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On the Nonexistence of a Strong Solution in the Boundary Problem for a Sticky Brownian Motion

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We prove that a sticky Brownian motion is not measurable with respect to a driving Wiener process thereby verifying Skorokhod's conjecture.
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We consider the stochastic differential equation

$$d\xi_t = aI_{[\xi_t=0]}dt + I_{[\xi_t>0]}dW_t \qquad \xi_0 = X_0 \geqslant 0, \tag{1}$$

where W is a Wiener process and $0 \leqslant a \leqslant \infty$.

It is convenient to represent (1) as a boundary problem with the departure rate of a general form (giving by the way the exact meaning to the condition $a = \infty$).

For the nonnegative measurable function a_t, $t \geqslant 0$, the process ξ is a solution of the boundary problem if

a) $\xi_0 = X_0$,

b) $\xi_t \geqslant 0,\ t \geqslant 0$,

c) $I_{[\xi_t>0]}d\xi_t = I_{[\xi_t>0]}dW_t$,

d) $d\zeta_t = I_{[\xi_t=0]}d\xi_t \geqslant 0$, (2)

e) $d\zeta_t = a_t I_{[a_t<\infty]} d\wedge_t$,

f) $I_{[a_t=\infty]}d\wedge_t = 0$,

where $\wedge_t = \int_0^t I_{[\xi_s=0]}ds,\ t \geqslant 0$.

It is well known that the equation (1) has a unique weak solution ([1]; for more general boundary problems see [2]).

For the extremal cases $a = 0$ (absorbtion) and $a = \infty$ (instantaneous reflection) the problem (2) admits the strong solutions expressed explicitly by

$$X_t^0 = X_0 + W_{\tau\wedge t},\ \tau = \min\{s \geqslant 0,\ X_0 + W_s = 0\}, \tag{3}$$

and

$$X_t^* = \max(X_0 + W_t,\ \max_{0 \leqslant s \leqslant t}(W_t - W_s)). \tag{4}$$

In the intermediate case with the finite positive rate of departure from the boundary, however, the sticky (or slowly reflecting) Brownian motion ξ is not representable as a functional of the (driving)

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THEOREM. *The equation* (1) *does not admit a strong solution for* $0<a<\infty$.

We prove this statement by constructing a sequence $\xi^n, n\geq 1$, of solutions of (2) adopted to W, which converges to the solution of (1) in the stable topology ([3]), but diverges in the strong sense (pathwise). It follows from this that the solution ξ of (1) is nonmeasurable with respect to W. To this end we proceed in several stages.

1. THE UNIQUENESS OF THE SOLUTION MEASURE OF (1).

DEFINITION. A probability μ on the Borel σ-algebra $B(C^2)$ of the product space $C^2=C\times C$ of two dimentional continuous functions $(x,y)=(x_t,y_t)$, $t\geq 0$, is called the solution measure of (2) if

$$\mu\{(x,y): x_t = x_0 + a\int_0^t I_{[x_s=0]}ds + \int_0^t I_{[x_s>0]}dy_s\} = 1, \tag{5}$$

$$\mu\{(x,y): y\in B\} = P^W(B),\ B\in B(C), \tag{6}$$

where P^W is a Wiener measure.

Denote by $\mu^1,\mu^2,\mu^1(\cdot 1\cdot),\mu^2(\cdot 1\cdot)$ the marginal and conditional distributions corresponding to μ:

$$\mu'(B) = \mu(B\times C),\ \mu^2(B) = \mu(C\times B);\ B\in B(C)$$

$$\mu'(B|y) = \mu\{x\in B|y\},\ \mu^2(B|x) = \mu(y\in B|x);\ x,y\in C.$$

The following construction of the solution measure is presented in [1]. Let P^x be the (unique) distribution of the weak solution of (1), and define $\bar{\mu}$ as an image measure corresponding to the mapping φ from the probability space $(C^2,B(C^2), P^X\times P^W)$ into the space $(C^2,B(C^2))$ expressed by

$$\varphi(x,y) = (x_t,\ x_t - x_0 - a\int_0^t I_{[X_s=0]}ds + \int_0^t I_{[X_s=0]}dy_s),\ t\geq 0.$$

It is easily seen that $\bar{\mu}$ is a solution measure of (1) characterized by the property

$$\bar{\mu}\{\int_0^t I_{[X_s=0]}dy_s\in B|x\} = Q_x(B),\ B\in B(C),\ \text{a.s;} \tag{7}$$

where, for each $x\in C$, Q_x is the distribution of a Gaussian process with independent increments expressable as

$$\int_0^t I_{[X_s=0]}dW_s$$

with a Wiener process W.

The property (7) is equivalent to the relation

$$E^{\bar{\mu}}f(x)\exp(i\int_0^t C_s I_{[X_s=0]}dy_s) = E^{\bar{\mu}}f(x)\exp(-\tfrac{1}{2}\int_0^t C_s^2 I_{[X_s=0]}ds), \tag{8}$$

satisfied for any bounded measurable functions

$$f: (C,B(C))\to(R^{(1)}, B(R^{(1)}))$$

$$C: (R_+^{(1)},B(R_+^{(1)})\to(R^{(1)},B(R^{(1)}))$$

and $t\geq 0$.

LEMMA. *The equation* (1) *admits an unique solution measure.*

PROOF. It is sufficient to show only that for any solution measure of (1) the relation (8) takes place.

By the uniqueness of $\mu^1=P^X$ as a solution of the Martingale problem corresponding to (1), it follows (see [4]) that every (bounded) measurable function

$$\phi: (C, B(C)) \to (R^{(1)}, B(R^{(1)}))$$

is expressable as a stochastic integral

$$\phi(x) = E^{\mu}\phi(x) + \int_0^{\infty} g_s(dx_s - aI_{[X_s=0]}ds), \ P^X a.s.$$

with some nonanticipative functional $g_t = g_t(X)$.
Taking

$$\phi(x) = f(x)\exp(-\tfrac{1}{2}\int_0^t C_s^2 I_{[X_s=0]}ds)$$

and denoting

$$\rho_t = \exp(i\int_0^t C_s I_{[X_s=0]}dy_s + \tfrac{1}{2}\int_0^t C_s^2 I_{[X_s=0]}d_s),$$

we have

$$E^{\mu}f(x)\exp(i\int_0^t C_s I_{[X_s=0]}dy_s) = E^{\mu}\phi(x)\rho_t = E^{\mu}\phi(x)E^{\mu}\rho_t +$$

$$+E^{\mu}\int_0^t g_s(dx_s - aI_{[x_s=0]}ds)\rho_t = E^{\mu}\phi(x) + E^{\mu}\int_0^t g_s(dx_s - aI_{[x_s=0]}ds) +$$

$$+E^{\mu}\int_0^t i\, g_s C_s I_{[x_s \gtrless 0]} I_{[x_s=0]}ds = E^{\mu}\phi(x) \square$$

2. THE COMPACTNESS OF THE CLASS OF MEASURE SOLUTIONS OF (2)

Let M be the class of all solution measures corresponding to the (uniquely solvable) boundary problems (2) with arbitrary departure rates (a_t), $t \geqslant 0$. We shall consider the convergence

$$u_n \xrightarrow{*} \mu, \ n \to \infty$$

of measures μ_n, μ on $B(C^2)$ determined by the convergence of integrals

$$E^{\mu}f_n(x,y) \to E^{\mu}f(x,y), \ n \to \infty$$

for each t and each bounded measurable in y and continuous in x function on f depending on first t coordinates $(f(x,y) = f(x_s, y_s; s \leqslant t))$.

Denote by M^s the subclass of M corresponding to the boundary problems (2) having unique strong solutions. If $\mu \in M^s$, then there exists some mapping

$$\varphi^{\mu}: (C, B(C)) \to (C, B(C))$$

such that for each A, $B \in B(C)$

$$\mu(A \times B) = \mu^2(B \cap (\varphi^{\mu})^{-1}(A)) = \int_{\{y \in B,\ \varphi(y) \in A\}} \mu^2(dy).$$

Evidently $\mu^2 = P^W$, and

$$\xi_t = \varphi_t^{\mu}(y) \ P^W \ a.s. \ 0 \leqslant t, \tag{9}$$

represents the solution of (2) as a functional of the driving process.

LEMMA 2. ([3]) a). *The weak compactness of the class* $(\mu^1 : \mu \in M)$ *implies the (stable) compactness (corresponding to the convergence* $\overset{*}{\to}$*) of M.*

$$b)\ \text{If } \mu_n \in M^s,\ \mu_n \overset{*}{\to} \mu,\ n\to\infty,\ \mu \in M^S,$$

then

$$\sup_{0\leq s\leq t} |\varphi_s^{\mu_n} - \varphi_s^{\mu}| \overset{P^W}{\to} 0,\ n\to\infty,\ t\geq 0.$$

Note that the class M^s consists of the solution measures corresponding to the class a^+ of functions (a_t), $t\to 0$ which are stepwise constant taking on only two values 0 and ∞.

LEMMA 3. *The class M is (relatively) compact.*

PROOF. Let $\mu \in M$ with some (a_t), $t\geq 0$. Consider the time change $\tau_t = \inf(s : s - \wedge_s = t)$ and the process

$$X_t^{*\mu} = X_{\tau_t}.$$

It is clear from

$$\int_0^t I_{[X_s^{*\mu}=0]} ds = \int_0^t I_{[X_{\tau_s}=0]} ds = \int_0^t I_{[X_{\tau_s}=0]} d(\tau_s - \wedge_{\tau_s})$$

$$= \int_0^t I_{[X_{\tau_s}=0]}(1 - I_{[X_{\tau_s}=0]}) d\tau_s = 0,$$

that the distribution of $X^{*\mu}$ coincides with the distribution of an instantaneous reflecting Brownian motion X^* stopped at some random time.

We have for each $0\leq h\leq 1$ and $t\geq 0$, using $|\tau_t - \tau_s| \leq |t-s|$,

$$\Delta_\epsilon^\mu(h) = \mu^1\{X: \sup_{0\leq s<u<s+h\leq t} |X_s - X_u| > \epsilon\} \leq$$

$$\leq P^{X^*}\{X: \sup_{0\leq s<u<s+h\leq t} |X_s - X_u| > \epsilon\} = \Delta_\epsilon(h).$$

Thus the condition of weak (relative) compactness of the class $\{\mu^1 : \mu \in M\}$, ([5]), follows from the continuity of the process X^*.

$$\lim_{h\to 0} \sup_{\mu\in M} \Delta_\epsilon^\mu(h) \leq \lim_{n\to 0} \Delta_\epsilon(h) = 0.$$

Hence the assertion is true by Lemma 2.a) □

3. THE CONDITIONS OF THE CONVERGENCE TO THE MEASURE SOLUTION OF (1)

LEMMA 4. *Let* μ *be the (unique) measure solution of (1) and let* $\mu_n \in M$, *with* μ_n^1 *having strong Markovian property,* $n\geq 1$. *Then, if for each* $t>0, \lambda>0$

$$L_t^n(\lambda) = E^{\mu_n}\Big(\int_t^\infty \exp(-\lambda(s-t)) d\zeta_s | x_t = 0\Big) \to \frac{a}{\lambda + \sqrt{2\lambda}\, a},\ n\to\infty \tag{10}$$

$$M_t^n(\lambda) = E^{\mu_n}\Big(\int_t^\infty \exp(-\lambda(s-t)) I_{[X_s=0]} ds | x_t = 0\Big) \to \frac{1}{\lambda + \sqrt{2\lambda}\, a},\ n\to\infty$$

then $\mu_n \overset{*}{\to} \mu$.

PROOF. If $\mu_n^1 \overset{W}{\to} \mu^1$, then for each limit point $\tilde{\mu}$ of the sequence μ_n we have for each $\Gamma>0$

$$\eta = \eta(x,y) = \sup_{0\leq s<t\leq TS\leq u\leq t} \inf X_u)(\sup_{S\leq v\leq t} |(X_\sigma - X_s) - y_v - y_s)|) = 0 \ \tilde{\mu} \text{ a.s.} \tag{11}$$

In fact $\eta=0$ is equivalent to the property that for each $0\leq\leq s<t\leq T$, with $X_u>0$, $S\leq u\leq t$) the equality $Y_u - Y_s = X_u - X_s$, $s\leq u\leq t$, takes place. From c) in (2) it follows that $\eta=0\mu_n$ a.s. $n\geq 1$, and, hence, to obtain (11) it is sufficient to take limit in $E^{\mu_n}\min(\eta\wedge C)$ for each $C>0$, using the continuity of η on C^2.

Thus, as the conditions a), b), d), e), f) in (2) are expressed only in terms of the marginal distribution μ^1, we have $\tilde{\mu}=\mu$.

Hence it remains to prove that the convergence (10) implies the convergence of finite dimensional distributions corresponding to μ_n^1, $n\geq 1$.

It is not difficult to calculate the Laplace transforms

$$L_t(\lambda) = L_0(\lambda) = E^\mu(\int_t^\infty \exp(-\lambda(s-t))\, d\zeta_s \,|x_t=0)$$

$$M_t(\lambda) = M_0(\lambda) = E^\mu(\int_t^\infty \exp(-\lambda(s-t)) I_{[X_s=0]} ds \,|x_t=0)$$

by solving the boundary problem

$$\lambda U(x,\Lambda) = \frac{1}{2} U_{xx}(x,\lambda),\ x>0, \tag{12}$$

$$\lambda U(x,\lambda) = aU_{xx}(x,\lambda) + 1,\ x=0,$$

for

$$U(x,\lambda) = E^\mu(\int_0^\infty \exp(-\lambda s) I_{[X_s=0]} ds | x_0 = x).$$

From (12) we obtain

$$L_t(\lambda) = aM_t(\lambda) = aM_0(\lambda),\ M_0(\lambda) = \frac{1}{\lambda+\sqrt{2\lambda}a}.$$

Further, for each $n\rightarrow 1$, the conditional Laplace transform

$$\phi_n(s,t,\lambda,x) = E^{\mu_n}(exp(-\lambda x_t)|x_s=x)$$

satisfies the equation (for each $t>s, x\geq 0$)

$$\phi_n(s,t,\lambda,x) = e^{-\lambda x} - \lambda E^{\mu_n}(\zeta_t - \zeta_s|x_s=x) + \frac{\lambda^2}{2}\int_s^t \phi_n(s,u,\lambda,x)du$$

$$-\frac{\lambda^2}{2}E^{\mu_n}(\wedge_t - \wedge_s | x_s = x),$$

which gives

$$\phi_n(s,t,\lambda,x) = e^{-\lambda x+\lambda^2}(t-s) - E^{\mu_n}(\int_s^t \exp(\frac{\lambda^2}{2}(+-u))(\lambda d\zeta_u + \frac{\lambda^2}{2} d\wedge_u)|x_s=x).$$

Besides, by the strong Markovian property, we have

$$E^{\mu_n}(\zeta_t - \zeta_s|x_s=x) = \int_s^t e_u^n(t)k(x,u-s)du,$$

$$E^{\mu_n}(\wedge_t - \wedge_s|x_s=x) = \int_s^t m_u^n(t)k(x,u-s)du,$$

where, for $t>u$,

$$l_u^n(t) = E^{\mu_n}(\zeta_t - \zeta_u | x_u = 0), \; m_u^n(t) = E^{\mu_n}(\wedge_t - \wedge | x_u = 0),$$

and $k(x,t)$ is the distribution density function of the random moment

$$\tau = \min(t \geqslant s : x_t = 0)$$

with the condition $x_s = x$. Evidently k does not depend on $n \geqslant 1$ and s, and, in terms of the Wiener process W,

$$k(x,t) = \frac{d}{dt} P(x + \inf_{0 \leqslant s \leqslant t} W_s < 0) = \frac{d}{dt} P(|W_t| > x).$$

Thus the convergence of $l_t^n(s)$ and $m_t^n(s)$ (or the convergence (10) of their Laplace transforms $L_t^n(\lambda)$ and $M_t^n(\lambda)$) is sufficient for the convergence of the conditional Laplace transforms ϕ_n □

4. THE NECESSARY CONDITION FOR THE STRONG CONVERGENCE

Let $\mu_n \in M^s$, and let

$$X_t^n = \varphi_t^{\mu_n}(y)$$

denote the strong solutions of (2).

LEMMA 5. *If for each $t>0, \lambda>0$*

$$P^W(\sup_{0 \leqslant s \leqslant t} |X_s^n - X_s^m| > \epsilon) \to 0, \; n,m \to \infty$$

then

$$E^W \int_0^\infty \exp(-\lambda s) |I_{[X_s^n = 0]} - I_{[X_s^m = 0]}| ds \to 0, \; n,m \to \infty.$$

PROOF. From (2) we have

$$E^W(X_t^n - X_t^m)^2 = 2E^W \int_0^t (X_s^n - X_s^m) d(\zeta_s^n - \zeta_s^m) + E^W \int_0^t (I_{[X_s^n > 0]} - I_{[X_s^m > 0]})^2 ds.$$

Thus

$$E^W \int_0^t |I_{[X_s^n = 0]} - I_{[X_s^m = 0]}| ds \leqslant E^W(\sup_{0 \leqslant s \leqslant t} |X_s^n - X_s^m|)^2 +$$

$$+ 2\left[E^W(\sup_{0 \leqslant s \leqslant t} |X_s^n - X_s^m|)^2 E^W(\zeta_t^n + \zeta_t^m)^2\right]^{\frac{1}{2}}. \tag{13}$$

Now it is sufficient to apply the fact that the instantaneous reflecting process X^* is maximal in the class of all strong solutions ([6]) and, so $X_t^n \leqslant X_t^*$ a.s. Thus using the estimations

$$\sup_{0 \leqslant s \leqslant t} |X_s^n - X_s^m| \leqslant 2 \sup_{0 \leqslant s \leqslant t} X_s^*, \; E^W(\zeta_t^n)^2 \leqslant 2E^W[(X_t^n)^2 + t] \leqslant 2E^W[(X_t^*)^2 + t],$$

we obtain that the right-hand side of (13) converges to zero □

5. THE DESCRIPTION OF THE APPROXIMATIONS AND THE WEAK CONVERGENCE

Consider for each $\Delta>0$, $0 \leqslant \alpha \leqslant 1$, $0<\delta<\Delta$, the unique strong solution (expressed as a functional of a Wiener process W)

$$\xi_t(\Delta, \alpha, \delta) = \varphi_t^{\mu}(W),$$

with the departure rate (a_t), $t \geqslant 0$ of the form

$$a_t = 0, \text{ for } t \in [k\Delta, k\Delta + \alpha\Delta] \cup [k\Delta + \alpha\Delta + \delta, (k+1)\Delta[$$

$$a_t = \infty, \text{ for } t \in]k\Delta + \alpha\Delta, k\Delta + \alpha\Delta + \delta[, k \geqslant 0$$

Thus the process $\xi(\Delta, \alpha, \delta)$ is everywhere absorbing except subintervals of length δ disposed at the one and the same positions inside the intervals $[k\Delta, (k+1)\Delta[$, $k = 0, 1, \ldots$

LEMMA 6. *For each* $0 \leqslant \alpha < 1$ *and* $0 < c < \infty$ *the sequence* $\xi^n = \xi(\frac{1}{n}, \alpha, \frac{c^2}{n^2})$ *converges weakly to the solution of* (1) *with* $a = \sqrt{\frac{\pi}{2}}c$.

PROOF. It is sufficient to verify the conditions (10).

Consider first the case $\alpha = 0$. It is easy to notice that the functions $L_t^{\Delta,\delta}(\lambda)$ and $M_t^{\Delta,\delta}(\lambda)$ corresponding to the solution $\xi(\Delta, 0, \delta)$ satisfy the relations

$$L_t^{\Delta,\delta}(\lambda) = L_0^{\Delta,\delta}(\lambda), \; k\Delta + \delta \leqslant t \leqslant (k+1)\Delta, \; k \geqslant 0, \tag{14}$$

$$M_t^{\Delta,\delta}(\lambda) = M_0^{\delta}(\lambda)\exp(([\frac{t}{\Delta}]+1)\Delta - t) + \frac{1}{\lambda}(\exp(([\frac{t}{\Delta}]+1)\Delta - t) - 1),$$

where $[\frac{t}{\Delta}]$ is a largest integer of $\frac{t}{\Delta}$.

To derive the recurrent equations for $M_0^{\Delta,\delta}(\lambda)$ and $L_0^{\Delta,\delta}(\lambda)$ introduce the moment

$$\tau = \min(s \geqslant \delta, \; \xi_s(\Delta, 0, \delta) = 0).$$

Suppose $\xi_0(\Delta, 0, \delta) = 0$. Then, by definition, the representations (3) and (4) give

$$\xi_t(\Delta, 0, \delta) = \sup_{0 \leqslant s \leqslant t}(W_t - W_s), \; 0 \leqslant t \leqslant \sigma$$

$$\xi_t(\Delta, 0, \Delta) - \xi_\delta(\Delta, 0, \delta) = W_t - W_\delta, \; \delta \leqslant t \leqslant \tau.$$

Thus (taking into consideration that the random variables $\sup_{0 \leqslant s \leqslant t} W_s$ and $|W_t|$ have the same distribution) we obtain

$$F^{\Delta,\delta}(t) = P^W\{\tau < t | \zeta_0(\Delta, 0, \delta) = 0\} = P^W\{\inf_{\delta \leqslant s \leqslant t+\delta}(W_s - W_\delta) \leqslant -\sup_{0 \leqslant s \leqslant \delta}(W_\delta - W_s)\} =$$

$$= P\{|W_{t+\delta} - W_\delta| \geqslant |W_\delta|\} = \frac{2}{\pi t}\int_0^\infty \int_y^\infty \exp(-\frac{s^2}{2t} - \frac{y^2}{2\delta})dxdy = \frac{2}{\pi}arctg(\sqrt{\frac{t}{\delta}}) \tag{15}$$

Consider, for convenience,

$$\tilde{M}_t^{\Delta,\delta}(\lambda) = \frac{1}{\lambda} - M_t^{\Delta,\delta}(\lambda) = E^\mu(\int_t^\infty \exp(-\lambda(s-t))I_{[x_s > 0]}ds | x_t = 0).$$

We have

$$\tilde{M}_0^{\Delta,\delta}(\lambda) = E\int_0^\tau \exp(-\lambda t)dt + E\exp(-\lambda\tau)\tilde{M}_\tau^{\Delta,\delta}(\lambda) = \int_0^\infty \frac{1}{\lambda}(1 - \exp(-\lambda t))F^{\Delta,\delta}(dt) +$$

$$+ \tilde{M}_0^{\Delta,\delta}(\lambda)E\exp(-\lambda([\frac{\tau}{\Delta}]+1)\Delta)I_{[\tau - [\frac{\tau}{\Delta}]\Delta \geqslant \delta]} + E\exp(-\tau\lambda)I_{[\tau - [\frac{\tau}{\Delta}]\Delta \leqslant \delta]}\tilde{M}_\tau^{\Delta,\delta}(\lambda).$$

Thus

$$\tilde{M}_0^{\Delta,\delta}(\lambda) = [\phi_1(\Delta,\delta,\lambda) + \phi_2(\Delta,\delta,\lambda)] \angle (1 - \phi_3(\Delta,\delta,\lambda)), \tag{16}$$

where

$$\phi_1(\Delta,\delta,\lambda) = \int_0^\infty \frac{1}{\lambda}(1-\exp(-\lambda t))F^{\Delta,\delta}(dt),$$

$$\phi_2(\Delta,\delta,\lambda) = E(\exp(-\lambda\tau)I_{[\tau-[\frac{\tau}{\Delta}]\Delta\leqslant\delta]}\tilde{M}_\tau^{\Delta,\delta}(\lambda),$$

$$\phi_3(\Delta,\delta,\lambda) = E\exp(-\lambda\Delta([\frac{\tau}{\Delta}]+1))I_{[\tau-[\frac{\tau}{\Delta}]\Delta\geqslant\delta]}.$$

For large X we shall use the estimation

$$|arctg(x) - \frac{\pi}{2} + \frac{1}{x}| \leqslant 2(\frac{1}{x})^3. \tag{17}$$

Using (15) and (17), since obviously $\tilde{M}_t^{\Delta,\delta}(\lambda) \leqslant \frac{1}{\lambda}$, we have for $\delta = c^2\Delta^2$

$$\phi_2(\Delta,c^2\Delta^2,\lambda) \leqslant \sum_{k=1}^\infty (F^{\Delta,\delta}(k\Delta+c^2\Delta^2)-F^{\Delta,\delta}(k\Delta))\exp(-k\lambda\Delta) =$$

$$= \frac{2}{\pi\lambda}\sum_{k=1}^\infty \exp(-k\Delta\lambda)[arctg(\frac{\sqrt{k+c^2\Delta}}{c^2\Delta}) - arctg(\sqrt{\frac{k}{c^2}}\Delta) \leqslant$$

$$\leqslant \frac{2c\sqrt{\Delta}}{\pi\lambda}\sum_{k=1}^\infty \exp(-k\lambda\Delta) \angle \sqrt{\frac{1}{k+c^2\Delta}} - \sqrt{\frac{1}{k}}| + 0(\Delta^{3/2}) \leqslant \frac{2c\sqrt{\Delta}}{\pi\lambda}\sum_{k=1}^\infty \frac{c^2\Delta}{2k^{3/2}} \tag{18}$$

$$+0(\Delta^{3/2}) = 0(\Delta^{3/2}).$$

Further, applying

$$\lim_{\Delta\to 0}\sqrt{\Delta}\sum_{k=1}^\infty \exp(-\lambda k\Delta)\frac{1}{\sqrt{k}} = \sqrt{\frac{\pi}{\lambda}}$$

we obtain

$$\phi_3(\Delta,c^2\Delta^2,\lambda) = \sum_{k=1}^\infty (1-\exp(-\lambda k\Delta))(F^{\Delta,\delta}(k\Delta)-F^{\Delta,\delta}((k-1)\Delta) =$$

$$= (1-\exp(-\lambda\Delta))\sum_{k=1}^\infty \exp(-\lambda k\Delta)F^{\Delta,\delta}(k\Delta) = \exp(-\lambda\Delta) -$$

$$-(1-\exp(-\lambda\Delta))[\sum_{k=1}^\infty \exp(-\lambda k\Delta)\sqrt{\frac{\Delta}{k}}\frac{2c}{\pi} + 0(\Delta)] = \exp(-\lambda\Delta) -$$

$$-(1-\exp(-\lambda\Delta))[2\sqrt{\frac{1}{\pi\lambda}}c + 0(\Delta)] = 1-\lambda\Delta-2c\sqrt{\frac{\lambda}{\pi}} + o(\Delta). \tag{19}$$

As for the expression ϕ_1, it is easily calculated that

$$\phi_1(\Delta,C^2\Delta^2,\lambda) = \frac{\Delta c}{\lambda\pi}\int_0^\infty (1-\exp(-\lambda t))\frac{dt}{+}\frac{1}{2}(t+C^2\Delta^2) =$$

$$= \frac{\Delta c}{\lambda\pi}[\int_0^\infty (1-\exp(-\lambda\frac{t))dt}{t^{3/2}} + 0(\Delta)] = \frac{2\Delta c}{\sqrt{\pi\lambda}} + 0(\Delta). \tag{20}$$

Thus using (18), (19) and (20) we obtain

$$\lim_{\Delta\geqslant 0}\tilde{M}_0^{\Delta,\delta}(\lambda) = \sqrt{\frac{2}{\pi}}\frac{c\sqrt{2\lambda}}{\lambda(\lambda+\sqrt{2\lambda}c\sqrt{\frac{2}{\pi}})},$$

and hence

$$\lim_{\Delta\to 0} M_0^{\Delta,\delta} = \frac{1}{\lambda+\sqrt{2\lambda}\,a}$$

with

$$a = c\sqrt{\frac{2}{\pi}}.$$

Analogeously, from the decomposition

$$L_0^{\Delta,\delta}(\lambda) = E\int_0^{\tau} \exp(-\lambda t)d\zeta_t + E\exp(-\lambda\tau)L_\tau^{\Delta,\delta}(\lambda),$$

we can derive, using the relation (14), and applying the same arguments as before, that

$$L_0^{\Delta,\delta}(\lambda) = E\int_0^{\tau} e^{-\lambda t}d\zeta_t(\Delta(\lambda+\sqrt{2\lambda}\sqrt{\frac{2}{\pi}}c))^{-1} + o(1).$$

Besides

$$E\int_0^{\tau}\exp(-\lambda t)d\zeta_t = E\int_0^{\delta}\exp(-\lambda t)d\zeta_t = E\zeta_\delta + o(\Delta) =$$

$$= E\sup_{0\leqslant s\leqslant\delta} W_s + 0(\Delta) = \Delta\sqrt{\frac{2}{\pi}}c + o(\Delta)$$

Thus

$$\lim_{\Delta\to 0} L_0^{\Delta,\delta} = \frac{a}{\lambda+a\sqrt{2\lambda}}$$

with

$$a = c\sqrt{\frac{2}{\pi}}.$$

From (14) it is evident that (10) is true for all $t\geqslant 0$.
Finally, it is easyly seen that

$$L_{t+\alpha\Delta}^{\Delta,\alpha,\delta}(\lambda) = L_t^{\Delta,0,\delta}(\lambda),\ M_{t+\alpha\Delta}^{\Delta,\alpha,\delta}(\lambda) = M_t^{\Delta,0,\delta}(\lambda),$$

and, thus (10) is true for each $0\leqslant\alpha<1$, $t\geqslant 0$. □

6. THE STRONG NONCONVERGENCE AND THE PROOF OF THE THEOREM

Consider now the sequence ξ^n defined for $m\geqslant 1$ as follows:

$$\xi^n = \xi(\frac{1}{2m}, 0, \frac{c^2}{(2m)^2}), \text{ as } n = 2m,$$

$$\xi^n = \xi(\frac{1}{2m}, \frac{1}{2}, \frac{c^2}{(2m)^2}), \text{as } n = 2m+1.$$

LEMMA 7. *For the sequence ξ^n defined above*

$$\overline{\lim_{n\to\infty}} E\int_0^{\infty} \exp(-\lambda s)|I_{[\xi_s^n=0]} - I_{[\xi_s^{n+1}=0]}|ds > 0.$$

PROOF. Obviously

$$E \int_0^\infty exp(-\lambda s)|_{[\xi_s^n=0]} - I_{[\xi_s^{n+1}=0]} ds \geqslant E \int_0^\infty \exp(-\lambda t) I_{[\xi_s^n=0]} ds -$$

$$-E \int_0^\infty \exp(-\lambda s) I[\max(\xi_s^n, \xi_1^{n+1}) = 0] ds. \tag{20}$$

Let us consider the process $\bar{\xi}$ which is the strong solution of (2) with

$$a_t = \infty, \text{ for } t \in [k\Delta, k+\Delta+\delta] \cup [k\Delta + \frac{\Delta}{2}, k\Delta + \frac{\Delta}{2} + \delta],$$

$$a_t = 0, \text{ otherwise.}$$

It is easyly seen that (with $\delta < \frac{\Delta}{2}$)

$$\bar{\xi}_t = \max(\xi_t(\Delta,0,\delta), \xi_t(\Delta, \frac{1}{2}, \delta)) = \xi_t(\frac{\Delta}{2}, 0, \delta).$$

Thus, for $\delta = c^2\Delta^2, \Delta = (2m)^{-1}$

$$E \int_0^\infty \exp(-\lambda s)|I_{[\xi_s^n=0]} - I_{[\xi_s^{n+1}=0]}| ds \geqslant M_0^{\Delta, c^2\Delta^2} \to$$

$$\to \frac{1}{\lambda + a\sqrt{2\lambda}} - \frac{1}{\lambda + a^1\sqrt{2\lambda}}, \ \Delta \to 0,$$

where

$$a = c\sqrt{\frac{2}{\pi}}, \ a^1 = 2c\sqrt{\frac{2}{\pi}} = 2a. \ \square$$

Combining now the statements of lemmas and taking into consideration the necessary condition for the strong convergence in assertion b) of Lemma 2, we obtain the proof of the theorem.

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