag{4.54}$$

as $n \to \infty$. Here we have used (4.42) and assumption (4.7). Hence (4.43) is proved. This completes the proof of Theorem 4.1. □

Proof of Theorem 4.2 To establish (4.11), it suffices to check that (4.42) and (4.43) remain valid, when $\xrightarrow{p}$ is replaced by $\xrightarrow{c}$, as $n \to \infty$, for each s at which λ is continuous and positive.

First, we prove that the l.h.s. of (4.42) converges completely to $\lambda(s)$, as $n \to \infty$. Following the structure of the proof of Theorem 1.1, it suffices to check that the probabilities appearing on the l.h.s. of (4.45) and (4.46) are summable, for each $\epsilon > 0$. We shall prove that the probability appearing on the l.h.s. of (4.45) is summable; the proof of the other case is similar.

In view of (4.47), it suffices now to show that the probabilities appearing on the l.h.s. of (4.48), (4.49), and (4.50), are summable, for each $\epsilon > 0$. The proof of the probability on the l.h.s. of (4.48) is summable is exactly the same as the proof of (4.39). Since, by assumption (4.10), we have

$$|\hat{\jmath}_{k_n}| \leq \frac{|W_n|}{\tau}(1 + o_c(1)),$$

as $n \to \infty$, (for any r.v. Y_n we write $Y_n = o_c(1)$ to denote that Y_n converges completely to zero, as $n \to \infty$), then by assumption (4.10) once more, we have that the probability on the l.h.s. of (4.49) is summable, for each $\epsilon > 0$. It remains to prove that the probability on the l.h.s. of (4.50) is summable. We only consider the case that $\hat{\tau}_n \geq \tau$ and $\hat{\jmath}_{k_n}, \bar{\jmath}_{k_n}$ are both positive; the proofs for the other seven cases are similar. An application of inequality (4.53), by using now assumption (4.10), yields that

$$|\hat{\jmath}_{k_n} - \bar{\jmath}_{k_n}| \leq \left(\bar{\jmath}_{k_n} + \tfrac{1}{2}\right)(\tau + o_c(1))^{-1} |\hat{\tau}_n - \tau| \ ,$$

as $n \to \infty$. Since $\bar{\jmath}_{k_n} = \mathcal{O}(|W_n|)$, as $n \to \infty$, by assumption (4.10) once more, we have that the probability on the l.h.s. of (4.50) is summable. Hence we have proved (4.42) with $\xrightarrow{p}$ replaced by $\xrightarrow{c}$.

Next we prove (4.43) with $\xrightarrow{p}$ replaced by $\xrightarrow{c}$. First note that, the l.h.s. of (4.43) is the same as the l.h.s. of (4.54). Because we have that the l.h.s. of (4.42) converges completely to $\lambda(s)$, as $n \to \infty$, by assumption (4.10) and Lemma A.5 (see Appendix), we also have that the l.h.s. of (4.54) converges completely to zero, which of course implies that the l.h.s. of (4.43) converges completely to zero, as $n \to \infty$. This completes the proof of Theorem 4.2. □

4.3 Statistical properties

In this section we focus on statistical properties of our estimator, i.e. we compute the bias, variance, and mean squared error (MSE) of $\hat{\lambda}_n$. We refer to section 3.3 for a more precise description of the type of assumption we will need for the estimator $\hat{\tau}_n$ of τ.

4.3.1 Results

Theorem 4.5 *Suppose that λ is periodic and locally integrable. If, in addition, (4.3) and (4.4) hold true, and*

$$|W_n| \, |\hat{\tau}_n - \tau| = \mathcal{O}\left(\delta_n \frac{k_n}{|W_n|}\right) \tag{4.55}$$

with probability 1 as $n \to \infty$, for some fixed sequence $\delta_n \downarrow 0$ as $n \to \infty$, then

$$\mathbf{E}\hat{\lambda}_n(s) \;\to\; \lambda(s) \tag{4.56}$$

as $n \to \infty$, for each s at which λ is continuous and positive.

Theorem 4.6 *Suppose that λ is periodic and locally integrable. If, in addition, (4.3), (4.4) and (4.55) hold, then*

$$Var\left(\hat{\lambda}_n(s)\right) \;\to\; 0 \tag{4.57}$$

as $n \to \infty$, for each s at which λ is continuous and positive.

Theorem 4.7 *Suppose that λ is periodic and locally integrable. If, in addition, (4.3) and (4.4) hold true, and*

$$|W_n| \, |\hat{\tau}_n - \tau| = \mathcal{O}\left(\frac{\delta_n k_n^{1/2}}{|W_n|}\right) \tag{4.58}$$

with probability 1 as $n \to \infty$, for some fixed sequence $\delta_n \downarrow 0$ as $n \to \infty$, then we have

$$Var\left(\hat{\lambda}_n(s)\right) = \frac{\lambda^2(s)}{k_n} + o\left(\frac{1}{k_n}\right) \tag{4.59}$$

as $n \to \infty$, for each s at which λ is continuous and positive.

Theorem 4.8 *Suppose that λ is periodic and locally integrable, (4.3) and (4.4) hold true, and*

$$|W_n|\,|\hat{\tau}_n - \tau| = \mathcal{O}\left(\delta_n \frac{k_n^3}{|W_n|^3}\right) \tag{4.60}$$

with probability 1 as $n \to \infty$, for some fixed sequence $\delta_n \downarrow 0$ as $n \to \infty$. If, in addition, λ has finite second derivative λ'' at s, then

$$\begin{aligned} \mathbf{E}\hat{\lambda}_n(s) &= \lambda(s) + \frac{\tau^2 \lambda''(s) k_n^2}{24\lambda^2(s)|W_n|^2} + o\left(\frac{k_n^2}{|W_n|^2}\right) \\ &\quad + \mathcal{O}\left(\frac{1}{|W_n|^{1/2-\epsilon_0}} + \frac{1}{k_n}\right), \end{aligned} \tag{4.61}$$

as $n \to \infty$, for each s at which λ is continuous and positive, where ϵ_0 is an arbitrary small positive real number.

Note that, the r.h.s. of (4.61) yields an asymptotic approximation for the bias of $\hat{\lambda}_n(s)$ provided $k_n \to \infty$ faster than $|W_n|^{3/4+\epsilon_0}$ for some arbitrary small $\epsilon_0 > 0$; otherwise the $\mathcal{O}(|W_n|^{\epsilon_0 - 1/2} + k_n^{-1})$ remainder term will dominate. However, the optimal choice of k_n (cf. (4.64)) satisfies this restriction.

Corollary 4.9 *Suppose that λ is periodic, locally integrable, (4.3) and (4.4) hold.*

(i) If, in addition, (4.55) holds true, then

$$MSE\left(\hat{\lambda}_n(s)\right) = Var\left(\hat{\lambda}_n(s)\right) + Bias^2\left(\hat{\lambda}_n(s)\right) \to 0 \tag{4.62}$$

as $n \to \infty$, for each s at which λ is continuous and positive.

(ii) If (4.60) hold true, and λ has finite second derivative λ'' at s, then

$$\begin{aligned} MSE\left(\hat{\lambda}_n(s)\right) &= \frac{\lambda^2(s)}{k_n} + \frac{\tau^4(\lambda''(s))^2 k_n^4}{576\lambda^4(s)|W_n|^4} + o\left(\frac{1}{k_n}\right) \\ &\quad + o\left(\frac{k_n^4}{|W_n|^4}\right) + \mathcal{O}\left(|W_n|^{\epsilon_0 - 1}\right) \end{aligned} \tag{4.63}$$

as $n \to \infty$, for each s at which λ is continuous and positive, where ϵ_0 is an arbitrary small positive real number.

The first statement of Corollary 4.9 is implied by Theorems 4.5 and 4.6, while its second statement is due to Theorems 4.7 and 4.8.

Now, we consider the r.h.s. of (4.63). By minimizing the sum of the first and second term of (4.63) (the leading terms for the variance and the squared bias), we obtain the optimal choice of k_n, which is given by

$$k_n = \left[\frac{144\lambda^6(s)}{\tau^4 \left(\lambda''(s)\right)^2} \right]^{1/5} |W_n|^{4/5}. \tag{4.64}$$

With this choice of k_n, the optimal rate of decrease of $MSE(\hat{\lambda}_n(s))$ is of order $O(|W_n|^{-4/5})$ as $n \to \infty$; and also in this important special case both (4.58) and (4.60) reduce to the same condition

$$|W_n| \, |\hat{\tau}_n - \tau| = \mathcal{O}\left(\delta_n |W_n|^{-3/5}\right) \tag{4.65}$$

with probability 1 as $n \to \infty$, for some fixed sequence $\delta_n \downarrow 0$ as $n \to \infty$.

Remark 4.2 The formulas (4.59), (4.61), (4.63), and (4.64) resemble closely corresponding ones in the 'classical' nearest neighbor density estimation for one dimensional case. To see this, let us consider for moment estimation of a density f, proportional to the intensity function λ and having support in $[0, \tau]$. For simplicity, we consider here only the (unrealistic) case where we know $\theta\tau$, where $\theta\tau = \int_0^\tau \lambda(s)ds$ (we assume here that $\theta > 0$). Then we have that $f(s) = \lambda(s)(\theta\tau)^{-1}$, for all $s \in [0, \tau]$. Consequently, the quantity $\hat{f}_n(s) = \hat{\lambda}_n(s)(\theta\tau)^{-1}$ can be viewed as an estimate of f at a given point s. Since $\lambda(s) = f(s)\theta\tau$, we also have that $\lambda''(s) = f''(s)\theta\tau$, for all $s \in (0, \tau)$. From (4.59), we can compute $Var(\hat{f}_n(s))$ as follows

$$\begin{aligned} Var\left(\hat{f}_n(s)\right) &= Var\left(\frac{\hat{\lambda}_n(s)}{\theta\tau}\right) = \frac{1}{(\theta\tau)^2}\frac{(f(s)\theta\tau)^2}{k_n} + o\left(\frac{1}{k_n}\right) \\ &= \frac{f^2(s)}{k_n} + o\left(\frac{1}{k_n}\right), \end{aligned} \tag{4.66}$$

as $n \to \infty$. Note that the r.h.s. of (4.66) is the same as the well known asymptotic approximation to the variance in nearest neighbor density estimator for one dimensional case. From (4.61), we have that

$$\begin{aligned} \mathbf{E}\hat{f}_n(s) &= \mathbf{E}\frac{\hat{\lambda}_n(s)}{\theta\tau} \\ &= \frac{\lambda(s)}{\theta\tau} + \frac{\tau^2 f''(s)\theta\tau k_n^2}{24\theta\tau(f(s)\theta\tau)^2|W_n|^2} + o\left(\frac{k_n^2}{|W_n|^2}\right) \end{aligned}$$

$$
\begin{aligned}
&+\mathcal{O}\left(\frac{1}{|W_n|^{1/2-\epsilon_0}}+\frac{1}{k_n}\right)\\
=\;& f(s)+\frac{f''(s)k_n^2}{24f^2(s)(\theta|W_n|)^2}+o\left(\frac{k_n^2}{|W_n|^2}\right)\\
&+\mathcal{O}\left(\frac{1}{|W_n|^{1/2-\epsilon_0}}+\frac{1}{k_n}\right), \qquad (4.67)
\end{aligned}
$$

as $n \to \infty$. Note that, due to our 'increasing domain asymptotic framework', the number of observations $X(W_n)$ in a given window W_n is random. However, it is easy to check that $\mathbf{E}X(W_n) \sim \theta|W_n|$. Hence, it seems appropriate to compare $\theta|W_n|$ with the 'sample size n' in the 'classical' density estimation case. If we replace $\theta|W_n|$ on the r.h.s. of (4.67) by n, the r.h.s. of (4.67) indeed reduces to the well-known expression for the asymptotic approximation to the bias in nearest neighbor density estimation. From (4.66) and (4.67), we also can find formulas for $MSE(\hat{f}_n(s))$ and optimal choice of k_n, when estimating f. These expressions also reduce to the corresponding ones in nearest neighbor density estimation, if we replace $\theta|W_n|$ by n. For example, the formula for $MSE(\hat{f}_n(s))$ reduces to 'one dimensional case' of formula (26) in Fukunaga and Hostetler (1973) (cf. also Mack and Rosenblatt (1979) and Prakasa Rao (1983)). □

Remark 4.3 Since $\hat{\lambda}_n(s) = 0$ if $X(W_n) < k_n$, we have that

$$\mathbf{E}\hat{\lambda}_n(s)\mathbf{I}(X(W_n) < k_n) = Var(\hat{\lambda}_n(s)\mathbf{I}(X(W_n) < k_n)) = 0.$$

This implies

$$\mathbf{E}\hat{\lambda}_n(s) = \mathbf{E}\hat{\lambda}_n(s)\mathbf{I}(X(W_n) \geq k_n),$$

and

$$Var(\hat{\lambda}_n(s)) = Var(\hat{\lambda}_n(s))\mathbf{I}(X(W_n) \geq k_n).$$

Hence, in all of our proofs in this subsection, we only need to consider the case $X(W_n) \geq k_n$ (cf (4.12)). □

4.3.2 Proofs

We begin with a simple lemma, which we will need in our proofs.

Lemma 4.10 *If (4.4) and (4.55) hold true, then we have with probability 1 that*

$$|\hat{s}_{(k_n)} - s| = |\bar{s}_{(k_n)} - s| + \mathcal{O}\left(\delta_n \frac{k_n}{|W_n|}\right) \qquad (4.68)$$

as $n \to \infty$, provided $X(W_n) \geq k_n$.

Proof: Similar to (4.47), we can write

$$\begin{aligned}(\hat{s}_{(k_n)} - s) &= (s_{(k_n)} + \hat{\jmath}_{k_n}\hat{\tau}_n - s) \\ &= (s_{(k_n)} + \bar{\jmath}_{k_n}\tau - s) + (\hat{\jmath}_{k_n}\hat{\tau}_n - \bar{\jmath}_{k_n}\tau) \\ &= (\bar{s}_{(k_n)} - s) + \hat{\jmath}_{k_n}(\hat{\tau}_n - \tau) + \tau(\hat{\jmath}_{k_n} - \bar{\jmath}_{k_n}). \end{aligned} \tag{4.69}$$

First we will show that the second term on the r.h.s. of (4.69) is of order $\mathcal{O}(\delta_n k_n |W_n|^{-1})$ with probability 1, as $n \to \infty$. To do this, we argue as follows. By (4.55), there exists a positive constant C such that we have with probability 1

$$|\hat{\tau}_n - \tau| \le C\delta_n k_n |W_n|^{-2}. \tag{4.70}$$

Since $s \in W_n$, by (4.4) and (4.70), we have with probability 1 that $|\hat{\jmath}_{k_n}| = \mathcal{O}(|W_n|)$ as $n \to \infty$. Combining this order bound and (4.70), we then have with probability 1 that the second term on the r.h.s. of (4.69) is of order $\mathcal{O}(\delta_n k_n |W_n|^{-1})$ as $n \to \infty$.

Next we will show that the third term on the r.h.s. of (4.69) is of order $\mathcal{O}(\delta_n k_n |W_n|^{-1})$ with probability 1, as $n \to \infty$. Here we only give the proof for the case $\hat{\tau}_n \ge \tau$ and $\hat{\jmath}_{k_n}$, $\bar{\jmath}_{k_n}$ are both positive; because the proofs of the other seven cases are similar. Recall (4.53). Since $s \in W_n$, we have that $\bar{\jmath}_{k_n} = \mathcal{O}(W_n)$ as $n \to \infty$. Then, by (4.70) and (4.53), we have with probability 1 that the third term on the r.h.s. of (4.69) is of order $\mathcal{O}(\delta_n k_n |W_n|^{-1})$ as $n \to \infty$. Therefore we have that

$$(\hat{s}_{(k_n)} - s) = (\bar{s}_{(k_n)} - s) + \mathcal{O}\left(\delta_n \frac{k_n}{|W_n|}\right)$$

as $n \to \infty$. By the triangle inequality, we have

$$\begin{aligned}&|\bar{s}_{(k_n)} - s| - \left|\mathcal{O}\left(\delta_n k_n |W_n|^{-1}\right)\right| \le |\hat{s}_{(k_n)} - s| \\ &\le |\bar{s}_{(k_n)} - s| + \left|\mathcal{O}\left(\delta_n k_n |W_n|^{-1}\right)\right|\end{aligned}$$

which implies this lemma. This completes the proof of Lemma 4.10. □

Proof of Theorem 4.5

By Remark 4.3, the l.h.s. of (4.56) is equal to

$$\begin{aligned}&\frac{k_n}{2|W_n|}\mathbf{E}\frac{\hat{\tau}_n}{|\hat{s}_{(k_n)} - s|}\mathbf{I}(X(W_n) \ge k_n) \\ &= \frac{\tau k_n}{2|W_n|}\mathbf{E}\frac{1}{|\hat{s}_{(k_n)} - s|}\mathbf{I}(X(W_n) \ge k_n) \\ &\quad + \frac{k_n}{2|W_n|}\mathbf{E}\frac{(\hat{\tau}_n - \tau)}{|\hat{s}_{(k_n)} - s|}\mathbf{I}(X(W_n) \ge k_n). \end{aligned} \tag{4.71}$$

We will prove (4.56) by showing that the first term on the r.h.s. of (4.71) is equal to $\lambda(s) + o(1)$ as $n \to \infty$, while its second term is of order $o(1)$ as $n \to \infty$.

First we consider the first term on the r.h.s. of (4.71). For each n, let A_n denote the set of all integers m_n, where $C_{1,n} \leq m_n \leq C_{2,n}$, with $C_{1,n}$ and $C_{2,n}$ are given respectively by (4.21) and (4.22). Let $A_n^c = [k_n, \infty) \setminus A_n$. Then, the expectation in the first term on the r.h.s. of (4.71) can be computed as follows

$$
\begin{aligned}
&= \mathbf{E}\left(\mathbf{E}\left(\frac{1}{|\hat{s}_{(k_n)} - s|}\mathbf{I}(X(W_n) \geq k_n)\middle| X(W_n) = m\right)\right) \\
&= \sum_{m_n \in A_n} \left(\mathbf{E}\left(\frac{1}{|\hat{s}_{(k_n)} - s|}\middle| X(W_n) = m_n\right)\right) \mathbf{P}(X(W_n) = m_n) \\
&+ \sum_{m=k_n}^{C_{1,n}-1} \left(\mathbf{E}\left(\frac{1}{|\hat{s}_{(k_n)} - s|}\middle| X(W_n) = m\right)\right) \mathbf{P}(X(W_n) = m) \\
&+ \sum_{m=C_{2,n}+1}^{\infty} \left(\mathbf{E}\left(\frac{1}{|\hat{s}_{(k_n)} - s|}\middle| X(W_n) = m\right)\right) \mathbf{P}(X(W_n) = m). \qquad (4.72)
\end{aligned}
$$

First we consider the first term on the r.h.s. of (4.72). To begin with, we first consider this term with $|\hat{s}_{(k_n)} - s|$ replaced by $|\bar{s}_{(k_n)} - s|$, where $|\bar{s}_{(k_n)} - s|$ is defined as in the paragraph preceding (4.14). Recall that, conditionally given $X(W_n) = m_n \in A_n$, $|\bar{s}_{(k_n)} - s|$ has the same distribution as $H_n^{-1}(Z_{k_n:m_n})$, where $Z_{k_n:m_n}$ denotes the k_n-th order statistics of a sample $Z_1, \ldots, Z_{m_n}$ of size m_n from the uniform $(0,1)$ distribution. First we write the expectation appearing in the first term on the r.h.s. of (4.72) as

$$
\begin{aligned}
&\mathbf{E}\left(\frac{1}{|\hat{s}_{(k_n)} - s|}\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}| \leq \epsilon_n \frac{k_n}{m_n}\right)\middle| X(W_n) = m_n\right) \\
&+\mathbf{E}\left(\frac{1}{|\hat{s}_{(k_n)} - s|}\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}| > \epsilon_n \frac{k_n}{m_n}\right)\middle| X(W_n) = m_n\right), \qquad (4.73)
\end{aligned}
$$

for some sequence of positive real numbers $\epsilon_n \downarrow 0$ as $n \to \infty$, and $\tilde{Z}_{k_n:m_n} = Z_{k_n:m_n} - \mathbf{E}Z_{k_n:m_n} = Z_{k_n:m_n} - k_n/(m_n+1)$. By a similar argument as in (4.34) (cf. also (4.35)), conditionally given $X(W_n) = m_n$, we have

$$
\begin{aligned}
&|\bar{s}_{(k_n)} - s|\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}| \leq \epsilon_n \frac{k_n}{m_n}\right) \\
&\stackrel{d}{=} \left\{\frac{\theta\tau k_n}{2\lambda(s)(m_n+1)} + o\left(\frac{k_n}{|W_n|}\right) + \left(\frac{\theta\tau}{2\lambda(s)}\right)\tilde{Z}_{k_n:m_n} + o\left(\tilde{Z}_{k_n:m_n}\right)\right\} \\
&\cdot \mathbf{I}\left(|\tilde{Z}_{k_n:m_n}| \leq \epsilon_n \frac{k_n}{m_n}\right)
\end{aligned}
$$

$$= \left\{ \frac{\tau k_n}{2\lambda(s)|W_n|} + o\left(\frac{k_n}{|W_n|}\right) \right\} \mathbf{I}\left(|\tilde{Z}_{k_n:m_n}| \le \epsilon_n \frac{k_n}{m_n} \right), \quad (4.74)$$

as $n \to \infty$. Combining (4.74) and Lemma 4.10, conditionally given $X(W_n) = m_n$, we then have

$$\begin{aligned} &|\hat{s}_{(k_n)} - s| \mathbf{I}\left(|\tilde{Z}_{k_n:m_n}| \le \epsilon_n \frac{k_n}{m_n} \right) \\ &\stackrel{d}{=} \left\{ \frac{\tau k_n}{2\lambda(s)|W_n|} + o\left(\frac{k_n}{|W_n|}\right) \right\} \mathbf{I}\left(|\tilde{Z}_{k_n:m_n}| \le \epsilon_n \frac{k_n}{m_n} \right) \\ &= \left\{ \frac{\tau k_n}{2\lambda(s)|W_n|} (1 + o(1)) \right\} \mathbf{I}\left(|\tilde{Z}_{k_n:m_n}| \le \epsilon_n \frac{k_n}{m_n} \right), \end{aligned} \quad (4.75)$$

as $n \to \infty$. By by Lemma A.4 (see Appendix), there exists a positive constant C_0 such that

$$\mathbf{P}\left(|\tilde{Z}_{k_n:m_n}| > \epsilon_n \frac{k_n}{m_n} \right) \le 2\exp\left\{ -C_0 \epsilon_n^2 k_n \right\} \le 2\exp\left\{ -C_0 k_n^{1/2} \right\}, \quad (4.76)$$

as $n \to \infty$, provided $\epsilon_n^{-1} = o(k_n^{1/4})$ as $n \to \infty$ (cf. also the r.h.s. of (4.41) with ϵ replaced by ϵ_n). Throughout this proof, we take $\epsilon_n^{-1} = o(k_n^{1/4})$ as $n \to \infty$. From (4.76), since $k_n \to \infty$ which implies the r.h.s. of (4.76) is $o(1)$ as $n \to \infty$, we obtain

$$\mathbf{P}\left(|\tilde{Z}_{k_n:m_n}| \le \epsilon_n \frac{k_n}{m_n} \right) = 1 - o(1), \quad (4.77)$$

as $n \to \infty$. By (4.75) and (4.77), we can compute the following conditional expectation

$$\begin{aligned} &\mathbf{E}\left(\frac{1}{|\hat{s}_{(k_n)} - s|} \mathbf{I}\left(|\tilde{Z}_{k_n:m_n}| \le \epsilon_n \frac{k_n}{m_n} \right) \middle| X(W_n) = m_n \right) \\ &= \mathbf{E}\, \frac{1}{(\tau k_n)(2\lambda(s)|W_n|)^{-1}(1 + o(1))} \mathbf{I}\left(|\tilde{Z}_{k_n:m_n}| \le \epsilon_n \frac{k_n}{m_n} \right) \\ &= \mathbf{E}\, \frac{2\lambda(s)|W_n|}{\tau k_n} (1 + o(1))\, \mathbf{I}\left(|\tilde{Z}_{k_n:m_n}| \le \epsilon_n \frac{k_n}{m_n} \right) \\ &= \frac{2\lambda(s)|W_n|}{\tau k_n} + o\left(\frac{|W_n|}{k_n} \right), \end{aligned} \quad (4.78)$$

as $n \to \infty$.

Next we consider the second term of (4.73). First note that

$$\begin{aligned} &\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}| > \epsilon_n \frac{k_n}{m_n} \right) = \mathbf{I}\left(Z_{k_n:m_n} > \frac{k_n}{m_n + 1} + \epsilon_n \frac{k_n}{m_n} \right) \\ &+ \mathbf{I}\left(Z_{k_n:m_n} < \frac{k_n}{m_n + 1} - \epsilon_n \frac{k_n}{m_n} \right). \end{aligned}$$

For the case $Z_{k_n:m_n} > \frac{k_n}{m_n+1} + \epsilon_n \frac{k_n}{m_n}$, by Lemma 4.10 and (4.31), conditionally given $X(W_n) = m_n$, we have

$$\begin{aligned}
|\hat{s}_{(k_n)} - s| &= |\bar{s}_{(k_n)} - s| + o\left(\frac{k_n}{|W_n|}\right) = H_n^{-1}(Z_{k_n:m_n}) + o\left(\frac{k_n}{|W_n|}\right) \\
&\geq H_n^{-1}\left(\frac{k_n}{m_n+1}\right) + o\left(\frac{k_n}{|W_n|}\right) = H^{-1}\left(\frac{k_n}{m_n+1} + \mathcal{O}(|W_n|^{-1})\right) + o\left(\frac{k_n}{|W_n|}\right) \\
&= \frac{\theta\tau k_n}{2\lambda(s)(m_n+1)} + o\left(\frac{k_n}{|W_n|}\right) \geq \frac{\tau k_n}{4\lambda(s)|W_n|},
\end{aligned}$$

for sufficiently large n. Hence, for sufficiently large n, conditionally given $X(W_n) = m_n$, we have

$$\begin{aligned}
&\frac{1}{|\hat{s}_{(k_n)} - s|}\mathbf{I}\left(Z_{k_n:m_n} > \frac{k_n}{m_n+1} + \epsilon_n \frac{k_n}{m_n}\right) \\
&\leq \frac{4\lambda(s)|W_n|}{\tau k_n}\mathbf{I}\left(Z_{k_n:m_n} > \frac{k_n}{m_n+1} + \epsilon_n \frac{k_n}{m_n}\right),
\end{aligned} \tag{4.79}$$

which in combination with (4.76), implies

$$\mathbf{E}\left(\frac{1}{|\hat{s}_{(k_n)} - s|}\mathbf{I}\left(Z_{k_n:m_n} > \frac{k_n}{m_n+1} + \epsilon_n \frac{k_n}{m_n}\right)\middle| X(W_n) = m_n\right) = o\left(\frac{|W_n|}{k_n}\right) \tag{4.80}$$

as $n \to \infty$. Next we will show

$$\mathbf{E}\left(\frac{1}{|\hat{s}_{(k_n)} - s|}\mathbf{I}\left(Z_{k_n:m_n} < \frac{k_n}{m_n+1} - \epsilon_n \frac{k_n}{m_n}\right)\middle| X(W_n) = m_n\right) = o\left(\frac{|W_n|}{k_n}\right) \tag{4.81}$$

as $n \to \infty$. By Lemma 4.10, the fact that $|\bar{s}_{(k_n)} - s| = H_n^{-1}(Z_{k_n:m_n})$, and an application of mean value theorem, together with a little calculation showing that $H_n^{-1'}(\xi_n) = (\theta\tau)(2\lambda(s))^{-1} + o(1)$ as $n \to \infty$, for any (random) point $\xi_n \in (Z_{k_n:m_n}, k_n(m_n+1)^{-1})$, whenever $\mathbf{I}(Z_{k_n:m_n} < k_n(m_n+1)^{-1} - \epsilon_n k_n m_n^{-1}) = 1$, shows that $|\hat{s}_{(k_n)} - s| = ((\theta\tau)(2\lambda(s))^{-1} + o(1))Z_{k_n:m_n} + o(k_n|W_n|^{-1})$, as $n \to \infty$. Since $\mathbf{E}Z_{k_n:m_n}^{-2} = \mathcal{O}(m_n^2 k_n^{-2})$ as $n \to \infty$, by an application of Cauchy-Schwarz inequality and (4.76), we can easily completes the proof of (4.81). Combining (4.78), (4.80) and (4.81), we have

$$\mathbf{E}\left(\frac{1}{|\hat{s}_{(k_n)} - s|}\middle| X(W_n) = m_n\right) = \frac{2\lambda(s)|W_n|}{\tau k_n} + o\left(\frac{|W_n|}{k_n}\right) \tag{4.82}$$

as $n \to \infty$. By an exponential bound for the Poisson probabilities (Lemma A.1), we know that (cf. also (4.24))

$$\mathbf{P}\left(X(W_n) \in A_n^c\right) \leq \mathcal{O}(1)\exp\left(-\frac{a_n^2}{2+o(1)}\right), \tag{4.83}$$

which is $o(1)$ as $n \to \infty$, since $a_n \to \infty$ as $n \to \infty$. This implies

$$\mathbf{P}\left(X(W_n) \in A_n\right) = (1 - o(1)), \tag{4.84}$$

as $n \to \infty$. By (4.82) and (4.84), the first term on the r.h.s. of (4.72) is equal to

$$\left(\frac{2\lambda(s)|W_n|}{\tau k_n} + o\left(\frac{|W_n|}{k_n}\right)\right)\mathbf{P}\left(X(W_n) \in A_n\right)$$
$$= \frac{2\lambda(s)|W_n|}{\tau k_n} + o\left(\frac{|W_n|}{k_n}\right), \tag{4.85}$$

as $n \to \infty$.

Next we consider the second and third term on the r.h.s. of (4.72). First, for any integer $m \in \{[k_n, C_{1,n}) \cup (C_{2,n}, \infty)\}$, we write the expectation appearing in this term as (4.73) with m_n replaced by m. For any integer $m \in \{[k_n, C_{1,n}) \cup (C_{2,n}, \infty)\}$, similar to that in (4.74) with m_n replaced by m, we have a stochastic expansion for $|\bar{s}_{(k_n)} - s|\mathbf{I}(|\tilde{Z}_{k_n:m}| \leq \epsilon_n k_n m^{-1})$, conditionally given $X(W_n) = m$, as follows

$$|\bar{s}_{(k_n)} - s|\mathbf{I}\left(|\tilde{Z}_{k_n:m}| \leq \epsilon_n \frac{k_n}{m}\right) \stackrel{d}{=}$$
$$\left\{\frac{\theta\tau k_n}{2\lambda(s)m} + o\left(\frac{k_n}{m}\right) + \mathcal{O}\left(\frac{1}{|W_n|}\right) + \mathcal{O}(\tilde{Z}_{k_n:m})\right\}\mathbf{I}\left(|\tilde{Z}_{k_n:m}| \leq \epsilon_n \frac{k_n}{m}\right)$$
$$= \left\{\frac{\theta\tau k_n}{2\lambda(s)m} + o\left(\frac{k_n}{m}\right) + \mathcal{O}\left(\frac{1}{|W_n|}\right)\right\}\mathbf{I}\left(|\tilde{Z}_{k_n:m}| \leq \epsilon_n \frac{k_n}{m}\right), \tag{4.86}$$

as $n \to \infty$. Combining (4.86) and Lemma 4.10, conditionally given $X(W_n) = m$, we have

$$|\hat{s}_{(k_n)} - s|\mathbf{I}\left(|\tilde{Z}_{k_n:m}| \leq \epsilon_n \frac{k_n}{m}\right)$$
$$\stackrel{d}{=} \left\{\frac{\theta\tau k_n}{2\lambda(s)m} + o\left(\frac{k_n}{m}\right) + o\left(\frac{k_n}{|W_n|}\right)\right\}\mathbf{I}\left(|\tilde{Z}_{k_n:m}| \leq \epsilon_n \frac{k_n}{m}\right), \tag{4.87}$$

as $n \to \infty$. Note that (4.76) and (4.77) remain hold true when $m_n \in A_n$ is now replaced by $m \in \{[k_n, C_{1,n}) \cup (C_{2,n}, \infty)\}$. By a similar argument as the one used to prove (4.80) and (4.81), but with $m_n \in A_n$ replaced by $m \in \{[k_n, C_{1,n}) \cup (C_{2,n}, \infty)\}$, conditionally given $X(W_n) = m$, we have

$$\mathbf{E}\frac{1}{|\hat{s}_{(k_n)} - s|}\mathbf{I}\left(|\tilde{Z}_{k_n:m}| > \epsilon_n \frac{k_n}{m}\right) = \mathcal{O}\left(\frac{m}{k_n} + \frac{|W_n|}{k_n}\right) \tag{4.88}$$

as $n \to \infty$. Then, by (4.77) with m_n replaced by $m \in [k_n, C_{1,n})$ and (4.87), in combination with (4.88), we have

$$\mathbf{E}\left(\frac{1}{|\hat{s}_{(k_n)} - s|}\middle| X(W_n) = m\right) = \mathcal{O}\left(\frac{|W_n|}{k_n}\right), \tag{4.89}$$

as $n \to \infty$, uniformly for all $m \in [k_n, C_{1,n})$. By (4.89), the second term on the r.h.s. of (4.72) is equal to $\mathcal{O}(|W_n|k_n^{-1})\mathbf{P}(X(W_n) \in [k_n, C_{1,n}))$. Since by (4.83) we have $\mathbf{P}(X(W_n) \in [k_n, C_{1,n})) \leq \mathbf{P}(X(W_n) \in A_n^c) = o(1)$, as $n \to \infty$, this term is of order $o(|W_n|k_n^{-1})$ as $n \to \infty$.

For any $m \in (C_{2,n}, \infty)$, since $m > (\theta|W_n|) + (\theta|W_n|)^{1/2}a_n$ (for some sequence $a_n \to \infty$ and $a_n = o(|W_n|^{1/2})$), we may have the absolute value of the third term on the r.h.s. of (4.87) is bigger than its first term. If the first term on the r.h.s. of (4.87) is the leading term, a similar argument as the one used to prove (4.82) shows that

$$\mathbf{E}(|\hat{s}_{(k_n)} - s|^{-1}\mathbf{I}(|\tilde{Z}_{k_n:m}| \leq \epsilon_n k_n m^{-1})|X(W_n) = m) = \mathcal{O}(mk_n^{-1}),$$

as $n \to \infty$. If the third term on the r.h.s. of (4.87) is the leading term, then there exists a sequence $c_n \to 0$ as $n \to \infty$, such that this term can be written as $c_n k_n |W_n|^{-1}$ with $|c_n| > (\theta\tau|W_n|)/(2\lambda(s)m)$. For this case, a similar argument as the one used to prove (4.82) shows that $\mathbf{E}(|\hat{s}_{(k_n)} - s|^{-1}\mathbf{I}(|\tilde{Z}_{k_n:m}| \leq \epsilon_n k_n m^{-1})|X(W_n) = m) = \mathcal{O}(|W_n|k_n^{-1}c_n^{-1})$ as $n \to \infty$. Since $|c_n| > (\theta\tau|W_n|)/(2\lambda(s)m)$ which implies $|c_n^{-1}| < (2\lambda(s)m)/(\theta\tau|W_n|)$, we also have

$$\mathbf{E}(|\hat{s}_{(k_n)} - s|^{-1}\mathbf{I}(|\tilde{Z}_{k_n:m}| \leq \epsilon_n k_n m^{-1})|X(W_n) = m) = \mathcal{O}(mk_n^{-1}),$$

as $n \to \infty$. A similar argument also holds true when the first and third terms on the r.h.s. of (4.87) are of the same order. Combining this result with (4.88), uniformly in $m \in (C_{2,n}, \infty)$, we have

$$\mathbf{E}\left(\frac{1}{|\hat{s}_{(k_n)} - s|}\middle| X(W_n) = m\right) = \mathcal{O}\left(\frac{m}{k_n}\right), \tag{4.90}$$

as $n \to \infty$. By (4.90), the third term on the r.h.s. of (4.72) can be computed as follows

$$\begin{aligned}
&\mathcal{O}\left(\frac{1}{k_n}\right) \sum_{m=C_{2,n}+1}^{\infty} m\,\mathbf{P}\left(X(W_n) = m\right) \\
&= \mathcal{O}\left(\frac{1}{k_n}\right) \mathbf{E}X(W_n)\mathbf{I}(X(W_n) > C_{2,n}) \\
&\leq \mathcal{O}\left(\frac{1}{k_n}\right) \left(\mathbf{E}X^2(W_n)\right)^{1/2} \mathbf{P}^{1/2}(X(W_n) > C_{2,n}) = o\left(\frac{|W_n|}{k_n}\right), \qquad (4.91)
\end{aligned}$$

as $n \to \infty$, because by periodicity of λ we have $(\mathbf{E}X^2(W_n))^{1/2} = \mathcal{O}(|W_n|)$ as $n \to \infty$, and by (4.83) we have $\mathbf{P}^{1/2}(X(W_n) > C_{2,n}) \leq \mathbf{P}^{1/2}(X(W_n) \in A_n^c) = o(1)$, as $n \to \infty$.

Since the first term on the r.h.s. of (4.72) is equal to the r.h.s. of (4.85), while the other terms are of order $o(|W_n|k_n^{-1})$ as $n \to \infty$, we then have

$$\mathbf{E}\,\frac{1}{|\hat{s}_{(k_n)} - s|}\mathbf{I}(X(W_n) \geq k_n) = \frac{2\lambda(s)|W_n|}{\tau k_n} + o\left(\frac{|W_n|}{k_n}\right), \tag{4.92}$$

as $n \to \infty$, which implies the first term on the r.h.s. of (4.71) is equal to $\lambda(s) + o(1)$ as $n \to \infty$.

Next we show that the second term on the r.h.s. of (4.71) is of order $o(1)$ as $n \to \infty$. By (4.70) and (4.92), the absolute value of this term does not exceed

$$\frac{C\delta_n k_n^2}{2|W_n|^3}\mathbf{E}\,\frac{1}{|\hat{s}_{(k_n)} - s|}\mathbf{I}(X(W_n) \geq k_n) = \frac{C\delta_n k_n^2}{2|W_n|^3}\mathcal{O}\left(\frac{|W_n|}{k_n}\right)$$
$$= \mathcal{O}\left(\frac{\delta_n k_n}{|W_n|^2}\right) = o(1)$$

as $n \to \infty$. This completes the proof of Theorem 4.5. □

Proof of Theorem 4.6

By Remark 4.3, we can write

$$Var\left(\hat{\lambda}_n(s)\right) = Var\left(\hat{\lambda}_n(s)\mathbf{I}(X(W_n) \geq k_n)\right)$$
$$= \mathbf{E}\left(\hat{\lambda}_n(s)\mathbf{I}(X(W_n) \geq k_n)\right)^2 - \left(\mathbf{E}\hat{\lambda}_n(s)\mathbf{I}(X(W_n) \geq k_n)\right)^2. \tag{4.93}$$

By Remark 4.3 and Theorem 4.5, we have $\mathbf{E}\hat{\lambda}_n(s)\mathbf{I}(X(W_n) \geq k_n) = \mathbf{E}\hat{\lambda}_n(s) = \lambda(s) + o(1)$ as $n \to \infty$. This implies the second term on the r.h.s. of (4.93) is equal to $-\lambda^2(s) + o(1)$ as $n \to \infty$. Then, to prove this theorem, it suffices to show that the first term on the r.h.s. of (4.93) is equal to $\lambda^2(s) + o(1)$ as $n \to \infty$. To do this we argue as follows. The first term on the r.h.s. of (4.93) is equal to

$$\mathbf{E}\left(\frac{\hat{\tau}_n k_n}{2|W_n||\hat{s}_{(k_n)} - s|}\mathbf{I}(X(W_n) \geq k_n)\right)^2$$
$$= \mathbf{E}\left(\frac{\tau k_n}{2|W_n||\hat{s}_{(k_n)} - s|}\mathbf{I}(X(W_n) \geq k_n)\right)^2$$
$$+ \mathbf{E}\left(\frac{(\hat{\tau}_n - \tau) k_n}{2|W_n||\hat{s}_{(k_n)} - s|}\mathbf{I}(X(W_n) \geq k_n)\right)^2$$
$$+ 2\mathbf{E}\left(\frac{\tau k_n}{2|W_n||\hat{s}_{(k_n)} - s|}\right)\left(\frac{(\hat{\tau}_n - \tau) k_n}{2|W_n||\hat{s}_{(k_n)} - s|}\right)\mathbf{I}(X(W_n) \geq k_n). \tag{4.94}$$

We will show that the first term on the r.h.s. of (4.94) is equal to $\lambda^2(s) + o(1)$ as $n \to \infty$, while its second and third terms are of order $o(1)$ as $n \to \infty$.

First we consider the first term on the r.h.s. of (4.94). This term can be written as

$$\begin{aligned} &\frac{\tau^2 k_n^2}{4|W_n|^2}\mathbf{E}\left(\frac{1}{|\hat{s}_{(k_n)} - s|}\mathbf{I}(X(W_n) \geq k_n)\right)^2 \\ &= \frac{\tau^2 k_n^2}{4|W_n|^2}\mathbf{E}\left\{\mathbf{E}\left(\left(\frac{1}{|\hat{s}_{(k_n)} - s|}\mathbf{I}(X(W_n) \geq k_n)\right)^2 \middle| X(W_n) = m\right)\right\}. \end{aligned} \tag{4.95}$$

Expectation of the quantity within curly brackets on the r.h.s. of (4.95) can be computed as follows

$$\begin{aligned} &\sum_{m_n \in A_n}\left\{\mathbf{E}\left(\left(\frac{1}{|\hat{s}_{(k_n)} - s|}\right)^2 \middle| X(W_n) = m_n\right)\right\}\mathbf{P}\left(X(W_n) = m_n\right) \\ &+ \sum_{m=k_n}^{C_{1,n}-1}\left\{\mathbf{E}\left(\left(\frac{1}{|\hat{s}_{(k_n)} - s|}\right)^2 \middle| X(W_n) = m\right)\right\}\mathbf{P}\left(X(W_n) = m\right) \\ &+ \sum_{C_{2,n}+1}^{\infty}\left\{\mathbf{E}\left(\left(\frac{1}{|\hat{s}_{(k_n)} - s|}\right)^2 \middle| X(W_n) = m\right)\right\}\mathbf{P}\left(X(W_n) = m\right). \end{aligned} \tag{4.96}$$

First we consider the first term of (4.96). The expectation appearing in this term can be written as

$$\begin{aligned} &\mathbf{E}\left(\left(\frac{1}{|\hat{s}_{(k_n)} - s|}\right)^2 \mathbf{I}\left(|\tilde{Z}_{k_n:m_n}| \leq \epsilon_n \frac{k_n}{m_n}\right) \middle| X(W_n) = m_n\right) \\ &+\mathbf{E}\left(\left(\frac{1}{|\hat{s}_{(k_n)} - s|}\right)^2 \mathbf{I}\left(|\tilde{Z}_{k_n:m_n}| > \epsilon_n \frac{k_n}{m_n}\right) \middle| X(W_n) = m_n\right), \end{aligned}$$

where ϵ_n a sequence of positive real numbers converging to zero and $\epsilon_n^{-1} = o(k_n^{1/4})$, as $n \to \infty$. By (4.75) and (4.77), we can compute the following conditional expectation

$$\begin{aligned} &\mathbf{E}\left(\left(\frac{1}{|\hat{s}_{(k_n)} - s|}\right)^2 \mathbf{I}\left(|\tilde{Z}_{k_n:m_n}| \leq \epsilon_n \frac{k_n}{m_n}\right) \middle| X(W_n) = m_n\right) \\ &= \mathbf{E}\frac{1}{(\tau^2 k_n^2)(4\lambda^2(s)|W_n|^2)^{-1}(1 + o(1))^2}\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}| \leq \epsilon_n \frac{k_n}{m_n}\right) \\ &= \frac{4\lambda^2(s)|W_n|^2}{\tau^2 k_n^2}(1 + o(1))\mathbf{P}\left(|\tilde{Z}_{k_n:m_n}| \leq \epsilon_n \frac{k_n}{m_n}\right) \end{aligned}$$

$$= \frac{4\lambda^2(s)|W_n|^2}{\tau^2 k_n^2} + o\left(\frac{|W_n|^2}{k_n^2}\right), \qquad (4.97)$$

as $n \to \infty$. By (4.79) and (4.76), together with a similar argument as the one used to prove (4.81), we have

$$\mathbf{E}\left(\left(\frac{1}{|\hat{s}_{(k_n)} - s|}\right)^2 \mathbf{I}\left(|\tilde{Z}_{k_n:m_n}| > \epsilon_n \frac{k_n}{m_n}\right) \middle| X(W_n) = m_n\right)$$
$$= o\left(\frac{|W_n|^2}{k_n^2}\right), \qquad (4.98)$$

as $n \to \infty$. By (4.97) and (4.98), we have

$$\mathbf{E}\left(\left(\frac{1}{|\hat{s}_{(k_n)} - s|}\right)^2 \middle| X(W_n) = m_n\right) = \frac{4\lambda^2(s)|W_n|^2}{\tau^2 k_n^2} + o\left(\frac{|W_n|^2}{k_n^2}\right), \qquad (4.99)$$

as $n \to \infty$. By (4.99), the first term of (4.96) is equal to

$$\left(\frac{4\lambda^2(s)|W_n|^2}{\tau^2 k_n^2} + o\left(\frac{|W_n|^2}{k_n^2}\right)\right) \mathbf{P}(X(W_n) \in A_n)$$
$$= \frac{4\lambda^2(s)|W_n|^2}{\tau^2 k_n^2} + o\left(\frac{|W_n|^2}{k_n^2}\right), \qquad (4.100)$$

as $n \to \infty$, because by (4.84) we have $\mathbf{P}(X(W_n) \in A_n) = 1 - o(1)$ as $n \to \infty$.

Next we consider the second and third term of (4.96). By a similar argument as the one used to compute the expectation in (4.99), but with m_n replaced by m, we have that

$$\mathbf{E}\left(\left(\frac{1}{|\hat{s}_{(k_n)} - s|}\right)^2 \middle| X(W_n) = m\right) = \mathcal{O}\left(\frac{|W_n|^2}{k_n^2}\right), \qquad (4.101)$$

as $n \to \infty$, uniformly for all $m \in [k_n, C_{1,n})$, and

$$\mathbf{E}\left(\left(\frac{1}{|\hat{s}_{(k_n)} - s|}\right)^2 \middle| X(W_n) = m\right) = \mathcal{O}\left(\frac{m^2}{k_n^2}\right), \qquad (4.102)$$

as $n \to \infty$, for each $m \in (C_{2,n}, \infty)$ (cf. also the argument used to prove (4.90) to handle possibility that the first term on the r.h.s. of (4.87) is of smaller order than its third term when $m \in (C_{2,n}, \infty)$). By (4.101), the second term of (4.96) is equal to $\mathcal{O}(|W_n|^2 k_n^{-2})\mathbf{P}(X(W_n) \in [k_n, C_{1,n}))$. Since by (4.83) we have $\mathbf{P}(X(W_n) \in [k_n, C_{1,n})) \leq \mathbf{P}(X(W_n) \in A_n^c) = o(1)$, as $n \to \infty$, this term is of order $o(|W_n|^2 k_n^{-2})$ as $n \to \infty$. By (4.102), the third term of (4.96) can be computed as follows

$$\begin{aligned}
&\mathcal{O}\left(\frac{1}{k_n^2}\right) \sum_{m=C_{2,n}+1}^{\infty} m^2\, \mathbf{P}(X(W_n) = m) \\
&= \mathcal{O}\left(\frac{1}{k_n^2}\right) \mathbf{E}X^2(W_n)\mathbf{I}(X(W_n) > C_{2,n}) \\
&\leq \mathcal{O}\left(\frac{1}{k_n^2}\right) \left(\mathbf{E}X^4(W_n)\right)^{1/2} \mathbf{P}^{1/2}(X(W_n) > C_{2,n}) = o\left(|W_n|^2 k_n^{-2}\right),
\end{aligned}$$

as $n \to \infty$, because by periodicity of λ we have $(\mathbf{E}X^4(W_n))^{1/2} = \mathcal{O}(|W_n|^2)$ as $n \to \infty$, and by (4.83) we have $\mathbf{P}^{1/2}(X(W_n) > C_{2,n}) \leq \mathbf{P}^{1/2}(X(W_n) \in A_n^c) = o(1)$, as $n \to \infty$.

Since the first term of (4.96) is equal to the r.h.s. of (4.100), while its second and third terms are of order $o(|W_n|^2 k_n^{-2})$ as $n \to \infty$, we then have

$$\mathbf{E}\left(\frac{1}{|\hat{s}_{(k_n)} - s|}\mathbf{I}(X(W_n) \geq k_n)\right)^2 = \frac{4\lambda^2(s)|W_n|^2}{\tau^2 k_n^2} + o\left(\frac{|W_n|^2}{k_n^2}\right), \qquad (4.103)$$

as $n \to \infty$, which implies the quantity in (4.95) is equal to $\lambda^2(s) + o(1)$ as $n \to \infty$. Hence, the first term on the r.h.s. of (4.94) is equal to $\lambda^2(s) + o(1)$ as $n \to \infty$.

It remains to show that the second and third term on the r.h.s. of (4.94) are of order $o(1)$ as $n \to \infty$. By (4.70) and (4.103), sum of the second term and the absolute value of the third term on the r.h.s. of (4.94) does not exceed

$$\begin{aligned}
&\left(\frac{C^2\delta_n^2 k_n^4}{4|W_n|^6} + \frac{C\tau\delta_n k_n^3}{2|W_n|^4}\right) \mathbf{E}\left(\frac{1}{|\hat{s}_{(k_n)} - s|}\mathbf{I}(X(W_n) \geq k_n)\right)^2 \\
&= \left(\frac{C^2\delta_n^2 k_n^4}{4|W_n|^6} + \frac{C\tau\delta_n k_n^3}{2|W_n|^4}\right) \mathcal{O}\left(\frac{|W_n|^2}{k_n^2}\right) = \mathcal{O}\left(\frac{\delta_n^2 k_n^2}{|W_n|^4} + \frac{\delta_n k_n}{|W_n|^2}\right) \\
&= o(1),
\end{aligned}$$

as $n \to \infty$. This completes the proof of Theorem 4.6. □

Proof of Theorem 4.7

Since we want to prove (4.59) instead of (4.57), it is not enough now to use the result from Theorem 4.5 to simplify the expression for $Var(\hat{\lambda}_n(s))$. Hence, instead of writing $Var(\hat{\lambda}_n(s))$ as in (4.93), here we have to directly compute $Var(\hat{\lambda}_n(s))$ as below. By Remark 4.3 we have

$$Var(\hat{\lambda}_n(s)) = Var(\hat{\lambda}_n(s)\mathbf{I}(X(W_n) \geq k_n)).$$

Then, the l.h.s. of (4.59) can be written as

$$\begin{aligned} & Var\left(\frac{\hat{\tau}_n k_n}{2|W_n||\hat{s}_{(k_n)}-s|}\mathbf{I}(X(W_n)\geq k_n)\right) \\ = \quad & Var\left(\frac{\tau k_n}{2|W_n||\hat{s}_{(k_n)}-s|}\mathbf{I}(X(W_n)\geq k_n)\right) \\ + \quad & Var\left(\frac{(\hat{\tau}_n-\tau) k_n}{2|W_n||\hat{s}_{(k_n)}-s|}\mathbf{I}(X(W_n)\geq k_n)\right) \\ + \quad & 2\ Cov\left(\frac{\tau k_n}{2|W_n||\hat{s}_{(k_n)}-s|}, \frac{(\hat{\tau}_n-\tau) k_n}{2|W_n||\hat{s}_{(k_n)}-s|}\right)\mathbf{I}(X(W_n)\geq k_n). \end{aligned} \tag{4.104}$$

The first term on the r.h.s. of (4.104) can be written as

$$\begin{aligned} & \frac{\tau^2 k_n^2}{4|W_n|^2} Var\left(\frac{1}{|\bar{s}_{(k_n)}-s|}\mathbf{I}(X(W_n)\geq k_n)\right) \\ & +\frac{\tau^2 k_n^2}{4|W_n|^2} Var\left(\left(\frac{1}{|\hat{s}_{(k_n)}-s|}-\frac{1}{|\bar{s}_{(k_n)}-s|}\right)\mathbf{I}(X(W_n)\geq k_n)\right) \\ & +\frac{\tau^2 k_n^2}{2|W_n|^2} Cov\left(\frac{1}{|\bar{s}_{(k_n)}-s|}, \frac{1}{|\hat{s}_{(k_n)}-s|}-\frac{1}{|\bar{s}_{(k_n)}-s|}\right)\mathbf{I}(X(W_n)\geq k_n). \end{aligned} \tag{4.105}$$

We will prove this theorem by showing that the first term of (4.105) is equal to $\lambda^2(s)k_n^{-1}+o(k_n^{-1})$ as $n\to\infty$, while the second and third terms of (4.105), as well as the second and third terms on the r.h.s. of (4.104) are of order $o(k_n^{-1})$ as $n\to\infty$.

First we show that the first term of (4.105) is equal to $\lambda^2(s)k_n^{-1}+o(k_n^{-1})$ as $n\to\infty$. The variance appearing in this term can be computed as follows

$$\begin{aligned} & \mathbf{E}\left(Var\left(\frac{1}{|\bar{s}_{(k_n)}-s|}\mathbf{I}(X(W_n)\geq k_n)\middle| X(W_n)=m\right)\right) \\ & +Var\left(\mathbf{E}\left(\frac{1}{|\bar{s}_{(k_n)}-s|}\mathbf{I}(X(W_n)\geq k_n)\middle| X(W_n)=m\right)\right). \end{aligned} \tag{4.106}$$

Similar to (4.92) (note that the r.h.s. of (4.74) is equal to the r.h.s. of (4.75)), we also have

$$\mathbf{E}\,\frac{1}{|\bar{s}_{(k_n)}-s|}\mathbf{I}(X(W_n)\geq k_n)=\frac{2\lambda(s)|W_n|}{\tau k_n}+o\left(\frac{|W_n|}{k_n}\right), \tag{4.107}$$

as $n\to\infty$, which is deterministic. Hence, the second term of (4.106) is equal to zero. The first term of (4.106) is equal to

$$\sum_{m_n\in A_n}\left\{Var\left(\frac{1}{|\bar{s}_{(k_n)}-s|}\middle| X(W_n)=m_n\right)\right\}\mathbf{P}\left(X(W_n)=m_n\right)$$

$$+\sum_{m=k_n}^{C_{1,n}-1}\left\{Var\left(\frac{1}{|\bar{s}_{(k_n)}-s|}\middle|X(W_n)=m\right)\right\}\mathbf{P}\left(X(W_n)=m\right)$$
$$+\sum_{C_{2,n}+1}^{\infty}\left\{Var\left(\frac{1}{|\bar{s}_{(k_n)}-s|}\middle|X(W_n)=m\right)\right\}\mathbf{P}\left(X(W_n)=m\right). \tag{4.108}$$

First we consider the first term of (4.108). The variance appearing in this term is equal to

$$\begin{aligned}&Var\left(\frac{1}{|\bar{s}_{(k_n)}-s|}\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}|\le\epsilon_n\frac{k_n}{m_n}\right)\middle|X(W_n)=m_n\right)\\&+Var\left(\frac{1}{|\bar{s}_{(k_n)}-s|}\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}|>\epsilon_n\frac{k_n}{m_n}\right)\middle|X(W_n)=m_n\right)\\&+2\,Cov\left\{\left(\frac{1}{|\bar{s}_{(k_n)}-s|}\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}|\le\epsilon_n\frac{k_n}{m_n}\right),\right.\right.\\&\qquad\left.\left.\frac{1}{|\bar{s}_{(k_n)}-s|}\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}|>\epsilon_n\frac{k_n}{m_n}\right)\right)\middle|X(W_n)=m_n\right\}.\end{aligned}\tag{4.109}$$

To compute the first term of (4.109), we argue as follows. From (4.74), conditionally given $X(W_n)=m_n$, we have the following stochastic expansion

$$\begin{aligned}&|\bar{s}_{(k_n)}-s|\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}|\le\epsilon_n\frac{k_n}{m_n}\right)\\&\stackrel{d}{=}\left\{\frac{\tau k_n}{2\lambda(s)|W_n|}\left(1+o(1)+\left(\frac{\theta|W_n|}{k_n}+o\left(\frac{|W_n|}{k_n}\right)\right)\tilde{Z}_{k_n:m_n}\right)\right\}\\&\quad\cdot\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}|\le\epsilon_n\frac{k_n}{m_n}\right).\end{aligned}\tag{4.110}$$

By (4.110), and by noting that $|W_n|k_n^{-1}\tilde{Z}_{k_n:m_n}\mathbf{I}|\tilde{Z}_{k_n:m_n}|\le\epsilon_n k_n m_n^{-1})=o(1)$ as $n\to\infty$, we can obtain the following stochastic expansion

$$\begin{aligned}&\left(\frac{1}{|\bar{s}_{(k_n)}-s|}\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}|\le\epsilon_n\frac{k_n}{m_n}\right)\middle|X(W_n)=m_n\right)\\=\;&\frac{2\lambda(s)|W_n|}{\tau k_n}\left\{\frac{1}{\{1+o(1)+(\theta|W_n|k_n^{-1}+o(|W_n|k_n^{-1}))\tilde{Z}_{k_n:m_n}\}}\right\}\\&\cdot\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}|\le\epsilon_n\frac{k_n}{m_n}\right)\\=\;&\frac{2\lambda(s)|W_n|}{\tau k_n}\left\{1+o(1)-\left(\frac{\theta|W_n|}{k_n}+o\left(\frac{|W_n|}{k_n}\right)\right)\tilde{Z}_{k_n:m_n}\right.\end{aligned}$$

$$+\mathcal{O}\left(\frac{|W_n|^2}{k_n^2}\right)\tilde{Z}^2_{k_n:m_n}\Big\}\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}|\le \epsilon_n\frac{k_n}{m_n}\right)$$

$$= \left\{\frac{2\lambda(s)|W_n|}{\tau k_n}+o\left(\frac{|W_n|}{k_n}\right)-\left(\frac{2\lambda(s)\theta|W_n|^2}{\tau k_n^2}+o\left(\frac{|W_n|^2}{k_n^2}\right)\right)\tilde{Z}_{k_n:m_n}\right.$$
$$\left.+\mathcal{O}\left(\frac{|W_n|^3}{k_n^3}\right)\tilde{Z}^2_{k_n:m_n}\right\}\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}|\le \epsilon_n\frac{k_n}{m_n}\right). \qquad (4.111)$$

By (4.111), we can compute the following conditional variance

$$Var\left(\frac{1}{|\bar{s}_{(k_n)}-s|}\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}|\le \epsilon_n\frac{k_n}{m_n}\right)\Big| X(W_n)=m_n\right)$$
$$=\left(\frac{2\lambda(s)\theta|W_n|^2}{\tau k_n^2}+o\left(\frac{|W_n|^2}{k_n^2}\right)\right)^2 Var\left(\tilde{Z}_{k_n:m_n}\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}|\le \epsilon_n\frac{k_n}{m_n}\right)\right)$$
$$+\mathcal{O}\left(\frac{|W_n|^6}{k_n^6}\right)Var\left(\tilde{Z}^2_{k_n:m_n}\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}|\le \epsilon_n\frac{k_n}{m_n}\right)\right)$$
$$+\mathcal{O}\left(\frac{|W_n|^5}{k_n^5}\right)Cov\left\{\left(\tilde{Z}_{k_n:m_n},\tilde{Z}^2_{k_n:m_n}\right)\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}|\le \epsilon_n\frac{k_n}{m_n}\right)\right\}. \qquad (4.112)$$

The variance appearing in the first term on the r.h.s. of (4.112) can be computed as follows

$$= Var\left(\tilde{Z}_{k_n:m_n}-\tilde{Z}_{k_n:m_n}\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}|> \epsilon_n\frac{k_n}{m_n}\right)\right)$$
$$= Var\left(\tilde{Z}_{k_n:m_n}\right)+Var\left(\tilde{Z}_{k_n:m_n}\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}|> \epsilon_n\frac{k_n}{m_n}\right)\right)$$
$$-\,2\,Cov\left(\tilde{Z}_{k_n:m_n},\tilde{Z}_{k_n:m_n}\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}|> \epsilon_n\frac{k_n}{m_n}\right)\right). \qquad (4.113)$$

The first term on the r.h.s. of (4.113) is equal to (cf. Reiss (1989), p. 45)

$$\frac{k_n(m_n-k_n+1)}{(m_n+1)^2(m_n+2)}=\frac{k_n}{\theta^2|W_n|^2}+o\left(\frac{k_n}{|W_n|^2}\right),$$

as $n\to\infty$. A simple calculation (using formula (1.7.4) of Reiss (1989, p. 45) shows that

$$\mathbf{E}\tilde{Z}^4_{k_n:m_n}=\mathcal{O}\left(k_n^2|W_n|^{-4}\right) \qquad (4.114)$$

as $n\to\infty$. By (4.76), (4.114) and an application of Cauchy-Schwarz inequality, we have that the second term on the r.h.s. of (4.113) does not exceed

$$\mathbf{E}\left(\tilde{Z}_{k_n:m_n}\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}|> \epsilon_n\frac{k_n}{m_n}\right)\right)^2=o\left(\frac{k_n}{|W_n|^2}\right),$$

as $n \to \infty$. Another application of Cauchy-Schwarz inequality shows that the third term on the r.h.s. of (4.113) is of order $o(k_n|W_n|^{-2})$ as $n \to \infty$. Combining all these results, uniformly in $m_n \in A_n$, we find that the first term on the r.h.s. of (4.112) is equal to

$$\left(\frac{4\lambda^2(s)\theta^2|W_n|^4}{\tau^2 k_n^4} + o\left(\frac{|W_n|^4}{k_n^4}\right)\right)\left(\frac{k_n}{\theta^2|W_n|^2} + o\left(\frac{k_n}{|W_n|^2}\right)\right)$$
$$= \frac{4\lambda^2(s)|W_n|^2}{\tau^2 k_n^3} + o\left(\frac{|W_n|^2}{k_n^3}\right),$$

as $n \to \infty$. By (4.114), the second term on the r.h.s. of (4.112) does not exceed

$$\mathcal{O}\left(\frac{|W_n|^6}{k_n^6}\right)\mathbf{E}\tilde{Z}^4_{k_n:m_n} = \mathcal{O}\left(\frac{|W_n|^2}{k_n^4}\right) = o\left(\frac{|W_n|^2}{k_n^3}\right),$$

as $n \to \infty$, since $k_n \to \infty$ as $n \to \infty$. An application of Cauchy-Schwarz inequality also shows that the third term on the r.h.s. of (4.112) is of order $o(|W_n|^2 k_n^{-2})$ as $n \to \infty$. Hence we have

$$Var\left(\frac{1}{|\bar{s}_{(k_n)} - s|}\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}| \leq \epsilon_n \frac{k_n}{m_n}\right)\middle| X(W_n) = m_n\right)$$
$$= \frac{4\lambda^2(s)|W_n|^2}{\tau^2 k_n^3} + o\left(\frac{|W_n|^2}{k_n^3}\right), \tag{4.115}$$

as $n \to \infty$, uniformly in $m_n \in A_n$.

Next we consider the second term of (4.109). First note that (4.79) remains valid if $|\hat{s}_{(k_n)} - s|$ replaced by $|\bar{s}_{(k_n)} - s|$. Then, by (4.79) with $|\hat{s}_{(k_n)} - s|$ replaced by $|\bar{s}_{(k_n)} - s|$ and (4.76), by noting that the r.h.s. of (4.76) is of order $o(k_n^{-1})$ as $n \to \infty$, and in combination with an argument like the one used to prove (4.81), we have

$$\mathbf{E}\left(\left(\frac{1}{|\bar{s}_{(k_n)} - s|}\right)^2 \mathbf{I}\left(|\tilde{Z}_{k_n:m_n}| > \epsilon_n \frac{k_n}{m_n}\right)\middle| X(W_n) = m_n\right) = o\left(\frac{|W_n|^2}{k_n^3}\right),$$

as $n \to \infty$. This implies the second term of (4.109) is of order $o(|W_n|^2 k_n^{-3})$ as $n \to \infty$. The third term of (4.109) is equal to zero. Combining all of our results, uniformly in $m_n \in A_n$, we have that

$$Var\left(\frac{1}{|\bar{s}_{(k_n)} - s|}\middle| X(W_n) = m_n\right) = \frac{4\lambda^2(s)|W_n|^2}{\tau^2 k_n^3} + o\left(\frac{|W_n|^2}{k_n^3}\right), \tag{4.116}$$

as $n \to \infty$. By (4.116), the quantity in the first term of (4.108) can be computed as follows

$$\left(\frac{4\lambda^2(s)|W_n|^2}{\tau^2 k_n^3} + o\left(\frac{|W_n|^2}{k_n^3}\right)\right)\mathbf{P}\left(X(W_n) \in A_n\right)$$
$$= \left(\frac{4\lambda^2(s)|W_n|^2}{\tau^2 k_n^3} + o\left(\frac{|W_n|^2}{k_n^3}\right)\right), \tag{4.117}$$

as $n \to \infty$, because by (4.84) we have $\mathbf{P}(X(W_n) \in A_n) = 1 - o(1)$ as $n \to \infty$.

Next we consider the second and third terms of (4.108). Sum of these two terms does not exceed

$$\sum_{m=k_n}^{C_{1,n}-1} \left\{ \mathbf{E}\left(\left(\frac{1}{|\bar{s}_{(k_n)} - s|} \right)^2 \middle| X(W_n) = m \right) \right\} \mathbf{P}\left(X(W_n) = m\right)$$
$$+ \sum_{C_{2,n}+1}^{\infty} \left\{ \mathbf{E}\left(\left(\frac{1}{|\bar{s}_{(k_n)} - s|} \right)^2 \middle| X(W_n) = m \right) \right\} \mathbf{P}\left(X(W_n) = m\right).$$

By a similar argument as the one used to prove (4.101) and (4.102), but with (4.87) now replaced by (4.86), and also by noting that (4.88) remains valid if we replace $|\hat{s}_{(k_n)} - s|$ by $|\bar{s}_{(k_n)} - s|$, we have (4.101) and (4.102) with $|\hat{s}_{(k_n)} - s|$ replaced by $|\bar{s}_{(k_n)} - s|$. Now we look at the upper bound for $\mathbf{P}(X(W_n) \in A_n^c)$ as given in (4.83). By (4.83), we can write the following

$$\mathbf{P}\left(X(W_n) \in A_n^c\right) \leq \mathcal{O}\left(\frac{1}{k_n^2}\right) \exp\left(-\frac{a_n^2}{2 + o(1)} + 2\log k_n\right),$$

as $n \to \infty$. By choosing now the sequence a_n such that $a_n^2/3 \to \infty$ faster than $2\log k_n$, we then have that

$$\mathbf{P}\left(X(W_n) \in A_n^c\right) = o\left(k_n^{-2}\right), \tag{4.118}$$

as $n \to \infty$, which implies

$$\mathbf{P}\left(X(W_n) \in A_n\right) = 1 - o\left(k_n^{-2}\right), \tag{4.119}$$

as $n \to \infty$. Then, by a similar argument as the one used to handle the second and third term of (4.96), but with $|\hat{s}_{(k_n)} - s|$ in (4.101) and (4.102) now replaced by $|\bar{s}_{(k_n)} - s|$ and also now we use (4.118) as the upper bound for $\mathbf{P}(X(W_n) \in [k_n, C_{1,n}))$ and $\mathbf{P}(X(W_n) > C_{2,n})$ so that $\mathbf{P}^{1/2}(X(W_n) > C_{2,n}) = o(k_n^{-1})$ as $n \to \infty$, we find that the second and third term of (4.108) are of order $o(|W_n|^2 k_n^{-3})$ as $n \to \infty$.

Since the first term of (4.108) is equal to the r.h.s. of (4.117), while its second and third terms are of order $o(|W_n|^2 k_n^{-3})$ as $n \to \infty$, we then have

$$\mathbf{E}\left(Var\left(\frac{1}{|\bar{s}_{(k_n)} - s|} \mathbf{I}(X(W_n) \geq k_n) \right) \right) = \frac{4\lambda^2(s)|W_n|^2}{\tau^2 k_n^3} + o\left(\frac{|W_n|^2}{k_n^3}\right),$$

as $n \to \infty$. Since the second term of (4.106) is equal to zero, we then have the first term of (4.105) is equal to $\lambda^2(s)k_n^{-1} + o(k_n^{-1})$ as $n \to \infty$.

Next, we prove that the second term of (4.105) is of order $o(k_n^{-1})$ as $n \to \infty$. To do this, it suffices to check

$$\mathbf{E}\left(\left(\frac{1}{|\hat{s}_{(k_n)} - s|} - \frac{1}{|\bar{s}_{(k_n)} - s|}\right)^2 \mathbf{I}(X(W_n) \geq k_n)\right) = o\left(\frac{|W_n|^2}{k_n^3}\right), \quad (4.120)$$

as $n \to \infty$. The l.h.s. of (4.120) can be computed as follows

$$\begin{aligned}
&\mathbf{E}\left\{\mathbf{E}\left(\left(\frac{1}{|\hat{s}_{(k_n)} - s|} - \frac{1}{|\bar{s}_{(k_n)} - s|}\right)^2 \mathbf{I}(X(W_n) \geq k_n) \middle| X(W_n) = m\right)\right\} \\
&= \sum_{m_n \in A_n} \mathbf{E}\left\{\left(\frac{1}{|\hat{s}_{(k_n)} - s|} - \frac{1}{|\bar{s}_{(k_n)} - s|}\right)^2 \middle| X(W_n) = m_n\right\} \mathbf{P}\left(X(W_n) = m_n\right) \\
&+ \sum_{m=k_n}^{C_{1,n}-1} \mathbf{E}\left\{\left(\frac{1}{|\hat{s}_{(k_n)} - s|} - \frac{1}{|\bar{s}_{(k_n)} - s|}\right)^2 \middle| X(W_n) = m\right\} \mathbf{P}\left(X(W_n) = m\right) \\
&+ \sum_{C_{2,n}+1}^{\infty} \mathbf{E}\left\{\left(\frac{1}{|\hat{s}_{(k_n)} - s|} - \frac{1}{|\bar{s}_{(k_n)} - s|}\right)^2 \middle| X(W_n) = m\right\} \mathbf{P}\left(X(W_n) = m\right).
\end{aligned} \quad (4.121)$$

First consider the second and third term on the r.h.s. of (4.121). Recall that, by a similar argument as the one used to prove (4.101) and (4.102), we have (4.101) and (4.102) with $|\hat{s}_{(k_n)} - s|$ replaced by $|\bar{s}_{(k_n)} - s|$. By (4.101) and (4.101) with $|\hat{s}_{(k_n)} - s|$ replaced by $|\bar{s}_{(k_n)} - s|$, we obtain

$$\mathbf{E}\left(\left(\frac{1}{|\hat{s}_{(k_n)} - s|} - \frac{1}{|\bar{s}_{(k_n)} - s|}\right)^2 \middle| X(W_n) = m\right) = \mathcal{O}\left(\frac{|W_n|^2}{k_n^2}\right), \quad (4.122)$$

as $n \to \infty$, uniformly for all $m \in [k_n, C_{1,n})$. By (4.102) and (4.102) with $|\hat{s}_{(k_n)} - s|$ replaced by $|\bar{s}_{(k_n)} - s|$, we obtain

$$\mathbf{E}\left(\left(\frac{1}{|\hat{s}_{(k_n)} - s|} - \frac{1}{|\bar{s}_{(k_n)} - s|}\right)^2 \middle| X(W_n) = m\right) = \mathcal{O}\left(\frac{m^2}{k_n^2}\right), \quad (4.123)$$

as $n \to \infty$, uniformly for all $m \in (C_{2,n}, \infty)$. Then, by a similar argument as the one used to handle the second and third term of (4.96) (cf. also the argument following (4.102)), provided we are now using (4.118) as the upper bound for $\mathbf{P}(X(W_n) \in [k_n, C_{1,n}))$ and $\mathbf{P}(X(W_n) > C_{2,n})$ so that $\mathbf{P}^{1/2}(X(W_n) > C_{2,n}) = o(k_n^{-1})$ as $n \to \infty$, we find that the second and third term of (4.121) are of order $o(|W_n|^2 k_n^{-3})$ as $n \to \infty$.

Next we consider the first term on the r.h.s. of (4.121). The expectation appearing in this term can be written as

$$\mathbf{E}\left\{\left(\frac{1}{|\hat{s}_{(k_n)} - s|} - \frac{1}{|\bar{s}_{(k_n)} - s|}\right)^2 \mathbf{I}\left(|\tilde{Z}_{k_n:m_n}| \leq \epsilon_n \frac{k_n}{m_n}\right) \middle| X(W_n) = m_n\right\}$$

$$+\mathbf{E}\left\{\left(\frac{1}{|\hat{s}_{(k_n)}-s|}-\frac{1}{|\bar{s}_{(k_n)}-s|}\right)^2\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}|>\epsilon_n\frac{k_n}{m_n}\right)\middle| X(W_n)=m_n\right\}. \quad (4.124)$$

First we consider the second term of (4.124). By a similar argument as the one used to prove (4.99), we have

$$\mathbf{E}\left(\left(\frac{1}{|\hat{s}_{(k_n)}-s|}\right)^4\middle| X(W_n)=m_n\right)=\mathcal{O}\left(\frac{|W_n|^4}{k_n^4}\right), \quad (4.125)$$

as $n\to\infty$. Similarly, we also have (4.125) with $|\hat{s}_{(k_n)}-s|$ replaced by $|\bar{s}_{(k_n)}-s|$. Then, by Cauchy-Schwarz inequality, (4.125), (4.125) with $|\hat{s}_{(k_n)}-s|$ replaced by $|\bar{s}_{(k_n)}-s|$, (4.76), and by using the fact that the r.h.s. of (4.76) is $o(k_n^{-2})$ as $n\to\infty$, we obtain that the second term of (4.124) is of order $o(|W_n|^2k_n^{-3})$ as $n\to\infty$. Next we consider the first term of (4.124). By Lemma 4.10 with condition (4.55) replaced by (4.58), conditionally given $X(W_n)=m_n$, we have

$$\begin{aligned}
&|\hat{s}_{(k_n)}-s|\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}|\le\epsilon_n\frac{k_n}{m_n}\right)\\
&=\left\{|\bar{s}_{(k_n)}-s|+\mathcal{O}\left(\delta_nk_n^{1/2}|W_n|^{-1}\right)\right\}\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}|\le\epsilon_n\frac{k_n}{m_n}\right)\\
&=|\bar{s}_{(k_n)}-s|\left(1+|\bar{s}_{(k_n)}-s|^{-1}\mathcal{O}\left(\delta_nk_n^{1/2}|W_n|^{-1}\right)\right)\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}|\le\epsilon_n\frac{k_n}{m_n}\right),
\end{aligned} \quad (4.126)$$

as $n\to\infty$. By (4.74), conditionally given $X(W_n)=m_n$, we have with probability 1
$|\bar{s}_{(k_n)}-s|^{-1}\mathbf{I}(|\tilde{Z}_{k_n:m_n}|\le\epsilon_nk_nm_n^{-1})=\mathcal{O}(|W_nk_n^{-1})\mathbf{I}(|\tilde{Z}_{k_n:m_n}|\le\epsilon_nk_nm_n^{-1})$, which implies $|\bar{s}_{(k_n)}-s|^{-1}\mathcal{O}(\delta_nk_n^{1/2}|W_n|^{-1})\mathbf{I}(|\tilde{Z}_{k_n:m_n}|\le\epsilon_nk_nm_n^{-1})$ $=o(1)\mathbf{I}(|\tilde{Z}_{k_n:m_n}|\le\epsilon_nk_nm_n^{-1})$ as $n\to\infty$. Then, by (4.74) and (4.126), conditionally given $X(W_n)=m_n$, we obtain the following stochastic expansion

$$\begin{aligned}
&\frac{1}{|\hat{s}_{(k_n)}-s|}\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}|\le\epsilon_n\frac{k_n}{m_n}\right)\\
=&\left\{\frac{1}{|\bar{s}_{(k_n)}-s|}+\frac{1}{|\bar{s}_{(k_n)}-s|^2}\mathcal{O}\left(\delta_nk_n^{1/2}|W_n|^{-1}\right)\right\}\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}|\le\epsilon_n\frac{k_n}{m_n}\right)\\
=&\left\{\frac{1}{|\bar{s}_{(k_n)}-s|}+\mathcal{O}\left(\delta_n|W_n|k_n^{-3/2}\right)\right\}\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}|\le\epsilon_n\frac{k_n}{m_n}\right)
\end{aligned} \quad (4.127)$$

as $n\to\infty$. By (4.127), conditionally given $X(W_n)=m_n$, we have

$$\left(\frac{1}{|\hat{s}_{(k_n)}-s|}-\frac{1}{|\bar{s}_{(k_n)}-s|}\right)^2\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}|\le\epsilon_n\frac{k_n}{m_n}\right)$$

$$= \mathcal{O}\left(\delta_n^2 |W_n|^2 k_n^{-3}\right) \mathbf{I}\left(|\tilde{Z}_{k_n:m_n}| \le \epsilon_n \frac{k_n}{m_n}\right),$$

with probability 1 as $n \to \infty$, which implies the first term of (4.124) is of order $o(|W_n|^2 k_n^{-3})$ as $n \to \infty$. Combining this order bound with the order bound of the second term of (4.124), we obtain

$$\mathbf{E}\left\{\left(\frac{1}{|\hat{s}_{(k_n)} - s|} - \frac{1}{|\bar{s}_{(k_n)} - s|}\right)^2 \middle| X(W_n) = m_n\right\} = o\left(\frac{|W_n|^2}{k_n^3}\right), \quad (4.128)$$

as $n \to \infty$, uniformly for all $m_n \in A_n$. Substitutuing (4.128) in to the first term on the r.h.s. of (4.121), we find that this term is of order $o(|W_n|^2 k_n^{-3})$, as $n \to \infty$. Since the second and third term on the r.h.s. of (4.121) are also of order $o(|W_n|^2 k_n^{-3})$ as $n \to \infty$, we have (4.120), which implies that the second term of (4.105) is of order $o(k_n^{-1})$, as $n \to \infty$.

Next we consider the third term of (4.105). Since the first term of (4.105) is $\mathcal{O}(k_n^{-1})$ and its second term is $o(k_n^{-1})$ as $n \to \infty$, an application of Cauchy-Schwarz inequality shows that the third term of (4.105) is of order $o(k_n^{-1})$ as $n \to \infty$.

It remains to prove the second and third term on the r.h.s. of (4.104) are of order $o(k_n^{-1})$ as $n \to \infty$. By (4.70) and (4.103), the second term on the r.h.s. of (4.104) does not exceed

$$\begin{aligned}
&\mathbf{E}\left(\frac{(\hat{\tau}_n - \tau)k_n}{2|W_n||\hat{s}_{(k_n)} - s|}\mathbf{I}(X(W_n) \ge k_n)\right)^2 \\
&\le \mathbf{E}\left(\frac{C\delta_n k_n^2}{2|W_n|^3|\hat{s}_{(k_n)} - s|}\mathbf{I}(X(W_n) \ge k_n)\right)^2 \\
&= \frac{C^2\delta_n^2 k_n^4}{2|W_n|^6}\mathbf{E}\left(\frac{1}{|\hat{s}_{(k_n)} - s|}\mathbf{I}(X(W_n) \ge k_n)\right)^2 \\
&= \frac{C^2\delta_n^2 k_n^4}{2|W_n|^6}\mathcal{O}\left(\frac{|W_n|^2}{k_n^2}\right) = o\left(\frac{1}{k_n}\right),
\end{aligned}$$

as $n \to \infty$. Hence, the second term on the r.h.s. of (4.104) is of order $o(k_n^{-1})$ as $n \to \infty$. Since the first term on the r.h.s. of (4.104) is of order $\mathcal{O}(k_n^{-1})$ as $n \to \infty$, an application of Cauchy-Schwarz inequality shows that the third term on the r.h.s. of (4.104) is of order $o(k_n^{-1})$ as $n \to \infty$. This completes the proof of Theorem 4.7. □

Proof of Theorem 4.8

We will prove this theorem by following the outline of the proof of Theorem 4.5. By Remark 4.3, we can write the l.h.s. of (4.61) as the quantity in (4.71). We will show the first term on the r.h.s. of (4.71) can be written as the r.h.s. of (4.61), while its second term is of order $o(k_n^2|W_n|^{-2})$ as $n \to \infty$.

First, we show the second term on the r.h.s. of (4.71) is of order $o(k_n^2|W_n|^{-2})$ as $n \to \infty$. By (4.60), there exists a positive constant C such that we have with probability 1

$$|\hat{\tau}_n - \tau| \leq C\delta_n k_n^3 |W_n|^{-4}. \tag{4.129}$$

Then, by (4.129) and (4.92), the second term on the r.h.s. of (4.71) does not exceed

$$\frac{C\delta_n k_n^4}{2|W_n|^5}\mathbf{E}\,\frac{1}{|\hat{s}_{(k_n)} - s|}\mathbf{I}(X(W_n) \geq k_n) = \frac{C\delta_n k_n^4}{2|W_n|^5}\mathcal{O}\left(\frac{|W_n|}{k_n}\right)$$
$$= \mathcal{O}\left(\frac{\delta_n k_n^3}{|W_n|^4}\right) = o\left(\frac{k_n^2}{|W_n|^2}\right),$$

as $n \to \infty$.

It remains to show the first term on the r.h.s. of (4.71) can be written as the r.h.s. of (4.61). Recall that the expectation appearing in this term can be written as the one in (4.72). First we will show that the second and third term on the r.h.s. of (4.72) multiplied by $k_n|W_n|^{-1}$ are of order $\mathcal{O}(k_n^{-1})$, as $n \to \infty$. By (4.89), (4.90), and (4.118) (cf. also the argument following (4.89) and (4.90), but now we use (4.118)), we find that the second and third term on the r.h.s. of (4.72) multiplied by $k_n|W_n|^{-1}$ are respectively of order $o(k_n^{-2})$ and $o(k_n^{-1})$, which are $\mathcal{O}(k_n^{-1})$, as $n \to \infty$. It remains to show the first term on the r.h.s. of (4.72) multiplied by $(\tau k_n)/(2|W_n|)$, which is

$$\frac{\tau k_n}{2|W_n|}\sum_{m_n \in A_n}\left\{\mathbf{E}\left(\frac{1}{|\hat{s}_{(k_n)} - s|}\middle| X(W_n) = m_n\right)\right\}\mathbf{P}(X(W_n) = m_n)$$
$$= \frac{\tau k_n}{2|W_n|}\sum_{m_n \in A_n}\left\{\mathbf{E}\left(\frac{1}{|\hat{s}_{(k_n)} - s|}\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}| \leq \epsilon_n \frac{k_n}{m_n}\right)\middle| X(W_n) = m_n\right)\right\}$$
$$\mathbf{P}(X(W_n) = m_n)$$
$$+\frac{\tau k_n}{2|W_n|}\sum_{m_n \in A_n}\left\{\mathbf{E}\left(\frac{1}{|\hat{s}_{(k_n)} - s|}\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}| > \epsilon_n \frac{k_n}{m_n}\right)\middle| X(W_n) = m_n\right)\right\}$$
$$\mathbf{P}(X(W_n) = m_n), \tag{4.130}$$

can be written as the r.h.s. of (4.61). Here, as before, ϵ_n is a sequence of positive real numbers converging to zero and $\epsilon_n^{-1} = o(k_n^{1/4})$ as $n \to \infty$.

First we show that the second term on the r.h.s. of (4.130) is $\mathcal{O}(k_n^{-1})$ as $n \to \infty$. By (4.79), (4.76), and an argument similar to the one used to prove (4.81), but now we use the fact that the r.h.s. of (4.76) is $o(k_n^{-2})$

as $n \to \infty$, we obtain

$$\mathbf{E}\left(\frac{1}{|\hat{s}_{(k_n)} - s|}\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}| > \epsilon_n \frac{k_n}{m_n}\right)\middle| X(W_n) = m_n\right) = o\left(\frac{|W_n|}{k_n^2}\right) \quad (4.131)$$

as $n \to \infty$. Substituting (4.131) into the second term on the r.h.s. of (4.130), we find that the second term on the r.h.s. of (4.130) is $o(k_n^{-1})$ as $n \to \infty$.

It remains to show that the first term on the r.h.s. of (4.130) can be written as the r.h.s. of (4.61). By a similar argument as the one used to prove (4.127), but we now use condition (4.60) instead of (4.58), conditionally given $X(W_n) = m_n$, we obtain the following stochastic expansion

$$\begin{aligned}&\frac{1}{|\hat{s}_{(k_n)} - s|}\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}| \le \epsilon_n \frac{k_n}{m_n}\right)\\ &= \left\{\frac{1}{|\bar{s}_{(k_n)} - s|} + \mathcal{O}\left(\delta_n \frac{k_n}{|W_n|}\right)\right\}\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}| \le \epsilon_n \frac{k_n}{m_n}\right), \quad (4.132)\end{aligned}$$

as $n \to \infty$. By (4.132), we have that the expectation appearing in the first term on the r.h.s. of (4.130) is equal to

$$\mathbf{E}\left(\frac{1}{|\bar{s}_{(k_n)} - s|}\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}| \le \epsilon_n \frac{k_n}{m_n}\right)\middle| X(W_n) = m_n\right) + o\left(\frac{k_n}{|W_n|}\right) \quad (4.133)$$

as $n \to \infty$, uniformly for all $m_n \in A_n$. Substituting (4.133) into the first term on the r.h.s. of (4.130), this term reduces to

$$\begin{aligned}&\frac{\tau k_n}{2|W_n|}\sum_{m_n \in A_n}\left(\mathbf{E}\left(\frac{1}{|\bar{s}_{(k_n)} - s|}\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}| \le \epsilon_n \frac{k_n}{m_n}\right)\middle| X(W_n) = m_n\right)\right)\\ &\qquad \cdot \mathbf{P}(X(W_n) = m_n) + o\left(\frac{k_n^2}{|W_n|^2}\right), \quad (4.134)\end{aligned}$$

as $n \to \infty$.

It remains to show that the first term of (4.134) can be written as the r.h.s. of (4.61). Recall that $|\bar{s}_{(k_n)} - s|$ has the same distribution as $H_n^{-1}(Z_{k_n:m_n})$ and consider a modified stochastic expansion of $H_n^{-1}(Z_{k_n:m_n})$ as given in (4.33), but with $\mathbf{I}(|\tilde{Z}_{k_n:m_n}| \le \epsilon_n)$ replaced by $\mathbf{I}(|\tilde{Z}_{k_n:m_n}| \le \epsilon_n k_n m_n^{-1})$. By (4.32), the sum of the second and third term on the r.h.s. of the modified (4.33) is equal to $\tilde{Z}_{k_n:m_n}\mathcal{O}(1)$, as $n \to \infty$. In order to have an appropriate expansion of the first term on the r.h.s. of the modified (4.33), we need to compute the second and third (right hand) derivative $H^{-1''}(0)$ and $H^{-1'''}(0)$, besides $H^{-1}(0)$ and $H^{-1'}(0)$, which from the proof of (4.34) we already know that $H^{-1}(0) = 0$ and

$H^{-1'}(0) = \theta\tau(2\lambda(s))^{-1}$. $H^{-1''}(0)$ can be computed as follows

$$\begin{aligned} H^{-1''}(0) &= \frac{d}{dt}\left(H^{-1'}(t)\right)_{t=0} = \frac{d}{dt}\left(\frac{1}{h(H^{-1}(t))}\right)_{t=0} \\ &= -\left.\frac{h'(H^{-1}(t))H^{-1'}(t)}{h^2(H^{-1}(t))}\right|_{t=0} = -\frac{h'(0)}{h^3(0)} = 0, \end{aligned}$$

since $h'(0) = 0$ while $h(0) = 2\lambda(s)(\theta\tau)^{-1} \neq 0$. A simple calculation shows that $h''(0) = 2\lambda''(s)(\theta\tau)^{-1}$. Then we can compute $H^{-1'''}(0)$ as follows

$$\begin{aligned} H^{-1'''}(0) &= \frac{d}{dt}\left(-\frac{h'(H^{-1}(t))H^{-1'}(t)}{h^2(H^{-1}(t))}\right)_{t=0} = -\frac{h''(0)}{h^4(0)} \\ &= -\left(\frac{2\lambda''(s)}{\theta\tau}\right)\left(\frac{\theta\tau}{2\lambda(s)}\right)^4 = -\frac{\theta^3\tau^3\lambda''(s)}{8\lambda^4(s)}. \end{aligned}$$

Note here that $h'(0)$ and $h''(0)$ denote respectively the first and second (right hand) derivative of h at 0. Then we can write an expansion of the first term on the r.h.s. of the modified (4.33) as follows

$$\begin{aligned} & H^{-1}\left(\frac{k_n}{m_n+1} + \mathcal{O}\left(\frac{1}{|W_n|}\right)\right) = H^{-1}\left(\frac{k_n}{m_n} + \mathcal{O}\left(\frac{1}{|W_n|}\right)\right) \\ = \;& 0 + \left(\frac{k_n}{m_n} + \mathcal{O}\left(\frac{1}{|W_n|}\right)\right) H^{-1'}(0) + 0 \\ & + \frac{1}{6}\left(\frac{k_n}{m_n} + \mathcal{O}\left(\frac{1}{|W_n|}\right)\right)^3 H^{-1'''}(0)(1+o(1)) \\ = \;& \left(\frac{k_n}{m_n} + \mathcal{O}\left(\frac{1}{|W_n|}\right)\right)\frac{\theta\tau}{2\lambda(s)} \\ & - \frac{\theta^3\tau^3\lambda''(s)}{48\lambda^4(s)}\left(\frac{k_n}{m_n} + \mathcal{O}\left(\frac{1}{|W_n|}\right)\right)^3 (1+o(1)), \end{aligned} \tag{4.135}$$

as $n \to \infty$. The first term on the r.h.s. of (4.135) can be written as

$$\begin{aligned} & \left(\frac{k_n}{\theta|W_n|} + \mathcal{O}\left(\frac{a_n k_n}{|W_n|^{3/2}}\right) + \mathcal{O}\left(\frac{1}{|W_n|}\right)\right)\frac{\theta\tau}{2\lambda(s)} \\ = \;& \frac{\tau k_n}{2\lambda(s)|W_n|} + \mathcal{O}\left(\frac{a_n k_n}{|W_n|^{3/2}}\right) + \mathcal{O}\left(\frac{1}{|W_n|}\right), \end{aligned} \tag{4.136}$$

as $n \to \infty$. To simplify the second term on the r.h.s. of (4.135) we argue as follows. Since $a_n = o(|W_n|^{1/2})$ as $n \to \infty$, we have that

$$\begin{aligned} & \left(\frac{k_n}{m_n} + \mathcal{O}\left(\frac{1}{|W_n|}\right)\right)^3 = \left(\frac{k_n}{\theta|W_n|} + \mathcal{O}\left(\frac{a_n k_n}{|W_n|^{3/2}}\right) + \mathcal{O}\left(\frac{1}{|W_n|}\right)\right)^3 \\ = \;& \left(\frac{k_n}{\theta|W_n|} + o\left(\frac{k_n}{|W_n|}\right)\right)^3 = \frac{k_n^3}{\theta^3|W_n|^3} + o\left(\frac{k_n^3}{|W_n|^3}\right), \end{aligned}$$

as $n \to \infty$. Hence, the second term on the r.h.s. of (4.135) can be written as

$$
\begin{aligned}
&-\frac{\theta^3\tau^3\lambda''(s)}{48\lambda^4(s)}\left(\frac{k_n^3}{\theta^3|W_n|^3}+o\left(\frac{k_n^3}{|W_n|^3}\right)\right)(1+o(1))\\
&=-\frac{\tau^3\lambda''(s)k_n^3}{48\lambda^4(s)|W_n|^3}+o\left(\frac{k_n^3}{|W_n|^3}\right), \qquad (4.137)
\end{aligned}
$$

as $n \to \infty$. By (4.135) with its first and second term replaced respectively by the r.h.s. of (4.136) and (4.137), in combination with the fact that the sum of the second and third term on the r.h.s. of the modified (4.33) is equal to $\mathcal{O}(1)\tilde{Z}_{k_n:m_n}$ as $n \to \infty$, we can write the modified (4.33) as the following stochastic expansion

$$
\begin{aligned}
&H_n^{-1}(Z_{k_n:m_n})\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}|\le\epsilon_n\frac{k_n}{m_n}\right)\\
&= \left\{\frac{\tau k_n}{2\lambda(s)|W_n|}-\frac{\tau^3\lambda''(s)k_n^3}{48\lambda^4(s)|W_n|^3}+o\left(\frac{k_n^3}{|W_n|^3}\right)+\mathcal{O}\left(\frac{a_nk_n}{|W_n|^{3/2}}+\frac{1}{|W_n|}\right)\right.\\
&\quad\left.+\mathcal{O}(1)\tilde{Z}_{k_n:m_n}\right\}\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}|\le\epsilon_n\frac{k_n}{m_n}\right)\\
&= \frac{\tau k_n}{2\lambda(s)|W_n|}\left\{1-\frac{\tau^2\lambda''(s)k_n^2}{24\lambda^3(s)|W_n|^2}+o\left(\frac{k_n^2}{|W_n|^2}\right)+\mathcal{O}\left(\frac{a_n}{|W_n|^{1/2}}+\frac{1}{k_n}\right)\right.\\
&\quad\left.+\mathcal{O}\left(\frac{|W_n|}{k_n}\right)\tilde{Z}_{k_n:m_n}\right\}\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}|\le\epsilon_n\frac{k_n}{m_n}\right), \qquad (4.138)
\end{aligned}
$$

as $n \to \infty$. By (4.138), we can compute the following conditional expectation

$$
\begin{aligned}
&\mathbf{E}\left(\frac{1}{|\bar{s}_{(k_n)}-s|}\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}|\le\epsilon_n\frac{k_n}{m_n}\right)\middle| X(W_n)=m_n\right)\\
&= \mathbf{E}\left(\frac{1}{H_n^{-1}(Z_{k_n:m_n})}\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}|\le\epsilon_n\frac{k_n}{m_n}\right)\right)\\
&= \mathbf{E}\,\frac{2\lambda(s)|W_n|}{\tau k_n}\left\{1+\frac{\tau^2\lambda''(s)k_n^2}{24\lambda^3(s)|W_n|^2}+o\left(\frac{k_n^2}{|W_n|^2}\right)+\mathcal{O}\left(\frac{k_n^4}{|W_n|^4}\right)\right.\\
&\quad\left.+\mathcal{O}\left(\frac{a_n}{|W_n|^{1/2}}+\frac{1}{k_n}\right)+\mathcal{O}\left(\frac{|W_n|}{k_n}\right)\tilde{Z}_{k_n:m_n}+\mathcal{O}\left(\frac{|W_n|^2}{k_n^2}\right)\tilde{Z}^2_{k_n:m_n}\right\}\\
&\quad\cdot\mathbf{I}\left(|\tilde{Z}_{k_n:m_n}|\le\epsilon_n\frac{k_n}{m_n}\right)\\
&= \left\{\frac{2\lambda(s)|W_n|}{\tau k_n}+\frac{\tau\lambda''(s)k_n}{12\lambda^2(s)|W_n|}+o\left(\frac{k_n}{|W_n|}\right)+\mathcal{O}\left(\frac{a_n|W_n|^{1/2}}{k_n}+\frac{|W_n|}{k_n^2}\right)\right\}\\
&\quad\cdot\left(1-o\left(\frac{1}{k_n^2}\right)\right)\\
&= \frac{2\lambda(s)|W_n|}{\tau k_n}+\frac{\tau\lambda''(s)k_n}{12\lambda^2(s)|W_n|}+o\left(\frac{k_n}{|W_n|}\right)+\mathcal{O}\left(\frac{a_n|W_n|^{1/2}}{k_n}+\frac{|W_n|}{k_n^2}\right), \qquad (4.139)
\end{aligned}
$$

as $n \to \infty$, uniformly for all $m_n \in A_n$. Note that, to get the r.h.s. of (4.139) we have used the facts $\mathbf{P}(|\tilde{Z}_{k_n:m_n}| \le \epsilon_n k_n m_n^{-1}) = 1 - o(k_n^{-2})$ (note that the r.h.s. of (4.76) is of order $o(k_n^{-2})$ as $n \to \infty$), $\mathbf{E}\tilde{Z}_{k_n:m_n} = 0$, $\mathbf{E}\tilde{Z}^2_{k_n:m_n} = \mathcal{O}(k_n|W_n|^{-2})$ so that $\mathcal{O}(|W_n|^3 k_n^{-3})\mathbf{E}\tilde{Z}^2_{k_n:m_n} = \mathcal{O}(|W_n| k_n^{-2})$, and the term of order $\mathcal{O}(k_n^3|W_n|^{-3})$ can be written as $o(k_n|W_n|^{-1})$ as $n \to \infty$, because of (4.4). Choose now $a_n = |W_n|^{\epsilon_0}$ for arbitrary small $\epsilon_0 > 0$. Substituting (4.139) into the first term of (4.134), this term reduces to

$$\begin{aligned}
&\frac{\tau k_n}{2|W_n|}\left\{\frac{2\lambda(s)|W_n|}{\tau k_n} + \frac{\tau\lambda''(s)k_n}{12\lambda^2(s)|W_n|} + o\left(\frac{k_n}{|W_n|}\right)\right. \\
&\quad\left. + \mathcal{O}\left(\frac{|W_n|^{1/2+\epsilon_0}}{k_n} + \frac{|W_n|}{k_n^2}\right)\right\}\mathbf{P}(X(W_n) \in A_n) \\
&= \lambda(s) + \frac{\tau^2\lambda''(s)k_n^2}{24\lambda^2(s)|W_n|^2} + o\left(\frac{k_n^2}{|W_n|^2}\right) + \mathcal{O}\left(\frac{1}{|W_n|^{1/2-\epsilon_0}} + \frac{1}{k_n}\right), \qquad (4.140)
\end{aligned}$$

as $n \to \infty$, since by (4.119) we have $\mathbf{P}(X(W_n) \in A_n) = 1 - o(k_n^{-2})$, as $n \to \infty$. This completes the proof of Theorem 4.8. □

4.4 Comparison of nearest neighbor and kernel type estimators

Consider a special case of the kernel type estimator $\hat{\lambda}_{n,K}$ studied in chapter 3 namely the one with a uniform kernel, i.e., $K(u) = 1/2$ for all $u \in [-1, 1]$, and zero otherwise (cf. (3.5)). In this case, the asymptotic approximations to the variance, bias, and MSE of $\hat{\lambda}_{n,K}$ as well as the optimal choice of h_n (cf. (3.57), (3.59), (3.61) and (3.62)) can be simplified as below.

Suppose that λ is periodic and locally integrable, and K is the uniform kernel on $[-1, 1]$.

(i) If $h_n \downarrow 0$, $|W_n|h_n \to \infty$, and $|W_n||\hat{\tau}_n - \tau| = \mathcal{O}(\delta_n|W_n|^{-1})$ with probability 1 as $n \to \infty$, for some fixed sequence $\delta_n \downarrow 0$ as $n \to \infty$, then we have

$$Var\left(\hat{\lambda}_{n,K}(s)\right) = \frac{\tau\lambda(s)}{2|W_n|h_n} + o\left(\frac{1}{|W_n|h_n}\right), \qquad (4.141)$$

as $n \to \infty$, provided s is a Lebesgue point of λ.

(ii) If $h_n \downarrow 0$ and $|W_n||\hat{\tau}_n - \tau| = \mathcal{O}(\delta_n h_n^3)$ with probability 1 as $n \to \infty$, for some fixed sequence $\delta_n \downarrow 0$ as $n \to \infty$, and λ has finite second

derivative λ'' at s, then

$$\mathbf{E}\hat{\lambda}_{n,K}(s) = \lambda(s) + \frac{\lambda''(s)}{6}h_n^2 + o(h_n^2) + O(|W_n|^{-1}), \quad (4.142)$$

as $n \to \infty$.

(iii) If conditions in (i) and (ii) are satisfied, then we have

$$MSE\left(\hat{\lambda}_{n,K}(s)\right) = \frac{\tau\lambda(s)}{2|W_n|h_n} + \frac{1}{36}\left(\lambda''(s)\right)^2 h_n^4 + o\left(|W_n|^{-1}h_n^{-1}\right) + o\left(h_n^4\right), \quad (4.143)$$

as $n \to \infty$.

(iv) By minimizing the leading term of the $MSE(\hat{\lambda}_{n,K}(s))$ (cf. (4.143)), we obtain the optimal choice of h_n, which is given by

$$h_n = \left[\frac{9\tau\lambda(s)}{2\left(\lambda''(s)\right)^2}\right]^{\frac{1}{5}} |W_n|^{-\frac{1}{5}}. \quad (4.144)$$

Next note that our nearest neighbor estimator $\hat{\lambda}_n(s)$, with optimal choice of k_n given in (4.64), yields the following approximation to the variance and bias,

$$Var\left(\hat{\lambda}_n(s)\right) = \frac{\tau^{4/5}(\lambda''(s))^{2/5}(\lambda(s))^{4/5}}{(144)^{1/5}}|W_n|^{-4/5} + o\left(|W_n|^{-4/5}\right), \quad (4.145)$$

and

$$\mathbf{E}\hat{\lambda}_n(s) = \lambda(s) + \frac{\tau^{2/5}(\lambda''(s))^{1/5}(\lambda(s))^{2/5}}{24(144)^{-2/5}}|W_n|^{-2/5} + o\left(|W_n|^{-2/5}\right), \quad (4.146)$$

as $n \to \infty$, provided λ has finite second derivative λ'' at s, $\lambda(s) > 0$, and (4.65) holds true. Similarly, the uniform kernel estimator $\hat{\lambda}_{n,K}(s)$, with optimal choice of h_n given in (4.144), yields the following approximation to the variance and bias,

$$Var\left(\hat{\lambda}_{n,K}(s)\right) = \frac{\tau^{4/5}(\lambda''(s))^{2/5}(\lambda(s))^{4/5}}{2(9/2)^{1/5}}|W_n|^{-4/5} + o\left(|W_n|^{-4/5}\right), \quad (4.147)$$

and

$$\mathbf{E}\hat{\lambda}_{n,K}(s) = \lambda(s) + \frac{\tau^{2/5}(\lambda''(s))^{1/5}(\lambda(s))^{2/5}}{6(9/2)^{-2/5}}|W_n|^{-2/5} + o\left(|W_n|^{-2/5}\right), \quad (4.148)$$

as $n \to \infty$, provided λ has finite second derivative λ'' at s and (4.65) holds true. Note that $(144)^{1/5} = 2(9/2)^{1/5}$ and $24(144)^{-2/5} = 6(9/2)^{-2/5}$, i.e. $\hat{\lambda}_n$ and $\hat{\lambda}_{n,K}$ have the same asymptotic approximations to the variance and bias, which also implies that the two estimators have the same asymptotic approximation to the MSE. This is in agreement with the comparison made by Mack and Rosenblatt (1979) for the density estimation case. Note also that the estimator $\hat{\lambda}_n$ requires condition $\lambda(s) > 0$ which is not needed for $\hat{\lambda}_{n,K}$.

Note also that, if we compare (4.59) and (4.61) with (4.141) and (4.142), we see that the role of τ and $|W_n|$ is different in the asymptotic approximations to the variance and bias of $\hat{\lambda}_n$ compared to those of $\hat{\lambda}_{n,K}$. For the nearest neighbor estimate, the bias is proportional to $\tau^2|W_n|^{-2}$ while the variance does not depend on either τ or $|W_n|$. In the case of the kernel estimate we have the opposite situation, i.e. the variance is proportional to $\tau|W_n|^{-1}$, while the bias does not depend on either τ or $|W_n|$.

Chapter 5

Estimation of the period

5.1 Introduction

This chapter is concerned with estimation of the period τ, using only a single realization $X(\omega)$ of the cyclic Poisson process X observed in W_n.

Let Θ denote the parameter space, $\tau \in \Theta$, and let Θ be an open interval in $\mathbf{R}^+$. A 'nonparametric' estimator $\hat{\tau}_n$ of τ is obtained as follows: for any $\delta \in \Theta$, define

$$Q_n(\delta) \quad = \quad \frac{1}{|W_n|} \sum_{i=1}^{N_{n\delta}} \left(X(U_{\delta,i}) - \frac{1}{N_{n\delta}} \sum_{j=1}^{N_{n\delta}} X(U_{\delta,j}) \right)^2, \tag{5.1}$$

where $N_{n\delta} = [|W_n|/\delta]$, which denotes the (maximum) number of adjacent disjoint intervals $U_{\delta,i}$ of length δ in the window W_n. We suppose that W_n is a closed interval, and let a_n and b_n denote its left- and right-end points, that is $W_n = [a_n, b_n]$. For convenience we shall require that the $U_{\delta,i}$'s are intervals of the form $[a_n + r + (i-1)\delta, a_n + r + i\delta)$, for some $r \in [0, (|W_n| - \delta N_{n\delta})]$. Otherwise the specific choice of r is free and basically no importance (cf. the paragraph following (5.179) in section 5.6). Now we may define

$$\hat{\tau}_{n,1} = \arg\min_{\delta \in \Theta} Q_n(\delta). \tag{5.2}$$

Clearly, τ also can be estimated more generally, as follows: first estimate $k\tau$, for some positive integer k satisfying $k = k_n = o(|W_n|)$, by $k\hat{\tau}_{n,k}$, which is given by

$$k\hat{\tau}_{n,k} = \arg\min_{\delta \in \Theta_k} Q_n(\delta), \tag{5.3}$$

where $\hat{\tau}_{n,k}$ denotes the resulting estimator of τ. Here $\Theta_k = (\tau_{k,0}, \tau_{k,1})$ is an open interval, such that no other multiple of τ than $k\tau$ is contained in

Θ_k. Of course $\Theta_1 = \Theta$. The restriction on the parameter space Θ_k that no other multiple of τ than $k\tau$ is contained in Θ_k of course requires some prior information about the value of $k\tau$.

Since $Q_n(\delta)$ may have 'flat parts', $\hat{\tau}_{n,k}$ is not uniquely determined by (5.3). However, it can be shown that our results (all Theorems below) remain valid whatever specific choice of $k\hat{\tau}_{n,k}$ is made (cf. also Lemma 5.14 in section 5.6).

In all theorems in this chapter, we have to restrict the range of k-values. In fact we require $k = k_n \sim |W_n|^c$, for some $0 \le c < \frac{1}{3}$. The reason for this is easy to explain. Let us write

$$Q_n(\delta) = \tilde{Q}_n(\delta) + \Lambda_n(\delta),$$

where $\tilde{Q}_n(\delta) = Q_n(\delta) - \mathbf{E}Q_n(\delta)$ and $\Lambda_n(\delta) = \mathbf{E}Q_n(\delta)$. By Lemmas 5.16 and 5.17 (see section 5.6) we have respectively $\Lambda_n(\delta) = \mathcal{O}(k^{-1})$ and $\tilde{Q}_n(\delta) = \mathcal{O}(k^{1/2}|W_n|^{-1/2})$ as $n \to \infty$, whenever $\delta \in \Theta_k$, and both order bounds are sharp. In order that our estimator $k\hat{\tau}_{n,k}$ of $k\tau$ is consistent, we need that the deterministic part of $Q_n(\delta)$ dominates its purely random part. But, this requirement automatically leads to the restriction $c < \frac{1}{3}$.

We conclude this section with the following remark. Throughout this chapter, we will assume that

$$\lambda \text{ is not constant a.e.}[\nu], \tag{5.4}$$

that is, there does not exist a positive constant λ_0 such that $\lambda(s) = \lambda_0$, for all $s \in \mathbf{R} \setminus N$, with $\nu(N) = 0$. Note that (5.4) implies $\theta > 0$ (cf. (1.12)). For cyclic λ with period τ, i.e. $\lambda(s + \tau) = \lambda(s)$ for all $s \in \mathbf{R}$ and some $\tau \in \mathbf{R}^+$, the failure of (5.4) would directly imply that

$$\mu_\tau(B) = \int_B \lambda(s)ds = \lambda_0 \nu(B),$$

for any value of τ and any Borel set B; here μ_τ denotes the mean measure corresponding to a cyclic λ with period τ. Hence, to ensure that τ is identifiable, i.e. $\mu_\tau \ne \mu_{\tau'}$ for every $\tau \ne \tau'$, we need (5.4) to hold. Note that, for every σ-finite mean measure μ, there exists, on a given probability space $(\Omega, \mathcal{A}, \mathbf{P})$, a unique Poisson process X with mean measure equal to μ. Identifiability is a necessary condition for the existence of consistent estimators: if τ is not identifiable, then a consistent estimator of τ can not exist.

5.2 Results

Suppose that, for each integer k satisfying $k = k_n = o(|W_n|)$ as $n \to \infty$, $\Theta_k = (\tau_{k,0}, \tau_{k,1})$ is an open interval, where $\tau_{k,0}$ and $\tau_{k,1}$ are known elements of $\mathbf{R}^+$, and such that no other multiple of τ than $k\tau$ is contained in Θ_k. Throughout we also assume that λ satisfies the condition: *if there exists a $t \in (0, \tau)$ such that, for each $n \geq 1$,*

$$\int_{U_{t,i}} \lambda(s)ds = t\theta \ \text{ for all } i,\ i = 1, \ldots, N_{nt}, \ \text{ then}$$

$$\nu(\{r : \int_{U_{t,i}} \lambda(s)ds = t\theta;\ i = 1, \ldots, N_{nt}\}) = 0, \tag{5.5}$$

with $U_{t,i} = [a_n + r + (i-1)t, a_n + r + it)$, $i = 1, \ldots, N_{nt}$, and $N_{nt} = [\frac{|W_n|}{t}]$ as before. We refer to (5.179) and the discussion following it for more details. Note that condition (5.5) is only violated in exceptional cases (cf. (5.180)). Note also that condition (5.5) implies (5.4).

In the first two theorems of this chapter we establish consistency (Theorem 5.1) and a slow rate of consistency (Theorem 5.2) using only the assumption that λ is bounded. The resulting rate of convergence is not the best possible. Note that we only assume boundedness of λ and that our proof of Theorem 5.2 is nothing but a refinement of our proof of Theorem 5.1. Because Theorem 5.2 covers Theorem 5.1 as a special case when $\gamma = 0$, we only prove Theorem 5.2.

Theorem 5.1 *Suppose that λ is periodic (with period τ) and bounded. In addition, we assume that (5.5) is satisfied, and $k = k_n \sim |W_n|^c$, for some $0 \leq c < \frac{1}{3}$. Then we have*

$$(\hat{\tau}_{n,k} - \tau) \stackrel{p}{\to} 0,$$

as $n \to \infty$. If, in addition, for each $\epsilon > 0$ we have

$$\sum_{n=1}^{\infty} \exp\left\{-\epsilon |W_n|^{\frac{1}{2}(1+c)}\right\} < \infty,$$

then

$$(\hat{\tau}_{n,k} - \tau) \stackrel{c}{\to} 0,$$

as $n \to \infty$.

Theorem 5.2 *Suppose that λ is periodic (with period τ) and bounded. In addition, we assume that (5.5) is satisfied, and $k = k_n \sim |W_n|^c$, for some $0 \leq c < \frac{1}{3}$. If $\gamma < \frac{1}{4} + \frac{c}{4}$, then we have*

$$|W_n|^{\gamma} (\hat{\tau}_{n,k} - \tau) \stackrel{p}{\to} 0, \tag{5.6}$$

as $n \to \infty$. If, in addition, for each $\epsilon > 0$ we have

$$\sum_{n=1}^{\infty} \exp\left\{-\epsilon |W_n|^{\min(1+c-4\gamma, \frac{1}{2}(1+c-2\gamma))}\right\} < \infty, \tag{5.7}$$

then

$$|W_n|^{\gamma} (\hat{\tau}_{n,k} - \tau) \xrightarrow{c} 0, \tag{5.8}$$

as $n \to \infty$.

Note that for the case $\gamma < c$, condition (5.7) is superfluous (cf. the argument following (5.21)). For the case $\gamma = 0$ (cf. Theorem 5.1), the quantity $\min(1 + c - 4\gamma, \frac{1}{2}(1 + c - 2\gamma))$ reduces to $\frac{1}{2}(1 + c)$.

In the following Theorem we show that, under an additional smoothness assumption on λ, we have a faster rate of convergence of our estimators $\hat{\tau}_{k,n}$. The rate $|W_n|^{-\gamma}$, $\gamma < \frac{1}{2}$ obtained here is the natural one, when estimating euclidean parameters.

Theorem 5.3 *Suppose that λ is periodic and Lipschitz (of order 1). In addition, we assume that (5.5) is satisfied. Then, for each positive integer k satisfying $k = k_n \sim |W_n|^c$ for some $0 \le c < \frac{1}{3}$ and for any $\gamma < \frac{1}{2}$, we have*

$$|W_n|^{\gamma} (\hat{\tau}_{k,n} - \tau) \xrightarrow{p} 0, \tag{5.9}$$

as $n \to \infty$.

The Lipschitz condition on λ is needed to obtain an appropriate stochastic expansion for $Q_n(\delta)$ (cf. Lemma 5.12), which we need to obtain a rate of convergence of order $|W_n|^{-\gamma}$, for $\gamma < \frac{1}{2}$ and to establish asymptotic normality.

Perhaps somewhat surprisingly, $\hat{\tau}_{k,n}$ is *not* asymptotically normally distributed. However a slight modification of $\hat{\tau}_{k,n}$, has, properly normalized, asymptotically normal distribution. Now we define our modified estimator $\hat{\tau}^*_{k,n}$ of τ. For each positive integer k satisfying $k = k_n = o(|W_n|)$, $\hat{\tau}^*_{k,n}$ is given by

$$\hat{\tau}^*_{k,n} = \frac{1}{k} \arg\min_{\delta \in \Theta_k} Q^*_n(\delta),$$

where for any $\delta \in \Theta_k$,

$$Q^*_n(\delta) = Q_n(\delta) + \frac{X(W_n \setminus W_{N_n\delta})}{|W_n|}.$$

That is we add, for each δ, a correction term $X(W_n \setminus W_{N_{n\delta}})|W_n|^{-1}$ to $Q_n(\delta)$. Note that this term is nothing but $|W_n|^{-1}$ times the number of points in a realization of X inside the window W_n, which are not used in the construction of $Q_n(\delta)$.

In the following theorem we show that, for each positive integer k satisfying $k = o(|W_n|^{1/3})$, our estimator $\hat{\tau}^*_{k,n}$ of τ is approximately normally distributed.

Theorem 5.4 *Suppose that λ is periodic and Lipschitz. In addition, we assume that (5.5) is satisfied. Then, for each positive integer k satisfying $k = k_n \sim |W_n|^c$ for some $0 \le c < \frac{1}{3}$, we have*

$$|W_n|^{1/2}\left(\hat{\tau}^*_{k,n} - \tau\right) - N\left(0, \sigma_k^2\right) = o_p\left(k^{-1/2}\right), \tag{5.10}$$

as $n \to \infty$, where

$$\sigma_k^2 = \frac{\tau^3\,\theta}{\int_0^\tau (\lambda(s)-\theta)^2 ds} + \frac{\tau^4\,\theta^3}{4(2\theta k\tau + 1)(\int_0^\tau (\lambda(s)-\theta)^2 ds)^2},$$

and $N(0, \sigma_k^2)$ denotes a normal r.v. with mean zero and variance σ_k^2.

Note that σ_k^2 decreases as $\int_0^\tau (\lambda(s)-\theta)^2 ds$ increases, i.e. when λ becomes less flat. This is as one would expect. A similar phenomenon was noted by Hall et. al. (2000) for their estimation of the period in a nonparametric regression context.

Let us now comment briefly on our results. First of all we indicate how to get rid of the unpleasant requirement on the parameter space Θ_k mentioned above. Set $m = \mathcal{O}(|W_n|^c)$ with $c < \frac{1}{3}$. It is easy to check from the proofs given in this chapter that

$$Q_n(\delta) = \Lambda_n(\delta) + \tilde{Q}_n(\delta) \overset{p}{\sim} \Lambda_n(\delta) \tag{5.11}$$

with $\Lambda_n(\delta) = \mathbf{E}Q_n(\delta)$, as $n \to \infty$ and $|\delta - k\tau| \to 0$, uniformly for all $1 \le k \le m$. Here $a_n \overset{p}{\sim} b_n$ means that $a_n/b_n \overset{p}{\to} 1$. Relation (5.11) also hold true, with $\overset{p}{\sim}$ replaced by $\overset{c}{\sim}$, where $a_n \overset{c}{\sim} b_n$ means that $\sum_{n=1}^\infty \mathbf{P}(|a_n/b_n - 1| > \epsilon) < \infty$, for each $\epsilon > 0$. This fact directly implies that we may define the estimators $\hat{\tau}_{n,1}, \ldots, \hat{\tau}_{n,m}$ of τ alternatively to be the first m locations of the local minima of the function $Q_n(\delta)$, for $\delta \in (0, |W_n|)$. Inspection of the graph of $Q_n(\delta)$ on the set $(0, |W_n|)$ will give us the m-dimensional vector $(\hat{\tau}_{n,1}, \ldots, \hat{\tau}_{n,m})$; the k-th component $(1 \le k \le m)$ is the τ-estimate corresponding to the k-th local minimum of $Q_n(\delta)$, that is the one obtained through (5.3), by minimizing over the parameter space

Θ_k. Hence the requirement on Θ_k that $k\tau \in \Theta_k$ but no other multiple of τ is contained in Θ_k becomes superfluous.

Let us now describe how to apply the results of this chapter to check the conditions on τ-estimation, which were needed in chapters 3 and 4. By way of an example, let us verify the condition (3.8) of Theorem 3.1, which was needed to obtain (weak) consistency of $\hat{\lambda}_{n,K}(s)$, when we take $\hat{\tau}_n$ to be our estimator (5.3). Let us assume that $h_n = |W_n|^{-b}$, where $0 < b < 1$ (because of (3.2) and (3.7)). Then, condition (3.8) can be written as

$$|W_n|^{1+b}|\hat{\tau}_n - \tau| \xrightarrow{p} 0 \tag{5.12}$$

as $n \to \infty$, for some $0 < b < 1$. Now suppose that we assume that λ is Lipschitz. Then by taking $\hat{\tau}_n = \hat{\tau}_{n,k}$, we have that, for each positive integer k satisfying $k = k_n \sim |W_n|^c$ for some $0 \leq c < \frac{1}{3}$ and for any $\gamma < \frac{1}{2}$,

$$|W_n|^{\gamma}|\hat{\tau}_n - \tau| \xrightarrow{p} 0 \tag{5.13}$$

as $n \to \infty$. Since $\gamma < \frac{1}{2}$ and $1 + b > 1$, (5.13) is a much weaker statement than (5.12). However we can do the following. First we construct the estimator $\hat{\tau}_n$ of τ by using the whole information about X in the window W_n, and then we use only the information about X in a (smaller) window $W_{0,n} \subset W_n$ of size $\sim |W_n|^{\gamma/(1+b)}$, to construct a consistent estimator $\hat{\lambda}_{n,K}(s)$ of $\lambda(s)$.

Next, let us verify the condition (3.53) of Theorem 3.7, which was needed to obtain asymptotic unbiasedness of $\hat{\lambda}_{n,K}(s)$, when we take $\hat{\tau}_n$ to be our estimator (5.3). Let us take $\delta_n = |W_n|^{-\epsilon_0}$, for some arbitrary small positive real number ϵ_0. Then, condition (3.53) requires that there exists a constant $C > 0$ and a positive integer n_0 such that

$$\mathbf{P}\left(|W_n|^{1+b+\epsilon_0}|\hat{\tau}_n - \tau| > C\right) = 0,$$

for all $n \geq n_0$. Since we know $|\Theta_k| < 2\tau$, we have that $|k\hat{\tau}_{n,k} - k\tau| < 2\tau$, with probability 1, for all $n \geq 1$. Then, by taking $\hat{\tau}_n = \hat{\tau}_{n,k}$ with $k \sim |W_n|^c$, $c < \frac{1}{3}$ and arbitrary close to $\frac{1}{3}$, we have that

$$\mathbf{P}\left(|W_n|^{\gamma}|\hat{\tau}_n - \tau| > 2\tau\right) = 0,$$

with $\gamma < \frac{1}{3}$, for all $n \geq 1$. Again one may proceed by estimating τ on W_n and compute $\hat{\lambda}_{n,K}(s)$ on a smaller window $W_{0,n}$ of appropriate size.

Better methods for estimating τ with high accuracy are of course desirable. Vere-Jones (1982) obtains an almost sure rate of order $o(n^{-1})$ where $(0, n)$ denotes the observation interval, provided λ admits a Fourier series with coefficients which are monotone decreasing, a condition which seems to be rather restrictive in our nonparametric framework. He also obtained a rate $\gamma < \frac{3}{2}$ for his estimator in the parametric model he considers (cf. section 1.4). In Hall et. al. (2000) the problem of estimating the period of τ was investigated in a somewhat different context, namely in a nonparametric periodic regression model. Under strong conditions on the regression function and assuming rather precise prior knowledge about τ, these authors obtain a rate of convergence $\gamma < \frac{3}{2}$. The estimation method investigated by Hall et. al. (2000) is somewhat more sophisticated than ours, though both are based on a least squares approach. It seems worthwhile to investigate their method in our Poisson process setting, but this is clearly outside the scope of the present study.

5.3 Proof of Theorem 5.2

First we prove (5.6). To check (5.6), we must show, for any $\gamma < \frac{1}{4} + \frac{c}{4}$ and for each $\epsilon > 0$,

$$\mathbf{P}(|\hat{\tau}_{n,k_n} - \tau| > \epsilon|W_n|^{-\gamma}) \rightarrow 0, \tag{5.14}$$

as $n \rightarrow \infty$. Since $k_n \sim |W_n|^c$, proving (5.14) is equivalent to proving

$$\mathbf{P}(|k_n\hat{\tau}_{n,k_n} - k_n\tau| > \epsilon|W_n|^{c-\gamma}) \rightarrow 0, \tag{5.15}$$

as $n \rightarrow \infty$. Since $|\Theta_k| < 2\tau$, we have with probability 1 that $|k_n\hat{\tau}_{n,k_n} - k_n\tau| < 2\tau$. Hence (5.15) automatically holds true if $\gamma < c$. So, it remains to check (5.15) only for the case $\gamma \geq c$.

The basic idea of the proof is a classical one (cf. e.g. Guyon (1995), page 119-120) and involves the modulus of continuity of the Q_n-process. Let $B_{n,k} = (k_n\tau - \epsilon|W_n|^{c-\gamma}, k_n\tau + \epsilon|W_n|^{c-\gamma})$, an open interval with centre $k_n\tau$ and of length $2\epsilon|W_n|^{c-\gamma}$. We then have

$$\{|k_n\hat{\tau}_{n,k} - k_n\tau| > \epsilon|W_n|^{c-\gamma}\} \subset \{\inf_{\delta\in\Theta_k\setminus B_{n,k}} Q_n(\delta) \leq Q_n(k_n\tau)\},$$

which implies that

$$\mathbf{P}(|k_n\hat{\tau}_{n,k} - k_n\tau| > \epsilon|W_n|^{c-\gamma}) \leq \mathbf{P}\left(\inf_{\delta\in\Theta_k\setminus B_{n,k}} Q_n(\delta) \leq Q_n(k_n\tau)\right). \tag{5.16}$$

For each k, let $(\tau_{k,0}, \tau_{k,1}]$ be partitioned into $L = L_n$ disjoint subintervals $(\delta_{i-1}, \delta_i]$, each of length $\eta = \eta_n$, where $\eta = (\tau_{k,1} - \tau_{k,0})/L$, for all $i = 1, 2, \ldots, L$. Hence we have $\tau_{k,0} = \delta_0 < \delta_1 < \delta_2 < \cdots < \delta_L = \tau_{k,1}$. For the purpose of our proofs we take $L_n = |W_n|^\beta$, for some $\beta > 0$. Now the modulus of continuity of the Q_n-process, $W_{n,\eta,k} =$

$\sup_{\delta,\delta'\in\Theta_k;\ |\delta-\delta'|\le\eta} |Q_n(\delta) - Q_n(\delta')|$, can easily seen to be at most equal to $3W^{\eta}_{n,L,k}$, where

$$W^{\eta}_{n,L,k} = \max_{1\le i\le L} \sup_{\delta_{i-1}<\delta\le\delta_i} |Q_n(\delta) - Q_n(\delta_i)| . \tag{5.17}$$

Then we have that

$$\inf_{\delta\in\Theta_k\backslash B_{n,k}} Q_n(\delta) \ge \min_{i,\ \delta_i\in\Theta_k\backslash B_{n,k}} Q_n(\delta_i) - W^{\eta}_{n,L,k},$$

which implies the r.h.s. of (5.16) does not exceed

$$\begin{aligned}
& \mathbf{P}\left(\min_{i,\ \delta_i\in\Theta_k\backslash B_{n,k}} Q_n(\delta_i) - W^{\eta}_{n,L,k} \le Q_n(k_n\tau)\right) \\
\le\ & \mathbf{P}\left(\{\min_{i,\ \delta_i\in\Theta_k\backslash B_{n,k}} Q_n(\delta_i) - W^{\eta}_{n,L,k} \le Q_n(k_n\tau)\} \cap \{W^{\eta}_{n,L,k} < \alpha_n\}\right) \\
& + \mathbf{P}\left(W^{\eta}_{n,L,k} \ge \alpha_n\right) \\
\le\ & \mathbf{P}\left(\min_{i,\ \delta_i\in\Theta_k\backslash B_{n,k}} Q_n(\delta_i) - Q_n(k_n\tau) < \alpha_n\right) + \mathbf{P}\left(W^{\eta}_{n,L,k} \ge \alpha_n\right),
\end{aligned}$$

for any $\alpha_n > 0$. Then, to prove (5.14), it suffices to show, for all sufficiently small $\alpha_n > 0$,

$$\mathbf{P}\left(\min_{i,\ \delta_i\in\Theta_k\backslash B_{n,k}} Q_n(\delta_i) - Q_n(k_n\tau) < \alpha_n\right) \to 0, \tag{5.18}$$

and

$$\mathbf{P}\left(W^{\eta}_{n,L,k} \ge \alpha_n\right) \to 0, \tag{5.19}$$

as $n \to \infty$.

First we consider (5.18). Recall that $\Lambda_n(.) = \mathbf{E}Q_n(.)$ and $\tilde{Q}_n(.) = Q_n(.) - \mathbf{E}Q_n(.)$. By the Bonferroni inequality, the probability on the l.h.s. of (5.18) does not exceed

$$\begin{aligned}
& \sum_{i,\ \delta_i\in\Theta_k\backslash B_{n,k}} \mathbf{P}\left(Q_n(\delta_i) - Q_n(k_n\tau) < \alpha_n\right) \\
=\ & \sum_{i,\ \delta_i\in\Theta_k\backslash B_{n,k}} \mathbf{P}\left(\tilde{Q}_n(\delta_i) - \tilde{Q}_n(k_n\tau) < \alpha_n - (\Lambda_n(\delta_i) - \Lambda_n(k_n\tau))\right).
\end{aligned} \tag{5.20}$$

By part (i) of Lemma 5.16 for the case $\epsilon_n = |W_n|^{c-\gamma}$ (cf. section 5.6), we have that there exist $\alpha_0 > 0$ and positive integer n_0 such that

$$\Lambda_n(\delta_i) - \Lambda_n(k_n\tau) > \alpha_0 k_n^{-1}|W_n|^{2c-2\gamma} \sim \alpha_0|W_n|^{c-2\gamma}$$

for all $\delta_i \in \Theta_{k_n} \setminus B_{n,k_n}$ and all $n \geq n_0$. Note that here we require the assumption that λ satisfy (5.5). By choosing $\alpha_n = \alpha |W_n|^{c-2\gamma}$ with $0 < \alpha \leq \alpha_0/2$, we have that $\alpha_n - (\Lambda_n(\delta_i) - \Lambda_n(k_n\tau))$ is strictly negative, so that $|\alpha_n - (\Lambda_n(\delta_i) - \Lambda_n(k_n\tau))| \geq \alpha |W_n|^{c-2\gamma}$ for all $n > n_0$ and all δ_i outside set B_{n,k_n}. Then, the probability on the r.h.s. of (5.20) does not exceed

$$\sum_{i,\ \delta_i \in \Theta_{k_n} \setminus B_{n,k_n}} \mathbf{P}\left(\left| \tilde{Q}_n(\delta_i) - \tilde{Q}_n(k_n\tau) \right| > \alpha |W_n|^{c-2\gamma} \right)$$
$$\leq \sum_{i=1}^{L_n} \mathbf{P}\left(\left| \tilde{Q}_n(\delta_i) \right| > \frac{\alpha}{2} |W_n|^{c-2\gamma} \right) + L_n \mathbf{P}\left(\left| \tilde{Q}_n(k_n\tau) \right| > \frac{\alpha}{2} |W_n|^{c-2\gamma} \right).$$

Since we restrict attention to the case $\gamma \geq c$, we have that $1 + c - 4\gamma \leq 1 - 2\gamma$. By condition $\gamma < \frac{1}{4} + \frac{c}{4}$, we have that $1 + c - 4\gamma > 0$. Then, by (1.2) and Lemma 5.5, we have (5.18). Note here that, since $\gamma \geq c$, the condition $\gamma < \frac{1}{4} + \frac{c}{4}$ automatically gives the restriction $c < \frac{1}{3}$.

Next we consider (5.19) with $\alpha_n = \alpha |W_n|^{c-2\gamma}$. In order to apply Lemma 5.6, we require $\gamma < \frac{1}{2} - \frac{c}{2}$. But, condition $\gamma < \frac{1}{4} + \frac{c}{4}$ and $c < \frac{1}{3}$, implies $\gamma < \frac{1}{2} - \frac{c}{2}$. Obviously we also have that $1 + c - 2\gamma > 0$. Then, by (1.2) and Lemma 5.6 we have that the probability on the l.h.s. of (5.19) with $\alpha_n = \alpha |W_n|^{c-2\gamma}$ converges to zero, as $n \to \infty$. This completes the proof of (5.6).

Next we prove (5.8). To verify this assertion we must show, for any $\gamma < \frac{1}{4} + \frac{c}{4}$ and for each $\epsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbf{P}(|\hat{\tau}_{n,k_n} - \tau| > \epsilon |W_n|^{-\gamma}) < \infty. \tag{5.21}$$

Recall that, since $|\Theta_k| < 2\tau$, we have with probability 1 that $|k_n\hat{\tau}_{n,k_n} - k_n\tau| < 2\tau$. Hence (5.21) automatically holds true if $\gamma < c$. So, it remains to check (5.21) only for the case $\gamma \geq c$. Following the structure of proof of (5.6), to prove (5.21) we see that it suffices to check that, for all sufficiently small $\alpha > 0$,

$$\sum_{n=1}^{\infty} \left\{ \sum_{i=1}^{L_n} \mathbf{P}\left(\left| \tilde{Q}_n(\delta_i) \right| > \alpha |W_n|^{c-2\gamma} \right) \right.$$
$$\left. + L_n \mathbf{P}\left(\left| \tilde{Q}_n(k_n\tau) \right| > \alpha |W_n|^{c-2\gamma} \right) \right\} < \infty, \tag{5.22}$$

and

$$\sum_{n=1}^{\infty} \mathbf{P}\left(W^{\eta}_{n,L,k} \geq \alpha |W_n|^{c-2\gamma} \right) < \infty, \tag{5.23}$$

with $L_n = |W_n|^\beta$, for some $\beta > 0$. To check (5.22), we will apply Lemma 5.5. Since we restrict attention to the case $\gamma \geq c$, we have $1 + c - 4\gamma \leq 1 - 2\gamma$. Since $\gamma < \frac{1}{4} + \frac{c}{4}$, we have $1 + c - 4\gamma > 0$. Then, by condition (5.7) and Lemma 5.5, we have (5.22). Recall that $\gamma < \frac{1}{4} + \frac{c}{4}$ and $c < \frac{1}{3}$ implies $\gamma < \frac{1}{2} - \frac{c}{2}$. Choose $\beta > 2$. Then, by condition (5.7) and Lemma 5.6 we have (5.23). This completes the proof of Theorem 5.2. □

Next we state and prove the two lemmas which were needed in our proof of Theorem 5.2. Lemma 5.5 will imply an exponential bound for the probability appearing on the l.h.s. of (5.18), while Lemma 5.6 gives a similar bound for the probability appearing on the l.h.s. of (5.19).

Lemma 5.5 *Suppose that λ is periodic (with period τ) and locally integrable. In addition, we assume that $k = k_n \sim |W_n|^c$, for some $0 \leq c < 1$. If $0 \leq \gamma < \frac{1}{4} + \frac{c}{4}$, then for each $\epsilon > 0$, there exists (large) constants C and n_0 such that*

$$\sum_{i=1}^{L_n} \mathbf{P}\left(|\tilde{Q}_n(\delta_i)| > \epsilon |W_n|^{c-2\gamma}\right) \leq C \exp\left\{-\alpha_\epsilon |W_n|^{\psi(c,\gamma)}\right\}, \tag{5.24}$$

and

$$L_n \mathbf{P}\left(|\tilde{Q}_n(k_n\tau)| > \epsilon |W_n|^{c-2\gamma}\right) \leq C \exp\left\{-\alpha_\epsilon |W_n|^{\psi(c,\gamma)}\right\}, \tag{5.25}$$

with $\delta_i \in \Theta_k$ and $L_n = |W_n|^\beta$ for some $\beta \geq 0$, for all $n \geq n_0$, where α_ϵ is a positive real number depending on ϵ, and $\psi(c,\gamma) = \min(1+c-4\gamma, 1-2\gamma)$.

Note that, in order to have $\psi(c,\gamma) > 0$ so that this lemma is useful, we require $\gamma < \min(\frac{1}{4} + \frac{c}{4}, \frac{1}{2})$. Since $c < 1$, the restriction on γ reduces to $0 \leq \gamma < \frac{1}{4} + \frac{c}{4}$.

Proof: Here we only prove (5.24), since the proof of (5.25) is similar and easier. To prove (5.24) we argue as follows. First note that $Q_n(\delta)$ in (5.1) can also be written as

$$Q_n(\delta) = \frac{1}{|W_n|} \sum_{j=1}^{N_{n\delta}} X^2(U_{\delta,j}) - \frac{X^2(W_{N_{n\delta}})}{|W_n| N_{n\delta}}, \tag{5.26}$$

so that

$$\tilde{Q}_n(\delta) = \frac{1}{|W_n|} \sum_{j=1}^{N_{n\delta}} X^2(\widetilde{U_{\delta,j}}) - \frac{1}{|W_n| N_{n\delta}} X^2(\widetilde{W_{N_{n\delta}}}). \tag{5.27}$$

For any Borel set B, we can write

$$\begin{aligned}
\widetilde{X^2(B)} &= X^2(B) - \mathbf{E}X^2(B) = \left(\tilde{X}(B) + \mathbf{E}X(B)\right)^2 - \mathbf{E}X^2(B) \\
&= \tilde{X}^2(B) + 2\tilde{X}(B)\mathbf{E}X(B) + (\mathbf{E}X(B))^2 - \mathbf{E}X^2(B) \\
&= \tilde{X}^2(B) + 2\tilde{X}(B)\mathbf{E}X(B) - \mathbf{E}X(B) \\
&= \widetilde{\tilde{X}^2(B)} + 2\tilde{X}(B)\int_B \lambda(s)ds. \qquad (5.28)
\end{aligned}$$

By (5.28), $\tilde{Q}_n(\delta)$ can be written as

$$\begin{aligned}
\tilde{Q}_n(\delta) &= \frac{1}{|W_n|}\sum_{j=1}^{N_{n\delta}} \widetilde{\tilde{X}^2(U_{\delta,j})} - \frac{1}{|W_n|N_{n\delta}}\widetilde{\tilde{X}^2(W_{N_{n\delta}})} \\
&+ \frac{2}{|W_n|}\left(\sum_{j=1}^{N_{n\delta}} \tilde{X}(U_{\delta,j})\int_{U_{\delta,i}} \lambda(s)ds - \frac{\tilde{X}(W_{N_{n\delta}})}{N_{n\delta}}\sum_{j=1}^{N_{n\delta}}\int_{U_{\delta,j}} \lambda(s)ds\right). \qquad (5.29)
\end{aligned}$$

Since λ is locally integrable and for each integer k we have $|\Theta_k| \leq 2\tau$, which implies for each $\delta \in \Theta_k$ we have $|\delta - k\tau| \leq 2\tau$, then, uniformly in j ($j = 1, 2, \ldots, N_{n\delta}$), we have

$$\int_{U_{\delta,j}} \lambda(s)ds = \int_{U_{k\tau,j}} \lambda(s)ds + \mathcal{O}(1) = k\theta\tau + \mathcal{O}(1), \qquad (5.30)$$

as $n \to \infty$, uniformly for all $\delta \in \Theta_k$. Now we consider the third term on the r.h.s. of (5.29). By (5.30), this term reduces to

$$\begin{aligned}
&\frac{2}{|W_n|}\left(\sum_{j=1}^{N_{n\delta}} \tilde{X}(U_{\delta,j})\left(k\theta\tau + \mathcal{O}(1)\right) - \tilde{X}(W_{N_{n\delta}})\left(k\theta\tau + \mathcal{O}(1)\right)\right) \\
&= \mathcal{O}\left(\frac{1}{|W_n|}\right)\tilde{X}(W_{N_{n\delta}}),
\end{aligned}$$

as $n \to \infty$. Hence we have

$$\begin{aligned}
\tilde{Q}_n(\delta) &= \frac{1}{|W_n|}\sum_{j=1}^{N_{n\delta}} \widetilde{\tilde{X}^2(U_{\delta,j})} - \frac{1}{|W_n|N_{n\delta}}\widetilde{\tilde{X}^2(W_{N_{n\delta}})} \\
&+ \mathcal{O}\left(\frac{1}{|W_n|}\right)\tilde{X}(W_{N_{n\delta}}), \qquad (5.31)
\end{aligned}$$

as $n \to \infty$.

By (5.31) with δ replaced by δ_i, to prove (5.24), it suffices to show

$$\sum_{i=1}^{L_n} \mathbf{P}\left(\left|\sum_{j=1}^{N_{n\delta_i}} \widetilde{\tilde{X}^2(U_{\delta_i,j})}\right| > \frac{\epsilon}{3}|W_n|^{1+c-2\gamma}\right) \leq \frac{C}{3}\exp\left\{-\alpha_\epsilon |W_n|^{\psi(c,\gamma)}\right\}, \quad (5.32)$$

$$\sum_{i=1}^{L_n} \mathbf{P}\left(\left|\widetilde{\tilde{X}^2(W_{N_{n\delta_i}})}\right| > \frac{\epsilon}{3}|W_n|^{1+c-2\gamma}N_{n\delta_i}\right) \le \frac{C}{3}\exp\left\{-\alpha_\epsilon|W_n|^{\psi(c,\gamma)}\right\}, \quad (5.33)$$

and

$$\sum_{i=1}^{L_n} \mathbf{P}\left(\left|\tilde{X}(W_{N_{n\delta_i}})\right| > \frac{\epsilon}{3}|W_n|^{1+c-2\gamma}\right) \le \frac{C}{3}\exp\left\{-\alpha_\epsilon|W_n|^{\psi(c,\gamma)}\right\}. \quad (5.34)$$

First we consider (5.32). The idea is to apply Lemma A.2 (see Appendix) with $K = k_n^2$. To do this, we first have to find an appropriate upper bound, that is the r.h.s. of (6.2) when $Y_j = \widetilde{\tilde{X}^2(U_{\delta_i,j})}$ and $K = k_n^2$. By (5.202) we obtain

$$\begin{aligned} \mathbf{E}\exp\left\{\left(\widetilde{\tilde{X}^2(U_{\delta_i,j})}\right)^2\right\} &= \sum_{l=0}^{\infty}\mathbf{E}\frac{\left(\widetilde{\tilde{X}^2(U_{\delta_i,j})}\right)^{2l}}{l!} = \sum_{l=0}^{\infty}\frac{\mathcal{O}(k_n^{2l})}{l!} \\ &\le C_1\sum_{l=0}^{\infty}\frac{\left(k_n^2\right)^l}{l!} = C_1\exp\left\{k_n^2\right\}, \end{aligned} \quad (5.35)$$

uniformly in j, where C_1 is a positive constant. This easily implies

$$\max_{j=1,\ldots,N_{n\delta_i}} k_n^2\left(\mathbf{E}\exp\left\{\left(\widetilde{\tilde{X}^2(U_{\delta_i,j})}\right)^2 k_n^{-2}\right\} - 1\right) \le C_2k_n^2, \quad (5.36)$$

for some positive constant C_2. Then, by Lemma A.2, and by noting that $k_n \sim |W_n|^c$ and $N_{n\delta_i} \sim |W_n|^{1-c}$ up to a constant uniformly in i, the probability appearing on the l.h.s. of (5.32) does not exceed

$$2\exp\left\{-\frac{\epsilon^2|W_n|^{2+2c-4\gamma}}{72(k_n^2 + C_2k_n^2)N_{n\delta_i}}\right\} \le 2\exp\left\{-\epsilon^2C_3|W_n|^{1+c-4\gamma}\right\}, \quad (5.37)$$

uniformly in i, for some positive constant C_3. By (5.37), the l.h.s. of (5.32) does not exceed

$$\begin{aligned} &2\exp\left\{\beta\log|W_n| - \epsilon^2C_3|W_n|^{1+c-4\gamma}\right\} \\ &\le 2\exp\left\{-\frac{C_3}{2}\epsilon^2|W_n|^{1+c-4\gamma}\right\}, \end{aligned} \quad (5.38)$$

for sufficiently large n. Since $\psi(c,\gamma) \le 1 + c - 4\gamma$, by (5.38), we obtain (5.32).

Next we prove (5.33). Since $N_{n\delta_i} \sim |W_n|^{1-c}$ and $\mathbf{E}X(W_{N_{n\delta_i}}) \sim |W_n|$ up to a constant, uniformly in i, for sufficiently large n, a simple calculation shows that the probability appearing on the l.h.s. of (5.33) does not exceed

$$\mathbf{P}\left(\left|\tilde{X}^2(\widetilde{W_{N_{n\delta_i}}})\right| > \epsilon C_4 |W_n|^{2-2\gamma}\right)$$
$$\leq \mathbf{P}\left(\frac{|X(W_{N_{n\delta_i}}) - \mathbf{E}X(W_{N_{n\delta_i}})|}{\sqrt{\mathbf{E}X(W_{N_{n\delta_i}})}} > \epsilon^{1/2} C_5 |W_n|^{1/2-\gamma}\right), \quad (5.39)$$

where C_4 and C_5 are some positive constants. Note here that, in order to have (5.39) holds true, we require that $\gamma < \frac{1}{2}$. But, this requirement is implied by condition $\gamma < \frac{1}{4} + \frac{c}{4}$ with $c < 1$. By Lemma A.1, the r.h.s. of (5.39) does not exceed

$$2\exp\left\{-\frac{\epsilon C_5^2 |W_n|^{1-2\gamma}}{2 + \epsilon^{1/2} C_5 |W_n|^{1/2-\gamma} (\mathbf{E}X(W_{N_{n\delta_i}}))^{-1/2}}\right\}$$
$$\leq 2\exp\left\{-\epsilon C_6 |W_n|^{1-2\gamma}\right\}, \quad (5.40)$$

uniformly in i, for some positive constant C_6 and for sufficiently large n. By (5.40), the l.h.s. of (5.33) does not exceed

$$2\exp\left\{\beta \log|W_n| - \epsilon C_6 |W_n|^{1-2\gamma}\right\}$$
$$\leq 2\exp\left\{-\frac{\epsilon C_6}{2}|W_n|^{1-2\gamma}\right\}, \quad (5.41)$$

for sufficiently large n. Since $\psi(c,\gamma) \leq 1-2\gamma$, by (5.41), we obtain (5.33).

Next we prove (5.34). Since $\mathbf{E}X(W_{N_{n\delta_i}}) \sim |W_n|$ up to a constant, the probability appearing on the l.h.s. of (5.34) does not exceed

$$\mathbf{P}\left(\frac{|X(W_{N_{n\delta_i}}) - \mathbf{E}X(W_{N_{n\delta_i}})|}{\sqrt{\mathbf{E}X(W_{N_{n\delta_i}})}} > \epsilon C_7 |W_n|^{1/2+c-2\gamma}\right)$$
$$\leq 2\exp\left\{-\frac{\epsilon^2 C_7^2 |W_n|^{1+2c-4\gamma}}{2 + \epsilon C_7 |W_n|^{1/2+c-2\gamma} (\mathbf{E}X(W_{N_{n\delta_i}}))^{-1/2}}\right\}$$
$$\leq 2\exp\left\{-\frac{\epsilon^2 C_7^2 |W_n|^{1+2c-4\gamma}}{2 + \epsilon C_8 |W_n|^{c-2\gamma}}\right\},$$
$$\leq 2\exp\left\{-\epsilon^2 C_9 |W_n|^{1+2c-4\gamma}\right\} + 2\exp\left\{-\epsilon C_{10} |W_n|^{1+c-2\gamma}\right\}, \quad (5.42)$$

uniformly in i, by Lemma A.1, where $C_7 - C_{10}$ are some positive constants. Note that the first term on the r.h.s. of (5.42) corresponds to the case $c \leq 2\gamma$, while its second term corresponds to the case $c > 2\gamma$. By (5.42), the l.h.s. of (5.34) does not exceed

$$2\exp\left\{\beta \log|W_n| - \epsilon^2 C_9 |W_n|^{1+2c-4\gamma}\right\}$$
$$+ 2\exp\left\{\beta \log|W_n| - \epsilon C_{10} |W_n|^{1+c-2\gamma}\right\}$$

$$\leq 2\exp\left\{-\frac{\epsilon^2 C_9}{2}|W_n|^{1+2c-4\gamma}\right\}$$
$$+ 2\exp\left\{-\frac{\epsilon C_{10}}{2}|W_n|^{1+c-2\gamma}\right\}, \tag{5.43}$$

for sufficiently large n. Obviously $\psi(c,\gamma) \leq \min(1+2c-4\gamma, 1+c-2\gamma)$. Then, by (5.43), we obtain (5.34). Hence we have proved (5.24). This completes the proof of Lemma 5.5. □.

Lemma 5.6 *Suppose that λ is periodic (with period τ) and bounded. In addition, we assume that $k = k_n \sim |W_n|^c$, for some $0 \leq c < 1$. If $0 \leq \gamma < \frac{1}{2} - \frac{c}{2}$, then for each $\epsilon > 0$, there exists (large) positive constants C and n_0 such that*

$$\mathbf{P}\left(W^{\eta}_{n,L,k_n} \geq \epsilon |W_n|^{c-2\gamma}\right) \leq C\exp\left\{-\alpha_\epsilon |W_n|^{\frac{1}{2}(1+c-2\gamma)}\right\}, \tag{5.44}$$

with $L_n = |W_n|^\beta$ for some $\beta > 2$, for all $n \geq n_0$, where α_ϵ is a positive real number depending on ϵ.

Note that we require λ is bounded in this lemma (cf. the argument preceding (5.78)).

Proof: Recall that $\delta \in \Theta_k$ and $Q_n(\delta)$ can be written as that in (5.26). Then, by the triangle-inequality, we have that

$$W^{\eta}_{n,L,k} \leq \frac{1}{|W_n|}\max_{1\leq i\leq L}\sup_{\delta_{i-1}<\delta\leq\delta_i}\left|\sum_{j=1}^{N_{n\delta}} X^2(U_{\delta,j}) - \sum_{j=1}^{N_{n\delta_i}} X^2(U_{\delta_i,j})\right|$$
$$+\frac{1}{|W_n|}\max_{1\leq i\leq L}\sup_{\delta_{i-1}<\delta\leq\delta_i}\left|\frac{X^2(W_{N_{n\delta}})}{N_{n\delta}} - \frac{X^2(W_{N_{n\delta_i}})}{N_{n\delta_i}}\right|. \tag{5.45}$$

A simple calculation shows that the second term on the r.h.s. of (5.45) does not exceed

$$\frac{2\tau_{k,1}X(W_n)}{|W_n||[a_n+\tau_{k,1}, b_n-\tau_{k,1}]|}X(W_n\setminus[a_n+\tau_{k,1}, b_n-\tau_{k,1}])$$
$$+ \frac{\tau_{k,1}^2 X^2(W_n)}{|W_n||[a_n+\tau_{k,1}, b_n-\tau_{k,1}]|^2}\max_{1\leq i\leq L}\sup_{\delta_{i-1}<\delta\leq\delta_i}|N_{n\delta_i}-N_{n\delta}|. \tag{5.46}$$

Choose $\beta > 1$ so that $|W_n|L_n^{-1} \to 0$ as $n \to \infty$. Then we have

$$|N_{n\delta_i} - N_{n\delta}| \leq \left|\frac{|W_n|}{\delta_i} - \frac{|W_n|}{\delta}\right| + 2 = \frac{|\delta-\delta_i||W_n|}{\delta_i\delta} + 2 \sim |W_n|L_n^{-1}k^{-2} + 2$$

as $n \to \infty$. Then, for sufficiently large n, we have that

$$\max_{1\le i\le L} \sup_{\delta_{i-1}<\delta\le\delta_i} |N_{n\delta_i} - N_{n\delta}| \le 3. \tag{5.47}$$

Note also that $|[a_n + \tau_{k,1}, b_n - \tau_{k,1}]| \sim |W_n|$ and $\tau_{k,1} \sim |W_n|^c$ up to a constant. Then, for sufficiently large n, the quantity in (5.46) does not exceed

$$\begin{aligned} &C_1|W_n|^{c-2}X(W_n)X(W_n \setminus [a_n + \tau_{k,1}, b_n - \tau_{k,1}]) \\ &+C_2|W_n|^{2c-3}X^2(W_n), \end{aligned} \tag{5.48}$$

where C_1 and C_2 are some positive constants.

Next we consider the first term on the r.h.s. of (5.45). This term does not exceed

$$\begin{aligned} &\frac{1}{|W_n|}\max_{1\le i\le L} \sup_{\delta_{i-1}<\delta\le\delta_i} \left|\sum_{j=1}^{N_{n\delta_i}} \left(X^2(U_{\delta,j}) - X^2(U_{\delta_i,j})\right)\right| \\ + &\frac{1}{|W_n|}\max_{1\le i\le L} \sup_{\delta_{i-1}<\delta\le\delta_i} \sum_{j=N_{n\delta_i}+1}^{N_{n\delta_{i-1}}} X^2(U_{\delta,j})I(j \le N_{n\delta}). \end{aligned} \tag{5.49}$$

For each j, $j = N_{n\delta_i}+1, \ldots, N_{n\delta}$, we can find two disjoint adjacent intervals U_{δ_i,l_j} and U_{δ_i,l_j+1} such that $U_{\delta,j} \subset (U_{\delta_i,l_j} \cup U_{\delta_i,l_j+1})$, except perhaps one interval U_{δ_i,l_j+1} which corresponds to the interval $U_{\delta,j}$ near the end point of W_n, may has part outside W_n. Then, by a simple monotonicity argument, the second term of (5.49) does not exceed

$$\frac{2}{|W_n|}\max_{1\le i\le L} \sum_{j=N_{n\delta_i}+1}^{N_{n\delta_{i-1}}} \left(X^2(U_{\delta_i,l_j}) + X^2(U_{\delta_i,l_j+1})\right). \tag{5.50}$$

Next, consider the first term of (5.49). Let us number the intervals $U_{\delta,j}$ and $U_{\delta_i,j}$ for $j = 1, \ldots, N_{n\delta_i}$ from left to right, and let $U_{\delta,1}$ and $U_{\delta_i,1}$ having the same left-end-point. For each j, $j = 1, \ldots, N_{n\delta_i}$, let $\bar{U}_{j\eta,j}$ denotes interval of length $j\eta$ having the same right-end-point as $U_{\delta_i,j}$, and let also $\bar{U}_{0,0}$ be a point at the left-end-point of $U_{\delta_i,1}$. We need this construction because of the the geometric situation. Then, for each j, we have that $U_{\delta,j} \subseteq (U_{\delta_i,j} \cup \bar{U}_{(j-1)\eta,j-1})$ and $(U_{\delta_i,j} \setminus \bar{U}_{j\eta,j}) \subseteq U_{\delta,j}$. This implies

$$\begin{aligned} &\sum_{j=1}^{N_{n\delta_i}} \left(X(U_{\delta_i,j}) - X(\bar{U}_{j\eta,j})\right)^2 \le \sum_{j=1}^{N_{n\delta_i}} X^2(U_{\delta,j}) \\ \le &\sum_{j=1}^{N_{n\delta_i}} \left(X(U_{\delta_i,j}) + X(\bar{U}_{(j-1)\eta,j-1})\right)^2. \end{aligned}$$

Now, for each j, let $\bar{U}_{\tau_{k,1},j}$ denotes interval of length $\tau_{k,1}$, having the same centre as $U_{\delta_i,j}$. Then, the first term of (5.49) does not exceed

$$\begin{aligned}
&\frac{1}{|W_n|}\max_{1\le i\le L}\left|\sum_{j=1}^{N_{n\delta_i}} X^2(\bar{U}_{j\eta,j}) - 2\sum_{j=1}^{N_{n\delta_i}} X(U_{\delta_i,j})X(\bar{U}_{j\eta,j})\right| \\
&+\frac{1}{|W_n|}\max_{1\le i\le L}\left|\sum_{j=1}^{N_{n\delta_i}} X^2(\bar{U}_{(j-1)\eta,j-1}) + 2\sum_{j=1}^{N_{n\delta_i}} X(U_{\delta_i,j})X(\bar{U}_{(j-1)\eta,j-1})\right| \\
\le\ &\frac{2}{|W_n|}\sum_{j=1}^{N_{n\tau_{k,0}}} X^2(\bar{U}_{j\eta,j}) + \frac{2}{|W_n|}\sum_{j=1}^{N_{n\tau_{k,0}}} X(\bar{U}_{\tau_{k,1},j})X(\bar{U}_{j\eta,j}) \\
&+\frac{2}{|W_n|}\sum_{j=1}^{N_{n\tau_{k,0}}} X(\bar{U}_{\tau_{k,1},j})X(\bar{U}_{(j-1)\eta,j-1}),
\end{aligned} \tag{5.51}$$

where we put $X(\bar{U}_{j\eta,j}) = 0$ and $X(\bar{U}_{\tau_{k,1},j}) = 0$ if $j > N_{n\delta_i}$.

Therefore, for sufficiently large n, $W^{\eta}_{n,L,k}$ does not exceed sum of the quantity in (5.48), (5.50), and the r.h.s. of (5.51). Recall our notation $\hat{\theta}_n = X(W_n)|W_n|^{-1}$ (cf. (2.1)). Then, to prove (5.44), it suffices now to show, for sufficiently large n,

$$\begin{aligned}
&\mathbf{P}\left(\hat{\theta}_n X(W_n\setminus[a_n+\tau_{k,1}, b_n-\tau_{k,1}]) \ge \frac{\epsilon}{6C_1}|W_n|^{1-2\gamma}\right) \\
&\le \frac{C}{6}\exp\left\{-\alpha_\epsilon |W_n|^{\frac{1}{2}(1+c-2\gamma)}\right\},
\end{aligned} \tag{5.52}$$

$$\mathbf{P}\left(X^2(W_n) \ge \frac{\epsilon}{6C_2}|W_n|^{3-c-2\gamma}\right) \le \frac{C}{6}\exp\left\{-\alpha_\epsilon |W_n|^{\frac{1}{2}(1+c-2\gamma)}\right\}, \tag{5.53}$$

$$\begin{aligned}
&\mathbf{P}\left(\max_{1\le i\le L}\sum_{j=N_{n\delta_i}+1}^{N_{n\delta_{i-1}}} (X^2(U_{\delta_i,l_j}) + X^2(U_{\delta_i,l_j+1})) \ge \frac{\epsilon}{12}|W_n|^{1+c-2\gamma}\right) \\
&\le \frac{C}{6}\exp\left\{-\alpha_\epsilon |W_n|^{\frac{1}{2}(1+c-2\gamma)}\right\},
\end{aligned} \tag{5.54}$$

$$\mathbf{P}\left(\sum_{j=1}^{N_{n\tau_{k,0}}} X^2(\bar{U}_{j\eta,j}) \ge \frac{\epsilon}{12}|W_n|^{1+c-2\gamma}\right) \le \frac{C}{6}\exp\left\{-\alpha_\epsilon |W_n|^{\frac{1}{2}(1+c-2\gamma)}\right\}, \tag{5.55}$$

$$\mathbf{P}\left(\sum_{j=1}^{N_{n\tau_{k,0}}} X(\bar{U}_{\tau_{k,1},j})X(\bar{U}_{j\eta,j}) \ge \frac{\epsilon}{12}|W_n|^{1+c-2\gamma}\right)$$

$$\leq \frac{C}{6} \exp\left\{-\alpha_\epsilon |W_n|^{\frac{1}{2}(1+c-2\gamma)}\right\}, \tag{5.56}$$

and

$$\mathbf{P}\left(\sum_{j=1}^{N_{n\tau_{k,0}}} X(\bar{U}_{\tau_{k,1},j})X(\bar{U}_{(j-1)\eta,j-1}) \geq \frac{\epsilon}{12}|W_n|^{1+c-2\gamma}\right)$$
$$\leq \frac{C}{6} \exp\left\{-\alpha_\epsilon |W_n|^{\frac{1}{2}(1+c-2\gamma)}\right\}. \tag{5.57}$$

Here we will only give the proofs of (5.52), (5.53), (5.54), and (5.56), since proofs of (5.55) and (5.57) are similar to and easier than the proof of (5.56).

First we consider (5.52). Let $A_{n,\tau_{k,1}} = W_n \setminus [a_n + \tau_{k,1}, b_n - \tau_{k,1}] = [a_n, a_n + \tau_{k,1}) \cup (b_n - \tau_{k,1}, b_n]$, and note that $|A_{n,\tau_{k,1}}| = 2\tau_{k,1} \sim |W_n|^c$ up to a constant, as $n \to \infty$. Then, the l.h.s. of (5.52) does not exceed

$$\mathbf{P}\left(\hat{\theta}_n X(A_{n,\tau_{k,1}}) > \frac{\epsilon}{6C_1}|W_n|^{1-2\gamma} \ \cap \ \hat{\theta}_n \leq 2\theta\right)$$
$$+\mathbf{P}\left(\hat{\theta}_n X(A_{n,\tau_{k,1}}) > \frac{\epsilon}{6C_1}|W_n|^{1-2\gamma} \ \cap \ \hat{\theta}_n > 2\theta\right)$$
$$\leq \mathbf{P}\left(2\theta X(A_{n,\tau_{k,1}}) > \frac{\epsilon}{6C_1}|W_n|^{1-2\gamma}\right) + \mathbf{P}\left(\hat{\theta}_n > 2\theta\right). \tag{5.58}$$

From the proof of Lemma 2.3 it can easily inferred that

$$\mathbf{P}\left(\hat{\theta}_n > 2\theta\right) \leq \mathbf{P}\left(\left|\hat{\theta}_n - \theta\right| > \theta\right) \leq 2\exp\left\{-\frac{\theta|W_n|}{18}\right\}. \tag{5.59}$$

Since $\frac{1}{2}(1+c-2\gamma) < 1$, for sufficiently large n, the r.h.s. of (5.59) does not exceed $\frac{C}{12}\exp\left\{-\alpha_\epsilon|W_n|^{\frac{1}{2}(1+c-2\gamma)}\right\}$. Then, to prove (5.52), it remains to show

$$\mathbf{P}\left(X(A_{n,\tau_{k,1}}) > \frac{\epsilon}{12\theta C_1}|W_n|^{1-2\gamma}\right) \leq \frac{C}{12}\exp\left\{-\alpha_\epsilon |W_n|^{\frac{1}{2}(1+c-2\gamma)}\right\}. \tag{5.60}$$

To verify (5.60), we argue as follows. The l.h.s. of (5.60) does not exceed

$$\mathbf{P}\left(\frac{|X(A_{n,\tau_{k_n,1}}) - \mathbf{E}X(A_{n,\tau_{k_n,1}})|}{\sqrt{\mathbf{E}X(A_{n,\tau_{k_n,1}})}}\right.$$
$$\left. > \frac{\epsilon|W_n|^{1-2\gamma}}{12\theta C_1\sqrt{\mathbf{E}X(A_{n,\tau_{k_n,1}})}} - \sqrt{\mathbf{E}X(A_{n,\tau_{k_n,1}})}\right). \tag{5.61}$$

Since $|A_{n,\tau_{k_n,1}}| \sim |W_n|^c$, we have $\mathbf{E}X(A_{n,\tau_{k_n,1}}) \sim |W_n|^c$ up to a constant, as $n \to \infty$. This implies $|W_n|^{1-2\gamma}(\mathbf{E}X(A_{n,\tau_{k_n,1}}))^{-1/2} \sim |W_n|^{1-2\gamma-c/2}$

and $(\mathbf{E}X(A_{n,\tau_{k_n},1}))^{1/2} \sim |W_n|^{c/2}$, up to a constant. In order to have the probability in (5.61) converges to zero, we require $1 - 2\gamma - \frac{c}{2} > \frac{c}{2}$, which is equivalent to the condition $\gamma < \frac{1}{2} - \frac{c}{2}$. (Hence, this is a necessary condition to prove convergence of the probability in (5.61)). By this condition, for sufficiently large n, there exists a positive constant C_3 such that the probability in (5.61) does not exceed

$$\begin{aligned}
&\mathbf{P}\left(\frac{|X(A_{n,\tau_{k_n},1}) - \mathbf{E}X(A_{n,\tau_{k_n},1})|}{\sqrt{\mathbf{E}X(A_{n,\tau_{k_n},1})}} > \epsilon C_3 |W_n|^{1-2\gamma-c/2}\right) \\
&\leq 2\exp\left\{-\frac{\epsilon^2 C_3^2 |W_n|^{2-4\gamma-c}}{2 + \epsilon C_3 |W_n|^{1-2\gamma-c/2}(\mathbf{E}X(A_{n,\tau_{k_n},1}))^{-1/2}}\right\} \\
&\leq 2\exp\left\{-\frac{\epsilon^2 C_3^2 |W_n|^{2-4\gamma-c}}{2 + \epsilon C_4 |W_n|^{1-2\gamma-c}}\right\}, \qquad (5.62)
\end{aligned}$$

by Lemma A.1, where C_4 is a positive constant. Here we have used Lemma A.1 (see Appendix) . Since $\gamma < \frac{1}{2} - \frac{c}{2}$ which implies $1-2\gamma-c > 0$, for sufficiently large n, there exists a positive constant C_5 such that the r.h.s. of (5.62) does not exceed

$$2\exp\left\{-\epsilon C_5 |W_n|^{1-2\gamma}\right\} \leq 2\exp\left\{-\epsilon C_5 |W_n|^{\frac{1}{2}(1+c-2\gamma)}\right\}. \qquad (5.63)$$

We have the l.h.s. of (5.63) does not exceed its r.h.s. because, by condition $\gamma < \frac{1}{2} - \frac{c}{2}$, we have $\frac{1}{2}(1 + c - 2\gamma) < 1 - 2\gamma$. Hence, we have (5.60). Therefore, we have proved (5.52).

Next we prove (5.53). The probability appearing on the l.h.s. of (5.53) is equal to

$$\begin{aligned}
&\mathbf{P}\left(X(W_n) > \frac{\epsilon^{1/2}}{(6C_2)^{1/2}} |W_n|^{3/2-\gamma-c/2}\right) \\
&\leq \mathbf{P}\left(\frac{|X(W_n) - \mathbf{E}X(W_n)|}{\sqrt{\mathbf{E}X(W_n)}} > \frac{\epsilon^{1/2}|W_n|^{3/2-\gamma-c/2}}{(6C_2)^{1/2}\sqrt{\mathbf{E}X(W_n)}} - \sqrt{\mathbf{E}X(W_n)}\right). \qquad (5.64)
\end{aligned}$$

By the condition $\gamma < \frac{1}{2} - \frac{c}{2}$, which implies $1 - \gamma - \frac{c}{2} > \frac{1}{2}$, we have that $(\mathbf{E}X(W_n))^{1/2}$ is of smaller order than $|W_n|^{3/2-\gamma-c/2}(\mathbf{E}X(W_n))^{-1/2}$. Then by a similar argument as the one in (5.62), for sufficiently large n, there exists a positive constant C_6 such that the r.h.s. of (5.64) does not exceed

$$\begin{aligned}
&\mathbf{P}\left(\frac{|X(W_n) - \mathbf{E}X(W_n)|}{\sqrt{\mathbf{E}X(W_n)}} > \epsilon^{1/2} C_6 |W_n|^{1-\gamma-c/2}\right) \\
&\leq 2\exp\left\{-\frac{\epsilon C_6^2 |W_n|^{2-2\gamma-c}}{2 + \epsilon^{1/2} C_6 |W_n|^{1-\gamma-c/2}(\mathbf{E}X(W_n))^{-1/2}}\right\} \\
&\leq 2\exp\left\{-\frac{\epsilon C_6^2 |W_n|^{2-2\gamma-c}}{2 + \epsilon^{1/2} C_7 |W_n|^{1/2-\gamma-c/2}}\right\}, \qquad (5.65)
\end{aligned}$$

by Lemma A.1, where C_7 is a positive constant. Since $\gamma < \frac{1}{2} - \frac{c}{2}$ which implies $\frac{1}{2} - \gamma - \frac{c}{2} > 0$, we have $|W_n|^{1/2-\gamma-c/2} \to \infty$ as $n \to \infty$. Then, the r.h.s. of (5.65) does not exceed

$$2\exp\left\{-\epsilon^{1/2}C_8|W_n|^{3/2-\gamma-c/2}\right\} \le 2\exp\left\{-\epsilon^{1/2}C_8|W_n|^{\frac{1}{2}(1+c-2\gamma)}\right\}, \quad (5.66)$$

for some positive constant C_8, since $\frac{1}{2}(1+c-2\gamma) < \frac{3}{2} - \gamma - \frac{c}{2}$.

Next we prove (5.54). By Bonferroni inequality, the l.h.s. of (5.54) does not exceed

$$\begin{aligned}
&\sum_{i=1}^{L_n}\mathbf{P}\left(\sum_{j=N_{n\delta_i}+1}^{N_{n\delta_{i-1}}}\left(X^2(U_{\delta_i,l_j}) + X^2(U_{\delta_i,l_j+1})\right) > \frac{\epsilon}{12}|W_n|^{1+c-2\gamma}\right)\\
\le\; &\sum_{i=1}^{L_n}\mathbf{P}\left(\sum_{j=N_{n\delta_i}+1}^{N_{n\delta_{i-1}}} X^2(U_{\delta_i,l_j}) > \frac{\epsilon}{24}|W_n|^{1+c-2\gamma}\right)\\
+\; &\sum_{i=1}^{L_n}\mathbf{P}\left(\sum_{j=N_{n\delta_i}+1}^{N_{n\delta_{i-1}}} X^2(U_{\delta_i,l_j+1}) > \frac{\epsilon}{24}|W_n|^{1+c-2\gamma}\right). \qquad (5.67)
\end{aligned}$$

Then to prove (5.54), it suffices to show that each term on the r.h.s. of (5.67) does not exceed

$$\frac{C}{12}\exp\left\{-\alpha_\epsilon|W_n|^{\frac{1}{2}(1+c-2\gamma)}\right\}. \quad (5.68)$$

Here we only give the proof that the first term on the r.h.s. of (5.67) does not exceed the quantity in (5.68), since the proof of the other term is similar and require the same condition.

To check the first term on the r.h.s. of (5.67) does not exceed the quantity in (5.68), we argue as follows. Because of (5.47) we know that $|N_{n\delta_{i-1}} - N_{n\delta_i}| \le 3$, for sufficiently large n. Here we only consider the case $N_{n\delta_{i-1}} = N_{n\delta_i} + 3$, because the other two cases ($N_{n\delta_{i-1}} = N_{n\delta_i} + 2$ and $N_{n\delta_{i-1}} = N_{n\delta_i} + 1$) can be treated similarly. For this case, the first term on the r.h.s. of (5.67) does not exceed

$$\begin{aligned}
&\sum_{i=1}^{L_n}\mathbf{P}\left(X^2(U_{\delta_i,l_{N_{n\delta_i}+1}}) > \frac{\epsilon}{72}|W_n|^{1+c-2\gamma}\right)\\
&+\sum_{i=1}^{L_n}\mathbf{P}\left(X^2(U_{\delta_i,l_{N_{n\delta_i}+2}}) > \frac{\epsilon}{72}|W_n|^{1+c-2\gamma}\right)\\
&+\sum_{i=1}^{L_n}\mathbf{P}\left(X^2(U_{\delta_i,l_{N_{n\delta_i}+3}}) > \frac{\epsilon}{72}|W_n|^{1+c-2\gamma}\right) \qquad (5.69)
\end{aligned}$$

Then, to prove that the first term on the r.h.s. of (5.67) does not exceed the quantity in (5.68), it suffices to show that each term of (5.69) does not exceed 1/3 of the quantity in (5.68). Here we only give the proof showing that the first term of (5.69) does not exceed 1/3 of the quantity in (5.68), because the proofs for the other terms are similar and require the same condition.

The probability appearing in the first term of (5.69) is equal to

$$\begin{aligned}
&\mathbf{P}\left(X(U_{\delta_i,l_{N_{n\delta_i}}+1}) > \sqrt{\frac{\epsilon}{72}}|W_n|^{1/2+c/2-\gamma}\right) \\
\leq\ &\mathbf{P}\left(\frac{|X(U_{\delta_i,l_{N_{n\delta_i}}+1}) - \mathbf{E}X(U_{\delta_i,l_{N_{n\delta_i}}+1})|}{\sqrt{\mathbf{E}X(U_{\delta_i,l_{N_{n\delta_i}}+1})}}\right. \\
&\left. > \frac{\epsilon^{1/2}|W_n|^{1/2+c/2-\gamma}}{\sqrt{72\mathbf{E}X(U_{\delta_i,l_{N_{n\delta_i}}+1})}} - \sqrt{\mathbf{E}X(U_{\delta_i,l_{N_{n\delta_i}}+1})}\right). \qquad (5.70)
\end{aligned}$$

Note that $(\mathbf{E}X(U_{\delta_i,l_{N_{n\delta_i}}+1}))^{1/2} \sim |W_n|^{c/2}$ and $|W_n|^{1/2+c/2-\gamma}(\mathbf{E}X(U_{\delta_i,l_{N_{n\delta_i}}+1}))^{-1/2} \sim |W_n|^{1/2-\gamma}$ up to a constant, as $n \to \infty$. Since $\gamma < \frac{1}{2} - \frac{c}{2}$ so that $\frac{c}{2} < \frac{1}{2} - \gamma$, $(\mathbf{E}X(U_{\delta_i,l_{N_{n\delta_i}}+1}))^{1/2}$ is of smaller order than $|W_n|^{1/2+c/2-\gamma}(\mathbf{E}X(U_{\delta_i,k_{N_{n\delta_i}}+1}))^{-1/2}$. Then for sufficiently large n, there exists a positive constant C_9 such that the probability on the r.h.s. of (5.70) does not exceed

$$\begin{aligned}
&\mathbf{P}\left(\frac{|X(U_{\delta_i,l_{N_{n\delta_i}}+1}) - \mathbf{E}X(U_{\delta_i,l_{N_{n\delta_i}}+1})|}{\sqrt{\mathbf{E}X(U_{\delta_i,l_{N_{n\delta_i}}+1})}} > \epsilon^{1/2}C_9|W_n|^{1/2-\gamma}\right) \\
&\leq 2\exp\left\{-\frac{\epsilon C_9^2|W_n|^{1-2\gamma}}{2+\epsilon^{1/2}C_9|W_n|^{1/2-\gamma}(\mathbf{E}X(U_{\delta_i,l_{N_{n\delta_i}}+1}))^{-1/2}}\right\} \\
&\leq 2\exp\left\{-\frac{\epsilon C_9^2|W_n|^{1-2\gamma}}{2+\epsilon^{1/2}C_{10}|W_n|^{1/2-\gamma-c/2}}\right\}, \qquad (5.71)
\end{aligned}$$

by Lemma A.1, where C_{10} is a positive constant. Since $\gamma < \frac{1}{2} - \frac{c}{2}$, we have that $|W_n|^{1/2-\gamma-c/2} \to \infty$ as $n \to \infty$. Then, for sufficiently large n, there exists a positive constant C_{11} such that the r.h.s. of (5.71) does not exceed $2\exp\{-\epsilon^{1/2}C_{11}|W_n|^{1/2-\gamma+c/2}\}$. Note that we choose $L_n = |W_n|^\beta$ for some constant $\beta > 1$. Then, the first term of (5.69) does not exceed

$$\begin{aligned}
&2\exp\left\{\beta\log|W_n| - \epsilon^{1/2}C_{11}|W_n|^{\frac{1}{2}(1+c-2\gamma)}\right\} \\
&\leq 2\exp\left\{-\epsilon^{1/2}\frac{C_{11}}{2}|W_n|^{\frac{1}{2}(1+c-2\gamma)}\right\}, \qquad (5.72)
\end{aligned}$$

for sufficiently large n. Hence, we have proved (5.54).

Finally we prove (5.56). First note that

$$\sum_{j=1}^{N_{n\tau_k,0}} X(\bar{U}_{\tau_k,1,j})X(\bar{U}_{j\eta,j})$$

$$\leq \left(|W_n|^{\epsilon_0} \max_{j,\ 1\leq j\leq N_{n\tau_{k_n},0}} X(\bar{U}_{\tau_{k_n},1,j})\right)\left(|W_n|^{-\epsilon_0}\sum_{j=1}^{N_{n\tau_{k_n},0}} X(\bar{U}_{j\eta,j})\right), \quad (5.73)$$

where ϵ_0 is a real number such that $0 < \epsilon_0 < 1 + c - 2\gamma$. The optimal choice of ϵ_0 will be determined later. By (5.73), a simple calculation shows that the l.h.s. of (5.56) does not exceed

$$\mathbf{P}\left(\max_{j,\ 1\leq j\leq N_{n\tau_{k_n},0}} X(\bar{U}_{\tau_{k_n},1,j}) > \frac{\epsilon}{12}|W_n|^{1+c-2\gamma-\epsilon_0}\right)$$

$$+\mathbf{P}\left(\sum_{j=1}^{N_{n\tau_{k_n},0}} X(\bar{U}_{j\eta,j}) > |W_n|^{\epsilon_0}\right). \quad (5.74)$$

Then, to prove (5.56), it suffices to show that each term of (5.74) does not exceed the quantity in (5.68).

First we show that the first term of (5.74) does not exceed the quantity in (5.68). By Bonferroni inequality, the first term of (5.74) does not exceed

$$\sum_{j=1}^{N_{n\tau_{k_n},0}} \mathbf{P}\left(X(\bar{U}_{\tau_{k_n},1,j}) > \frac{\epsilon}{12}|W_n|^{1+c-2\gamma-\epsilon_0}\right)$$

$$\leq \sum_{j=1}^{N_{n\tau_{k_n},0}} \mathbf{P}\left(\frac{|X(\bar{U}_{\tau_{k_n},1,j}) - \mathbf{E}X(\bar{U}_{\tau_{k_n},1,j})|}{\sqrt{\mathbf{E}X(\bar{U}_{\tau_{k_n},1,j})}}\right.$$

$$\left. > \frac{\epsilon|W_n|^{1+c-2\gamma-\epsilon_0}}{12\sqrt{\mathbf{E}X(\bar{U}_{\tau_{k_n},1,j})}} - \sqrt{\mathbf{E}X(\bar{U}_{\tau_{k_n},1,j})}\right). \quad (5.75)$$

Note that $(\mathbf{E}X(\bar{U}_{\tau_{k_n},1,j}))^{1/2} \sim |W_n|^{c/2}$ up to a constant, as $n \to \infty$. Now choose $\epsilon_0 < 1 - 2\gamma$. Then we have $1 + \frac{c}{2} - 2\gamma - \epsilon_0 > \frac{c}{2}$, which implies that $(\mathbf{E}X(\bar{U}_{\tau_{k_n},1,j}))^{1/2}$ is smaller than $|W_n|^{1+c-2\gamma-\epsilon_0}(\mathbf{E}X(\bar{U}_{\tau_{k_n},1,j}))^{-1/2}$, for sufficiently large n. Then for sufficiently large n, there exists a positive constant C_{12} such that the r.h.s. of (5.75) does not exceed

$$\sum_{j=1}^{N_{n\tau_{k_n},0}} \mathbf{P}\left(\frac{|X(\bar{U}_{\tau_{k_n},1,j}) - \mathbf{E}X(\bar{U}_{\tau_{k_n},1,j})|}{\sqrt{\mathbf{E}X(\bar{U}_{\tau_{k_n},1,j})}} > \epsilon C_{12}|W_n|^{1+c/2-2\gamma-\epsilon_0}\right)$$

$$\leq 2 \sum_{j=1}^{N_{n\tau_{k_n},0}} \exp\left\{-\frac{\epsilon^2 C_{12}^2 |W_n|^{2+c-4\gamma-2\epsilon_0}}{2+\epsilon C_{12}|W_n|^{1+c/2-2\gamma-\epsilon_0}(\mathbf{E}X(\bar{U}_{\tau_{k_n},1,j}))^{-1/2}}\right\}$$

$$\leq 2N_{n\tau_{k_n},0} \exp\left\{-\frac{\epsilon^2 C_{12}^2 |W_n|^{2+c-4\gamma-2\epsilon_0}}{2+\epsilon C_{13}|W_n|^{1-2\gamma-\epsilon_0}}\right\}, \qquad (5.76)$$

by Lemma A.1, where C_{13} is a positive constant. Note that $N_{n\tau_{k_n},0} \sim |W_n|^{1-c}$ up to a constant, as $n \to \infty$. Since $\epsilon_0 < 1-2\gamma$, we have $|W_n|^{1-2\gamma-\epsilon_0} \to \infty$ as $n \to \infty$. Then, for sufficiently large n, there exists a positive constant C_{14} such that the r.h.s. of (5.76) does not exceed

$$\begin{aligned}
&2N_{n\tau_{k_n},0} \exp\left\{-\epsilon C_{14}|W_n|^{1+c-2\gamma-\epsilon_0}\right\} \\
&= 2\exp\left\{\log N_{n\tau_{k_n},0} - \epsilon C_{14}|W_n|^{1+c-2\gamma-\epsilon_0}\right\} \\
&\leq 2\exp\left\{-\frac{\epsilon C_{14}}{2}|W_n|^{1+c-2\gamma-\epsilon_0}\right\} \\
&= 2\exp\left\{-\frac{\epsilon C_{14}}{2}|W_n|^{\frac{1}{2}(1+c-2\gamma)}\right\}, \qquad (5.77)
\end{aligned}$$

by choosing $\epsilon_0 = \frac{1}{2}(1+c-2\gamma)$. Note that, since $\gamma < \frac{1}{2} - \frac{c}{2}$, this choice of ϵ_0 satisfies the condition $\epsilon_0 < 1-2\gamma$. Hence, we have proved that the first term of (5.74) does not exceed the quantity in (5.68).

Next we prove that the second term of (5.74), with $\epsilon_0 = \frac{1}{2}(1+c-2\gamma)$, does not exceed the quantity in (5.68). To do this, we argue as follows. Since $\beta > 1$ we have $\eta_n|W_n| \to 0$ as $n \to \infty$. This implies that, for sufficiently large n, the intervals $\bar{U}_{j\eta,j}$ and $\bar{U}_{k\eta,k}$ are disjoint, provided $j \neq k$. Let $A_{n,\eta} = \cup_{j=1}^{N_{n\tau_{k_n},0}} \bar{U}_{j\eta,j}$. Then we have $\sum_{j=1}^{N_{n\tau_{k_n},0}} X(\bar{U}_{j\eta,j}) = X(A_{n,\eta})$. Recall that $\eta = \eta_n \sim L_n^{-1} = |W_n|^{-\beta}$, and $N_{n\tau_{k_n},0} \sim |W_n|^{1-c}$ up to a constant, as $n \to \infty$. Then, we have

$$|A_{n,\eta}| = \sum_{j=1}^{N_{n\tau_{k_n},0}} j\eta = \eta \frac{N_{n\tau_{k_n},0}(N_{n\tau_{k_n},0}+1)}{2} \sim |W_n|^{2-2c-\beta},$$

up to a constant. Now we choose $\beta > 2-2c$ so that $|A_{n,\eta}| \downarrow 0$ as $n \to \infty$. This, together with the assumption that λ is bounded, implies $\mathbf{E}X(A_{n,\eta}) = \mathcal{O}(1)$, as $n \to \infty$. Note that only here we require the boundedness assumption on λ. Now, the probability in the second term of (5.74) with $\epsilon_0 = \frac{1}{2}(1+c-2\gamma)$, can be written as

$$\begin{aligned}
&\mathbf{P}\left(X(A_{n,\eta}) > |W_n|^{1/2+c/2-\gamma}\right) \\
&\leq \mathbf{P}\left(\frac{|X(A_{n,\eta}) - \mathbf{E}X(A_{n,\eta})|}{\sqrt{\mathbf{E}X(A_{n,\eta})}} > \frac{|W_n|^{1/2+c/2-\gamma}}{\sqrt{\mathbf{E}X(A_{n,\eta})}} - \sqrt{\mathbf{E}X(A_{n,\eta})}\right). \qquad (5.78)
\end{aligned}$$

Since $\gamma < \frac{1}{2} - \frac{c}{2}$, we have that $|W_n|^{1/2+c/2-\gamma} \to \infty$ as $n \to \infty$. Then, for sufficiently large n, the r.h.s. of (5.78) does not exceed

$$\begin{aligned} &\mathbf{P}\left(\frac{|X(A_{n,\eta}) - \mathbf{E}X(A_{n,\eta})|}{\sqrt{\mathbf{E}X(A_{n,\eta})}} > \frac{|W_n|^{1/2+c/2-\gamma}}{2\sqrt{\mathbf{E}X(A_{n,\eta})}}\right) \\ &\leq 2\exp\left\{-\frac{|W_n|^{1+c-2\gamma}}{4\mathbf{E}X(A_{n,\eta})\left(2 + (|W_n|^{1/2+c/2-\gamma})(2\mathbf{E}X(A_{n,\eta}))^{-1}\right)}\right\} \\ &\leq 2\exp\left\{-C_{15}|W_n|^{\frac{1}{2}(1+c-2\gamma)}\right\}, \end{aligned} \tag{5.79}$$

where C_{15} is a positive constant. Hence we have that the second term of (5.74) does not exceed the quantity in (5.68). Therefore, we have proved (5.56). This completes the proof of Lemma 5.5. □

5.4 Proof of Theorem 5.3

To prove (5.9), we have to show, for any $\gamma < \frac{1}{2}$ and each $\epsilon > 0$,

$$\mathbf{P}\left(|W_n|^{\gamma}\,|\hat{\tau}_{k,n} - \tau| > \epsilon\right) \to 0, \tag{5.80}$$

as $n \to \infty$. To prove (5.80), we argue as follows. For each integer k satisfying $k = k_n = o(|W_n|)$, define

$$k\hat{\tau}_{k,n,s} = \arg\min_{\delta \in \Theta_{k,n}} Q_n(\delta), \tag{5.81}$$

where $\Theta_{k,n} = (k\tau - \epsilon_n, k\tau + \epsilon_n)$ and ϵ_n is an arbitrary sequence of positive real numbers such that $\epsilon_n \to 0$ as $n \to \infty$. Then we have that the probability on the l.h.s. of (5.80) is equal to

$$\begin{aligned} &\mathbf{P}\left(\frac{|W_n|^{\gamma}}{k}\,|k\hat{\tau}_{k,n} - k\tau| > \epsilon\right) \\ &\leq \mathbf{P}\left(\frac{|W_n|^{\gamma}}{k}\,|k\hat{\tau}_{k,n} - k\tau| > \epsilon \;\wedge\; k\hat{\tau}_{k,n} = k\hat{\tau}_{k,n,s}\right) + \mathbf{P}\left(k\hat{\tau}_{k,n} \neq k\hat{\tau}_{k,n,s}\right) \\ &\leq \mathbf{P}\left(|W_n|^{\gamma-c}\,|k\hat{\tau}_{k,n,s} - k\tau| > \epsilon\right) + \mathbf{P}\left(k\hat{\tau}_{k,n} \neq k\hat{\tau}_{k,n,s}\right). \end{aligned}$$

Then, to prove (5.80), it suffices to check

$$\mathbf{P}\left(k\hat{\tau}_{k,n} \neq k\hat{\tau}_{k,n,s}\right) \to 0, \tag{5.82}$$

and for any $\epsilon > 0$,

$$\mathbf{P}\left(|W_n|^{\gamma-c}\,|k\hat{\tau}_{k,n,s} - k\tau| > \epsilon\right) \to 0, \tag{5.83}$$

as $n \to \infty$.

First we prove (5.82) by the following lemma.

Lemma 5.7 *Suppose that λ is periodic and bounded. In addition, we assume that λ satisfy (5.5). Then for each positive integer k satisfying $k = k_n \sim |W_n|^c$ for some $0 \leq c < \frac{1}{3}$, we have*

$$\mathbf{P}\left(k\hat{\tau}_{k,n} \neq k\hat{\tau}_{k,n,s}\right) \to 0, \tag{5.84}$$

as $n \to \infty$, where $\hat{\tau}_{k,n}$ and $\hat{\tau}_{k,n,s}$ is given respectively by (5.3) and (5.81).

Proof: The probability on the l.h.s. of (5.84) is equal to

$$\mathbf{P}\left(|k\hat{\tau}_{k,n} - k\tau| > \epsilon_n\right),$$

with $\epsilon_n \downarrow 0$ as given in (5.81) and the definition of $\Theta_{k,n}$ following it. Hence, to prove this lemma, it suffices to check (5.15) for the case $\gamma = c$ and $\epsilon = \epsilon_n$. Repeating the argument in the proof of (5.15), for this case, we find that it suffices to show, for all sufficiently small $\alpha_n > 0$,

$$\mathbf{P}\left(\min_{i,\ \delta_i \in \Theta_k \setminus B_{k_n,\epsilon_n}} Q_n(\delta_i) - Q_n(k_n\tau) < \alpha_n\right) \to 0, \tag{5.85}$$

and

$$\mathbf{P}\left(W^{\eta}_{n,L,k} \geq \alpha_n\right) \to 0, \tag{5.86}$$

as $n \to \infty$, where $B_{k_n,\epsilon_n} = (k_n\tau - \epsilon_n, k_n\tau + \epsilon_n)$.

First we consider (5.85). Following the proof of (5.18), the probability on the l.h.s. of (5.85) does not exceed the quantity in (5.20) with $B_{n,k}$ replaced by B_{k_n,ϵ_n}. By part (i) of Lemma 5.16 (cf. section 5.6), we have that there exist $\alpha_0 > 0$ and positive integer n_0 such that

$$\Lambda_n(\delta_i) - \Lambda_n(k_n\tau) > \alpha_0\epsilon_n^2 k_n^{-1} \sim \alpha_0\epsilon_n^2|W_n|^{-c}$$

for all $\delta_i \in \Theta_{k_n} \setminus B_{k_n,\epsilon_n}$ and all $n \geq n_0$. Note that here we require the assumption that λ satisfy (5.5). By choosing $\alpha_n = \alpha\epsilon_n^2|W_n|^{-c}$ with $0 < \alpha \leq \alpha_0/2$, we have that $\alpha_n - (\Lambda_n(\delta_i) - \Lambda_n(k_n\tau))$ is strictly negative, so that $|\alpha_n - (\Lambda_n(\delta_i) - \Lambda_n(k_n\tau))| \geq \alpha\epsilon_n^2|W_n|^{-c}$ for all $n > n_0$ and all δ_i outside set B_{k_n,ϵ_n}. Then, the probability on the r.h.s. of (5.20) with B_{n,k_n} replaced by B_{k_n,ϵ_n}, does not exceed

$$\sum_{i,\ \delta_i \in \Theta_{k_n} \setminus B_{k_n,\epsilon_n}} \mathbf{P}\left(\left|\tilde{Q}_n(\delta_i) - \tilde{Q}_n(k_n\tau)\right| > \alpha\epsilon_n^2|W_n|^{-c}\right)$$

$$\leq \sum_{i=1}^{L_n} \mathbf{P}\left(\left|\tilde{Q}_n(\delta_i)\right| > \frac{\alpha}{2}\epsilon_n^2|W_n|^{-c}\right) + L_n\mathbf{P}\left(\left|\tilde{Q}_n(k_n\tau)\right| > \frac{\alpha}{2}\epsilon_n^2|W_n|^{-c}\right) \tag{5.87}$$

Now we want to apply Lemma 5.5. Inspection of the proof of Lemma 5.5, we see that α_ϵ in this lemma is equal to $\min(C_1\epsilon, C_2\epsilon^2)$, where C_1, C_2 are some positive constants. Then, by Lemma 5.5 for the case $c = \gamma$ and $\epsilon = \epsilon_n^2$, we have the following result:
Suppose that λ is periodic (with period τ) and locally integrable. Then, for each integer k satisfying $k = k_n \sim |W_n|^c$, for some $0 \leq c < \frac{1}{3}$, and for each sequence $\epsilon_n \downarrow 0$, there exists (large) constants C_3 and n_0 such that

$$\sum_{i=1}^{L_n} \mathbf{P}\left(|\tilde{Q}_n(\delta_i)| > \epsilon_n^2 |W_n|^{-c}\right) \leq C_3 \exp\left\{-C_4 \epsilon_n^4 |W_n|^{1-3c}\right\}, \tag{5.88}$$

with $\delta_i \in \Theta_k$ and $L_n = |W_n|^\beta$ for some $\beta \geq 0$, for all $n \geq n_0$, where C_4 is a positive constant.

Now choose $\epsilon_n \to 0$ such that $\epsilon_n |W_n|^{(1-3c)/4} \to \infty$ as $n \to \infty$. Since $c < \frac{1}{3}$ which implies $1 - 3c > 0$, by (1.2) and (5.88), we have that the r.h.s. of (5.87) converges to zero as $n \to \infty$. Hence, we have proved (5.85).

Next we prove (5.86), with $\alpha_n = \alpha\epsilon_n^2 |W_n|^{-c}$. To do this, we will apply Lemma 5.6. Inspection of the proof of Lemma 5.6, we see that α_ϵ in this lemma is equal to $\min(C_5\epsilon^{1/2}, C_6\epsilon, C_7)$, where $C_5 - C_7$ are some positive constants. Then by Lemma 5.6 for the case $c = \gamma$ and $\epsilon = \epsilon_n^2$, we have the following result:
Suppose that λ is periodic (with period τ) and bounded. Then, for each integer k satisfying $k = k_n \sim |W_n|^c$ for some $0 \leq c < \frac{1}{3}$, and for each sequence $\epsilon_n \downarrow 0$ there exists (large) positive constants C_8 and n_0 such that

$$\mathbf{P}\left(W_{n,L,k_n}^\eta \geq \epsilon_n^2 |W_n|^{-c}\right) \leq C_8 \exp\left\{-C_9 \epsilon_n^2 |W_n|^{\frac{1}{2}(1-c)}\right\}, \tag{5.89}$$

with $L_n = |W_n|^\beta$ for some $\beta > 2$, for all $n \geq n_0$, where C_9 is a positive constant. (Note that $\gamma = c$ and $\gamma < \frac{1}{2} - \frac{c}{2}$ implies $c < \frac{1}{3}$).

Next we choose $\epsilon_n \to 0$ such that $\epsilon_n |W_n|^{\frac{1}{4}(1-c)} \to \infty$ as $n \to \infty$. But, since $(1 - 3c) \leq (1 - c)$ for $c \geq 0$, this requirement is implied by $\epsilon_n |W_n|^{(1-3c)/4} \to \infty$ as $n \to \infty$, which was already needed to establish (5.85). Then, by (1.2) and (5.89), we have (5.86). This completes the proof of Lemma 5.7. □

It remains to show (5.83). To prove (5.83) we require Lemmas 5.8 - 5.12. Recall our notation $\Lambda_n(\cdot) = \mathbf{E}Q_n(\cdot)$ and $\tilde{Q}_n(\cdot) = Q_n(\cdot) - \mathbf{E}Q_n(\cdot)$. As before, for any r.v. Y with finite expectation we denote $Y - \mathbf{E}Y$ by $\tilde{Y}$. We begin with establishing a stochastic expansion for $\tilde{Q}_n(\delta)$.

Lemma 5.8 *Suppose that λ is periodic and Lipschitz. Then, for any positive integer k satisfying $k = o(|W_n|)$ and for any δ in a neighborhood of $k\tau$, we have a stochastic expansion of $\tilde{Q}_n(\delta)$ as follows*

$$\tilde{Q}_n(\delta) = \frac{(\delta - k\tau)}{|W_n|^{1/2}} A_{n,\delta} + \frac{1}{|W_n|^{1/2}} B_{n,\delta} + O_p\left(\frac{(\delta - k\tau)^2}{|W_n|^{1/2}}\right) + \mathcal{O}_p\left(\frac{k}{|W_n|}\right), \quad (5.90)$$

as $n \to \infty$ and $|\delta - k\tau| \to 0$, where

$$A_{n,\delta} = \frac{2}{|W_n|^{1/2}} \sum_{i=1}^{N_{n\delta}} \tilde{X}(U_{\delta,i}) \left(c_{i,\delta,k\tau} - \bar{c}_{\cdot,\delta,k\tau}\right), \quad (5.91)$$

$$B_{n,\delta} = \frac{1}{|W_n|^{1/2}} \sum_{i=1}^{N_{n\delta}} \widetilde{\tilde{X}^2(U_{\delta,i})}, \quad (5.92)$$

$$c_{i,\delta,k\tau} = \lambda(a_n + r + (i-1)(\delta - k\tau)), \quad \text{and} \quad (5.93)$$

$$\bar{c}_{\cdot,\delta,k\tau} = (N_{n\delta})^{-1} \sum_{i=1}^{N_{n\delta}} c_{i,\delta,k\tau}. \quad (5.94)$$

Proof: First recall that (cf. (5.29))

$$\begin{aligned} \tilde{Q}_n(\delta) &= \frac{1}{|W_n|} \sum_{i=1}^{N_{n\delta}} \widetilde{\tilde{X}^2(U_{\delta,i})} - \frac{1}{|W_n| N_{n\delta}} \widetilde{\tilde{X}^2(W_{N_{n\delta}})} \\ &+ \frac{2}{|W_n|} \sum_{i=1}^{N_{n\delta}} \tilde{X}(U_{\delta,i}) \int_{U_{\delta,i}} \lambda(s)ds - \frac{2\tilde{X}(W_{N_{n\delta}}) \int_{W_{N_{n\delta}}} \lambda(s)ds}{|W_n| N_{n\delta}}. \end{aligned} \quad (5.95)$$

Note that the first term on the r.h.s. of (5.95) is equal to the second term on the r.h.s. of (5.90). The variance of the second term on the r.h.s. of (5.95) is equal to

$$\begin{aligned} & \frac{1}{|W_n|^2 N_{n\delta}^2} \mathbf{E}\left(\widetilde{\tilde{X}^2(W_{N_{n\delta}})}\right)^2 \\ = & \frac{1}{|W_n|^2 N_{n\delta}^2} \left\{ \mathbf{E}\tilde{X}^4(W_{N_{n\delta}}) - \left(\mathbf{E}\tilde{X}^2(W_{N_{n\delta}})\right)^2 \right\} \\ = & \frac{1}{|W_n|^2 N_{n\delta}^2} \left(\int_{W_{N_{n\delta}}} \lambda(s)ds + 2\left(\int_{W_{N_{n\delta}}} \lambda(s)ds\right)^2 \right) = \mathcal{O}\left(\frac{k^2}{|W_n|^2}\right), \end{aligned}$$

as $n \to \infty$. Hence, by Chebyshev's inequality, this term is of order $\mathcal{O}_p(k|W_n|^{-1})$ as $n \to \infty$. It remains to show that the sum of the third and fourth term on the r.h.s. of (5.95) is equal to the first term on the r.h.s. of (5.90) plus a remainder term of lower order.

Since λ is Lipschitz, we have (5.189). By (5.185) and (5.189), uniformly in i ($i = 1, 2, \ldots, N_{n\delta}$), we can write

$$\int_{U_{\delta,i}} \lambda(s)ds = k\theta\tau + (\delta - k\tau)\ c_{i,\delta,k\tau} + \mathcal{O}((\delta - k\tau)^2), \tag{5.96}$$

with $c_{i,\delta,k\tau}$ as in (5.93). By (5.96), the third term on the r.h.s. of (5.95) can be written as follows

$$\frac{2k\theta\tau}{|W_n|}\tilde{X}(W_{N_{n\delta}}) + \frac{2(\delta - k\tau)}{|W_n|}\sum_{i=1}^{N_{n\delta}} c_{i,\delta,k\tau}\tilde{X}(U_{\delta,i}) + \mathcal{O}\left(\frac{(\delta - k\tau)^2}{|W_n|^{1/2}}\right), \tag{5.97}$$

as $n \to \infty$, since by Chebyshev's inequality, we have $\sum_{i=1}^{N_{n\delta}} \tilde{X}(U_{\delta,i}) = \tilde{X}(W_{N_{n\delta}}) = \mathcal{O}_p(|W_n|^{1/2})$, as $n \to \infty$. Similarly, we write the last term on the r.h.s. of (5.95) as

$$-\frac{2k\theta\tau}{|W_n|}\tilde{X}(W_{N_{n\delta}}) - \frac{2(\delta - k\tau)}{|W_n|}\frac{\tilde{X}(W_{N_{n\delta}})}{N_{n\delta}}\sum_{i=1}^{N_{n\delta}} c_{i,\delta,k\tau} + \mathcal{O}\left(\frac{(\delta - k\tau)^2}{|W_n|^{1/2}}\right), \tag{5.98}$$

as $n \to \infty$. The sum of the quantities in (5.97) and (5.98) is equal to

$$\frac{2(\delta - k\tau)}{|W_n|}\sum_{i=1}^{N_{n\delta}} c_{i,\delta,k\tau}\left(\tilde{X}(U_{\delta,i}) - \frac{\tilde{X}(W_{N_{n\delta}})}{N_{n\delta}}\right) + \mathcal{O}\left(\frac{(\delta - k\tau)^2}{|W_n|^{1/2}}\right), \tag{5.99}$$

as $n \to \infty$. Note that the first term of (5.97) cancels against the first term of (5.98). A simple calculation shows that the first term of (5.99) is equal to $(\delta - k\tau)|W_n|^{-1/2}A_{n,\delta}$, while its second term is equal to the third term on the r.h.s. of (5.90). This completes the proof of Lemma 5.8. □

In the next two lemmas - Lemmas 5.9 and 5.10 - we obtain an asymptotic approximation to the variance of $A_{n,\delta}$ and $B_{n,\delta}$ respectively and also show that these terms are approximately normally distributed. The covariance between $A_{n,\delta}$ and $B_{n,\delta}$, and their joint normality is established in Lemma 5.11.

Lemma 5.9 *Consider $A_{n,\delta}$ as given in (5.91), and suppose that λ is periodic and bounded, and (5.4) holds.*

(i) Then for any positive integer k satisfying $k = o(|W_n|)$ and for any δ in a neighborhood of $k\tau$ we have

$$\frac{A_{n,\delta}}{\sigma(A_{n,\delta})} - N_1(0,1) \xrightarrow{p} 0, \tag{5.100}$$

as $n \to \infty$, where $N_1(0,1)$ denotes a standard normal r.v.

(ii) If in addition, λ is Lipschitz, then for any positive integer k satisfying $k = o(|W_n|)$ and for any δ in a neighborhood of $k\tau$ with $|W_n|k^{-1}|\delta - k\tau| \to \infty$ as $n \to \infty$, we have

$$\begin{aligned} Var(A_{n,\delta}) &= \frac{4\theta}{\tau}\int_0^\tau (\lambda(s)-\theta)^2 ds + \mathcal{O}\left(\frac{k}{|W_n|}\right) + \mathcal{O}(|\delta - k\tau|) \\ &\quad + \mathcal{O}\left(k|W_n|^{-1}|\delta - k\tau|^{-1}\right), \end{aligned} \tag{5.101}$$

as $n \to \infty$ and $|\delta - k\tau| \to 0$.

Proof: First we prove part (i) of this lemma. Note that for all $i \neq j$, $i, j = 1, \ldots, N_{n\delta}$ we have that $\tilde{X}(U_{\delta,i})$ and $\tilde{X}(U_{\delta,j})$ are independent. Furthermore, for each i, $i = 1, \ldots, N_{n\delta}$ and any δ in the neighborhood of $k\tau$, we also have $\mathbf{E}\tilde{X}(U_{\delta,i})\,(c_{i,\delta,\tau} - \bar{c}_{\cdot,\delta,\tau}) = 0$ and $Var(\tilde{X}(U_{\delta,i})(c_{i,\delta,\tau} - \bar{c}_{\cdot,\delta,\tau})) = (c_{i,\delta,\tau} - \bar{c}_{\cdot,\delta,\tau})^2 \int_{U_{\delta,i}} \lambda(s)ds$, which is finite, because λ is bounded. Now we notice that

$$\left(\sum_{i=1}^{N_{n\delta}} (c_{i,\delta,\tau} - \bar{c}_{\cdot,\delta,\tau})^2 \int_{U_{\delta,i}} \lambda(s)ds\right)^2 = \mathcal{O}(|W_n|^2),$$

as $n \to \infty$. Then, proving part (i) of this lemma, it suffices to check that the Lyapounov's condition

$$\sum_{i=1}^{N_{n\delta}} \mathbf{E}\left(\tilde{X}(U_{\delta,i})\,(c_{i,\delta,\tau} - \bar{c}_{\cdot,\delta,\tau})\right)^4 = o(|W_n|^2) \tag{5.102}$$

holds, as $n \to \infty$ (cf. Serfling (1980), p. 30). To prove (5.102) note that

$$\begin{aligned} &\sum_{i=1}^{N_{n\delta}} \mathbf{E}\left(\tilde{X}(U_{\delta,i})\,(c_{i,\delta,\tau} - \bar{c}_{\cdot,\delta,\tau})\right)^4 = \sum_{i=1}^{N_{n\delta}} (c_{i,\delta,\tau} - \bar{c}_{\cdot,\delta,\tau})^4\,\mathbf{E}\left(\tilde{X}(U_{\delta,i})\right)^4 \\ &= \sum_{i=1}^{N_{n\delta}} (c_{i,\delta,\tau} - \bar{c}_{\cdot,\delta,\tau})^4 \left(\int_{U_{\delta,i}} \lambda(s)ds + 3\left(\int_{U_{\delta,i}} \lambda(s)ds\right)^2\right) \\ &= \mathcal{O}(|W_n|k) = o(|W_n|^2) \end{aligned}$$

as $n \to \infty$, because $k = o(|W_n|)$ as $n \to \infty$. Hence we have (5.102), which implies part (i) of this lemma. Note that (5.4) implies that $\sigma(A_{n,\delta})$ is bounded away from zero, for all large n.

Next we prove part (ii) of this lemma. By (5.185), for each i, we can write $\int_{U_{\delta,i}} \lambda(s)ds = k\theta\tau + \mathcal{O}(|\delta - k\tau|)$, as $|\delta - k\tau| \to 0$ uniformly in n.

Since $\tilde{X}(U_{\delta,i})$ and $\tilde{X}(U_{\delta,j})$ are independent for all $i \neq j$; $i,j = 1,\ldots,N_{n\delta}$, we can compute $Var(A_{n,\delta})$ as follows.

$$\begin{aligned} Var(A_{n,\delta}) &= \frac{4}{|W_n|}\sum_{i=1}^{N_{n\delta}} \int_{U_{\delta,i}} \lambda(s)ds \ \left(c_{i,\delta,k\tau} - \bar{c}_{\cdot,\delta,k\tau}\right)^2 \\ &= \frac{4}{|W_n|}\sum_{i=1}^{N_{n\delta}} (k\theta\tau + \mathcal{O}(|\delta - k\tau|))\left(c_{i,\delta,k\tau} - \bar{c}_{\cdot,\delta,k\tau}\right)^2 \\ &= \frac{4k\theta\tau}{|W_n|}\sum_{i=1}^{N_{n\delta}} \left(c_{i,\delta,k\tau} - \bar{c}_{\cdot,\delta,k\tau}\right)^2 + \mathcal{O}\left(\frac{|\delta - k\tau|}{k}\right), \qquad (5.103) \end{aligned}$$

as $|\delta - k\tau| \to 0$ uniformly in n.

Let $J_{\delta,k\tau} = \left[\frac{\tau}{|\delta - k\tau|}\right]$, and an approximation for $J^{-1}_{\delta,k\tau}$ is given in (5.191). Since $|W_n|k^{-1}|\delta - k\tau| \to \infty$, we have $J_{\delta,k\tau} = o(N_{n\delta})$ as $n \to \infty$ and as $|\delta - k\tau| \to 0$. Because the intensity function λ is periodic with period τ, we can simplify $\bar{c}_{\cdot,\delta,k\tau}$ as follows

$$\begin{aligned} &\bar{c}_{\cdot,\delta,k\tau} \\ &= \frac{1}{N_{n\delta}}\sum_{i=1}^{N_{n\delta}} \lambda(a_n + r + (i-1)(\delta - k\tau)) \\ &= \frac{1}{N_{n\delta}}\left(\left[\frac{N_{n\delta}}{J_{\delta,k\tau}}\right]\sum_{i=1}^{J_{\delta,k\tau}} \lambda(a_n + r + (i-1)(\delta - k\tau)) + \mathcal{O}(J_{\delta,k\tau})\right) \\ &= \frac{1}{N_{n\delta}}\frac{N_{n\delta}}{J_{\delta,k\tau}}\sum_{i=1}^{J_{\delta,k\tau}} \lambda(a_n + r + (i-1)(\delta - k\tau)) + \mathcal{O}\left(\frac{J_{\delta,k\tau}}{N_{n\delta}}\right) \\ &= \left(\frac{|\delta - k\tau|}{\tau} + \mathcal{O}\left((\delta - k\tau)^2\right)\right)\sum_{i=1}^{J_{\delta,k\tau}} \lambda(a_n + r + (i-1)(\delta - k\tau)) + \mathcal{O}\left(\frac{J_{\delta,k\tau}}{N_{n\delta}}\right) \\ &= \frac{1}{\tau}\sum_{i=1}^{J_{\delta,k\tau}} |\delta - k\tau|\lambda(a_n + r + (i-1)(\delta - k\tau)) + \mathcal{O}\left(|\delta - k\tau|\right) + \mathcal{O}\left(\frac{J_{\delta,k\tau}}{N_{n\delta}}\right) \\ &= \frac{1}{\tau}\left(\int_0^\tau \lambda(s)ds + \mathcal{O}(|\delta - k\tau|)\right) + \mathcal{O}\left(|\delta - k\tau|\right) + \mathcal{O}\left(\frac{J_{\delta,k\tau}}{N_{n\delta}}\right) \\ &= \theta + \mathcal{O}\left(|\delta - k\tau|\right) + \mathcal{O}\left(\frac{k}{|W_n||\delta - k\tau|}\right), \qquad (5.104) \end{aligned}$$

as $n \to \infty$ and $|\delta - k\tau| \to 0$. Here we have used the assumption that λ is Lipschitz, and the error for the Riemann approximation is incorporated in the $\mathcal{O}(|\delta - k\tau|)$ remainder term. Replacing $\bar{c}_{\cdot,\delta,k\tau}$ on the r.h.s. of (5.103) by $\theta + \mathcal{O}(|\delta - k\tau|) + \mathcal{O}(k|W_n|^{-1}|\delta - k\tau|^{-1})$, the r.h.s. of (5.103) is equal to

$$\frac{4k\theta\tau N_{n\delta}}{|W_n|}\frac{1}{N_{n\delta}}\sum_{i=1}^{N_{n\delta}}\left(\lambda(a_n+r+(i-1)(\delta-k\tau))-\theta\right)^2$$
$$+\mathcal{O}(|\delta-k\tau|)+\mathcal{O}\left(k|W_n|^{-1}|\delta-k\tau|^{-1}\right), \tag{5.105}$$

as $n\to\infty$ and $|\delta-k\tau|\to 0$. Now we make the following approximation

$$\frac{4k\theta\tau N_{n\delta}}{|W_n|}=\frac{4k\theta\tau}{|W_n|}\left(\frac{|W_n|}{\delta}+\mathcal{O}(1)\right)$$
$$=4k\theta\tau\left(\frac{1}{k\tau}+\mathcal{O}\left(\frac{|\delta-k\tau|}{k^2}\right)\right)+\mathcal{O}\left(\frac{k}{|W_n|}\right)$$
$$=4\theta+\mathcal{O}\left(\frac{|\delta-k\tau|}{k}\right)+\mathcal{O}\left(\frac{k}{|W_n|}\right), \tag{5.106}$$

as $n\to\infty$ and $|\delta-k\tau|\to 0$. Substituting (5.192) and (5.106) into the first term on the r.h.s. of (5.105), then we get (5.101). This completes the proof of Lemma 5.9. □

Lemma 5.10 *Consider $B_{n,\delta}$ as given in (5.92), and suppose that λ is periodic and locally integrable, and (5.4) holds.*

(i) Then for any positive integer k satisfying $k=o(|W_n|)$ and for any δ in a neighborhood of $k\tau$ we have

$$\frac{B_{n,\delta}}{\sigma(B_{n,\delta})}-N_2(0,1)\xrightarrow{p}0, \tag{5.107}$$

as $n\to\infty$, where $N_2(0,1)$ denotes a standard normal r.v.

(ii) If in addition, λ is Lipschitz, then for any positive integer k satisfying $k=o(|W_n|)$ and for any δ in a neighborhood of $k\tau$ with $|W_n|k^{-1}|\delta-k\tau|\to\infty$ as $n\to\infty$, we have

$$Var(B_{n,\delta})=2\theta^2\delta+\theta+\mathcal{O}\left(\frac{k^2}{|W_n|}\right)+\mathcal{O}\left((\delta-k\tau)^2\right), \tag{5.108}$$

as $n\to\infty$ and $|\delta-k\tau|\to 0$.

Proof: First we prove part (i) of this lemma. Note that for all $i\neq j$, $i,j=1,\ldots,N_{n\delta}$ we have that $\widetilde{\tilde{X}^2(U_{\delta,i})}$ and $\widetilde{\tilde{X}^2(U_{\delta,j})}$ are independent. Note also that, for each i, $i=1,\ldots,N_{n\delta}$ and any δ in the neighborhood of $k\tau$, we also have $\mathbf{E}\widetilde{\tilde{X}^2(U_{\delta,i})}=0$ and an easy calculation shows that

$$Var\left(\widetilde{\tilde{X}^2(U_{\delta,i})}\right)=\int_{U_{\delta,i}}\lambda(s)ds+2\left(\int_{U_{\delta,i}}\lambda(s)ds\right)^2, \tag{5.109}$$

which is finite. By (5.109) we have that

$$\left(\sum_{i=1}^{N_{n\delta}} Var\left(\widetilde{\tilde{X}^2(U_{\delta,i})}\right)\right)^2 = \mathcal{O}(|W_n|^2k^2),$$

as $n \to \infty$. Then, to prove part (i) of this lemma, it suffices to show that the Lyapounov's condition

$$\sum_{i=1}^{N_{n\delta}} \mathbf{E}\left(\widetilde{\tilde{X}^2(U_{\delta,i})}\right)^4 = o(|W_n|^2k^2) \tag{5.110}$$

holds, as $n \to \infty$ (cf. Serfling (1980), p. 30). To check (5.110), we first compute, for each i, the following quantity

$$\begin{aligned}
\mathbf{E}\left(\widetilde{\tilde{X}^2(U_{\delta,i})}\right)^4 &= \mathbf{E}\left(\tilde{X}^2(U_{\delta,i}) - \int_{U_{\delta,i}} \lambda(s)ds\right)^4 \\
&= \mathbf{E}\tilde{X}^8(U_{\delta,i}) - 4\int_{U_{\delta,i}} \lambda(s)ds\mathbf{E}\tilde{X}^6(U_{\delta,i}) + 6\left(\int_{U_{\delta,i}} \lambda(s)ds\right)^2 \mathbf{E}\tilde{X}^4(U_{\delta,i}) \\
&\quad -4\left(\int_{U_{\delta,i}} \lambda(s)ds\right)^3 \mathbf{E}\tilde{X}^2(U_{\delta,i}) + \left(\int_{U_{\delta,i}} \lambda(s)ds\right)^4.
\end{aligned} \tag{5.111}$$

By Haight (1967) page 7, for any δ in the neighborhood of $k\tau$, we have that

$$\mathbf{E}\tilde{X}^4(U_{\delta,i}) = \mathcal{O}(k^2), \quad \mathbf{E}\tilde{X}^6(U_{\delta,i}) = \mathcal{O}(k^3), \quad \text{and} \quad \mathbf{E}\tilde{X}^8(U_{\delta,i}) = \mathcal{O}(k^4),$$

as $n \to \infty$, uniformly in i. Hence, the quantity in (5.111) is of order $\mathcal{O}(k^4)$ as $n \to \infty$, uniformly in i. Because $N_{n\delta} = \mathcal{O}(|W_n|k^{-1})$, the l.h.s. of (5.110) is of order $\mathcal{O}(|W_n|k^3)$, which is $o(|W_n|^2k^2)$ as $n \to \infty$, because $k = o(|W_n|)$ as $n \to \infty$. Hence we have (5.110), which implies part (i) of this lemma. Note that (5.4) ensures that $\theta > 0$, and hence $\sigma(B_{n,\delta})$ is positive for large n.

Next we prove part (ii) of this lemma. Since $B_{n,\delta}$ is a sum of independent random variables with expectation zero, by (5.109), its variance is equal to

$$Var(B_{n,\delta}) = \frac{1}{|W_n|}\sum_{i=1}^{N_{n\delta}} \int_{U_{\delta,i}} \lambda(s)ds + \frac{2}{|W_n|}\sum_{i=1}^{N_{n\delta}} \left(\int_{U_{\delta,i}} \lambda(s)ds\right)^2. \tag{5.112}$$

By Lemma 5.15 (see section 5.6), the first term on the r.h.s. of (5.112) is equal to

$$\frac{(|W_n| + \mathcal{O}(k))}{|W_n|}\frac{1}{|W_{N_{n\delta}}|}\int_{W_{N_{n\delta}}} \lambda(s)ds = \theta + \mathcal{O}\left(\frac{k}{|W_n|}\right), \tag{5.113}$$

as $n \to \infty$. Since λ is Lipschitz, for each i we can write $\int_{U_{\delta,i}} \lambda(s)ds$ as that in (5.96). By (5.96) and by noting that $c_{i,\delta,k\tau} = \lambda(a_n+r+(i-1)(\delta-k\tau))$, the second term on the r.h.s. of (5.112) can be written as

$$\begin{aligned} &\frac{2}{|W_n|}\sum_{i=1}^{N_{n\delta}}\left(k\theta\tau + (\delta-k\tau)\lambda(a_n+r+(i-1)(\delta-k\tau)) + \mathcal{O}((\delta-k\tau)^2)\right)^2 \\ &= \frac{2k^2\theta^2\tau^2 N_{n\delta}}{|W_n|} + \frac{4k\theta\tau(\delta-k\tau)N_{n\delta}}{|W_n|}\frac{1}{N_{n\delta}}\sum_{i=1}^{N_{n\delta}}\lambda(a_n+r+(i-1)(\delta-k\tau)) \\ &+ \frac{2(\delta-k\tau)^2 N_{n\delta}}{|W_n|}\frac{1}{N_{n\delta}}\sum_{i=1}^{N_{n\delta}}\lambda^2(a_n+r+(i-1)(\delta-k\tau)) + \mathcal{O}((\delta-k\tau)^2), \end{aligned} \tag{5.114}$$

as $|\delta-k\tau| \to 0$. For any δ in the neighborhood of $k\tau$ we can write

$$\frac{1}{\delta} = \frac{1}{k\tau} - \frac{(\delta-k\tau)}{k^2\tau^2} + \mathcal{O}\left(\frac{(\delta-k\tau)^2}{k^3}\right), \tag{5.115}$$

as $|\delta-k\tau| \to 0$. By (5.115), the first term on the r.h.s. of (5.114) can be written as

$$\begin{aligned} &\frac{2k^2\theta^2\tau^2}{|W_n|}\left(\frac{|W_n|}{\delta} + \mathcal{O}(1)\right) \\ &= 2k^2\theta^2\tau^2\left(\frac{1}{k\tau} - \frac{(\delta-k\tau)}{k^2\tau^2} + \mathcal{O}\left(\frac{(\delta-k\tau)^2}{k^3}\right)\right) + \mathcal{O}\left(\frac{k^2}{|W_n|}\right) \\ &= 2\theta^2 k\tau - 2\theta^2(\delta-k\tau) + \mathcal{O}\left(\frac{(\delta-k\tau)^2}{k}\right) + \mathcal{O}\left(\frac{k^2}{|W_n|}\right), \end{aligned} \tag{5.116}$$

as $n \to \infty$ and $|\delta-k\tau| \to 0$.

Next we consider the second term on the r.h.s. of (5.114). By (5.104), the second term on the r.h.s. of (5.114) can be computed as follows

$$\begin{aligned} &\frac{4k\theta\tau(\delta-k\tau)}{|W_n|}\left(\frac{|W_n|}{\delta} + \mathcal{O}(1)\right)\left(\theta + \mathcal{O}(|\delta-k\tau|) + \mathcal{O}\left(\frac{k}{|W_n||\delta-k\tau|}\right)\right) \\ &= \frac{4\theta^2 k\tau(\delta-k\tau)}{\delta} + \mathcal{O}((\delta-k\tau)^2) + \mathcal{O}\left(\frac{k|\delta-k\tau|}{|W_n|}\right) + \mathcal{O}\left(\frac{k}{|W_n|}\right) \\ &= 4\theta^2 k\tau(\delta-k\tau)\left(\frac{1}{k\tau} + \mathcal{O}\left(\frac{|\delta-k\tau|}{k^2}\right)\right) + \mathcal{O}((\delta-k\tau)^2) + \mathcal{O}\left(\frac{k}{|W_n|}\right) \\ &= 4\theta^2(\delta-k\tau) + \mathcal{O}((\delta-k\tau)^2) + \mathcal{O}\left(\frac{k}{|W_n|}\right), \end{aligned} \tag{5.117}$$

as $n \to \infty$ and $|\delta-k\tau| \to 0$.

Next we consider the third term on the r.h.s. of (5.114). By a similar argument as the one in (5.104), we also have

$$\begin{aligned}\frac{1}{N_{n\delta}}\sum_{i=1}^{N_{n\delta}}\lambda^2(a_n+r+i(\delta-k\tau)) &= \frac{1}{\tau}\int_0^{\tau}\lambda^2(s)ds+\mathcal{O}\left(|\delta-k\tau|\right)\\ &\quad +\mathcal{O}\left(k|W_n|^{-1}|\delta-k\tau|^{-1}\right), \qquad (5.118)\end{aligned}$$

as $n\to\infty$ and $|\delta-k\tau|\to 0$. To verify (5.118) we have used the condition $|W_n|k^{-1}|\delta-k\tau|\to\infty$ as $n\to\infty$ and the Lipschitz condition on λ (which implies that λ^2 is also Lipschitz). By (5.118), the third term on the r.h.s. of (5.114) can be computed as follows

$$\begin{aligned}&\frac{2(\delta-k\tau)^2N_{n\delta}}{|W_n|}\left(\frac{1}{\tau}\int_0^{\tau}\lambda^2(s)ds+\mathcal{O}\left(|\delta-k\tau|\right)+\mathcal{O}\left(\frac{k}{|W_n||\delta-k\tau|}\right)\right)\\ &=\frac{2(\delta-k\tau)^2}{|W_n|}\mathcal{O}\left(\frac{|W_n|}{k}\right)\mathcal{O}\left(1+\frac{k}{|W_n||\delta-k\tau|}\right)\\ &=\mathcal{O}\left(\frac{(\delta-k\tau)^2}{k}\right)+\mathcal{O}\left(\frac{|\delta-k\tau|}{|W_n|}\right), \qquad (5.119)\end{aligned}$$

as $|\delta-k\tau|\to 0$ and as $n\to\infty$. Combining (5.113), (5.116), (5.117), and (5.119), we obtain

$$\begin{aligned}Var(B_{n,\delta}) &= 2\theta^2k\tau+\theta+2\theta^2(\delta-k\tau)+\mathcal{O}\left(\frac{k^2}{|W_n|}\right)+\mathcal{O}\left((\delta-k\tau)^2\right)\\ &= 2\theta^2\delta+\theta+\mathcal{O}\left(\frac{k^2}{|W_n|}\right)+\mathcal{O}\left((\delta-k\tau)^2\right),\end{aligned}$$

as $n\to\infty$ and $|\delta-k\tau|\to 0$. Hence we have (5.108). This completes the proof of Lemma 5.10. □

Lemma 5.11 *Consider $A_{n,\delta}$ and $B_{n,\delta}$ as given in (5.91) and (5.92), and suppose that λ is periodic and bounded, and (5.4) holds.*

(i) Then for any positive integer k satisfying $k=o(|W_n|)$ and for any δ in a neighborhood of $k\tau$ we have

$$\left(\frac{A_{n,\delta}}{\sigma(A_{n,\delta})},\frac{B_{n,\delta}}{\sigma(B_{n,\delta})}\right)-N\left(0,0,1,1,\rho(A_{n,\delta},B_{n,\delta})\right)\xrightarrow{p}0, \qquad (5.120)$$

as $n\to\infty$, where ρ denotes the correlation coefficient.

(ii) If in addition, λ is Lipschitz, then for any positive integer k satisfying $k=o(|W_n|)$ and for any δ in a neighborhood of $k\tau$ with

$|W_n|k^{-1}|\delta - k\tau| \to \infty$ as $n \to \infty$, we have

$$Cov(A_{n,\delta}, B_{n,\delta}) = \frac{2(\delta - k\tau)}{k\tau}\left(\frac{1}{\tau}\int_0^\tau \lambda^2(s)ds - \theta^2\right) + \mathcal{O}\left(\frac{k}{|W_n|}\right)$$
$$+\mathcal{O}\left((\delta - k\tau)^2\right) + \mathcal{O}\left(k|W_n|^{-1}|\delta - k\tau|^{-1}\right), \qquad (5.121)$$

as $n \to \infty$ and $|\delta - k\tau| \to 0$.

Proof: First we prove part (i) of this lemma by an application of the Cramer-Wold device. For any real numbers d_1 and d_2, define

$$Y_{n,\delta} = d_1 A_{n,\delta} + d_2 B_{n,\delta}.$$

Then, to prove part (i) of this lemma, it suffices to check

$$|Y_{n,\delta} - N(0, Var(Y_{n,\delta}))| \xrightarrow{p} 0, \qquad (5.122)$$

as $n \to \infty$. To prove (5.122), we argue as follows. By its definition, we can write

$$\begin{aligned} Y_{n,\delta} &= \frac{2d_1}{|W_n|^{1/2}}\sum_{i=1}^{N_{n\delta}} \tilde{X}(U_{\delta,i})\left(c_{i,\delta,k\tau} - \bar{c}_{\cdot,\delta,k\tau}\right) + \frac{d_2}{|W_n|^{1/2}}\sum_{i=1}^{N_{n\delta}} \widetilde{\tilde{X}^2(U_{\delta,i})} \\ &= \frac{1}{|W_n|^{1/2}}\sum_{i=1}^{N_{n\delta}}\left\{2d_1\tilde{X}(U_{\delta,i})\left(c_{i,\delta,k\tau} - \bar{c}_{\cdot,\delta,k\tau}\right) + d_2\widetilde{\tilde{X}^2(U_{\delta,i})}\right\}. \end{aligned}$$

For each i ($i = 1, \ldots N_{n\delta}$), let

$$Y_{n,\delta,i} = 2d_1\tilde{X}(U_{\delta,i})\left(c_{i,\delta,k\tau} - \bar{c}_{\cdot,\delta,k\tau}\right) + d_2\widetilde{\tilde{X}^2(U_{\delta,i})}.$$

Then, for any $i \neq j$; $i, j = 1, \ldots N_{n\delta}$, we have $Y_{n,\delta,i}$ and $Y_{n,\delta,j}$ are independent, $\mathbf{E}Y_{n,\delta,i} = 0$ for all i, and $Var(Y_{n,\delta,i})$ can be computed as follows

$$\begin{aligned} Var(Y_{n,\delta,i}) &= 4d_1^2(c_{i,\delta,k\tau} - \bar{c}_{\cdot,\delta,k\tau})^2\mathbf{E}\tilde{X}^2(U_{\delta,i}) + d_2^2\mathbf{E}\left(\widetilde{\tilde{X}^2(U_{\delta,i})}\right)^2 \\ &+4d_1d_2(c_{i,\delta,k\tau} - \bar{c}_{\cdot,\delta,k\tau})\mathbf{E}\tilde{X}(U_{\delta,i})\widetilde{\tilde{X}^2(U_{\delta,i})} \\ &= 4d_1^2(c_{i,\delta,k\tau} - \bar{c}_{\cdot,\delta,k\tau})^2\int_{U_{\delta,i}}\lambda(s) + d_2^2\left\{\int_{U_{\delta,i}}\lambda(s) + 2\left(\int_{U_{\delta,i}}\lambda(s)\right)^2\right\} \\ &+4d_1d_2(c_{i,\delta,k\tau} - \bar{c}_{\cdot,\delta,k\tau})\int_{U_{\delta,i}}\lambda(s) = \mathcal{O}(k^2), \end{aligned}$$

as $n \to \infty$, uniformly for all δ in the neighborhood of $k\tau$. (Here we have used the assumption that λ is bounded and Haight (1967), p. 7). This implies

$$\left(\sum_{i=1}^{N_{n\delta}} Var(Y_{n,\delta,i})\right)^2 = \mathcal{O}\left(|W_n|^2 k^2\right),$$

as $n \to \infty$. Then to prove (5.122), it suffices to check that the Lyapounov's condition

$$\sum_{i=1}^{N_{n\delta}} \mathbf{E}\left(Y_{n,\delta,i}\right)^4 = o\left(|W_n|^2 k^2\right) \tag{5.123}$$

holds, as $n \to \infty$ (cf. Serfling (1980), p. 30). To prove (5.123), we first note that, for each i,

$$\begin{aligned}\mathbf{E}\left(Y_{n,\delta,i}\right)^4 &= \mathbf{E}\left(2C_1\tilde{X}(U_{\delta,i})(c_{i,\delta,k\tau} - \bar{c}_{\cdot,\delta,k\tau}) + C_2\widetilde{\tilde{X}^2(U_{\delta,i})}\right)^4 \\ &= \mathcal{O}\left(\mathbf{E}\left(\tilde{X}(U_{\delta,i})\right)^4 + \mathbf{E}\left(\widetilde{\tilde{X}^2(U_{\delta,i})}\right)^4\right), \end{aligned} \tag{5.124}$$

because λ is bounded. We know that $\mathbf{E}(\tilde{X}(U_{\delta,i}))^4 = \mathcal{O}(k^2)$ as $n \to \infty$, uniformly in i. From proof of Lemma 5.10, we know that the quantity in (5.111) is of order $\mathcal{O}(k^4)$ as $n \to \infty$, uniformly in i. Hence, the quantity in (5.124) is of order $\mathcal{O}(k^4)$ as $n \to \infty$, uniformly in i, which implies the l.h.s. of (5.123) is of order $\mathcal{O}(|W_n|k^3)$, which is $o(|W_n|^2k^2)$ as $n \to \infty$, because $k = o(|W_n|)$ as $n \to \infty$. Then (5.123) holds true, which implies part (i) of this lemma. Note that (5.4) implies that both $\sigma(A_{n,\delta})$ and $\sigma(B_{n,\delta})$ are bounded away from zero, for all large n.

Next we prove part (ii) of this lemma. Since $\mathbf{E}A_{n,\delta} = \mathbf{E}B_{n,\delta} = 0$, we have that

$$\begin{aligned} &Cov(A_{n,\delta}, B_{n,\delta}) = \mathbf{E}A_{n,\delta}B_{n,\delta} \\ &= \frac{2}{|W_n|}\mathbf{E}\left(\sum_{i=1}^{N_{n\delta}} c_{i,\delta,k\tau}\tilde{X}(U_{\delta,i})\right)\left(\sum_{j=1}^{N_{n\delta}} \widetilde{\tilde{X}^2(U_{\delta,j})}\right) \\ &\quad - \frac{2\bar{c}_{\cdot,\delta,k\tau}}{|W_n|}\mathbf{E}\sum_{i=1}^{N_{n\delta}} \tilde{X}(U_{\delta,i})\sum_{j=1}^{N_{n\delta}} \widetilde{\tilde{X}^2(U_{\delta,j})}. \end{aligned} \tag{5.125}$$

The first term on the r.h.s. of (5.125) is equal to

$$\frac{2}{|W_n|}\left\{\sum_{i=1}^{N_{n\delta}} c_{i,\delta,k\tau}\mathbf{E}\tilde{X}(U_{\delta,i})\widetilde{\tilde{X}^2(U_{\delta,i})}\right\}$$

$$= \frac{2}{|W_n|} \sum_{i=1}^{N_{n\delta}} c_{i,\delta,k\tau} \mathbf{E} \tilde{X}(U_{\delta,i}) \left(\tilde{X}^2(U_{\delta,i}) - \int_{U_{\delta,i}} \lambda(s)ds \right)$$

$$= \frac{2}{|W_n|} \sum_{i=1}^{N_{n\delta}} c_{i,\delta,k\tau} \mathbf{E} \tilde{X}^3(U_{\delta,i}) = \frac{2}{|W_n|} \sum_{i=1}^{N_{n\delta}} c_{i,\delta,k\tau} \int_{U_{\delta,i}} \lambda(s)ds, \quad (5.126)$$

where we used the fact that $\mathbf{E}\tilde{X}^3(U_{\delta,i}) = \mathbf{E}X(U_{\delta,i})$ (cf. (1.3-17) of Haight (1967), p. 7). Similarly, the second term on the r.h.s. of (5.125) is equal to

$$- \frac{2\bar{c}_{\cdot,\delta,k\tau}}{|W_n|} \left\{ \sum_{i=1}^{N_{n\delta}} \mathbf{E} \tilde{X}(U_{\delta,i}) \widetilde{X^2(U_{\delta,i})} \right\}$$

$$= - \frac{2\bar{c}_{\cdot,\delta,k\tau}}{|W_n|} \sum_{i=1}^{N_{n\delta}} \mathbf{E} \tilde{X}(U_{\delta,i}) \left(\tilde{X}^2(U_{\delta,i}) - \int_{U_{\delta,i}} \lambda(s)ds \right)$$

$$= - \frac{2\bar{c}_{\cdot,\delta,k\tau}}{|W_n|} \sum_{i=1}^{N_{n\delta}} \mathbf{E} \tilde{X}^3(U_{\delta,i}) = - \frac{2\bar{c}_{\cdot,\delta,k\tau}}{|W_n|} \sum_{i=1}^{N_{n\delta}} \int_{U_{\delta,i}} \lambda(s)ds. \quad (5.127)$$

By (5.126), (5.127), and (5.93), we can write

$$Cov(A_{n,\delta}, B_{n,\delta}) = \frac{2}{|W_n|} \sum_{i=1}^{N_{n\delta}} \int_{U_{\delta,i}} \lambda(s)ds \;\; \lambda(a_n + r + (i-1)(\delta - k\tau))$$
$$- \frac{2\bar{c}_{\cdot,\delta,k\tau}}{|W_n|} \sum_{i=1}^{N_{n\delta}} \int_{U_{\delta,i}} \lambda(s)ds. \quad (5.128)$$

First, we consider the first term on the r.h.s. of (5.128). By (5.96), this term can be simplified to get

$$\frac{2k\theta\tau}{|W_n|} \sum_{i=1}^{N_{n\delta}} \lambda(a_n + r + (i-1)(\delta - k\tau))$$
$$+ \frac{2(\delta - k\tau)}{|W_n|} \sum_{i=1}^{N_{n\delta}} \lambda^2(a_n + r + (i-1)(\delta - k\tau)) + \mathcal{O}\left((\delta - k\tau)^2 k^{-1}\right), \quad (5.129)$$

By a similar argument as the one in (5.104), but with (5.191) is now replaced by

$$J_{\delta,k\tau}^{-1} = \frac{|\delta - k\tau|}{\tau} + b(\delta, k\tau)\frac{(\delta - k\tau)^2}{\tau^2} + \mathcal{O}\left(|\delta - k\tau|^3\right),$$

as $|\delta - k\tau| \to 0$ uniformly in n, where $b(\delta, k\tau) = \frac{\tau}{|\delta - k\tau|} - [\frac{\tau}{|\delta - k\tau|}]$, we have

$$\frac{1}{N_{n\delta}} \sum_{i=1}^{N_{n\delta}} \lambda(a_n + r + (i-1)(\delta - k\tau))$$

$$= \frac{1}{\tau}\int_0^{\tau}\lambda(s)ds + \frac{b(\delta,k\tau)|\delta-k\tau|}{\tau^2}\int_0^{\tau}\lambda(s)ds + \mathcal{O}\left((\delta-k\tau)^2\right) + \mathcal{O}\left(\frac{J_{\delta,k\tau}}{N_{n\delta}}\right)$$
$$= \theta + \frac{\theta b(\delta,k\tau)|\delta-k\tau|}{\tau} + \mathcal{O}\left((\delta-k\tau)^2\right) + \mathcal{O}\left(\frac{k}{|W_n||\delta-k\tau|}\right), \tag{5.130}$$

as $n\to\infty$ and $|\delta-k\tau|\to 0$. Note that, to verify (5.130) we have used the condition $|W_n|k^{-1}|\delta-k\tau|\to\infty$ as $n\to\infty$, and the Lipschitz condition on λ. Hence, the first term on the r.h.s. of (5.129) is equal to

$$\frac{2k\theta\tau}{|W_n|}N_{n\delta}\left(\theta + \frac{\theta b(\delta,k\tau)|\delta-k\tau|}{\tau} + \mathcal{O}\left((\delta-k\tau)^2\right) + \mathcal{O}\left(\frac{k}{|W_n||\delta-k\tau|}\right)\right)$$
$$= 2\theta^2 - \frac{2\theta^2(\delta-k\tau)}{k\tau} + \frac{2\theta^2 b(\delta,k\tau)|\delta-k\tau|}{\tau} + \mathcal{O}\left((\delta-k\tau)^2\right)$$
$$+\mathcal{O}\left(\frac{k}{|W_n|}\right) + \mathcal{O}\left(k|W_n|^{-1}|\delta-k\tau|^{-1}\right), \tag{5.131}$$

as $n\to\infty$ and $|\delta-k\tau|\to 0$. Similar to that in (5.130), we also have

$$\frac{1}{N_{n\delta}}\sum_{i=1}^{N_{n\delta}}\lambda^2(a_n+r+(i-1)(\delta-k\tau))$$
$$= \frac{1}{\tau}\int_0^{\tau}\lambda^2(s)ds + \mathcal{O}\left(|\delta-k\tau|\right) + \mathcal{O}\left(\frac{k}{|W_n||\delta-k\tau|}\right), \tag{5.132}$$

as $n\to\infty$ and $|\delta-k\tau|\to 0$. Here we also have used the condition $|W_n|k^{-1}|\delta-k\tau|\to\infty$ as $n\to\infty$, and the Lipschitz condition on λ. By (5.132), the second term on the r.h.s. of (5.129) is equal to

$$\frac{2(\delta-k\tau)}{|W_n|}\left(\frac{|W_n|}{\delta}+\mathcal{O}(1)\right)$$
$$\left(\frac{1}{\tau}\int_0^{\tau}\lambda^2(s)ds + \mathcal{O}\left(|\delta-k\tau|\right) + \mathcal{O}\left(\frac{k}{|W_n||\delta-k\tau|}\right)\right)$$
$$= \frac{2(\delta-k\tau)}{k\tau^2}\int_0^{\tau}\lambda^2(s)ds + \mathcal{O}\left(\frac{(\delta-k\tau)^2}{k}\right) + \mathcal{O}\left(\frac{1}{|W_n|}\right). \tag{5.133}$$

Substituting (5.131) and (5.133) into (5.129), we then have that the first term on the r.h.s. of (5.128) is equal to

$$2\theta^2 - \frac{2\theta^2(\delta-k\tau)}{k\tau} + \frac{2\theta^2 b(\delta,k\tau)|\delta-k\tau|}{\tau} + \frac{2(\delta-k\tau)}{k\tau^2}\int_0^{\tau}\lambda^2(s)ds$$
$$+\mathcal{O}\left((\delta-k\tau)^2\right) + \mathcal{O}\left(\frac{k}{|W_n|}\right) + \mathcal{O}\left(\frac{k}{|W_n||\delta-k\tau|}\right), \tag{5.134}$$

as $n\to\infty$ and $|\delta-k\tau|\to 0$.

Next we consider the second term on the r.h.s. of (5.128). By its definition, $\bar{c}_{\cdot,\delta,k\tau}$ is equal to the quantity in (5.130). By (5.186), we have

$$\begin{aligned}
&\frac{2}{|W_n|}\sum_{i=1}^{N_{n\delta}}\int_{U_{\delta,i}}\lambda(s)ds \\
&= \frac{2}{|W_n|}\left(\frac{|W_n|}{\delta}+\mathcal{O}(1)\right)\left(k\theta\tau+\theta(\delta-k\tau)+\mathcal{O}\left(\frac{k|\delta-k\tau|}{|W_n|}\right)\right) \\
&= 2\theta+\mathcal{O}\left(\frac{(\delta-k\tau)^2}{k}\right)+\mathcal{O}\left(\frac{k}{|W_n|}\right). \qquad (5.135)
\end{aligned}$$

By (5.130) and (5.135), the second term on the r.h.s. of (5.128) is equal to

$$-2\theta^2-\frac{2\theta^2 b(\delta,k\tau)|\delta-k\tau|}{\tau}+\mathcal{O}\left((\delta-k\tau)^2\right)+\mathcal{O}\left(\frac{k}{|W_n|}\right), \qquad (5.136)$$

as $n\to\infty$ and $|\delta-k\tau|\to 0$.

Combining (5.134) and (5.136), we obtain (5.121). Note that, the sum of the second and fourth term of (5.134) is equal to the leading term on the r.h.s. of (5.121), while the first and third term of (5.134) cancel with the first and second term of (5.136). Hence we have proved part (ii) of this lemma. This completes the proof of Lemma 5.11. □

Lemma 5.11 tells us that $(A_{n,\delta}/\sigma(A_{n,\delta}), B_{n,\delta}/\sigma(B_{n,\delta}))\xrightarrow{d} N((0,0),\mathbf{I})$, as $n\to\infty$ and $|\delta-k\tau|\to 0$, where $N((0,0),\mathbf{I})$ denote bivariate normal with mean zero vector and identity covariance matrix. Note that

$$\begin{aligned}
\frac{Cov(A_{n,\delta},B_{n,\delta})}{\sigma(A_{n,\delta})\sigma(B_{n,\delta})} &= \mathcal{O}\left(\frac{|\delta-k\tau|}{k^{3/2}}\right)+\mathcal{O}\left(\frac{k^{1/2}}{|W_n|}\right)+\mathcal{O}\left(\frac{(\delta-k\tau)^2}{k^{1/2}}\right) \\
&\quad+\mathcal{O}\left(\frac{k^{1/2}}{|W_n||\delta-k\tau|}\right)=o(1),
\end{aligned}$$

as $n\to\infty$ and $|\delta-k\tau|\to 0$, by the assumption $|W_n|k^{-1}|\delta-k\tau|\to\infty$ as $n\to\infty$. The bivariate normal vector $N((0,0),\mathbf{I})$ can also be expressed as $(N_1(0,1),N_2(0,1))$, where $N_1(0,1)$ and $N_2(0,1)$ are two independent standard normal r.v.'s.

Using the results given in Lemmas 5.8 - 5.11, as well as that in part (ii) of Lemma 5.16, we obtain a stochastic expansion for $Q_n(\delta)$ in the following lemma. Note that here we add the requirement $k=o(|W_n|^{1/2})$, otherwise the quadratic approximation to $\Lambda_n(\delta)$ (cf. Lemma 5.16) will not dominate the $\mathcal{O}_p((\delta-k\tau)^2|W_n|^{-1/2})$ random error term in (5.137).

Lemma 5.12 *Suppose that λ is periodic and Lipschitz, and (5.4) holds. Then, for any positive integer k satisfying $k = o(|W_n|^{1/2})$ and for any δ in a neighborhood of $k\tau$ with $|W_n|k^{-1}|\delta - k\tau| \to \infty$ as $n \to \infty$, we have*

$$\begin{aligned} Q_n(\delta) &= \frac{(\delta - k\tau)^2}{k\tau^2} \int_0^\tau (\lambda(s) - \theta)^2 ds \\ &+ \frac{(\delta - k\tau)}{|W_n|^{1/2}} \left\{ \left(\frac{4\theta}{\tau} \int_0^\tau (\lambda(s) - \theta)^2 ds \right)^{1/2} N_1(0,1) + \frac{\theta^{3/2}}{(2\theta k\tau + 1)^{1/2}} N_2(0,1) \right\} \\ &+ \frac{(2\theta^2 k\tau + \theta)^{1/2}}{|W_n|^{1/2}} N_2(0,1) + \theta + \mathcal{O}_p\left(\frac{k}{|W_n|}\right) + \mathcal{O}_p\left(\frac{|\delta - k\tau| k^{1/2}}{|W_n|}\right) \\ &+ \mathcal{O}_p\left(\frac{(\delta - k\tau)^2}{|W_n|^{1/2}}\right) + \mathcal{O}\left(\frac{|\delta - k\tau|^3}{k}\right), \end{aligned} \tag{5.137}$$

as $n \to \infty$ and $|\delta - k\tau| \to 0$.

Proof: From (5.101) we see that (cf. also (5.4))

$$\begin{aligned} \sigma(A_{n,\delta}) &= \left(\frac{4\theta}{\tau} \int_0^\tau (\lambda(s) - \theta)^2 ds \right)^{1/2} \\ &\quad \left(1 + \mathcal{O}\left(\frac{k}{|W_n|}\right) + \mathcal{O}(|\delta - k\tau|) + \mathcal{O}\left(\frac{k}{|W_n||\delta - k\tau|}\right)\right)^{1/2} \\ &= \left(\frac{4\theta}{\tau} \int_0^\tau (\lambda(s) - \theta)^2 ds \right)^{1/2} + \mathcal{O}\left(\frac{k}{|W_n|}\right) + \mathcal{O}(|\delta - k\tau|) \\ &\quad + \mathcal{O}\left(\frac{k}{|W_n||\delta - k\tau|}\right), \end{aligned} \tag{5.138}$$

as $n \to \infty$ and $|\delta - k\tau| \to 0$. By (5.100) and (5.138), we can write

$$\begin{aligned} \frac{(\delta - k\tau)}{|W_n|^{1/2}} A_{n,\delta} &= \frac{(\delta - k\tau)}{|W_n|^{1/2}} \left(\frac{4\theta}{\tau} \int_0^\tau (\lambda(s) - \theta)^2 ds \right)^{1/2} N_1(0,1) \\ &\quad + \mathcal{O}_p\left(\frac{k}{|W_n|^{3/2}}\right) + \mathcal{O}_p\left(\frac{|\delta - k\tau| k^{1/2}}{|W_n|}\right), + \mathcal{O}_p\left(\frac{(\delta - k\tau)^2}{|W_n|^{1/2}}\right), \end{aligned} \tag{5.139}$$

as $n \to \infty$ and $|\delta - k\tau| \to 0$. The error in replacing $A_{n,\delta}/\sigma(A_{n,\delta})$ by $N_1(0,1)$ is of order $\mathcal{O}_p(k^{1/2}|W_n|^{-1/2})$, which follows easily from the Berry-Esseen bound for $A_{n,\delta}/\sigma(A_{n,\delta})$. This error is incorporated in the $\mathcal{O}_p(|\delta - k\tau| k^{1/2}|W_n|^{-1})$ term of (5.139). From (5.108) we see that (cf. (5.4))

$$\sigma(B_{n,\delta}) = \left(2\theta^2\delta + \theta + \mathcal{O}\left(\frac{k^2}{|W_n|}\right) + \mathcal{O}\left((\delta - k\tau)^2\right)\right)^{1/2}$$

$$
\begin{aligned}
&= \left(2\theta^2\delta+\theta\right)^{1/2}\left(1+\mathcal{O}\left(\frac{k}{|W_n|}\right)+\mathcal{O}\left(\frac{(\delta-k\tau)^2}{k}\right)\right)^{1/2}\\
&= \left(2\theta^2\delta+\theta\right)^{1/2}\left(1+\mathcal{O}\left(\frac{k}{|W_n|}\right)+\mathcal{O}\left(\frac{(\delta-k\tau)^2}{k}\right)\right)\\
&= \left(2\theta^2\delta+\theta\right)^{1/2}+\mathcal{O}\left(\frac{k^{3/2}}{|W_n|}\right)+\mathcal{O}\left(\frac{(\delta-k\tau)^2}{k^{1/2}}\right), \qquad (5.140)
\end{aligned}
$$

as $n \to \infty$ and $|\delta - k\tau| \to 0$. The first term on the r.h.s. of (5.140) can be written as

$$
\left(2\theta^2k\tau+\theta\right)^{1/2}+\frac{\theta^2(\delta-k\tau)}{(2\theta^2k\tau+\theta)^{1/2}}-\frac{\theta^4(\delta-k\tau)^2}{2(2\theta^2k\tau+\theta)^{3/2}}+\mathcal{O}\left(\frac{|\delta-k\tau|^3}{k^{5/2}}\right). \qquad (5.141)
$$

By (5.107), (5.140), and (5.141), we can write

$$
\begin{aligned}
&\frac{B_{n,\delta}}{|W_n|^{1/2}}\\
&= \frac{1}{|W_n|^{1/2}}\left[\left(2\theta^2k\tau+\theta\right)^{1/2}+\frac{\theta^2(\delta-k\tau)}{(2\theta^2k\tau+\theta)^{1/2}}-\frac{\theta^4(\delta-k\tau)^2}{2(2\theta^2k\tau+\theta)^{3/2}}\right]N_2(0,1)\\
&+\mathcal{O}_p\left(\frac{k}{|W_n|}\right)+\mathcal{O}_p\left(\frac{(\delta-k\tau)^2}{|W_n|^{1/2}k^{1/2}}\right), \qquad (5.142)
\end{aligned}
$$

as $n \to \infty$ and $|\delta - k\tau| \to 0$. Here we have used the fact that $|\delta - k\tau|^3 k^{-5/2}|W_n|^{-1/2}$ is of smaller order than $(\delta-k\tau)^2 k^{-1/2}|W_n|^{-1/2}$, and $\mathcal{O}_p(k^{3/2}|W_n|^{-3/2})$ is of smaller order than $\mathcal{O}_p(k^{1/2}|W_n|^{-1/2})$. The error in replacing $B_{n,\delta}/\sigma(B_{n,\delta})$ by $N_2(0,1)$ is of order $\mathcal{O}_p(k^{1/2}|W_n|^{-1/2})$, which follows easily from the Berry-Esseen bound for $B_{n,\delta}/\sigma(B_{n,\delta})$. This error is incorporated in the $\mathcal{O}_p(k|W_n|^{-1})$ term of (5.142). Substituting (5.139) and (5.142) into the r.h.s. of (5.90), we then get

$$
\begin{aligned}
&\tilde{Q}_n(\delta)\\
&= \frac{(\delta-k\tau)}{|W_n|^{1/2}}\left\{\left(\frac{4\theta}{\tau}\int_0^\tau(\lambda(s)-\theta)^2ds\right)^{1/2}N_1(0,1)+\frac{\theta^{3/2}}{(2\theta k\tau+1)^{1/2}}N_2(0,1)\right\}\\
&+\frac{(2\theta^2k\tau+\theta)^{1/2}}{|W_n|^{1/2}}N_2(0,1)+\mathcal{O}_p\left(\frac{|\delta-k\tau|k^{1/2}}{|W_n|}\right)+\mathcal{O}_p\left(\frac{(\delta-k\tau)^2}{|W_n|^{1/2}}\right)\\
&+\mathcal{O}_p\left(\frac{k}{|W_n|}\right), \qquad (5.143)
\end{aligned}
$$

as $n \to \infty$ and $|\delta - k\tau| \to 0$. Here we have also used the fact that the error in replacing $(A_{n,\delta}/\sigma(A_{n,\delta}), B_{n,\delta}/\sigma(B_{n,\delta}))$ by $(N_1(0,1), N_2(0,1))$ is again of order $\mathcal{O}(k^{1/2}|W_n|^{-1/2})$ (cf. Corollary 17.2 of Bhattacharya and Rao (1976)). Combining (5.183) and (5.143), we obtain (5.137). This completes the proof of Lemma 5.12. □

Now we continue the proof of Theorem 5.3, that is we want to show (5.83). To establish (5.83), we can restrict our considerations to $Q_n(\delta)$ only for $\delta \in \Theta_{k,n}$, i.e., δ such that $(\delta - k\tau) \to 0$ as $n \to \infty$, so that we can apply Lemma 5.12. First note that, since $|\Theta_{k,n}| \downarrow 0$ as $n \to \infty$, (5.83) automatically holds true if $\gamma \leq c$. So, it remains to check (5.83) only for the case $\gamma > c$. Let $\bar{Q}_n(\delta)$ denotes the leading term on the r.h.s. of (5.137), that is

$$\begin{aligned}\bar{Q}_n(\delta) &= \frac{(\delta - k\tau)^2}{k\tau^2} \int_0^\tau (\lambda(s) - \theta)^2 ds \\ &+ \frac{(\delta - k\tau)}{|W_n|^{1/2}} \left\{ \left(\frac{4\theta}{\tau} \int_0^\tau (\lambda(s) - \theta)^2 ds \right)^{1/2} N_1(0,1) + \frac{\theta^{3/2}}{(2\theta k\tau + 1)^{1/2}} N_2(0,1) \right\} \\ &+ \frac{(2\theta^2 k\tau + \theta)^{1/2}}{|W_n|^{1/2}} N_2(0,1) + \theta. \end{aligned} \tag{5.144}$$

Note that $\bar{Q}_n(\delta)$ is a quadratic function of $(\delta - k\tau)$. Define the auxiliary quantity $\bar{\tau}_{k,n}$ by

$$k\bar{\tau}_{k,n,s} = \arg \min_{\delta \in \Theta_{k,n}} \bar{Q}_n(\delta). \tag{5.145}$$

Minimizing $\bar{Q}_n(\delta)$ w.r.t. $(\delta - k\tau)$ yields a stochastic expansion for $(\bar{\tau}_{k,n,s} - \tau)$:

$$\begin{aligned}&|W_n|^{1/2} (\bar{\tau}_{k,n,s} - \tau) \\ &= -\left\{ \frac{\tau^{3/2}\, \theta^{1/2}\, N_1(0,1)}{\left(\int_0^\tau (\lambda(s) - \theta)^2 ds\right)^{1/2}} + \frac{\tau^2\, \theta^{3/2}\, N_2(0,1)}{2(2\theta k\tau + 1)^{1/2} \int_0^\tau (\lambda(s) - \theta)^2 ds} \right\}. \end{aligned} \tag{5.146}$$

Formula (5.146) directly implies that, for any $\gamma < \frac{1}{2}$, we have

$$|W_n|^\gamma (\bar{\tau}_{k,n,s} - \tau) \xrightarrow{p} 0,$$

as $n \to \infty$. Then, to prove (5.83), it now clearly suffices to show, for any $\gamma < \frac{1}{2}$, that

$$k\left(\hat{\tau}_{k,n,s} - \bar{\tau}_{k,n,s}\right) = o_p\left(\frac{k}{|W_n|^\gamma}\right), \tag{5.147}$$

as $n \to \infty$.

To verify (5.147) we argue as follows. By (5.137) and (5.144), we know that

$$Q_n(\delta) = \bar{Q}_n(\delta) + R_n(\delta),$$

where

$$R_n(\delta) = \mathcal{O}_p\left(\frac{|\delta - k\tau|k^{1/2}}{|W_n|}\right) + \mathcal{O}_p\left(\frac{(\delta - k\tau)^2}{|W_n|^{1/2}}\right) + \mathcal{O}\left(\frac{|\delta - k\tau|^3}{k}\right)$$
$$+\mathcal{O}_p\left(\frac{k}{|W_n|}\right), \qquad (5.148)$$

will be a remainder term of lower order. To check that this term is indeed negligible for our present purposes, we note that, a simple calculation using (5.144) yields

$$\bar{Q}_n\left(\delta + o_p\left(\frac{k}{|W_n|^\gamma}\right)\right)$$
$$= \bar{Q}_n(\delta) + o_p\left(\frac{k}{|W_n|^{\gamma+1/2}}\right) + o_p\left(\frac{|\delta - k\tau|}{|W_n|^\gamma}\right) + o_p\left(\frac{k}{|W_n|^{2\gamma}}\right), \qquad (5.149)$$

as $n \to \infty$ and as $|\delta - k\tau| \to 0$. Note that if $(\delta - k\tau) = \mathcal{O}(k|W_n|^{-\gamma})$, then $|\delta - k\tau|^3 k^{-1} = o(|\delta - k\tau||W_n|^{-\gamma})$, since $\gamma > c$ which implies $k|W_n|^{-\gamma} \downarrow 0$ as $n \to \infty$. Since $k = o(|W_n|^{1/2})$ and $\gamma < \frac{1}{2}$, we also have $|\delta - k\tau|k^{1/2}|W_n|^{-1} = o(|\delta - k\tau||W_n|^{-\gamma})$, $(\delta - k\tau)^2|W_n|^{-1/2} = o(|\delta - k\tau||W_n|^{-\gamma})$, and $k|W_n|^{-1} = o(k|W_n|^{-2\gamma})$ as $n \to \infty$. Hence, since $R_n(\delta)$, i.e. the r.h.s. of (5.148), is at most of order

$$o_p\left(\frac{k}{|W_n|^{\gamma+1/2}}\right) + o_p\left(\frac{|\delta - k\tau|}{|W_n|^\gamma}\right) + o_p\left(\frac{k}{|W_n|^{2\gamma}}\right)$$

provided $|\delta - k\tau| = \mathcal{O}(k|W_n|^{-\gamma})$ and $\gamma < \frac{1}{2}$, which does not exceed the remainder term on the r.h.s. of (5.149), we have proved (5.147). The requirement $|\delta - k\tau| = \mathcal{O}(k|W_n|^{-\gamma})$ is automatically satisfied, since here it suffices to consider $|\delta - k\tau| = o(k|W_n|^{-\gamma})$. Therefore we have proved (5.83). This completes the proof of Theorem 5.3. □

5.5 Proof of Theorem 5.4

Before presenting the proof of asymptotic normality of our modified estimator $\hat{\tau}^*_{k,n}$ of τ, we will first explain why we need to modify our original estimator $\hat{\tau}_{k,n}$.

Inspection of the latter part of the proof of Theorem 5.3 shows that the transition from $\hat{\tau}_{k,n,s}$ to $\bar{\tau}_{k,n,s}$ (cf.(5.147)) will fail to give us asymptotic normality of $|W_n|^{1/2}(\hat{\tau}_{k,n} - \tau)$. If we replace $o_p(k|W_n|^{-\gamma})$ by $o_p(k|W_n|^{-1/2})$ in (5.147), we will arrive at (5.149) with $\gamma = 1/2$. But this order bound 'just' fails to be of the same order as $R_n(\delta)$. Though, since $|\delta - k\tau|^3 k^{-1} = o(|\delta - k\tau||W_n|^{-1/2})$ and $(\delta - k\tau)^2|W_n|^{-1/2} = o(k|W_n|^{-1})$, provided $|\delta -$

$k\tau| = o((k|W_n|^{-1/2})^{1/2})$ as $n \to \infty$, the first and second terms on the r.h.s. of (5.148) poses no problem, the last term $\mathcal{O}_p(k|W_n|^{-1})$ in (5.148) is 'just' too big, it should $o_p(k|W_n|^{-1})$ as $n \to \infty$. To remedy this, we require a slight modification to the function $Q_n(\delta)$, and hence of the estimator of the period.

From the proofs of Lemma 5.16 and lemmas 5.8 - 5.12, we know that the term $\theta + \mathcal{O}_p(k|W_n|^{-1})$ on the r.h.s. of (5.137) is due to the error in replacing $B_{n,\delta}/\sigma(B_{n,\delta})$ by $N_2(0,1)$ on the r.h.s. of (5.142), and the sum of the following three terms, namely, the second term on the l.h.s. of (5.175), the second term on the l.h.s. of (5.176), and the second term on the r.h.s. of (5.95). Note, however, an easy computation shows that the error due to replacing $B_{n,\delta}/\sigma(B_{n,\delta})$ by $N_2(0,1)$ can be written as

$$\mathcal{O}_p\left(k|W_n|^{-1}\right) + \mathcal{O}_p\left(|\delta - k\tau||W_n|^{-1}\right), \tag{5.150}$$

as $n \to \infty$, where the first term of (5.150) does not depend on δ, while its second term, which depends on δ, is of smaller order. Hence, the error due to replacing $B_{n,\delta}/\sigma(B_{n,\delta})$ by $N_2(0,1)$ poses no problem.

It remains to treat the $\theta + \mathcal{O}_p(k|W_n|^{-1})$ term due to the sum of the three term mentioned above. Note that the $\mathcal{O}_p(k|W_n|^{-1})$ term here is of exact order. To handle this, first note that the second term on the l.h.s. of (5.176) is equal to the expectation of the second term on the r.h.s. of (5.95), before centering. Hence, the sum of these three terms can be written as

$$\frac{\mathbf{E}X(W_{N_{n\delta}})}{|W_n|} - \frac{\tilde{X}^2(W_{N_{n\delta}})}{|W_n|N_{n\delta}}. \tag{5.151}$$

For each positive integer k such that $k = k_n = o(|W_n|)$, we define the function

$$Q_n^{**}(\delta) = Q_n^*(\delta) - \frac{X(W_n)}{|W_n|} = Q_n(\delta) - \frac{X(W_{N_{n\delta}})}{|W_n|}.$$

Since the difference between $Q_n^*(\delta)$ and $Q_n^{**}(\delta)$ does not depend on δ, our modified estimator $\hat{\tau}_{k,n}^*$ can also be viewed as

$$\hat{\tau}_{k,n}^* = \frac{1}{k} \arg \min_{\delta \in \Theta_k} Q_n^{**}(\delta).$$

In fact we will use the latter definition of $\hat{\tau}_{k,n}^*$ to prove Theorem 5.4.

Proof of Theorem 5.4:

We will prove Theorem 5.4 by showing that, for each integer k satisfying $k = o(|W_n|^{1/3})$, we have

$$\begin{aligned}
&|W_n|^{1/2}\left(\hat{\tau}^*_{k,n} - \tau\right) \\
&= -\left\{\frac{\tau^{3/2}\,\theta^{1/2}\,N_1(0,1)}{\left(\int_0^\tau (\lambda(s)-\theta)^2 ds\right)^{1/2}} + \frac{\tau^2\,\theta^{3/2}\,N_2(0,1)}{2(2\theta k\tau+1)^{1/2}\int_0^\tau (\lambda(s)-\theta)^2 ds}\right\} \\
&\quad + o_p(k^{-1/2}), \qquad (5.152)
\end{aligned}$$

as $n \to \infty$, where $N_1(0,1)$ and $N_2(0,1)$ are independent standard normal random variables. To prove this, we argue as follows. For each integer k, define

$$k\hat{\tau}^*_{k,n,s} = \arg\min_{\delta\in\Theta_{k,n}} Q_n^{**}(\delta),$$

where $\Theta_{k,n} = (k\tau - \epsilon_n, k\tau + \epsilon_n)$ and ϵ_n is an arbitrary sequence of positive real numbers such that $\epsilon_n \to 0$ as $n \to \infty$. Then, to prove (5.10), it suffices to show

$$\begin{aligned}
&|W_n|^{1/2}\left(\hat{\tau}^*_{k,n,s} - \tau\right) \\
&= -\left\{\frac{\tau^{3/2}\,\theta^{1/2}\,N_1(0,1)}{\left(\int_0^\tau (\lambda(s)-\theta)^2 ds\right)^{1/2}} + \frac{\tau^2\,\theta^{3/2}\,N_2(0,1)}{2(2\theta k\tau+1)^{1/2}\int_0^\tau (\lambda(s)-\theta)^2 ds}\right\} \\
&\quad + o_p(k^{-1/2}), \qquad (5.153)
\end{aligned}$$

and

$$|W_n|^{1/2}\left(\hat{\tau}^*_{k,n} - \hat{\tau}^*_{k,n,s}\right) = o_p(k^{-1/2}), \qquad (5.154)$$

as $n \to \infty$.

First we consider (5.153). To establish (5.153), we can restrict our considerations to $Q_n^{**}(\delta)$ only for $\delta \in \Theta_{k,n}$, i.e., δ such that $(\delta - k\tau) \to 0$ as $n \to \infty$, so that we can apply the result in Lemma 5.12. By its definition, we can write $Q_n^{**}(\delta)$ as the r.h.s. of (5.137), provided the term $\theta + \mathcal{O}_p(k|W_n|^{-1})$, which is equal to the sum of the quantity in (5.150) and (5.151), is now replaced by the sum of the quantity in (5.150) and (5.151) minus $X(W_{N_{n\delta}})|W_n|^{-1}$, which is equal to

$$\begin{aligned}
&-\tilde{X}(W_{N_{n\delta}})|W_n|^{-1} - \tilde{X}^2(W_{N_{n\delta}})|W_n|^{-1}N_{n\delta}^{-1} + \mathcal{O}_p\left(\frac{k}{|W_n|}\right) + \mathcal{O}_p\left(\frac{|\delta - k\tau|}{|W_n|}\right) \\
&= -\frac{\theta k\tau}{|W_n|}N_3^2(0,1) - \frac{\theta^{1/2}}{|W_n|^{1/2}}N_3(0,1) + \mathcal{O}_p\left(\frac{k}{|W_n|}\right) + \mathcal{O}_p\left(\frac{|\delta - k\tau|}{|W_n|}\right) \\
&\quad + \mathcal{O}_p\left(\frac{k}{|W_n|^{3/2}} + \frac{k^2}{|W_n|^2}\right), \qquad (5.155)
\end{aligned}$$

as $n \to \infty$ and $|\delta - k\tau| \to 0$, by Lemma 5.13. Note that the $\mathcal{O}_p(k|W_n|^{-1})$ term on the r.h.s. of (5.155) does not depend on δ, and the $\mathcal{O}_p(k^{3/2}|W_n|^{-3/2} + k^{1/2}|W_n|^{-1})$ term of (5.161) is already incorporated in this term. Combining this fact, for each k satisfying $k = o(|W_n|^{1/2})$, we can write $Q_n^{**}(\delta)$ as follows

$$\begin{aligned} Q_n^{**}(\delta) &= \frac{(\delta - k\tau)^2}{k\tau^2} \int_0^\tau (\lambda(s) - \theta)^2 ds \\ &+ \frac{(\delta - k\tau)}{|W_n|^{1/2}} \left\{ \left(\frac{4\theta}{\tau} \int_0^\tau (\lambda(s) - \theta)^2 ds \right)^{1/2} N_1(0,1) + \frac{\theta^{3/2}}{(2\theta k\tau + 1)^{1/2}} N_2(0,1) \right\} \\ &+ \frac{(2\theta^2 k\tau + \theta)^{1/2}}{|W_n|^{1/2}} N_2(0,1) - \frac{\theta k\tau}{|W_n|} N_3^2(0,1) - \frac{\theta^{1/2}}{|W_n|^{1/2}} N_3(0,1) + \mathcal{O}_p\left(\frac{k}{|W_n|}\right) \\ &+ \mathcal{O}_p\left(\frac{|\delta - k\tau| k^{1/2}}{|W_n|}\right) + \mathcal{O}_p\left(\frac{(\delta - k\tau)^2}{|W_n|^{1/2}}\right) + \mathcal{O}_p\left(\frac{k}{|W_n|^{3/2}}\right) + \mathcal{O}\left(\frac{|\delta - k\tau|^3}{k}\right), \end{aligned} \tag{5.156}$$

as $n \to \infty$ and $|\delta - k\tau| \to 0$, with the $\mathcal{O}_p(k|W_n|^{-1})$ term does not depend on δ.

Similar to $\bar{Q}_n(\delta)$, let $\bar{Q}_n^{**}(\delta)$ denotes the leading term on the r.h.s. of (5.156) plus the $\mathcal{O}_p(k|W_n|^{-1})$ remainder term which does not depend on δ. Note that $\bar{Q}_n^{**}(\delta)$ is a quadratic function of $(\delta - k\tau)$. Since the difference between $\bar{Q}_n(\delta)$ and $\bar{Q}_n^{**}(\delta)$ is only in the constant term (the term which does not involve $(\delta - k\tau)$), we have that

$$\arg \min_{\delta \in \Theta_{k,n}} \bar{Q}_n^{**}(\delta) = \arg \min_{\delta \in \Theta_{k,n}} \bar{Q}_n(\delta) = k\bar{\tau}_{k,n,s}$$

(cf. (5.145)). By (5.146), to prove (5.153), it suffices to show that

$$k\left(\hat{\tau}_{k,n,s}^{*} - \bar{\tau}_{k,n,s}\right) = o_p\left(\frac{k^{1/2}}{|W_n|^{1/2}}\right), \tag{5.157}$$

as $n \to \infty$. To check (5.157) we argue as follows. Similar to the proof of (5.147), we have

$$Q_n^{**}(\delta) = \bar{Q}_n^{**}(\delta) + R_n^{**}(\delta),$$

where

$$\begin{aligned} R_n^{**}(\delta) &:= \mathcal{O}_p\left(\frac{|\delta - k\tau| k^{1/2}}{|W_n|}\right) + \mathcal{O}_p\left(\frac{(\delta - k\tau)^2}{|W_n|^{1/2}}\right) + \mathcal{O}_p\left(\frac{k}{|W_n|^{3/2}}\right) \\ &+ \mathcal{O}\left(\frac{|\delta - k\tau|^3}{k}\right), \end{aligned} \tag{5.158}$$

as $n \to \infty$ and as $|\delta - k\tau| \to 0$, which will be a remainder term of lower order. To verify that this term is indeed negligible for our present purposes, we note that, a simple calculation similar to the one in (5.149), yields

$$\bar{Q}_n^{**}\left(\delta + o_p\left(\frac{k^{1/2}}{|W_n|^{1/2}}\right)\right) = \bar{Q}_n^{**}(\delta) + o_p\left(\frac{|\delta - k\tau|}{k^{1/2}|W_n|^{1/2}}\right) + o_p\left(\frac{k^{1/2}}{|W_n|}\right) \quad (5.159)$$

as $n \to \infty$ and as $|\delta - k\tau| \to 0$. Now we notice that $|\delta - k\tau|k^{1/2}|W_n|^{-1} = o(|\delta - k\tau|k^{-1/2}|W_n|^{-1/2})$ provided $k = o(|W_n|^{1/2})$, and $k|W_n|^{-3/2} = o(k^{1/2}|W_n|^{-1})$ provided $k = o(|W_n|)$, as $n \to \infty$. We also have $(\delta - k\tau)^2|W_n|^{-1/2} = o(k^{1/2}|W_n|^{-1})$ and $|\delta - k\tau|^3 k^{-1} = o(k^{1/2}|W_n|^{-1})$, provided $(\delta - k\tau) = \mathcal{O}(k|W_n|^{-1/2})$ and $k = o(|W_n|^{1/3})$, as $n \to \infty$. Hence, since $R_n^{**}(\delta)$, i.e. the r.h.s. of (5.158), is at most of order

$$o_p\left(\frac{k^{1/2}}{|W_n|}\right) + o_p\left(\frac{|\delta - k\tau|}{k^{1/2}|W_n|^{1/2}}\right),$$

which does not exceed the remainder terms on the r.h.s. of (5.159), we have proved (5.157). Therefore we have proved (5.153).

Next we consider (5.154). To prove (5.154), we have to show, for each $\epsilon > 0$,

$$\mathbf{P}\left(\left|k\hat{\tau}_{k,n}^* - k\hat{\tau}_{k,n,s}^*\right| \geq \frac{\epsilon k^{1/2}}{|W_n|^{1/2}}\right) \to 0, \quad (5.160)$$

as $n \to \infty$. The probability on the l.h.s. of (5.160) does not exceed $\mathbf{P}(k\hat{\tau}_{k,n}^* \neq k\hat{\tau}_{k,n,s}^*)$. Hence, by an argument similar to the one employed in the proof of Lemma 5.7, with $Q_n(\delta)$ now replaced by $Q_n^{**}(\delta)$, we obtain $\mathbf{P}(k\hat{\tau}_{k,n}^* \neq k\hat{\tau}_{k,n,s}^*) \to 0$ as $n \to \infty$, which implies (5.160), and hence (5.154). Note that, to prove (5.160) we again require $k = o(|W_n|^{1/3})$ as $n \to \infty$. This completes the proof of Theorem 5.4. □

To conclude this section, we state and prove Lemma 5.13, which was needed in the proof of Theorem 5.4.

Lemma 5.13 *Suppose that λ is periodic and locally integrable. Then, for any positive integer k satisfying $k = o(|W_n|)$ and for any δ in a neighborhood of $k\tau$, we have*

$$\frac{\tilde{X}^2(W_{N_{n\delta}})}{|W_n|N_{n\delta}} + \frac{\tilde{X}(W_{N_{n\delta}})}{|W_n|} = \frac{\theta k\tau}{|W_n|}N_3^2(0,1) + \frac{\theta^{1/2}}{|W_n|^{1/2}}N_3(0,1)$$
$$+\mathcal{O}_p\left(\frac{k^{3/2}}{|W_n|^{3/2}} + \frac{k^{1/2}}{|W_n|}\right) + \mathcal{O}_p\left(\frac{|\delta - k\tau|}{|W_n|}\right) + \mathcal{O}_p\left(\frac{k}{|W_n|^{3/2}} + \frac{k^2}{|W_n|^2}\right), \quad (5.161)$$

as $n \to \infty$ and $|\delta - k\tau| \to 0$ with the $\mathcal{O}_p(k^{3/2}|W_n|^{-3/2} + k^{1/2}|W_n|^{-1})$ term does not depend on δ, where $N_3(0,1)$ denotes a standard normal r.v.

Proof: Let $\bar{Y}_{n,\delta} = \tilde{X}(W_{N_{n\delta}})|W_n|^{-1/2} = \sum_{i=1}^{N_{n\delta}} \tilde{X}(U_{\delta,i})$. First, we will show that

$$\left|\bar{Y}_{n,\delta} - N(0, Var(\bar{Y}_{n,\delta}))\right| \xrightarrow{p} 0, \tag{5.162}$$

as $n \to \infty$, where

$$Var\left(\bar{Y}_{n,\delta}\right) = |W_n|^{-1} \int_{W_{N_{n\delta}}} \lambda(s)ds = \theta + \mathcal{O}(k|W_n|^{-1}) \tag{5.163}$$

as $n \to \infty$, uniformly in δ. For any $i \neq j$; $i, j = 1, \ldots N_{n\delta}$, we have $\tilde{X}(U_{\delta,i})$ and $\tilde{X}(U_{\delta,j})$ are independent, $\mathbf{E}\tilde{X}(U_{\delta,i}) = 0$ for all i, and $Var(\tilde{X}(U_{\delta,i})) = \int_{U_{\delta,i}} \lambda(s)ds = \mathcal{O}(k)$ as $n \to \infty$, uniformly in δ. Hence we have

$$\left(\sum_{i=1}^{N_{n\delta}} Var(\tilde{X}(U_{\delta,i}))\right)^2 = \mathcal{O}(|W_n|^2),$$

as $n \to \infty$. Then to prove (5.162), it suffices to check that the Lyapounov's condition

$$\sum_{i=1}^{N_{n\delta}} \mathbf{E}\left(\tilde{X}(U_{\delta,i})\right)^4 = o\left(|W_n|^2\right), \tag{5.164}$$

holds, as $n \to \infty$ (cf. Serfling (1980), p. 30). To prove (5.164), we argue as follows. For each i, we have $\mathbf{E}(\tilde{X}(U_{\delta,i}))^4 = \mathcal{O}(k^2)$ as $n \to \infty$, uniformly in i. Hence, the quantity on the l.h.s. of (5.164) is of order $\mathcal{O}(|W_n|k)$, which is $o(|W_n|^2)$ as $n \to \infty$, because $k = o(|W_n|)$ as $n \to \infty$. Hence we have (5.162).

By (5.163) we can compute

$$\sigma(\bar{Y}_{n,\delta}) = \left(\theta + \mathcal{O}(k|W_n|^{-1})\right)^{1/2} = \theta^{1/2} + \mathcal{O}(k|W_n|^{-1}), \tag{5.165}$$

as $n \to \infty$, uniformly in δ. By (5.162) and (5.165), we can write

$$\bar{Y}_{n,\delta} = \tilde{X}(W_{N_{n\delta}})|W_n|^{-1/2} = \theta^{1/2}N(0,1) + \mathcal{O}_p(k|W_n|^{-1}), \tag{5.166}$$

as $n \to \infty$, uniformly in δ. By (5.166), we can write the l.h.s. of (5.161) as follows

$$\begin{aligned}
&\frac{1}{N_{n\delta}}\left(\frac{\tilde{X}(W_{N_{n\delta}})}{|W_n|^{1/2}}\right)^2 + \frac{1}{|W_n|^{1/2}}\left(\frac{\tilde{X}(W_{N_{n\delta}})}{|W_n|^{1/2}}\right) \\
&= \frac{1}{N_{n\delta}}\left(\theta^{1/2}N(0,1) + \mathcal{O}_p(k|W_n|^{-1})\right)^2 \\
&\quad + \frac{1}{|W_n|^{1/2}}\left(\theta^{1/2}N(0,1) + \mathcal{O}_p(k|W_n|^{-1})\right) \\
&= \frac{\theta}{N_{n\delta}}N^2(0,1) + \frac{\theta^{1/2}}{|W_n|^{1/2}}N(0,1) + \mathcal{O}_p\left(\frac{k}{|W_n|^{3/2}} + \frac{k^2}{|W_n|^2}\right),
\end{aligned} \tag{5.167}$$

as $n \to \infty$. For any δ in a neighborhood of $k\tau$, we have the following approximation

$$\begin{aligned}\frac{1}{N_{n\delta}} &= \frac{1}{|W_n|\delta^{-1}(1+\mathcal{O}(k|W_n|^{-1}))} = \frac{\delta}{|W_n|}\left(1+\mathcal{O}\left(\frac{k}{|W_n|}\right)\right)\\ &= \frac{k\tau}{|W_n|} + \mathcal{O}\left(\frac{|\delta - k\tau|}{|W_n|}\right) + \mathcal{O}\left(\frac{k^2}{|W_n|^2}\right), \qquad (5.168)\end{aligned}$$

as $n \to \infty$ and $|\delta - k\tau| \to 0$. Substituting (5.168) into the first term on the r.h.s. of (5.167), we obtain (5.161). The error in replacing $\bar{Y}_{n,\delta}/\sigma(\bar{Y}_{n,\delta})$ by $N_3(0,1)$ is of order $\mathcal{O}_p(k^{1/2}|W_n|^{-1/2})$, which follows easily from the Berry-Esseen bound for $\bar{Y}_{n,\delta}/\sigma(\bar{Y}_{n,\delta})$. Furthermore, we can split this error in to $\mathcal{O}_p(k^{1/2}|W_n|^{-1/2})+\mathcal{O}_p(|\delta-k\tau|k^{-1/2}|W_n|^{-1/2})$, as $n \to \infty$ and $|\delta-k\tau| \to 0$, where the $\mathcal{O}_p(k^{1/2}|W_n|^{-1/2})$ term does not depend on δ. This error is incorporated in the $\mathcal{O}_p(k^{3/2}|W_n|^{-3/2} + k^{1/2}|W_n|^{-1}) + \mathcal{O}_p(|\delta - k\tau||W_n|^{-1})$ term of (5.161). This completes the proof of Lemma 5.13. □

5.6 Some technical lemmas

Let $\Theta_k = (\tau_{k,0}, \tau_{k,1})$ where $(k-1)\tau < \tau_{k,0} < k\tau < \tau_{k,1} < (k+1)\tau$, and $k = k_n = o(|W_n|)$ as $n \to \infty$. Note that we always can write $\tau_{k,0} = k\tau_0$ and $\tau_{k,1} = k\tau_1$ for some positive constants τ_0 and τ_1.

First we take a brief look at the fact that $\hat{\tau}_{n,k}$ is not uniquely determined by (5.3). For each k, define

$$\begin{aligned}k\hat{\tau}_{n-,k} &= \inf\left(\arg\min_{\delta\in\Theta_k} Q_n(\delta)\right), \quad \text{and}\\ k\hat{\tau}_{n+,k} &= \sup\left(\arg\min_{\delta\in\Theta_k} Q_n(\delta)\right).\end{aligned}$$

In the following lemma we show that the difference between $k\hat{\tau}_{n-,k}$ and $k\hat{\tau}_{n+,k}$, i.e. the length of the 'lowest flat part' of $Q_n(\delta)$, is negligible for our purposes. So, any minimizer of (5.3) will do. Throughout this thesis, for any r.v. Y_n, we write $Y_n = \mathcal{O}_c(1)$ to denote that Y_n is bounded completely, as $n \to \infty$; that is there exists constant $M > 0$ such that $\sum_{n=1}^{\infty} \mathbf{P}(|Y_n| > M) < \infty$.

Lemma 5.14 *Let λ be periodic (with period τ) and locally integrable. Then, for any fixed $k = o(|W_n|)$,*

$$\frac{|W_n|}{k}(\hat{\tau}_{n+,k} - \hat{\tau}_{n-,k}) = \frac{|W_n|}{k^2}(k\hat{\tau}_{n+,k} - k\hat{\tau}_{n-,k}) = \mathcal{O}_c(1), \qquad (5.169)$$

as $n \to \infty$.

Proof: Consider two points $\hat{\delta}_{a,n}$ and $\hat{\delta}_{b,n}$ in the parameter space Θ_k, where $\hat{\delta}_{a,n} = k\hat{\tau}_{n-,k}$, and $\hat{\delta}_{b,n} \geq \hat{\delta}_{a,n}$ such that

$$\frac{|W_n|}{\hat{\delta}_{a,n}} = \frac{|W_n|}{\hat{\delta}_{b,n}} + 1,$$

for all $n \geq 1$. If, for each $n \geq 1$, a change in the value of $Q_n(\delta)$ is only caused by a change in the value of $N_{n\delta}$, then we would have that $(k\hat{\tau}_{n+,k} - k\hat{\tau}_{n-,k}) = (\hat{\delta}_{b,n} - \hat{\delta}_{a,n})$. But in fact, for each $n \geq 1$, a change in the value of $Q_n(\delta)$ can also be caused by a change in the value of $X(U_{\delta,i})$ for at least one i, $i = 1, 2, \ldots, N_{n\delta}$. Hence, we have that, for each $n \geq 1$,

$$(k\hat{\tau}_{n+,k} - k\hat{\tau}_{n-,k}) \leq (\hat{\delta}_{b,n} - \hat{\delta}_{a,n}).$$

Then to prove (5.169), it suffices to show

$$\sum_{n=1}^{\infty} \mathbf{P}\left(\frac{|W_n|}{k^2}(\hat{\delta}_{b,n} - \hat{\delta}_{a,n}) > C_0\right) < \infty, \tag{5.170}$$

for some constant C_0. Now note that

$$\frac{|W_n|}{\hat{\delta}_{a,n}} = \frac{|W_n|}{\hat{\delta}_{b,n}} + 1 \iff (\hat{\delta}_{b,n} - \hat{\delta}_{a,n}) = \frac{\hat{\delta}_{a,n}\hat{\delta}_{b,n}}{|W_n|}.$$

Since $\tau_{k,0} \leq \hat{\delta}_{a,n} \leq \tau_{k,1}$ and $\tau_{k,0} \leq \hat{\delta}_{b,n} \leq \tau_{k,1}$, we then have that $\hat{\delta}_{a,n}\hat{\delta}_{b,n} \leq \tau_{k,1}^2 = k^2\tau_1^2$. Hence we have that, with probability 1,

$$(\hat{\delta}_{b,n} - \hat{\delta}_{a,n}) \leq \frac{k^2\tau_1^2}{|W_n|}, \iff \frac{|W_n|}{k^2}(\hat{\delta}_{b,n} - \hat{\delta}_{a,n}) \leq \tau_1^2.$$

By choosing now $C_0 > \tau_1^2$, we then of course have (5.170), which also implies this lemma. This completes the proof of Lemma 5.14. □

Remark 5.1 By Lemma 5.14, in order to have that the statement like

$$|W_n|^{\gamma}\left(\hat{\tau}_{n,k} - \tau\right) \xrightarrow{c} 0$$

as $n \to \infty$, remains true whatever specific choice of $k\hat{\tau}_{n,k}$ is made, we require that $|W_n|k^{-1} > |W_n|^{\gamma}$, which is equivalent to $k < |W_n|^{1-\gamma}$. With $k \sim |W_n|^c$, this condition reduces to $\gamma < 1 - c$.

From the proof of Lemma 5.14, we can see that actually we have a stronger result than the statement in Lemma 5.14, namely we have the following. *Suppose that λ is periodic with period τ and locally integrable. If $k =$*

$k_n = o(|W_n|)$ as $n \to \infty$, then there exists a large positive constant C_0 and n_0 such that

$$\mathbf{P}\left(\frac{|W_n|}{k}(\hat{\tau}_{n+,k} - \hat{\tau}_{n-,k}) \leq C_0\right) = 1,$$

for all $n \geq n_0$.

In Lemma 5.15 we state (and prove) a well-known result on (almost) periodic functions, phrased in a form appropriate for our purposes. Recall that $W_{N_{n\delta}}$ denotes the union of all disjoint intervals $U_{\delta,i}$ of length δ in the window W_n, and θ denotes the 'global intensity' of the inhomogeneous cyclic Poisson process X.

Lemma 5.15 *Suppose that λ is periodic with period τ and locally integrable. Then, for any $\delta \in \Theta_k$ with $k = k_n = o(|W_n|)$, we have*

$$\frac{1}{|W_{N_{n\delta}}|}\int_{W_{N_{n\delta}}} \lambda(s)ds = \theta + \mathcal{O}\left(\frac{k}{|W_n|}\right), \tag{5.171}$$

as $n \to \infty$, uniformly in $\delta \in \Theta_k$.

Proof: Let $R_{n,\delta,k\tau} = W_{N_{n\delta}} \setminus W_{N_{nk\tau}}$ if $W_{N_{nk\tau}} \subseteq W_{N_{n\delta}}$, and $R_{n,\delta,k\tau} = W_{N_{nk\tau}} \setminus W_{N_{n\delta}}$ if $W_{N_{n\delta}} \subset W_{N_{nk\tau}}$. Since the intensity function λ is periodic with period τ, we have $|W_{N_{nk\tau}}|^{-1}\int_{W_{N_{nk\tau}}} \lambda(s)ds = \theta$, for all $|W_n| \geq k\tau$. Then to establish (5.171), it suffices to prove

$$|W_{N_{nk\tau}}||W_{N_{n\delta}}|^{-1} = 1 + \mathcal{O}\left(k|W_n|^{-1}\right) \tag{5.172}$$

and

$$|W_{N_{n\delta}}|^{-1}\int_{R_{n,\delta,k\tau}} \lambda(s)ds = \mathcal{O}\left(k|W_n|^{-1}\right), \tag{5.173}$$

as $n \to \infty$, uniformly in $\delta \in \Theta_k$.

First we prove (5.172). By its definition, we have

$$\frac{|W_{N_{nk\tau}}|}{|W_{N_{n\delta}}|} = \frac{k\tau N_{nk\tau}}{\delta N_{n\delta}} = \frac{(k\tau)\left(\frac{|W_n|}{k\tau} - \mathcal{O}(1)\right)}{\delta\left(\frac{|W_n|}{\delta} - \mathcal{O}(1)\right)} = \frac{|W_n| - \mathcal{O}(k)}{|W_n| - \mathcal{O}(k)}$$
$$= 1 + \mathcal{O}\left(k|W_n|^{-1}\right),$$

as $n \to \infty$, uniformly in $\delta \in \Theta_k$, since $k = k_n = o(|W_n|)$ as $n \to \infty$. Hence we have (5.172).

Next we prove (5.173). Recall that $W_n = [a_n, b_n]$. Since $\delta \in \Theta_k$ and $k\tau \in \Theta_k$, we have that $[a_n + \tau_{k,1}, b_n - \tau_{k,1}] \subseteq W_{N_{n\delta}} \subseteq W_n$ and $[a_n + \tau_{k,1}, b_n - \tau_{k,1}] \subseteq W_{N_{nk\tau}} \subseteq W_n$. Then, by its definition, we have $|R_{n,\delta,k\tau}| \leq |W_n \setminus [a_n + \tau_{k,1}, b_n - \tau_{k,1}]| = 2\tau_{k,1} \leq 2\tau(k+1)$. This implies $\int_{R_{n,\delta,k\tau}} \lambda(s)ds \leq 2\theta\tau(k+1)$. Since $|W_{N_{n\delta}}| \sim |W_n|$ as $n \to \infty$, uniformly in $\delta \in \Theta_k$, we then have (5.173). This completes the proof of Lemma 5.15. □

We note in passing that Lemma 5.15 contain Lemma 2.1 as a special case.

Recall that $\Lambda_n(\delta) = \mathbf{E}Q_n(\delta)$. First we simplify the expression for $\Lambda_n(\delta)$, which will be useful to illustrate the necessity of condition (5.5) and to prove Lemma 5.16. From (5.26) we can compute $\Lambda_n(\delta)$ as follows.

$$\Lambda_n(\delta) = \frac{1}{|W_n|}\sum_{i=1}^{N_{n\delta}} \mathbf{E}X^2(U_{\delta,i}) - \frac{1}{|W_n|N_{n\delta}}\mathbf{E}X^2(W_{N_{n\delta}}). \tag{5.174}$$

The first term on the r.h.s. of (5.174) can be simplified as follows

$$\begin{aligned} &\frac{1}{|W_n|}\sum_{i=1}^{N_{n\delta}}\left(\int_{U_{\delta,i}}\lambda(s)ds\right)^2 + \frac{1}{|W_n|}\int_{W_{N_{n\delta}}}\lambda(s)ds \\ &= \frac{1}{|W_n|}\sum_{i=1}^{N_{n\delta}}\left(\int_{U_{\delta,i}}\lambda(s)ds\right)^2 + \theta + \mathcal{O}\left(\frac{k}{|W_n|}\right), \end{aligned} \tag{5.175}$$

as $n \to \infty$. Here we have used Lemma 5.15. The second term on the r.h.s. of (5.174) is equal to

$$\begin{aligned} &-\frac{1}{|W_n|N_{n\delta}}\left(\sum_{i=1}^{N_{n\delta}}\int_{U_{\delta,i}}\lambda(s)ds\right)^2 - \frac{1}{|W_n|N_{n\delta}}\int_{W_{N_{n\delta}}}\lambda(s)ds \\ &= -\frac{1}{|W_n|N_{n\delta}}\left(\sum_{i=1}^{N_{n\delta}}\int_{U_{\delta,i}}\lambda(s)ds\right)^2 - \mathcal{O}\left(\frac{k}{|W_n|}\right), \end{aligned} \tag{5.176}$$

as $n \to \infty$, by Lemma 5.15. By (5.175) and (5.176), we have

$$\begin{aligned} \Lambda_n(\delta) \quad = \quad & \frac{1}{|W_n|}\sum_{i=1}^{N_{n\delta}}\left(\int_{U_{\delta,i}}\lambda(s)ds - \frac{1}{N_{n\delta}}\sum_{j=1}^{N_{n\delta}}\int_{U_{\delta,j}}\lambda(s)ds\right)^2 + \theta \\ & + \mathcal{O}\left(\frac{k}{|W_n|}\right), \end{aligned} \tag{5.177}$$

as $n \to \infty$, uniformly in $\delta \in \Theta_k$. The expression for $\Lambda_n(\delta)$ in (5.177) can further simplified, by another application of Lemma 5.15, to obtain

$$\Lambda_n(\delta) = \frac{1}{|W_n|}\sum_{i=1}^{N_{n\delta}}\left(\int_{U_{\delta,i}}\lambda(s)ds - \delta\theta\right)^2 + \theta + \mathcal{O}\left(\frac{k}{|W_n|}\right),$$

as $n \to \infty$, uniformly in $\delta \in \Theta_k$.

In order to prove our consistency result (cf. also Theorem 5.2), we will require that if $\Lambda_n(\delta)$ attains its minimum value $\theta + \mathcal{O}(k|W_n|^{-1})$, this implies that $\delta = k\tau$ (otherwise (5.181) may fail). In other words, we assume that: *if for each* $\delta \in \Theta_k$ *and each* $r \in [0, (|W_n| - \delta N_{n\delta})]$, *we have*

$$\int_{U_{\delta,i}}\lambda(s)ds = \delta\theta \text{ for all } i,\ i = 1, \dots N_{n\delta}, \text{ then } \delta = k\tau, \tag{5.178}$$

where $U_{\delta,i} = [a_n + r + (i-1)\delta, a_n + r + i\delta)$. Note, however, that (5.178) need not to hold for each value of r. In fact we can weaken condition (5.178) slightly, and replace it by condition (5.5), that is: *if there exists* $t \in (0,\tau)$ *such that, for each* $n \geq 1$,

$$\int_{U_{t,i}}\lambda(s)ds = t\theta \ \text{ for all } i,\ i = 1, \dots, N_{nt}, \ \text{ then}$$

$$\nu(\{r : \int_{U_{t,i}}\lambda(s)ds = t\theta;\ i = 1, \dots, N_{nt}\}) = 0, \tag{5.179}$$

with $U_{t,i} = [a_n + r + (i-1)t, a_n + r + it)$, $i = 1, \dots, N_{nt}$, and $N_{nt} = [\frac{|W_n|}{t}]$ as before.

First we note that condition (5.5) implies (5.4). For any λ satisfying (5.5), the choice of r in the construction of $Q_n(\delta)$ is basically free, i.e. for ν-almost every $r \in [0, (|W_n| - \delta N_{n\delta})]$, Lemma 5.16 holds true and hence all of our theorems in this chapter.

In connection with the condition (5.5), here we present two examples of λ, the first example satisfies condition (5.5), while the second one does not. For the first example, we consider intensity function λ of the form $\lambda(s) = \cos(s) + 1$, which is cyclic with period $\tau = 2\pi$. Here we have (sufficiently large n), $\int_{U_{\pi,i}}\lambda(s)ds = \pi\theta$, for all $i = 1, \dots, N_{n\pi}$ (where $U_{\pi,i} = [a_n + r + (i-1)\pi, a_n + r + i\pi)$ and $N_{n\pi} = [\frac{|W_n|}{\pi}]$), if and only if we take $a_n + r \in \{j\pi; j \in \mathbf{Z}\}$, where $\mathbf{Z}$ denotes the set of integers. Since $\nu(\{j\pi; j \in \mathbf{Z}\} = 0$, λ in this example satisfies condition (5.5).

In the second example, we consider intensity function λ of the form

$$\lambda(s) = \begin{cases} s - [s], & \text{if } s \in B_1 \\ \frac{1}{2}, & \text{if } s \in B_2 \\ 1 - s + [s], & \text{if } s \in B_3 \end{cases} \tag{5.180}$$

where $B_1 = \cup_{i\in\mathbf{Z}}\{[0,1)+2i\}$, $B_2 = \cup_{i\in\mathbf{Z}}\{[1,2)+4i\}$, $B_3 = \cup_{i\in\mathbf{Z}}\{[3,4)+4i\}$, and we have $B_1 \cup B_2 \cup B_3 = \mathbf{R}$. Clearly λ is cyclic with period 4 and $\theta = \frac{1}{2}$. Here, for each $a_n + r \in B_1$ and each (sufficiently large) n, we have that $\int_{U_{2,i}} \lambda(s)ds = 2\theta$, for all $i = 1, \ldots, N_{n2}$, where $U_{2,i} = [a_n + r + 2(i-1), a_n + r + 2i)$ and $N_{n2} = [\frac{|W_n|}{2}]$. Since $\nu(B_1) > 0$, λ in this example does not satisfy condition (5.5).

For each k, let $B_{k,\epsilon_n} = (k\tau - \epsilon_n, k\tau + \epsilon_n)$, where ϵ_n is an arbitrary sequence of positive real numbers such that $B_{k,\epsilon_n} \subset \Theta_k$, for all n. The sequence ϵ_n may or may not converges to zero, as $n \to \infty$.

Lemma 5.16 *Suppose that λ is periodic (with period τ) and locally integrable. In addition we assume that λ satisfy (5.5).*

(i) Then, for any sequence of positive real numbers ϵ_n such that $\epsilon_n^{-1} = o(|W_n|k_n^{-1})$ as $n \to \infty$, and for each integer k_n such that $k_n = o(|W_n|^{1/2}\epsilon_n)$, there exists $\alpha_0 > 0$ and positive integer n_0 such that

$$\Lambda_n(\delta) - \Lambda_n(k_n\tau) > \epsilon_n^2 \alpha_0 k_n^{-1}, \tag{5.181}$$

for all $\delta \in \Theta_k \setminus B_{k,\epsilon_n}$ and all $n \geq n_0$. In addition, for each k satisfying $k = o(|W_n|)$, we have

$$\Lambda_n(k\tau) = \theta + \mathcal{O}\left(k|W_n|^{-1}\right), \tag{5.182}$$

$n \to \infty$.

(ii) If, in addition, λ is Lipschitz, then for any positive integer k satisfying $k = o(|W_n|^{1/2})$ and for any δ in the neighborhood of $k\tau$ such that $|W_n|k^{-1}|\delta - k\tau| \to \infty$ as $n \to \infty$, we have

$$\begin{aligned}\Lambda_n(\delta) &= \frac{(\delta - k\tau)^2}{k\tau^2}\int_0^\tau (\lambda(s)-\theta)^2 ds + \theta + \mathcal{O}\left(\frac{k}{|W_n|}\right) \\ &\quad + \mathcal{O}\left(|\delta - k\tau|^3 k^{-1}\right),\end{aligned} \tag{5.183}$$

where the $O(|\delta - \tau|^3 k^{-1})$ term holds true as $|\delta - k\tau| \to 0$, uniformly in n, and the $\mathcal{O}(k|W_n|^{-1})$ term holds true as $n \to \infty$ uniformly for all δ in the neighborhood of $k\tau$.

Proof: First we prove part (i) of this lemma. By (5.177) we have

$$\begin{aligned}\Lambda_n(\delta) - \Lambda_n(k_n\tau) &= \frac{1}{|W_n|}\sum_{i=1}^{N_{n\delta}}\left(\int_{U_{\delta,i}}\lambda(s)ds - \frac{1}{N_{n\delta}}\sum_{j=1}^{N_{n\delta}}\int_{U_{\delta,j}}\lambda(s)ds\right)^2 \\ &\quad + \mathcal{O}(k|W_n|^{-1})\end{aligned} \tag{5.184}$$

as $n \to \infty$, uniformly in $\delta \in \Theta_k$. Here we certainly need condition (5.5), because otherwise the first term on the r.h.s. of (5.184) may equal to zero when δ is not a multiple of τ. In this case, (5.181) can not hold true.

Now we consider the first term on the r.h.s. of (5.177). Let we number the interval $U_{\delta,i}$ from left to right by $1, 2, \ldots, N_{n\delta}$. Recall that $W_n \setminus W_{N_{n\delta}}$ may consists of two separate parts, namely the parts on the left-end-point and the right-end-point of W_n. Total length of these two parts is $|W_n| - [\frac{|W_n|}{\delta}]$. Let $r = r_{n\delta}$, $0 \le r < \delta$, denotes the length of part of $W_n \setminus W_{N_{n\delta}}$ which is on the left-end-point of W_n. Then, for each i, $i = 1, 2, \ldots, N_{n\delta}$, the interval $U_{\delta,i}$ can be written as $[a_n+r+(i-1)\delta, a_n+r+i\delta)$. Hence, for each i, $i = 1, \ldots, N_{n\delta}$, we can write

$$\int_{U_{\delta,i}} \lambda(s)ds = \int_{a_n+r+(i-1)\delta}^{a_n+r+i\delta} \lambda(s)ds = k\theta\tau + \int_{a_n+r+(i-1)(\delta-k\tau)}^{a_n+r+i(\delta-k\tau)} \lambda(s)ds. \quad (5.185)$$

We also have that

$$\begin{aligned}
\frac{1}{N_{n\delta}} \sum_{i=1}^{N_{n\delta}} \int_{U_{\delta,i}} \lambda(s)ds &= k\theta\tau + \frac{1}{N_{n\delta}} \int_{a_n+r}^{a_n+r+N_{n\delta}(\delta-k\tau)} \lambda(s)ds \\
&= k\theta\tau + \left(\frac{\delta}{|W_n|} + \mathcal{O}(|W_n|^{-2})\right)\left(\frac{|W_n|(\delta-k\tau)}{\delta\tau} \int_0^\tau \lambda(s)ds + \mathcal{O}(|\delta-k\tau|)\right) \\
&= k\theta\tau + \theta(\delta-k\tau) + \mathcal{O}(k|W_n|^{-1}|\delta-k\tau|), \qquad (5.186)
\end{aligned}$$

where the $\mathcal{O}(|W_n|^{-2})$ term holds true as $n \to \infty$, uniformly in δ in the neighborhood of $k\tau$, the $\mathcal{O}(|\delta-k\tau|)$ term is valid as $|\delta-k\tau| \to 0$, uniformly in n, and $\mathcal{O}(k|W_n|^{-1}|\delta-k\tau|)$, term holds true as $n \to \infty$ and $|\delta-k\tau| \to 0$.

By (5.185) and (5.186), using the fact that $|\delta-k\tau| = \mathcal{O}(1)$ as $n \to \infty$, uniformly in δ, (5.184) can be written as

$$\begin{aligned}
\Lambda_n(\delta) - \Lambda_n(k_n\tau) &= \frac{1}{|W_n|} \sum_{i=1}^{N_{n\delta}} \left(\int_{-n+r+(i-1)(\delta-k\tau)}^{-n+r+i(\delta-k\tau)} \lambda(s)ds - \theta(\delta-k_n\tau) \right)^2 \\
&\quad + \mathcal{O}(k|W_n|^{-1})
\end{aligned}$$

as $n \to \infty$ uniformly in $\delta \in \Theta_k$. Let $J_{\delta,k\tau} = [\frac{\tau}{|\delta-k\tau|}]$ for all $\delta \in \Theta_k \setminus B_{k,\epsilon_n}$. Since λ is periodic (with period τ) and $\epsilon_n^{-1} = o(|W_n|k_n^{-1})$ so that for any $\delta \in \Theta_k \setminus B_{k,\epsilon_n}$ we have $N_{n\delta}J_{\delta,k\tau}^{-1} \to \infty$ as $n \to \infty$, we then have

$$\begin{aligned}
&\Lambda_n(\delta) - \Lambda_n(k_n\tau) \\
&\ge \frac{1}{|W_n|}\left[\frac{N_{n\delta}}{J_{\delta,k\tau}}\right] \sum_{i=1}^{J_{\delta,k\tau}} \left(\int_{a_n+r+(i-1)(\delta-k\tau)}^{a_n+r+i(\delta-k\tau)} \lambda(s)ds - \theta(\delta-k_n\tau) \right)^2 + \mathcal{O}\left(\frac{k}{|W_n|}\right) \\
&\ge \frac{1}{2\delta J_{\delta,k\tau}} \sum_{i=1}^{J_{\delta,k\tau}} \left(\int_{a_n+r+(i-1)(\delta-k\tau)}^{a_n+r+i(\delta-k\tau)} \lambda(s)ds - \theta(\delta-k_n\tau) \right)^2 + \mathcal{O}\left(\frac{k}{|W_n|}\right),
\end{aligned}$$

as $n \to \infty$ uniformly in $\delta \in \Theta_k \setminus B_{k,\epsilon_n}$. Note that, since any multiple of τ, except $k_n\tau$, is not contained in Θ_k, we have that $|\delta - k_n\tau| < \tau$ for all $\delta \in \Theta_k$. Because λ is not constant a.e. w.r.t. ν (cf. (5.4)), then the integrals $\int_{a_n+r+(i-1)(\delta-k\tau)}^{a_n+r+i(\delta-k\tau)} \lambda(s)ds$ for $i = 1, 2, \ldots, J_{\delta,k\tau}$ are not all identical. For each i, we define a constant c_i, where

$$c_i = (\delta - k_n\tau)^{-1} \int_{a_n+r+(i-1)(\delta-k\tau)}^{a_n+r+i(\delta-k\tau)} \lambda(s)ds.$$

Then we have

$$\Lambda_n(\delta) - \Lambda_n(k_n\tau) \geq (\delta - k_n\tau)^2 \frac{1}{2\delta J_{\delta,k\tau}} \sum_{i=1}^{J_{\delta,k\tau}} (c_i - \theta)^2 + \mathcal{O}\left(\frac{k}{|W_n|}\right), \tag{5.187}$$

as $n \to \infty$ uniformly in $\delta \in \Theta_k$. Since the integrals $\int_{a_n+r+(i-1)(\delta-k\tau)}^{a_n+r+i(\delta-k\tau)} \lambda(s)ds$ for $i = 1, 2, \ldots, J_{\delta,k\tau}$ are not all identical, we also have that the constant c_i's for $i = 1, 2, \ldots, J_{\delta,k\tau}$ are not all identical. This implies there exists $\epsilon_0 > 0$ such that $J_{\delta,k\tau}^{-1} \sum_{i=1}^{J_{\delta,k\tau}} (c_i - \theta)^2 > \epsilon_0$. Now note also that $(\delta - k_n\tau)^2 \geq \epsilon_n^2$ for all $\delta \in \Theta_k \setminus B_{k,\epsilon_n}$, and $\delta \leq \tau_{k,1}$. Then, for all $\delta \in \Theta_k \setminus B_{k,\epsilon_n}$, the first term on the r.h.s. of (5.187) is greater or equal to $(\epsilon_n^2\epsilon_0)/(2\tau_{k,1}) = (\epsilon_n^2\epsilon_0)/(2k_n\tau_1)$. Since $k_n = o(|W_n|^{1/2}\epsilon_n)$ and the $\mathcal{O}(k|W_n|^{-1})$ remainder term on r.h.s. of (5.187) holds true as $n \to \infty$ uniformly in $\delta \in \Theta_k$, there exists large real number n_0 such that the absolute value of this term does not exceed $(\epsilon_n^2\epsilon_0)/(4k_n\tau_1)$ for all $n \geq n_0$. By choosing now $\alpha_0 = (\epsilon_0)/(4\tau_1)$, we then get part (i) of this lemma.

Next we prove (5.182). By (5.177) with δ replaced by $k\tau$ and by noting that for each i ($i = 1, \ldots, N_{nk\tau}$) we have $\int_{U_{k\tau,i}} \lambda(s)ds = k\theta\tau$, we then get

$$\begin{aligned}\Lambda_n(k\tau) = \frac{N_{nk\tau}k\theta\tau}{|W_n|} - \frac{k\theta\tau}{|W_n|} &= \frac{k\theta\tau}{|W_n|}\left[\frac{|W_n|}{k\tau}\right] - \frac{k\theta\tau}{|W_n|} \\ = \theta - \frac{k\theta\tau}{|W_n|}\left\{\left(\frac{|W_n|}{k\tau} - \left[\frac{|W_n|}{k\tau}\right]\right) + 1\right\} &= \theta + \mathcal{O}\left(\frac{k}{|W_n|}\right)\end{aligned} \tag{5.188}$$

as $n \to \infty$, since $0 \leq (\frac{|W_n|}{k\tau} - [\frac{|W_n|}{k\tau}]) < 1$.

Next we prove part (ii) of this lemma. Since λ is Lipschitz, then we can write,

$$\begin{aligned}\int_{a_n+r+(i-1)(\delta-k\tau)}^{a_n+r+i(\delta-k\tau)} \lambda(s)ds &= (\delta - k\tau)\lambda(a_n + r + (i-1)(\delta - k\tau)) \\ &\quad + \mathcal{O}((\delta - k\tau)^2),\end{aligned} \tag{5.189}$$

as $|\delta - k\tau| \to 0$. Combining (5.185), (5.186), and (5.189), for any δ in the neighborhood of $k\tau$, we can write the first term on the r.h.s. of (5.177) as follows

$$\frac{(\delta-k\tau)^2}{|W_n|}\sum_{i=1}^{N_{n\delta}}\left(\lambda(a_n+r+(i-1)(\delta-k\tau))-\theta+\mathcal{O}(|\delta-k\tau|)+\mathcal{O}(k|W_n|^{-1})\right)^2$$
$$=\frac{(\delta-k\tau)^2N_{n\delta}}{|W_n|}\frac{1}{N_{n\delta}}\sum_{i=1}^{N_{n\delta}}(\lambda(a_n+r+(i-1)(\delta-k\tau))-\theta)^2+\mathcal{O}\left(\frac{|\delta-k\tau|^3}{k}\right)$$
$$+\mathcal{O}\left((\delta-k\tau)^2|W_n|^{-1}\right), \tag{5.190}$$

where the $\mathcal{O}(|\delta-\tau|^3/k)$ term holds true as $|\delta-k\tau|\to 0$, uniformly in n, and the $\mathcal{O}((\delta-k\tau)^2|W_n|^{-1})$ term holds true as $n\to\infty$ and $|\delta-k\tau|\to 0$.

Recall our notation $J_{\delta,k\tau}=[\frac{\tau}{|\delta-k\tau|}]$, and note also that

$$J_{\delta,k\tau}^{-1}=\frac{|\delta-k\tau|}{\tau}+\mathcal{O}\left((\delta-k\tau)^2\right), \tag{5.191}$$

as $|\delta-k\tau|\to 0$. Since $|W_n|k^{-1}|\delta-k\tau|\to\infty$, we have $J_{\delta,k\tau}=o(N_{n\delta})$ as $n\to\infty$ and as $|\delta-k\tau|\to 0$. Then, we can compute the following quantity

$$\frac{1}{N_{n\delta}}\sum_{i=1}^{N_{n\delta}}(\lambda(a_n+r+(i-1)(\delta-k\tau))-\theta)^2$$
$$=\frac{1}{N_{n\delta}}\left\{\left[\frac{N_{n\delta}}{J_{\delta,k\tau}}\right]\sum_{i=1}^{J_{\delta,k\tau}}(\lambda(a_n+r+(i-1)(\delta-k\tau))-\theta)^2+\mathcal{O}(J_{\delta,k\tau})\right\}$$
$$=\frac{1}{N_{n\delta}}\frac{N_{n\delta}}{J_{\delta,k\tau}}\sum_{i=1}^{J_{\delta,k\tau}}(\lambda(a_n+r+(i-1)(\delta-k\tau))-\theta)^2+\mathcal{O}\left(\frac{J_{\delta,k\tau}}{N_{n\delta}}\right)$$
$$=\left(\frac{|\delta-k\tau|}{\tau}+\mathcal{O}\left((\delta-k\tau)^2\right)\right)\sum_{i=1}^{J_{\delta,k\tau}}(\lambda(a_n+r+(i-1)(\delta-k\tau))-\theta)^2$$
$$+\mathcal{O}\left(\frac{J_{\delta,k\tau}}{N_{n\delta}}\right)$$
$$=\frac{1}{\tau}\sum_{i=1}^{J_{\delta,k\tau}}|\delta-k\tau|\,(\lambda(a_n+r+(i-1)(\delta-k\tau))-\theta)^2+\mathcal{O}\left(|\delta-k\tau|\right)+\mathcal{O}\left(\frac{J_{\delta,k\tau}}{N_{n\delta}}\right)$$
$$=\frac{1}{\tau}\int_0^{\tau}(\lambda(s)-\theta)^2ds+\mathcal{O}\left(|\delta-k\tau|\right)+\mathcal{O}\left(\frac{J_{\delta,k\tau}}{N_{n\delta}}\right), \tag{5.192}$$

as $|\delta-k\tau|\to 0$ uniformly in n. The idea here is, since λ is periodic with period τ, $\lambda(a_n+r+(i-1)(\delta-k\tau))$ for $i=1,\ldots,N_{n\delta}$ can be divided into $[N_{n\delta}J_{\delta,k\tau}^{-1}]$ blocks, and within each block we have indexes $i=1,\ldots,J_{\delta,k\tau}$. The error due to this approximation is of order $\mathcal{O}(J_{\delta,k\tau})$, as $|\delta-k\tau|\to 0$

uniformly in n. Note that the error for the Riemann approximation is incorporated in the $\mathcal{O}(|\delta - k\tau|)$ remainder term. By (5.192) and by noting that $N_{n\delta} = |W_n|\delta^{-1} + \mathcal{O}(1)$ as $n \to \infty$, the first term on the r.h.s. of (5.190) can be written as follows

$$\begin{aligned}
&\frac{(\delta - k\tau)^2}{|W_n|}\left(\frac{|W_n|}{\delta} + \mathcal{O}(1)\right)\frac{1}{\tau}\int_0^\tau (\lambda(s) - \theta)^2 ds \\
&+\frac{(\delta - k\tau)^2 N_{n\delta}}{|W_n|}\left(\mathcal{O}\left(|\delta - k\tau|\right) + \mathcal{O}\left(\frac{J_{\delta,k\tau}}{N_{n\delta}}\right)\right) \\
&= \frac{(\delta - k\tau)^2}{\delta\tau}\int_0^\tau (\lambda(s) - \theta)^2 ds + \mathcal{O}\left(\frac{(\delta - k\tau)^2}{|W_n|}\right) + \mathcal{O}\left(\frac{|\delta - k\tau|^3}{k}\right) \\
&+ \mathcal{O}\left(\frac{|\delta - k\tau|}{|W_n|}\right) \\
&= \frac{(\delta - k\tau)^2}{k\tau^2}\int_0^\tau (\lambda(s) - \theta)^2 ds + \mathcal{O}\left(\frac{|\delta - k\tau|^3}{k}\right) + \mathcal{O}\left(\frac{|\delta - k\tau|}{|W_n|}\right), \qquad (5.193)
\end{aligned}$$

where the $\mathcal{O}(|\delta - k\tau|^3/k)$ term holds true as $|\delta - k\tau| \to 0$, uniformly in n, and the $\mathcal{O}(|\delta - \tau||W_n|^{-1})$ term holds true as $n \to \infty$ and $|\delta - k\tau| \to 0$. Substituting the r.h.s. of (5.193) into the first term on the r.h.s. of (5.190), and subsequently substituting the r.h.s. of (5.190) into the first term on the r.h.s. of (5.177), we then get (5.183). This completes the proof of Lemma 5.16. □

Lemma 5.17 *Suppose that λ is periodic and locally integrable. Then for each positive integer k satisfying $k = o(|W_n|)$ as $n \to \infty$ and for any positive integer m, we have*

$$\mathbf{E}\left(\tilde{Q}_n(\delta)\right)^{2m} = \mathcal{O}\left(\frac{k_n^m}{|W_n|^m}\right), \qquad (5.194)$$

as $n \to \infty$, uniformly for all $\delta \in \Theta_k$.

Proof: Recall that $\tilde{Q}_n(\delta)$ can be written as that in (5.31). First we will show that, for any sequence of intervals A_n such that $|A_n| \to \infty$ as $n \to \infty$ and for any fixed positive integer m, we have

$$\mathbf{E}\left(\widetilde{\tilde{X}^2(A_n)}\right)^{2m} = \mathcal{O}\left(|A_n|^{2m}\right), \qquad (5.195)$$

as $n \to \infty$. To verify (5.195) we argue as follows. Proving (5.195) is equivalent to proving

$$\mathbf{E}\left(\frac{\widetilde{\tilde{X}^2(A_n)}}{|A_n|}\right)^{2m} = \mathbf{E}\left(\frac{\tilde{X}^2(A_n)}{|A_n|} - \frac{\int_{A_n}\lambda(s)ds}{|A_n|}\right)^{2m} = \mathcal{O}(1), \qquad (5.196)$$

as $n \to \infty$. By Lemma 2.1, we have $|A_n|^{-1} \int_{A_n} \lambda(s)ds \to \theta = \mathcal{O}(1)$ as $n \to \infty$. Then, to prove (5.196) it suffices to check, for any positive integer m,

$$\mathbf{E}\left(\frac{\tilde{X}^2(A_n)}{|A_n|}\right)^{2m} = \mathbf{E}\left(\frac{\tilde{X}(A_n)}{|A_n|^{1/2}}\right)^{4m} = \mathcal{O}(1), \tag{5.197}$$

as $n \to \infty$. A simple calculation shows that, for any positive integer m, we have $\mathbf{E}(\tilde{X}(A_n))^{4m} = \mathcal{O}(|A_n|^{2m})$ as $n \to \infty$, which implies (5.197).

Now we proceed the proof of (5.194). By (5.31), to prove (5.194), it suffices to check

$$\mathbf{E}\left(|W_n|^{-1}\sum_{i=1}^{N_{n\delta}} \widetilde{\tilde{X}^2(U_{\delta,i})}\right)^{2m} = \mathcal{O}\left(k_n^m |W_n|^{-m}\right), \tag{5.198}$$

$$\mathbf{E}\left(|W_n|^{-1} N_{n\delta}^{-1} \widetilde{\tilde{X}^2(W_{N_{n\delta}})}\right)^{2m} = \mathcal{O}\left(k_n^m |W_n|^{-m}\right), \tag{5.199}$$

$$\mathbf{E}\left(\mathcal{O}\left(|W_n|^{-1}\right) \tilde{X}(W_{N_{n\delta}})\right)^{2m} = \mathcal{O}\left(k_n^m |W_n|^{-m}\right), \tag{5.200}$$

as $n \to \infty$, uniformly for all $\delta \in \Theta_k$.

First we consider (5.199) and (5.200). Note that $|W_{N_{n\delta}}| \leq |W_n|$ and $|W_{N_{n\delta}}| \sim |W_n|$ as $n \to \infty$. Application of (5.195) with $A_n = W_{N_{n\delta}}$ yields that the l.h.s. of (5.199) is of order

$$\mathcal{O}\left(|W_n|^{-2m} N_{n\delta}^{-2m} |W_{N_{n\delta}}|^{2m}\right) = \mathcal{O}\left(k_n^{2m} |W_n|^{-2m}\right)$$

as $n \to \infty$, uniformly for all $\delta \in \Theta_k$, which implies (5.199), since $k_n = o(|W_n|)$ as $n \to \infty$. By (5.197) with $A_n = W_{N_{n\delta}}$, the l.h.s. of (5.200) can be written as

$$\mathbf{E}\left\{\mathcal{O}\left(\frac{|W_{N_{n\delta}}|^{1/2}}{|W_n|}\right)\left(\frac{\tilde{X}(W_{N_{n\delta}})}{|W_{N_{n\delta}}|^{1/2}}\right)\right\}^{2m} = \mathcal{O}\left(|W_n|^{-m}\right)$$

as $n \to \infty$, uniformly for all $\delta \in \Theta_k$, which implies (5.200).

Next we prove (5.198). The l.h.s. of (5.198) can be written as

$$|W_n|^{-2m} \sum_{i_1=1}^{N_{n\delta}} \cdots \sum_{i_{2m}=1}^{N_{n\delta}} \mathbf{E}\widetilde{\tilde{X}^2(U_{\delta,i_1})} \cdots \widetilde{\tilde{X}^2(U_{\delta,i_{2m}})}. \tag{5.201}$$

Now we distinguish $2m$ cases, namely: case (1) where all indices are the same, until case ($2m$) where all indices are different. Next we split the quantity in (5.201) into $2m$ terms, where each term corresponds to one of the $2m$ cases. Since, for each i, $\mathbf{E}\widetilde{\tilde{X}^2(U_{\delta,i})} = 0$, all terms corresponding to case $(m+1)$ until case $(2m)$ are equal to zero. Now we indicate how to treat the other m cases. By an application of (5.195) with $A_n = U_{\delta,i}$ and by noting that $|U_{\delta,i}| = \mathcal{O}(k_n)$ as $n \to \infty$ uniformly in i, we have that, for any fixed positive integer m,

$$\mathbf{E}\left(\widetilde{\tilde{X}^2(U_{\delta,i})}\right)^{2m} = \mathcal{O}\left(k_n^{2m}\right), \tag{5.202}$$

as $n \to \infty$, uniformly in i. By (5.202), the term corresponding to case (1) can be computed as follows

$$\begin{aligned} |W_n|^{-2m} \sum_{i=1}^{N_{n\delta}} \mathbf{E}\left(\widetilde{\tilde{X}^2(U_{\delta,i})}\right)^{2m} &= \mathcal{O}\left(|W_n|^{-2m} N_{n\delta} k_n^{2m}\right) \\ &= \mathcal{O}\left(k_n^{2m-1} |W_n|^{-(2m-1)}\right) \end{aligned} \tag{5.203}$$

as $n \to \infty$, uniformly for all $\delta \in \Theta_k$. The term corresponding to case (m), when there are m different pairs with the same index (this will be the leading term), is equal to

$$\begin{aligned} & |W_n|^{-2m} \sum_{i_1=1}^{N_{n\delta}} \cdots \sum_{i_m=1}^{N_{n\delta}} \mathbf{E}\left(\widetilde{\tilde{X}^2(U_{\delta,i_1})}\right)^2 \cdots \mathbf{E}\left(\widetilde{\tilde{X}^2(U_{\delta,i_m})}\right)^2 \\ &= \mathcal{O}\left(|W_n|^{-2m} N_{n\delta}^m k_n^{2m}\right) = \mathcal{O}\left(k_n^m |W_n|^{-m}\right) \end{aligned} \tag{5.204}$$

as $n \to \infty$, uniformly for all $\delta \in \Theta_k$. By a similar argument as the one given in (5.203) and (5.204), we find that the order of the other $m-2$ terms do not exceed $\mathcal{O}(k_n^m |W_n|^{-m})$ as $n \to \infty$, uniformly for all $\delta \in \Theta_k$. This completes the proof of Lemma 5.17. □

Remark 5.2 By Lemma 5.8 (cf. also (5.143)), it is evident that the order bound in (5.194) is sharp.

Appendix

In this Appendix we present some well-known results which we use in the proofs of our theorems.

Lemma A. 1 *Let X be a Poisson r.v. with $\mathbf{E}X > 0$. Then, for any $\epsilon > 0$, we have*

$$\mathbf{P}\left(\frac{|X - \mathbf{E}X|}{(\mathbf{E}X)^{1/2}} > \epsilon\right) \leq 2\exp\left\{-\frac{\epsilon^2}{2 + \epsilon(\mathbf{E}X)^{-1/2}}\right\}. \tag{6.1}$$

Proof: We refer to Reiss (1993, p. 222). □

The following lemma is concerned with an exponential probability inequality for sum of independent r.v's with expectation zero.

Lemma A. 2 *Suppose that $Y_1, Y_2, \ldots, Y_m$ are independent random variables with expectation zero and with*

$$\max_{i=1,\ldots,m} K\left(\mathbf{E}e^{|Y_i|^2/K} - 1\right) \leq C_0 \tag{6.2}$$

for some positive real numbers K and C_0, then for any $\alpha > 0$,

$$\mathbf{P}\left(\left|\sum_{j=1}^{m} Y_j\right| \geq \alpha\right) \leq 2\exp\left\{-\frac{\alpha^2}{8m(K + C_0)}\right\}. \tag{6.3}$$

Proof: We refer to van de Geer (2000, p. 127-128). □

The following lemma is concerned with the Laplace transform of a Poisson process X.

Lemma A. 3 *Let X be a Poisson process on real line* $\mathbf{R}$ *with mean measure μ and intensity function λ. Then*

$$\mathbf{E}\exp\left\{\int_{\mathbf{R}} f(x)\,(X(dx)-\lambda(x)dx)\right\}$$
$$=\exp\left\{\int_{\mathbf{R}}\left[e^{f(x)}-1-f(x)\right]\lambda(x)dx\right\}. \qquad (6.4)$$

provided $\int_{\mathbf{R}}|f(x)|\lambda(x)dx<\infty$ *and* $\int_{\mathbf{R}}|e^{f(x)}-1-f(x)|\lambda(x)dx<\infty$.

Proof: We refer to Kutoyants (1998, p. 18-20). □

An exponential bound for 'intermediate' uniform order statistics is given in the following lemma.

Lemma A. 4 *Let k_n and m_n, $n=1,2,\ldots$ be sequences of positive integers, and $Z_{k_n:m_n}$ denote the k_n-th order statistic of a random sample of size m_n from the uniform distribution on $(0,1)$. If $k_n/m_n\downarrow 0$ as $m_n\to\infty$, then for each $\alpha_n>0$ such that $\alpha_n^{-1}=o(m_n k_n^{-1/2})$ and $\alpha_n=\mathcal{O}(k_n^{1/2})$, there exists a positive absolute constant C_0 and a (large) positive integer n_0 such that*

$$\mathbf{P}\left(\left|Z_{k_n:m_n}-\frac{k_n}{m_n+1}\right|\left(\frac{m_n}{k_n/(m_n+1)\,(1-k_n/(m_n+1))}\right)^{1/2}\geq\alpha_n\right)$$
$$\leq 2\exp\left\{-C_0\alpha_n^2\right\}, \qquad (6.5)$$

for all $n\geq n_0$.

Proof: A slight modification of the proof of Lemma A2.1. of Albers, Bickel, and van Zwet (1976) gives our bound. □

Lemma A. 5 *For real valued r.v.'s X_n and Y_n, if $X_n\stackrel{c}{\to}\alpha$, for some constant α, and $Y_n\stackrel{c}{\to}0$, then we have $X_nY_n\stackrel{c}{\to}0$, as $n\to\infty$.*

Proof: To prove this lemma, we must show, for each $\epsilon>0$,

$$\sum_{n=1}^{\infty}\mathbf{P}\left(|X_nY_n|>\epsilon\right)<\infty. \qquad (6.6)$$

To check (6.6), we argue as follows. By definition of $X_n\stackrel{c}{\to}\alpha$ and $Y_n\stackrel{c}{\to}0$ as $n\to\infty$, we have for each $\epsilon>0$,

$$\sum_{n=1}^{\infty}\mathbf{P}\left(|X_n-\alpha|>\epsilon\right)<\infty, \qquad (6.7)$$

and

$$\sum_{n=1}^{\infty} \mathbf{P}\left(|Y_n| > \epsilon\right) < \infty. \tag{6.8}$$

The probability appearing on the l.h.s. of (6.6) does not exceed

$$\begin{aligned} &\mathbf{P}\left(|X_n Y_n - Y_n \alpha| + |Y_n \alpha| > \epsilon\right) \\ &\leq \mathbf{P}\left(|Y_n||X_n - \alpha| > \frac{\epsilon}{2}\right) + \mathbf{P}\left(|\alpha||Y_n| > \frac{\epsilon}{2}\right). \end{aligned} \tag{6.9}$$

By (6.8) we have the second term on the r.h.s. of (6.9) is summable. Hence, to prove (6.6), it remains to show that the first term on the r.h.s. of (6.9) is summable. This term is equal to

$$\begin{aligned} &\mathbf{P}\left(|Y_n||X_n - \alpha| > \frac{\epsilon}{2} \cap |Y_n| \leq 1\right) \\ &+\mathbf{P}\left(|Y_n||X_n - \alpha| > \frac{\epsilon}{2} \cap |Y_n| > 1\right) \\ &\leq \mathbf{P}\left(|X_n - \alpha| > \frac{\epsilon}{2}\right) + \mathbf{P}\left(|Y_n| > 1\right). \end{aligned} \tag{6.10}$$

By (6.7) and (6.8), we have the r.h.s. of (6.10) is summable. This completes the proof of Lemma A.5. □

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Summary

In this thesis we study nonparametric estimation of the *global intensity, the intensity function at a given point (local intensity)*, and the *period* of a cyclic Poisson point process, using only *a single realization* of the cyclic Poisson process observed in an interval (called window).

We begin with a general introduction in chapter 1. The basic properties of an inhomogeneous Poisson process are presented in section 1.1, and a description of a cyclic Poisson process is given in section 1.2. Finally we give an overview of the thesis in section 1.3, and discuss some related work in section 1.4.

In chapter 2 we propose and study an estimator of the global intensity. Asymptotic properties of the proposed estimator are presented. If the intensity function is periodic and locally integrable, our estimator is shown to be asymptotically unbiased, and weakly and strongly consistent in estimating the global intensity, as the size of window expands (cf. section 2.2). Finally, in section 2.3, we establish asymptotic normality and a bootstrap CLT for our estimator.

Nonparametric estimation of the intensity function λ at a given point s is studied in chapters 3 and 4. In chapter 3 we propose and study kernel type estimators, while a nearest neighbor type estimator is proposed and investigated in chapter 4. Suppose that λ is periodic, locally integrable, and s is a Lebesgue point of λ. Then, under some assumptions on the kernel function and the rate of convergence of the estimator of the period, we show that our kernel type estimator of λ at s is weakly and strongly consistent, as the size of window expands (cf. section 3.2). Asymptotic approximations to the variance and the bias of the kernel type estimator are obtained, under additional conditions on λ, in section 3.3.

Parallel to chapter 3, in chapter 4 we discuss asymptotic properties of our nearest neighbor estimator of λ at a given point s. Suppose that λ is periodic, locally integrable, and s is a point at which λ is continuous and positive. Then, under an appropriate assumption on the rate of conver-

gence of the estimator of the period, we show that our nearest neighbor estimator is weakly and strongly consistent, as the size of window expands (cf. section 4.2). Asymptotic approximations to the variance and the bias of the nearest neighbor estimator are obtained, under an additional condition on λ, in section 4.3.

A nonparametric estimator of the period is proposed and investigated in chapter 5. If the intensity function λ is periodic and bounded, then the estimator is shown to be weakly and strongly consistent in estimating the period, as the size of window expands. Furthermore, rates of convergence (in probability as well as almost surely) of order $o(|W_n|^{-\gamma})$, as the size of window $|W_n|$ expands, with $\gamma < \frac{1}{3}$ are obtained. If, in addition, the intensity function λ is assumed to satisfy a Lipschitz condition, we obtain a rate of convergence of order $o_p(|W_n|^{-\gamma})$ with $\gamma < \frac{1}{2}$. Asymptotic normality of a slight modification of our original estimator, properly normalized, is also established.

Samenvatting

In dit proefschrift bestuderen we niet parametrische schattingsmethoden voor de globale intensiteit θ, de intensiteitsfunctie $\lambda(s)$ in een gegeven punt s (de locale intensiteit), en de periode τ van een cyclisch Poisson punt proces, voor het geval dat slechts één realizatie van het cyclische Poisson proces is waargenomen in een begrensd interval W.

In hoofdstuk 1 definiëren we allereerst inhomogene en cyclische Poisson puntprocessen (secties 1.1 en 1.2); in sectie 1.3 geven we een samenvatting van de voornaamste resultaten van dit onderzoek; sectie 1.4 bevat een kort overzicht van verwante literatuur.

In hoofdstuk 2 bestuderen we asymptotische eigenschappen van een eenvoudige niet parametrische schatter van de globale intensiteit, zoals asymptotische zuiverheid, zwakke en sterke convergentie en asymptotische normaliteit.

Niet parametrische schattingsmethoden voor de intensiteitfunctie λ in een gegeven punt s - de locale intensiteit - vormen het onderwerp van de hoofdstukken 3 en 4. In hoofdstuk 3 bestuderen we 'kern schatters' voor de locale intensiteit, en in hoofdstuk 4 onderzoeken we een schatter gebaseerd op een 'nabijgelegen' waarneming. Indien λ cyclisch is en lokaal integreerbaar, en s een Lebesgue punt van λ is, dan bewijzen we sterke en zwakke consistentie van onze kernschatter voor $\lambda(s)$, mits de periode τ van het cyclische Poisson proces voldoende nauwkeurig geschat kan worden. Asymptotische benaderingen voor de variantie en de onzuiverheid van kernschatters worden ook bepaald, onder additionele aannames voor λ.

In hoofdstuk 4 bestuderen we een schatter gebaseerd op een 'nabijgelegen' waarneming. Indien λ cyclisch is en lokaal integreerbaar, en s een punt is waar λ continu en positief is, dan bewijzen we sterke en zwakke consistentie van onze schatter voor $\lambda(s)$, mits de periode van het cyclische Poisson proces voldoende nauwkeurig geschat kan worden. Asymptotische benaderingen voor de variantie en de onzuiverheid van de schatter gebaseerd op (een) 'nabijgelegen' waarneming worden ook

bepaald, onder additionele aannames voor λ.

In hoofdstuk 5 bestuderen we het probleem de periode τ van het cyclische Poisson proces te schatten. Indien λ cyclisch is en begrensd, dan bewijzen we dat onze schatter - een nieuwe eenvoudige nietparametrische schatter van de periode τ - sterk en zwak consistent is; ook geven we resultaten voor de snelheid van convergentie. Indien λ bovendien een Lipschitz functie is, dan kunnen we de resultaten over de snelheid van convergentie nog wat verscherpen. Ook bewijzen we dat onze schatter voor τ, mits enigszins gemodificeerd, asymptotisch normaal verdeeld is.

Curriculum Vitae

I Wayan Mangku was born in Ababi, Bali, Indonesia, on March 5, 1962. He received a Sarjana degree in Statistics from Bogor Agricultural University, Bogor, Indonesia, in 1985.

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