# A PROBLEM RELATED TO THE APPROXIMATION OF π BY ARCHIMEDES/HUYGENS

JUAN ARIAS DE REYNA AND JAN VAN DE LUNE

*Dedicated to Herman J. J. te Riele on the occasion of his retirement from the CWI in January 2012*

## 1. The problem and its origin.

It is well known that Archimedes approximated $2\pi$ ( := the length of the circumference of a circle having radius $r = 1$ ) by the lengths of inscribed and circumscribed regular $n$-gons.

Denoting the length of such an inscribed $n$-gon by $\ell_n$ and that of a circumscribed one by $L_n$ we have

$$\ell_n = 2n \sin\frac{2\pi}{2n} \qquad \text{and} \qquad L_n = 2n\tan\frac{2\pi}{2n}. \tag{1}$$

Huygens considered the question: Which of $\ell_n$ and $L_n$ is the best approximation of $2\pi$ and to what extent ?

It should be clear that $\ell_n < 2\pi < L_n$. So, for a suitable $\lambda \in (0, 1)$ one should take $2\pi = \lambda\ell_n + (1-\lambda)L_n$.

From this it is easily seen that the best $\lambda$ would be $\lambda = \frac{1}{\left(1+\frac{2\pi-\ell_n}{L_n-2\pi}\right)}$.

So, one should consider the ratio $\frac{2\pi-\ell_n}{L_n-2\pi}$, or as we actually did

$$\frac{L_n - 2\pi}{2\pi - \ell_n} = \frac{\tan\frac{\pi}{n} - \frac{\pi}{n}}{\frac{\pi}{n} - \sin\frac{\pi}{n}}.$$

Writing $x := \frac{\pi}{n}$ we are thus led to consider the ( even ) function $Q(x) := \frac{\tan x - x}{x - \sin x}$ for $x$ close to 0.

It was known to Huygens that $Q(x) > 2$, and using l'Hôpital's rule it is easily seen that $\lim_{x\to 0} Q(x) = 2$.

Consequently one should ( in this context ) approximate $2\pi$ by $\frac{2}{3}\ell_n + \frac{1}{3}L_n$. Also note that $\frac{2}{3}\ell_n + \frac{1}{3}L_n > 2\pi$.

( A similar analysis holds for the areas $a_n$ and $A_n$ of the $n$-gons. )

For us it was just a matter of curiosity to have a closer look at the coefficients in the power series of the function $\frac{\tan x - x}{x - \sin x}$ for $x$ close to 0.

Invoking *Mathematica* we found ( for various values of nMax ) for example:

```
nMax = 30; (*  For example  *)
Normal[Series[(Tan[x] - x)/(x - Sin[x]), {x, 0, nMax}]]
```

$$2+\frac{9\,x^{2}}{10}+\frac{513\,x^{4}}{1400}+\frac{297\,x^{6}}{2000}+\frac{2\,595\,081\,x^{8}}{43\,120\,000}+\frac{136\,726\,449\,x^{10}}{5\,605\,600\,000}+\frac{7\,757\,835\,963\,x^{12}}{784\,784\,000\,000}+$$
$$\frac{4\,810\,522\,436\,537\,x^{14}}{1\,200\,719\,520\,000\,000}+\frac{228\,184\,846\,967\,215\,909\,x^{16}}{140\,532\,212\,620\,800\,000\,000}+\frac{924\,798\,350\,722\,118\,597\,x^{18}}{1\,405\,322\,126\,208\,000\,000\,000}+$$
$$\frac{423\,613\,976\,567\,459\,270\,644\,897\,x^{20}}{1\,588\,323\,173\,482\,805\,760\,000\,000\,000}+\frac{1\,716\,842\,780\,515\,524\,728\,374\,151\,x^{22}}{15\,883\,231\,734\,828\,057\,600\,000\,000\,000}+$$
$$\frac{126\,064\,430\,908\,322\,638\,705\,746\,667\,x^{24}}{2\,877\,667\,867\,251\,201\,024\,000\,000\,000\,000}+\frac{15\,852\,808\,185\,558\,085\,074\,420\,916\,349\,x^{26}}{892\,881\,980\,908\,641\,884\,160\,000\,000\,000\,000}+$$
$$\frac{6\,162\,379\,696\,360\,573\,178\,218\,943\,175\,357\,313\,x^{28}}{856\,398\,823\,168\,714\,776\,773\,222\,400\,000\,000\,000\,000}+\frac{324\,677\,394\,542\,156\,500\,969\,976\,683\,473\,676\,127\,x^{30}}{111\,331\,847\,011\,932\,920\,980\,518\,912\,000\,000\,000\,000\,000}$$

and ( observing that all coefficients turned out to be positive ) arrived at the *conjecture* that all coefficients $c_n$ in the power series expansion

$$\frac{\tan x - x}{x - \sin x} = \sum_{n=0}^{\infty} c_n\, x^{2n}$$

are strictly positive indeed.

We thus ran into the problem: If true, how can this be proved ?

## 2. A proof of the conjecture.

$Q(x)$ is a meromorphic function on the complex plane. Its poles are those of $\tan x$ at the points $x = (2n+1)\pi/2$ with $n$ an integer and at the zeros of $x - \sin x$, except $x = 0$, which is a removable singularity of $Q(x)$.

We consider the square $R = [-2\pi, 2\pi]^2$. Inside this square there are only four poles of $Q(x)$ : at the points $\pm\frac{\pi}{2}$ and $\pm\frac{3\pi}{2}$. To see this it suffices to show that $x - \sin x$ has only one ( triple ) zero inside $R$. This can be proved formally by computing the variation of the argument of $x - \sin x$ when moving along the rim of the rectangle with vertices at $\pm 2\pi \pm iT$ with $T$ a big real number.

We compute the residues

$$\operatorname*{Res}_{x=\pi/2} Q(x) = \frac{2}{2-\pi}, \qquad \operatorname*{Res}_{x=-\pi/2} Q(x) = -\frac{2}{2-\pi},$$
$$\operatorname*{Res}_{x=3\pi/2} Q(x) = -\frac{2}{2+3\pi}, \qquad \operatorname*{Res}_{x=-3\pi/2} Q(x) = \frac{2}{2+3\pi}.$$

Hence, we may write

$$(2) \quad Q(x) = \frac{8}{\pi(\pi-2)}\frac{1}{1-4x^2/\pi^2} + \frac{8}{3\pi(2+3\pi)}\frac{1}{1-4x^2/9\pi^2} + h(x).$$

where $h$ is analytic on $R$.

We thus find the following value for $c_n$

$$(3)\quad c_n = \frac{8}{\pi(\pi-2)}\Big(\frac{2}{\pi}\Big)^{2n} + \frac{8}{3\pi(2+3\pi)}\Big(\frac{2}{3\pi}\Big)^{2n} + d_n,$$

$$\text{with}\quad d_n = \frac{1}{2\pi i}\int_{\partial R}\frac{h(z)}{z^{2n+1}}\,dz.$$

For $x\in\partial R$ we have $|Q(x)-h(x)|\le\frac14$. In fact for $|x|>2\pi$ we have

$$|Q(x)-h(x)| \le \frac{8}{\pi(\pi-2)}\frac{1}{16-1} + \frac{8}{3\pi(2+3\pi)}\frac{1}{16/9-1} = 0.244234\ldots$$

Also, for $x\in\partial R$ we will show that $|Q(x)|\le 2$. Since $Q$ is even, we only have to bound $Q(2\pi+iy)$ and $Q(x+2\pi i)$ for $|y|<2\pi$ and $|x|<2\pi$.

First for $y$ real and $|y|<2\pi$ we have

$$Q(2\pi+iy) = \frac{-2\pi + i(\tanh y - y)}{2\pi + i(y-\sinh y)}.$$

Then $|Q(2\pi+iy)|\le 2$ is equivalent to

$$4\pi^2 + (\tanh y - y)^2 < 16\pi^2 + 4(\sinh y - y)^2$$

or

$$\tanh^2 y - 2y\tanh y < 12\pi^2 + 3y^2 + 4\sinh^2 y - 8y\sinh y.$$

So $|Q(2\pi+iy)|\le 2$ follows from the two elementary inequalities: $\tanh^2 y<1$ and $8y\sinh y < 2+3y^2+4\sinh^2 y$.

On the other side of the rectangle, for $-2\pi<x<2\pi$ we have

$$|Q(x+2\pi i)| = \Big|\frac{\tan(x+2\pi i) - x - 2\pi i}{x+2\pi i - \sin(x+2\pi i)}\Big| \le \frac{\coth 2\pi + |x+2\pi i|}{\sinh 2\pi - |x+2\pi i|}$$

$$\le \frac{\coth 2\pi + 2\sqrt{2}\pi}{\sinh 2\pi - 2\sqrt{2}\pi} = 0.0381898\ldots$$

It follows that on $\partial R$ we have $|h(x)|\le|Q(x)-h(x)|+|Q(x)|\le 3$, so that

$$|d_n| \le \frac{1}{2\pi}\int_{\partial R}\frac{3}{(2\pi)^{2n+1}}|dz| \le 24(2\pi)^{-2n-1}.$$

Hence with $|\theta|\le 1$

$$(4)\qquad c_n = \frac{8}{\pi(\pi-2)}\Big(\frac{2}{\pi}\Big)^{2n} + \frac{8}{3\pi(2+3\pi)}\Big(\frac{2}{3\pi}\Big)^{2n} + \theta\cdot 24(2\pi)^{-2n-1}.$$

Since $8/(\pi(\pi-2))$ is about $2.23064\ldots$ we have

$$c_n > 2\Big(\frac{2}{\pi}\Big)^{2n} - 24\Big(\frac{1}{2\pi}\Big)^{2n+1} > 0 \qquad \text{for all } n\ge 1$$

completing our proof.

## 3. Further observations.

In the previous Section we proved that in the power series expansion

$$\frac{\tan x - x}{x - \sin x} = \sum_{n=0}^{\infty} c_n\, x^{2n}$$

all $c_n$ are positive.

Writing $\tan x = \sum_{n=1}^{\infty} t_n\, x^{2n-1}$ and $\sin x = \sum_{n=1}^{\infty} s_n\, x^{2n-1}$ we defined

$$T := \sum_{n=1}^{N} t_n\, x^{2n-1} \quad \text{and} \quad S := \sum_{n=1}^{N} s_n\, x^{2n-1}$$

and observed ( using Mathematica ) the following:
The coefficients $q_n$ in the power series expansion

$$\frac{\tan x - T}{S - \sin x} = \sum_{n=0}^{\infty} q_n\, x^{2n}$$

(1) are all positive if $N \equiv 1 \pmod 2$
(2) are all negative if $N \equiv 0 \pmod 2$.

We have no proof for this and leave a proof ( or refutation ) as a challenge to the interested reader. One may want to try things out by means of the following program.

```
In[1]:=
n = 3; (*  Also try some other  n ∈ N  *)
T = Normal[Series[Tan[x], {x, 0, 2 n - 1}]];
S = Normal[Series[Sin[x], {x, 0, 2 n - 1}]];
Print["f = ", f = (Tan[x] - T)/(S - Sin[x])];
nTerms = 24;  (*  For example  *)
Normal[Series[f, {x, 0, nTerms}]]
```

$$f = \frac{-x - \frac{x^3}{3} - \frac{2x^5}{15} + \mathrm{Tan}[x]}{x - \frac{x^3}{6} + \frac{x^5}{120} - \mathrm{Sin}[x]}$$

Out[4]=

$$272 + 114 x^2 + \frac{6101 x^4}{132} + \frac{890\,149 x^6}{47\,520} + \frac{26\,000\,961\,209 x^8}{3\,424\,861\,440} + \frac{64\,491\,289\,360\,457 x^{10}}{20\,960\,152\,012\,800} + \frac{30\,254\,970\,559\,608\,601 x^{12}}{24\,262\,182\,114\,508\,800} + \frac{208\,883\,539\,141\,611\,618\,143 x^{14}}{413\,311\,124\,757\,080\,309\,760} + \frac{7\,710\,587\,768\,733\,558\,650\,509\,987 x^{16}}{37\,644\,377\,242\,874\,874\,612\,940\,800} + \frac{28\,124\,851\,654\,909\,083\,303\,025\,556\,651 x^{18}}{338\,799\,395\,185\,873\,871\,516\,467\,200\,000} + \frac{1\,995\,115\,035\,944\,689\,724\,814\,158\,752\,505\,297 x^{20}}{59\,300\,735\,738\,173\,875\,479\,270\,286\,950\,400\,000} + \frac{59\,091\,732\,921\,225\,317\,488\,043\,271\,690\,096\,506\,747 x^{22}}{4\,333\,697\,767\,745\,746\,820\,025\,072\,570\,335\,232\,000\,000} + \frac{3\,507\,664\,213\,216\,293\,552\,099\,055\,375\,264\,386\,853\,121 x^{24}}{634\,730\,927\,560\,495\,429\,381\,430\,471\,959\,353\,753\,600\,000}$$

A similar analysis of the inner and outer areas $a_n$ and $A_n$ leads to "similar" observations.

Facultad de Matemáticas, Universidad de Sevilla,
Apdo. 1160, 41080-Sevilla, Spain
*E-mail address*: arias@us.es

Langebuorren 49, 9074 CH Hallum, The Neterlands
(Formerly at CWI, Amsterdam )
*E-mail address*: j.vandelune@hccnet.nl