

# A NOTE ON PÓLYA'S OBSERVATION CONCERNING LIOUVILLE'S FUNCTION

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*Dedicated to Herman J. J. te Riele on the occasion of his retirement from the  
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ABSTRACT. We show that a certain weighted mean of the Liouville function  $\lambda(n)$  is negative. In this sense, we can say that the Liouville function is negative “on average”.

## 1. INTRODUCTION

For  $n \in \mathbb{N}$  let  $n = \prod_{p|n} p^{e_p(n)}$  be the canonical prime factorization of  $n$  and let  $\Omega(n) := \sum_{p|n} e_p(n)$ . Here (as always in this paper)  $p$  is prime. Thus,  $\Omega(n)$  is the total number of prime factors of  $n$ , counting multiplicities. For example:  $\Omega(1) = 0$ ,  $\Omega(2) = 1$ ,  $\Omega(4) = 2$ ,  $\Omega(6) = 2$ ,  $\Omega(8) = 3$ ,  $\Omega(16) = 4$ ,  $\Omega(60) = 4$ , etc.

Define Liouville's multiplicative function  $\lambda(n) = (-1)^{\Omega(n)}$ . For example  $\lambda(1) = 1$ ,  $\lambda(2) = -1$ ,  $\lambda(4) = 1$ , etc. The Möbius function  $\mu(n)$  may be defined to be  $\lambda(n)$  if  $n$  is square-free, and 0 otherwise.

It is well-known, and follows easily from the Euler product for the Riemann zeta-function  $\zeta(s)$ , that  $\lambda(n)$  has the Dirichlet generating function

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)}$$

for  $\operatorname{Re}(s) > 1$ . This provides an alternative definition of  $\lambda(n)$ .

Let  $L(n) := \sum_{k \leq n} \lambda(k)$  be the summatory function of the Liouville function; similarly  $M(n) := \sum_{k \leq n} \mu(k)$  for the Möbius function.

The topic of this note is closely related to Pólya's conjecture [12, 1919] that  $L(n) \leq 0$  for  $n \geq 2$ .

Pólya verified this for  $n \leq 1500$  and Lehmer [9, 1956] checked it for  $n \leq 600\,000$ . However, Ingham [5, 1942] cast doubt on the plausibility of Pólya's conjecture by showing that it would imply not only the Riemann Hypothesis and simplicity of the zeros of  $\zeta(s)$ , but also the linear dependence over the rationals of the imaginary parts of the zeros

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$\rho$  of  $\zeta(s)$  in the upper half-plane. Ingham cast similar doubt on the Mertens conjecture  $|M(n)| \leq \sqrt{n}$ , which was subsequently disproved in a remarkable *tour de force* by Odlyzko and te Riele [11, 1985]. More recent results and improved bounds were given by Kotnik and te Riele [7, 2006]; see also Kotnik and van de Lune [6, 2004].

In view of Ingham's results, it was no surprise when Haselgrove showed [2, 1958] that Pólya's conjecture is false. He did not give an explicit counter-example, but his proof suggested that  $L(u)$  might be positive in the vicinity of  $u \approx 1.8474 \times 10^{361}$ .

Sherman Lehman [8, 1960] gave an algorithm for calculating  $L(n)$  similar to Meissel's [10, 1885] formula for the prime-counting function  $\pi(x)$ , and found the counter-example  $L(906\,180\,359) = +1$ .

Tanaka [14, 1980] found the smallest counter-example  $L(n) = +1$  for  $n = 906\,150\,257$ . Walter M. Lioen and Jan van de Lune [*circa* 1994] scanned the range  $n \leq 2.5 \times 10^{11}$  using a fast sieve, but found no counter-examples beyond those of Tanaka. More recently, Borwein, Ferguson and Mossinghoff [1, 2008] showed that  $L(n) = +1$  160 327 for  $n = 351\,753\,358\,289\,465$ .

Humphries [3, 4] showed that, under certain plausible but unproved hypotheses (including the Riemann Hypothesis), there is a limiting logarithmic distribution of  $L(n)/\sqrt{n}$ , and numerical computations show that the logarithmic density of the set  $\{n \in \mathbb{N} | L(n) < 0\}$  is approximately 0.99988. Humphries' approach followed that of Rubinstein and Sarnak [13], who investigated "Chebyshev's bias" in prime "races".

Here we show in an elementary manner, and without any unproved hypotheses, that  $\lambda(n)$  is (in a certain sense) "negative on average". To prove this, all that we need are some well-known facts about Mellin transforms, and the functional equation for the Jacobi theta function (which may be proved using Poisson summation). Our main result is:

**Theorem 1.** *There exists a positive constant  $c$  such that for every (fixed)  $N \in \mathbb{N}$*

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{e^{n\pi x} + 1} = -\frac{c}{\sqrt{x}} + \frac{1}{2} + O(x^N) \quad \text{as } x \downarrow 0.$$

Thus, a weighted mean of  $\{\lambda(n)\}$ , with positive weights initially close to a constant (1/2) and becoming small for  $n \gg 1/x$ , is negative for  $x < x_0$  and tends to  $-\infty$  as  $x \downarrow 0$ .

In the final section we mention some easy results on the Möbius function  $\mu(n)$  to contrast its behaviour with that of  $\lambda(n)$ .

2. PROOF OF THEOREM 1

We prove Theorem 1 in three steps.

*Step 1.* For  $x > 0$ ,

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{e^{n\pi x} - 1} = \phi(x) = \frac{\theta(x) - 1}{2},$$

where

$$\phi(x) := \sum_{k=1}^{\infty} e^{-k^2\pi x}, \quad \theta(x) := \sum_{k \in \mathbb{Z}} e^{-k^2\pi x}.$$

*Step 2.* For  $x > 0$ ,

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{e^{n\pi x} + 1} = \phi(x) - 2\phi(2x).$$

*Step 3.* Theorem 1 now follows from the functional equation

$$\theta(x) = \frac{1}{\sqrt{x}} \theta\left(\frac{1}{x}\right)$$

for the Jacobi theta function  $\theta(x)$ .

*Proof of Theorem 1.*

(1) In the following, we assume that  $\operatorname{Re}(s) > 1$ , so the Dirichlet series and integrals are absolutely convergent, and interchanging the orders of summation and integration is easy to justify.

As mentioned above, it is well-known that

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \prod_p (1 + p^{-s})^{-1} = \prod_p \frac{1 - p^{-s}}{1 - p^{-2s}} = \frac{\zeta(2s)}{\zeta(s)}.$$

Define

$$f(x) := \sum_{n=1}^{\infty} \frac{\lambda(n)}{e^{nx} - 1}, \quad (x > 0).$$

We will use the well known fact that if two sufficiently well-behaved functions (such as ours below) have the same Mellin transform then the functions are equal.

The Mellin transform of  $f(x)$  is

$$\begin{aligned} F(s) &:= \int_0^\infty f(x)x^{s-1} dx = \int_0^\infty \sum_{n=1}^\infty \frac{\lambda(n)}{e^{nx} - 1} x^{s-1} dx \\ &= \sum_{n=1}^\infty \lambda(n) \int_0^\infty \frac{x^{s-1}}{e^{nx} - 1} dx = \left( \sum_{n=1}^\infty \frac{\lambda(n)}{n^s} \right) \times \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx \\ &= \frac{\zeta(2s)}{\zeta(s)} \times \zeta(s)\Gamma(s) = \zeta(2s)\Gamma(s). \end{aligned}$$

We also have

$$\begin{aligned} \int_0^\infty \phi\left(\frac{x}{\pi}\right)x^{s-1} dx &= \int_0^\infty \left( \sum_{n=1}^\infty e^{-n^2x} \right) x^{s-1} dx \\ &= \left( \sum_{n=1}^\infty \frac{1}{n^{2s}} \right) \times \int_0^\infty e^{-x} x^{s-1} dx = \zeta(2s)\Gamma(s), \end{aligned}$$

so the Mellin transforms of  $f(x)$  and of  $\phi(x/\pi)$  are identical. Thus  $f(x) = \phi(x/\pi)$ . Replacing  $x$  by  $\pi x$ , we see that

$$\sum_{n=1}^\infty \frac{\lambda(n)}{e^{n\pi x} - 1} = \sum_{k=1}^\infty e^{-k^2\pi x},$$

completing the proof of step (1).

(2) Observe that

$$\frac{1}{e^{n\pi x} + 1} = \frac{1}{e^{n\pi x} - 1} - \frac{2}{e^{2n\pi x} - 1},$$

so, from step (1),

$$\sum_{n=1}^\infty \frac{\lambda(n)}{e^{n\pi x} + 1} = \phi(x) - 2\phi(2x).$$

(3) Using the functional equation for  $\theta(x)$ , we easily find that

$$\phi(x) - 2\phi(2x) = -\frac{c}{\sqrt{x}} + \frac{1}{2} + \frac{1}{\sqrt{x}} \left( \phi\left(\frac{1}{x}\right) - \sqrt{2}\phi\left(\frac{1}{2x}\right) \right)$$

with  $c = (\sqrt{2} - 1)/2 > 0$ , proving our claim, since the “error” term is bounded by  $\phi(1/x)/\sqrt{x} \sim \exp(-\pi/x)/\sqrt{x} = O(x^N)$  as  $x \downarrow 0$  (for any fixed exponent  $N$ ).  $\square$

3. REMARKS ON THE MÖBIUS FUNCTION

We give some further applications of the identity

$$(*) \quad \frac{1}{z+1} = \frac{1}{z-1} - \frac{2}{z^2-1}$$

that we used (with  $z = e^{n\pi x}$ ) in proving step (2) above.

**Lemma 2.** For  $|x| < 1$ , we have

$$\sum_{n=1}^{\infty} \mu(n) \frac{x^n}{x^n + 1} = x - 2x^2.$$

*Proof.* Assume that  $|x| < 1$ . It is well known that

$$\sum_{n=1}^{\infty} \mu(n) \frac{x^n}{1 - x^n} = x,$$

in fact this “Lambert series” identity is equivalent to the Dirichlet series identity  $\sum \mu(n)/n^s = 1/\zeta(s)$ . Writing  $y = 1/x$ , we have

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{y^n - 1} = 1/y.$$

It follows on taking  $z = y^n$  in our identity (\*) that

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{y^n + 1} = \sum_{n=1}^{\infty} \frac{\mu(n)}{y^n - 1} - 2 \sum_{n=1}^{\infty} \frac{\mu(n)}{y^{2n} - 1} = y^{-1} - 2y^{-2}.$$

Replacing  $y$  by  $1/x$  gives the result. □

**Corollary 3.**

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{2^n + 1} = 0.$$

*Proof.* Take  $x = 1/2$  in Lemma 2. □

It follows from Lemma 2 that

$$\lim_{x \uparrow 1} \sum_{n=1}^{\infty} \mu(n) \frac{x^n}{x^n + 1} = -1,$$

so that one might say that in this sense  $\mu(n)$  is negative on average. However, this is much weaker than what we showed in Theorem 1 for  $L(n)$ , where the corresponding sum tends to  $-\infty$ . The “complex-analytic” reason for this difference is that  $\zeta(2s)/\zeta(s)$  has a pole (with negative residue) at  $s = 1/2$ , but  $1/\zeta(s)$  is regular at  $s = 1$ .

## REFERENCES

- [1] P. BORWEIN, R. FERGUSON AND M. J. MOSSINGHOFF, *Sign changes in sums of the Liouville function*, Math. Comp. **77** (2008), 1681–1694.
- [2] C. B. HASELGROVE, *A disproof of a conjecture of Pólya*, Mathematika **5** (1958), 141–145.
- [3] P. B. HUMPHRIES, *The Summatory Function of Liouville's Function and Pólya's Conjecture*, Honours Thesis, Department of Mathematics, The Australian National University, Canberra, October 2010.
- [4] P. B. HUMPHRIES, *The Distribution of Weighted Sums of the Liouville's Function and Pólya's Conjecture*, arXiv:1108.1524, August 2011.
- [5] A. E. INGHAM, *On two conjectures in the theory of numbers*, Amer. J. of Mathematics **64** (1942), 313–319.
- [6] T. KOTNIK AND J. VAN DE LUNE *On the order of the Mertens function*, Experimental Mathematics **13**:4 (2004), 473–481.
- [7] T. KOTNIK AND H. J. J. TE RIELE *The Mertens conjecture revisited*, Proc. ANTS 2006, Lecture Notes in Computer Science **4076** (2006), 156–167.
- [8] R. S. LEHMAN, *On Liouville's function*, Math. Comp. **14** (1960), 311–320.
- [9] D. H. LEHMER, *Extended computation of the Riemann zeta function*, Mathematika **3** (1956), 102–108.
- [10] E. D. F. MEISSEL, *Berechnung der Menge von Primzahlen, welche innerhalb der ersten Milliarde natürlicher Zahlen vorkommen*, Math. Ann. **25** (1885), 251–257.
- [11] A. M. ODLYZKO AND H. J. J. TE RIELE, *Disproof of the Mertens conjecture*, J. für die reine und angewandte Mathematik **357** (1985), 138–160.
- [12] G. PÓLYA, *Verschiedene Bemerkungen zur Zahlentheorie*, Jahresbericht der deutschen Math.-Vereinigung **28** (1919), 31–40.
- [13] M. RUBINSTEIN AND P. SARNAK, *Chebyshev's bias*, Experimental Mathematics **3** (1994), 173–197.
- [14] M. TANAKA, *A numerical investigation on cumulative sum of the Liouville function*, Tokyo J. Math. **3** (1980), 187–189.

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