ECONOMETRIC INSTITUTE

A TUTORIAL INTRODUCTION TO
DIFFERENTIABLE MANIFOLDS AND VECTOR FIELDS
AND
A SHORT TUTORIAL ON LIE ALGEBRAS

M. HAZEWINKEL

REPRINT SERIES no. 297

This article appeared in "Stochastic Systems: The Mathematics of Filtering and Identification and Applications", D. Reidel, Publ. Comp., Dordrecht, Holland (1981).

ERASMUS UNIVERSITY ROTTERDAM - P.O. BOX 1738 - 3000 DR ROTTERDAM - THE NETHERLANDS

A TUTORIAL INTRODUCTION TO DIFFERENTIABLE MANIFOLDS AND VECTOR FIELDS

Michiel Hazewinkel

Dept. Math., Erasmus Univ. Rotterdam

In this tutorial I try by means of several examples to illustrate the basic definitions and concepts of differentiable manifolds. There are few proofs (not that there are ever many at this level of the theory). This material should be sufficient to understand the use made of these concepts in the other contributions in this volume, or, at least, it should help in explaining the terminology employed.

1. INTRODUCTION AND A FEW MOTIVATIONAL REMARKS

Roughly an n-dimensional differentiable manifold is a gadget which locally looks like $\mathbb{R}^n$ but globally perhaps not; A precise definition is given below in section 2. Examples are the sphere and the torus, which are both locally like $\mathbb{R}^2$ but differ globally from $\mathbb{R}^2$ and from each other.

Such objects often arise naturally when discussing problems in analysis (e.g. differential equations) and elsewhere in mathematics and its applications. A few advantages which may come about by doing analysis on manifolds rather than just on $\mathbb{R}^n$ are briefly discussed below.

M. Hazewinkel and J. C. Willems (eds.), Stochastic Systems: The Mathematics of Filtering and Identification and Applications, 77–93.

1.1 Coordinate freeness ("Diffeomorphisms"). A differentiable manifold can be viewed as consisting of pieces of $\mathbb{R}^n$ which are glued together in a smooth (= differentiable) manner. And it is on the basis of such a picture that the analysis (e.g. the study of differential equations) often proceeds. This brings more than a mere extension of analysis on $\mathbb{R}^n$ to analysis on spheres, tori, projective spaces and the like; it stresses the "coordinate free approach", i.e. the formulation of problems and concepts in terms which are invariant under (nonlinear) smooth coordinate transformations and thus also helps to bring about a better understanding even of analysis on $\mathbb{R}^n$. The more important results, concepts and definitions tend to be "coordinate free".

1.2 Analytic continuation. A convergent power series in one complex variable is a rather simple object. It is considerably more difficult to obtain an understanding of the collection of all analytic continuations of a given power series, especially because analytic continuation along a full circle may yield a different function value than the initial one. The fact that the various continuations fit together to form a Riemann surface (a certain kind of 2-dimensional manifold usually different from $\mathbb{R}^2$) was a major and most enlightening discovery which contributes a great deal to our understanding.

1.3 Submanifolds. Consider an equation $\dot{x} = f(x)$ in $\mathbb{R}^n$. Then it often happens, especially in problems coming from mechanics, that the equation is such that it evolves in such a way that certain quantities (e.g. energy, angular momentum) are conserved. Thus the equation really evolves on a subset $\{x \in \mathbb{R}^n | E(x) = c\}$ which is often a differentiable submanifold. Thus it could happen that $\dot{x} = f(x)$, f smooth, is constrained to move on a 2-sphere which then immediately tells us that there is an equilibrium point.

Also one might meet 2 seemingly different equations, say, one in $\mathbb{R}^4$ and one in $\mathbb{R}^3$ (perhaps both intended as a description of the same process) of which the first has two conserved quantities and the second one. It will then be important to decide whether the surfaces on which the equations evolve are diffeomorphic, i.e. the same after a suitable invertible transformation and whether the equations on these submanifolds correspond under these transformations.

1.4 Behaviour at infinity. Consider a differential equation in the plane $\dot{x} = P(x,y)$, $\dot{y} = Q(x,y)$ where P and Q are relatively prime polynomials. To study the behavior of the paths far out in the plane and such things as solutions escaping to infinity and coming back, Poincaré already completed the plane to real projective 2-space (an example of a differential manifold). Also the projective plane is by no means the only smooth manifold compatifying $\mathbb{R}^2$ and it will be of some importance for the behaviour of the equation near infinity whether the "right" compactification to which the equation can be extended will be a projective 2-space, a sphere or a torus, or, ..., or, whether no such compactification exists at all. A good example of a set of equations which are practically impossible to analyse completely without bringing in manifolds are the matrix Riccati equations (which naturally live on Grassmann manifolds (which also gives in this case a very considerable saving in the number of dimensions needed)).

1.5 Avoiding confusion between different kinds of objects. Consider an ordinary differential equation $\dot{x} = f(x)$ on $\mathbb{R}^n$, where $f(x)$ is a function $\mathbb{R}^n \to \mathbb{R}^n$. When one now tries to generalize this idea of a differential equation to a differential equation on a manifold one discovers that $\dot{x}$ and hence $f(x)$ are a different kind of object; they are not functions, but as we shall see, they are vectorfields; in other words under a nonlinear change of coordinates they transform in a different way than functions do.

2. DIFFERENTIABLE MANIFOLDS

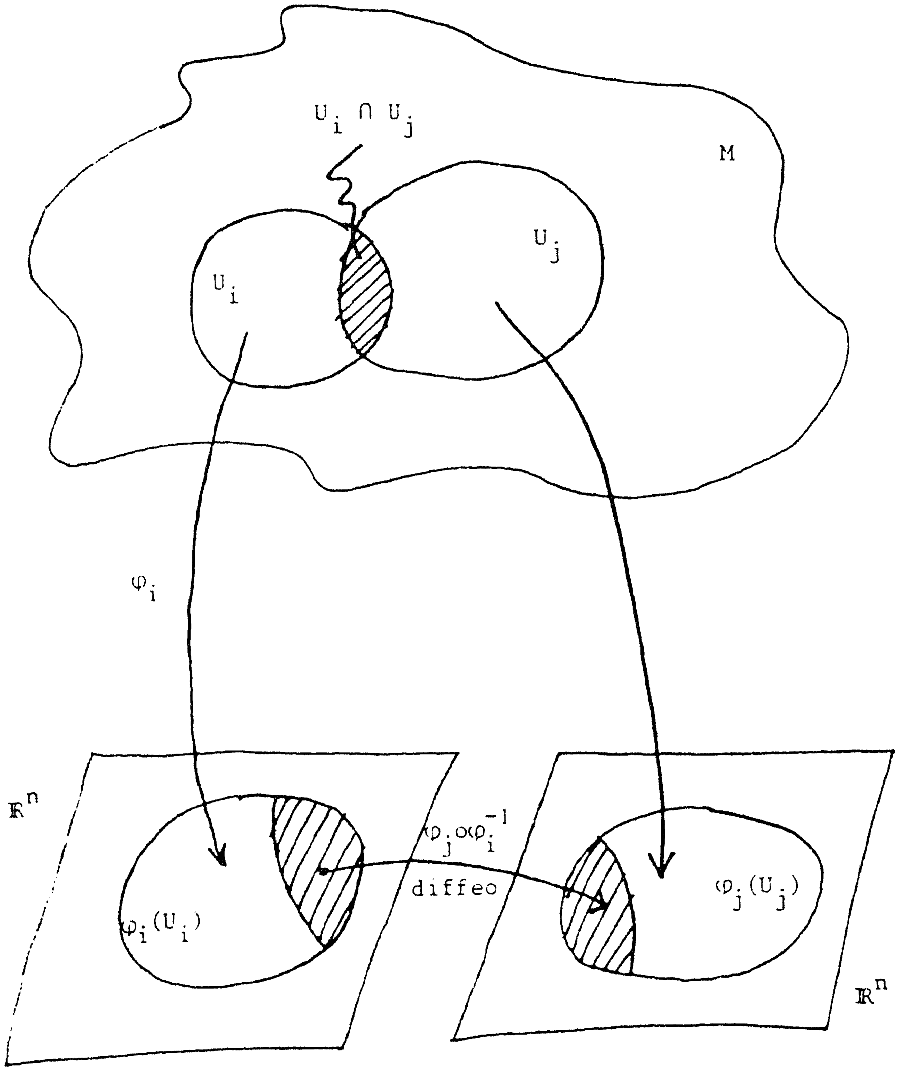

fig.1. Pictorial definition of a differentiable manifold.

Let U be an open subset of $\mathbb{R}^n$, e.g. an open ball. A function $f: U \to \mathbb{R}$ is said to be C^∞ or smooth if all partial derivatives (any order) exist at all $x \in U$. A mapping $\mathbb{R}^n \supset U \to \mathbb{R}^m$ is smooth if all components are smooth; $\varphi: U \to V$, $U \subset \mathbb{R}^n, V \subset \mathbb{R}^n$ is called a diffeomorphism if φ is 1 - 1, onto, and both φ and φ^{-1} are smooth.

As indicated above a smooth n-dimensional manifold is a gadget consisting of open pieces of $\mathbb{R}^n$ smoothly glued together. This gives the above pictorial definition of a smooth n-dimensional manifold M (fig.1).

2.1 <u>Example</u>. The circle $S^1 = \{(x_1,x_2) | x_1^2 + x_2^2 = 1\} \subset \mathbb{R}^2$

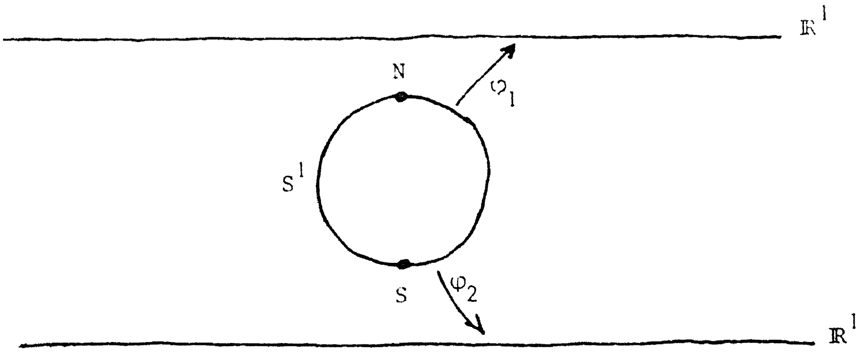

fig.2. Example: the circle

$U_1 = S^1 \setminus \{S\}$, $U_2 = S^1 \setminus \{N\}$ so $U_1 \cup U_2 = S^1$. The "coordinate charts" φ_1 and φ_2 are given by

$$\varphi_1(x_1,x_2) = \frac{x_1}{1+x_2} \quad , \quad \varphi_2(x_1,x_2) = \frac{x_1}{1-x_2}$$

Thus $\varphi_1(U_1 \cap U_2) = \mathbb{R}\setminus\{0\}$, $\varphi_2(U_1 \cap U_2) = \mathbb{R}\setminus\{0\}$ and the map

$\varphi_2 \circ \varphi_1^{-1}: \mathbb{R}\setminus\{0\} \to \mathbb{R}\setminus\{0\}$ is given by $x \mapsto x^{-1}$ which is a diffeomorphism.

2.2 Formal definition of a differentiable manifold.
The data are
- M, a Hausdorff topological space
- A covering $\{U_i\}_{i\in I}$ by open subsets of M
- Coordinate maps $\varphi_i: U_i \to \varphi_i(U_i) \subset \mathbb{R}^n$, $\varphi_i(U_i)$ open in $\mathbb{R}^n$.

These data are subject to the following condition
- $\varphi_i \circ \varphi_j^{-1}: \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ is a diffeomorphism

Often one also adds the requirement that M be paracompact. We shall however disregard these finer points; nor shall we need them in this volume.

2.3 Constructing differentiable manifolds 1: embedded manifolds. Let M be a subset of $\mathbb{R}^N$. Suppose for every $x \in M$ there exists an open neighbourhood $U \subset \mathbb{R}^n$ and a smooth function ψ: $U \to \mathbb{R}^N$ mapping U homeomorphically onto an open neighbourhood V of x in M. Suppose moreover that the Jacobian matrix of ψ has rank n at all $u \in U$. Then M is a smooth manifold of dimension n. (Exercise: the coordinate neighbourhoods are the V's and the coordinate maps are the ψ^{-1}; use the implicit function theorem). Virtually the same arguments show that if $\varphi: U \to \mathbb{R}^k$, $U \subset \mathbb{R}^{n+k}$, is a smooth map and the rank of the Jacobian matrix $J(f)(x)$ is k for all $x \in \varphi^{-1}(0)$, then $\varphi^{-1}(0)$ is a smooth n-dimensional manifold. We shall not pursue this approach but concentrate instead on:

2.4 Constructing differentiable manifolds 2: gluing. Here the data are as follows
- an index set I
- for every $i \in I$ an open subset $U_i \subset \mathbb{R}^n$
- for every ordered pair (i,j) an open subset $U_{ij} \subset U_i$

- diffeomorphisms $\varphi_{ij}: U_{ij} \to U_{ji}$ for all $i,j \in I$

These data are supposed to satisfy the following compatibility conditions

- $U_{ii} = U_i$, $\varphi_{ii} = id$
- $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ (where appropriate)

(where the last identity is supposed to imply also that $\varphi_{ij}(U_{ij} \cap U_{ik}) \subset U_{jk}$ so that $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{jk} \cap U_{ji}$).

These are not all conditions but the present lecturer e.g. has often found it advantageous to stop right here so to speak, and to view a manifold simply as a collection of open subsets of $\mathbb{R}^n$ together with gluing data (coordinate transformation rules).

From the data given above one now defines an abstract topological space M by taking the disjoint union of the U_i and then identifying $x \in U_i$ and $y \in U_j$ iff $x \in U_{ij}$, $y \in U_{ji}$, $\varphi_{ij}(x) = y$. This gives a natural injection $U_i \to M$ with image U_i' say. Let $\varphi_i: U_i' \to U_i$ be the inverse map. Then this gives us a differentiable manifold M in the sense of definition 2.2 provided that M is Hausdorff and paracompact, and these are the conditions which must be added to the gluing compatibility conditions above.

2.5 Functions on a "glued manifold". Let M be a differentiable manifold obtained by the gluing process described in 2.4 above. Then a differentiable function $f: M \to \mathbb{R}$ consist simply of a collection of functions $f_i: U_i \to \mathbb{R}$ such that $f_j \circ \varphi_{ij} = f_i$ on U_{ij}, as illustrated in fig. 3.

Thus for example a function on the circle S^1, cf. figure 2, can be described either as a function of two variables restricted to S^1 or as two functions f_1, f_2 of one variable on U_1 and U_2 such that $f_1(x) = f_2(x^{-1})$. Obviously the latter approach can have considerable advantages.

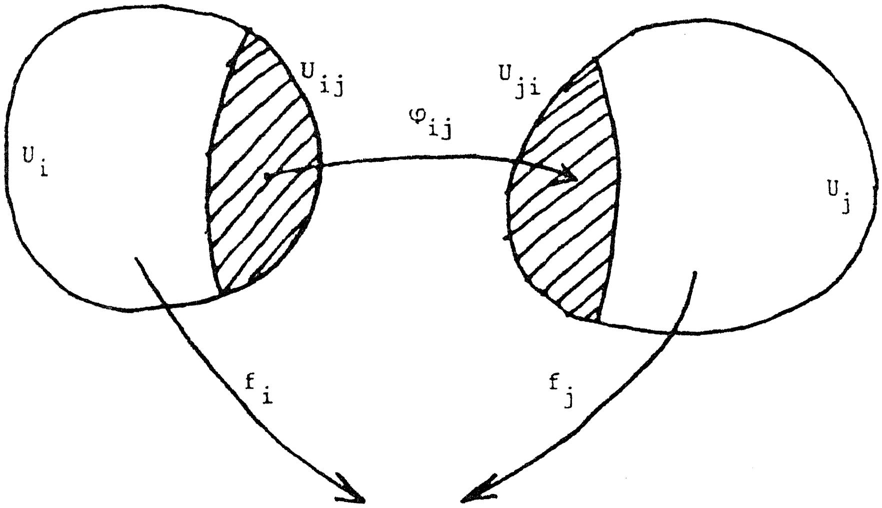

Fig.3. Functions on a glued manifold

2.6 Example of a 2 dimensional manifold: the Möbius band. The (open) Möbius band is obtained by taking a strip in $\mathbb{R}^2$ as indicated below in fig. 4 without its upper and lower edges and identifying the left hand and right hand edges as indicated

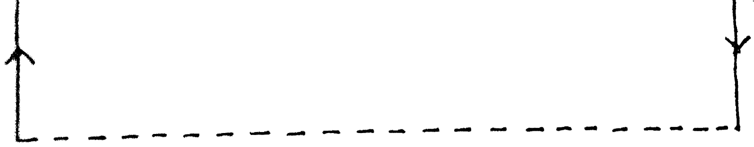

Fig.4. Construction of the Möbius band

The resulting manifold (as a submanifold of $\mathbb{R}^3$) looks something like the following figure 5.

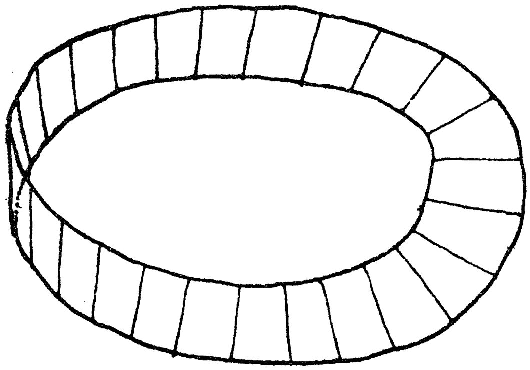

fig.5. The Möbius band

It is left as an exercise to the reader to cast this description in the form required by the gluing description of 2.4 above. The following pictorial description (fig. 6) will suffice.

fig.6. Gluing description of the Möbius band

2.7 Example: the 2-dimensional sphere. The picture in fig. 7 below shows how the 2-sphere $S^2 = \{(x_1,x_2,x_3) | x_1^2 + x_2^2 + x_3^2 = 1\}$ can be obtained by gluing two disks together. If the surface of the earth is viewed as a model for S^2 the first disk covers everything north of Capricorn and the second everything south of Cancer.

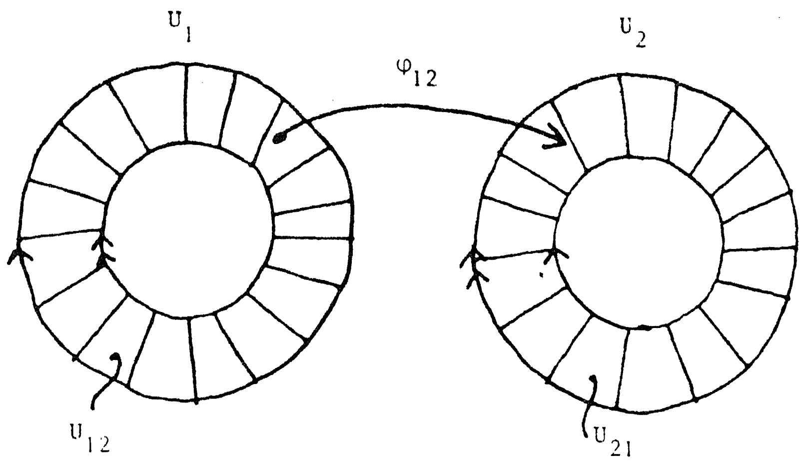

fig.7. Gluing description of the 2-sphere S^2

2.8 Morphisms of differentiable manifolds. Let M and N be differentiable manifolds obtained by the gluing process of section 2.4 above. Say M is obtained by gluing together open subsets U_i of $\mathbb{R}^n$ and N by gluing together open subsets V_j of $\mathbb{R}^m$. Then a smooth map $f: M \to N$ (a morphism) is given by specifying for all i, j an open subset $U_{ij} \subset U_i$ and a smooth map $f_{ij}: U_{ij} \to V_j$ such that $\bigcup_j U_{ij} = U_i$ and the f_{ij} are compatible under the identifications $\varphi_{ii'}: U_{ii'} \to U_{i'i}$, $\varphi_{jj'}: V_{jj'} \to V_{j'j}$, i.e. $f_{i'j'} \circ \varphi_{ii'} = \varphi_{jj'} \circ f_{ij}$ whenever appropriate. (Here the φ's are the gluing diffeomorphisms for M and the ψ's are the gluing diffeomorphisms for N).

2.9 Exercise: Show that the description of the circle S^1 as in 2.1 above gives an injective morphism $S^1 \to \mathbb{R}^2$.

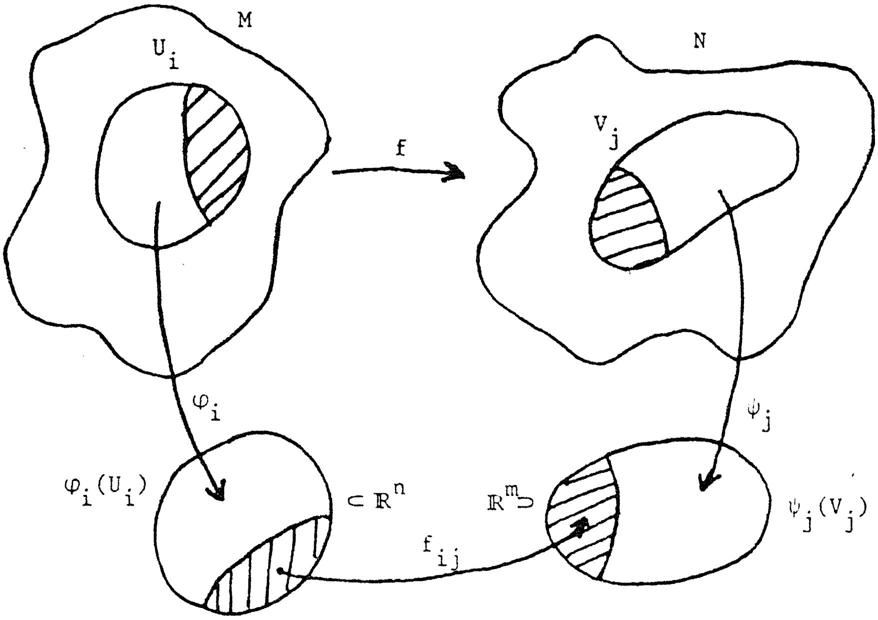

fig.8. Morphisms

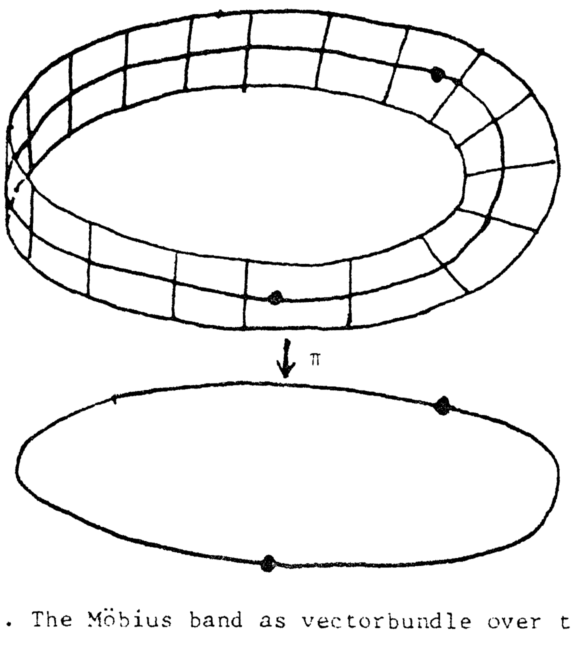

fig.9. The Möbius band as vectorbundle over the circle

3. DIFFERENTIABLE VECTORBUNDLES

Intuitively a vectorbundle over a space S is a family of vectorspaces parametrized by S. Thus for example the Möbius band of example 2.6 can be viewed as a family of open intervals in $\mathbb{R}$ parametrized by the circle, cf. fig. 9 above, and if we are willing to identify the open intervals with $\mathbb{R}$ this gives us a family of one dimensional vectorspaces parametrized by S^1 which locally (i.e. over small neighbourhoods in the base space S^1) looks like a product but globally is not equal to a product.

3.1 Formal definition of a differentiable vectorbundle. A differentiable vectorbundle of dimension m over a differentiable manifold M consists of a surjective morphism $\pi: E \to M$ of differentiable manifolds and a structure of an m-dimensional real vectorspace on $\pi^{-1}(x)$ for all $x \in M$ such that moreover there is for all $x \in M$ an open neighbourhood $U \subset M$ containing x and a diffeomorphism $\varphi_U: U \times \mathbb{R}^m \to \pi^{-1}(U)$ such that the following diagram commutes

$$\begin{array}{ccc} U \times \mathbb{R}^m & \xrightarrow{\ \varphi_U\ } & \pi^{-1}(U) \\ & \searrow \quad \swarrow_{\pi} & \\ & U & \end{array}$$

where the lefthand arrow is the projection on the first factor, and such that φ_U induces for every $y \in U$ an isomorphism $\{y\} \times \mathbb{R}^m \to \pi^{-1}(y)$ of real vectorspaces.

3.2 Constructing vectorbundles. The definition given above is not always particularly easy to assimilate. It simply means that a vectorbundle over M is obtained by taking an open covering $\{U_i\}$ of M and gluing together products $U_i \times \mathbb{R}^m$ by means of diffeomorphisms which are linear (i.e. vectorspace structure preserving) in the second coordinate. Thus an m-dimensional vectorbundle over M is given by the following data

- an open covering $\{U_i\}_{i\in I}$ of M.
- for every i,j a smooth map $\varphi_{ij}: U_i \cap U_j \to GL_m(\mathbb{R})$ where $GL_m(\mathbb{R})$ is the space of all invertible real $m \times m$ matrices considered as an open subset of $\mathbb{R}^{m^2}$. These data are subject to the following compatibility conditions
- $\varphi_{ii}(x) = I_m$, the identity matrix, for all $x \in U_i$
- $\varphi_{jk}(x)\ \varphi_{ij}(x) = \varphi_{ik}(x)$ for all $x \in U_i \cap U_j \cap U_k$

From these data E is constructed by taking the disjoint union of the $U_i \times \mathbb{R}^m$, $i \in I$ and identifying $(x,v) \in U_i \times \mathbb{R}^m$ with $(y,w) \in U_j \times \mathbb{R}^m$ if and only if $x = y$ and $\varphi_{ij}(x)v = w$. The morphism π is induced by the first coordinate projections $U_i \times \mathbb{R}^m \to U_i$.

3.3 Constructing vectorbundles 2. If the base manifold M is itself viewed as a smoothly glued together collection of open sets in $\mathbb{R}^n$ we can descripe the gluing for M and a vectorbundle E over M all at once. The combined data are than as follows

- open sets $U_i \times \mathbb{R}^m$, $U_i \subset \mathbb{R}^n$ for all $i \in I$
- open subsets $U_{ij} \subset U_i$ for all $i,j \in I$
- diffeomorphisms $\varphi_{ij}: U_{ij} \to U_{ji}$
- diffeomorphisms $\tilde{\varphi}_{ij}: U_{ij} \times \mathbb{R}^m \to U_{ji} \times \mathbb{R}^m$ of the form $(x,v) \mapsto (\varphi_{ij}(x), A_{ij}(x)v)$ where $A_{ij}(x)$ is an $m \times m$ invertible real matrix depending smoothly on x.

These data are then subject to the same compatibility conditions for the $\tilde{\varphi}_{ij}$'s (and hence the φ_{ij}) as described in 2.4 above.

3.4 Example: the tangent vectorbundle of a smooth manifold. Let the smooth manifold M be given by the data U_i, U_{ij}, φ_{ij} as in 2.4. Then the tangent bundle TM is given by the data

- $U_i \times \mathbb{R}^n$, $U_{ij} \times \mathbb{R}^n \subset U_i \times \mathbb{R}^n$
- $\tilde{\varphi}_{ij}: U_{ij} \times \mathbb{R}^n \to U_{ji} \times \mathbb{R}^n$, $\tilde{\varphi}_{ij}(x,v) = (\varphi_{ij}(x), J(\varphi_{ij})(x)v)$

where $J(\varphi_{ij})(x)$ is the Jacobian matrix of φ_{ij} at $x \in U_{ij}$.

Exercise: check that these gluing morphisms do indeed define a vectorbundle; i.e. check the compatibility (chain rule !)

3.5 Morphisms of vectorbundles. A morphism of vectorbundles from the vectorbundle $\pi: E \to M$ to the vectorbundle $\pi': E' \to M'$ is a pair of smooth maps $\tilde{f}: E \to E'$, $f: M \to M'$ such that $\pi' \circ \tilde{f} = f \circ \pi$ and such that the induced map $\tilde{f}_x: \pi^{-1}(x) \to \pi^{-1}(f(x))$ is a homomorphism of vectorspaces for all $x \in M$. We leave it to the reader to translate this into a local pieces and gluing data description.

As an example consider two manifolds M, N both described in terms of local pieces and gluing data. Let $f: M \to N$ be given in these terms by the $f_{ij}: U_{ij} \to V_j$ (cf. 2.8 above). Then the maps $\tilde{f}_{ij}: U_{ij} \times \mathbb{R}^n \to V_j \times \mathbb{R}^m$ defined by $\tilde{f}_{ij}(x,v) = (f_{ij}(x), J(f_{ij})(x)v)$ combine to define a morphism of vectorbundles $\tilde{f} = Tf: TM \to TN$.

4. VECTORFIELDS

A vectorfield on a manifold M assigns in a differentiable manner to every $x \in M$ a tangent vector at x, i.e. an element of the fibre $T_xM = \pi^{-1}(x)$ of the tangent bundle TM. Slightly more precisely this gives the

4.1 Definitions. Let $\pi: E \to M$ be a vectorbundle. Then a section of E is a smooth map $s: M \to E$ such that $\pi \circ s = \mathrm{id}$. A section of the tangent vectorbundle $TM \to M$ is called a vectorfield.

Suppose that M is given by a local pieces and gluing data description as in 2.4 above. Then a vectorfield s is given by "local sections" $s_i': U_i \to U_i \times \mathbb{R}^n$ of the form $s_i'(x) = (x, s_i(x))$, i.e. by a collection of functions $s_i: U_i \to \mathbb{R}^n$ such that $J(\varphi_{ij})(x)(s_i(x)) = s_j(x)$ for all $x \in U_{ij}$.

4.2 Derivations. Let A be an algebra over $\mathbb{R}$. Then a derivation is an $\mathbb{R}$-linear map $D: A \to A$ such that $D(fg) = (Df)g + f(Dg)$ for all $f,g \in A$.

4.3 Derivations and vectorfields. Now let M be a differentiable manifold and let S(M) be the $\mathbb{R}$-algebra of smooth functions $M \to \mathbb{R}$. Then every vectorfield s on M defines a derivation of S(M), (which assigns to a function f its derivative along s), which can be described as follows. Let M be given in terms of local pieces U_i and gluing data U_{ij}, φ_{ij}. Let $f: M \to \mathbb{R}$ and the section $s: M \to TM$ be given by the local functions $f_i: U_i \to \mathbb{R}$, $s_i: U_i \to \mathbb{R}^n$. Now define $g_i: U_i \to \mathbb{R}$ by the formula

$$g_i(x) = \sum_k s_i(x)_k \frac{\partial f_i}{\partial x_k}(x) \tag{4.4}$$

where $s_i(x)_k$ is the k-th component of the n-vector $s_i(x)$. It is now an easy exercise to check that $g_j(\varphi_{ij}(x)) = g_i(x)$ for all $x \in U_{ij}$ (because $(\varphi_{ij})(x)s_i(x) = s_j(x)$ for these x) so that the $g_i(x)$ combine to define a function $g = D_s(f): M \to \mathbb{R}$. This defines a map $D: S(M) \to S(M)$ which is seen to be a derivation. Inversely every derivation of S(M) arises in this way.

4.5 The Lie bracket of derivations and vectorfields. Let D_1, D_2 be derivations of an $\mathbb{R}$-algebra A. Then, as is easily checked, so is

$$[D_1, D_2] = D_1 D_2 - D_2 D_1$$

So if s_1, s_2 are vectorfields on M, then there is a vectorfield $[s_1, s_2]$ on M corresponding to the derivation $[D_{s_1}, D_{s_2}]$. This vectorfield is called the Lie bracket of s_1 and s_2 and $(s_1,s_2) \mapsto [s_1,s_2]$ defines a Lie algebra structure on the vectorspace V(M) of all vectorfields on M.

If M is given in terms of local pieces U_i and gluing data U_{ij},φ_{ij} then the Lie bracket operation can be described as follows. Let the vectorfields s and t be given by the local functions $s_i,t_i\colon U_i \to \mathbb{R}^n$. Then [s,t] is given by the local functions

$$\sum_j s_j \frac{\partial t_i}{\partial x_j} - \sum_j t_j \frac{\partial s_i}{\partial x_j}$$

4.6 <u>The $\frac{\partial}{\partial x}$ notation.</u> Let the vectorfield s: M → TM be given by the functions $s_i\colon U_i \to \mathbb{R}^n$. Then using the symbols $\frac{\partial}{\partial x_k}$ in first instance simply as labels for the coordinates in $\mathbb{R}^n$ we can write

$$s_i = \sum s_i(x)_k \frac{\partial}{\partial x_k} \tag{4.7}$$

This is a most convenient notation because as can be seen from (4.4) this gives precisely the local description of the differential operator (derivation) D_s associated to s.

4.7 <u>Differential equations on a manifold.</u> A differential equation on a manifold M is given by an equation

$$\dot{x} = s(x) \tag{4.8}$$

where s: M → TM is a vectorfield, i.e. a section of the tangentbundle. At every moment t, equation (4.8) tells us in which direction and how fast x(t) will evolve by specifying a tangent vector s(x(t)) at x(t).

Again it is often useful to take a local pieces and gluing data point of view. Then the differential equation (4.8) is given by a collection of differential equations $\dot{x} = s_i(x)$ in the

nse of the word on U_i where the functions $s_i(x)$ satisfy
:) $s_i(x) = s_j(x)$ for all $x \in U_{ij}$.
these terms a solution of the differential equation is
. collection of solutions of the local equations, i.e. a
on of maps $f_i: V_i \to U_i$, $V_i \subset \mathbb{R}(\geq 0)$ such that $\cup V_i = \mathbb{R}(\geq 0)$,
$= s_i(f_i(t))$ which fit together to define a morphism
$\to M$, i.e. such that $\varphi_{ij}(f_i(t)) = f_j(t)$ if $t \in V_i \cap V_j$.
more global terms a solution of (4.8) which passes
x_0 at time 0 is a morphism of smooth manifolds $f: \mathbb{R} \to M$
t $Tf: T\mathbb{R} \to TM$ satisfies $Tf(t,1) = s(f(t))$ for all $t \in \mathbb{R}$
itable subset of $\mathbb{R}$), i.e. Tf takes the vectorfield
.) $1: \mathbb{R} \to T\mathbb{R} = \mathbb{R} \times \mathbb{R}$, $t \mapsto (t,1)$ into the vectorfield (sec-
$M \to TM$.

<u>Conclusion.</u> Here, where it starts to get interesting, is,
g to a developing tradition in textbook writing, a good
stop.

A SHORT TUTORIAL ON LIE ALGEBRAS

Michiel Hazewinkel
Dept. Math., Erasmus Univ. Rotterdam
P.O. Box 1738,
3000 DR ROTTERDAM

This tutorial does not correspond to an actual oral lecture during the conference at Les Arcs in June, 1980. However, to improve accessibility and understandability of the material in this volume it seemed wise to include a small section on the basic facts and definitions concerning Lie algebras which play a role in control and nonlinear filtering theory. This is what these few pages attempt to do.

1. DEFINITION OF LIE ALGEBRAS. EXAMPLES. Let k be a field and V a vectorspace over k. (For the purpose of this volume it suffices to take $k = \mathbb{R}$ or (rarely) $k = \mathbb{C}$; the vectorspace V over k need not be finite dimensional). A Lie algebra structure on V is then a bilinear map (called brackett multiplication)

$$[\ ,\ ]: V \times V \to V \tag{1.1}$$

such that the two following conditions hold

$$[u,u] = 0 \text{ for all } u \in V \tag{1.2}$$

M. Hazewinkel and J. C. Willems (eds.), Stochastic Systems: The Mathematics of Filtering and Identification and Applications, 95–108.

(1.3) $[u,[v,w]] + [v,[w,u]] + [w,[u,v] = 0$ for all $u,v,w, \in V$.

The last identity is called the Jacobi identity. Of course the bilinearity of (1.1) means that $[au+bv,w] = a[u,w] + b[v,w]$, $[u,bv+cw] = b[u,v] + c[u,w]$. From (1.2) it follows that

(1.4) $$[u,v] = -[v,u]$$

by considering $[u+v,u+v] = 0$ and using bilinearity.

1.5. Example. The Lie algebra associated to an associative algebra.

Let A be an associative algebra over k. Now define a new multiplication (brackett) on A by the formula

(1.6) $$[v,w] = vw - wv, \quad w,v \in A$$

Then A with this new multiplication is a Lie algebra. (Exercise: check the Jacobi identity (1.3)).

1.9. Remark. In a certain precise sense all Lie algebras arise in this way. That is for every Lie algebra L there is an associative algebra A containing L such that $[u,v] = uv - vu$. I.e. every Lie algebra arises as a subspace of an associative algebra A which happens to be closed under the operation $(u,v) \to uv - vu$. Though this "universal enveloping algebra" construction is quite important it will play no role in the following and the remark is intended to make Lie algebras easier to understand for the reader.

1.7. Example. Let $M_n(k)$ be the associative algebra of all n x n matrices with coefficients in k. The associated Lie algebra is written $g\ell_n(k)$; i.e. $g\ell_n(k)$ is the n^2-dimensional vectorspace of all n x n matrices with the brackett multiplication $[A,B] = AB - BA$.

1.8. Example. Let $s\ell_n(k)$ denote the subspace of all n x n matrices of trace zero. Because $Tr(AB-BA) = 0$ for all n x n matrices A,B, we see that $[A,B] \in s\ell_n(k)$ if $A,B \in s\ell_n(k)$ giving us an (n^2-1)-dimensional sub Lie algebra of $g\ell_n(k)$.

1.10. Example. The Lie algebra of first order differential operators with C^∞-coefficients.

Let V_n be the space of all differential operators (on the space $F(\mathbb{R}^n)$ of C^∞-functions (i.e. arbitrarily often differentiable functions in $x_1,\ldots,x_n$)) of the form

$$X = \sum_{i=1}^{n} f_i(x_1,\ldots,x_n) \frac{\partial}{\partial x_i} \tag{1.11}$$

where the f_i, $i = 1, \ldots, n$ are C^∞-functions. Thus $X: F(\mathbb{R}^n) \to F(\mathbb{R}^n)$ is the operator $X(\phi) = \sum_{i=1}^{n} f_i \frac{\partial \phi}{\partial x_i}$. Now define a brackett operation on V_n by the formula

$$[X,Y] = \sum_{i,j} (f_i \frac{\partial g_j}{\partial x_i} \frac{\partial}{\partial x_j} - g_j \frac{\partial f_i}{\partial x_j} \frac{\partial}{\partial x_i}) \tag{1.12}$$

if $X = \sum f_i \frac{\partial}{\partial x_i}$, $Y = \sum g_j \frac{\partial}{\partial x_j}$. This makes V_n a Lie algebra. Check that $[X,Y](\phi) = X(Y(\phi)) - Y(X(\phi))$ for all $\phi \in F(\mathbb{R}^n)$.

1.13. Example. Derivations. Let A be any algebra (i.e. A is a vectorspace together with any bilinear map (multiplication) $A \times A \to A$: in particular A need not be associative). A derivation on A is a linear map $D: A \to A$ such that

(1.14) $$D(uv) = (Du)v + u(Dv)$$

For example let $A = \mathbb{R}[x]$ and D the operator $\frac{d}{dx}$. The D is a derivation. The operators (1.11) of the example above are derivations on $F(\mathbb{R}^n)$.

Let Der(A) be the vectorspace of all derivations. Define $[D_1,D_2] = D_1D_2 - D_2D_1$. Then $[D_1,D_2]$ is again a derivation and this brackett multiplication makes Der(A) a Lie algebra over k.

1.15. Example. The Weyl algebra W_1. Let W_1 be the vectorspace of all (any order) differential operators in one variable with polynomial coefficients. I.e. W_1 is the vectorspace with basis $x^i \frac{d^j}{dx^j}$, $i,j \in \mathbb{N} \cup \{0\}$. ($x^i$ is considered as the operator $f(x) \to x^i f(x)$). Consider W_1 as a space of operators acting, say, on k[x]. Composition of operators makes W_1 an associative algebra and hence gives W_1 also the structure of a Lie algebra. For example one has

$$[x \frac{d^2}{dx^2} , x^2 \frac{d}{dx}] = 3x^2 \frac{d^2}{dx^2} + 2x \frac{d}{dx} , \quad [x \frac{d}{dx}, x^i \frac{d^j}{dx^j}] = (i-j)x^i \frac{d^j}{dx^j}$$

1.16. Example. The oscillator algebra. Consider the four dimensional subspace of W_1 spanned by the four operators $\frac{1}{2}\frac{d^2}{dx^2} - \frac{1}{2}x^2$, x, $\frac{d}{dx}$, 1. One easily checks that (under the bracket multiplication of W_1)

(1.17)
$$[\tfrac{1}{2}\frac{d^2}{dx^2}-\tfrac{1}{2}x^2,x] = \frac{d}{dx}, \quad [\tfrac{1}{2}\frac{d^2}{dx^2}-\tfrac{1}{2}x^2,\frac{d}{dx}] = x, \quad [\frac{d}{dx},x] = 1$$
$$[\tfrac{1}{2}\frac{d^2}{dx^2}-\tfrac{1}{2}x^2,1] = [x,1] = [\frac{d}{dx},1] = 0$$

Thus this four dimensional subspace is a sub-Lie-algebra of W_1. It is called the <u>oscillator Lie algebra</u> (being intimately associated to the harmonic oscillator).

2. HOMOMORPHISMS, ISOMORPHISMS, SUBALGEBRAS AND IDEALS.

2.1. <u>Sub-Lie-algebras</u>. Let L be a Lie algebra over k and V a subvectorspace of L. If $[u,v] \in V$ for all $u,v \in L$. Then V is a sub-Lie-algebra of L. We have already seen a number of examples of this, e.g. the oscillator algebra of example 1.16 as a sub-Lie-algebra of the Weyl algebra W_1 and the Lie-algebra $s\ell_n(k)$ as a sub-Lie-algebra of $g\ell_n(k)$. Some more examples follow.

2.2. <u>The Lie-algebra</u> $so_n(k)$. Let $so_n(k)$ be the subspace of $g\ell_n(k)$ consisting of all matrices A such that $A + A^T = 0$ (where the upper ^{T} denotes transposes). Then if $A,B \in so_n(k)$

$[A,B] + [A,B]^T = AB - BA + (AB-BA)^T = A(B+B^T) - B(A+A^T) +$

$+ (B^T+B)A^T - (A^T+A)B^T = 0$ so that $[A,B] \in so_n(k)$. Thus $so_n(k)$ is a sub-Lie-algebra of $g\ell_n(k)$.

2.3. The Lie-algebra $t_n(k)$. Let $t_n(k)$ be the subspace of $g\ell_n(k)$ consisting of all upper triangular matrices. Because product and sum of upper triangular matrices are again upper triangular $t_n(k)$ is a sub-Lie-algebra of $g\ell_n(k)$.

2.4. <u>The Lie-algebra</u> $sp_n(k)$. Let Q be the 2n x 2n matrix

$Q = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Now let $sp_n(k)$ be the subspace of all 2n x 2n matrices A such that $AQ + QA^T = 0$. Then as above in example 2.2 one sees that $A,B \in sp_n(k) \Rightarrow [A,B] \in sp_n(k)$ so that $sp_n(k)$ is a sub-Lie-algebra of $g\ell_{2n}(k)$.

2.5. <u>Ideals</u>. Let L be a Lie-algebra over k. A subvectorspace $I \subset L$ with the property that for all $u \in I$ and all $v \in L$ we have $[u,v] \in I$ is called an ideal of L. An example is $s\ell_n(k) \subset g\ell_n(k)$, cf. example 1.8 above. Another example follows.

2.6. Example. The Heisenberg Lie-algebra. Consider the 3-dimensional subspace of W_1 spanned by the operators $x, \frac{d}{dx}, 1$. The formulas (1.17) show that this subspace is an ideal in the oscillator algebra.

2.7. Example. The centre of a Lie algebra. Let L be a Lie algebra. The centre of L is defined as the subset $Z(L) = \{z \in L | [u,z] = 0$ for all $u \in L\}$. Then Z(L) is a subvector space of L and in fact an ideal of L. As an example it is easy to check that the centre of $g\ell_n(k)$ consists of scalar multiples of the unit matrix I_n.

2.8. Homomorphisms and isomorphisms. Let L_1 and L_2 be two Lie algebras over k. A morphism of $\alpha: L_1 \to L_2$ vectorspaces (i.e. a k-linear map) is a homomorphism of Lie algebras if $\alpha[u,v] = \alpha(u),\alpha(v)$ for all $u,v \in L_1$. The homomorphism α is called an isomorphism if it is also an isomorphism of vectorspaces.

2.9. Example. Consider the following three first-order differential operators in two variables x,P

$$a = (1-p^2)\frac{\partial}{\partial p} - Px\frac{\partial}{\partial x}, \quad b = P\frac{\partial}{\partial x}, \quad c = \frac{\partial}{\partial x}$$

Then one easily calculates (cf. (1.9)) $[a,b] = c$, $[a,c] = b$, $[b,c] = 0$. Now define α from the oscillator algebra of example 1.16 to this 3-dimensional Lie algebra as the linear map $\frac{1}{2}\frac{d^2}{dx^2} - \frac{1}{2}x^2 \to a$, $x \to b$, $\frac{d}{dx} \to c$, $1 \to 0$. Then the formulas above and (1.17) show that α is a homomorphism of Lie algebras.

2.10. Kernel of a homomorphism. Let $\alpha: L_1 \to L_2$ be a homomorphism of Lie algebras. Let $Ker(\alpha) = \{u \in L_1 | \alpha(u) = 0\}$. Then $Ker(\alpha)$ is an ideal in L_1.

2.11. Quotient Lie algebras. Let L be a Lie algebra and I an ideal in L. Consider the quotient vector space L/I and the quotient morphisms of vector spaces $L \overset{\alpha}{\to} L/I$. For all $\bar{u},\bar{v} \in L/I$ choose $u,v \in L$ such that $\alpha(u) = \bar{u}$, $\alpha(v) = \bar{v}$. Now define $[\bar{u},\bar{v}] = \alpha[u,v]$. Check that this does not depend on the choice of u,v.

This then defines a Lie-algebra structure on L/I and $\alpha: L \twoheadrightarrow L/I$ becomes a homomorphism of Lie-algebras.

2.12. Image of a homomorphism. Let $\alpha: L_1 \to L_2$ be a homomorphism of Lie algebras. Let $\mathrm{Im}(\alpha) = \alpha(L_1) = \{u \in L_2 \mid \exists v \in L_1,\ \alpha(v) = u\}$. Then $\mathrm{Im}\alpha$ is a sub-Lie-algebra of L_2 and α induces an isomorphism $L_1/\mathrm{Ker}(\alpha) \simeq \mathrm{Im}(\alpha)$.

2.13. Exercise. Consider the 3-dimensional vector space of all real upper triangular 3 x 3 matrices with zero's on the diagonal. Show that this a sub-Lie-algebra of $g\ell_3(\mathbb{R})$, and show that it is isomorphic to the 3-dimensional Heisenberg-Lie-algebra of example 2.6 but that it is not isomorphic to the 3-dimensional Lie-algebra $s\ell_2(\mathbb{R})$ of example 1.8.

2.14. Exercise. Show that the four operators x^2, $\frac{d^2}{dx^2}$, $x\frac{d}{dx}$, 1 span a 4-dimensional subalgebra of W_1, and show that this 4-dimensional Lie algebra contains a three dimensional Lie algebra which is isomorphic to $s\ell_2(\mathbb{R})$.

2.15. Exercise. Show that the six operators x^2, $\frac{d^2}{dx^2}$, x, $\frac{d}{dx}$, $x\frac{d}{dx}$, 1 space a six dimensional sub-Lie-algebra of W_1. Show that x, $\frac{d}{dx}$, 1 space a 3-dimensional ideal in this Lie-algebra and show that the corresponding quotient algebra is $s\ell_2(\mathbb{R})$.

3. LIE ALGEBRAS OF VECTORFIELDS.

Let M be a C^∞-manifold (cf. the tutorial on manifolds and vectorfields in this volume). Intuitively a vectorfield on M specifies a tangent vector t(m) at every point $m \in M$. Then given a C^∞-function f on M we can for each $m \in M$ take the derivation of f at m in the direction t(m), giving us a new function g on

M. This can be made precise in varying ways; e.g. as follows.

3.1. The Lie algebra of vectorfields on a manifold M. Let M be a C^∞-manifold, and let F(M) be the $\mathbb{R}$-algebra (pointwise additior and multiplications) of all smooth (= C^∞) functions f: M → $\mathbb{R}$. By definition a C^∞-vectorfield on M is a derivation X: F(M) → F(M). The Lie algebra of derivations of F(M) cf. example 1.13, i.e. the Lie-algebra of smooth vectorfields on M, is denoted V(M).

3.2. Derivations and vectorfields. Now let $M = \mathbb{R}^n$ so that F(M) is simply the $\mathbb{R}$-algebra of C^∞-functions in $x_1, \ldots, x_n$. Then it is not difficult to show that every derivation X: $F(\mathbb{R}^n) \to F(\mathbb{R}^n)$ is necessarily of the form

$$X = \sum_{i=1}^{n} g_i \frac{\partial}{\partial x_i} \tag{3.3}$$

with $g_i \in F(\mathbb{R}^n)$. For a proof cf. [4, Ch.I,§2]. The correspondin vectorfield on $\mathbb{R}^n$ now assigns to $x \in \mathbb{R}^n$ the tangent vector $(g_1(x), \ldots, g_n(x))^T$.

On an arbitrary manifold we have representations (3.3) locally around every point and these expressions turn out to be compatible in precisely the x way needed to define a vectorfield as described in the tutorial on manifolds and vectorfields in this volume [3].

3.4. Homomorphisms of Lie algebras of vectorfields. Let M and N be C^∞-manifolds and let α: L → V(N) be a homomorphism of Lie algebras where L is a sub-Lie-algebra of V(M). Let ϕ: M → N be a smooth map. Then α and ϕ are said to be compatible if

$$\phi^*(\alpha(X)f) = X(\phi^*(f)) \quad \text{for all } f \in F(N) \tag{3.5}$$

where ϕ^* is the homomorphism of algebras F(N) → F(M), $f \to \phi^*(f) = f \circ \phi$.

In terms of the Jacobian of ϕ (cf.[3]), this means that

(3.6) $$J(\phi)(X_m) = \alpha(X)_{\phi(m)}$$

where X_m is the tangent vector at m of the vectorfield X.

If $\phi : M \to N$ is an isomorphism of C^∞-manifolds there is always precisely one homomorphism of Lie-algebras $\alpha : V(M) \to V(N)$ compatible with ϕ (which is then an isomorphism). It is defined (via formula (3.5)) by

(3.7) $$\alpha(X)(f) = (\phi^*)^{-1}X(\phi^* f), \quad f \in F(N)$$

3.8. <u>Isotropy subalgebras</u>. Let L be a sub-Lie-algebra of V(M) and let m M. The isotropy subalgebra L_m of L at m consists of all vectorfields in L whose tangent vector in m is zero, or, equivalently

(3.9) $$L_m = \{X \in L | Xf(m) = 0 \quad \text{all } f \in F(M)\}$$

Now suppose that $\alpha: L \to V(N)$ and $\phi: M \to N$ are compatible in the sense of 3.4 above. Then it follows easily from (3.5) that

(3.10) $$\alpha(L_m) \subset V(N)_{\phi(m)}$$

i.e. α isotropy subalgebras into isotropy subalgebras. Inversely if we restrict our attention to analytic vectorfields then condition (3.10) on α at m implies that locally there exists a ϕ which is compatible with α [7].

4. SIMPLE, NILPOTENT AND SOLVABLE LIE ALGEBRAS.

4.1. Nilpotent Lie algebras let L be a Lie-algebra over k.

The descending central seems of L is defined inductively by

$$(4.2) \qquad C^1L = L,\ C^{i+1}L = [L, C^iL],\ i \geq 1$$

It is easy to check that the C^iL are ideals. The Lie algebra L is called nilpotent if $C^nL = \{0\}$ for n big enough.

For each $x \in L$ we have the endomorphism adx: $L \to L$ defined by $y \to [x,j]$. It is now a theorem that if L is finite dimensional then L is nilpotent iff the endomorphisms adx are nilpotent for all $x \in L$. Whence the terminology.

4.3. Solvable Lie algebras. The derived seems of Lie algebras of a Lie algebra L is defined inductively by

$$(4.4) \qquad D^1L = L,\ D^{i+1}L = [D^iL, D^iL]\ ,\ i \geq 1$$

It is again easy to check that the D^iL are ideals. The Lie algebra L is called solvable if $D^nL = \{0\}$ for n large enough.

4.5. Examples. The Heisenberg Lie algebra of example 2.6 is nilpotent. The Oscillator algebra of example 1.16 is solvable but not nilpotent. The sub-Lie-algebra of W_1 with vector-space basis $x^2, \frac{d^2}{dx^2}, x, \frac{d}{dx}, 1, x\frac{d}{dx}$ is neither nilpotent, nor solvable. The Lie-algebra $t_n(k)$ of example 2.3 is solvable and in a way is typical of finite dimensional solvable Lie algebras in the sense that if k is algebraically closed (e.g. k = C), then every finite dimensional solvable Lie algebra over k is isomorphic to a sub-Lie-algebra of some $t_n(k)$.

4.6. Exercise. Show that sub-Lie-algebras and quotient-Lie-algebras of solvable Lie algebras (resp. nilpotent Lie algebras) are solvable (resp. nilpotent).

4.7. Abelian Lie-algebras. A lie algebra L is called abelian if $[L,L] = \{0\}$, i.e. if every brackett product is zero.

4.8. Simple Lie-algebras. A Lie algebra L is called simple if it is not abelian and if it has no other ideals than 0 and L. (Given the second condition the first one only rules out the zero- and one-dimensional Lie algebras). These simple-Lie-algebras and the abelian ones are in a very precise sense the basic building blocks of all Lie algebras.

The finite dimensional simple Lie algebras over $\mathbb{C}$ have been classified. They are the Lie algebras $sl_n(\mathbb{C})$, $sp_n(\mathbb{C})$, $so_n(\mathbb{C})$ of examples 1.8, 2.4 and 2.2 above and five additional exceptional Lie algebras. For infinite dimensional Lie algebras things are more complicated. The socalled filtered, primitive, transitive simple Lie algebras have also been classified (cf. e.g. [2]). One of these is the Lie-algebra $\hat{V}_n$ of all formal vector fields $\Sigma f_i(x_1,\ldots,x_n)\frac{\partial}{\partial x_i}$, where the $f_i(x)$ are (possibly non converging) formal power series in $x_1,\ldots, x_n$. This class of infinite dimensional simple Lie algebras by no means exhausts all possibilities. E.g. the quotient-Lie-algebras $W_n/\mathbb{R}.1$ are simple and non-isomorphic to any of those just mentioned.

4.9. Exercise. Let $V_{alg}(\mathbb{R}^n)$ be the Lie algebra of all differential operators (vector fields) of the form $\Sigma f_i(x_1,\ldots,x_n)\frac{\partial}{\partial x_i}$ with $f_i(x_1,\ldots,x_n)$ polynomial. Prove that $V_{alg}(\mathbb{R}^n)$ is simple.

5. REPRESENTATIONS.

Let L be a Lie algebra over k and M a vectorspace over k. A representation of L in M is a homomorphism of Lie algebras.

$$\rho : L \to \text{End}_k(M) \tag{5.1}$$

where $\text{End}_k(M)$ is the vectorspace of all k-linear maps $M \to M$ which is of course given the Lie algebra structure

$[A,B] = AB - BA$. Equivalently a representation of L in M consists of a k-bilinear map

$$(5.2) \qquad \sigma : L \times M \to M$$

such that, writing xm for $\sigma(x,m)$, we have $x,y\ m = x(ym) - y(xm)$ for all $x,y \in L$, $m \in M$. The relation between the two definitions is of course $\sigma(x,m) = \rho(x)(m)$.

Instead of speaking of a representation of L in M we also speak (equivalently) of the L-module M.

5.3. <u>Example</u>. The Lie algebra $gl_n(k)$ of all $n \times n$ matrices naturally acts on k^n by $(A,v) \to Av \in k^n$ and this defines a representation $gl_n(k) \times k^n \to k^n$. The Lie algebra V(M) of vectorfields on a manifold M acts (by its definition) on F(M) and this is a representation of V(M). A quite important theorem concerning the existence of representations is

5.4. <u>Ado's theorem</u>. Cf. e.g. [1,§7]. If k is a field of characteristic zero, e.g. $k = \mathbb{R}$ or C and L is finite dimensional then there is a faithful representation $\rho: L \to End(k^n)$ for some n. (Here faithful means that ρ is injective).

Thus every finite dimensional Lie algebra L over k (of characteristic zero) can be viewed as a subalgebra of some $gl_n(k)$, and this subalgebra can then be viewed as a more concrete matrix "representation" of the "abstract" Lie algebra L.

5.5. Realizing Lie-algebras in V(M). A question of some importance for filtering theory is when a LIe algebra L can be realized as a sub-Lie-algebra of V(M), i.e. when L can be represented in F(M) by means of derivations of several papers in this volume for a discussion of the relevance of this problem. For finite dimensional Lie algebras Ado's theorem gives the answer because $(a_{ij}) \to \Sigma a_{ij} x_i \frac{\partial}{\partial x_j}$ defines an injective homomorphism of Lie-algebras $gl_n(\mathbb{R}) \to V(\mathbb{R}^n)$ (Exercise: check this)

6. LIE ALGEBRAS AND LIE GROUPS.

6.1. Lie groups. A (finite dimensional) Lie group is a finite dimensional smooth manifold G together with smooth maps $G \times G \to G$, $(x,y) \to xy$, $G \to G$, $x \to x^{-1}$ and a distinguished element $e \in G$ which make G a group. An example is the open subset of $\mathbb{R}^{n^2}$ consisting of all invertible n x n matrices with the usual matrix multiplication.

6.2. Left invariant vectorfields and the Lie algebra of a Lie group.

Let G be a Lie group. Let for all $g \in G$, $L_g: G \to G$ be the smooth map $x \to gx$. A vectorfield $X \in V(G)$ is called left invariant if $X(L_g^*f) = XL_g^*(Xf)$ for all functions f on G. Or, equivalently, if $J(L_g)X_x = X_{gx}$ for all $x \in G$, cf. section 3.4 above. Especially from the last condition it is easy to see that $X \to X_e$ defines an isomorphism between the vectorspace of left invariant vectorfields on G and the tangent space of G at e. Now the brackett product of two left invariant vectorfields is easily seen to be left invariant again so the tangent space of G at e (which is $\mathbb{R}^n$ if G is n-dimensional) inherits a Lie algebra structure. This is the Lie algebra Lie(G) of the Lie group G. It reflects so to speak the infinitesimal structure of G. A main reason for the importance of Lie algebras in many parts of mathematics and its applications is that this construction is reversible to a great extent making it possible to study Lie groups by means of their Lie algebras.

6.3. Exercise. Show that the Lie algebra of the Lie group $GL_n(\mathbb{R})$ of invertible real n x n matrices is the Lie algebra $gl_n(\mathbb{R})$.

7. POSTSCRIPT.

The above is a very rudimentary introduction to Lie algebras Especially the topic "Lie algebras and Lie groups" also called "Lie theory" has been given very little space, in spite of the fact that it is likely to become of some importance in filtering (integration of a representation of a Lie algebra to a representation of a LIe (semi)group). The books [1, 4, 5, 6, 8] are all recommended for further material. My personal favourite (but by no means the easiest) is [4]; [6] is a classic and in its present incarnation very good value indeed.

REFERENCES.

1. A. Bourbaki, Groupes et algèbres de Lie, Chap. 1: Algèbres de Lie, Hermann, 1960.
2. M. Demazure, Classification des algèbres de Lie filtres, Sém. Bourbaki 1966/1967, Exp. 326, Benjamin, 1967.
3. M. Hazewinkel, Tutorial on Manifolds and Vectorfields. This volume.
4. S. Helgason, Differential Geometry, Lie Groups and Symmetric Spaces, Acad. Pr., 1978.
5. J.E. Humphreys, Introduction to Lie Algebras and Representation Theory, Springer 1972.
6. N. Jacobson, Lie Algebras, Dover reprint, 1980.
7. A.J. Krener, On the Equivalence of Control Systems and the Linearization of nonlinear Systems SIAM J. Control 11(1973), 670-676.
8. J.P. Serre, Lie Algebras and Lie Groups, Benjamin 1965.