on some nonlinear problems arising in the physics of ionized gases

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CONTENTS

GENERAL INTRODUCTION

The problem which is studied in this thesis has its origin in the physics of ionized gases. One considers an assembly of ions and electrons. The electrons, which are highly mobile, are described by a time and space dependent density n_e. The ions, which are heavy and slow, are considered not to move on the time scale of interest; they are described by a time-independent density f. One wants to determine the electron density for a given ion density, with the extra condition that the total number of electrons is also given. The physical laws which determine this problem are

(i) Coulomb's law, which gives the electric field (or the electric potential) in terms of the charge densities.

(ii) A constitutive equation which gives the local electron current in terms of the local electric field and the electron density gradient.

(iii) The continuity equation, which links the local rate of change of the electron density to the electron current.

These basic laws are stated mathematically in Chapter 1. On several occasions we shall generalize our equations to arbitrary spatial dimension n, where we use the generalization of Coulomb's law. One can consider either the limited problem of finding the stationary electron density, or the full problem with nonstationary solutions. Both are investigated in this thesis.

Given these basic laws we show that these problems can take various forms, depending on which physical quantity one takes as the unknown function. The most obvious one is the electron density n_e but it is also possible to take as the unknown function the electric field p due to the electrons or the electric potential u. If one takes as unknown the electric potential, one obtains the following problem for the stationary state

$$\overline{\text{BVP}} \quad \begin{cases} -\Delta u + e^{u/\varepsilon} = f & \text{in } \Omega \\ \int_\Omega e^{u(x)/\varepsilon} dx = C \\ u|_{\partial\Omega} = \text{constant (but unknown)} \end{cases}$$

where Ω is a domain in $\mathbb{R}^n$ and ε is a positive constant proportional to the temperature; the quantity $e^{u/\varepsilon}$ corresponds to the density of the electrons and the integral condition expresses the fact that the total charge of the electrons is a given positive constant C. One assumes furthermore that the domain Ω is surrounded by an electrical conductor which implies the condition $u|_{\partial\Omega}$ = constant.

An alternative formulation of this problem is the minimization of the free energy

$$\overline{\text{VP}^*} \quad \begin{cases} \inf \ \varepsilon \int_\Omega \operatorname{div} p \ln \operatorname{div} p + \frac{1}{2} \int (g-p)^2 \\ \text{such that } \int_\Omega \operatorname{div} p = C \end{cases}$$

Here g denotes the electric field created by the ions and is given and one wants to solve for p. These functions are related to n_e, u and f by $\operatorname{div} p = n_e = e^{u/\varepsilon}$ and $\operatorname{div} g = f$. A short derivation by physical arguments of the problems $\overline{\text{BVP}}$ and $\overline{\text{VP}^*}$ is given in Appendix 2 of Chapter 5.

Because our interests are mathematical, we propose to consider as well the larger class of problems

$$\text{BVP} \quad \begin{cases} -\Delta u + h(\frac{u}{\varepsilon}) = f & \text{in } \Omega \\ \int_\Omega h(\frac{u(x)}{\varepsilon}) dx = C \\ u|_{\partial\Omega} = \text{constant (but unknown)} \end{cases}$$

where $h: \mathbb{R} \to \mathbb{R}$ is a given continuous, strictly increasing function and the constant C satisfies the compatibility condition

$$h(-\infty)|\Omega| < C < h(+\infty)|\Omega|$$

where $|\Omega|$ denotes the measure of Ω.

The partial differential equation in BVP, whose left-hand-side is the sum of the Laplacian and of a monotone operator is similar to equations in problems studied by BENILAN, BREZIS & CRANDALL [5], CRANDALL & EVANS [14] and VASQUEZ [41]. Also, if one substitutes $n_e = \operatorname{div} p$ in $\overline{VP}^*$, one makes more apparent the similarity between $\overline{VP}^*$ and the Thomas-Fermi model (see for instance LIEB & SIMON [36], ARTHURS & ROBINSON [3] and BENILAN & BREZIS [4]). The most striking feature of these problems, when Ω is unbounded, is that the existence of a solution may depend on the dimension [41] and on whether the parameter C lies above or below a critical value [36], [4]; we shall see, at least in one special case, that such a threshold phenomenon also occurs with $\overline{BVP}$ and $\overline{VP}^*$.

Another interesting feature of BVP is that it is a singular perturbation problem. We shall study it in its general form in the case that Ω is bounded and we shall show that as $\varepsilon \downarrow 0$ (physically: the low temperature limit) the solution u_ε of BVP converges to the solution of a free boundary problem. Related Dirichlet problems have been studied by BRAUNER & NICOLAENKO [6], [7]; they also use problems similar to BVP to approximate free boundary problems characterized by elliptic variational inequalities [8]. FRANK & VAN GROESEN [21] and FRANK & WENDT [22] consider related inhomogeneous Dirichlet problems and study in particular the coincidence set of the limit problem.

The main questions which have been motivating our work are:

(i) What are the conditions which insure existence and non-existence of a solution for BVP, and, if BVP has a solution, is it unique?

(ii) Is the solution u_ε of BVP stable, when considered as the steady state solution of a suitable evolution problem?
Also, how does the solution of the evolution problem behave in the case that BVP does not have a solution?

(iii) What is the asymptotic behaviour of u_ε as $\varepsilon \downarrow 0$?

(iv) In what sense are $\overline{BVP}$ and $\overline{VP}^*$ equivalent?
More generally, if one associates a variational problem with BVP, what does its dual problem (in the sense of EKELAND & TÉMAM [19]) look like?

This work does not pretend to answer thoroughly all the above questions but we have used them as a guideline and we may solve in the future some points that are left open here. We now give a more detailed overview of the contents of this thesis.

A special case of physical interest is that of a filamentary discharge between two electrodes, considered by MARODE [37] and MARODE, BASTIEN & BAKKER [38]. By means of numerical methods, these authors study a system of moment equations which describe the motion of particles (charged ions, neutrals and electrons) in the filament of the discharge. Because there is cylindrical symmetry in the experimental situation, one works with two space variables, the distance r to the axis and the height z, and a time variable t.

In Chapter 1 we propose a physical model which simplifies the experimental situation. In particular we suppose that the radial dimension of the discharge is much smaller than its longitudinal dimension and thus that all the quantities involved depend only on r and t. The steady state problem is then given by the cylindrically symmetric version of $\overline{\mathrm{BVP}}$, namely

$$\begin{cases} -\dfrac{1}{r}\dfrac{\partial}{\partial r}\, r\, \dfrac{\partial u}{\partial r} + e^{u/\varepsilon} = f & \text{for } r \in [0,R] \\ \dfrac{\partial u}{\partial r}(0) = 0 & \\ \int_0^R e^{u/\varepsilon} r\, dr = C. & \end{cases}$$

Rather than studying this problem, we study the two-point boundary value problem which one obtains after transforming from u and f to two new functions y and g:

$$y(x) = \int_0^{x^{1/2}} e^{u(r)/\varepsilon} r\, dr$$

and

$$g(x) = \int_0^{x^{1/2}} f(r) r\, dr.$$

The transformed problem reads:

$$P(\varepsilon,R)\begin{cases} \varepsilon x y'' + (g(x) - y)y' = 0 & \text{for } x \in (0,R) \\ y(0) = 0, \quad y(R) = C. & \end{cases}$$

In the experimental situation R is very large and hence the case $R = \infty$ is of interest. In Chapter 1 we also derive the evolution problem corresponding to $P(\varepsilon,R)$ namely

$$P\begin{cases} v_t = \varepsilon x v_{xx} + (g(x)-v)v_x & \text{on } D = (0,\infty)\times(0,T) \\ v(0,t) = 0 & \text{for } t \in [0,T] \\ v(x,0) = \psi(x) & \text{for } x \in (0,\infty) \end{cases}$$

where the initial function ψ is nondecreasing and such that $\psi(0) = 0$ and $\psi(\infty) = C$. Finally we summarize in Chapter 1 in physical terms the results obtained in Chapters 2 - 4. We discuss in particular the escape of electrons to infinity above a critical temperature and the boundary layer exhibited by the electron density near zero temperature. The references [9] of Chapter 1 can be supplemented with more recent ones on two-dimensional Coulomb systems with circular symmetry [1], [10], [29], [30], [39].

We remark that the notation used in this introduction does not always coincide with the notation of the following chapters. Since these were originally separate articles, the same symbol denotes sometimes different quantities.

In Chapter 2 we study the two-point boundary value problem $P(\varepsilon,R)$, in which we suppose that g satisfies the hypothesis

$$H_g\colon\ g \in C^2(\mathbb{R}_+),\quad g(0) = 0,\quad g'(x) > 0 \ \text{ and } \ g''(x) < 0 \qquad \text{for all } x \geq 0,$$

and suppose that $C \in (0,g(\infty))$ and $R > x_0 := g^{-1}(C)$. It turns out that $P(\varepsilon,R)$ has a unique solution y which is monotonic in ε and in R. Interesting from both the physical and the mathematical point of view are the regions of the parameters where ε is small and R is large.

We first study the limiting behaviour of y when R tends to infinity and ε is kept fixed and obtain the following results: as $R \to \infty$, y converges uniformly on compact subsets to a function $\bar{y}$. If $\varepsilon \leq g(\infty) - C$, $\bar{y}$ coincides with the unique solution of $P(\varepsilon,\infty)$ and the convergence is uniform on $[0,\infty)$; on the other hand if $\varepsilon > g(\infty) - C$, $P(\varepsilon,\infty)$ has no solution and $\bar{y}$ is characterized as follows: it satisfies the differential equation in $P(\varepsilon,\infty)$, the boundary condition $\bar{y}(0) = 0$ and the condition at infinity $y(\infty) = \max(g(\infty) - \varepsilon, 0)$.

The physical problem corresponding to $P(\varepsilon,\infty)$ is essentially two-dimensional; a mathematical formulation of the same problem in n dimensions is given by the differential equation

$$\varepsilon x^{2(n-1)/n} y'' + (g(x) - y)y' = 0$$

and similar conditions at $x = 0$ and at $x = \infty$. Using methods like those of Chapter 2 one can show that this problem has a unique solution if $n = 1$ and no solution if $n \geq 3$.

We then analyse the limiting behaviour of y as ε tends to zero and R is kept fixed. As $\varepsilon \downarrow 0$, y converges uniformly in x to the function $\tilde{y}(x) = \min(g(x),C)$ and its derivative y' converges uniformly to $\tilde{y}'$ on compact subsets of $[0,R]$ which do not contain the point x_0. At this point an interior layer occurs. Using the method of matched asymptotic expansions as presented for instance by VAN HARTEN [26], we derive uniform approximations for y and y'.

Finally we consider the problem $P(\varepsilon,R)$ with much weaker hypotheses on the function g, namely

$$g \in C^1([0,R]),\ g(0) = 0,\ g(R) \geq C$$
$$g \text{ has finitely many local extrema on } [0,R].$$

Also in this case, $P(\varepsilon,R)$ turns out to have a unique solution which converges uniformly to a limit function as $\varepsilon \downarrow 0$: this limit function is continuous and consists of pieces where it is equal to $g(x)$ and pieces where it is constant. However, there are cases where at this stage we are not able to determine the limit completely.

The methods used in Chapter 2 are based on the maximum principle and on finding lower and upper solutions; in the case that g is not monotonic we also use arguments borrowed from the theory of dynamical systems. The fact that we cannot always completely characterize the limit of the solution as $\varepsilon \downarrow 0$ has led us to study $P(\varepsilon,R)$ be means of a variational method which we shall present in Chapter 4.

Problems related to $P(\varepsilon,R)$ have been considered by HALLAM & LOPER [25] and in cases where bifurcation occurs by CLÉMENT & PELETIER [11], [12], HOWES & PARTER [27], KEDEM, PARTER & STEUERWALT [34] and KOPELL & PARTER [35]. Also related are linear problems with turning points studied by GRASMAN & MATKOWSKI [24], KAMIN [31], [32], [33], DEVINATZ & FRIEDMAN [15] and SCHUSS [40].

A natural idea, after the investigation of Problem $P(\varepsilon,R)$ is to analyse the stability of its solution when considered as a steady state solution of the evolution problem P. In Chapter 3 we study the limiting behaviour of the solution v of Problem P as $t \to \infty$. We suppose that the function g satisfies the hypothesis H_g given above and that the initial function ψ satisfies the hypothesis H_ψ :

(i) ψ is continuous, with piece-wise continuous derivative on $[0,\infty)$;
(ii) $\psi(0) = 0$ and $\psi(\infty) = C$;
(iii) there exists a constant $M_\psi \geq g'(0)$ such that $0 \leq \psi'(x) \leq M_\psi$ at all points x where ψ' is defined.

When studying Problem P the difficulty is twofold: the parabolic equation in Problem P is degenerate at the origin and the coefficient of u_{xx} becomes unbounded as $x \to \infty$.

To begin with, we prove a comparison theorem. The method we apply is inspired by results of ARONSON & WEINBERGER [2] and makes use of a maximum principle due to COSNER [13]. The uniqueness of the solution of Problem P is a direct consequence of the comparison theorem.

Then we prove that P has a classical solution v which satisfies furthermore the condition

$$(*) \qquad v(\infty,t) = C \qquad \text{for } t \in [0,T],\ T < \infty$$

To do so, we first prove that property for certain related uniformly parabolic problems. We then deduce that P has a generalized solution, in a certain sense and finally we show that this solution is in fact a classical solution; we use here arguments taken from VAN DUYN [16], [17] and GILDING & PELETIER [23]. To prove that v satisfies condition (*) we construct a suitable lower solution.

We then investigate the behaviour of v as $t \to \infty$ and show that it converges towards the function $\bar{y} = \lim_{R\to\infty} y$.

Finally we analyse the rate of convergence of v towards its steady state. If g tends to infinity fast enough, $\bar{y}$ turns out to be exponentially stable; our proof follows the same line as that of FIFE & PELETIER [20]. In the more general case that $\varepsilon < g(\infty)-C$, we use a method of IL'IN & OLEINIK [28] and VAN DUYN & PELETIER [18] to derive that v converges algebraically fast towards its steady state.

Also considered is the limit case $\varepsilon \downarrow 0$: as $\varepsilon \downarrow 0$, v converges to the generalized solution $\bar{v}$ of the corresponding hyperbolic problem and as $t \to \infty$, $\bar{v}$ converges algebraically fast to its limit.

In Chapter 4, we return to the problem of determining the limit as $\varepsilon \downarrow 0$ of the solution y of the steady state problem $P(\varepsilon,R)$. In order to keep the proofs less technical while retaining the essential features of the problem we choose to analyse the following simplified version of $P(\varepsilon,R)$

$$\mathrm{BVP}^* \begin{cases} \varepsilon y'' + (g-y)y' = 0 \\ y(0) = 0, \quad y(1) = 1 \end{cases}$$

where the function $g \in L^2(0,1)$ is given. The existence of a solution y_ε of BVP^* is proven by applying Schauder's fixed point theorem. Alternatively BVP^* can be rewritten as the abstract equation

$$(\varepsilon A + I)y = g$$

where A is a maximal monotone operator on $L^2(0,1)$; it is also equivalent to a variational problem related to $\overline{\mathrm{VP}^*}$. We use these equivalent formulations of BVP* to show that as $\varepsilon \downarrow 0$ y_ε converges strongly in $L^2(0,1)$ to a limit y_0, which is the projection of g on $\overline{D(A)}$. Finally we give a more concrete form to the characterization of y_0: we present sufficient conditions for a function to be the limit and we show, by means of examples, how these criteria can be used in some concrete cases.

In Chapter 5 we study Problem BVP in the case that Ω is a bounded domain and we suppose that f is a distribution in $H^{-1}(\Omega)$. In order to prove that BVP has a unique solution u_ε which belongs to the direct sum of $H_0^1(\Omega)$ and the constant functions on Ω, we rewrite it as the subdifferential equation $\partial V_\varepsilon(u) = 0$ where V_ε is a proper, strictly convex, lower semicontinuous and coercive functional. A technical difficulty in doing so is due to the fact that we do not impose any growth condition on the nonlinear function h; we overcome it by using results from BREZIS [9] and duality theory. We remark here that BVP can be interpreted as a problem of the class $\mathcal{A}u+\mathcal{B}_\varepsilon u = f$, where $\mathcal{A}$ and $\mathcal{B}_\varepsilon$ are maximal monotone operators on $L_2(\Omega) \times \mathbb{R}$.

We then show that as $\varepsilon \downarrow 0$ u_ε converges to a limit function u_0; the main ingredients of the proof are the fact that V_ε is monotonic in ε and uniformly coercive. The limit function u_0 can be characterized as the solution of an operator inclusion relation if h is bounded and as the solution of a variational inequality if either $h(+\infty) = +\infty$ or $h(-\infty) = -\infty$. Remarkable is the fact that u_0 depends only on C, f and $h(\pm\infty)$.

Since we know a variational form of BVP, it is natural to introduce a dual formulation; to do so we follow closely EKELAND & TÉMAM [19]. In the case of the physical problem it turns out that $\overline{\mathrm{VP}^*}$ is precisely the dual problem corresponding to $\overline{\mathrm{BVP}}$. In the general case the dual problem is equivalent to a problem of the form

$$(\varepsilon A + I)p = g$$

where A is a maximal monotone operator on $(L^2(\Omega))^n$ and g is related to f by $\operatorname{div} g = f$.

Finally we suppose that $f \in L^\infty(\Omega)$. Then u_ε and u_0 belong to $W^{2,p}(\Omega)$ for each $p \geq 1$ and u_ε converges weakly to u_0 in $W^{2,p}_{loc}(\Omega)$. Thus either one has convergence in $W^{2,p}(\Omega)$ or a boundary layer develops in the neighbourhood of $\partial\Omega$ as $\varepsilon \downarrow 0$. We present criteria in terms of the data f, $h(\pm\infty)$ and C from which it can be decided in many cases which of these two possibilities actually occurs.

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CHAPTER 1

RIGOROUS RESULTS ON A TIME-DEPENDENT INHOMOGENEOUS COULOMB GAS PROBLEM

ABSTRACT

We report results obtained by rigorous analysis of a nonlinear differential equation for the electron density n_e in a specific type of electrical discharge. The problem is essentially two-dimensional. We discuss in particular (i) the escape of electrons to infinity above a critical temperature; and (ii) the boundary layer exhibited by n_e near zero temperature.

KEY WORDS & PHRASES: *singularly perturbed nonlinear two-point boundary value problem; nonlinear parabolic equation degenerate at the origin in one space dimension; Coulomb gas; pre-breakdown discharge in an ionized gas between two electrodes*

In a filamentary discharge studied by Marode et al. [1,2] electrons and ions are produced with number densities n_e and n_i, respectively. The charged particles move in a background of neutrals. The discharge area is cylindrical and has its radial dimension much smaller than its longitudinal dimension. Since to a good approximation the physical situation is cylindrically symmetric, it suffices to consider a two-dimensional cross section perpendicular to the cylinder axis, in which all quantities involved are functions only of the distance r to the axis. As the ions are heavy and slow, $n_i(r,t) \equiv n_i(r)$ may be regarded as fixed on the time scale of interest. For the density $n_e(r,t)$ Marode et al. [3] use the following three equations:
(i) Coulomb's law

$$\frac{1}{r}\frac{\partial}{\partial r} rE(r,t) = 4\pi e[n_i(r) - n_e(r,t)] \tag{1}$$

where E is the electric field and -e the electron charge;
(ii) a constitutive equation for the current density j(r), consisting of a drift term and a diffusion term,

$$j(r,t) = e\mu n_e(r,t)E(r,t) + eD\frac{\partial n_e(r,t)}{\partial r} \tag{2}$$

where μ is the electron mobility and D the diffusion constant; and
(iii) the continuity equation

$$e\frac{\partial n_e(r,t)}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r} rj(r,t). \tag{3}$$

Both E and j are radially directed.

From Eqs. (1) - (3) a nonlinear partial differential equation for a single function can be derived. To this end we set [4]

$$u(x,t) = \int_0^{\sqrt{x}} \rho n_e(\rho,t)d\rho, \tag{4a}$$

$$g(x) = \int_0^{\sqrt{x}} \rho n_i(\rho)d\rho. \tag{4b}$$

Upon employing for the diffusion constant the Einstein relation $D = k_B T\mu/e$ (where k_B is Boltzmann's constant and T the electron temperature), putting $\varepsilon = k_B T/(2\pi e^2)$, and absorbing a factor $8\pi\mu e$ in the time scale we deduce that u satisfies

$$u_t = \varepsilon x u_{xx} + (g-u)u_x, \tag{5}$$

$$u(0,t) = 0. \tag{6}$$

By its definition $g(0) = 0$. Typically, as r increases, $n_i(r)$ rapidly falls off to zero, and hence $g(x)$ attains a limit value $g(\infty)$. The nonlinear term in Eq. (5) represents the interaction between the electrons. Without it, this equation would reduce to a linear one studied by McCauley [5] and describing the Brownian motion of a pair of opposite two-dimensional charges in each other's field. As it stands, Eq. (5) is rather reminiscent of the nonlinear equations occurring in the Thomas-Fermi theory of the atom (see, e.g., ref. [6]).

In the experimental situation that we are describing the total charge in the discharge area is positive and conserved in time. This is expressed by

$$u(\infty,t) = N_e \qquad \text{for } 0 \le t < \infty \tag{7}$$

with $0 \le N_e < g(\infty)$. One of the authors has investigated [4,7,8], by rigorous mathematical methods, the solution of Eqs. (5) and (6) for a given initial distribution $u(x,0) = u_0(x)$ and subject to condition (7) on the total charge. Here we present the main results in physical language.

1. We take g concave and in $C^2([0,\infty))$. Then at given ε (i.e. at given temperature), there exists [4] a unique stationary solution $u_{st}(x)$ if the total number of electrons N_e is such that $N_e \le g(\infty) - \varepsilon$. In particular, when $\varepsilon \ge g(\infty)$, thermal motion prevents any electrons to be bound to the fixed ionic background. The existence of such a critical temperature is characteristic of two-dimensional Coulomb systems [9]. The main mathematical tools in treating the stationary problem are maximum principle arguments and the construction of upper and lower solutions.

2. The solution u_{st}, when it exists, has the following properties [4].

(i) It belongs to $C^2([0,\infty))$. It is strictly increasing, concave, and bounded from above by the function $\min(g(x),N_e)$. As $x \to \infty$, $u_{st}(x)$ approaches its limiting value N_e at least fast enough so that

$$n_e(r) \le n_e(r_1)\left(\frac{r^2}{x_1}\right)^{-\frac{1}{\varepsilon}[g(x_1)-N_e]}, \qquad r \to \infty, \tag{8}$$

where $r_1^2 \equiv x_1 > 0$ is arbitrary. Such power law decay is again typical of Coulomb systems in two dimensions.

(ii) As $\varepsilon \downarrow 0$, $u_{st}(x)$ converges to $\min(g(x),N_e)$ uniformly on $[0,\infty)$, and we have for the zero temperature limit of the electron density

$$\lim_{\varepsilon\downarrow 0} n_e(r) = \begin{cases} n_i(r) & r < r_0 \\ 0 & r > r_0 \end{cases} \tag{9}$$

where the critical radius r_0 is defined by the relation $g(r_0) = N_e$. At small ε there is a transition layer of width $\sim \varepsilon^{\frac{1}{2}}$, located at r_0, analogous to a Debye shielding length [3]. A uniformly valid approximate stationary solution for $\varepsilon \ll 1$ is given in [4]. It is obtained by the method of matched asymptotic expansions.

3. We consider now the time evolution problem of Eqs. (5) and (6). Suppose that the initial condition u_0 is sufficiently smooth, nondecreasing, with bounded derivative, and with $u_0(0) = 0$ and $u_0(\infty) = N_e$. Mathematically one has to find a way to deal with the degeneracy of the parabolic equation (5) in the origin. In [7] this is done via a sequence of regularized problems. The following is shown.

(i) The time evolution problem has a unique solution $u(x,t)$ such that u and u_x are bounded. In fact it satisfies $0 \le u(x,t) \le N_e$, it is nondecreasing in x for all t, and for each $t \ge 0$ we have $u(\infty,t) = N_e$.

(ii) In order to discuss the behavior of $u(x,t)$ as $t \to \infty$ we consider the function $\bar{u}_{st}$ which satisfies the steady state equation and has boundary values $\bar{u}_{st}(0) = 0$ and

$$\bar{u}_{st}(\infty) = \begin{cases} N_e & \text{if } N_e \le g(\infty) - \varepsilon & (10a) \\ g(\infty) - \varepsilon & \text{if } 0 < g(\infty) - \varepsilon < N_e & (10b) \\ 0 & \text{otherwise} & (10c) \end{cases}$$

We know from section 1 that $\bar{u}_{st}$ exists and is unique. In particular, in the case of Eq. (10c), $\bar{u}_{st}(x) \equiv 0$. Our result is that the solution $u(x,t)$ of the evolution problem converges to $\bar{u}_{st}(x)$ as $t \to \infty$, uniformly on all compact subsets of $[0,\infty)$; in the case of Eq. (10a) the convergence is actually uniform on $[0,\infty)$. The proofs are based upon the use of upper and lower solutions of the stationary problem and on a comparison theorem. Thus we have proved that all the electrons stay attached to the ions for $t \le \infty$ at temperatures such that $\varepsilon \le g(\infty) - N_e$ (case (10a)). If the temperature rises above this critical value, then some of the electrons diffuse away to infinity (case (10b)), and if it rises above a second critical value, viz. $\varepsilon = g(\infty)$, then all electrons escape to infinity (case (10c)).

(iii) For the case of Eq. (10a) (with the inequality strictly satisfied) we have derived results about the rate of convergence of u to $\bar{u}_{st}$. Let the initial state have the property that $N_e - u_0(x) \le N_e(x_1/x)^\nu$ for some $x_1, \nu > 0$ satisfying $\varepsilon \le (\nu+1)^{-1}[g(x_1) - N_e]$. Then $u(x,t)$ converges to $\bar{u}_{st}(x)$ at least as fast as $t^{-1/(2p)}$ with $p = [1/\nu] + 1$, for all finite x. Furthermore, if $\nu > 1$ and $\varepsilon < \frac{1}{2}[g(\infty) - N_e]$, then u converges to $\bar{u}_{st}$ at least as fast as $t^{-\frac{1}{2}}$.

4. *Negative regions in the background charge density.* We have considered an interesting modification of the above problem obtained by also allowing negative ions to be present in the fixed background [8].
This leads to a function g which can assume minima and maxima. We studied the stationary state on a bounded domain $[0,R]$ with boundary condition $u_{st}(R) = N_e$. For non-monotone g it is nontrivial to find the zero temperature ($\varepsilon \to 0$) limit of $u_{st}(x)$ (and thus of $n_e(r)$), since the solution of the reduced differential equation (i.e. the one obtained by setting $\varepsilon = 0$) is no longer unique. To solve this problem we observe that for $\varepsilon > 0$ the solution $u_{st}(x;\varepsilon)$ minimizes the free energy functional

$$F_\varepsilon[u] = \varepsilon \int_0^R u_x \ell n u_x dx + \tfrac{1}{2} \int_0^R \frac{(g-u)^2}{x} dx, \tag{11}$$

which is readily recognized as the sum of an entropy and an electrostatic energy term.

In [8] two alternative methods were used to study the minimization of F_ε: one based on the theory of maximal monotone operators and one on duality theory. Both yield

$$\lim_{\varepsilon \downarrow 0} u_{st}(x;\varepsilon) = \inf_{0 \le u \le N_e, u' \ge 0} \tfrac{1}{2} \int_0^R \frac{(g-u)^2}{x} dx, \tag{12}$$

i.e. the limit solution of the differential equation is the physically expected minimum energy configuration. The function $u_{st}(x;0)$ is continuous [10] and can be characterized as follows: there exist intervals $[a_1,b_1]$, $[a_2,b_2],\dots,[a_s,b_s]$, $s \ge 0$, where $u_{st}(x;0)$ takes constant values $c_1,c_2,\dots,c_s$, respectively, and where, therefore, $n_e(r) = 0$. Outside those intervals $u_{st}(x;0) = g(x)$. The constants a_i, b_i, c_i, $i = 1,2,\dots,s$, can be shown, finally, to be uniquely determined by the set of implicit inequalities

$$\left.\begin{aligned} &\int_x^{b_i} \frac{c_i - g(\xi)}{\xi} d\xi \ge 0 \quad \text{if} \quad c_i \ne N_e \qquad &(13a) \\ &\int_{a_i}^{x} \frac{c_i - g(\xi)}{\xi} d\xi \le 0 \quad \text{if} \quad c_i \ne 0 \qquad &(13b) \end{aligned}\right\} \text{ for all } x \in [a_i,b_i],\ i=1,2,\dots,s.$$

To verify this characterization of $u_{st}(x;0)$, one checks [8] that this function satisfies a variational inequality related to the minimization problem (12). In particular, if $0 < c_i < N_e$, we have the equal area construction $\int_{a_i}^{b_i}(c_i-g(\xi))\xi^{-1}d\xi = 0$. The interpretation is that the points $x = a_i$ and $x = b_i$ are at equal potential and separated by a potential barrier. Eqs. (13) may serve as the basis for a numerical algorithm to compute a_i, b_i, c_i.

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CHAPTER 2

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A SINGULAR BOUNDARY VALUE PROBLEM ARISING IN A PRE-BREAKDOWN GAS DISCHARGE*

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Abstract. We consider the nonlinear two-point boundary value problem $\varepsilon xy''+(g(x)-y)y'=0$, $y(0)=0$, $y(R)=k$, where g is a given function. We prove that the problem has a unique solution and we study the limiting behavior of this solution as $R\to\infty$ and as $\varepsilon\downarrow 0$.

Furthermore, we show how a so-called pre-breakdown discharge in an ionized gas between two electrodes can be described by an equation of this form, and we interpret the results physically.

1. Introduction. In this paper we study the two-point boundary value problem

$$\varepsilon xy''+(g(x)-y)y'=0, \qquad x\in(0,R), \tag{1.1}$$

in which R is a positive number, which may be infinite, and g a given function, which satisfies the hypotheses

$$H_g\colon g\in C^2(\mathbb{R}_+), \qquad g(0)=0, \qquad g'(x)>0 \quad \text{and} \quad g''(x)<0 \quad \text{for all } x\geqq 0.$$

We are interested in solutions of (1.1) which satisfy the boundary conditions

$$y(0)=0, \tag{1.2}$$

$$y(R)=k \tag{1.3}$$

in which $k\in(0,g(\infty))$ and $R>x_0$, x_0 being the (unique) root of the equation $g(x)=k$.

In § 2 we shall sketch how problem (1.1)–(1.3) arises in the study of electrical discharges in an ionized gas. It will appear that y' and g' are measures for, respectively, the electron and ion densities, and that the parameter ε is proportional to the temperature of the gas.

In § 3 we begin the mathematical analysis of problem (1.1)–(1.3). We derive some a priori estimates and then prove the existence of a solution. Subsequently, in § 4 we prove that the solution is unique.

The main objective of this paper is the study of the dependence of the solution on the parameters ε and R. In § 4 we prove that the solution is a monotone function of ε and R. From the physical point of view the interesting regions of the parameters are small ε and large R. In § 5 we analyze the limiting behavior of the solution when R tends to infinity and ε is kept fixed. It turns out that the solution converges uniformly in x to a function $\bar{y}$ which satisfies (1.1)–(1.2) and the limiting form of (1.3), i.e., $\bar{y}(\infty)=k$, if and only if $\varepsilon\leqq g(\infty)-k$. If on the other hand, this inequality is violated, then the solution converges uniformly on compact sub-sets to a function $\bar{y}$ which satisfies (1.1)–(1.2) and $\bar{y}(\infty)=\max\{g(\infty)-\varepsilon, 0\}$. In particular this implies that $\bar{y}$ is identically zero if $\varepsilon\geqq g(\infty)$.

In § 6 we analyze the limiting behavior of the solution when ε tends to zero and R is kept fixed. It turns out that the solution y converges uniformly for $x\in[0,R]$ to the function $\tilde{y}(x)=\min\{g(x),k\}$, but that its derivative y' converges uniformly to $\tilde{y}'$ only on compact subsets of $[0,R]$ which do not contain the transition point x_0.

In § 7 we discuss in greater detail the behavior of y' near the point x_0 as $\varepsilon\downarrow 0$. By the standard method of matched asymptotic expansions we formally obtain in § 8 an approximation y_a. In § 9 we prove that for each $n>1$

$$y-y_a=O(\varepsilon^{n+1/2}), \qquad y'-y_a'=O(\varepsilon^{n-1/2}) \quad \text{as } \varepsilon\downarrow 0,$$

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uniformly on $[0, R]$, where n counts the number of terms included in the approximation. In this part of our treatment of the singular perturbation problem we derived much inspiration from reading bits and pieces of van Harten's thesis [9].

Since the limits $\varepsilon \downarrow 0$ and $R \to \infty$ (for $\varepsilon \leqq g(\infty)-k$) are interchangeable, the two separate limits give a complete picture of the limiting behavior with respect to both parameters.

Finally, in § 10, we consider problem (1.1)–(1.3) under the much weaker condition on g:

$$\tilde{H}_g\colon g \in C^1([0, R]), \qquad g(0)=0, \qquad g(R) \geqq k,$$

$$g \text{ has only finitely many local extrema on } [0, R].$$

Again, the existence and uniqueness of a solution $y(x;\varepsilon)$ is established and it is shown that $y'>0$. In addition

$$y(x;\varepsilon) \to u(x) \quad \text{as } \varepsilon \downarrow 0,$$

uniformly on $[0, R]$, where the function u, which is continuous, consists of pieces where $u(x)=g(x)$ and pieces where $u(x)$ is a constant. The arguments we employ here are borrowed from the theory of dynamical systems and are somewhat unusual in this context.

Problems like the one treated in this paper have also been considered by Hallam and Loper [8], Howes and Parter [11] (also see Howes [10]), Clément and Emmerth [4] and Clément and Peletier [5]. Both of the first two papers deal with one particular equation and the second two papers deal with concave solutions y_ε of a general class of equations. In all of these $\lim_{\varepsilon \downarrow 0} y_\varepsilon$ is determined. In this paper we do the same by the method of upper and lower solutions, which was also used by Howes and Parter, and in addition we give precise estimates of the behavior of y_ε and y'_ε as $\varepsilon \downarrow 0$.

2. Physical background.

2.1. An electrical discharge. Marode et al. [14] consider an ionized gas between two electrodes in which the ions and electrons are present with densities $n_i(r)$ and $n_e(r)$ respectively, where $r=(x_1, x_2, x_3)$. The ions are heavy and slow, and the density $n_i(r)$ may therefore be regarded as fixed. The electrons are highly mobile and assume a spatial distribution in thermal equilibrium with the ions. The problem is then to find $n_e(r)$ for given $n_i(r)$.

A special situation of practical interest is a so-called pre-breakdown discharge which spreads out in filamentary form (cf. Gallimberti [7] and Marode [13]). In this situation there is cylindrical symmetry about the x_3-axis and the particle densities depend on $\rho := (x_1^2+x_2^2)^{1/2}$ only. Using Coulomb's law and a constitutive equation for the electric current, which contains both a diffusion and a conduction term, Marode et al. [14] derived that the electron density $n_e(\rho)$ should satisfy the equation

$$-\frac{\varepsilon}{2}\frac{1}{\rho}\frac{d}{d\rho}\left(\frac{\rho}{n_e(\rho)}\frac{d}{d\rho}n_e(\rho)\right) = n_i(\rho) - n_e(\rho), \tag{2.1}$$

where ε is a combination of physical constants which is proportional to the temperature. In addition n_e has to satisfy the boundary condition

$$\frac{dn_e}{d\rho}(0) = 0 \tag{2.2}$$

and the condition

$$\int_0^{\infty} \{n_i(\rho) - n_e(\rho)\}\, \rho d\rho = N > 0, \tag{2.3}$$

where N is a measure for the excess of ions.

In the experiment the ions are concentrated near the center of the discharge. Hence we shall take for n_i a function which decreases monotonically to zero as ρ tends to infinity. In this paper we study the solution n_e of (2.1)–(2.3) and in particular its behavior as $\varepsilon \downarrow 0$.

In order to cast (2.1) in a more convenient form, we make the change of variable

$$x = \rho^2 \tag{2.4}$$

and we define the new dependent variable

$$y(x) = \int_0^{x^{1/2}} n_e(s) s\, ds. \tag{2.5}$$

Thus, $y(x)$ represents the number of electrons contained in a cylinder of unit height and radius $x^{1/2}$. Analogously, we define

$$g(x) = \int_0^{x^{1/2}} n_i(s) s\, ds. \tag{2.6}$$

If we now multiply (2.1) by ρ, integrate from $\rho = 0$ to $\rho = x^{1/2}$ and use (2.4)–(2.6) we obtain (1.1). The boundary condition (1.2) is implied by (2.5) and the boundary condition (1.3), with $R = \infty$, follows from (2.3):

$$y(\infty) = k := g(\infty) - N,$$

where clearly $k \in (0, g(\infty))$.

2.2. The two-dimensional Coulomb gas. Equation (1.1) describes the equilibrium distribution of electrons interacting, via the Coulomb potential, with themselves and with a fixed positive background in a two-dimensional geometry. Theoretically one can generalize Coulomb's law to a space of arbitrary dimension d and then the corresponding equation would become

$$\varepsilon x^{2((d-1)/d)} y'' + (g(x) - y) y' = 0 \tag{2.7}$$

in which ε is again a positive constant which is proportional to the temperature.

The behavior of an assembly of charges depends on the competition between the electrostatic forces, which tend to bind positive and negative charges together, and the thermal motion which drives them apart. By physical arguments one can show that for $d > 2$ the thermal motion wins: at no nonzero temperature are the electrons bound to the ions. For $d < 2$, the electrostatic forces win, and whatever the temperature the charges are bound together (see Chui and Weeks [3]).

For the model problem consisting of (2.7) supplemented with the boundary conditions (1.2) and (1.3), with $R = \infty$, we find these matters reflected in the fact that for arbitrary positive ε, no solution exists when $d > 2$ whereas, on the contrary, a unique solution exists when $d < 2$. One can prove this along the lines indicated in § 5.

The marginal case $d = 2$ is of greatest interest. Presumably there is a critical value of the temperature at which a transition occurs from bound to unbound charges and recently there has been much interest in the precise nature of this transition (see Kosterlitz and Thouless [12]).

In our study of the two-dimensional case we find indeed, in § 5, a critical value of ε (and hence of the temperature)

$$\varepsilon_1 = g(\infty) - k = N$$

at which the nature of the solution n_e changes, corresponding to the loss (towards infinity) of part of the negative charge. Beyond a still higher value of ε:

$$\varepsilon_2 = g(\infty)$$

there appears to be no solution, indicating that the negative charge is no longer bound to the positive background.

2.3. Low temperatures. We also have studied the equations in the low temperature regime, i.e. for $\varepsilon \downarrow 0$. Physically one then expects all the electrons to gather in the region of lowest energy, that is in the center of the ion distribution. Indeed we have found that for $\varepsilon \downarrow 0$ the solution of (2.1) exhibits transition behavior

$$\lim_{\varepsilon \downarrow 0} n_e(\rho) = \begin{cases} n_i(\rho), & \rho < \rho_0, \\ 0, & \rho > \rho_0, \end{cases}$$

where ρ_0 is determined by the boundary condition (2.3). There appears to be a transition layer of width of order $\varepsilon^{1/2}$ which, according to Marode et al. [14], has the form of a Debye shielding length.

3. A priori estimates and the existence of a solution. In this section we consider the problem (1.1)–(1.3) for fixed values of the parameters ε and R. By a solution we shall mean a function $y \in C^2([0, R])$ which satisfies (1.1)–(1.3). We first derive some a priori estimates for a solution and its first two derivatives. Subsequently we prove that a solution actually exists by constructing an upper and lower solution and by verifying the appropriate Nagumo condition.

THEOREM 3.1. *Let y be a solution; then for all $x \in (0, R)$*

(i) $0 < y(x) < \min\{g(x), k\}$;

(ii) $0 < y'(x) < g'(0)$;

(iii) $-(g'(0))^2/\varepsilon < y''(x) < 0$.

Proof. Let us first prove that $y'(x) > 0$ for all $x \in (0, R)$. Suppose that $y'(x_1) = 0$ for some $x_1 > 0$; then the standard uniqueness theorem for ordinary differential equations implies that $y(x) = y(x_1)$ for all x. Since this is not compatible with the two boundary conditions we conclude that y' is sign-definite. Invoking the boundary conditions once more, we see that the sign has to be positive.

The positivity of y' implies that $0 < y(x) < k$ for $x \in (0, R)$. Next we shall prove that $y(x) < g(x)$. We begin by observing that this inequality holds for $x \geqq x_0$. Suppose there is an interval $[x_1, x_2] \subset [0, x_0]$ such that $y - g$ is strictly positive in the interior of $[x_1, x_2]$ and $y(x_1) - g(x_1) = y(x_2) - g(x_2) = 0$. Then $y'(x_2) \leqq g'(x_2) < g'(x_1) \leqq y'(x_1)$. On the other hand (1.1) implies that $y''(x) > 0$ for $x \in (x_1, x_2)$ and hence $y'(x_2) = y'(x_1) + \int_{x_1}^{x_2} y''(\xi)\, d\xi > y'(x_1)$. So our assumption must be false since it leads to a contradiction. Thus, $y(x) \leqq g(x)$. Now, let us suppose that $y(x_1) = g(x_1)$ for some $x_1 > 0$, then necessarily $y'(x_1) = g'(x_1)$. However, because $y''(x_1) = 0$ (by (1.1)) and $g''(x_1) < 0$, this would imply that $y(x) > g(x)$ in a right-hand neighborhood of x_1, which is impossible. Hence the inequality is strict for $x \in (0, R]$, and this completes the proof of (i).

From (i), $y'(x) > 0$ and (1.1) we deduce that $y''(x) < 0$ for $x \in (0, R)$. Hence $y'(x) < y'(0) \leqq g'(0)$ for $x \in (0, R)$ which completes the proof of (ii).

Finally, we note that H_g implies that $g(x) \geqq g'(0)x$ and hence that $y''(x) = (\varepsilon x)^{-1}(y(x) - g(x))y'(x) > -(\varepsilon x)^{-1} g(x) g'(0) \geqq -\varepsilon^{-1}(g'(0))^2$. This proves property (iii). □

THEOREM 3.2. *There exists a function $y \in C^2([0, R])$ which satisfies* (1.1)–(1.3).

Proof. We define two functions α and β by $\alpha(x) := 0$ and $\beta(x) := g(x)$ for $x \in [0, R]$. Moreover, we define a function f by $f(x, y, y') := (\varepsilon x)^{-1}(y - g(x))y'$. Then $\alpha''(x) = 0 \geqq 0 = f(x, \alpha(x), \alpha'(x))$ and $\beta''(x) = g''(x) < 0 = f(x, \beta(x), \beta'(x))$ for $x \in (0, R)$. Hence α and β are, respectively, a lower and an upper solution of (1.1). The existence of a solution now follows from [1, Thm. 1. 5.1] if we can show that f satisfies a Nagumo condition with respect to the pair α, β. This amounts to finding a positive continuous function h on $[0, \infty)$ such that $|f(x, y, y')| \leqq h(|y'|)$ for all $x \in [0, R]$, $\alpha(x) \leqq y \leqq \beta(x)$ and $y' \in \mathbb{R}$ and, furthermore, such that

$$\int_{R^{-1}\beta(R)}^{\infty} \frac{s}{h(s)}\, ds > \beta(R),$$

cf. [1, Def. 1.4.1]. The function h defined by $h(s) := \varepsilon^{-1} g'(0)(s+1)$ satisfies all these conditions. □

4. A comparison theorem. In order to emphasize that we are going to study the dependence of a solution on the parameters ε and R, we introduce the notation $P(\varepsilon, R)$ for the problem (1.1)–(1.3). The main result of this section is a comparison theorem which is proved by standard maximum principle arguments. As corollaries we obtain that the solution is unique and that it depends in a monotone fashion on both ε and R.

THEOREM 4.1. *Let y_i be a solution of $P(\varepsilon_i, R_i)$ for $i = 1, 2$ and suppose that $R_2 \geqq R_1 > x_0$ and $\varepsilon_2 \geqq \varepsilon_1$. Then $y_1(x) \geqq y_2(x)$ for $0 < x < R_1$. Moreover, if one of the inequalities for the parameters is strict, then so is the inequality for the solutions.*

Proof. Let the function m be defined by $m(x) := y_1(x) - y_2(x)$. Suppose that m achieves a nonpositive minimum on $(0, R_1)$, i.e. suppose that for some $x_1 \in (0, R_1)$, $m(x_1) \leqq 0$, $m'(x_1) = 0$ and $m''(x_1) \geqq 0$. By subtracting the equation for y_2 from the one for y_1 we obtain

$$\varepsilon_1 x_1 m''(x_1) - (\varepsilon_2 - \varepsilon_1) x_1 y_2''(x_1) - y_1'(x_1) m(x_1) = 0.$$

However, all the terms on the left-hand side of this equality are nonnegative and if either $\varepsilon_2 > \varepsilon_1$ or $m(x_1) < 0$ at least one of them is positive. If $\varepsilon_1 = \varepsilon_2$ and $m(x_1) = 0$ then the uniqueness theorem for ordinary differential equations implies that $m(x) = 0$ for all $x \in [0, R_1]$, which cannot be true if $R_2 > R_1$. So we see that m cannot achieve a negative minimum and that m cannot become zero on $(0, R_1)$ if one of the inequalities for the parameters is strict. Since $m(0) = 0$ and $m(R_1) \geqq 0$ this proves the theorem. □

COROLLARY 4.2. *The problem $P(\varepsilon, R)$ has one and only one solution.*

Proof. We know that at least one solution exists (Theorem 3.2). Let both y_1 and y_2 satisfy $P(\varepsilon, R)$, then Theorem 4.1 implies that $y_1(x) \geqq y_2(x)$ but likewise that $y_2(x) \geqq y_1(x)$. Hence, $y_1(x) = y_2(x)$ for $x \in [0, R]$. □

COROLLARY 4.3. *Let $y = y(x; \varepsilon, R)$ be the solution of $P(\varepsilon, R)$. Then y is a monotone decreasing function of ε for each $R > x_0$ and each $x \in (0, R)$ and y is a monotone decreasing function of R for each $\varepsilon > 0$ and each $x \in (0, R)$.*

5. The limiting behavior as $R \to \infty$. In this section we study the limiting behavior as $R \to \infty$ of the solution $y = y(x; \varepsilon, R)$ of the problem $P(\varepsilon, R)$. Since y is a bounded and monotone function of R, the definition $\bar{y}(x; \varepsilon) := \lim_{R \to \infty} y(x; \varepsilon, R)$ makes sense for all $x, \varepsilon > 0$. This definition implies at once that $\bar{y}(0, \varepsilon) = 0$ and that $\bar{y}$ is a nondecreasing function of x and a nonincreasing function of ε.

From the estimates in Theorem 3.1 we obtain, via the Arzela–Ascoli theorem, that both $y(\cdot\,;\varepsilon,R)$ and $y'(\cdot\,;\varepsilon,R)$ converge uniformly on compact subsets. Invoking (1.1) we see that the same must be true for $y''(\cdot\,;\varepsilon,R)$. It follows that $\bar{y}(\cdot\,;\varepsilon)$ belongs to $C^2(\mathbb{R}_+)$ and satisfies (1.1). Now it remains to determine $\bar{y}(\infty,\varepsilon)$. We will estimate $\bar{y}(\infty,\varepsilon)$ from below by constructing a more subtle lower solution for y. But first we prove a result which can be used to estimate $\bar{y}(\infty,\varepsilon)$ from above.

LEMMA 5.1. *Let* $z\in C^2(\mathbb{R}_+)$ *satisfy* (1.1) *and* $z(0)=0$. *Suppose that* $z(\infty):=\lim_{x\to\infty} z(x)$ *exists and satisfies* $0<z(\infty)<\infty$. *Then* $z(\infty)\leqq g(\infty)-\varepsilon$.

Proof. Both z and z' are positive on $(0,\infty)$ (cf. the proof of Theorem 3.1). For the purpose of contradiction, let us suppose that $z(\infty)>g(\infty)-\varepsilon$. Let x_1 be such that $\beta:=\varepsilon^{-1}(z(x_1)-g(\infty))>-1$. Then $z(x)-g(x)\geqq z(x_1)-g(\infty)=\varepsilon\beta$ for all $x\geqq x_1$. Integrating (1.1) twice from x_1 to x we obtain

$$z(x)=z(x_1)+z'(x_1)\int_{x_1}^{x}\exp\left(\int_{x_1}^{\xi}\frac{z(\eta)-g(\eta)}{\varepsilon\eta}\,d\eta\right)d\xi.$$

Thus, for $x\geqq x_1$,

$$z(x)\geqq z'(x_1)\int_{x_1}^{x}\exp\left(\beta\ln\frac{\xi}{x_1}\right)d\xi=\frac{x_1z'(x_1)}{\beta+1}\left(\left(\frac{x}{x_1}\right)^{\beta+1}-1\right).$$

Since $\beta+1>0$ this would imply that $z(x)\to\infty$ as $x\to\infty$. Hence the assumption that $z(\infty)>g(\infty)-\varepsilon$ must be false. □

We define a function $s=s(x;\lambda,x_1,\nu)$ by

$$s(x;\lambda,x_1,\nu):=\lambda\left(1-\left(\frac{x}{x_1}\right)^{-\nu}\right)\tag{5.1}$$

and we investigate which conditions for the parameters λ, x_1 and ν guarantee that $s''\geqq f(x,s,s')$ for $x\geqq x_1$ (recall that $f(x,y,y')=(\varepsilon x)^{-1}(y-g(x))y'$). A simple computation shows that this inequality holds indeed for all $x\geqq x_1$ if and only if $g(x_1)-\lambda-\varepsilon\nu-\varepsilon\geqq 0$, or equivalently, $\nu\leqq\varepsilon^{-1}(g(x_1)-\lambda)-1$. The latter inequality can be satisfied for some *positive* value of ν if and only if $\lambda<g(x_1)-\varepsilon$. In its turn this inequality can be satisfied for sufficiently large x_1 and some *positive* value of λ if and only if $g(\infty)-\varepsilon>0$.

We now have all the ingredients at hand to prove the following theorem.

THEOREM 5.2.

(i) *If* $\varepsilon\leqq g(\infty)-k$ *then* $\bar{y}(\infty,\varepsilon)=k$ *and* $\lim_{R\to\infty}\sup_{0\leqq x\leqq R}|y(x;\varepsilon,R)-\bar{y}(x;\varepsilon)|=0$;

(ii) *if* $g(\infty)-k<\varepsilon<g(\infty)$ *then* $\bar{y}(\infty;\varepsilon)=g(\infty)-\varepsilon$;

(iii) *if* $\varepsilon\geqq g(\infty)$ *then* $\bar{y}(x;\varepsilon)=0$ *for all* $x\geqq 0$.

Proof. (i) For any $\lambda<k$ we can choose x_1 such that $\lambda<g(x_1)-\varepsilon$ and subsequently ν such that $0<\nu\leqq\varepsilon^{-1}(g(x_1)-\lambda)-1$. For these values of the parameters, s is a lower solution on the interval $[x_1,R]$. The function t defined by $t(x):=k$ is an upper solution and f satisfies a Nagumo condition with respect to the pair s, t and the interval $[x_1,R]$. It follows that the inequality

$$s(x;\lambda,x_1,\nu)\leqq y(x;\varepsilon,R)\leqq k,$$

which holds for $x=x_1$ and for $x=R$, actually is satisfied for all $x\in[x_1,R]$. By taking first the limit $R\to\infty$ and then the limit $x\to\infty$ we obtain

$$\lambda\leqq\bar{y}(\infty;\varepsilon)\leqq k.$$

Since this inequality holds for $\lambda < k$, necessarily $\bar{y}(\infty, \varepsilon) = k$. This result and the monotonicity of y with respect to x together imply that the convergence of y to $\bar{y}$ is in fact uniform in x (we refer to [6, Lemma 2.4] for the proof of this statement).

(ii) If $g(\infty) - k < \varepsilon < g(\infty)$, we can make s into a lower solution by a suitable choice of x_1 and ν if and only if $\lambda < g(\infty) - \varepsilon$. The argument we used in the proof of (i) now shows that $\bar{y}(\infty; \varepsilon) \geqq g(\infty) - \varepsilon$. On the other hand, Lemma 5.1 implies that $\bar{y}(\infty; \varepsilon) \leqq g(\infty) - \varepsilon$. So $\bar{y}(\infty; \varepsilon) = g(\infty) - \varepsilon$.

(iii) From Lemma 5.1 we deduce that no solution of (1.1) with a positive limit at infinity can exist if $\varepsilon \geqq g(\infty)$. Hence $\bar{y}(\infty; \varepsilon) = 0$ and consequently $\bar{y}(x; \varepsilon) = 0$ for all $x \geqq 0$. □

The results of this section are at the same time results concerning the existence and nonexistence of a solution of the problem $P(\varepsilon, \infty)$ defined by (1.1), (1.2) and $\lim_{x\to\infty} y(x) = k$. By exactly the same arguments which we used before one can derive the bounds of Theorem 3.1 and one can show that there exists at most one solution of $P(\varepsilon, \infty)$. For convenience we formulate this result in the following theorem.

THEOREM 5.3. *There exists a function $y \in C^2(\mathbb{R}_+)$ which satisfies* (1.1), (1.2) *and the condition* $\lim_{x\to\infty} y(x) = k$ *if and only if $\varepsilon \leqq g(\infty) - k$. If it exists, it is unique and it satisfies the inequalities given in Theorem* 3.1.

6. The limiting behavior as $\varepsilon \downarrow 0$. Throughout this section $R > x_0$ will be fixed and we will suppress the dependence on R in the notation, because it is inessential. The solution y of (1.1)–(1.3) is a bounded and monotone function of ε and we define $\tilde{y}(x) := \lim_{\varepsilon\downarrow 0} y(x; \varepsilon)$. From Theorem 3.1(i) and (ii) and the Arzela–Ascoli theorem we deduce that $\tilde{y}$ is continuous and that in fact

$$\lim_{\varepsilon\downarrow 0} \sup_{0\leqq x\leqq R} |\tilde{y}(x) - y(x; \varepsilon)| = 0.$$

THEOREM 6.1. $\tilde{y}(x) = \min\{g(x), k\}$.

Proof. From Theorem 3.1(i) we know that $\tilde{y}(x) \leqq \min\{g(x), k\}$. Take any $x < x_0$, then $y(x) < k$. We claim that this implies that $\lim_{\varepsilon\downarrow 0} \inf y'(x; \varepsilon) > 0$. Indeed, suppose that the sequence $\{\varepsilon_i\}$ is such that $\varepsilon_i \downarrow 0$ and $y'(x; \varepsilon_i) \downarrow 0$ as $i \to \infty$, then by taking the limit $i \to \infty$ in the relation

$$k = y(R; \varepsilon_i) = y(x; \varepsilon_i) + \int_x^R y'(\xi; \varepsilon_i)\, d\xi \leqq y(x; \varepsilon_i) + (R - x) y'(x; \varepsilon_i),$$

we arrive at the conclusion that $\tilde{y}(x) \geqq k$, which is impossible.

Integrating (1.1) from 0 to x we obtain

$$\varepsilon(y'(x; \varepsilon) - y'(0; \varepsilon)) = \int_0^x \frac{y(\xi; \varepsilon) - g(\xi)}{\xi} y'(\xi, \varepsilon)\, d\xi. \tag{6.1}$$

Suppose that $x < x_0$ and $\max_{0\leqq\xi\leqq x} |\tilde{y}(\xi) - g(\xi)| > 0$; then, since $g'(0) > y'(\xi; \varepsilon) \geqq y'(x; \varepsilon)$ for $0 < \xi \leqq x$ and $\lim_{\varepsilon\downarrow 0} \inf y'(x; \varepsilon) > 0$, the right-hand side of (6.1) is bounded away from zero as $\varepsilon \downarrow 0$. However, this is impossible since the left-hand side tends to zero as $\varepsilon \downarrow 0$. So $\tilde{y}(x) = g(x)$ for all $x < x_0$, and by continuity $\tilde{y}(x_0) = k$. The function $\tilde{y}$, being the limit of monotone functions, is monotone nondecreasing. Hence $\tilde{y}(x) \geqq k$ for $x > x_0$ and consequently $\tilde{y}(x) = k$ for $x > x_0$. □

By taking $\varepsilon = 0$ in (1.1) we obtain the reduced equation

$$(g(x) - y)y' = 0. \tag{6.2}$$

The limiting function $\tilde{y}$ satisfies the boundary conditions (1.2) and (1.3) and (6.2) except at the point $x = x_0$, where $\tilde{y}'$ is not defined. Motivated in part by the physical application (cf. § 2) we shall now investigate the limiting behavior of $y'(x;\varepsilon)$ as $\varepsilon \downarrow 0$. It will then become even more apparent that $x = x_0$ is an exceptional point. The following lemma is needed in the proof of Theorem 6.3, but it is of some interest in itself.

LEMMA 6.2. *Let $\delta > 0$ be arbitrary. For any $\varepsilon_0 > 0$ there exists an $M > 0$ such that $0 < g(x) - y(x;\varepsilon) < M\varepsilon x$ for all $x \in [0, x_0 - \delta]$ and all $\varepsilon \in (0, \varepsilon_0)$.*

Proof. Let $\delta > 0$ and $\varepsilon_0 > 0$ arbitrary. We define

$$m(\varepsilon) := \min_{x_0-\delta \leqq x \leqq x_0 - \frac{1}{2}\delta} \{g(x) - y(x;\varepsilon)\}.$$

Then there exist positive constants C_i, $i = 1, 2, 3$, such that for $\varepsilon \in (0, \varepsilon_0)$

$$m(\varepsilon) \leqq C_1 \int_{x_0-\delta}^{x_0-\delta/2} (g(\xi) - y(\xi;\varepsilon))\, d\xi$$

$$\leqq C_2 \int_{x_0-\delta}^{x_0-\delta/2} \frac{g(\xi) - y(\xi;\varepsilon)}{\xi} y'(\xi;\varepsilon)\, d\xi \leqq C_3\varepsilon$$

(see the proof of Theorem 6.1 and in particular formula (6.1)). Let the function $v = v(x;\varepsilon)$ be defined by $v(x;\varepsilon) := g(x) - y(x;\varepsilon) - M\varepsilon x$, where the constant $M > 0$ is still at our disposal. Then v satisfies the equation

$$\varepsilon x v'' - y'(x;\varepsilon) v = \varepsilon x (g''(x) + M y'(x;\varepsilon))$$

and consequently $\varepsilon x v'' - \mu v > 0$ if $M > \gamma\mu^{-1}$, $\varepsilon \in (0, \varepsilon_0)$ and $x \in (0, x_0 - \frac{1}{2}\delta]$, where the positive numbers γ and μ are defined by

$$\gamma := -\inf_{0 < x \leqq x_0 - \frac{1}{2}\delta} g''(x)$$

and

$$\mu := \inf_{0 < \varepsilon < \varepsilon_0} y'\left(x_0 - \frac{\delta}{2};\varepsilon\right).$$

So if $M > \gamma\mu^{-1}$ and $\varepsilon \in (0, \varepsilon_0)$, then v cannot assume a nonnegative maximum on $(0, x_0 - \frac{1}{2}\delta)$. Let $x(\varepsilon)$ be such that $g(x) - y(x;\varepsilon)$ achieves its minimum on the set $[x_0 - \delta, x_0 - \frac{1}{2}\delta]$ in the point $x = x(\varepsilon)$. Then $v(x(\varepsilon);\varepsilon) = m(\varepsilon) - M\varepsilon x(\varepsilon) < 0$ if $M > (x_0 - \delta)^{-1} C_3$. Since $v(0;\varepsilon) = 0$, this implies that for $M > \max\{\gamma\mu^{-1}, (x_0-\delta)^{-1}C_3\}$, $v(x;\varepsilon) < 0$ for $x \in (0, x(\varepsilon))$ and a fortiori for $x \in (0, x_0 - \delta)$. □

THEOREM 6.3. *Let $\delta > 0$ be arbitrary. Then*

(i) $\lim_{\varepsilon \downarrow 0} \sup_{0 \leqq x \leqq x_0 - \delta} |g'(x) - y'(x;\varepsilon)| = 0$;

(ii) $\lim_{\varepsilon \downarrow 0} \sup_{x_0 + \delta \leqq x \leqq R} |y'(x;\varepsilon)| = 0$.

Proof. (i) From (1.1), Theorem 3.1(ii) and Lemma 6.2 we deduce that $-g'(0)M < y''(x;\varepsilon) < 0$ for $x \in [0, x_0 - \delta]$ and $\varepsilon \in (0, \varepsilon_0)$. By the Arzela–Ascoli theorem this implies that the limit set of $\{y'(\cdot\,;\varepsilon) \mid \varepsilon > 0\}$ as $\varepsilon \downarrow 0$ is nonempty in $C([0, x_0 - \delta])$. The result now follows from the fact that y tends to g on $[0, x_0 - \delta]$ as $\varepsilon \downarrow 0$.

(ii) Integrating (1.1) from $x_0 + \frac{1}{2}\delta$ to x we obtain

$$\varepsilon(y'(x;\varepsilon) - y'(x_0 + \tfrac{1}{2}\delta;\varepsilon)) = \int_{x_0+\frac{1}{2}\delta}^{x} \frac{y(\xi;\varepsilon) - g(\xi)}{\xi} y'(\xi;\varepsilon)\, d\xi.$$

For $x \in [x_0 + \delta, R]$ the right-hand side is smaller than $\frac{1}{2}\delta R^{-1}(k - g(x_0 + (\delta/2)))y'(x;\varepsilon)$. Consequently $0 < y'(x;\varepsilon) < 2g'(0)\varepsilon R\delta^{-1}(g(x_0 + (\delta/2)) - k)^{-1}$. □

In the next section we shall concentrate on a formal approximation for y and y' in the neighborhood of $x = x_0$.

In § 5 it was shown that the problem $P(\varepsilon, \infty)$ has a unique solution for ε sufficiently small. The analysis of this section can be repeated, mutatis mutandis, to derive the analogous results concerning the limiting behavior of this solution as $\varepsilon \downarrow 0$. In particular this implies that the limits $\varepsilon \downarrow 0$ and $R \to \infty$ are interchangeable.

7. The transition layer. In Theorem 6.3 we have shown that y' converges nonuniformly on the interval $[0, R]$ as $\varepsilon \downarrow 0$. This feature is typical for a singular perturbation problem. In this section we use the standard method of the stretching of a variable to obtain more information about the behavior of y' near the transition point $x = x_0$.

By the stretching of the variable x near x_0 we mean the introduction of a local coordinate ξ according to $x = x_0 + \varepsilon^\alpha \xi$. At the same time we introduce a local dependent variable η according to

$$y(x) = g(x_0) + \varepsilon^\beta \eta(\xi).$$

If we make these substitutions in the equation, and subsequently only retain the terms of lowest order in ε, it depends on the values of α and β what the resulting equation will be. One easily verifies that the choice $\alpha = \beta = \frac{1}{2}$ leads to a significant equation, namely to

$$x_0 \eta_1'' + (\xi g'(x_0) - \eta_1)\eta_1' = 0, \tag{7.1}$$

where we have introduced the subscript 1 to indicate that we consider in fact a first approximation. To this equation we add the condition that its solution should match the limits of y to the left and to the right of x_0, respectively, up to the appropriate order in $\sqrt{\varepsilon}$. This amounts to the conditions

$$\begin{aligned} &\eta_1(\xi) = g'(x_0)\xi + o(1) \quad \text{as } \xi \to -\infty, \\ &\eta_1(\xi) = o(1), \quad \text{as } \xi \to +\infty. \end{aligned} \tag{7.2}$$

A straightforward application of the maximum principle (see Theorem 4.1) shows that the problem (7.1)–(7.2), which we shall denote by Π_1, admits at most one solution.

The problem Π_1 is nonautonomous. However, if we set $\eta_1' = z_1$, divide the equation by z_1 and then differentiate it, we formally obtain an autonomous problem, which we denote by $\tilde{\Pi}_1$:

$$x_0\left(\frac{z_1'}{z_1}\right)' + g'(x_0) - z_1 = 0, \tag{7.3}$$

$$\begin{aligned} &z_1(\xi) = g'(x_0) + o(1) \quad \text{as } \xi \to -\infty, \\ &z_1(\xi) = o(1) \quad \text{as } \xi \to +\infty. \end{aligned} \tag{7.4}$$

One should note that, at least formally up to first order in $\sqrt{\varepsilon}$, z_1 describes the shape of y' in the neighborhood of x_0. In the remainder of this section we shall discuss the existence of a family of solutions of problem $\tilde{\Pi}_1$, and we shall show how this family can be used to obtain the solution of problem Π_1.

One way to handle problem $\tilde{\Pi}_1$ is to write (7.3) as a two-dimensional first order system and analyze the trajectories in the phase plane. It turns out that the singular point $(z_1, z_1') = (g'(x_0), 0)$ is a saddle point and that one branch of the unstable manifold lies in the half-plane $z_1' < 0$ and enters the (singular) singular point $(0, 0)$. Hence $\tilde{\Pi}_1$ has a one-parameter family of strictly decreasing solutions, where the parameter describes simply the translation of one particular solution.

However, it so happens that $\tilde{\Pi}_1$ can be solved explicitly for ξ in terms of z_1. To this end we put

$$z_1 = g'(x_0)e^v \quad \text{and} \quad \xi' = \sqrt{\frac{2g'(x_0)}{x_0}}\,\xi.$$

Then $v = v(\xi')$ has to satisfy

$$2v'' + 1 - e^v = 0,$$
$$v(-\infty) = 0, \qquad v(+\infty) = -\infty,$$

and we obtain, after multiplication by v' and one integration,

$$(v')^2 + v - e^v = -1$$

and finally

$$\xi' = \int_v^C \frac{dw}{\sqrt{e^w - w - 1}}, \tag{7.5}$$

where the parameter C corresponds to the free translation parameter. From this expression we easily obtain the asymptotic behavior of the solutions:

$$z_1(\xi) \sim g'(x_0) + \exp\left(\frac{\sqrt{g'(x_0)}}{x_0}(\xi - C)\right), \qquad \xi \to -\infty,$$

$$z_1(\xi) \sim g'(x_0) \exp\left(-\frac{g'(x_0)}{2x_0}(\xi - C)^2\right), \qquad \xi \to +\infty.$$

As candidates for a solution of Π_1 we take the functions

$$\psi(\xi, C) = \int_\infty^\xi \tilde{z}_1(\tau + C)\, d\tau = \int_\infty^{\xi + C} \tilde{z}_1(\tau)\, d\tau,$$

where $\tilde{z}_1$ is the particular solution of $\tilde{\Pi}_1$ which satisfies $\tilde{z}_1(0) = \frac{1}{2}g'(x_0)$ (or, in other words, which corresponds with $C = \frac{1}{2}g'(x_0)$ in (7.5)). Using (7.3) we obtain after some manipulation

$$(x_0\psi'' + (\xi g'(x_0) - \psi)\psi')' = \frac{\psi''}{\psi'}(x_0\psi'' + (\xi g'(x_0) - \psi)\psi'),$$

where primes denote differentiation with respect to ξ and where we have suppressed the dependence on C in the notation. Hence

$$x_0\psi'' + (\xi g'(x_0) - \psi)\psi' = K_1\psi'.$$

Furthermore, we deduce from $\tilde{\Pi}_1$ that

$$\psi(\xi; C) = g'(x_0)\xi + K_2 + o(1), \qquad \xi \to -\infty.$$

Since ψ''/ψ' tends to zero as $\xi \to -\infty$ it follows that $K_2 = -K_1$.

Of course the constants K_1 and K_2 depend on C and it remains to show that we can choose C in such a way that they both become zero. We observe that

$$K_1(C) = x_0\frac{\psi''(0; C)}{\psi'(0; C)} - \psi(0; C)$$
$$= x_0\frac{\tilde{z}_1'(C)}{\tilde{z}_1(C)} - \int_\infty^C \tilde{z}_1(\tau)\, d\tau.$$

From the known asymptotic behavior of $\tilde{z}_1$ we deduce that K_1 tends to $\pm\infty$ as C tends to $\mp\infty$. Moreover

$$\frac{dK_1}{dC}(C)=x_0\left(\frac{\tilde{z}_1'}{\tilde{z}_1}\right)'(C)-\tilde{z}_1(C)=-g'(x_0)<0.$$

Thus, K_1 is a strictly decreasing function with range $(-\infty, \infty)$ and we conclude that there exists a unique value of C, C_1 say, such that $K_1(C)=0$. Consequently $\eta_1 := \psi(\cdot\,; C_1)$ is the solution of problem Π_1. Furthermore, the properties of $\tilde{z}_1$ imply that (i) η_1 is negative, strictly increasing and concave, (ii) $\eta_1(\xi)\to 0$ faster than exponentially as $\xi\to+\infty$, (iii) the function $\eta_1(\xi)-g'(x_0)\xi$, as well as all its derivatives, converge exponentially to zero as $\xi\to-\infty$.

The idea of singular perturbation theory is that $\tilde{z}_1(\cdot+C_1)$ describes the transition of y' near $x=x_0$ for small values of ε, and that one can approximate y' uniformly on $[0, R]$ by using the building-stones $\tilde{z}_1(\cdot+C_1)$ and $\tilde{y}'$. In the following sections we shall elaborate this idea and we shall prove its correctness. It turns out that this will require the construction of at least five terms in a uniform asymptotic expansion. Since for us, as for many mathematicians, five is almost equal to infinity we shall first discuss the construction of a complete asymptotic expansion.

8. Matched asymptotic expansions. Throughout this and the next section we shall assume that $g\in C^\infty([0, R])$.

On the interval $[0, x_0-\delta]$ we look for an asymptotic expansion of the form

$$y(x)=\sum_{n=0}^{\infty}\varepsilon^n y_n(x). \tag{8.1}$$

We find that $y_0(x)=g(x)$ and that y_n is defined recursively by

$$y_n(x)=(y_0'(x))^{-1}\left\{xy_{n-1}''(x)-\sum_{k=1}^{n-1}y_k(x)y_{n-k}'(x)\right\},\qquad n\geqq 1. \tag{8.2}$$

In order to calculate the matching conditions for the transition layer expansion, we expand each y_n in a Taylor series

$$y_n(x)=\sum_{k=0}^{\infty}(\sqrt{\varepsilon})^k\frac{y_n^{(k)}(x_0)}{k!}\xi^k,$$

where, as before, $\xi=(x-x_0)/\sqrt{\varepsilon}$. If we substitute this in the expansion for y and rearrange the resulting expression by collecting terms with like powers of $\sqrt{\varepsilon}$, we obtain

$$y(x)=\sum_{m=0}^{\infty}(\sqrt{\varepsilon})^m u_m(\xi) \tag{8.3}$$

where, by definition,

$$u_m(\xi)=\sum_{n=0}^{[m/2]}\frac{y_n^{(m-2n)}(x_0)}{(m-2n)!}\xi^{m-2n}. \tag{8.4}$$

On the interval $[x_0+\delta, R]$ one can also introduce a series expansion in powers of ε, but it will quickly turn out that all the terms, except the one of zero'th order which is k, are zero.

Next we introduce the transition layer expansion

$$y(x)=\sum_{n=0}^{\infty}(\sqrt{\varepsilon})^n\eta_n(\xi), \tag{8.5}$$

where $\eta_0(\xi) = g(x_0)$ and η_1 is the solution of the problem Π_1 discussed in § 7. Substitution in the equation yields an equation for each η_n. Together with the matching condition which is obtained by formal identification of (8.5), as $\xi \to -\infty$, with (8.3), this yields for $n \geqq 2$ a *linear* problem Π_n defined recursively by

$$\begin{aligned} &x_0\eta_n'' + (g'(x_0)\xi - \eta_1)\eta_n' - \eta_1'\eta_n = q_n, \\ &\eta_n(\xi) = u_n(\xi) + o(1) \quad \text{as } \xi \to -\infty, \\ &\eta_n = o(1) \quad \text{as } \xi \to +\infty, \end{aligned} \tag{8.6}$$

where

$$q_n(\xi) := -\frac{g^{(n)}(x_0)}{n!}\xi^n\eta_1' - \xi\eta_{n-1}'' - \sum_{k=2}^{n-1} \eta_{n+1-k}'\left(\frac{g^{(k)}(x_0)}{k!}\xi^k - \eta_k\right). \tag{8.7}$$

As before the maximum principle implies that problem Π_n can have at most one solution. In order to discuss the existence of a solution we first rewrite the equation by making use of (7.1) for η_1:

$$x_0\left(\frac{\eta_n'}{\eta_1'}\right)' - \eta_n = \frac{q_n}{\eta_1'}.$$

Introducing $z_1 := \eta_1'$, $\zeta_n := (z_1)^{-1}\eta_n'$ and $h_n := ((z_1)^{-1}q_n)'$, we obtain by differentiation

$$x_0\zeta_n'' - z_1\zeta_n = h_n. \tag{8.8}$$

At this point it is important to observe that we know a particular solution of the homogeneous equation $x_0\phi'' - z_1\phi = 0$, namely

$$\phi(\xi) := \frac{z_1'(\xi)}{z_1(\xi)} \tag{8.9}$$

(one can verify this by differentiation of (7.3)). Hence we can construct solutions of (8.8) through the method of variation of constants, and we find

$$\zeta_n(\xi; C) = \frac{\phi(\xi)}{x_0}\int_0^{\xi} \phi^{-2}(\tau)\int_{-\infty}^{\tau} \phi(\sigma)h_n(\sigma)\,d\sigma\,d\tau + C\phi(\xi) \tag{8.10}$$

(note that we do not consider the general solution of the homogeneous equation since only ϕ has the right asymptotic behavior as $\xi \to -\infty$). For any C, the function defined in (8.10) is of polynomial growth as $\xi \to +\infty$ and behaves like $g'(x_0)u_n'$ as $\xi \to -\infty$. The last statement can be verified by working out the consistency relations between q_n and u_n which follow from the identity

$$x_0u_n'' - g'(x_0)u_n = -\frac{g^{(n)}(x_0)}{n!}\xi^n g'(x_0) - \xi u_{n-1}'' - \sum_{k=2}^{n-1} u_{n+1-k}'\left(\frac{g^{(k)}(x_0)}{k!}\xi^k - u_k\right)$$

and by making use of the known asymptotic behavior of ϕ.

Finally, we define

$$\eta_n(\xi; C) = \int_{\infty}^{\xi} z_1(\tau)\zeta_n(\tau; C)\,d\tau = \eta_n(\xi; 0) + C\eta_1'(\xi). \tag{8.11}$$

Then $\eta_n(\xi; C) = u_n(\xi) + B_n + g'(x_0)C + o(1)$, $\xi \to -\infty$, where B_n is some number, which does not depend on C. It follows that there exists a unique constant, say C_n, for which the matching condition is satisfied and consequently $\eta_n(\xi; C_n)$ is the unique solution of the problem Π_n. This completes the construction of the transition layer expansion.

To conclude this section we construct a uniform approximation of formal order $2n+1$ in $\sqrt{\varepsilon}$. We introduce two C^∞-functions H and J defined on $\mathbb{R}$ (so-called cut-off functions) with the following properties

$$H(x)=\begin{cases}0 & \text{if } |x-x_0|\geqq \delta_1,\\ 1 & \text{if } |x-x_0|\leqq \dfrac{\delta_1}{2},\end{cases}$$

$$J(x)=\begin{cases}0 & \text{if } |x|\leqq \delta_2,\\ 1 & \text{if } |x|\geqq 2\delta_2,\end{cases}$$

where δ_1 and δ_2 are suitable constants which do not depend on ε. Then the formal approximation $y_a(x)$ is defined by

$$y_a(x)=\begin{cases} J\left(\dfrac{x-x_0}{\sqrt{\varepsilon}}\right)\sum\limits_{m=1}^{n}\varepsilon^m y_m(x)+H(x)\sum\limits_{m=1}^{2n+1}(\sqrt{\varepsilon})^m\left(\eta_m\left(\dfrac{x-x_0}{\sqrt{\varepsilon}}\right)\right.\\ \qquad\qquad\left.-J\left(\dfrac{x-x_0}{\sqrt{\varepsilon}}\right)u_m\left(\dfrac{x-x_0}{\sqrt{\varepsilon}}\right)\right) & \text{for } x\leqq x_0,\\ J\left(\dfrac{x-x_0}{\sqrt{\varepsilon}}\right)k(1-H(x))+H(x)\sum\limits_{m=1}^{2n+1}(\sqrt{\varepsilon})^m\eta_m\left(\dfrac{x-x_0}{\sqrt{\varepsilon}}\right) & \text{for } x\geqq x_0.\end{cases}\tag{8.12}$$

Apart from the cut-off functions this formula is the usual one, expressing a uniform approximation as the sum of approximations in the different regions minus the matching terms, which are contained in two approximations and hence should be subtracted in order to avoid double counting. The cut-off functions are used to achieve two ends: the approximation should satisfy the boundary conditions and it should be smooth at $x=x_0$. Moreover, the cut-off functions are harmless in the sense that they are multipled by factors which are small (if ε is small) in regions where the cut-off functions are different from one. In the next section we shall prove that y_a and y_a' are indeed uniform approximations of y and y' up to the order $\varepsilon^{n+(1/2)}$ and $\varepsilon^{n-(1/2)}$, respectively.

9. A proof of the validity of the formal construction. We begin by deriving an estimate for the difference

$$z(x):=y(x)-y_a(x).\tag{9.1}$$

It follows from the equation for y and from the construction of y_a that z satisfies

$$\begin{gathered}\varepsilon\times z''+(g-y)z'-y'z+zz'=r,\\ z(0)=0,\qquad z(R)=0,\end{gathered}\tag{9.2}$$

where the remainder term r, defined by

$$r(x):=-(\varepsilon xy_a''+(g-y_a)y_a'),\tag{9.3}$$

can be shown, after an elaborate computation, to satisfy

$$r(x)=O(x\varepsilon^n)\quad\text{as } \varepsilon\downarrow 0 \text{ and/or } x\downarrow 0.\tag{9.4}$$

If we multiply the equation for z by z and integrate from 0 to R we obtain after some integrations by parts and an application of the Cauchy–Schwarz inequality

$$\varepsilon\int_0^R x(z'(x))^2\,dx+\frac{1}{2}\int_0^R (g'(x)+y'(x))z^2(x)\,dx\leqq\|z\|\|r\|,$$

where $\|\cdot\|$ denotes the L_2-norm. Since $g'(x)+y'(x)\geqq g'(R)$ this implies, first of all, that

$$\|z\|\leqq\frac{2}{g'(R)}\|r\|$$

and hence that

$$\varepsilon\int_0^R x(z'(x))^2\,dx+\frac{g'(R)}{2}\|z\|^2\leqq\frac{2}{g'(R)}\|r\|^2.$$

Now, fix $\delta\in(0,x_0)$. The estimate above is easily translated into an estimate for the $H^1(\delta,R)$-norm of z, where H^1 denotes the usual Sobolev space of L_2-functions which have a generalized derivative belonging to L_2. Thus, by the continuous imbedding of H^1 into the space of continuous functions we obtain

$$|z(x)|\leqq C(\varepsilon^{-1}\|r\|^2)^{1/2}\leqq C\varepsilon^{n-1/2},\qquad \delta\leqq x\leqq R,$$

where C depends on δ. Having established this estimate on the interval $[\delta,R]$, we can extend it to the interval $[0,R]$ by means of the maximum principle in exactly the same way as we proved Lemma 6.2.

Next, it is advantageous to take explicitly into account the dependence on the parameter n, which counts the number of terms included in the approximation. So putting $z=z_n$ we write the estimate obtained so far as

$$|z_n(x)|\leqq Cx\varepsilon^{n-1/2},\qquad 0\leqq x\leqq R,\quad n\in\mathbb{N}.$$

Then, observing that

$$|z_{n+1}(x)-z_n(x)|\leqq Cx\varepsilon^{n+1},$$

we deduce the sharper estimate

$$|z_n(x)|\leqq|z_n(x)-z_{n+1}(x)|+|z_{n+1}(x)|\leqq Cx\varepsilon^{n+1/2}.$$

(This is the familiar "throwing away" of terms which are needed in the proof, but do not contribute to the result.) We state this as a theorem.

THEOREM 9.1. *There exist constants $\varepsilon_0>0$ and $C>0$ such that*

$$|y(x)-y_a(x)|\leqq Cx\varepsilon^{n+1/2}$$

for $0<\varepsilon<\varepsilon_0$ and $0\leqq x\leqq R$.

Our next objective is to show that the derivative of y_a is a good approximation for the derivative of y (recall that y_a is more or less constructed through the integration of its derivative, and that in our application the derivative is the function which has a direct physical meaning). Our proof will be based on the following interpolation inequality.

LEMMA 9.2. *There exist constants $\mu_0>0$ and $D>0$ such that for any $\phi\in C^2([0,R])$ and each $\mu\in(0,\mu_0)$*

$$\sup|\phi'(x)|\leqq D\{\mu\sup|\phi''(x)|+\mu^{-1}\sup|\phi(x)|\},$$

where the suprema are taken over the interval $[0,R]$.

Proof. See Besjes [2]. The proof is based on a result to be found in Miranda [15, 33, III, p. 149]. □

THEOREM 9.3. *There exist constants $\varepsilon_0>0$ and $C>0$ such that*

$$|y'(x)-y_a'(x)|\leqq C\varepsilon^{n-1/2}$$

for $0<\varepsilon<\varepsilon_0$ and $0\leqq x\leqq R$.

Proof. From the equation for z (see (9.2)) we deduce that

$$|z''(x)| \leqq \varepsilon^{-1}\left\{\left|\frac{r(x)}{x}\right| + C_1|z'(x)| + C_2\left|\frac{z(x)}{x}\right|\right\},$$

where

$$C_1 := \sup_{0 \leqq x \leqq R} \frac{g(x)-y(x)}{x}, \qquad C_2 := \sup_{0 \leqq x \leqq R} |y_a'(x)|.$$

Next we apply Lemma 9.2 with $\mu = \varepsilon(2C_1D)^{-1}$ to obtain

$$\sup|z''(x)| \leqq 2\varepsilon^{-1}\left\{\sup\left|\frac{r(x)}{x}\right| + 2(C_1D)^2\varepsilon^{-1}\sup|z(x)| + C_2 \sup\left|\frac{z(x)}{x}\right|\right\}.$$

By Theorem 9.1 and the estimate (9.4) this implies that

$$\sup|z''(x)| = O(\varepsilon^{n-3/2}).$$

Then a second application of Lemma 9.2, this time with $\mu = \varepsilon$, leads to the desired result. □

10. Some remarks about the case where g is neither everywhere increasing nor everywhere concave. In this section we shall discuss some extensions of our results to equations in which the conditions on the function g are considerably relaxed. In fact we shall merely assume that g satisfies the following hypotheses

$$\tilde{H}_g\colon g \in C^1([0, R]), \qquad g(0) = 0, \quad g(R) \geqq k,$$
$$g \text{ has only finitely many local extrema on } [0, R].$$

Thus, in particular the sign conditions on g' and g'' are dropped.

First of all we observe that the existence of a solution of (1.1)–(1.3) can be proved as in Theorem 3.2 by using zero as a lower solution and G as an upper solution, where G is any increasing, concave and smooth function such that $G(0) = 0$ and $G(x) \geqq g(x)$ on $[0, R]$.

As before we find that if $y = y(x; \varepsilon)$ is a solution then $y' > 0$ and hence sign y'' = sign $(y - g)$; subsequently, reasoning along the lines indicated in the proofs of Theorem 3.1 one can show that for any $\varepsilon > 0$,

$$0 < y'(x; \varepsilon) \leqq \sup_{0 \leqq \xi \leqq R} g'(\xi). \tag{10.1}$$

This in turn enables one to prove by means of the maximum principle that (1.1)–(1.3) can have at most one solution, and that the mapping $\varepsilon \mapsto y(\cdot\,; \varepsilon)$ is continuous from $\mathbb{R}_+$ into $C = C([0, R])$.

By (10.1) the set $\{y(\cdot\,; \varepsilon) | \varepsilon > 0\}$ is a precompact subset of C. Let X denote its limit set, as $\varepsilon \downarrow 0$, in C. Taking into account the continuity with respect to ε, we conclude that X is a nonempty, compact and connected subset of C (see Sell [16, p. 20]).

Any element u of X is a nondecreasing function with $u(0) = 0$ and $u(R) = k$. Our first objective is to give further characteristics of the elements of X.

LEMMA 10.1. *Let $u \in X$. Then there exist a nonempty, open set A and a closed set B such that*

(i) *$u(x) = g(x)$ if $x \in A$,*

(ii) *u is constant on each connected component of B,*

(iii) *$A \cap B = \varnothing$, $A \cup B = [0, R]$.*

Proof. Since $u \in X$, there exists a sequence $\{\varepsilon_n\}$ such that as $n \to \infty$, $\varepsilon_n \downarrow 0$ and $y(\cdot\,;\varepsilon_n) \to u$ strongly in C. By (10.1) $\{y(\cdot\,;\varepsilon_n)\}$ is bounded in $H^1 = H^1(0, R)$ and hence it is possible to pick a subsequence, again denoted by $\{\varepsilon_n\}$, such that as $n \to \infty$, $y(\cdot\,;\varepsilon_n) \to u$ weakly in H^1.

Next, we multiply (1.1) by an arbitrary function $\phi \in H^1$, integrate from 0 to R, integrate the first term by parts and let n tend to infinity. This yields the identity

$$\int_0^R (g(x) - u(x))u'(x)\phi(x)\,dx = 0,$$

whence

$$(g(x) - u(x))u'(x) = 0 \quad \text{a.e. on } [0, R]. \tag{10.2}$$

Define the sets A and B by

$$A = \{x \in [0, R] \mid u = g \text{ in a neighborhood of } x\}, \qquad B = [0, R]\backslash A,$$

then clearly $u'(x) = 0$ a.e. on B. In view of the continuity of g and u the sets A and B have all the properties listed in the lemma. □

LEMMA 10.2. *Let $u \in X$ and let I be a connected component of B such that $I \subset (0, R)$. Then*

$$\int_I \frac{u(x) - g(x)}{x}\,dx = 0. \tag{10.3}$$

Before proving this lemma, we prove an auxiliary result.

LEMMA 10.3. *Suppose that, as $n \to \infty$, $\varepsilon_n \downarrow 0$ and $y(x;\varepsilon_n) \to g(x)$ uniformly on $[a, b] \subset [0, R]$. Then*

$$\varepsilon_n \ln y'(x;\varepsilon_n) \to 0 \quad \text{as } n \to \infty$$

uniformly on $[a, b]$.

Proof. Choose a subinterval $[c, d]$ of $[a, b]$ and a positive constant $\delta > 0$ such that $g'(x) \geqq \delta$ on $[c, d]$. Define for each $n \geqq 1$, a point $\xi_n \in [c, d]$ such that

$$y'(\xi_n;\varepsilon_n) = \max\{y'(x;\varepsilon_n) \mid c \leqq x \leqq d\}.$$

Then it follows that there exists an $N_1 \geqq 1$ such that

$$y'(\xi_n;\varepsilon_n) \geqq \tfrac{1}{2}\delta \quad \text{for } n \geqq N_1.$$

If we divide (1.1) by xy' and integrate from ξ_n to x we obtain

$$\varepsilon_n \ln y'(x;\varepsilon_n) = \varepsilon_n \ln y'(\xi_n;\varepsilon_n) + \int_{\xi_n}^{x} \frac{y(\tau;\varepsilon_n) - g(\tau)}{\tau}\,d\tau.$$

Since the right-hand side tends to zero as $n \to \infty$, the same must be true for the left-hand side and the result follows. □

Proof of Lemma 10.2. Let $I = (e, f)$, where, by assumption, $0 < e < f < R$. Manipulating as above we obtain

$$\varepsilon_n \ln y'(e;\varepsilon_n) - \varepsilon_n \ln y'(f;\varepsilon_n) = \int_e^f \frac{y(\tau, \varepsilon_n) - g(\tau)}{\tau}\,d\tau.$$

Applying Lemma 10.3 to a left-hand neighborhood of e and to a right-hand neighborhood of f, we deduce that the left-hand side of this identity tends to zero as $n \to \infty$. So taking the limit $n \to \infty$ leads to the desired result. □

We now collect the information we have obtained about an arbitrary element u of X: u is a continuous, nondecreasing function with $u(0)=0$ and $u(R)=k$, which is composed out of pieces where $u(x)=g(x)$ and pieces where $u(x)$ is constant. Moreover, if I is a maximal interval on which u is constant, and I does not contain 0 or R, then (10.3) has to be satisfied. For convenience of formulation we shall call the set of functions having all these characteristics Y.

Our next objective is to show that Y is finite. First we shall illustrate our approach by discussing one example in full detail.

Consider a function g satisfying $\tilde{H}_g$ and such that g' vanishes at only two points b and c, b being a local maximum and c a local minimum. Assume that $0<b<c<R$ and $0<g(c)<g(b)<k$. Let g_1^{-1} denote the inverse of g on $[0, b]$ and g_2^{-1} the inverse of g on $[c, R]$.

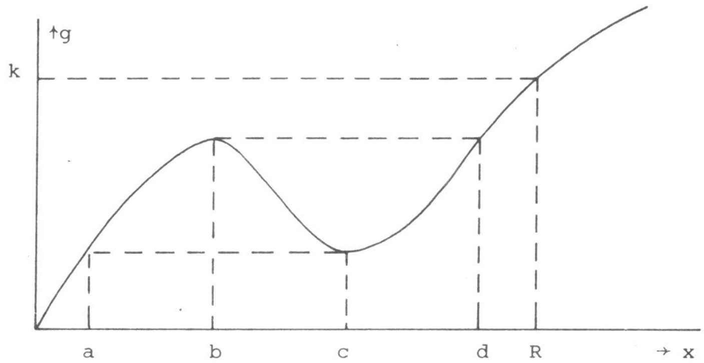

FIG. 1

Define two points a and b by

$$a=g_1^{-1}(g(c)), \qquad d=g_2^{-1}(g(b)).$$

Then $g([a, b])=g([c, d])$. (See Fig. 1.)

On $[a, b]$ we define a mapping F by

$$F(x)=\int_x^{g_2^{-1}(g(x))} \frac{g(x)-g(\tau)}{\tau}\, d\tau.$$

Then on (a, b)

$$F'(x)=g'(x)\int_x^{g_2^{-1}(g(x))} \frac{d\tau}{\tau}>0$$

and $F(a)<0$, $F(b)>0$. Consequently F has a unique zero on $[a, b]$.

Let w be an arbitrary element of Y. Then w has to coincide with g on $[0, a]$ and $[d, g_2^{-1}(k)]$ and it has to be equal to k on $[g_2^{-1}(k), R]$. Since w is nondecreasing the inverse function of w must "jump" from a point on $[a, b]$ to a point on $[c, d]$. In view of (10.3) this jump can only take place at the *unique* zero of F. Thus Y consists of one and only one element.

Returning to a general function g satisfying $\tilde{H}_g$ we define E to be the set of local maxima and minima of g and D to be the closure of the set $\{x \mid g$ is increasing in a neighborhood of x}. Let D_c be one of the finitely many connected components of D. The set $g^{-1}(E) \cap D_c$ is finite. Take two successive points α_0 and β_0 in this set. To $[\alpha_0, \beta_0]$ there correspond finitely many disjunct intervals $[\alpha_i, \beta_i] \subset D$ such that $\alpha_i > \alpha_0$ and $g([\alpha_0, \beta_0]) = g([\alpha_i, \beta_i])$. Define g_i^{-1} on $[g(\alpha_0), g(\beta_0)]$ as the inverse of g with range in $[\alpha_i, \beta_i]$. On $[\alpha_0, \beta_0]$ we define mappings F_i by

$$F_i(x) = \int_x^{g_i^{-1}(g(x))} \frac{g(x) - g(\tau)}{\tau} \, d\tau.$$

Since F_i is monotone, it has at most one zero.

As already noted above the condition (10.3) implies that a point where the inverse function of an element of Y makes a jump should be a zero of some F_i for some connected component D_c of D and some pair of points α_0, β_0. Hence the set of possible "jump" points is finite and likewise the set Y is finite.

Thus X, being a subset of Y, must be discrete. Because it is also connected it can only consist of a single element. Consequently $y(\cdot\,; \varepsilon)$ converges in C to this function as $\varepsilon \downarrow 0$. We summarize the results in the following theorem.

THEOREM 10.4. *There exists a function* $u \in Y$ *such that*

$$\lim_{\varepsilon \downarrow 0} y(x; \varepsilon) = u(x), \quad \textit{uniformly on } [0, R].$$

In some cases the conditions determine the limit uniquely. For instance, this happens in the example we discussed at length and, more generally if the local extrema are ordered in such a way that with each connected component of D there corresponds precisely one possible "jump" point. In other cases our analysis is not constructive in the sense that, although we have shown that convergence occurs as $\varepsilon \downarrow 0$, we are not able to describe the limit completely. (See Fig. 2.) We intend to investigate whether this ambiguity can be resolved by using variational principles. See note added in proof.

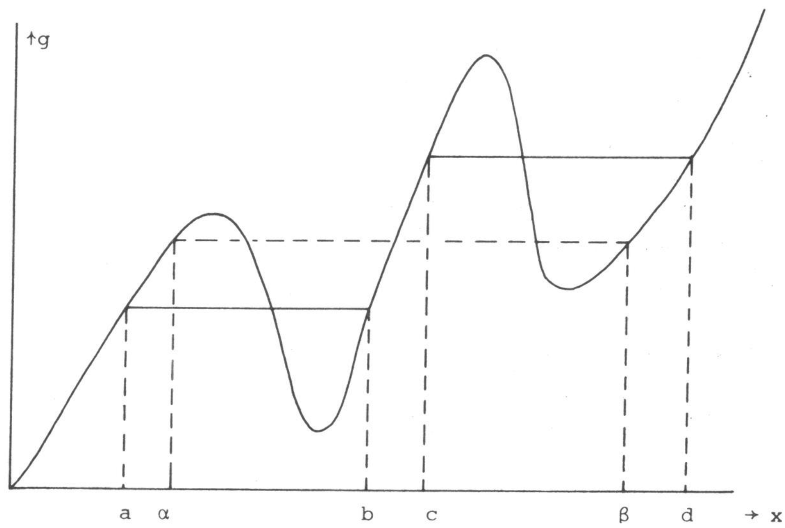

FIG. 2. *Two possible configurations: separate jumps* ($a - b$, $c - d$) *or a two-in-one jump* ($\alpha - \beta$).

In conclusion we remark that the hypothesis $g(R) \geqq k$ was made in order to obtain the uniform convergence on $[0, R]$. If $g(R) < k$ the solution will exhibit boundary layer behavior near the right endpoint. However, outside a small neighborhood of this endpoint, the solution will behave in exactly the same way as we have shown for the case $g(R) > k$.

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Note added in proof. It has been possible indeed to resolve the ambiguity connected with the limit $\varepsilon \to 0$ by means of a variational formulation of the problem (O. Diekmann and D. Hilhorst, *How many jumps? Variational characterization of the limit solution of a singular perturbation problem*, Proceedings of the Fourth Scheveningen Conference on Differential Equations, 1979, Springer, to appear).

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CHAPTER 3

A NONLINEAR EVOLUTION PROBLEM ARISING IN THE PHYSICS OF IONIZED GASES

ABSTRACT

We consider a Coulomb gas in a special experimental situation: the pre-breakdown gas discharge between two electrodes. The equation for the negative charge density can be formulated as a nonlinear parabolic equation degenerate at the origin. We prove the existence and uniqueness of the solution as well as the asymptotic stability of its unique steady state. Also some results are given about the rate of convergence.

KEY WORDS & PHRASES: *nonlinear parabolic equation degenerate at the origin in one space dimension; pre-breakdown discharge in an ionized gas between two electrodes*

1. INTRODUCTION

In this paper we study the nonlinear evolution problem,

$$P \begin{cases} u_t = \varepsilon x u_{xx} + (g(x)-u)u_x & \text{on } D = (0,\infty) \times (0,T) \\ u(0,t) = 0 & \text{for } t \in [0,T] \\ u(x,0) = \psi(x) & \text{for } x \in (0,\infty). \end{cases}$$

where ε is a positive constant, g is a given function which satisfies the hypothesis H_g: $g \in C^2([0,\infty))$; $g(0) = 0$; $g'(x) > 0$ and $g''(x) < 0$ for all $x \geq 0$ and the initial function ψ satisfies the hypothesis H_ψ:

(i) ψ is continuous, with piecewise continuous derivative on $[0,\infty)$;

(ii) $\psi(0) = 0$ and $\psi(\infty) = K \in (0,g(\infty))$;

(iii) there exists a constant $M_\psi \geq g'(0)$ such that $0 \leq \psi'(x) \leq M_\psi$ at all points x where ψ' is defined.

In section 2, we briefly describe how the problem arises in physics and give the derivation of the equations.

In section 3, we present maximum principles for certain linear and nonlinear problems related to P; the uniqueness of the solution of P follows directly from those principles.

In section 4, we prove that P has a classical solution which satisfies furthermore the condition

$$(*) \qquad u(\infty,t) = K \qquad \text{for } t \in [0,T],\ T < \infty.$$

The methods used here are inspired by those of VAN DUYN [7],[8] and GILDING & PELETIER [13]. We also consider the limit case $\varepsilon \downarrow 0$ and prove that u tends to the generalized solution of the corresponding hyperbolic problem.

We then investigate the behaviour of u as $t \to \infty$ and prove that it converges towards the unique solution $\emptyset$ of the problem P_0 defined as follows

$$P_0 \begin{cases} \varepsilon\, x\, \emptyset'' + (g(x)-\emptyset)\emptyset' = 0 \\ \emptyset(0) = 0 \qquad \emptyset(\infty) = \lambda_0 =: \min(\max(g(\infty)-\varepsilon,0),K) \end{cases}$$

Qualitative properties of $\emptyset$ have been extensively studied by DIEKMANN, HILHORST & PELETIER [6]. Here we analyse its stability. In section 5, following a method of ARONSON & WEINBERGER [2] based on the knowledge about lower and upper solutions for the steady state problem P_0, we prove that $\emptyset$ is asymptotically stable.

In section 6 we investigate the rate of convergence of u towards its steady state. The function $\emptyset$ turns out to be exponentially stable when the function g grows fast enough to infinity as $x \to \infty$; the proof, based on constructing upper and lower solutions for the function $u - \emptyset$, follows the same lines as that of FIFE & PELETIER [10]. We also consider the case when g increases less fast and show that provided that $\varepsilon < g(\infty) - K$ and that $\emptyset$ converges algebraically fast to K as $x \to \infty$, the function $u - \emptyset$ decays algebraically fast; this is done by obtaining first that property for a weighted integral of $u - \emptyset$ according to a method of IL'IN & OLEINIK [14] and VAN DUYN & PELETIER [9]. Finally we consider the corresponding hyperbolic problem and obtain a similar result of algebraic convergence.

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2. PHYSICAL DERIVATION OF THE EQUATIONS

The physical context of the present problem has been described in some detail by DIEKMANN, HILHORST & PELETIER [6]. Here we shall summarize it again and explain how one can obtain the time evolution problem P.

One considers an ionized gas between two electrodes in which the ions and electrons are present with densities $n_i(\vec{r})$ and $n_e(\vec{r},t)$ respectively, where $\vec{r} = (x_1,x_2,x_3)$. The ions are heavy and slow and the density $n_i(\vec{r})$ may therefore be regarded as fixed. The electrons are highly mobile. The problem is then to find $n_e(\vec{r},t)$ for given $n_i(\vec{r})$ and in particular to find out whether given an initial electron distribution the electrons stabilize and if so to evaluate the time needed for such a stabilization.

A special situation of practical interest is a so-called pre-breakdown discharge which spreads out in filamentary form (cf. MARODE [17] and MARODE, BASTIEN & BAKKER [18]). In this situation there is cylindrical symmetry about the x_3-axis and the particle densities depend on $r = (x_1^2 + x_2^2)^{\frac{1}{2}}$ only. We thus have effectively a two-dimensional Coulomb gas with circular symmetry. The starting equations are

(i) Coulomb's law for the electric field E,

$$\frac{1}{r}\frac{\partial}{\partial r} r\, E = -C_d(n_e - n_i) \tag{2.1}$$

where C_d is a fixed constant;

(ii) a constitutive equation for the electric current j,

$$j = n_e \mu E + kT\frac{\partial n_e}{\partial r}, \tag{2.2}$$

in which the first term represents Ohm's law and the second term is due to thermal diffusion, μ being the mobility, k Boltzmann's constant and T the temperature; and

(iii) the continuity equation for the electron density,

$$\frac{\partial n_e}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r} rj. \tag{2.3}$$

If we set

$$u(x,t) = \int_0^{\sqrt{x}} n_e(r,t)\, r\, dr$$

and

$$g(x) = \int_0^{\sqrt{x}} n_i(r)\, r\, dr,$$

we obtain after redefining the constants the equation

$$u_t = \varepsilon x u_{xx} + (g(x) - u)u_x \tag{2.4}$$

where $\varepsilon = 2kT/(\mu C_d)$ and the boundary condition

$$(2.5) \qquad u(0,t) = 0.$$

Furthermore one makes the hypothesis that the total charge is positive and fixed, that is

$$\int_0^\infty (n_i(r) - n_e(r,t))r\,dr = N > 0$$

from which we deduce the boundary condition at infinity,

$$(2.6) \qquad u(\infty,t) = K := g(\infty) - N.$$

Clearly $K \in (0,g(\infty))$.

Equations (2.4) and (2.5) together with the initial condition

$$(2.7) \qquad u(x,0) = \psi(x)$$

constitute the mathematical formulation of the problem which we propose to study in this paper. Furthermore the condition (2.6) will turn out te be satisfied at all finite times t and also, for low enough values of the small parameter ε, at the time $t = \infty$. This latter property expresses the fact that all the electrons stay attached to the ions at low enough temperature; we shall also see that if the temperature rises above a critical value, then some of the electrons escape to infinity and if it rises even further above a second critical value, then all the electrons escape to infinity.

3. MAXIMUM PRINCIPLES FOR SOME DEGENERATE PARABOLIC OPERATORS - UNIQUENESS THEOREM

In this section we prove maximum principles for some linear and non-linear operators which have a degeneracy at the origin; these principles hold for functions $u \in C^{2,1}(D) \cap C(\overline{D})$ where $C^{2,1}(D)$ is the set of continuous functions on D with two continuous x-derivatives and one continuous t-derivative. It will follow easily from those maximum principles that P can have at most one solution $u \in C^{2,1}(D) \cap C(\overline{D})$ such that u_x is bounded in $\overline{D}$.

We begin by defining a linear operator L as follows

$$(3.1) \qquad Lu = \varepsilon x u_{xx} + b(x,t)u_x + c(x,t)u - u_t$$

where the functions b and c are continuous on D and such that the quantities $b/(1+x)$ and c are bounded on $\overline{D}$. First we consider the bounded domain $D_R :=$ $(0,R) \times (0,T)$, where R is a positive constant. In the same way as for a uniformly parabolic operator one can prove the following maximum principle which holds in fact for a much wider class of degenerate parabolic operators (see for example IPPOLITO [15] or COSNER [4])

THEOREM 3.1. *Suppose* $c \le 0$. *Let* $u \in C^{2,1}(D_R) \cap C(\overline{D}_R)$ *satisfy* $Lu \ge 0$ *on* $(0,R) \times (0,T]$. *Then if* u *has a positive maximum in* $\overline{D}_R$, *that maximum is attained on* $((0,R) \times \{0\}) \cup (\{0,R\} \times [0,T])$.

Next following a method due to ARONSON & WEINBERGER [2] we derive a comparison theorem for a class of nonlinear evolution problems.

THEOREM 3.2. *Let* u *and* $v \in C^{2,1}(D_R) \cap C(\overline{D}_R)$ *and suppose that either* u_x *or* v_x *is bounded on* $\overline{D}_R$. *Let* u *and* v *satisfy*

$$Lv - vv_x \ge Lu - uu_x \qquad \text{on } (0,R) \times (0,T]$$

and let

$$0 \le v \le u \le K \qquad \text{on } (0,R) \times \{0\} \text{ and } \{0,R\} \times [0,T].$$

Then $v \le u$ *in* $(0,R) \times (0,T]$.

PROOF. Let

$$w = (v-u)e^{-\alpha t}$$

where

$$\alpha = \max_{(x,t) \in \bar{D}} (c(x,t) - u_x(x,t))$$

(in the case where u_x is bounded). Then w satisfies

$$\varepsilon x w_{xx} + (b(x,t) - v)w_x + (c(x,t) - u_x - \alpha)w - w_t \geq 0$$

and

$$w \leq 0 \text{ on } (0,R) \times \{0\} \text{ and } \{0,R\} \times [0,T].$$

Thus we deduce from Theorem 3.1 that

$$w \leq 0 \text{ in } (0,R) \times (0,T]$$

which completes the proof of theorem 3.2. □

Now let us consider the unbounded domain D. To begin with we present a Phragmèn-Lindelöf principle which is a special case of a theorem due to COSNER [4].

THEOREM 3.3. *Suppose that* $b/(1+x)$ *and* c *are continuous and bounded in* $\bar{D}$. *Let* $u \in C^{2,1}(D) \cap C(\bar{D})$ *satisfy* $Lu \geq 0$ *on* $(0,\infty) \times (0,T]$ *and the growth condition*

$$\liminf_{R \to \infty} e^{-BR} [\max_{0 \leq t \leq T} u(R,t)] \leq 0 \tag{3.2}$$

for some positive constant B. *If* $u \leq 0$ *for* $t = 0$ *and on* $\{0\} \times [0,T]$, *then* $u \leq 0$ *in* $(0,\infty) \times (0,T]$.

Making use of Theorem 3.3 one can prove a comparison theorem on the unbounded domain D.

THEOREM 3.4. *Let* u *and* $v \in C^{2,1}(D) \cap C(\bar{D})$ *be such that either* u_x *and* v *or* u *and* v_x *are bounded on* $\bar{D}$ *and that*

$$|u(x,t)|, \ |v(x,t)| \leq C\, e^{B_1 x}$$

for some positive constants C *and* B_1 *and uniformly in* $t \in [0,T]$. *Suppose that*

$$Lv - vv_x \geq Lu - uu_x \qquad on\ (0,\infty) \times (0,T]$$

and that

$$0 \leq v \leq u \leq K \qquad on\ (0,\infty) \times \{0\}\ and\ \{0\} \times [0,T].$$

Then $v \leq u$ *in* $(0,\infty) \times (0,T]$.

Finally let us come to the question of uniqueness of the solution of problem P.

DEFINITION. We shall say that u is a classical solution of Problem P if it is such that (i) $u \in C^{2,1}(D) \cap C(\overline{D})$, (ii) u and u_x are bounded in $\overline{D}$, (iii) u satisfies the equation in D, (iv) u satisfies the initial and boundary conditions.

THEOREM 3.5. *Problem* P *can have at most one solution.*

PROOF. Apply Theorem 3.4 twice to deduce that if u and v are two such solutions then their difference $w = u-v$ satisfies $w \geq 0$ and $w \leq 0$ and thus $w \equiv 0$. □

4. EXISTENCE AND REGULARITY OF THE SOLUTION

In order to be able to prove the existence of a solution of the nonlinear degenerate parabolic problem P, we consider certain related nonlinear uniformly parabolic problems on bounded domains and observe that they have a unique solution; we then deduce that P has a generalized solution, in a certain sense. It finally turns out that this solution is in fact a classical solution of P and thus the unique solution of P and that it also satisfies condition (*). Finally we consider its limiting behaviour as $\varepsilon \downarrow 0$.

4.1. Existence

Let us first introduce some notation. Let $D_n := (0,n) \times (0,T)$. We denote by $C_{2+\alpha}([0,n])$ the space of functions v which are twice differentiable and such that v" is Hölder continuous on $[0,n]$ with exponent α. We also use the spaces $\overline{C}_\alpha(D_n)$, $\overline{C_{2+\alpha}}(D_n)$ and $C_{2+\alpha}(D_n)$, defined in FRIEDMAN [11] p. 62 and 63.

Consider the problem

$$
P_n \quad \begin{cases} u_t = \varepsilon(x+1/n)u_{xx} + (g(x)-u)u_x & \text{in } D_n \\ u(0,t) = 0 \qquad u(n,t) = K & t \in [0,T] \\ u(x,0) = \psi_n(x) & x \in (0,n). \end{cases}
$$

with $n \geq g^{-1}(K)$ and where ψ_n is such that

(i) $\psi_n \in C^\infty([0,\infty])$,

(ii) ψ_n satisfies H_ψ,

(iii) $\psi_n''(0) = 0$ and $\psi_n(x) = K$ for $x \in [n-1,\infty)$.

In what follows we shall denote by H_n properties (i) - (iii). The following theorem holds:

<u>THEOREM 4.1</u>. *There exists a unique solution* $u_n \in \overline{C}_{2+\alpha}(D_n)$ *of* P_n *for any* $\alpha \in (0,1)$; *furthermore* u_n *satisfies the inequalities*

$$
(4.1) \qquad 0 \leq u_n(x,t) \leq \min(M_{\psi_n} x, K)
$$

$$
(4.2) \qquad 0 \leq u_{nx}(x,t) \leq M_{\psi_n}
$$

for all $(x,t) \in \overline{D}_n$.

<u>PROOF</u>. The existence and uniqueness of $u_n \in \overline{C_{2+\alpha}}(D_n)$ is a consequence of Theorem 5.2 of LADYŽENSKAJA ([16] p. 564-565). The inequalities in (4.1) can be deduced by means of a comparison theorem analogous to theorem 3.2. From the linear theory (FRIEDMAN [11] p. 72) we deduce that the function $w := u_{nx} \in C_{2+\alpha}(D_n)$; thus $w \in C^{2,1}(D_n) \cap C(\overline{D}_n)$. Furthermore w satisfies

$$
(4.3) \qquad \begin{cases} w_t = \varepsilon(x+1/n)w_{xx} + (g(x)-u_n+\varepsilon)w_x + (g'(x)-w)w \\ 0 \leq w(0,t) \leq M_{\psi_n} \qquad\qquad 0 \leq w(n,t) \leq M_{\psi_n} \\ w(x,0) = \psi'_n(x). \end{cases}
$$

The bounds on the function $w(n,t)$ follow from the fact that the function $\max(0, M_{\psi_n}(x-n)+K)$ is a lower solution of the boundary value problem

$$\varepsilon(x+1/n)\phi'' + (g(x)-\phi)\phi' = 0$$

$$\phi(0) = 0 \qquad\qquad \phi(n) = K$$

and consequently a lower bound for u_n. Clearly the set

$$\{w \in C([0,n]) \text{ such that } 0 \le w(x) \le M_{\psi_n}\}$$

is invariant with respect to the problem (4.3) and thus the inequalities (4.2) are satisfied.

Next we deduce from theorem 4.1 the existence of solution of P. We begin by approximating the initial function ψ by a sequence of smooth functions $\{\psi_n\}$.

LEMMA 4.2. *Let the function* ψ *satisfy* H_ψ. *Then there exists a sequence* $\{\psi_n\}$ *which satisfies the properties* H_n *given at the beginning of this section with* $M_{\psi_n} = M_\psi$ *for all* n, *such that* $\psi_n \to \psi$ *as* $n \to \infty$, *uniformly on* $[0,\infty)$.

PROOF. Let $n_0 \ge g^{-1}(K)$ be such that for all $n \ge n_0$ the point x_{1n} defined by $M_\psi(x_{1n}-1/n) = \psi(x_{1n})$ is such that $1/n < x_{1n} \le n-2$ and that the point x_{2n} defined by $x_{2n} = n - 2 + (K-\psi(n-2))/M_\psi$ satisfies $n-2 < x_{2n} < n-1$. Also define

$$\psi_n^*(x) = \begin{cases} 0 & -\infty < x \le 1/n \\ M_\psi(x-1/n) & 1/n < x \le x_{1n} \\ \psi(x) & x_{1n} < x \le n-2 \\ M_\psi(x-n+2) + \psi(n-2) & n-2 < x \le x_{2n} \\ K & x_{2n} < x < +\infty \end{cases}$$

Note that for all x

$$|\psi_n^*(x) - \psi(x)| \le \max(M_\psi/n,\ K - \psi(n-2)).$$

Next introduce the function

$$\rho(x) = \begin{cases} 0 & \text{if } |x| \geq 1 \\ C\exp(1/(|x|^2-1)) & \text{if } |x| < 1 \end{cases}$$

where the constant C is such that $\int_{\mathbb{R}} \rho dx = 1$, and let

$$\rho_\delta(x) = \rho(x/\delta)/\delta.$$

Finally define

$$\psi_n(x) = \int_{\mathbb{R}} \rho_{\delta_n}(x-y)\psi_n^*(y)dy \qquad x \in [0,n]$$

with $\delta_n = \min(1/n, x_{1n}-1/n, n-2-x_{1n}, x_{2n}-n+2, n-1-x_{2n})/10$. We now show that ψ_n has the desired properties. Firstly $\psi_n \in C^\infty([0,n])$. The uniform convergence of $\{\psi_n\}$ to ψ follows from the continuity of ψ_n^*, uniformly in n and in x and the uniform convergence of ψ_n^* to ψ as $n \to \infty$. Finally properties (ii) and (iii) of H_n can be deduced for ψ_n from the fact that ψ also satisfies them. □

Next we prove the following theorem

THEOREM 4.3. *P has a unique classical solution. Furthermore this solution also satisfies condition* (*):

$$\lim_{x \to \infty} u(x,t) = K \quad \textit{for each } t \in (0,T].$$

PROOF. We rewrite the parabolic equation of Problem P_n as

$$u_t = \varepsilon(x+1/n)u_{xx} + c(x,t)u_x, \tag{4.4}$$

where

$$c(x,t) = g(x) - u_n(x,t).$$

From Theorem 4.1 we know that for all $(x',t),(x'',t) \in \overline{D}_n$ and for all $n \geq n_0$

$$|u_n(x',t) - u_n(x'',t)| \leq M_\psi|x'-x''|. \tag{4.5}$$

Now fix $I \geq n_0$; (4.4) and (4.5) enable us to apply a theorem of GILDING [12] about the Hölder continuity of solutions of parabolic equations and we obtain

$$|u_n(x,t') - u_n(x,t")| \leq C\ |t'-t"|^{\frac{1}{2}}$$

for all $n \geq I$ and for all $(x,t'),(x,t") \in \overline{D}_I$, with $|t'-t"| \leq 1$. Here the constant C depends on I but not on n. The set $\{u_n(x,t)\}_{n=I}^{\infty}$ is bounded and equicontinuous in D_I and thus there exists a continuous function $u_I(x,t)$ and a convergent subsequence $\{u_{n_k}(x,t)\}$ with $n_k \geq I$ such that $u_{n_k}(x,t) \to u_I(x,t)$ as $n_k \to \infty$, uniformly on $\overline{D}_I$. Then, by a diagonal process, it follows that there exists a function $u(x,t)$ defined on $\overline{D}$ and a convergent subsequence, denoted by $\{u_j(x,t)\}$ such that $u_j(x,t) \to u(x,t)$ as $j \to \infty$, pointwise on $\overline{D}$. Since this convergence is uniform on any bounded subset of $\overline{D}$, the limit function u is continuous on $\overline{D}$.

It remains to show that u is a solution of P; to that purpose we shall proceed in two steps: firstly we show that u is a generalized solution of P in a certain sense and then we conclude that it is in fact a classical solution. We shall say that u is a generalized solution of P if it has the following properties:

(i) u is continuous and uniformly bounded in $\overline{D}$;

(ii) $u(0,t) = 0$ for all $t \in [0,T]$;

(iii) u has a bounded generalized derivative with respect to x in D;

(iv) u satisfies the identity

$$\iint_D [u\phi_t - \varepsilon(xu_x-u)\phi_x - (g-u/2)u\phi_x - ug'\phi]dxdt + \int_0^\infty \psi(x)\phi(x,0)dx = 0 \tag{4.6}$$

for all $\phi \in C^1(\overline{D})$ which vanish for $x = 0$, large x and $t = T$.

Let us check that u satisfies those properties.

(i) We already know that u is continuous on $\overline{D}$ and furthermore, since $u(x,t) = \lim_{j\to\infty} u_j(x,t)$, we have that $0 \leq u \leq K$.

(ii) This property follows from a similar boundary condition in P_n.

(iii) Let ϕ be an admissible test function and let $L \geq n_0$ be such that supp $\phi \subset D_L$. Since $|u_{jx}|$ is uniformly bounded with respect to $j \geq L$ for all $(x,t) \in D_L$, it follows that there exists a subsequence $\{(u_{j_k})_x\}$ and a bounded function $p \in L^2(D_L)$ such that

$$(u_{j_k})_x \rightharpoonup p \quad \text{in } L^2(D_L) \quad \text{as } j_k \to \infty .$$

Now let $\zeta \in C_0^1(\overline{D}_L)$. Then

$$((u_{j_k})_x, \zeta) \to (p,\zeta) \quad \text{as } j_k \to \infty \tag{4.7}$$

where $(.,.)$ denotes the inner product in $L^2(D_L)$. But since $u_{j_k} \to u$ as $j_k \to \infty$, uniformly on $\overline{D}_L$, we have

$$(u_{j_k}, \zeta_x) \to (u,\zeta_x) \quad \text{as } j_k \to \infty. \tag{4.8}$$

Hence combining (4.7) and (4.8) we find that p is the generalized derivative of u.

(iv) Since u_{j_k} is a classical solution of P_n it follows that

$$\iint_{D_L} [u_{j_k}\phi_t - \varepsilon((x+1/j_k)(u_{j_k})_x - u_{j_k})\phi_x - (g-u_{j_k}/2)u_{j_k}\phi_x - u_{j_k}g'\phi]dxdt + \int_0^L \psi_{j_k}(x)\phi(x,0)\,dx = 0 \tag{4.9}$$

The sequences $\{u_{j_k}\}$ and $\{u_{j_k}{}^2\}$ converge to u and u^2, respectively, strongly in $L^2(D_L)$ as $j_k \to \infty$. Furthermore since $(u_{j_k})_x$ is uniformly bounded we have

$$\iint_{D_L} \frac{1}{j_k} (u_{j_k})_x\, \phi_x\, dxdt \to 0 \quad \text{as } j_k \to \infty.$$

Thus letting $j_k \to \infty$ we obtain (4.6). Because ϕ has been chosen arbitrarily, we may conclude that u is indeed a generalized solution of P.

It remains to show that u is a classical solution of P. One can do it by using a classical bootstrap argument (see for example GILDING & PELETIER [13]) to show that for whatever $\eta, L > 0$ there exists $\alpha(\eta,L) \in (0,1)$ such that

$$u \in \overline{C_{2+\alpha}}\,((\eta,L) \times (\eta,T)) \tag{4.10}$$

where α and $\|u\|_{\overline{C_{2+\alpha}}}$ may be estimated independently of T. In particular

$$u \in C^{2,1}(D) \cap C(\overline{D}).$$

Since furthermore u and u_x are uniformly bounded, u is a classical solution of Problem P and by theorem 3.5 it is the unique solution of P.

Finally let us analyze the behaviour of u for large x; since we have $0 \le u \le K$ and $u_x \ge 0$, $u(\infty,t) = \lim_{x\to\infty} u(x,t)$ is well defined for all $t \in [0,T]$ and such that $0 \le u(\infty,t) \le K$. Next we show that $u(\infty,t) \equiv K$ by constructing a time dependent lower solution for P. Consider the problem

$$(4.11) \qquad \begin{cases} u_t = \varepsilon x u_{xx} + (K-u)u_x \\ u(x_0,t) = 0 & x_0 \ge g^{-1}(K) \\ u(x,0) = \psi(x) \end{cases}$$

Since $u_x \ge 0$ we have that

$$\varepsilon x u_{xx} + (g(x)-u)u_x - u_t = \varepsilon x u_{xx} + (K-u)u_x - u_t + (g(x)-K)u_x$$

$$\ge \varepsilon x u_{xx} + (K-u)u_x - u_t \quad \text{for all } x \ge g^{-1}(K).$$

Thus a lower solution $\hat{u}$ of (4.11) with $\hat{u}_x \ge 0$ is also a lower solution of P on $[x_0,\infty) \times [0,T]$. We search such functions $\hat{u}_k$ which satisfy furthermore

$$\hat{u}_k(\infty,t) = K - k \quad \text{for all } t \in [0,T] \text{ and with } k \in (0,K).$$

Writing

$$\hat{v} = K - \hat{u}$$

this comes down to finding an upper solution $\hat{v}_k$ of

$$\begin{cases} v_t = \varepsilon x v_{xx} + v v_x \\ v(x_0,t) = K & v(\infty,t) = 0 \end{cases}$$

Next we look for such a function $\hat{v}_k$, also requiring that

$$\hat{v}_k(x,t) = \hat{f}_k(x/(t+1)).$$

Setting

$$\eta = x/(t+1)$$

one can easily derive that $\hat{f}_k$ should be an upper solution for the boundary value problem

$$\pi \begin{cases} \varepsilon\eta f'' + (f+\eta)f' = 0 \\ f(x_0) = K \quad f(\infty) = 0. \end{cases}$$

Let $x_0 > \max(\varepsilon, g^{-1}(K))$ and take

$$\hat{f}_k(\eta) = k + (K-k)(\eta/x_0)^{1-x_0/\varepsilon}.$$

One can check that indeed $\hat{f}_k$ is an upper solution for problem π and consequently that $\hat{u}_k(x,t) = K - \hat{f}_k(x/(t+1))$ is a lower solution for Problem P on the sector $\{t \geq 0,\ x \geq x_0(t+1)\}$ provided that x_0 is large enough. Since k can be chosen arbitrarily in (0,K) it follows that $u(\infty,t) = K$ for all $t < \infty$. □

4.2. The limiting behaviour as $\varepsilon \downarrow 0$.

In this section we study the limiting behaviour of the solution u of P as $\varepsilon \downarrow 0$. To begin with we consider the following hyperbolic problem

$$H \begin{cases} u_t = (g(x)-u)u_x & \text{in } D \\ u(x,0) = \psi(x) & \text{for all } x \in (0,\infty) \end{cases}$$

and make some heuristic considerations about the solution $\bar{u}$ of Problem H; they are due to WILDERS [23]. One possible configuration of g and ψ is drawn in Figure 1; the corresponding characteristics are represented in Figure 2.

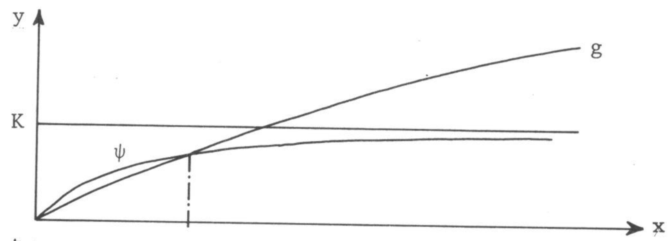

Fig. 1

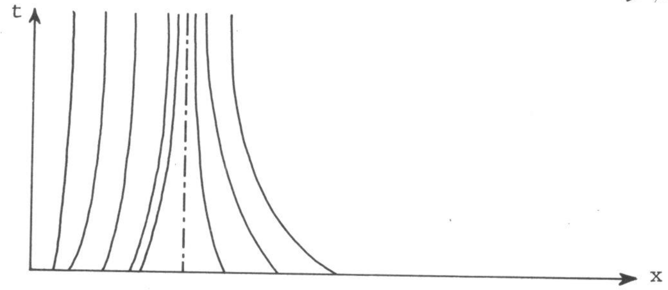

Fig. 2

Their equations are

$$\frac{dx}{dt} = -(g(x)-\psi(x(0))).$$

Along those characteristics $\bar{u}$ is constant, i.e. $\bar{u} = \psi(x(0))$. Also since $\psi(0) = 0$ it follows that the line $x = 0$ is the characteristic passing through the point $(0,0)$ and consequently that $\bar{u}$ automatically satisfies a boundary condition of the form $\bar{u}(0,t) = 0$. Next we deduce from the fact that ψ is nondecreasing that two characteristics do not intersect. Suppose that there exist two characteristics, issuing from the points $x = a$ and $x = b$ $(a < b)$ on the initial line, intersecting each other at the point $(x,t) = (x^*,t^*)$. Then if they would intersect transversally, we would have $-(g(x^*)-\psi(a)) > -(g(x^*)-\psi(b))$ and hence $\psi(a) > \psi(b)$ which is impossible. Now if the characteristics would be tangent to each other at the point (x^*,t^*) we would have $-(g(x^*)-\psi(a)) = -(g(x^*)-\psi(b))$ and consequently $\psi(a) = \psi(b)$; both characteristics would then be described by the same differential equation $\frac{dx}{dt} = -(g(x)-\psi(a))$ which, by the standard uniqueness theorem for ordinary differential equations, implies $a = b$. Finally we conclude that since the initial condition ψ is continuous and nondecreasing, no shock wave can occur and $\bar{u}(\cdot,t)$ is continuous at all times.

In [19] OLEINIK proved existence and uniqueness of the generalized solution of Cauchy problems and boundary value problems related to Problem H; but since the boundary line $x = 0$ is a characteristic for H (which is reflected in the relation $g(0)-\bar{u}(0,0) = 0$), Problem H does not satisfy all the assumptions made in [19]. This leads us to give here a proof of the existence of a solution of Problem H, by showing that the solution u of Problem P tends to a limit as $\varepsilon \downarrow 0$; the uniqueness is a consequence of [19]. Following lemmas 18 and 19 from [19] we say that $\bar{u}$ is a generalized solution of H if it satisfies

(i) $\bar{u}$ is bounded and measurable in $\bar{D}$.

(ii) $\dfrac{\bar{u}(x_1,t) - \bar{u}(x_2,t)}{x_1-x_2} \le M_\psi$ for all points $(x_1,t),(x_2,t) \in \bar{D}$

(iii) $\bar{u}$ satisfies the identity

$$\iint_D [\bar{u}\,\phi_t - (g-\bar{u}/2)\bar{u}\phi_x - \bar{u}g'\phi]dxdt + \int_0^\infty \psi(x)\phi(x,0)dx = 0 \tag{4.12}$$

for all $\phi \in C^1(\bar{D})$ which vanish for large x and $t = T$.

Next we shall prove the theorem

<u>THEOREM 4.4</u>. *The solution* $u(x,t)$ *of* P *tends uniformly on all compact subdomains of* D *to a limit* $\bar{u}$ *as* $\varepsilon \downarrow 0$, *where* $\bar{u}$ *is the unique generalized solution of* H. *The function* $\bar{u}$ *is furthermore continuous, nondecreasing in* x *at all times* $t \in [0,T]$ *and satisfies the boundary conditions* $\bar{u}(0,t) = 0$ *and* $\bar{u}(\infty,t) = K$.

Before proving theorem 4.4 let us introduce a class of upper and lower solutions for Problem P which depend neither on ε nor on time. They will turn out to be very useful both to prove that $\bar{u}(\infty,t) = K$ in theorem 4.4 and to study the asymptotic behaviour of u as $t \to \infty$ in the next sections. Next we define

$$s^+(x) := \min\,(M_\psi x, K)$$

and

$$s^-(x,\lambda,x_1,\nu) := \max(0,\lambda(1-(x/x_1)^{-\nu}))$$

where the constants $\lambda \in [0,K]$, $\nu > 0$ and $x_1 > 0$ are chosen in the following manner:

(a) if $\varepsilon < g(\infty)$, we choose $x_1 > 0$ so that

$$g(x_1) > \varepsilon,$$

then $\lambda > 0$ so that

$$\lambda < g(x_1) - \varepsilon$$

and finally $\nu > 0$ so that

$$\nu \le \varepsilon^{-1}(g(x_1) - \lambda) - 1 \tag{4.13}$$

(b) if $\varepsilon \ge g(\infty)$, we set $\lambda = 0$, which amounts to setting $s^- \equiv 0$. It is easily seen that s^- satisfies the inequality

$$\hat{\varepsilon}x(s^-)'' + (g-s^-)(s^-)' \ge 0 \quad \text{for all } x \in [0,\infty)\backslash\{x_1\},\ \hat{\varepsilon} \in (0,\varepsilon).$$

Thus if $\varepsilon < g(\infty)$, given any $\hat{\lambda} < \lambda_0 = \min(g(\infty) - \varepsilon, K)$, one can find $\hat{x}_1$ and $\hat{\nu}$ satisfying (4.13) and such that $s^-(.,\hat{\lambda},\hat{x}_1,\hat{\nu}) \le \psi$. Applying the comparison theorem 3.4 we deduce that $s^-(.,\hat{\lambda},\hat{x}_1,\hat{\nu}) \le u$ (and thus that $\lambda_0 \le u(\infty,t)$ for all $t \le \infty$). Similarly one can check that $u \le s^+$.

<u>PROOF of Theorem 4.4</u>. The uniqueness of the solution of Problem H can be proven along the same lines as in the proof of Theorem 1 and Lemma 21 of [19]. Next we show its existence. Fix $I \ge 1$. Since u and u_x are bounded uniformly in ε we deduce from GILDING [12] that u is equicontinuous on $\overline{D}_I$; thus there exists a subsequence $\{u_{\varepsilon_n}\}_{n=I}^{\infty}$ of u and a function $\overline{u}_I \in C(\overline{D}_I)$, such that $u_{\varepsilon_n} \to \overline{u}_I$ as $\varepsilon_n \downarrow 0$ uniformly in $\overline{D}_I$ and such that for all $\lambda < K$, one can find x_1 and ν satisfying (4.13) and $s^-(.,\lambda,x_1,\nu) \le \overline{u}_I(.,t) \le s^+(.)$. Then by a diagonal process, it follows that there exists a bounded continuous function $\overline{u}$ and a converging subsequence denoted by $\{u_{\varepsilon_k}\}$ such that $u_{\varepsilon_k} \to \overline{u}$ as $\varepsilon_k \downarrow 0$, pointwise on D and uniformly on all compact subsets of D. Since $0 \le (u_{\varepsilon_k})_x \le M_\psi$, $\overline{u}$ is nondecreasing in the x-direction and satisfies

(ii); $u_{\varepsilon_k}(0) = 0$ implies the same property for $\overline{u}$. The boundary condition $\overline{u}(\infty,t) = K$ follows from the inequalities $s^-(.,\lambda,x_1,\nu) \le \overline{u}(.,t) \le s^+(.)$ for all $\lambda < K$.

It remains to show that $\overline{u}$ is a generalized solution of H. Let $\phi \in C^1(\overline{D})$ vanish for large x and t = T and let $L \ge 1$ be such that ϕ vanishes in the neighbourhood of x = L and for x > L. Because the functions u_{ε_k} are classical solutions of P, we have

$$\iint\limits_{D_L} [u_{\varepsilon_k}\phi_t - \varepsilon_k(xu_{\varepsilon_k x} - u_{\varepsilon_k})\phi_x - (g - u_{\varepsilon_k}/2)u_{\varepsilon_k}\phi_x - u_{\varepsilon_k}g'\phi]dxdt$$

$$+ \int_0^L \psi(x)\phi(x,0)dx = 0$$

Now letting $\varepsilon_k \downarrow 0$, we deduce that $\overline{u}$ satisfies (4.12); because ϕ has been chosen arbitrarily we conclude that $\overline{u}$ is indeed the generalized solution of H and that $\{u_\varepsilon\}$ converges to $\overline{u}$ as $\varepsilon \downarrow 0$. □

5. ASYMPTOTIC STABILITY OF THE STEADY STATE

Adapting a method due to ARONSON & WEINBERGER [2] we investigate the stability of the solution $\emptyset$ of Problem P_0. To that purpose we consider the solution u of the corresponding evolution problem P; since its dependence on ψ plays a central rôle in what follows, we denote this solution by $u(x,t,\psi)$. We show that for all the functions ψ satisfying the hypothesis H_ψ given in the introduction we have that

$$u(x,t,\psi) \to \emptyset(x) \qquad \text{as } t \to \infty.$$

To begin with we prove two auxiliary lemmas.

<u>LEMMA 5.1.</u>

(i) *Let* $\varepsilon < g(\infty)$ *and* $\hat{\lambda},\hat{x}_1,\hat{\nu}$ *satisfy* (4.13). *The function* $u(x,t,s^-(.,\hat{\lambda},\hat{x}_1,\hat{\nu}))$ *is nondecreasing in time and such that*

(5.1) $$\lim_{t\to\infty} u(x,t,s^-(.,\hat{\lambda},\hat{x}_1,\hat{\nu})) = \phi_{\hat{\lambda}}(x)$$

where $\phi_{\hat\lambda}$ *is the unique solution of*

$$(5.2)\qquad \begin{cases} \varepsilon x\phi'' + (g(x) - \phi)\phi' = 0 \\ \phi(0) = 0 \qquad \phi(\infty) = \hat\lambda \end{cases}$$

(ii) *The function* $u(x,t,s^+)$ *is nonincreasing in time. Furthermore*

$$(5.3)\qquad \lim_{t\to\infty} u(x,t,s^+) = \emptyset.$$

PROOF. First note that it follows from the proofs in section 4 that Problem P with initial value $s^-(x,\hat\lambda,\hat x_1,\hat\nu)$ has a unique classical solution $u(x,t,s^-(.,\hat\lambda,\hat x_1,\hat\nu))$ with $u(\infty,t) = \hat\lambda$ for all $t \le \infty$. Applying repeatedly theorem 3.4 one can show that $u(x,t,s^-(.,\hat\lambda,\hat x_1,\hat\nu))$ is nondecreasing in time and that $u(x,t,s^+)$ is nonincreasing in time; it also follows from theorem 3.4 that

$$u(x,t,s^-(.,\hat\lambda,\hat x_1,\hat\nu)) \le \phi_{\hat\lambda}(x)$$

and that

$$u(x,t,s^+) \ge \emptyset(x).$$

Now for each x, $u(x,t,s^-(.,\hat\lambda,\hat x_1,\hat\nu))$ is nondecreasing in t and bounded from above. Therefore it has a limit $\tau^-(x)$ as $t \to \infty$ and one can use standard arguments (see for example ARONSON & WEINBERGER [2]) to show that $\tau^- \in C_{2+\alpha}((0,\infty)) \cap C([0,\infty))$ and satisfies the differential equation in (5.2) and the boundary conditions $\tau^-(0) = 0$ and $\tau^-(\infty) = \hat\lambda$. Finally since $\phi_{\hat\lambda}$ is the unique solution of Problem (5.2) we have that $\tau^- = \phi_{\hat\lambda}$. Similarly one can show that $u(x,t,s^+)$ converges to a function $\tau^+ \in C_{2+\alpha}((0,\infty)) \cap C([0,\infty))$ which satisfies the steady state equation, the boundary condition $\tau^+(0) = 0$ and the condition $\emptyset(\infty) \le \tau^+(\infty) \le K$. The fact that $\tau^+(\infty) = \emptyset(\infty)$ follows from [6, Lemma 5.1]. Consequently $\tau^+ = \emptyset$.

LEMMA 5.2. $\phi_{\hat\lambda}$ *is an increasing and continuous function of* $\hat\lambda$. *More precisely if* $\hat\lambda_1 \ge \hat\lambda_2$ *we have*

$$0 \le \phi_{\hat\lambda_1} - \phi_{\hat\lambda_2} \le \hat\lambda_1 - \hat\lambda_2.$$

PROOF. *Let* $m = \phi_{\hat\lambda_1} - \phi_{\hat\lambda_2}$. It satisfies the differential equation

$$\varepsilon x m'' + (g-\phi_{\hat\lambda_1})m' - \phi'_{\hat\lambda_2} m = 0$$

and the boundary conditions $m(0) = 0$ and $m(\infty) = \hat\lambda_1 - \hat\lambda_2 \geq 0$. Suppose that m attains a negative minimum at a certain point $\xi \in (0,\infty)$; then $m(\xi) < 0$, $m'(\xi) = 0$ and $m''(\xi) \geq 0$ which is in contradiction with $\varepsilon\xi m''(\xi) = \phi'_{\hat\lambda_2}(\xi)m(\xi)$. Thus $m \geq 0$. In the same way one can show that m cannot attain a positive maximum which implies $m \leq \hat\lambda_1 - \hat\lambda_2$.

Finally we are in a position to prove the following theorem

THEOREM 5.3. *Let* $\emptyset(x)$ *be the solution of* Problem P_0. *Suppose* ψ *satisfies the hypothesis* H_ψ, *then for each* $x \geq 0$

$$\lim_{t\to\infty} u(x,t,\psi) = \emptyset(x).$$

If $\varepsilon \leq g(\infty) - K$, *the convergence is uniform on* $[0,\infty)$; *if* $\varepsilon > g(\infty) - K$, *it is uniform on all compact intervals of* $[0,\infty)$.

PROOF. Since the functions u and u_x are bounded uniformly in t, we apply the Arzela-Ascoli theorem and a diagonal process to deduce that there exists a function $\tau \in C([0,\infty))$ and a sequence $\{u(t_n)\}$ with $u(t_n) = u(.,t_n,\psi)$ such that $u(t_n) \to \tau$ as $t_n \to \infty$, uniformly on all compact subsets of $[0,\infty)$. Let $\varepsilon < g(\infty)$; then for each $\hat\lambda < \lambda_0 = \min(g(\infty)-\varepsilon,K)$ one can find $\hat\nu$ and $\hat x_1$ satisfying (4.13) and such that $s^-(.,\hat\lambda,\hat x_1,\hat\nu) \leq \psi$. Applying Theorem 3.4 we obtain

$$u(x,t,s^-(.,\hat\lambda,\hat x_1,\hat\nu)) \leq u(x,t,\psi) \leq u(x,t,s^+). \tag{5.4}$$

Letting $t \to \infty$ in (5.4) and applying lemma 5.1 we obtain

$$\phi_{\hat\lambda} \leq \tau \leq \emptyset \qquad \text{for all } \hat\lambda < \lambda_0.$$

Next we deduce from lemma 5.2 that

$$\emptyset - \tau \leq \lambda_0 - \hat\lambda \qquad \text{for all } \hat\lambda < \lambda_0$$

and thus that $\tau = \emptyset$. If $\varepsilon \geq g(\infty)$ then the inequalities

$$0 \leq u(x,t,\psi) \leq u(x,t,s^+)$$

imply

$$0 \leq \tau \leq \emptyset = 0.$$

Thus also in this case we have that $\tau = \emptyset$. Finally we conclude that as $t \to \infty$, $u(.,t,\psi)$ converges to $\emptyset$, uniformly on all compact intervals of $[0,\infty)$. This convergence result can be made slightly stronger in the case that $\varepsilon \leq g(\infty) - K$: since then $\emptyset(\infty) = K$ and since u is nondecreasing in x one can apply Lemma 2.4 of DIEKMANN [5] to deduce that the convergence is uniform on $[0,\infty)$. □

6. RATE OF CONVERGENCE OF THE SOLUTION TOWARDS THE STEADY STATE

In this section we analyse the rate of convergence of the solution u of P towards its steady state $\emptyset$. The results which we are able to derive depend strongly on the behaviour of g as $x \to \infty$. If g tends to infinity fast enough, we can prove exponential convergence with a certain weighted norm. In the more general case when $\varepsilon < g(\infty) - K$ we find that the solution converges algebraically fast towards its steady state on all finite x-intervals. No results are available in the case $\varepsilon \geq g(\infty) - K$, which coincides with the physical situation when some (or all the) electrons escape to infinity.

We write

$$u(x,t,\psi) = \emptyset(x) + v(x,t).$$

Then v satisfies the problem

$$(6.1) \qquad \begin{cases} v_t = \varepsilon x v_{xx} + (g-\emptyset)v_x - \emptyset' v - v v_x \\ v(0,t) = 0 \\ v(x,0) = \psi(x) - \emptyset(x). \end{cases}$$

Now let us make the change of function

$$v(x,t) = \exp\left(-\int_0^x \frac{g(\zeta)-\phi(\zeta)}{2\varepsilon\zeta}\,d\zeta\right)\tilde{v}(x,t).$$

Problem (6.1) becomes

$$(6.2)\qquad \begin{cases} \tilde{v}_t = \varepsilon x\tilde{v}_{xx} - q(x)\tilde{v} + h(x,\tilde{v},\tilde{v}_x) \\ \tilde{v}(0,t) = 0 \\ \tilde{v}(x,0) = \exp\left(\int_0^x \frac{g(\zeta)-\phi(\zeta)}{2\varepsilon\zeta}\,d\zeta\right)(\psi(x)-\phi(x)) \end{cases}$$

where

$$q(x) = \frac{(g(x)-\phi(x))^2}{4\varepsilon x} + \frac{g'(x)+\phi'(x)}{2} - \frac{g(x)-\phi(x)}{2x}$$

and

$$h(x,\tilde{v},\tilde{v}_x) = -\exp\left(-\int_0^x \frac{g(\zeta)-\phi(\zeta)}{2\varepsilon\zeta}\,d\zeta\right)\tilde{v}\left(\tilde{v}_x - \frac{g(x)-\phi(x)}{2\varepsilon x}\,\tilde{v}\right).$$

In particular, there exists $M > 0$ such that

$$|h(x,\tilde{v},\tilde{v}_x)| \le M(\|\tilde{v}\|^2 + \|\tilde{v}_x\|^2) \qquad 0 < x < \infty.$$

where the notation $\|\cdot\|$ indicates the sup-norm.

In what follows we shall distinguish two cases: (i) the case when $\liminf_{x\to\infty} q(x) = \delta > 0$: this is so if $g(x) \ge C_0\sqrt{x}$ for all $x \ge x_2$ for some positive constants C_0 and x_2; (ii) the case when $\liminf_{x\to\infty} q(x) = 0$.

6.1. Case when g tends to infinity at least at fast as $\sqrt{x}$ for $x \to \infty$

The theorem we give next is very similar in its form and in its proof to a theorem of FIFE & PELETIER [10].

THEOREM 6.1. *Suppose that there exist constants $x_2, C_0 > 0$ such that*

$$(6.3)\qquad g(x) \ge C_0\sqrt{x} \qquad \textit{for all } x \ge x_2,$$

then there exist positive constants δ,μ,C *such that if*

$$\left\| \exp\left(\int_0^{\cdot} \frac{g(\zeta) - \phi(\zeta)}{2\varepsilon\zeta}\, d\zeta \right) (\psi - \phi) \right\| \le \delta$$

then

$$\left\| \exp\left(\int_0^{\cdot} \frac{g(\zeta) - \phi(\zeta)}{2\varepsilon\zeta}\, d\zeta \right) (u(.,t,\psi) - \phi) \right\| \le Ce^{-\mu t} \qquad t \ge 0$$

where the notation $\|.\|$ *indicates the sup-norm.*

PROOF. To begin with we note that with the hypothesis of Theorem 6.1 we have that $v(\infty,t) = 0$ (since $\varepsilon < g(\infty)-K$) or equivalently

$$\lim_{x\to\infty} \exp\left(- \int_0^{x} \frac{g(\zeta) - \phi(\zeta)}{2\varepsilon\zeta}\, d\zeta \right) \tilde{v}(x,t) = 0.$$

Next let us consider the boundary value problem

$$(6.4) \qquad \begin{aligned} &\varepsilon x w'' - (q(x)+\lambda)w = -\theta(\phi'(R)+\lambda)\min(\tilde{\phi}(x), (x/R)^{-\nu_0}\, \tilde{\phi}(R)) \\ &w(0) = 0 \end{aligned}$$

where

$$\tilde{\phi}(x) = \exp\left(\int_0^{x} \frac{g(\zeta) - \phi(\zeta)}{2\varepsilon\zeta}\, d\zeta \right) \phi(x).$$

The right hand side of the differential equation in (6.4) has been chosen in a special manner so that one can exhibit upper and lower solutions for a problem closely related to (6.4); more precisely we shall prove in the appendix that this problem has at least one solution $w \in C^2([0,\infty))$ with w,w' and w'' bounded such that

$$0 < w(x) \le \min(\tilde{\phi}(x), (x/R)^{-\nu_0}\, \tilde{\phi}(R))$$

for all constants $\nu_0 > 1$ provided that the constants $\theta \in (0,1)$, $R > 0$ and $\lambda < 0$ satisfy certain conditions. We adjust θ such that $\|w\| + \|w'\| \le 1$.

We are now in a position to prove theorem 6.1. Let

$$z(x,t) = \beta(w(x) + \gamma)e^{-\mu t},$$

in which β, γ and μ are positive constants still to be determined, and let

$$Mz = \varepsilon x z_{xx} - q(x)z + h(x,z,z_x) - z_t.$$

(i) The function q is positive for x near zero and, because of condition (6.3), also for large x; thus there exists $\overline{q}_0 > 0$ and $\zeta_1, \zeta_2 \in (0,\infty)$ such that $\overline{q}_0 = \min\{q(x) : x \in [0,\zeta_1] \cup [\zeta_2,\infty)\}$ is positive; therefore

$$Mz \le \beta e^{-\mu t}((\lambda+\mu)w + \gamma(-\overline{q}_0+\mu) + M\beta(1+\gamma)^2)$$

Choose

$$0 < \mu < \min(-\lambda,\overline{q}_0),$$

assume that γ is known (we shall specify it later) and choose

$$\beta = \frac{\gamma(\overline{q}_0-\mu)}{M(1+\gamma)^2}.$$

Then $Mz \le 0$ for all $x \in [0,\zeta_1] \cup [\zeta_2,\infty)$ and $t \ge 0$.

(ii) Let $\zeta_1 \le x \le \zeta_2$; since $w(x) > 0$ on $(0,\infty)$ and since w is continuous we have

$$m = \min\{w(x) : \zeta_1 \le x \le \zeta_2\} > 0.$$

Therefore

$$Mz \le \beta e^{-\mu t}((\lambda+\mu)m + \gamma(-\overline{q}+\mu) + M\beta(1+\gamma)^2)$$

where $\overline{q}$ is an arbitrary constant such that

$$\overline{q} < \min\{q(x) : x \in [0,\infty)\}.$$

Hence

$$Mz \le \beta e^{-\mu t}((\lambda+\mu)m + \gamma(-\overline{q}+\overline{q}_0)).$$

Therefore if we choose

$$\gamma = -\frac{\lambda+\mu}{-\overline{q}+\overline{q}_0}\, m$$

we have

$$Mz \le 0 \qquad \text{for } \zeta_1 \le x \le \zeta_2 \text{ and } t \ge 0.$$

Thus for the above choice of β, γ and μ the function z is an upper solution of the equation $M\tilde{v} = 0$. Let

$$\sup_{[0,\infty)} \tilde{v}(x,0) \leq \delta$$

where $\delta = \beta\gamma$. Then

$$\tilde{v}(x,0) \leq z(x,0) \qquad \text{for all } x \in [0,\infty)$$

and hence by theorem 3.4

$$\tilde{v}(x,t) \leq z(x,t) \qquad \text{for all } x \in [0,\infty),\ t \geq 0.$$

In a similar manner one can show that if

$$\inf_{[0,\infty)} \tilde{v}(x,0) \geq -\delta$$

then

$$\tilde{v}(x,t) \geq -z(x,t) \qquad \text{for all } x \in [0,\infty),\ t \geq 0.$$

Hence if

$$\|\tilde{v}(.,0)\| \leq \delta$$

then

$$\|\tilde{v}(.,t)\| \leq Ce^{-\mu t}$$

where we define

$$C = \beta(1+\gamma) = (1+1/\gamma)\delta. \quad \square$$

6.2. Algebraic decay rate in the case that $\varepsilon < g(\infty) - K$

Provided that $\varepsilon < g(\infty) - K$ and that the initial function ψ converges algebraically fast to K as $x \to \infty$, we prove that the solution u of P converges algebraically fast to the steady state solution $\emptyset$ for all finite values of x. To that purpose we show that a certain weighted space integral of the function $|u-\emptyset|^p$, for some integer $p \geq 1$, decays algebraically in time; a similar proof, with exponent $p = 1$, has been given for example by van DUYN & PELETIER [9].

THEOREM 6.2. *Provided that* $\varepsilon < g(\infty) - K$ *and that* $\psi \geq s^-(.,K,\overline{x}_1,\overline{\nu})$ *for some* $\overline{x}_1,\overline{\nu}$ *satisfying* (4.13) *with* $\lambda = K$, *we have that*

$$(6.5)\qquad \int_0^\infty (g'(x) + (p-1)\,\phi'(x))\,|u(x,t,\psi) - \phi(x)|^p\,dx \leq \left[\int_0^\infty ((s^+-\phi)^p + (\phi - s^-)^p)\,dx\right]/t$$

for all $t > 0$ *and* $p = [1/\overline{\nu}] + 1$.

PROOF. Since $|v(x,t)|^p \leq (s^+(x) - s^-(x,K,\overline{x}_1,\overline{\nu}))^p$, it follows that $\int_0^\infty (v(x,t))^p dx$ is defined for all $t \geq 0$. If $p \geq 2$ let us multiply the differential equation in (6.1) by v^{p-1} and integrate with respect to x; we obtain

$$\frac{d}{dt}\int_0^\infty \frac{v^p}{p}\,dx = \Big[\varepsilon x v_x\, v^{p-1}\Big]_0^\infty - \Big[\varepsilon\,\frac{v^p}{p}\Big]_0^\infty - \varepsilon(p-1)\int_0^\infty x v^{p-2}(v_x)^2\,dx$$
$$+ \Big[g\,\frac{v^p}{p}\Big]_0^\infty - \int_0^\infty (g' + \phi'(p-1))\,\frac{v^p}{p}\,dx - \Big[\phi\,\frac{v^p}{p} + \frac{v^{p+1}}{p+1}\Big]_0^\infty.$$

Since v tends to zero at least as fast as $x^{-\overline{\nu}}$ as $x \to \infty$, the equation above can be written in the simpler form

$$(6.6)\qquad \frac{d}{dt}\int_0^\infty \frac{v^p}{p}\,dx = \Big[\varepsilon x v_x v^{p-1}\Big]_0^\infty - \varepsilon(p-1)\int_0^\infty x v^{p-2}(v_x)^2 dx$$
$$- \int_0^\infty (g' + \phi'(p-1))\frac{v^p}{p}\,dx.$$

Now let us define the functions v^+ and v^- as the solutions of (6.1) with initial values $v^+(x,0) = s^+(x) - \phi(x)$ and $v^-(x,0) = s^-(x,K,\overline{x}_1,\overline{\nu}) - \phi(x)$ respectively. By Theorem 3.4 we know that $v^+ \geq 0$ and $v^- \leq 0$. Furthermore it follows from Lemma 5.1 that v^+ is nonincreasing in time and v^- nondecreasing. Of course both v^+ and v^- satisfy (6.6) and in order to simplify this expression we use the following lemma which we shall prove later.

LEMMA 6.3. *Let* $\varepsilon < g(\infty) - K$. *Then* $\lim_{x\to\infty} x\phi'(x) = 0$. *If furthermore* $\psi \geq s^-(.,K,\overline{x}_1,\overline{\nu})$ *for some* $\overline{x}_1,\overline{\nu}$ *satisfying* (4.13) *with* $\lambda = K$ (*we suppose furthermore that* $\overline{\nu} > 1$ *if* $\varepsilon < (g(\infty) - K)/2$) *and* $\psi \in C_{1,\alpha}([x_3,\infty))$ *for* *some* $\alpha, x_3 > 0$, *then* $\lim_{x\to\infty} xu_x(x,t) = 0$ *for all* $t \in (0,\infty)$.

From Lemma 6.3 and formula (6.6) we deduce that v^+ satisfies

$$\frac{d}{dt}\int_0^\infty \frac{(v^+)^p}{p}\,dx = -\varepsilon(p-1)\int_0^\infty x(v^+)^{p-2}\,(v_x^+)^2dx - \int_0^\infty (g' + \emptyset'(p-1))\frac{(v^+)^p}{p}dx.$$

If $p = 1$ similar calculations yield

$$\frac{d}{dt}\int_0^\infty v^+\,dx = -\int_0^\infty g'v^+\,dx.$$

Since $0 < g'(x) < g'(0)$ and $0 < \emptyset'(x) < \emptyset'(0)$ we have for all $p \geq 1$

$$\int_0^\infty (v^+(x,t))^p\,dx \geq \frac{1}{g'(0) + (p-1)\emptyset'(0)}\int_0^\infty (g'(x)$$

$$+ (p-1)\emptyset'(x))(v^+(x,t))^p\,dx$$

and thus

$$\int_0^\infty (g'(x) + (p-1)\emptyset'(x))(v^+(x,t))^p\,dx \leq (g'(0)+(p-1)\emptyset'(0))\int_0^\infty (v^+(x,0))^p\,dx$$

$$- (g'(0) + (p-1)\emptyset'(0))\int_0^t d\tau\int_0^\infty (g'(x)+(p-1)\emptyset'(x))(v^+(x,\tau))^p\,dx.$$

In what follows we apply the following lemma that we shall prove later.

LEMMA 6.4. *Let* $y \in C([0,\infty))$ *with* $y' \in L^1((0,\infty))$ *and* $y' \leq 0$ *such that*

(6.7) $$0 \leq y(t) \leq N - M\int_0^t y(\tau)\,d\tau$$

for some constants $N \geq 0$, $M > 0$. *Then*

(6.8) $$y(t) \leq N/(Mt).$$

Since the function $\int_0^\infty (g'(x) + (p-1)\emptyset'(x))(v^+(x,t))^p\,dx$ is continuous and nonincreasing (because v^+ is nonincreasing), we deduce from Lemma 6.4 that

$$\int_0^\infty (g'(x) + (p-1)\emptyset'(x))(v^+(x,t))^p dx \le \left(\int_0^\infty (v^+(x,0))^p dx\right)/t.$$

Similarly one can show that

$$\int_0^\infty (g'(x) + (p-1)\emptyset'(x))(-v^-(x,t))^p dx \le \left(\int_0^\infty (-v^-(x,0))^p dx\right)/t.$$

Formula (6.5) is then deduced from the fact that

$$|v(x,t)|^p \le \max((v^+(x,t))^p, (-v^-(x,t))^p) \le (v^+(x,t))^p + (-v^-(x,t))^p. \quad \square$$

PROOF of Lemma 6.3. We first show that $\lim_{x\to\infty} x\emptyset'(x) = 0$. Since

$$\varepsilon x\emptyset'(x) = \varepsilon\emptyset(x) - \int_0^x (g(\zeta) - \emptyset(\zeta))\emptyset'(\zeta)d\zeta \le \varepsilon K,$$

we have

$$0 \le x\emptyset'(x) \le K.$$

Furthermore

$$(x\emptyset')' = x\emptyset'' + \emptyset' = -\frac{g-\emptyset-\varepsilon}{\varepsilon}\emptyset' \le 0 \qquad \text{for x large enough.}$$

Since the function $x\emptyset'$ is bounded and decreasing for large x, we deduce that there exists $E \in [0,K]$ such that

$$\lim_{x\to\infty} x\emptyset'(x) = E$$

which implies

$$\emptyset(x) \sim E \ln x + C \qquad \text{as } x \to \infty.$$

Since

$$\lim_{x\to\infty} \emptyset(x) = K$$

we deduce that $E = 0$.

Next we show that $\lim_{x\to\infty} xu_x = 0$ by making use of Bernstein's argument, in a similar way as in ARONSON [1] and PELETIER & SERRIN [21].

Let

$$R_n = (n/2,\ 3n/2) \times (0,T],\ n > 3x_3$$

and let

$$\phi(r) = Nr(4-r)/3$$

where $N = \sup_{R_n} u - \inf_{R_n} u$. The function ϕ increases from 0 to N as r increases from 0 to 1. Note that $\phi'(r) = 2N(2-r)/3 > 0$ and $\phi''(r) = -2N/3 < 0$ and define a new function w such that

$$u = \inf_{R_n} u + \phi(w).$$

Then w satisfies the differential equation

$$w_t = \varepsilon x w_{xx} + \varepsilon x \frac{\phi''(w)}{\phi'(w)} (w_x)^2 + (g-\phi(w)-\inf_{R_n} u)w_x.$$

Set $p = w_x$ and differentiate the last equation with respect to x; we get

$$p_t = \varepsilon x p_{xx} + \varepsilon p_x + \varepsilon \frac{\phi''}{\phi'} p^2 + 2\varepsilon x \frac{\phi''}{\phi'} pp_x + \varepsilon x \left(\frac{\phi''}{\phi'}\right)' p^3$$

$$+ (g-\phi-\inf_{R_n} u)p_x + (g'-\phi' p)p$$

and thus

$$\tag{6.9} \tfrac{1}{2}(p^2)_t - \varepsilon x p p_{xx} = \varepsilon x \left(\frac{\phi''}{\phi'}\right)' p^4 + \varepsilon \left(\frac{\phi''}{\phi'} - \phi'\right) p^3$$

$$+ 2\varepsilon x \frac{\phi''}{\phi'} p^2 p_x + (g-\phi-\inf_{R_n} u+\varepsilon)pp_x + g'p^2.$$

Let $R_n^* = (3n/4,\ 5n/4) \times (0,T]$ and let $\zeta = 1-4(x-n)^2/n^2$. Set $z = \zeta^2 p^2$.

(i) If z attains its maximum value at the lower boundary of R_n we have

$$\sup_{R_n^*} z \le z(\tilde{x},0) \qquad \text{where } \tilde{x} \in [n/2, 3n/2].$$

Hence

$$\sup_{R_n^*} \zeta|w_x| \leq \zeta(\tilde{x})|w_x(\tilde{x},0)|.$$

Since $\zeta \geq 3/4$ in $(3n/4,5n/4)$ and since $u_x = \phi'(w)w_x$ we find

$$\sup_{R_n^*} |u_x| \leq \frac{4}{3}\,\frac{\sup \phi'}{\inf \phi'}\,|\psi'(\tilde{x})| \leq 8\, M_\psi/3.$$

(ii) If z attains its maximum value at an interior point $(\tilde{x},\tilde{t})$ of R_n we have at that point

$$(6.10)\qquad \begin{cases} z_x = 2\zeta\zeta' p^2 + 2\zeta^2 p p_x = 0 \\ \varepsilon x z_{xx} - z_t \leq 0. \end{cases}$$

The last inequality can be cast in the more explicit form

$$\zeta^2(\tfrac{1}{2}(p^2)_t - \varepsilon x p p_{xx}) \geq \varepsilon x(\zeta'^2 p^2 + \zeta\zeta'' p^2 + 4\zeta\zeta' p p_x + \zeta^2 p_x^2).$$

Using (6.9), (6.10) and the inequality

$$|4\zeta\zeta' p p_x| \leq \zeta^2 p_x^{\,2} + 4\zeta'^2 p^2$$

we obtain

$$-\zeta^2\varepsilon\left(\frac{\phi''}{\phi'}\right)' p^4 \leq \left(-2\varepsilon\zeta\zeta'\frac{\phi''}{\phi'} + \varepsilon\frac{\zeta^2}{x}\frac{\phi''}{\phi'} - \frac{\zeta^2}{x}\phi'\right)p^3$$

$$+\left(\zeta^2\frac{g'}{x} + 3\varepsilon\zeta'^2 - \varepsilon\zeta\zeta'' - \frac{g - \phi - \inf_{R_n} u + \varepsilon}{x}\zeta\zeta'\right)p^2.$$

Since $\left(\frac{\phi''}{\phi'}\right)' \leq -1/4$, this implies

$$2\zeta^2 p^4 \leq C_1 p^2 + \zeta\, C_2\, |p|^3$$

where the C_i's are positive and depend only on N and n. Since

$$\zeta\, C_2\, |p|^3 \le \zeta^2 p^4 + \frac{C_2^{\,2}}{4}\, p^2$$

it follows that

$$z(x,t) \le \max_{\overline{R}_n} (z(x,t)) \le C_1 + \frac{C_2^{\,2}}{4} \equiv C_3.$$

Therefore

$$\max_{\overline{R_n^*}} |w_x| \le 4\, C_3^{1/2} / 3.$$

Finally, $u_x = \phi'(w) w_x$ and $\phi' \le 4\,N/3$ imply that

$$\max_{\overline{R_n^*}} |u_x| \le 16N\, C_3^{1/2} / 9.$$

Note that $N \le \sup_{R_n}(K - s^-(x,K,\overline{x}_1,\overline{\nu}))$ (which behaves as $x^{-\overline{\nu}}$ where $\overline{\nu} > 0$) is furthermore such that $\overline{\nu} > 1$ if $\varepsilon < (g(\infty)-K)/2$.
Thus

$$\max_{\overline{R_n^*}} |u_x| \le 16\, C_3^{1/2} \sup_{R_n} (K - s^-(x,K,\overline{x}_1,\overline{\nu})) / 9. \tag{6.11}$$

If $\varepsilon < (g(\infty)-K)/2$, C_3 is bounded uniformly in n and we deduce that xu_x tends to zero as $x \to \infty$. If on the other hand $(g(\infty)-K)/2 \le \varepsilon < g(\infty)-K$, then we only have that $\overline{\nu} > 0$ in (6.11) and $\sup_{R_n} (K - s^-(x,K,\overline{x}_1,\overline{\nu}))$ tends to zero as $x \to \infty$ but then $C_3^{1/2}$ tends to zero as $1/x$ when $x \to \infty$ which also yields the result. □

<u>PROOF of Lemma 6.4</u>. Integrating by parts we get

$$\int_0^t y(\tau)\,d\tau = ty(t) - \int_0^t \tau y'(\tau)\,d\tau \ge ty(t).$$

Also we deduce from (6.7) that

$$\int_0^t y(\tau)\,d\tau \le N/M$$

and thus (6.8) follows. □

Next we deduce from theorem 6.2 that there is also pointwise convergence. More precisely we prove the following theorem.

THEOREM 6.5. *Provided that* $\varepsilon < g(\infty)-K$ *and that* $\psi \geq s^-(.,K,\bar{x}_1,\bar{\nu})$ *for some* $\bar{x}_1,\bar{\nu}$ *satisfying* (4.13) *with* $\lambda = K$, *we have that*

$$(6.12) \qquad \|(g'(.)+(p-1)\emptyset'(.))^{1/p}(u(.,t,\psi)-\emptyset)\| \leq C_\varepsilon/t^{\frac{1}{2}p} \qquad \textit{for all } t > 0$$

and $p = [1/\bar{\nu}]+1$, *where*

$$(6.13) \qquad C_\varepsilon = \left[2\left((K^{p-1}p^2+K^p\frac{p-1}{\varepsilon})(g'(0))^2+K^p\sup_{x\in[0,\infty)}|g''(x)|\right)\int_0^\infty((s^+-\emptyset)^p + (\emptyset-s^-)^p)\,dx\right]^{\frac{1}{2}p}.$$

In particular, if $\varepsilon < (g(\infty)-K)/2$ and $\bar{\nu} > 1$, then $p = 1$ and formulas (6.12) and (6.13) simplify as follows

$$(6.14) \qquad \|g'(.)(u(.,t,\psi)-\emptyset)\| \leq C/\sqrt{t} \qquad \text{for all } t > 0$$

where

$$C = \left[2\left((g'(0))^2+K\sup_{x\in[0,\infty)}|g''(x)|\right)\int_0^\infty(s^+(x)-s^-(x,K,\bar{x}_1,\bar{\nu}))\,dx\right]^{\frac{1}{2}}.$$

PROOF. To prove Theorem 6.5 we need the following auxiliary lemma:

LEMMA 6.6. *Let* ϕ *be defined for* $0 \leq x < \infty$ *and satisfy the conditions*

(i) $\phi(x) \geq 0$ *and* $\phi(0) = 0$;

(ii) ϕ *is Lipschitz continuous with constant* ℓ;

(iii) $\int_0^\infty \phi(x)\,dx \leq N$,

then

$$\sup_{0\leq x<\infty}|\phi(x)| \leq \sqrt{2N\ell}.$$

We omit here the demonstration of this lemma since the main ideas of the proof are given in the proof of Lemma 3 of PELETIER [20].

Now let us apply Lemma 6.6 to the function $(g' + (p-1)\emptyset')|u-\emptyset|^p$; it is nonnegative, equal to zero at the origin and its derivative is continuous by parts and bounded by

$$\{(K^{p-1}p^2 + K^p \frac{p-1}{\varepsilon})(g'(0))^2 + K^p \sup_{x\in[0,\infty)} |g''(x)|\}$$

at all points where it is defined. Finally the bound on its integral is given in theorem 6.2. Inequality (6.12) follows. □

6.3. Asymptotic behaviour of the solution $\overline{u}$ of the hyperbolic problem H as $t \to \infty$

THEOREM 6.7. *Let* ψ *satisfy* H_ψ *and be such that* $\psi \geq s^-(.,K,\overline{x}_1,\overline{\nu})$ *for some* $\overline{x}_1 > 0$, $\overline{\nu} > 1$ *satisfying* (4.13) *with* $\lambda = K$ *and define* $\overline{\emptyset}(x) = \min(g(x),K)$. *Then*

$$\|g'(.)(\overline{u}(.,t,\psi) - \overline{\emptyset})\| \leq C/\sqrt{t} \qquad \textit{for all } t > 0$$

where C *is the constant defined in Theorem* 6.5.

PROOF. Let $\varepsilon \in (0,(g(\infty) - K)/2) \downarrow 0$ in inequality (6.14), note that the constant C does not depend in ε and use the fact that $\emptyset$ converges to $\overline{\emptyset}$ uniformly on $[0,\infty)$ as $\varepsilon \downarrow 0$ (see [6]). □

APPENDIX

In what follows we shall prove the following theorem:

THEOREM A1. *There exists* $\theta \in (0,1)$, $R > 0$ *and* $\lambda < 0$ *such that the Cauchy Dirichlet problem* (6.4) *has at least one solution* $w \in C^2([0,\infty))$ *with* w,w',w'' *bounded and*

$$0 < w(x) \leq \min(\tilde{\emptyset}(x),\ (x/R)^{-\nu_0}\,\tilde{\emptyset}(R)) \qquad \textit{for all } x \in (0,\infty).$$

PROOF. Let $n \geq 1$ and consider the boundary value problem

(A1) $$\varepsilon(x+\frac{1}{n})\, w'' - (q_n(x)+\lambda)w = -\theta(\phi'(R)+\lambda)\min(\tilde{\phi}_n(x), (x/R)^{-\nu_0}\,\tilde{\phi}_n(R))$$

(A2) $$w(0) = 0$$

where

$$\tilde{\phi}_n(x) = \exp\left(\int_0^x \frac{g(\zeta)-\phi(\zeta)}{2\varepsilon(\zeta+1/n)}\,d\zeta\right)\phi(x)$$

and

$$q_n(x) = \frac{(g(x)-\phi(x))^2}{4\varepsilon(x+1/n)} + \frac{g'(x)+\phi'(x)}{2} - \frac{g(x)-\phi(x)}{2(x+1/n)}$$

$\nu_0 > 1$ is arbitrary and where the constants $\theta \in (0,1)$, $R > 0$ and $\lambda \in (-\phi'(R),0)$ satisfy some additional conditions which will be given later. Obviously zero is a lower solution for the differential equation in (A1). We shall now construct an upper solution. Firstly we deduce from the asymptotic behaviour of g that there exists $R_1 \geq 1$ and $q_0 > 0$ such that $q_n(x) \geq 2q_0$ for $x \geq R_1$. Also if $\lambda > \max(-q_0,-\phi'(R))$ and $\theta < (q_0+\lambda)/(\phi'(R)+\lambda)$, then the function $(x/R)^{-\nu_0}\,\tilde{\phi}_n(R)$ is an upper solution of the differential equation (A1) for $x \geq R := \max(R_1, 2\varepsilon\nu_0(\nu_0+1)/q_0)$. Next we note that $\tilde{\phi}_n$ is an upper solution of (A1) on $[0,R]$ and thus that $\min(\tilde{\phi}_n(x), (x/R)^{-\nu_0}\,\tilde{\phi}_n(R))$ is an upper solution of (A1) on $[0,\infty)$. Finally we conclude that there exists at least one solution $w_n \in C^2([0,\infty))$ of (A1), (A2) [3, Theorem 1.7.1], such that

$$0 \leq w_n(x) \leq \min(\tilde{\phi}_n(x),\ (x/R)^{-\nu_0}\,\tilde{\phi}_n(R))$$

which, since $\tilde{\phi}_n \leq \tilde{\phi}$, implies that

(A3) $$0 \leq w_n(x) \leq \min(\tilde{\phi}(x),\ (x/R)^{-\nu_0}\,\tilde{\phi}(R)).$$

Furthermore the inequalities (A3) and

(A4) $$|q_n(x)| \leq \frac{(g-\phi)^2}{4\varepsilon x} + \frac{g'+\phi'}{2}$$

yield, together with (A1),

$$|w_n''(x)| \le C \qquad \text{for all } x \in [0,\infty)$$

where $C > 0$ is independent of n. Now let us integrate (A1); we get

$$\text{(A5)} \quad w_n'(x) = w_n'(0) + \int_0^x \frac{(q_n(\zeta)+\lambda)w_n(\zeta) - \theta(\emptyset'(R)+\lambda)\min(\tilde{\emptyset}_n(\zeta), (\zeta/R)^{-\nu_0}\,\tilde{\emptyset}_n(R))d\zeta}{\varepsilon(\zeta+1/n)}$$

and using again (A3) and (A4) we obtain

$$|w_n'(x)| \le C \qquad \text{for all } x \in [0,\infty).$$

Using the Arzela-Ascoli theorem and a diagonal process, we deduce that there exists a function $w \in C^1([0,\infty))$ and a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $w_{n_k} \to w$ as $n_k \to \infty$, uniformly in $C^1([0,\infty))$ on all compact subsets of $[0,\infty)$. Also setting $n = n_k$ in (A5) and letting $n_k \to \infty$, we deduce that w satisfies the differential equation

$$\text{(A6)} \qquad \varepsilon x w'' - (q(x)+\lambda)w = -\theta(\emptyset'(R)+\lambda)\min(\tilde{\emptyset}(x), (x/R)^{-\nu_0}\,\tilde{\emptyset}(R))$$

and the boundary condition

$$w(0) = 0.$$

It follows from (A6) that $w \in C^2((0,\infty))$ and since

$$\lim_{x\to\infty} w''(x) = [(\emptyset'(0) + \lambda)w'(0) - \theta(\emptyset'(R)+\lambda)\,\tilde{\emptyset}'(0)]/\varepsilon$$

we deduce that in fact $w \in C^2([0,\infty))$. Finally the strict inequality $w > 0$ is proven by means of a maximum principle argument. □

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CHAPTER 4

HOW MANY JUMPS?
VARIATIONAL CHARACTERIZATION OF THE LIMIT SOLUTION OF A SINGULAR PERTURBATION PROBLEM

ABSTRACT

Using the theory of maximal monotone operators we describe the limiting behaviour, as $\varepsilon \downarrow 0$, of the solution y_ε of the nonlinear two-point boundary value problem $\varepsilon y'' + (g-y)y' = 0$, $y(0) = 0$, $y(1) = 1$, where g is a given function.

KEY WORDS & PHRASES: *singularly perturbed nonlinear two-point boundary value problem, maximal monotone operator, convex analysis*

1. INTRODUCTION

Consider the two-point boundary value problem

$$\varepsilon y'' + (g-y)y' = 0,$$

BVP

$$y(0) = 0, \quad y(1) = 1,$$

where $g \in L_2 = L_2(0,1)$ is a given function and $y \in H^2$ is unknown. As we shall show, there exists for each $\varepsilon > 0$ a unique solution y_ε, which is increasing. We are interested in the limiting behaviour of y_ε as $\varepsilon \downarrow 0$.

Motivated by a physical application we previously studied a similar problem in a joint paper with L.A. PELETIER [2]. Using the maximum principle as our main tool we were able to establish the existence of a unique limit solution y_0 under certain, physically reasonable, assumptions on the function g. In some cases we could characterize y_0 completely, in others, however, some ambiguity remained.

Here, inspired by the work of GRASMAN & MATKOWSKY [4], we shall resolve this ambiguity by using a variational formulation of the problem. The method we use is based on the theory of maximal monotone operators. It has been suggested to us by Ph. Clément.

During our investigation of BVP we experienced that it could serve as a fairly simple, yet nontrivial, illustration of concepts and methods from abstract functional analysis. In order to demonstrate this aspect of the problem we shall spell out our arguments in some more detail than is strictly necessary.

The organization of the paper is as follows. In Section 2 we prove, by means of Schauder's fixed point theorem, that BVP has a solution y_ε for each $\varepsilon > 0$. Moreover, we show that BVP is equivalent to an abstract equation AE, involving a maximal monotone operator A, and to a variational problem VP, involving a convex, lower semi-continuous functional W.

In Section 3 we exploit these formulations in the investigation of the limiting behaviour of y_ε as $\varepsilon \downarrow 0$. It turns out that y_ε converges in L_2 to a limit y_0. Moreover, y_0 is abstractly characterized as the projection (in L_2) of g on $\overline{\mathcal{D}(A)}$. We conclude this section with some results about uniform convergence under restrictive assumptions.

In Section 4 we give concrete form to the characterization of y_0. In particular we present sufficient conditions for a function to be y_0 and we show, by means of examples, how these criteria can be used in concrete cases. The first part of the title originated from Example 4.

In Section 5 we make various remarks about generalizations and limitations of our approach.

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2. THREE EQUIVALENT FORMULATIONS

In order to demonstrate the existence of a solution of BVP, let us first look at the auxiliary problem

$$u'' + (g-w)u' = 0,$$

$$u(0) = 0, \quad u(1) = 1,$$

where $w \in L_2$ is a given function. The solution of this linear problem is given explicitly by

$$u(x) = C(w) \int_0^x \exp\left(\int_0^\zeta (w(\xi) - g(\xi))d\xi\right)d\zeta$$

with

$$C(w) = \left(\int_0^1 \exp\left(\int_0^\zeta (w(\xi) - g(\xi))d\xi\right)d\zeta\right)^{-1}.$$

From this expression it can be concluded that $u' > 0$ and $0 \le u \le 1$. So if we write $u = Tw$, then T is a compact map of the closed convex set

$\{w \in L_2 \mid 0 \le w \le 1\}$ into itself and hence, by Schauder's theorem, T must have a fixed point. Clearly this fixed point corresponds to a solution of BVP. Thus we have proved

PROPOSITION 2.1. *For each* $\varepsilon > 0$ *there exists a solution* $y_\varepsilon \in H^2$ *of* BVP. *Moreover, any solution* $y \in H^2$ *satisfies* (i) $y' > 0$ *and* (ii) $0 \le y \le 1$.

The a priori knowledge that y' is positive allows us to divide the equation by y'. In this manner we are able to reformulate the boundary value problem as an equivalent abstract equation

$$\text{AE} \qquad (I + \varepsilon A)y = g$$

where the (unbounded, nonlinear) operator A: $\mathcal{D}(A) \to L_2$ is defined by

$$(2.1) \qquad Au = -\frac{u''}{u'} = -(\ell n\ u')'$$

with

$$(2.2) \qquad \mathcal{D}(A) = \{u \in L_2 \mid u \in H^2,\ u' > 0,\ u(0) = 0,\ u(1) = 1\}.$$

PROPOSITION 2.2. *The operator* A *is monotone. Hence the solution of* AE *(and* BVP*) is unique.*

PROOF. Let $u_i \in \mathcal{D}(A)$ for $i = 1,2$ then

$$\begin{aligned}(Au_1 - Au_2, u_1 - u_2) &= -\int ((\ell n\ u_1')' - (\ell n\ u_2')')(u_1 - u_2) \\ &= \int (\ell n\ u_1' - \ell n\ u_2')(u_1' - u_2') \ge 0\end{aligned}$$

(because $z \mapsto \ell n\ z$ is monotone on $(0,\infty)$; note that here and in the following we write $\int\phi$ to denote $\int_0^1 \phi(x)dx$.) Next, suppose $\varepsilon Ay_i = g - y_i$, $i = 1,2$, then $0 \le \varepsilon(Ay_1 - Ay_2, y_1 - y_2) = (g - y_1 - g + y_2, y_1 - y_2) = -\|y_1 - y_2\|^2$ and hence $y_1 = y_2$. $\square$

We recall that a monotone operator A defined on a Hilbert space H is is called *maximal monotone* if it admits no proper monotone extension (i.e., it is maximal in the sense of inclusion of graphs). It is well known that A is maximal monotone if and only if $\mathcal{R}(I + \varepsilon A) = H$ for each $\varepsilon > 0$ (see BRÉZIS [1]). In our case, with $H = L_2$ and A defined in (2.1), this is just a reformulation of the existence result Proposition 2.1. Consequently we know

PROPOSITION 2.3. A *is maximal monotone.*

In search for yet another formulation let us write the equation in the form

$$-\varepsilon(\ell n\ y')' + y - g = 0$$

Hence, for any $\phi \in H_0^1$,

$$\varepsilon \int \phi'(\ell n\ y' + 1) + \int \phi(y - g) = 0.$$

Motivated by this calculation we define a functional $W\colon L_2 \to \overline{\mathbb{R}}$ by

$$W(u) = \varepsilon\Psi(u) + \frac{1}{2}\|u - g\|^2 \tag{2.3}$$

where

$$\Psi(u) = \begin{cases} \int u'\ \ell n\ u' & \text{if } u \in \mathcal{D}(\Psi), \\ +\infty & \text{otherwise,} \end{cases} \tag{2.4}$$

and

$$\mathcal{D}(\Psi) = \{u \in L_2 \mid u \text{ is AC, } u' \geq 0,\ u'\ \ell n\ u' \in L_1,\ u(0) = 0,\ u(1) = 1\} \tag{2.5}$$

(here AC means absolutely continuous.) Also we define a variational problem

$$\text{VP} \qquad \operatorname{Inf}_{L_2} W.$$

We note that the mappings $z \mapsto z\ \ell n\ z$ and $z \mapsto z^2$ are (strictly) convex (on

$[0,\infty)$ and $(-\infty,\infty)$ respectively) and that W inherits this property because $\mathcal{D}(\Psi)$ is convex as well. Hence VP has at most one solution. For further use we observe that the convexity of $z \mapsto z \ln z$ implies, for $z \geq 0$ and $\zeta > 0$, the inequality

$$z \ln z - \zeta \ln \zeta \geq (1 + \ln \zeta)(z - \zeta).$$

PROPOSITION 2.4. y_ε *solves* VP.

PROOF. Firstly we note that $y_\varepsilon \in \mathcal{D}(\Psi)$. So for any $u \in \mathcal{D}(\Psi)$

$$\begin{aligned} W(u) - W(y_\varepsilon) &= \varepsilon \int (u' \ln u' - y_\varepsilon' \ln y_\varepsilon') + \tfrac{1}{2}\|u - g\|^2 - \tfrac{1}{2}\|y_\varepsilon - g\|^2 \\ &\geq \varepsilon \int (1 + \ln y_\varepsilon')(u' - y_\varepsilon') + \int (y_\varepsilon - g)(u - y_\varepsilon) \\ &= \int (-\varepsilon \frac{y_\varepsilon''}{y_\varepsilon'} + y_\varepsilon - g)(u - y_\varepsilon) = 0. \qquad \square \end{aligned}$$

We recall that the *subgradient* $\partial\Psi$ of the convex functional Ψ is defined by

$$\partial\Psi(u) = \{\zeta \in L_2 \mid \Psi(v) - \Psi(u) \geq (\zeta, v - u),\ \forall v \in \mathcal{D}(\Psi)\}.$$

A calculation like the one above shows that, for $u \in \mathcal{D}(A)$ and $v \in \mathcal{D}(\Psi)$,

$$\Psi(v) - \Psi(u) \geq (Au, v - u).$$

Hence $A \subset \partial\Psi$, but, since $\partial\Psi$ is monotone and A is *maximal* monotone, we must have $A = \partial\Psi$. Likewise it follows that $\partial W = \varepsilon A + I - g$. These observations should clarify the relation between VP and AE.

One can show that Ψ (and hence W as well) is lower semicontinuous and subsequently one can use this knowledge to give a direct variational proof of the existence of a solution of VP.

We summarize the main results of this section in the following theorem.

THEOREM 2.5. *The problems* BVP, AE *and* VP *are equivalent. In fact, for each*

$\varepsilon > 0$, *there exists* $y_\varepsilon \in \mathcal{D}(A)$ *which solves each problem and no problem admits any other solution.*

3. LIMITING BEHAVIOUR AS $\varepsilon \downarrow 0$

The fact that y_ε solves AE can be expressed as

$$y_\varepsilon = (I + \varepsilon A)^{-1} g.$$

Subsequently, the observation that A is maximal monotone provides a key to describing the limiting behaviour. For, it is known from the general theory of such operators (see BRÉZIS [1, Section II.4, in particular Th. 2.2]) that

$$\lim_{\varepsilon \downarrow 0} (I + \varepsilon A)^{-1} g = \mathrm{Proj}_{\overline{\mathcal{D}(A)}}\, g,$$

where the expression at the right-hand side denotes the projection (in the sense of the underlying Hilbert space, hence L_2 in this case) of g on the closed convex set $\overline{\mathcal{D}(A)}$, or, in other words,

$$\mathrm{Proj}_{\overline{\mathcal{D}(A)}}\, g = y_0$$

where y_0 denotes the unique solution of the variational problem

$$\mathrm{Min}_{\overline{\mathcal{D}(A)}}\, W_0$$

with

$$W_0(u) = \| u - g \|^2.$$

Below we shall give a proof of this result for this special case, using techniques as in Brézis' book, but exploiting the fact that A is the sub-differential of the functional Ψ.

<u>THEOREM 3.1.</u>

$$\lim_{\varepsilon \downarrow 0} \| y_\varepsilon - y_0 \| = 0.$$

PROOF. First of all we note that $\|y_\varepsilon\| \le 1$. We shall split the proof into three steps.

Step 1. Take any $z \in \mathcal{D}(A)$ then from

$$\Psi(y_\varepsilon) - \Psi(z) \ge (Az, y_\varepsilon - z)$$

it follows that

$$\liminf_{\varepsilon \downarrow 0} \varepsilon(\Psi(y_\varepsilon) - \Psi(z)) \ge 0.$$

Step 2. By definition,

$$0 \ge W(y_\varepsilon) - W(z) = \varepsilon(\Psi(y_\varepsilon) - \Psi(z)) + \frac{1}{2}\|g - y_\varepsilon\|^2 - \frac{1}{2}\|g - z\|^2.$$

Hence

$$\limsup_{\varepsilon \downarrow 0} \|g - y_\varepsilon\|^2 \le \|g - z\|^2, \qquad \forall z \in \mathcal{D}(A).$$

But then, in fact, the same must hold for all $z \in \overline{\mathcal{D}(A)}$.

Step 3. Since $\|y_\varepsilon\| \le 1$, $\{y_\varepsilon\}$ is weakly precompact in L_2. Take any $\{\varepsilon_n\}$ and $\tilde{y}$ such that $y_{\varepsilon_n} \rightharpoonup \tilde{y}$ in L_2, then

$$(*) \qquad \|g-\tilde{y}\|^2 \le \liminf_{n \to \infty} \|g-y_{\varepsilon_n}\|^2 \le \limsup_{n \to \infty} \|g-y_{\varepsilon_n}\|^2 \le \|g-z\|^2, \qquad \forall z \in \overline{\mathcal{D}(A)}.$$

Consequently $\tilde{y} = y_0$, which shows that the limit does not depend on the subsequence under consideration. Hence $y_\varepsilon \rightharpoonup y_0$. Finally, by taking $z = y_0$ in (*) it follows that in fact $y_\varepsilon \to y_0$. □

We note that

$$\overline{\mathcal{D}(A)} = \{u \in L_2 \mid u \text{ is nondecreasing, } 0 \le u \le 1\}.$$

So in general y_0 need not be continuous (nor does it need to satisfy the boundary conditions). However it is possible, as our next result shows, to establish uniform convergence to a continuous limit at the price of some conditions on g.

THEOREM 3.2. *Suppose* $g \in C^1$, $g(0) < 0$ *and* $g(1) > 1$. *Then* $y_0 \in C$ *and*

$$\lim_{\varepsilon \downarrow 0} \sup_{0 \le x \le 1} |y_\varepsilon(x) - y_0(x)| = 0.$$

PROOF. The idea is to derive a uniform bound for y'_ε. We know already that $y'_\varepsilon > 0$ and we are going to show that $y'_\varepsilon \le \sup g'$. To this end we first observe that $g(0) - y_\varepsilon(0) < 0$ and $g(1) - y_\varepsilon(1) > 0$, which, combined with the differential equation, shows that $y''_\varepsilon(0) > 0$ and $y''_\varepsilon(1) < 0$. Hence y'_ε assumes its maximum in an interior point, say $\bar{x}$. Next, differentiation of the differential equation followed by substitution of $y''_\varepsilon(\bar{x}) = 0$, $y'''_\varepsilon(\bar{x}) \le 0$, leads to the conclusion that $y'_\varepsilon(\bar{x}) \le g'(\bar{x})$. The uniform bound for y'_ε implies, by virtue of the Arzela-Ascoli theorem, that the limit set of $\{y_\varepsilon\}$ in the space of continuous functions is nonempty. Combination of this result with Theorem 3.1 leads to the desired conclusion. □

In Section 4 we shall show that y_0 can be calculated in many concrete examples. Quite often it will turn out that y_0 is continuous (or piecewise continuous). This motivates our next result.

THEOREM 3.3. *Suppose* y_0 *is continuous. Then* y_ε *converges to* y_0 *uniformly on compact subsets of* $(0,1)$.

PROOF. Let $I \subset (0,1)$ be a compact set. Put $\beta(\varepsilon) = \max\{y_\varepsilon(x) - y_0(x) \mid x \in I\}$ and let $\bar{x}(\varepsilon) \in I$ be such that $y_\varepsilon(\bar{x}(\varepsilon)) - y_0(\bar{x}(\varepsilon)) = \beta(\varepsilon)$. Suppose $\limsup_{\varepsilon \downarrow 0} \beta(\varepsilon) = \beta > 0$ and let $\{\varepsilon_n\}$ be such that $\beta(\varepsilon_n) \to \beta$ as $n \to \infty$. Choose $\delta \in (0, \delta_1)$, where δ_1 denotes the distance of 1 to I, such that $|y_0(x) - y_0(\xi)| \le \frac{1}{4}\beta$ if $|x - \xi| \le \delta$. Also, choose n_0 such that $\beta(\varepsilon_n) \ge \frac{3}{4}\beta$ for $n \ge n_0$. Then for $x \in [\bar{x}(\varepsilon_n), \bar{x}(\varepsilon_n) + \delta]$ and $n \ge n_0$ the following inequality holds:

$$y_{\varepsilon_n}(x) - y_0(x) \ge y_{\varepsilon_n}(\bar{x}(\varepsilon_n)) - y_0(\bar{x}(\varepsilon_n)) + y_0(\bar{x}(\varepsilon_n)) - y_0(x)$$

$$\ge \frac{3}{4}\beta - \frac{1}{4}\beta = \frac{1}{2}\beta.$$

However, this leads to

$$\|y_{\varepsilon_n} - y_0\|^2 \ge \frac{1}{4}\delta\beta^2$$

which is in contradiction with Theorem 3.1. Hence our assumption $\beta > 0$ must be false and we arrive at the conclusion that $\limsup_{\varepsilon\downarrow 0} \max\{y_\varepsilon(x) - y_0(x) \mid x \in I\} \le 0$. Essentially the same argument yields that $\liminf_{\varepsilon\downarrow 0} \min\{y_\varepsilon(x) - y_0(x) \mid x \in I\} \ge 0$. Taking both statements together yields the result. □

It should be clear that appropriate analogous results can be proved if y_0 is piece-wise continuous. In Theorem 3.3 the sense of convergence is sharpened "a posteriori", that is, once the continuity of y_0 is established by other means. Note that our proof exploits the uniform one-sided bound $y'_\varepsilon > 0$.

4. CALCULATION OF y_0

We recall that y_0 is the unique solution of the variational problem $\min_{\overline{\mathcal{D}(A)}} W_0$, where $W_0(u) := \|u-g\|^2$. It is well known (for instance, see EKELAND & TÉMAM [3, II, 2.1]) that one can equivalently characterize y_0 as the unique solution of the *variational inequality:*

(4.1) find $y \in \overline{\mathcal{D}(A)}$ such that $(y-g, v-y) \ge 0$, $\forall v \in \overline{\mathcal{D}(A)}$.

Already from the reduced differential equation $(g-y)y' = 0$, it can be guessed that y_0 is possibly composed out of pieces where it equals g and pieces where it equals a constant. Of course, if $y_0 = g$ in some open interval, g has to be nondecreasing in that interval. The characterization of y_0 by (4.1) can be used to find conditions on the "allowed" constants.

THEOREM 4.1. *Suppose* $y \in \overline{\mathcal{D}(A)}$ *has the following property: there exists a partition* $0 = x_0 < x_1 < \dots < x_{n-1} < x_n = 1$ *of* $[0,1]$ *and a subset* L *of* $\{0,1,\dots,n-1\}$ *such that:*

(i) *if* $i \notin L$ *then* $y(x) = g(x)$ *for* $x \in [x_i,x_{i+1}]$,

(ii) *if* $i \in L$ *then* $y(x) = C_i$ *for* $x \in [x_i,x_{i+1}]$ *and*

$$\int_x^{x_{i+1}} (C_i - g(\xi))d\xi \ge 0, \qquad \forall x \in [x_i,x_{i+1}], \text{ if } C_i \in [0,1),$$

$$\int_{x_i}^{x} (C_i - g(\xi))d\xi \le 0, \qquad \forall x \in [x_i,x_{i+1}], \text{ if } C_i \in (0,1],$$

(*so in particular, if* $C_i \in (0,1)$, $\int_{x_i}^{x_{i+1}}(C_i - g(\xi))d\xi = 0$).

Then $y = y_0$.

PROOF. According to (4.1) it is sufficient to check that

$$I(v) = \int (y - g)(v - y) \ge 0, \qquad \forall v \in \overline{\mathcal{D}(A)}.$$

In fact it is sufficient to check this for all $v \in \overline{\mathcal{D}(A)} \cap H^1$ (since this set is dense in $\overline{\mathcal{D}(A)}$ and I is continuous). We note that $I(v) = \Sigma_{i\in L} I_i(v)$, where

$$I_i(v) = \int_{x_i}^{x_{i+1}} (C_i - g(\xi))(v(\xi) - C_i)d\xi.$$

If $C_i = 0$ then

$$I_i(v) = -v(x_i)\int_{x_i}^{x_{i+1}} g(\xi)d\xi - \int_{x_i}^{x_{i+1}} v'(\xi)\int_{\xi}^{x_{i+1}} g(x)dxd\xi \ge 0.$$

If $C_i \in (0,1)$ then

$$I_i(v) = \int_{x_i}^{x_{i+1}} v'(\xi)\int_{\xi}^{x_{i+1}} (C_i - g(x))dxd\xi \ge 0.$$

If $C_i = 1$ then

$$I_i(v) = (v(x_{i+1}) - 1) \int_{x_i}^{x_{i+1}} (C_i - g(\xi))d\xi - \int_{x_i}^{x_{i+1}} v'(\xi) \int_{x_i}^{\xi} (C_i - g(x))dxd\xi \geq 0 .$$

Hence indeed $I(v) \geq 0$, $\forall v \in \overline{\mathcal{D}(A)} \cap H^1$. □

The sufficient conditions of the theorem can be used as a kind of algorithm to compute y_0 in concrete cases. We shall illustrate this idea by means of a number of examples (some of which are almost literally taken from [2]).

EXAMPLE 1. Suppose g is nondecreasing, then

$$y_0(x) = \begin{cases} 0 & \text{if } g(x) \leq 0, \\ g(x) & \text{if } 0 \leq g(x) \leq 1, \\ 1 & \text{if } g(x) \geq 1. \end{cases}$$

EXAMPLE 2. Suppose g is nonincreasing, then $y_0(x) = C$ with

$$C = \begin{cases} 0 & \text{if } \int g \leq 0, \\ \int g & \text{if } 0 \leq \int g \leq 1, \\ 1 & \text{if } \int g \geq 1. \end{cases}$$

EXAMPLE 3. Suppose that $g \in C^1$ is such that g' vanishes at only two points b and c, b being a local maximum and c a local minimum. Assume that $0 < b < c < 1$ and $0 < g(c) < g(b) < 1$. Let g_1^{-1} denote the inverse of g on $[0,b]$ and g_2^{-1} the inverse of g on $[c,1]$. Define two points a and d by

$$a = g_1^{-1}(g(c)), \qquad d = g_2^{-1}(g(b)).$$

Then $g([a,b]) = g([c,d])$. (See Figure 1).

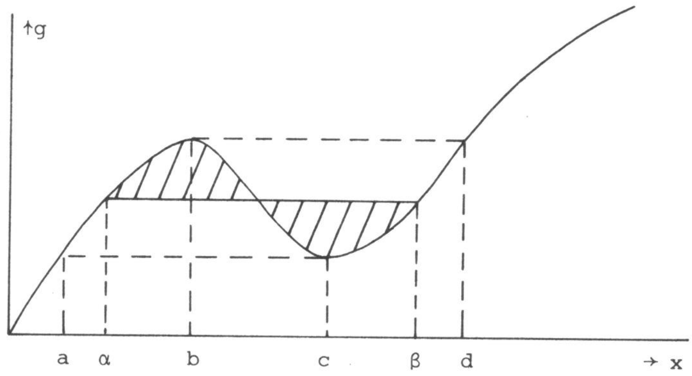

Figure 1

On [a,b] we define a mapping G by

$$G(x) = \int_{x}^{g_2^{-1}(g(x))} (g(x) - g(\xi))d\xi.$$

Then G(a) < 0, G(b) > 0 and on (a,b)

$$G'(x) = g'(x) \int_{x}^{g_2^{-1}(g(x))} d\xi > 0.$$

Consequently G has a unique zero on [a,b], say for $x = \alpha$. The function y_0 has the tendency to follow g as much as possible. However, it also has to be nondecreasing. So the inverse function of y_0 must "jump" from a point on [a,b] to a point on [c,d]. In view of Theorem 4.1 this jump can only take place between α and $\beta = g_2^{-1}(\alpha)$. We leave it to the reader to verify (by checking all requirements of Theorem 4.1) that

$$y_0(x) = \begin{cases} 0 & \text{if } x \le \alpha \text{ and } g(x) \le 0, \\ g(x) & \text{if } x \le \alpha \text{ and } g(x) \ge 0, \\ g(\alpha) & \text{if } \alpha \le x \le \beta, \\ g(x) & \text{if } x \ge \beta \text{ and } g(x) \le 1, \\ 1 & \text{if } x \ge \beta \text{ and } g(x) \ge 1. \end{cases}$$

It should be clear that the differentiability of g is not strictly necessary for our arguments to apply. In fact the monotonicity of G follows from straightforward geometrical considerations and the condition $G(\alpha) = 0$ has a corresponding interpretation (see Figure 1).

EXAMPLE 4. If g has more maxima and minima the construction of candidates for y_0 can be based on essentially the same idea as outlined in Example 3. However, it becomes more complicated since the number of possibilities becomes larger (see [2] for some more details). For instance, if g has a graph as shown in Figure 2, looking at zeroes of functions like G above leaves us with two possible candidates: one with two "jumps" (a-b,c-d) and one with a "two-in-one jump" (α-β).

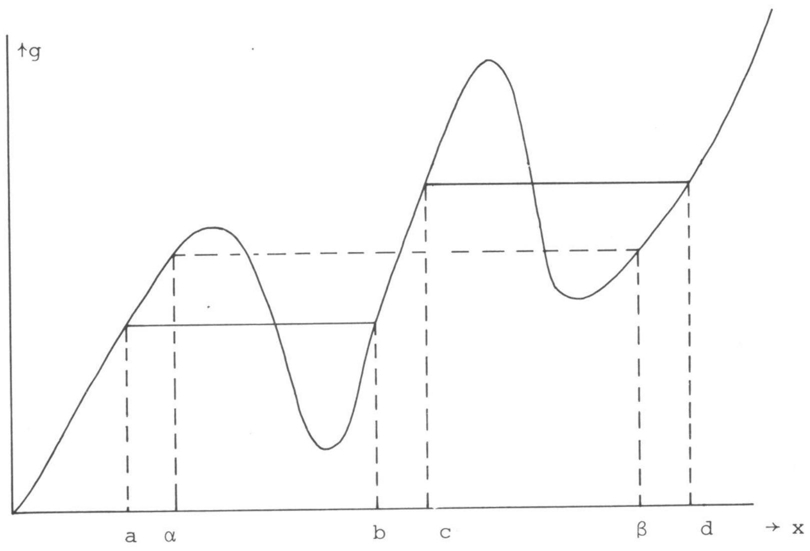

Figure 2

In [2] we were unable to decide in such a situation which was the actual limit. But now it can be read off from the picture that only the one with two "jumps" satisfies the requirements of Theorem 4.1, and hence this one must actually be y_0. (The other one corresponds to a saddle point of the functional W_0 restricted to $\overline{\mathcal{D}(A)}$.) It is in this sense that y_0 must have as many "jumps" as possible.

5. CONCLUDING REMARKS

(i) In all our examples y_0 satisfies the reduced equation $(g-y)y' = 0$. However this equation is by no means sufficient to characterize y_0 completely. Our analysis clearly shows that the reduced variational problem $\mathrm{Min}_{\overline{\mathcal{D}(A)}}\, W_0$ contains much more information than the reduced differential equation.

(ii) In [2] we were actually interested in a boundary value problem of the type

$$\varepsilon x y'' + (g-y)y' = 0, \qquad 0 < x < 1, \tag{5.1}$$

$$y(0) = 0, \quad y(1) = 1, \tag{5.2}$$

which arises from the assumption of radial symmetry in a two-dimensional geometry. This problem can be analysed in completely the same way as we did with BVP in this paper, by choosing as the underlying Hilbert space the weighted L_2-space corresponding to the measure $d\mu(x) = x^{-1}dx$. For instance, the operator $\tilde{A}$ defined by

$$(\tilde{A}u)(x) = -x\,\frac{u''(x)}{u'(x)}$$

with

$$\mathcal{D}(\tilde{A}) = \{u \in L^2(d\mu) \mid u' \in C(0,1],\ u' > 0,\ u(1) = 1,$$
$$i\,\frac{u''}{u'} \in L^2(d\mu)\}\,,$$

where i denotes the function $i(x) = x$, is clearly monotone in this space. The surjectivity of $I + \varepsilon\tilde{A}$ can be proved with the aid of an auxiliary problem and Schauder's fixed point theorem. (Note that some care is needed in checking that the functions which occur belong to the right space and that the solution operator is compact. This turns out to be all right. We refer to Martini's thesis [5] where related problems are treated in full detail.) Hence $\tilde{A}$ is maximal monotone. Subsequently it follows that, for given $g \in L_2(d\mu)$, the solution y_ε tends, as $\varepsilon \downarrow 0$, to a limit y_0 in $L_2(d\mu)$ and that y_0 is the projection in $L_2(d\mu)$ of g onto the closed convex set

$$\overline{\mathcal{D}(\tilde{A})} = \{u \in L_2(d\mu) \mid u \text{ is nondecreasing, } 0 \le u \le 1\}.$$

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CHAPTER 5

VARIATIONAL ANALYSIS
OF A PERTURBED FREE BOUNDARY PROBLEM

ABSTRACT

Using convex analysis we show that the solution u_ε of a nonlinear boundary value problem (depending on a parameter ε) converges to a limit u_0 as $\varepsilon \downarrow 0$. We characterize u_0 as the solution of a free boundary problem and we discuss some of its properties.

KEY WORDS & PHRASES : *nonlinear boundary value problem, integral condition, singular perturbation, convex analysis, duality theory, maximal monotone operator, free boundary problem*

1. INTRODUCTION

In this paper we study the nonlinear boundary value problem

$$\text{BVP}\quad \begin{cases} -\Delta u + h(\frac{u}{\varepsilon}) = f & \text{in } \Omega \\ \int_\Omega h(\frac{u(x)}{\varepsilon})dx = C & \\ u|_{\partial\Omega} \text{ is constant (but unknown)} & \end{cases}$$

where

(i) Ω is a bounded open subset of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$

(ii) ε is a small positive parameter

(iii) $h : \mathbb{R} \to \mathbb{R}$ is a given continuous, strictly monotone increasing function with $h(0) = 0$

(iv) f is a given distribution in $H^{-1}(\Omega)$

(v) C is a given constant which satisfies the compatibility condition

$$h(-\infty)|\Omega| < C < h(+\infty)|\Omega|.$$

Here $|\Omega|$ denotes the measure of Ω.

The motivation for studying BVP partly stems from the physics of ionized gases and in this respect we continue earlier work [15, 16, 21, 22]. We refer to [22] and Appendix 2 for a discussion of this connection.

Our basic tools are the calculus of variations, convex analysis and the maximum principle.

We prove that BVP admits for each $\varepsilon > 0$ a unique solution u_ε which converges as $\varepsilon \downarrow 0$ to a limit u_0. Moreover, we give a variational characterization of u_0 which narrows down to the conclusion that u_0 solves a free boundary problem.

Our findings fit in with those of BRAUNER & NICOLAENKO [7, 8] in their study of related Dirichlet problems (we certainly have been inspired by their paper). In this connection it is also worth mentioning the work of FRANK & VAN GROESEN [18] and FRANK & WENDT [19] which analyses in particular the coincidence set. In Appendix 1 we give the analysis of the homogeneous Dirichlet problem.

In the physical problem of Appendix 2 the parameter ε naturally appears in the same way as in BVP. In other situations one may arrive at the equation

$$-\varepsilon\Delta v + h(v) = f.$$

Then our results bear on $\varepsilon v_\varepsilon$ and $h(v_\varepsilon)$.

In a recent paper [9] BRAUNER & NICOLAENKO stress the following point. Suppose one wants to analyse some free boundary problem, then it may be possible to view this problem as the limit when $\varepsilon \downarrow 0$ of a problem like BVP (with ε occurring in the argument of a smooth function). This smooth regularization can be used to solve problems of existence, regularity and approximation and it forms an alternative version of the usual penalization method. (see also [6]).

After these general remarks, let us describe the contents of the paper in some more detail. We shall interpret BVP as the subdifferential equation $\partial V_\varepsilon(u) = 0$, where V_ε is a proper, strictly convex, lower semicontinuous and coercive functional defined on the direct sum of $H^1_0(\Omega)$ and the constant functions on Ω. This is rather easy if h satisfies certain growth restrictions. For the general case we heavily lean upon some results of BREZIS [11]. These and some other preliminaries are collected in section 2. The functional V_ε is defined in section 3 and from its properties we deduce the existence and uniqueness of a solution u_ε for each $\varepsilon > 0$.

The functional V_ε depends monotonously on ε and therefore has a well-defined limit V_0. Moreover, V_ε is coercive uniformly in ε and consequently we deduce in section 4 that as $\varepsilon \downarrow 0$ u_ε converges to u_0, the minimizer of V_0. The subdifferential ∂V_0 is multivalued. We find that u_0 satisfies an operator inclusion relation if h is bounded and a variational inequality if h is unbounded. We emphasize that the reduced problem is piecewise linear: u_0 depends only on f,C and $h(\pm\infty)$.

Problem BVP has the form

$$Lu + N\left(\frac{u}{\varepsilon}\right) = f$$

where both L and N are maximal monotone operators. The variational approach suggests the introduction of a dual formulation (in section 5) which turns out to be of the form

$$(\varepsilon A + I)p = g$$

where A is a maximal monotone operator on $(L_2(\Omega))^n$ with a special structure, and where g is related to f by div g = f. This gives some further insight into the convergence. The limit p_0 equals the projection of g onto the closed convex set $\overline{\mathcal{D}(A)}$. Duality theory yields a characterization of $\overline{\mathcal{D}(A)}$ by inequalities which seems difficult to obtain directly. Duality theory has been applied to related problems by ARTHURS & ROBINSON [4] and ARTHURS [3]. For the basic theory we refer to EKELAND & TEMAM [17]

In section 6 we assume $f \in L_\infty(\Omega)$. We employ maximum principle arguments and make some estimates. We prove that u_ε and u_0 belong to $W^{2,p}(\Omega)$ for each $p \geq 1$ and that u_ε converges weakly to u_0 in $W^{2,p}(\mathcal{O})$ for each $\mathcal{O}$ with $\bar{\mathcal{O}} \subset \Omega$. Either one has convergence in $W^{2,p}(\Omega)$ itself, or a boundary layer develops as $\varepsilon \downarrow 0$. We present criteria in terms of the data f, $h(\pm\infty)$ and C from which it can be decided in many cases which of these two possibilities actually occurs. In section 7 we briefly discuss the one-dimensional case.

Our analysis reveals that BVP and the homogeneous Dirichlet problem have exactly the same variational structure. In order to emphasize this point we analyse the latter problem in Appendix 1. Finally, we discuss the physical background of BVP in Appendix 2.

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2. PRELIMINARIES

In this section we collect some definitions and results from the literature which we will use later. We state these in the form we need, which is not always the most general.

Let B be a Banach space and B^* its dual. Let $F : B \to (-\infty,+\infty]$ be a proper (i.e. $F \not\equiv +\infty$), lower semicontinuous (ℓ.s.c.), convex functional. The polar (or conjugate) functional $F^* : B^* \to (-\infty,+\infty]$ is defined by

$$F^*(v^*) = \sup\{<u^*,u> - F(u) \mid u \in \mathcal{D}(F)\} \tag{2.1}$$

where

$$\mathcal{D}(F) = \{u \mid F(u) < +\infty\} \tag{2.2}$$

and where $<\cdot,\cdot>$ denotes the duality pairing between B^* and B. The subdifferential ∂F is a, possibly multivalued, mapping of X into X^* defined by

$$u^* \in \partial F(u) \text{ if and only if } F(v) - F(u) \geq <u^*,v-u>, \ \forall v \in B. \tag{2.3}$$

LEMMA 2.1.

$$u^* \in \partial F(u) \textit{ if and only if } F(u) + F^*(u^*) = <u^*,u>.$$

LEMMA 2.2.

$$u^* \in \partial F(u) \textit{ if and only if } u \in \partial F^*(u^*).$$

A convenient reference for these items is EKELAND & TEMAM [17].

If B is a Hilbert space one can identify B and B^* and then ∂F becomes a mapping of B into itself. It is well-known that ∂F is maximal monotone.

LEMMA 2.3. *Let* H *be a Hilbert space and* A *a maximal monotone operator on* H. *Then, for each* $\varepsilon > 0$, $(I+\varepsilon A)^{-1}$ *is a contraction defined on all of* H *and* $\lim_{\varepsilon \downarrow 0} (I+\varepsilon A)^{-1} h =$ *projection of* h *on* $\overline{\mathcal{D}(A)}$.

For this standard result we refer to BREZIS [10].

Let, as before, Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary. We shall write H_0^1, L_2 etc. to denote $H_0^1(\Omega), L_2(\Omega)$ etc. Also, we write $\int u$ to denote $\int_\Omega u(x)dx$.

Let $j : \mathbb{R} \to [0,+\infty]$ be a convex, l.s.c. function such that $j(0) = 0$. The convex, l.s.c. functional $J : H_0^1 \to [0,+\infty]$ is defined by

$$(2.4) \qquad J(u) = \begin{cases} \int j(u) & \text{if } j(u) \in L_1 \\ +\infty & \text{otherwise.} \end{cases}$$

The following two lemmas are special cases of results due to BREZIS [11].

<u>LEMMA 2.4</u>. *Suppose* $\mathcal{D}(j) = \mathbb{R}$ *then*

$$J^*(w) = \begin{cases} \int j^*(w) & \textit{if } w \in H^{-1} \cap L_1 \textit{ and } j^*(w) \in L_1 \\ +\infty & \textit{otherwise.} \end{cases}$$

<u>LEMMA 2.5</u>. *Suppose* $\mathcal{D}(j) = \mathbb{R}$ *then* $w \in \partial J(u)$ *if and only if* $w \in H^{-1} \cap L_1$, $w.u \in L_1$ *and* $w(x) \in \partial j(u(x))$ *for almost all* $x \in \Omega$.

Finally, we quote a special case of a result of BREZIS & BROWDER [12,13].

<u>LEMMA 2.6</u>. *Assume* $w \in H^{-1} \cap L_1$ *and* $u \in H_0^1$ *are such that* $w(x)u(x) \geq g(x)$ *for almost all* $x \in \Omega$ *and some* $g \in L_1$. *Then* $w.u \in L_1$ *and*

$$\langle w,u \rangle = \int w.u.$$

Here and in the following $\langle \cdot,\cdot \rangle$ denotes the duality pairing of H^{-1} and H_0^1. We observe that Lemma 2.6 implies that the condition $w.u \in L_1$ in Lemma 2.5 is automatically satisfied.

3. VARIATIONAL FORMULATION

Let X be the direct sum of H_0^1 and the constant functions: $X = H_0^1 \oplus \{c\}$. If u is some element of X, we write $u = \tilde{u} + u|_{\partial\Omega}$ for its decomposition. X is, provided with the topology inherited of H^1, a Hilbert space. Moreover, X is isomorphic to $H_0^1 \times \mathbb{R}$ and the H^1-norm is equivalent with the norm $\|\tilde{u}\|_{H_0^1} + |\, u|_{\partial\Omega} \,|$ on X. So we can realize the dual space X^* by

$$X^* = H^{-1} \times \mathbb{R}$$

the pairing being given by

$$<(w,k),u>_X = <w,\tilde{u}> + ku|_{\partial\Omega}.$$

Consider the functional W defined on X by

$$\text{(3.1)} \qquad W(u) = \begin{cases} \int H(u) - C\, u|_{\partial\Omega} & \text{if } H(u) \in L_1, \\ +\infty & \text{otherwise,} \end{cases}$$

where by definition

$$\text{(3.2)} \qquad H(y) = \int_0^y h(\eta)d\eta.$$

<u>LEMMA 3.1.</u>

$$W^*(w,k) = \begin{cases} \int H^*(w) & \textit{if } w \in L_1 \cap H^{-1},\ H^*(w) \in L_1 \textit{ and } \int w = k + C, \\ +\infty & \textit{otherwise.} \end{cases}$$

<u>PROOF</u>. The idea is to take first the supremum with respect to the H_0^1-component and to use Lemma 2.4.

$$\sup\{<w,\tilde{u}> + k\, u|_{\partial\Omega} - \int H(\tilde{u} + u|_{\partial\Omega}) + C\, u|_{\partial\Omega} \;\Big|\; \tilde{u} \in H_0^1,\ u|_{\partial\Omega} \in \mathbb{R}\}$$

$$= \begin{cases} \sup\{ \int H^*(w) - u|_{\partial\Omega} \int w + (k+C)\, u|_{\partial\Omega} \;\Big|\; u|_{\partial\Omega} \in \mathbb{R} \} \\ \qquad \text{if } w \in L_1 \cap H^{-1} \text{ and } H^*(w) \in L_1 \\ +\infty \qquad \text{otherwise} \end{cases}$$

$$= \begin{cases} \int H^*(w) & \text{if } w \in L_1 \cap H^{-1},\ H^*(w) \in L_1 \text{ and } \int w = k + C \\ +\infty & \text{otherwise.} \end{cases}$$

<u>LEMMA 3.2.</u>

$$\partial W(u) = \begin{cases} (h(u), \int h(u) - C) & \textit{if } h(u) \in H^{-1} \cap L_1 \\ \emptyset & \textit{otherwise.} \end{cases}$$

<u>PROOF</u>. (i) Let $(w,k) \in \partial W(u)$ then

$$W(\tilde{v} + v|_{\partial\Omega}) - W(\tilde{u} + u|_{\partial\Omega}) \geq \langle w, \tilde{v} - \tilde{u}\rangle + k(v-u)|_{\partial\Omega}$$

for all $\tilde{v} \in H_0^1$ and all $v|_{\partial\Omega} \in \mathbb{R}$. By first taking $v|_{\partial\Omega} = u|_{\partial\Omega}$, we see that necessarily w belongs to the subdifferential of the functional $\tilde{u} \to W(\tilde{u}+u|_{\partial\Omega})$ defined on H_0^1. Hence, by Lemma 2.5, $w = h(u)$ and $w \in L_1$. Next, a combination of Lemma 2.1 and Lemma 3.1 shows that necessarily $k = \int w - C = \int h(u) - C$.
(ii) Conversely, let $h(u) \in H^{-1} \cap L_1$. Since h is the derivative of H we have

$$H(v) - H(u) \geq h(u)(v-u) = h(u)(\tilde{v} - \tilde{u} + (v-u)|_{\partial\Omega}).$$

So if $H(v)$ and $H(u) \in L_1$, we can invoke Lemma 2.6 and conclude that $h(u)(\tilde{v} - \tilde{u}) \in L_1$ and that the integral equals the duality pairing. Integration of the inequality then yields, after adding a term $-C(v-u)|_{\partial\Omega}$,

$$W(v) - W(u) \geq \langle h(u), \tilde{v} - \tilde{u}\rangle + (\int h(u) - C)(v-u)|_{\partial\Omega}. \qquad \square$$

We remark that, by Lemma 2.2, $\partial H^* = h^{-1}$. So, since h is strictly monotone,

$$(3.3)\qquad H^*(y) = \int_0^y h^{-1}(\eta)d\eta.$$

Let $g \in (L_2)^n$ be such that div g = f. The functional $G : (L_2)^n \to \mathbb{R}$ defined by

$$(3.4)\qquad G(p) = \int (\tfrac{1}{2}p^2 + g.p)$$

is Fréchet-differentiable with derivative p+g. The polar functional $G^* : (L_2)^n \to \mathbb{R}$ is given by

$$(3.5)\qquad G^*(p) = \tfrac{1}{2}\int (p-g)^2$$

and its derivative is p-g.

We define the bounded linear mapping $T : X \to (L_2)^n$ by

$$(3.6)\qquad T\,u = -\text{ grad } u.$$

Its adjoint $T^* : (L_2)^n \to X^*$ is given by

$$(3.7)\qquad T^*p = (\text{div } p, 0).$$

Clearly the functional $u \mapsto G(-Tu)$ defined on X is differentiable with derivative $-T^*G'(-Tu) = (-\Delta u - f, 0)$.

Finally, let us put together the materials constructed above. Define $V_\varepsilon : X \to (-\infty, +\infty]$ by

$$(3.8)\qquad V_\varepsilon(u) = G(-Tu) + \varepsilon W(\tfrac{u}{\varepsilon}).$$

Then

$$(3.9)\qquad \partial V_\varepsilon(u) = \begin{cases} (-\Delta u - f + h(\frac{u}{\varepsilon}), \int h(\frac{u}{\varepsilon}) - C) & \text{if } h(\frac{u}{\varepsilon}) \in H^{-1} \cap L_1 \\ \emptyset & \text{otherwise} \end{cases}$$

and, consequently, the problem BVP is equivalent with the variational problem

$$\text{VP} \qquad \operatorname*{Inf}_{u\in X} V_\varepsilon(u).$$

<u>THEOREM 3.3</u>. VP *has a unique solution* u_ε.

<u>PROOF</u>. G is convex, W is strictly convex and both functionals are l.s.c. (by Fatou's lemma). It remains to verify that V_ε is coercive on X. It is convenient to rewrite the functional V_ε as

$$V_\varepsilon(u) = \int (\tfrac{1}{2}(\mathrm{grad}u)^2 + (g-a)\,.\,\mathrm{grad}u + \varepsilon\, H(\tfrac{u}{\varepsilon}) - \tfrac{C}{|\Omega|}\, u)$$

where $|\Omega|$ denotes the measure of Ω and a is such that $\mathrm{div}a = C|\Omega|^{-1}$ (for instance take $a = C(n|\Omega|)^{-1}\,(x_1,\ldots,x_n))$. Since $C|\Omega|^{-1} \in (h(-\infty),h(+\infty))$, there exist positive constants δ and M_1 such that

$$\varepsilon\, H(\tfrac{y}{\varepsilon}) - \tfrac{C}{|\Omega|}\, y \geq \delta|y| - M_1.$$

By the inequalities of Hölder and Poincaré there exists a positive constant $M_2 = M_2(\Omega)$ such that

$$\int |\tilde{u}| \leq \sqrt{|\Omega|}\, \|\tilde{u}\|_{L_2} \leq M_2 \|\mathrm{grad}\ \tilde{u}\|_{L_2} = M_2 \|\mathrm{grad}\ u\|_{L_2}.$$

Hence, using Hölder's inequality once more, we find

$$\begin{aligned} V_\varepsilon(u) &\geq \tfrac{1}{2}\|\mathrm{grad}\,u\|^2_{L_2} - \|g-a\|_{L_2}\, \|\mathrm{grad}\ u\|_{L_2} + \delta|\Omega|\,|\,u|_{\partial\Omega}| - \delta \int |\tilde{u}| - M_1 \\ &\geq \tfrac{1}{4}\|\mathrm{grad}\ u\|^2_{L_2} + \delta|\Omega|\,|\,u|_{\partial\Omega}| - M_3 \end{aligned}$$

for some constant M_3. It should be noted that the right hand side is independent of ε. □

4. LIMITING BEHAVIOUR OF u_ε AS $\varepsilon \downarrow 0$

In this section we show that u_ε converges as $\varepsilon \downarrow 0$. The limit u_0 is characterized as the unique solution of a variational problem. Equivalently one can characterize u_0 by an operator inclusion relation if h is bounded and by a variational inequality if h is unbounded. It turns out that u_0 depends only on $h(\pm\infty)$, f and C.

As $\varepsilon \downarrow 0$, the function $h(\frac{y}{\varepsilon})$ converges in the sense of graphs to the multivalued function

$$(4.1) \qquad h_0(y) = \begin{cases} h(+\infty), & y > 0 \\ [h(-\infty), h(+\infty)], & y = 0 \\ h(-\infty), & y < 0. \end{cases}$$

We define

$$(4.2) \qquad H_0(y) = \begin{cases} h(+\infty)y, & y > 0 \\ 0\ , & y = 0 \\ h(-\infty)y, & y < 0 \end{cases}$$

<u>LEMMA 4.1</u>. $\varepsilon\, H(\frac{y}{\varepsilon})$ *converges monotonously increasing to* $H_0(y)$.

<u>PROOF</u>. $h(\frac{\eta}{\varepsilon})$ increases towards $h_0(\eta)$ for $\eta > 0$ and decreases towards $h_0(\eta)$ for $\eta < 0$. Since $\varepsilon\, H(\frac{y}{\varepsilon}) = \int_0^y h(\frac{\eta}{\varepsilon})d\eta$ we can use Lebesgue's monotone convergence theorem. □

We note that, by Dini's theorem, the convergence is uniform on compact subsets if h is bounded and, for instance, uniform on compact subsets of $(-\infty,0]$ if $h(-\infty) > -\infty$ and $h(+\infty) = +\infty$. Motivated by Lemma 4.1 we define

$$(4.3) \qquad W_0(u) = \begin{cases} \int H_0(u) - C\, u|_{\partial\Omega} & \text{if } H_0(u) \in L_1 \\ +\infty & \text{otherwise} \end{cases}$$

and we introduce the reduced variational problem

$$\text{RVP} \quad \operatorname*{Inf}_{u \in X} \quad G(-Tu) + W_0(u).$$

Exactly as in the proof of Theorem 3.3 it follows that RVP has a solution. The functional $G(-Tu) + W_0(u)$ is convex, but not strictly convex. Still we have

LEMMA 4.2. RVP *has a unique solution* u_0.

PROOF. Since $G(\text{grad}\, u)$ is strictly convex on H_0^1, two minimizers can only differ by a constant. For arbitrary $u \in X$ define

$$\Omega_+(u) = \{x \mid u(x) > 0\}, \ \Omega_0(u) = \{x \mid u(x) = 0\}, \ \Omega_-(u) = \{x \mid u(x) < 0\}.$$

Then

$$\lim_{\delta \downarrow 0} \frac{1}{\delta}(W_0(u+\delta) - W_0(u)) = h(+\infty)|\Omega_+(u)| + h(+\infty)|\Omega_0(u)| + h(-\infty)|\Omega_-(u)| - C$$

and

$$\lim_{\delta \uparrow 0} \frac{1}{\delta}(W_0(u+\delta) - W_0(u)) = h(+\infty)|\Omega_+(u)| + h(-\infty)|\Omega_0(u)| + h(-\infty)|\Omega_-(u)| - C.$$

So if $W_0(u+\ell)$ is constant for $|\ell| \le \eta$ then necessarily for those values of ℓ

$$h(+\infty)|\Omega_+(u+\ell)| + h(+\infty)|\Omega_0(u+\ell)| + h(-\infty)|\Omega_-(u+\ell)| =$$

$$h(+\infty)|\Omega_+(u+\ell)| + h(-\infty)|\Omega_0(u+\ell)| + h(-\infty)|\Omega_-(u+\ell)| = C.$$

Since $h(+\infty) > h(-\infty)$ this implies that

$$\{x \mid -\eta \le u(x) \le \eta\}$$

has measure zero. Then, however, u has to be sign-definite (this follows, for instance, from the connection between Sobolev and Beppo Levi spaces; see DENY & LIONS [14]) and we arrive at the conclusion that either $h(+\infty)|\Omega| = C$ or $h(-\infty)|\Omega| = C$. Finally, the compatibility condition excludes both of these possibilities. $\square$

THEOREM 4.3.

$$\lim_{\varepsilon\downarrow 0} \|u_\varepsilon - u_0\|_X = 0.$$

PROOF.

Step 1. We know that V_ε is coercive uniformly in ε (see the proof of Theorem 3.3). Hence $\|u_\varepsilon\|_X \le M$ for some constant M independent of ε and, consequently, the weak limit set of $\{u_\varepsilon\}$ is nonempty.

Step 2. Suppose $u_{\varepsilon_n} \rightharpoonup \bar{u}$ as $n \to +\infty$ and suppose that $h(+\infty) = +\infty$. We claim that $\bar{u} \le 0$. Define $Q_0^\delta = \{x \mid \bar{u}(x) \ge \delta > 0\}$ and $Q_n^\delta = \{x \in Q_0^\delta \mid u_{\varepsilon_n}(x) \ge \tfrac{1}{2}\delta\}$. Then

$$\int |u_{\varepsilon_n} - \bar{u}|^2 \ge \int_{Q_0^\delta\setminus Q_n^\delta} |u_{\varepsilon_n} - \bar{u}|^2 \ge \frac{\delta^2}{4}\, |Q_0^\delta\setminus Q_n^\delta|.$$

Hence, since $u_{\varepsilon_n} \to \bar{u}$ strongly in L_2, necessarily $|Q_n^\delta| \to |Q_0^\delta|$. Furthermore,

$$\varepsilon_n \int H\left(\frac{u_{\varepsilon_n}}{\varepsilon_n}\right) \ge \varepsilon_n \int_{Q_n^\delta} H\left(\frac{\delta}{2\varepsilon_n}\right) = \varepsilon_n\, H\left(\frac{\delta}{2\varepsilon_n}\right) |Q_n^\delta|$$

Since $\varepsilon_n \int H\left(\frac{u_{\varepsilon_n}}{\varepsilon_n}\right)$ is bounded uniformly in n and since $\varepsilon_n H\left(\frac{\delta}{2\varepsilon_n}\right) \to +\infty$ as $n \to +\infty$, necessarily $|Q_n^\delta| \to 0$ as $n \to +\infty$. So we must have $|Q_0^\delta| = 0$. Since $\delta > 0$ was arbitrary we conclude that $\bar{u} \le 0$. Similarly, $h(-\infty) = -\infty$ implies $\bar{u} \ge 0$.

Step 3. Suppose $u_{\varepsilon_n} \rightharpoonup \bar{u}$ as $n \to +\infty$. We claim that $V_{\varepsilon_n}(u_{\varepsilon_n}) \to V_0(\bar{u})$. From $V_{\varepsilon_n}(u_{\varepsilon_n}) - V_{\varepsilon_n}(\bar{u}) \ge \langle \partial V_{\varepsilon_n}(\bar{u}), u_{\varepsilon_n} - \bar{u}\rangle_X$ we obtain, using step 2,

$$V_{\varepsilon_n}(u_{\varepsilon_n}) - V_{\varepsilon_n}(\bar{u}) \ge \int (\operatorname{grad} \bar{u} + g)(\operatorname{grad} u_{\varepsilon_n} - \operatorname{grad} \bar{u})$$

$$+ \int \left(h\left(\frac{\bar{u}}{\varepsilon_n}\right)(u_{\varepsilon_n} - \bar{u})\right) - C(u_{\varepsilon_n} - \bar{u})|_{\partial\Omega}$$

Since the right-hand side converges to zero as $n \to +\infty$ we find

$$\liminf_{n\to+\infty} V_{\varepsilon_n}(u_{\varepsilon_n}) \geq \lim_{n\to\infty} V_{\varepsilon_n}(\bar{u}) = V_0(\bar{u}).$$

On the other hand, since u_{ε_n} minimizes V_{ε_n} and since $V_\varepsilon(v)$ is, for fixed v, monotone with respect to ε (Lemma 4.1), we have

$$V_{\varepsilon_n}(u_{\varepsilon_n}) \leq V_{\varepsilon_n}(\bar{u}) \leq V_0(\bar{u}).$$

<u>Step 4</u>. Suppose $u_{\varepsilon_n} \rightharpoonup \bar{u}$ as $n \to +\infty$. Then

$$V_{\varepsilon_n}(u_{\varepsilon_n}) \leq V_{\varepsilon_n}(u_0) \leq V_0(u_0)$$

and therefore $V_0(\bar{u}) \leq V_0(u_0)$. Hence $\bar{u} = u_0$.

<u>Step 5</u>. We now know that u_0 is the only point in the weak limit set of $\{u_\varepsilon\}$ and thus $u_\varepsilon \rightharpoonup u_0$ as $\varepsilon \downarrow 0$. From

$$\varepsilon \int \left(H\left(\frac{u_\varepsilon}{\varepsilon}\right) - H\left(\frac{u_0}{\varepsilon}\right)\right) \geq \int h\left(\frac{u_0}{\varepsilon}\right)(u_\varepsilon - u_0)$$

and Step 2 we conclude that

$$\liminf_{\varepsilon\downarrow 0} \int \varepsilon H\left(\frac{u_\varepsilon}{\varepsilon}\right) \geq \int H_0(u_0).$$

It then follows from the weak l.s.c. of G and Step 3 that necessarily $\|\operatorname{grad} u_\varepsilon\|_{L_2} \to \|\operatorname{grad} u_0\|_{L_2}$ as $\varepsilon \downarrow 0$. Consequently u_ε converges in fact strongly in X to u_0. □

In order to get more information about u_0 we first determine W_0^* and ∂W_0. We write $u \geq 0$ for some $u \in X$ if and only if $u(x) \geq 0$ for almost all $x \in \Omega$. Let C denote the closed, convex, positive cone corresponding to this ordering. By duality C induces a cone C^* in X^*: we write $(w,k) \geq 0$ if and only if $\langle (w,k),u\rangle_X \geq 0$ for all $u \in C$. For any $u \in X$ we define $u_+ = \max(u,0)$ and $u_- = \max(-u,0)$. Then $u_+ \in X$, $u_- \in X$ and at least one of these belongs to H_0^1 (see, for instance, KINDERLEHRER & STAMPACCHIA [23, Ch. II, Proposition 5.3]).

In the following we slightly abuse notation. But let us agree upon the convention that any inequality in which a quantity $+\infty$ appears is trivially fulfilled.

LEMMA 4.4.

$$W_0^*(w,k) = \begin{cases} 0 \quad \text{if both} & (h(+\infty)-w, h(+\infty)|\Omega| - C - k) \in C^* \\ & (w - h(-\infty), k - h(-\infty)|\Omega| + C) \in C^* \\ +\infty \quad \text{otherwise.} \end{cases}$$

PROOF.

$$\begin{aligned} W_0^*(w,k) &= \sup\{<(w,k),u>_X - \int h(+\infty)u_+ + \int h(-\infty)u_- + C\, u|_{\partial\Omega} \mid u \in X\} \\ &= \sup\{<(w-h(+\infty), k - h(+\infty)|\Omega| + C),\ u_+>_X \\ &\qquad - <(w-h(-\infty), k - h(-\infty)|\Omega| + C), u_->_X \mid u \in X\}. \qquad \square \end{aligned}$$

LEMMA 4.5. *Suppose* $-\infty < h(-\infty) < h(+\infty) < +\infty$ *then*

$$\partial W_0(u) = \{(w,k) \mid w \in L_1,\ w(x) \in h_0(u(x)) \text{ for a.e. } x \in \Omega,\ k = \int w - C\}.$$

PROOF. (i) Suppose $(w,k) \in \partial W_0(u)$. As in the proof of Lemma 3.2 it follows that $w \in L_1$ and $w(x) \in h_0(u(x))$ a.e.. Let v_n be the solution of

$$\begin{cases} -\frac{1}{n}\Delta v_n + v_n = 0 \\ v_n|_{\partial\Omega} = 1. \end{cases}$$

Then $v_n \geq 0$ and, as $n \to \infty$, v_n converges strongly in L_2 to zero. By Lemmas 2.1 and 4.4 we know that

$$<(h(+\infty) - w,\ h(+\infty)|\Omega| - C - k),\ v_n>_X \geq 0$$

and

$$<(h(-\infty) - w,\ h(-\infty)|\Omega| - C - k),\ v_n>_X \leq 0.$$

Taking into account that $w \in L_\infty$ (since $w \in h_0(u)$), we rewrite these inequalities as

$$\int (h(+\infty) - w)(v_n - 1) + h(+\infty)|\Omega| - C - k \geq 0$$

and

$$\int (h(-\infty) - w)(v_n - 1) + h(-\infty)|\Omega| - C - k \leq 0.$$

Upon passing to the limit $n \to +\infty$ we find that $\int w - C - k \geq 0$ and $\int w - C - k \leq 0$.

(ii) is exactly the same as the second part of the proof of Lemma 3.2. □

<u>COROLLARY 4.6</u>. *Suppose* $-\infty < h(-\infty) < h(+\infty) < +\infty$ *then* RVP *is equivalent with the reduced boundary value problem*

$$\text{RBVP} \quad \begin{cases} \Delta u + f \in h_0(u) \\ \int (\Delta u + f) = C \\ u|_{\partial\Omega} \text{ is constant (but unknown).} \end{cases}$$

Finally, let us consider a function h which is unbounded. We concentrate on the case $h(-\infty) > -\infty$ and $h(+\infty) = +\infty$. From the proof of Theorem 4.3 we know that $u_0 \leq 0$. Consequently RVP is equivalent to minimizing a differentiable functional on the cone $-\mathcal{C}$ and, therefore, with the variational inequality:

$$\text{VI} \quad \begin{cases} \text{Find } u \in -\mathcal{C} \text{ such that for all } v \in -\mathcal{C} \\ \langle(-\Delta u + h(-\infty) - f,\ h(-\infty)|\Omega| - C),\ v - u\rangle_X \geq 0. \end{cases}$$

Unfortunately we cannot use Lemma 2.5 in this situation (see, however, [20]) but still we have

<u>LEMMA 4.7</u>. *Suppose* $h(-\infty) > -\infty$ *and* $h(+\infty) = +\infty$. *Then*

$$\partial W_0(u) = \begin{cases} \{(w,k) \mid (w-h(-\infty),\ k - h(-\infty)|\Omega| + C) \in C^* \textit{ and} \\ \quad <(w-h(-\infty),\ k-h(-\infty)|\Omega| + C), u>_X = 0\} \textit{ if } -u \in C \\ \emptyset \textit{ otherwise.} \end{cases}$$

PROOF. This follows directly from Lemma 2.1, Lemma 4.4 and the fact that W_0 is linear on the negative cone. □

5. THE DUAL FORMULATION

So far we have used polar functionals repeatedly, but we have not yet given a systematic presentation of duality theory as applied to our problem. This will be done now. We follow closely EKELAND & TEMAM [17, Ch. III, section 4, in particular Remarque 4.2].

The dual formulation of VP, corresponding to the splitting $V_\varepsilon(u) = G(-Tu) + \varepsilon W(\frac{u}{\varepsilon})$, is given by

$$VP^* \qquad \operatorname*{Inf}_{p\in(L_2)^n} \ \varepsilon\, W^*(T^*p) + G^*(p).$$

Since VP is stable (use [17, Proposition III.2.3]), VP^* has a (unique) solution p_ε. Furthermore, the infima are equal to each other and u_ε and p_ε are related by the so-called extremality relations

$$(5.1) \qquad T^*p_\varepsilon = \partial W\left(\frac{u_\varepsilon}{\varepsilon}\right)$$

$$(5.2) \qquad p_\varepsilon = \partial G(-Tu_\varepsilon).$$

By Lemma 3.2 and (3.4) these can be rewritten as

$$(5.3) \qquad \operatorname{div} p_\varepsilon = h\left(\frac{u_\varepsilon}{\varepsilon}\right) \text{ and } \int h\left(\frac{u_\varepsilon}{\varepsilon}\right) = C$$

$$(5.4) \qquad p_\varepsilon = g + \operatorname{grad} u_\varepsilon.$$

Note that g is not uniquely determined by div g = f but that (5.3) and (5.4) define $p_\varepsilon - g$ and div p_ε unambiguously. One can view (5.3) and (5.4) as a canonical splitting of BVP into first order equations. Indeed, elimination of p_ε leads to BVP. On the other hand, we can also eliminate u_ε to find the subdifferential equation satisfied by p_ε:

$$\varepsilon\, T(\partial W)^{-1}(T^* p_\varepsilon) + p_\varepsilon = g \tag{5.5}$$

or, more explicitly,

$$\mathrm{BVP}^* \quad \begin{cases} -\varepsilon\, \mathrm{grad}(h^{-1}(\mathrm{div}\, p_\varepsilon)) + p_\varepsilon = g \\ \int \mathrm{div}\, p_\varepsilon = C \\ h^{-1}(\mathrm{div}\, p_\varepsilon) \in X \end{cases}$$

By Lemmas 2.2, 3.2 and [17, Proposition I.5.7] the operator A from $(L_2)^n$ into itself defined by

$$\begin{cases} Ap = -\mathrm{grad}(h^{-1}(\mathrm{div}\, p)) \\ \mathcal{D}(A) = \{p \in (L_2)^n \mid \mathrm{div}\, p \in L_1,\ \int \mathrm{div}\, p = C,\ \mathrm{div}\, p = h(u) \text{ for} \\ \qquad \text{some } u \in X\} \end{cases} \tag{5.6}$$

is the subdifferential of the convex l.s.c. functional $p \mapsto W^*(T^*p)$. Consequently, A is maximal monotone. (See Weyer [26] for related results). Rewriting (5.5) as

$$(\varepsilon A + I)p_\varepsilon = g \tag{5.7}$$

and invoking Lemma 2.3, we find that p_ε converges, as $\varepsilon \downarrow 0$, strongly in $(L_2)^n$ to the projection of g onto $\overline{\mathcal{D}(A)}$. It does not seem easy to characterize $\overline{\mathcal{D}(A)}$ directly from (5.6). Therefore we use duality theory once more, but now for the reduced problem.

The dual formulation of RVP is given by

$$\text{RVP}^* \quad \inf_{p \in (L_2)^n} \quad W_0^*(T^* p) + G^*(p).$$

By (3.5) and Lemma 4.4 the solution of RVP^* is the projection of g onto the closed convex set

(5.8) $$Q = \{p \in (L_2)^n \mid (h(+\infty) - \operatorname{div} p,\ h(+\infty)|\Omega| - C) \in \mathcal{C}^*$$
$$\text{and } (\operatorname{div} p - h(-\infty),\ C - h(-\infty)|\Omega|) \in \mathcal{C}^*\}$$

Denoting the (unique) solution of RVP^* by p_0, we have the extremality relations

(5.9) $$T^* p_0 \in \partial W_0(u_0)$$

(5.10) $$p_0 = \partial G(-T u_0).$$

The second one, $p_0 = g + \operatorname{grad} u_0$, is identical to the extremality relation $p_\varepsilon = g + \operatorname{grad} u_\varepsilon$. Hence the fact that u_ε converges strongly in X to u_0, implies that p_ε converges strongly in $(L_2)^n$ to p_0. So we find that p_ε converges to a limit which is at the same time the projection of g onto $\overline{\mathcal{D}(A)}$ and onto Q. Since g is an arbitrary element of $(L_2)^n$, necessarily $\overline{\mathcal{D}(A)} = Q$. Thus we have shown that (5.8) gives an explicit characterization of $\overline{\mathcal{D}(A)}$.

The extremality relation (5.9) is easy to work with only in the case that h is bounded (see Lemmas 4.5 and 4.7). It then follows that RBVP is equivalent to (5.9) - (5.10). Likewise one can, by elimination of u_0, derive a subdifferential equation for p_0 similar to BVP^*.

If $h(-\infty) > -\infty$ and $h(+\infty) = +\infty$ we deduce from Lemma 4.7 that u_0 is the solution of the following variant of VI:

$$\begin{cases} \text{Find } u \in -\mathcal{C} \text{ such that} \\ \text{(i)} \quad \langle(-\Delta u + h(-\infty) - f,\ h(-\infty)|\Omega| - C),\ v\rangle_X \le 0, \ \forall v \in \mathcal{C}, \\ \text{(ii)} \quad \langle(-\Delta u + h(-\infty) - f,\ h(-\infty)|\Omega| - C),\ u\rangle_X = 0. \end{cases}$$

6. THE REDUCED PROBLEM AS A FREE BOUNDARY PROBLEM

In this section we assume that $f \in L_\infty$. We shall deal with the regularity of u_0 (and u_ε), with the free boundary value problem satisfied by u_0 and with sharp convergence results versus the occurrence of boundary layers. We shall write $C^{1,\alpha}$ to denote the Hölder space $C^{1,\alpha}(\bar{\Omega})$ and $W^{2,p}$ to denote the usual Sobolev space. We recall that $W^{2,p}$ is imbedded into $C^{1,\alpha}$ if $p(1-\alpha) \geq n$.

THEOREM 6.1. *If* h *is bounded,* u_ε *converges to* u_0 *weakly in* $W^{2,p}$ *for each* $p \geq 1$ *and strongly in* $C^{1,\alpha}$ *for each* $\alpha \in [0,1)$.

PROOF.

$$\|\Delta u_\varepsilon\|_{L_\infty} \leq \max\{-h(-\infty), h(+\infty)\} + \|f\|_{L_\infty}. \qquad \square$$

We can now interpret RBVP as a free boundary problem. The domain Ω consists of three subdomains:

$$\Omega_+ = \{x \in \Omega \mid u_0(x) > 0\} \text{ where } -\Delta u_0 + h(+\infty) = f \quad \text{a.e.}$$

$$\Omega_- = \{x \in \Omega \mid u_0(x) < 0\} \text{ where } -\Delta u_0 + h(-\infty) = f \quad \text{a.e.}$$

$$\Omega_0 = \{x \in \Omega \mid u_0(x) = 0\} \text{ which has to be a subset of}$$

$$\{x \in \Omega \mid h(-\infty) \leq f(x) \leq h(+\infty)\}.$$

These subdomains are unknown, possibly empty and such that

$$h(+\infty)|\Omega_+| + h(-\infty)|\Omega_-| + \int_{\Omega_0} f = C.$$

From the proof of Theorem 4.3 we know that $u_0 = 0$ if $h(\pm\infty) = \pm\infty$. So in that case we cannot have convergence in $W^{2,p}$ unless $\int f = C$.

Next, we concentrate on the most interesting case in which h is bounded from one and only one side. In the remaining part of this section we assume that $h(-\infty) > -\infty$ and $h(+\infty) = +\infty$. We emphasize that all theorems below have a counterpart in the case $h(-\infty) = -\infty$ and $h(+\infty) < +\infty$.

<u>THEOREM 6.2.</u> $u_\varepsilon \in W^{2,p}$ *for each* $p \geq 1$.

<u>PROOF</u>. We shall show that Δu_ε is bounded by finding an upper bound for u_ε. Let $\zeta \in H_0^1$ be the solution of $-\Delta\zeta + h(-\infty) = f$. Then, in fact, since $\Delta\zeta$ is bounded, we have $\zeta \in C^{1,\alpha}$. Define $\psi \in H_0^1$ by $\psi = u_\varepsilon - u_\varepsilon|_{\partial\Omega} - \zeta$. Then

$$\Delta\psi = \Delta u_\varepsilon - \Delta\zeta = h\left(\frac{u_\varepsilon}{\varepsilon}\right) - h(-\infty) \geq 0$$

and hence, by the weak maximum principle, $\psi \leq 0$. So u_ε is bounded from above by the bounded function $u_\varepsilon|_{\partial\Omega} + \zeta$. □

<u>THEOREM 6.3.</u> *If* $C \leq \int f$, u_ε *converges to* u_0 *weakly in* $W^{2,p}$ *for each* $p \geq 1$ *and strongly in* $C^{1,\alpha}$ *for each* $\alpha \in [0,1)$.

<u>PROOF</u>. We show that $h(\frac{u_\varepsilon}{\varepsilon})$ and hence Δu_ε is bounded. Choose $\delta > 0$ and define

$$\Omega_\varepsilon = \{x \in \Omega \,|\, h\left(\frac{u_\varepsilon(x)}{\varepsilon}\right) > \|f\|_{L_\infty} + \delta\}.$$

The points of $\partial\Omega_\varepsilon$ either belong to $\partial\Omega$ or are such that $h(\frac{u_\varepsilon(x)}{\varepsilon}) = \|f\|_{L_\infty} + \delta$. If $|\Omega_\varepsilon| \neq 0$ and $\partial\Omega_\varepsilon \cap \partial\Omega = \emptyset$, we find that simultaneously $\Delta u_\varepsilon > 0$ in Ω_ε and u_ε assumes, with respect to Ω_ε, its maximum in an interior point. Since this is impossible we conclude that either $|\Omega_\varepsilon| = 0$ or $\partial\Omega_\varepsilon \cap \partial\Omega \neq \emptyset$ and u_ε assumes its maximum at $\partial\Omega$ with $h(\frac{u_\varepsilon|_{\partial\Omega}}{\varepsilon}) > \|f\|_{L_\infty} + \delta$.

Suppose $|\Omega_\varepsilon| \neq 0$. Let $\tilde{\Omega}_\varepsilon$ be a domain with boundary $\partial\Omega \cup \Gamma$ and strictly contained in Ω_ε. We define $\tilde{u}_\varepsilon$ to be the solution of $\Delta\tilde{u} = \delta$, $\tilde{u}(x) = u_\varepsilon(x)$, $x \in \partial\tilde{\Omega}_\varepsilon$. Then $\tilde{u}_\varepsilon$ attains its maximum on $\partial\Omega$ and it follows from the Hopf maximum principle [24, Thm 7, p. 65] that $\frac{\partial\tilde{u}_\varepsilon}{\partial n}|_{\partial\Omega} > 0$. Also we have that $\Delta(\tilde{u}_\varepsilon - u_\varepsilon) = \delta - h(\frac{u_\varepsilon}{\varepsilon}) + f \leq 0$ and therefore $\tilde{u}_\varepsilon - u_\varepsilon \geq 0$ and, finally,

$$\frac{\partial u_\varepsilon}{\partial n}\Big|_{\partial\Omega} \geq \frac{\partial\tilde{u}_\varepsilon}{\partial n}\Big|_{\partial\Omega} > 0.$$

This leads to the contradiction

$$C - \int f = \int \Delta u_\varepsilon = \int_{\partial\Omega} \frac{\partial u_\varepsilon}{\partial n} > 0. \qquad \square$$

The proof above shows that, if $h(\frac{u_\varepsilon}{\varepsilon})$ blows up somewhere, it does so at the boundary. If $u_0|_{\partial\Omega} < 0$ this can not happen, so we also have

THEOREM 6.4. *If* $u_0|_{\partial\Omega} < 0$ *then* u_ε *converges to* u_0 *weakly in* $W^{2,p}$, $p \geq 1$, *and strongly in* $C^{1,\alpha}$, $\alpha \in [0,1)$.

THEOREM 6.5. $u_0 \in W^{2,p}$ *for each* $p \geq 1$.

PROOF. If $u_0|_{\partial\Omega} < 0$ we can apply Theorem 6.4. If $u_0|_{\partial\Omega} = 0$, then u_0 is completely characterized by the restriction of RVP to H_0^1. The result then follows, for instance, from Appendix 1. □

THEOREM 6.6. u_0 *is completely characterized by*

$$\begin{cases} -\Delta u_0 + h(-\infty) - f \leq 0 & \text{a.e.} \\ u_0 \leq 0 & \text{a.e.} \\ u_0(-\Delta u_0 + h(-\infty) - f) = 0 & \text{a.e.} \\ \int(\Delta u_0 + f) - C \leq 0 & \\ u_0|_{\partial\Omega}(\int(\Delta u_0+f) - C) = 0. & \end{cases}$$

PROOF. Because of Theorem 6.5 we can rewrite the variant of VI given at the end of section 5 in the form

$$\int(\Delta u_0 - h(-\infty) + f)v + (C - \int(\Delta u_0 + f))v|_{\partial\Omega} \geq 0, \qquad \forall v \in C,$$

$$\int(\Delta u_0 - h(-\infty) + f)u_0 + (C - \int(\Delta u_0 + f))u_0|_{\partial\Omega} = 0,$$

and from this formulation the result easily follows. □

If $\int f \geq C$ then Theorem 6.3 implies that actually $\int(\Delta u_0+f) = C$. We emphasize that $\int f < C$ does not preclude the possibility that $u_0|_{\partial\Omega} < 0$ and $\int(\Delta u_0+f) = C$. However, if $\int(\Delta u_0+f) < C$ we cannot have weak convergence in $W^{2,p}$. Next, we present some conditions on the data $h(-\infty)$, f and C under which this happens.

THEOREM 6.7. *Any of the three assumptions*

(i) $f(x) \le h(-\infty)$ a.e.

(ii) $f(x) \ge h(-\infty)$ a.e. *and* $\int f < C$

(iii) $\int_{\tilde\Omega} f < C$ *for all* $\tilde\Omega \subset \Omega$

implies that $\int(\Delta u_0 + f) < C$.

PROOF. (i) Let $v \in H_0^1$ be the solution of $\Delta v = h(-\infty) - f$. Then $v \le 0$ and $\int(\Delta v + f) = h(-\infty)|\Omega| < C$. By Theorem 6.6 $u_0 = v$.

(ii) Again by Theorem 6.6, $u_0 = 0$.

(iii)

$$\int(\Delta u_0 + f) = \int_{\bar\Omega} h(-\infty) + \int_{\Omega\setminus\bar\Omega} f = h(-\infty)|\bar\Omega| + \int_{\Omega\setminus\bar\Omega} f < C$$

where $\bar\Omega = \{x \mid u_0(x) < 0\}$. □

In the proof of Theorem 6.3 it was already shown that if u_ε displays a layer of rapid change somewhere, it certainly does so near to the boundary. Next we prove that it can do so *only* near to the boundary. The estimates below have been indicated to us by H. BREZIS.

THEOREM 6.8. *Assume* h *is* C^1. *Then* u_ε *converges to* u_0 *weakly in* $W^{2,p}(\mathcal{O})$ *for any open set* $\mathcal{O}$ *with* $\bar{\mathcal{O}} \subset \Omega$ *and any* $p \ge 1$.

PROOF.

Step 1. Since $h(y) > h(-\infty)$ we have

$$\int \left|h\left(\frac{u_\varepsilon}{\varepsilon}\right)\right| \le \int h\left(\frac{u_\varepsilon}{\varepsilon}\right) - 2h(-\infty)|\Omega| = C - 2h(-\infty)|\Omega|.$$

Step 2. Since u_ε is bounded uniformly in ε in H^1, it follows from the Sobolev imbedding theorem (see, for instance, ADAMS [1, p. 97]) that u_ε is bounded uniformly in ε in $L_r(\Omega)$, where $r = \frac{2n}{n-2}$ if $n > 2$ and $r \ge 1$ if $n \le 2$.

Step 3. (Proof by recursion). We suppose that $h(\frac{u_\varepsilon}{\varepsilon})$ is bounded uniformly in ε in $L_q(\mathcal{U}_1)$ for some $q \ge 1$ and $\mathcal{U}_1$ such that $\overline{\mathcal{U}_1} \subset \Omega$. Let ζ be a C^∞-function with compact support in $\mathcal{U}_1$. We multiply the differential equation by $|h(\frac{u_\varepsilon}{\varepsilon})|^{t-2} h(\frac{u_\varepsilon}{\varepsilon}) |\zeta|^t$ and we integrate. Thus we obtain

$$\int -\Delta u_\varepsilon \, |h(\frac{u_\varepsilon}{\varepsilon})|^{t-2} h(\frac{u_\varepsilon}{\varepsilon})|\zeta|^t + \int |h(\frac{u_\varepsilon}{\varepsilon})\zeta|^t \le \int |f\zeta| \, |h(\frac{u_\varepsilon}{\varepsilon})\zeta|^{t-1}.$$

Integrating the first term by parts and using the inequality $ab \le \frac{1}{\alpha} a^\alpha + \frac{1}{\beta} b^\beta$ with $a,b > 0$, $\alpha,\beta > 1$ and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, for the term at the right hand side we deduce

$$\frac{t-1}{\varepsilon} \int |\mathrm{grad}\, u_\varepsilon|^2 \, |\zeta|^t \, |h(\frac{u_\varepsilon}{\varepsilon})|^{t-2} \, h'(\frac{u_\varepsilon}{\varepsilon}) + \frac{1}{t} \int |h(\frac{u_\varepsilon}{\varepsilon})\zeta|^t$$

$$\le \frac{1}{t} \int |f\zeta|^t - \int |h(\frac{u_\varepsilon}{\varepsilon})|^{t-2} \, h(\frac{u_\varepsilon}{\varepsilon}) \, \mathrm{grad}\, u_\varepsilon \cdot \mathrm{grad}\, |\zeta|^t.$$

We observe that the first term at the left hand side is nonnegative (so we delete this term). Now let $\gamma(x) = |h(x)|^{t-2} h(x)$ and $\Gamma(x) = \int_0^x \gamma(\tau)d\tau$. Then $\Gamma(x) \le x\gamma(x)$ for all x and hence

$$-\int \gamma(\frac{u_\varepsilon}{\varepsilon}) \mathrm{grad}\, u_\varepsilon \cdot |\mathrm{grad}\, \zeta|^t = \varepsilon \int \Gamma(\frac{u_\varepsilon}{\varepsilon}) \Delta |\zeta|^t \le \int u_\varepsilon \gamma(\frac{u_\varepsilon}{\varepsilon}) \Delta |\zeta|^t$$

So finally

$$(6.1) \qquad \int |h(\frac{u_\varepsilon}{\varepsilon})\zeta|^t \le K_1 + K_2 \int_{\mathcal{U}_1} |u_\varepsilon| \, |h(\frac{u_\varepsilon}{\varepsilon})|^{t-1}.$$

We now distinguish different cases:

<u>1st case $q = 1$</u>. If $n > 2$, we choose $t = 1 + \frac{n+2}{2n}$ in (6.1) and apply Hölder's inequality with conjugate exponents $\frac{2n}{n-2}$ and $\frac{2n}{n+2}$; also using the results of Steps 1 and 2 we deduce that $\int |h(\frac{u_\varepsilon}{\varepsilon})\zeta|^t$ is bounded uniformly in ε. If $n \le 2$, we choose $t = 1 + \frac{r-1}{r}$ for some $r > 1$ and apply Hölder's inequality with conjugate exponents r and $\frac{r}{r-1}$ to obtain a similar result. So we know in both cases that $h(\frac{u_\varepsilon}{\varepsilon})$ is bounded uniformly in $L^t(\mathcal{U}_2)$ for some $t > 1$ and any open set $\mathcal{U}_2$ with $\overline{\mathcal{U}_2} \subset \mathcal{U}_1$. Consequently u_ε is bounded uniformly in $W^{2,t}(\mathcal{U}_2)$ (cf. AGMON [2]).

<u>2nd case $q > \frac{n}{2}$</u>. It follows from the Sobolev imbedding theorem that u_ε is bounded uniformly in ε in $L_\infty(\mathcal{U}_1)$. Choosing $t = q+1$ in (6.1), we deduce that $h(\frac{u_\varepsilon}{\varepsilon})$ is bounded uniformly in ε in $L^{q+1}(\mathcal{U}_2)$. The result of the theorem follows then from a bootstrap argument.

<u>3rd case $q \le \frac{n}{2}$</u>. By the Sobolev imbedding theorem u_ε is bounded uniformly in $L_{q^*}(\mathcal{U}_1)$ with $\frac{1}{q^*} = \frac{1}{q} - \frac{2}{n}$ (or $\frac{1}{q^*} = \frac{1}{q} - \alpha$ for any $\alpha \in (0,\frac{1}{q})$ if $q = \frac{2}{n}$). Let q^{**} be the conjugate exponent of q^* and choose $t = 1 + \frac{q}{q^{**}}$. Applying Hölder's inequality (with exponents q^* and q^{**}) to (6.1) we deduce that $h(\frac{u_\varepsilon}{\varepsilon})$ is bounded uniformly in $L^t(\mathcal{U}_2)$. Now a bootstrap argument either yields the result or leads to the 2nd case. □

7. THE ONE DIMENSIONAL CASE

Again we assume that $h(-\infty) > -\infty$ and $h(+\infty) = +\infty$. The results of section 5 imply that p_0 is the projection of g onto the set

$$\overline{\mathcal{D}(A)} = \{p \in L_2 \mid (p' - h(-\infty),\ C - h(-\infty)|\Omega|) \in C^*\}.$$

A simple calculation shows that, with $\Omega = (-1,+1)$,

$$\overline{\mathcal{D}(A)} \cap H^1 = \{p \mid p' \ge h(-\infty) \text{ and } p(1) - p(-1) \le C\}.$$

We found in section 6 that $p_0 \in \overline{\mathcal{D}(A)} \cap H^1$ if $f \in L_\infty$. So we can find p_0 by minimizing the L_2-distance to g subject to two constraints: an inequality for the derivative and a bound for the total variation. This is more or less a combinatorial problem which is rather easy to solve for some given smooth g, but whose general solution is cumbersome. We refer to [16, section 4] for a more detailed discussion of the symmetric case, noting that the result presented there covers the general case after some minor modifications. Finally, we remark that, once p_0 is found, u_0 can be calculated from the extremality relations.

APPENDIX 1. THE HOMOGENEOUS DIRICHLET PROBLEM

In this appendix we present some results about the problem

$$-\Delta u + h(\tfrac{u}{\varepsilon}) \ni f,$$

where by assumption h is the subdifferential of a convex, l.s.c. function $H : \mathbb{R} \to [0,\infty)$, with $H(0) = 0$ and $H(y) < +\infty$ for all $y \in \mathbb{R}$. Here $f \in H^{-1}$ is given and $u \in H_0^1$ is sought. We use some of the notation defined in the preceding pages and omit all proofs since these are similar to (and in fact easier than) those already given. In contravention of prior definitions we now have:

$$T : H_0^1 \to (L_2)^n, \qquad Tu = -\text{grad } u$$

$$T^*: (L_2)^n \to H^{-1}, \qquad T^*p = \text{div } p$$

$$W : H_0^1 \to [0,\infty], \qquad W(u) = \begin{cases} \int H(u) & \text{if } H(u) \in L_1 \\ +\infty & \text{otherwise.} \end{cases}$$

The problem can be rewritten as

$$\partial V_\varepsilon(u) \ni 0$$

where

$$V_\varepsilon(u) = G(-Tu) + \varepsilon\, W(\tfrac{u}{\varepsilon}).$$

It admits a unique solution u_ε which converges as $\varepsilon \downarrow 0$ strongly in H_0^1 to u_0, the unique solution of

$$\operatorname*{Inf}_{u \in H_0^1} \; G(-Tu) + W_0(u).$$

If h is bounded u_0 satisfies

$$-\Delta u + h_0(u) \ni f$$

and if, for instance, $h(-\infty) > -\infty$ and $h(+\infty) = +\infty$ then u_0 solves the variational inequality: find $u \le 0$ such that

$$\langle -\Delta u + h(-\infty) - f,\ v-u \rangle \ge 0, \qquad \forall v \le 0.$$

The dual formulation is obtained by the transformations

$$p = g - Tu$$
$$u \in \varepsilon h^{-1}(T^* p)$$
$$f = T^* g$$

and reads

$$\varepsilon\, T(h^{-1}(T^* p)) + p \ni g$$

or, equivalently,

$$(\varepsilon A + I)p \ni g$$

where $A : (L_2)^n \to (L_2)^n$ is defined by

$$Ap = T(h^{-1}(T^* p))$$

$$\mathcal{D}(A) = \{p \in (L_2)^n \mid T^* p \in L_1 \text{ and there exists } u \in H_0^1 \text{ such that } T^* p \in h(u)\}.$$

As $\varepsilon \downarrow 0$, p_ε converges to the projection of g onto

$$\overline{\mathcal{D}(A)} = \{p \in (L_2)^n \mid h(-\infty) \le T^* p \le h(+\infty)\},$$

where the inequalities are defined by the positive cone in H_0^1 and the duality of H_0^1 and H^{-1}.

If $f \in L_\infty$, u_ε converges to u_0 weakly in $W^{2,p}$ for each $p \ge 1$ and strongly in $C^{1,\alpha}$ for each $\alpha \in [0,1)$. This follows most easily from the observation that, by the maximum principle, u_ε equals the solution of the "truncated" problem

$$-\Delta u + \tilde{h}\left(\frac{u}{\varepsilon}\right) \ni f$$

where

$$\tilde{h}(y) = \begin{cases} \|f\|_{L_\infty} & \text{if } h(y) \geq \|f\|_{L_\infty} \\ h(y) & \text{if } -\|f\|_{L_\infty} \leq h(y) \leq \|f\|_{L_\infty} \\ -\|f\|_{L_\infty} & \text{if } h(y) \leq -\|f\|_{L_\infty}. \end{cases}$$

For sharper estimates under additional assumptions we refer to [7], [8], [5] and [25].

APPENDIX 2. THE PHYSICAL BACKGROUND OF THE PROBLEM

Consider a bounded domain Ω in $\mathbb{R}^2$ or $\mathbb{R}^3$ and a charge distribution inside Ω with two components:

(i) a fixed ionic charge density en_i

(ii) a mobile electronic charge density $-en_e$ such that

(A.1) $$\int n_e = N_e.$$

Here e is the unit charge, n_i and n_e are number densities and N_e is a number. N_e and n_i are given, but n_e is unknown.

Let the region outside Ω be a conductor. Then we have the condition

(A.2) the potential Φ is constant outside Ω.

Physically this condition is realized by the formation of a surface charge density which, however, will be of no further concern.

The equation for the potential Φ in Ω can be deduced from two physical laws:

(A.3) $$\Delta\Phi = -4\pi e(n_i - n_e),$$ Poisson's equation,

and

(A.4) $$n_e = K\, e^{\frac{e\Phi}{k_B T}},$$ Boltzmann's formula.

Here K is a normalization constant, T is the temperature of the system and k_B is Boltzmann's constant.

Substituting (A.4) into (A.3) and (A.1) we obtain the problem

$$\begin{cases} -\Delta\Phi + 4\pi e K\, e^{\frac{e\Phi}{k_B T}} = 4\pi e\, n_i \\ \\ K \displaystyle\int e^{\frac{e\Phi}{k_B T}} = N_e \\ \\ \Phi|_{\partial\Omega} \text{ is constant (but unknown)} \end{cases}$$

which, up to a renaming of the constants and variables, is the special case of BVP in which $h(y) = e^y - 1$.

Alternatively, one can argue that n_e should be such that the free energy F of the system be minimized under the constraint (A.1). The free energy is defined by

$$F = U - TS$$

where U is the electrostatic energy given by

$$U = \frac{1}{8\pi}\int (\text{grad } \Phi)^2,$$

T is the temperature and S the entropy given by

$$S = -\,k_B \int n_e \,\ln n_e .$$

So if E_i denotes the electric field created by the ions and E_e the electric field created by the electrons, it comes to solve the minimization problem

$$\operatorname*{Inf}_{E_e}\; k_B T \int \text{div } E_e \,\ln\,(\text{div } E_e) + \frac{1}{8\pi}\int (E_i - E_e)^2$$

subject to the constraint

$$\int \operatorname{div} E_e = N_e .$$

Clearly this problem corresponds to VP^*.

The main results of this paper concern the limiting behaviour of the potential Φ and the electrical field E_e due to the electrons, as the temperature T tends to zero. For instance, we find that at $\partial\Omega$ no boundary layer occurs if the total charge density $\int n_i$ of the ions exceeds N_e. In the limit $T \to 0$ there may be regions where electrons are absent. If such a region $\bar{\Omega}$ is strictly contained in Ω it necessarily must be such that $\int_{\bar{\Omega}} n_i = 0$. For such a region which extends up to $\partial\Omega$ there is a more complicated condition. If $n_i \geq 0$ and $\int n_i < N_e$, necessarily a boundary layer arises: the electrons are repelled against the conductor.

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SAMENVATTING

Het in dit proefschrift bestudeerde probleem vindt zijn oorsprong in de fysica van de geioniseerde gassen. Verscheidene aspecten van dit probleem worden bestudeerd in een vijftal artikelen die ieder als een hoofdstuk in dit proefschrift zijn opgenomen. Hieraan vooraf gaat een inleiding waarin de onderlinge samenhang wordt besproken.

Deze artikelen zijn:

1. *Rigorous results on a time dependent inhomogeneous Coulomb gas problem*, Phys. Lett. 84A (1981) 424-426, met H.J. Hilhorst en E. Marode.
2. *A singular boundary value problem arising in a pre-breakdown gas discharge*, SIAM J. Appl. Math. 39 (1980) 48-66, met O. Diekmann en L.A. Peletier.
3. *A nonlinear evolution problem arising in the physics of ionized gases*, SIAM J. Math. Anal. 13 (1982).
4. *How many jumps? Variational characterization of the limit solution of a singular perturbation problem*, in Geometrical Approaches to Differential Equations, R. Martini ed., Lecture Notes in Mathematics 810, Springer 1980, met O. Diekmann (in een gemodificeerde versie).
5. *Variational analysis of a perturbed free boundary problem*, voor publicatie aangeboden, met O. Diekmann.

In de inleiding laten we zien hoe de diverse in dit proefschrift bestudeerde problemen uit één fysisch model kunnen worden afgeleid. Cylindrische symmetrie in de experimentele situatie leidt tot de bestudering van een niet-lineaire parabolische differentiaalvergelijking in één tijd- en één ruimte-dimensie. Deze vergelijking is ontaard in de oorsprong. Met de stationaire oplossing correspondeert een gewone niet-lineaire differentiaalvergelijking. Zonder de aanname van cylindrische symmetrie kan het stationaire probleem geformuleerd worden als een partiële differentiaalvergelijking met een monotone niet-lineariteit.

In Hoofdstuk 1 worden de resultaten van de latere hoofdstukken 2 - 4 in fysische termen samengevat. We bediscussiëren in het bijzonder het ontsnappen van electronen naar het oneindige boven een kritieke temperatuur, en de grenslaag in de electronendichtheid bij lage temperatuur.

In Hoofdstuk 2 bestuderen we het tweepunts niet-lineaire randwaardeprobleem $\varepsilon xy'' + (g(x) - y)y' = 0$, $y(0) = 0$, $y(R) = k$, waarin de functie g voldoende glad, strikt stijgend, en strikt concaaf is. We tonen aan dat dit probleem een unieke oplossing y heeft die naar een limiet $\bar{y}$ convergeert als R naar oneindig gaat. Er blijkt eveneens dat voor $\varepsilon \downarrow 0$ y naar de limietfunctie $\min(g(x),k)$ convergeert. De belangrijkste wiskundige methoden bij de behandeling van dit probleem zijn maximumprincipe-argumenten en de constructie van boven- en onderoplossingen.

In Hoofdstuk 3 beschouwen we het niet-lineaire evolutieprobleem $u_t = \varepsilon xu_{xx} + (g(x) - u)u_x$, $u(0,t) = 0$, $u(x,0) = \psi(x)$, waarin de beginfunctie ψ een gladde niet-dalende functie is. We tonen aan dat dit probleem een unieke klassieke oplossing heeft die naar $\bar{y}$ convergeert als t naar oneindig gaat. We analyseren vervolgens de convergentiesnelheid. Als g voldoende snel naar oneindig gaat blijkt $\bar{y}$ exponentieel stabiel te zijn; in het meer algemene geval dat $\varepsilon < g(\infty) - k$ bewijzen we dat u algebraïsch snel naar zijn stationaire toestand convergeert.

In Hoofdstuk 4 gebruiken we de theorie van de maximale monotone operatoren om het limietgedrag voor $\varepsilon \downarrow 0$ te beschrijven van de oplossing van het niet-lineaire tweepunts randwaardeprobleem $\varepsilon y'' + (g-y)y' = 0$, $y(0) = 0$, $y(1) = 1$, waarin g een gegeven functie is. We geven een karakterisering van de limiet en presenteren enkele concrete voorbeelden.

In Hoofdstuk 5 bestuderen we een niet-lineair randwaardeprobleem met de partiële differentiaalvergelijking $-\Delta u + h(\frac{u}{\varepsilon}) = f$ waarin de niet-lineaire functie h strikt stijgend is. Met gebruik van convexe analyse bewijzen we dat dit probleem een unieke oplossing u_ε heeft en tonen aan dat voor $\varepsilon \downarrow 0$ u_ε convergeert naar een limiet u_0 die de oplossing is van een vrij randwaardeprobleem. Het mogelijke voorkomen van een grenslaag aan de rand van het domein wordt onderzocht.

CURRICULUM VITAE

De schrijfster van dit proefschrift werd op 25 november 1952 geboren te Saint-Maur-des-Fossés in Frankrijk. Zij behaalde in 1969 het einddiploma van het Lycée Paul Valéry te Parijs. Vervolgens vatte zij de wiskunde-studie op en behaalde in 1974 het diploma Maîtrise ès Sciences aan de Universiteit van Parijs-Zuid te Orsay, in de specialiteit toegepaste wiskunde. Het cursusjaar 1974-1975 bracht zij door aan het Massachusetts Institute of Technology, waar zij teaching assistent was en waar zij de graad van Master of Science behaalde. Terug in Frankrijk verrichtte zij toegepast wiskundig werk bij het Commissariat à l'Energie Atomique, waarvoor zij in 1977 het Doctorat de 3^{e} Cycle verkreeg aan de Universiteit te Orsay. Sinds 1 januari 1977 is de schrijfster als wetenschappelijk medewerkster verbonden aan het Mathematisch Centrum te Amsterdam.

De in dit proefschrift uitgewerkte ideeën werden gestimuleerd in vele discussies met prof.dr.ir. L.A. Peletier, prof.dr. R. Témam en prof.dr. Ph. Clément, en door dagelijkse samenwerking met dr. O. Diekmann.

STELLINGEN

BEHORENDE BIJ HET PROEFSCHRIFT

ON SOME NONLINEAR PROBLEMS ARISING IN

THE PHYSICS OF IONIZED GASES

VAN

DANIELLE HILHORST-GOLDMAN

2 DECEMBER 1981

I

Laat H een reële Hilbertruimte zijn met inwendig produkt $(.,.)$. Laten A, B en B_n maximale monotone operatoren zijn zodanig dat B_n naar B convergeert in de zin van de resolvente. Zij verder $B_{n,\lambda}$ de Yosida-approximatie van B_n. Veronderstel dat $(Au, B_{n,\lambda}u) \geq 0$ is voor alle $u \in D(A)$, alle $\lambda > 0$ en $n > 0$. Dan convergeert $A + B_n$ naar $A + B$ in de zin van de resolvente.

II

Brauner & Nicolaenko benaderen door variationele ongelijkheden gekarakteriseerde vrije-randwaardeproblemen door middel van homografe functies en leiden enige foutenschattingen af. Men kan soortgelijke schattingen verkrijgen via benaderingen door middel van monotoon stijgende continue functies die nul zijn in de oorsprong $x = 0$ en snel genoeg naar ± 1 convergeren voor $x \to \pm \infty$.

C.M. Brauner & B. Nicolaenko, te verschijnen in Advances in Mathematics.

III

Zij Ω een begrensd open deel van $\mathbb{R}^n$ en laten $f \in L^p(\Omega)$, $\psi \in W^{2,p}(\Omega)$ met $p > n$ gegeven functies zijn zodanig dat $\Delta\psi = 0$ en $\psi > 0$. Laat $h : \mathbb{R} \to \mathbb{R}$ een willekeurige Lipschitz-continue strikt monotoon stijgende begrensde functie zijn zodanig dat $h(0) = 0$. Als

$$\int_\Omega h(-\infty)\psi \, dx < C < \int_\Omega h(+\infty)\psi \, dx,$$

dan heeft het probleem

$$\begin{cases} -\Delta u + h(u) = f & \text{in } \Omega \\ \displaystyle\int_\Omega h(u)\psi \, dx = C & \\ u = \theta\psi & \text{op } \partial\Omega \end{cases}$$

waarin θ een onbekende constante is, een unieke oplossing $u \in W^{2,p}(\Omega)$. Deze stelling is geïnspireerd door het werk van Cipolatti.

R. Cipolatti, te verschijnen in C.R. Acad. Sc. Paris.

IV

Beschouw

$$F(\beta) = \sum_{k=1}^{\infty} \frac{\beta^k}{\Gamma(k\alpha)}$$

waarin $\alpha, \beta > 0$ zijn. De volgende scherpe schattingen gelden:

$$F(\beta) \le \frac{1}{\alpha}\, \beta^{\frac{1}{\alpha}}\, e^{\beta^{\frac{1}{\alpha}}} + \frac{\beta\Gamma(1-\alpha)}{\pi \sin \pi\alpha}$$

als $\alpha \le 1/2$ en

$$F(\beta) \le \frac{1}{\alpha}\, \beta^{\frac{1}{\alpha}}\, e^{\beta^{\frac{1}{\alpha}}} + \frac{1}{2\pi} \operatorname{cotg} \frac{\pi\alpha}{2}$$

als $\alpha \le 1$.

V

Het bepalen van het electrisch veld in de TOKAMAK leidt natuurlijkerwijze tot een probleem van optimale controle.

D. Hilhorst, Thèse de 3ème Cycle, Orsay 1977.

J.L. Lions, *Contrôle Optimal des Systèmes Gouvernés par des Equations aux Dérivées Partielles*, Dunod 1968.

VI

Zij $P(m,n)$ een waarschijnlijkheidsverdeling op $\mathbb{Z} \times \mathbb{Z}$ gegeven door

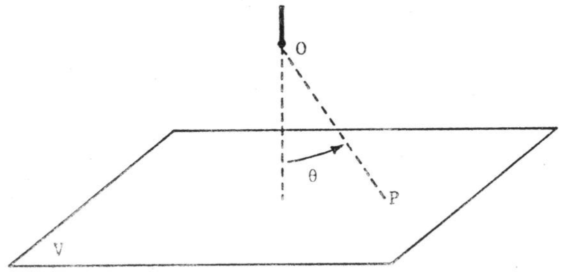
$$P(m,n) = A \exp\left(-\frac{\pi\sqrt{3}}{3}\{m^2 + n^2 + (m-n)^2\}\right)$$

waarin A de normeringsfactor is. Dan geldt voor het gemiddelde van $(m-n)^2$ t.o.v. deze verdeling

$$E[(m-n)^2] = \frac{\sqrt{3}}{6\pi}\ .$$

VII

De wet van Warburg voor electrische gasontladingen tussen een puntelectrode O en een vlakke electrode V stelt dat de stroomdichtheid in een punt P juist boven V evenredig is met $\cos^5\theta$, waarin θ de hoek is tussen OP en de normaal uit O op V.



De door deze wet geïmpliceerde cylindrische symmetrie wordt experimenteel slechts verkregen na middeling over een groot aantal asymmetrische ontladingen. Geen van de tot op heden gedane berekeningen verklaart deze wet.

E. Warburg, Wied. Ann. 67 (1899) 69.

B.L. Henson, J. Appl. Phys. 52 (1981) 3921.

VIII

De mogelijkheid op te bellen naar telefooncellen biedt voordelen.

IX

Uitzending van films in hun oorspronkelijke taal door de Franse televisie zou de kennis van vreemde talen van de kijkers zeer ten goede komen.