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ON ALMOST-FIXED-POINT THEORY

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1. INTRODUCTION

Let X be a topological space, $\mathcal{U}$ a finite covering of X (NB the words 'covering' and 'cover' are used interchangeably). We say that $(X, \mathcal{U})$ has the almost fixed point property for a class $\mathcal{F}$ of continuous maps $f: X \to X$ if for all $f \in \mathcal{F}$ there is an $x \in X$ and $U \in \mathcal{U}$ such that $x \in U$ and $f(x) \in U$, or, equivalently, if there is a $U \in \mathcal{U}$ such that $U \cap f(U) \neq \phi$.

For example if X is the euclidean plane and $\mathcal{U}$ a finite open covering by convex sets then $(X, \mathcal{U})$ has the almost fixed point property for all continuous maps. Cf. De Groot, De Vries, Van der Walt [2]. Other examples of almost fixed point theorems can be found in Klee [7], Halpern [6] and Gray, Vaughan [5].

It is fairly natural to restrict attention to finite coverings of X. Indeed if a space X is such that $(X, \mathcal{U})$ has the almost fixed point property for all open coverings $\mathcal{U}$, then X has the fixed point property. It is also fairly natural to concentrate somewhat on noncompact spaces X because if a compact space X is such that $(X, \mathcal{U})$ has the almost fixed point property for all (or a cofinal set of) finite coverings $\mathcal{U}$ then X has the fixed point property.

There is an extension of this result. Let X be a T_1-space and $f: X \to X$ a continuous map. Let w(X) be the Wallman compactification of X. There is an induced continuous map $w(f): w(X) \to w(X)$. The following two statements are then equivalent: (i) w(f) has a fixed point; (ii) for every finite open

covering $\mathcal{U}$ of X there is a $U \in \mathcal{U}$ such that $U \cap f(U) \neq \phi$. This follows from the fact that there is a one-one correspondence between finite coverings of X and finite coverings of w(X).

In this paper we develop what Thompson [11] calls an indirect theory. A main result is a Lefshetz-type almost fixed point theorem. We first define a certain kind of finite coverings called geometric coverings. The result then is: let X be a space with a closed geometric covering $\mathcal{C}$ and let $f: X \to X$. Then $L(f) = 0$ or for every finite open cover $\mathcal{U}$ which is refined by $\mathcal{C}$ there is a $U \in \mathcal{U}$ such that $U \cap f(U) \neq \phi$. Here L(f), the Lefshetz number, is defined in terms of compactly generated Čech homology.

The next step is then to find at least some examples of geometric coverings. In this direction we have e.g. the following results:

(i) A compact space X admits a weak semicomplex structure (cf. Thompson [10] for this notion) if and only if every finite open covering is geometric. (Spaces which admit a WSC structure include all compact polyhedra).

(ii) If X is a not necessarily compact normal space and $\mathcal{C}$ is a finite closed convexoid covering which admits a finite open refinement then $\mathcal{C}$ is geometric. Cf. 8.4.

This last result, the Lefshetz-type almost fixed point theorem, and a result on the existence of finite closed convex refinements of finite open convex coverings of euclidean spaces then combine to give a proof of the following almost fixed point theorem, conjectured by De Groot, cf. [2].

Let $\mathcal{U}$ be a finite open convex covering of $\mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$ continuous, then there is a $U \in \mathcal{U}$ such that $U \cap f(U) \neq \phi$.

2. COMPACTLY GENERATED ČECH HOMOLOGY

In this section we introduce some notation and give a short outline of the definitions of the (compactly generated) Čech homology groups. For more details cf e.g. Eilenberg Steenrod [3] and Spanier [9].

2.1. Some Notations and Conventions

If X is a topological space then $\mathrm{cov}^f(X)$ denotes the set of all finite open coverings of X. All coverings occurring in this paper (open or not) will be finite. If $\mathcal{C}$ is a finite covering of X and $\mathcal{A} \subset \mathcal{C}$ is a subset then $\cap\mathcal{A} = \bigcap_{A\in\mathcal{A}} A$.

A simplicial complex will be an abstract simplicial complex. All simplicial complexes will be finite. If S is a simplicial complex then S^n denotes its n-skeleton, ΔS is the chain complex with coefficients in $\mathbb{Q}$ associated to S and $H_k(S)$ is the k-th homology group of ΔS. The symbol I denotes the simplicial complex with two vertices e_0, e_1 and one 1-simplex $\{e_0, e_1\}$. If S and S_1 are two simplicial complexes then $S \times S_1$ is their cartesian product. The vertices of $S \times I$ are pairs (a, e_i) where a is a vertex of S, i = 0,1; we write a^i for (a, e_i), i = 0,1. With this notation the simplices of $S \times I$ can be described as follows: let $a_1, \ldots, a_s$ be an ordering of the vertices of S. Then a simplex of $S \times I$ is of the form $\{a^0_{i_1}, \ldots, a^0_{i_r}, a^1_{i_{r+1}}, \ldots, a^1_{i_k}\}$ where

$i_1 < \ldots < i_r \leq i_{r+1} < \ldots < i_k$.

Let $\mathcal{U} \in \mathrm{cov}^f(X)$, K a subspace of X; with $\mathcal{U}|K$ we denote the set of subsets of K of the form $U \cap K$, $U \in \mathcal{U}$, and $\check{C}(K, \mathcal{U})$ stands for the simplicial complex which is the nerve of $\mathcal{U}|K$; i.e. a typical simplex of $\check{C}(K, \mathcal{U})$ is a subset $\sigma = \{U_{i_1}, \ldots, U_{i_n}\}$ of $\mathcal{U}$ such that $U_{i_1} \cap \ldots \cap U_{i_n} \cap K \neq \phi$. The n-skeleton of $\check{C}(K, \mathcal{U})$ is denoted $\check{C}^n(K, \mathcal{U})$ and the k-th homology group of $\Delta\check{C}(K, \mathcal{U})$ is denoted $H_k(\mathcal{U}|K)$; if K = X we sometimes write $H_k(\mathcal{U})$ or $H_k(X, \mathcal{U})$ for $H_k(\mathcal{U}|X)$ and $\check{C}(\mathcal{U})$ for $\check{C}(X, \mathcal{U})$.

Let σ be a simplex of $\check{C}(K, \mathcal{U})$, $\mathcal{U} \in \mathrm{cov}^f(X)$, $K \subset X$. Then the support of σ, $\mathrm{supp}(\sigma)$, is defined as $\bigcup_{U\in\sigma} U \cap K$ and the reduced support of σ as

$\mathrm{rsupp}(\sigma) = \bigcap_{U\in\sigma} U \cap K$. A chain $c \in \Delta\check{C}(K, \mathcal{U})$ is said to be on a subspace $A \subset K$ if c is in the subchain complex $\Delta\check{C}(A, \mathcal{U}) \subset \Delta\check{C}(K, \mathcal{U})$ or, equivalently, if $\mathrm{rsupp}(\sigma) \cap A \neq \phi$ for all σ occurring in c (i.e. having nonzero coefficient in c).

2.2. Čech Homology of Compact Spaces.

Let K be a compact space, K' a closed (compact) subspace of K. Let $\mathcal{V}' \in \mathrm{cov}^f(K')$ and $\mathcal{V} \in \mathrm{cov}^f(K)$ then we say that $\mathcal{V}'$ refines $\mathcal{V}$ and write $\mathcal{V}' \leq \mathcal{V}$ if for every $V' \in \mathcal{V}'$ there is a $V \in \mathcal{V}$ such that $V' \subset V$. Choosing such a V for every $V' \in \mathcal{V}'$ defines a map of simplicial complexes $\check{C}(K', \mathcal{V}') \to \check{C}(K, \mathcal{V})$ and a homomorphism of chain complexes $\Delta\check{C}(K', \mathcal{V}') \to \Delta\check{C}(K, \mathcal{V})$ and induces a homomorphism $H_k(\mathcal{V}') \to H_k(\mathcal{V})$. All these maps are called refinement maps. There is usually more than one refinement map $\check{C}(K', \mathcal{V}') \to \check{C}(K, \mathcal{V})$ but they are all homotopic and hence induce the same homomorphism $H_k(\mathcal{V}') \to H_k(\mathcal{V})$.

Applying this with $K' = K$ and letting $\mathcal{V}$ run through $\mathrm{cov}^f(K)$ we obtain a projective system of groups and homomorphisms, $H_k(K,-)$, indexed by $\mathrm{cov}^f(K)$. The k-th Čech homology group of K is now defined as $\check{H}_k(K) = \lim_{\leftarrow} H_k(K,-)$. The canonical projection $\check{H}_k(K) \to H_k(\mathcal{U})$ is denoted $q^K_{\mathcal{U}}$.

Let $f: K_1 \to K_2$ be a continuous map of compact spaces. Every $\mathcal{U}_2 \in \mathrm{cov}^f(K_2)$ then gives rise to a $f^{-1}\mathcal{U}_2 = \mathcal{U}_1 \in \mathrm{cov}^f(K_1)$, $f^{-1}(\mathcal{U}_2) = \{f^{-1}U_2 \mid U_2 \in \mathcal{U}_2\}$. Assigning to a vertex $f^{-1}U_2$ the vertex U_2 defines a map of simplicial complexes $\check{C}(K_1, \mathcal{U}_1) \to \check{C}(K_2, \mathcal{U}_2)$ and induces a homomorphism $H_k(f) : H_k(\mathcal{U}_1) \to H_k(\mathcal{U}_2)$. Letting $\mathcal{U}_2$ run through $\mathrm{cov}^f(K_2)$ we obtain a homomorphism of projective systems $H_k(f): H_k(K_1,-) \to H_k(K_2,-)$ which in turn gives rise to a homomorphism $\check{H}_k(f): \check{H}_k(K_1) \to \check{H}_k(K_2)$.

Note that we have a commutative diagram

(2.2.1)
$$\begin{array}{ccc} \check{H}_k(K_1) & \xrightarrow{\check{H}_k(f)} & \check{H}_k(K_2) \\ \Big\downarrow q & & \Big\downarrow q \\ H_k(K_1, \mathcal{U}_1) & \longrightarrow & H_k(K_2, \mathcal{U}_2) \end{array}$$

whenever $\mathcal{U}_1$ refines $f^{-1}\mathcal{U}_2$ (where the lower horizontal homomorphism is the composite of a refinement map $H_k(K_1, \mathcal{U}_1) \to H_k(K_1, f^{-1}\mathcal{U}_2)$ and $H_k(f)\colon H_k(K_1, f^{-1}\mathcal{U}_2) \to H_k(K_2, \mathcal{U}_2)$.

The Čech homology theory on compact spaces (or more generally compact pairs) satisfies all the usual homology axioms, cf. Eilenberg Steenrod [3].

2.3. Compactly Generated Čech Homology

Now let X be a not necessarily compact topological space. We could of course again write down the definitions of 2.2 and thus define Čech homology groups of X based on finite covers. However, this homology theory does not satisfy the homotopy axiom (e.g. the Čech homology based on finite covers of $\mathbb{R}$ is not trivial). Instead we take compactly generated Čech homology, also called Čech homology with compact supports, which is defined as follows.

Let Γ be a cofinal collection of compact subsets of X, i.e. for every compact $K \subset X$, there is a $K' \in \Gamma$ such that $K \subset K'$. For each $K \in \Gamma$, write down $\check{H}_k(K)$: if $K_1 \subset K_2$, $K_1, K_2 \in \Gamma$ we have an induced homomorphism $\check{H}_k(K_1) \to \check{H}_k(K_2)$ giving us an injective system of homology groups indexed by Γ. We now define $H_k(X) = \varinjlim_{K \in \Gamma} \check{H}_k(K)$. This definition does not depend on Γ. If $f\colon X \to Y$ is a continuous map then for every $K \subset X$, K compact we have that $f(K) \subset Y$ is compact and hence we have an induced map $\check{H}_k(K) \to \check{H}_k(fK)$ and this gives us a homomorphism of inductive systems and, taking the limit, an induced homomorphism $\check{H}_k(f)\colon \check{H}_k(X) \to \check{H}_k(Y)$.

For compact spaces X these definitions agree with the ones from 2.2.

Let $\mathcal{U} \in \mathrm{cov}^f(X)$, i.e. $\mathcal{U}$ is a finite open cover of the (not necessarily compact) space X. Then there is a natural homomorphism $q^X_{\mathcal{U}} : \check{H}_k(X) \to H_k(\mathcal{U})$ which is defined as follows. Let $z \in \check{H}_k(X)$, then there is a compact K and a $z' \in \check{H}_k(K)$ such that z' is mapped onto z under the natural homomorphism $\check{H}_k(K) \to \check{H}_k(X)$. Enlarging K if necessary we can assume that $\check{C}(K, \mathcal{U}) = \check{C}(X, \mathcal{U})$. We now define $q^X_{\mathcal{U}}(z) = q^K_{\mathcal{U}}(z')$. This does not depend on K.

Let f: X → Y be a continuous map, $\mathcal{V} \in \mathrm{cov}^f(Y)$, $\mathcal{U} \in \mathrm{cov}^f(X)$ and suppose that $\mathcal{U} \leq f^{-1}\mathcal{V}$, then we have a commutative diagram

$$\begin{array}{ccc} \check{H}_k(X) & \xrightarrow{\check{H}_k(f)} & \check{H}_k(Y) \\ \downarrow{\scriptstyle q^X_{\mathcal{U}}} & & \downarrow{\scriptstyle q^Y_{\mathcal{V}}} \\ H_k(\mathcal{U}) & \longrightarrow & H_k(\mathcal{V}) \end{array}$$

where the lower horizontal homomorphism is defined in the obvious way.

2.4. Reduced Homology Groups.

The chain complexes $\Delta\check{C}(K,\mathcal{U})$ carry a natural augmentation. The homology groups of the augmented complex are the reduced homology groups denoted $\tilde{H}_k(K,\mathcal{U})$. Replacing $H_k(K,\mathcal{U})$ with $\tilde{H}_k(K,\mathcal{U})$ everywhere in 2.2 and 2.3 then defines reduced Čech homology groups $\check{\tilde{H}}_k(X)$, $\check{\tilde{H}}_k(K)$

2.5. Lefshetz Theorem.

Let X be a compact space, $\mathcal{U} \in \mathrm{cov}^f(X)$. Then there is a $\mathcal{V} \in \mathrm{cov}^f(X)$ which refines $\mathcal{U}$ such that

$$\mathrm{Im}(\check{H}(X) \to H(\mathcal{U})) = \mathrm{Im}(H(\mathcal{V}) \to H(\mathcal{U}))$$

This follows directly from the fact that the $H(\mathcal{U})$ are finite dimensional vector spaces over $\mathbb{Q}$.
The same result holds for reduced homology (for the same reason).

3. GEOMETRIC COVERS.

In this section we define and discuss the main technical tool of this paper, the notion of what we like to call a geometric cover.

3.1. Definition of Geometric Covers.

Let X be a topological space. A finite (not necessarily open) cover $\mathcal{C}$ of X is *geometric in dimension* $\leq n$ *with respect to compactly generated Čech homology* if there exist

(i) a cofinal collection Γ of compact subsets of X

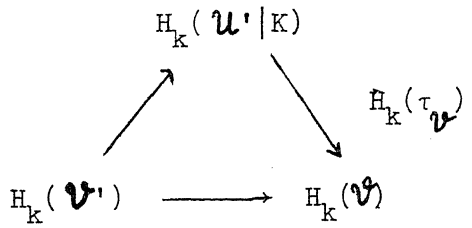
(ii) a map $\gamma : \Gamma \to \Gamma$ such that

(3.1.1) $\qquad K \subset \gamma(K)$ for all $K \in \Gamma$

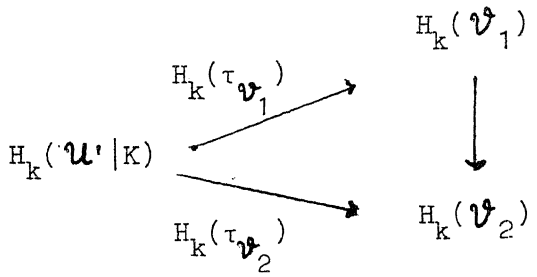
(iii) a finite open refinement $\mathcal{U}'$ of $\mathcal{C}$

(iv) for every $K \in \Gamma$ and $\mathcal{V} \in \mathrm{cov}^f(\gamma K)$ such that $\mathcal{V} \leq \mathcal{U}'$ an augmentation preserving chain map $\tau_{\mathcal{V}} : \Delta\check{C}^{n+1}(K, \mathcal{U}') \to \Delta\check{C}(\gamma K, \mathcal{V})$ such that the following conditions are satisfied

(3.1.2) (Factorization property). If $k \leq n$, $K \in \Gamma$, $\mathcal{V} \in \mathrm{cov}^f(\gamma K)$, $\mathcal{V} \leq \mathcal{U}'$, then there is a $\mathcal{V}' \in \mathrm{cov}^f(K)$, $\mathcal{V}' \leq \mathcal{V}$ such that the following diagram commutes (where the two unlabelled arrows are induced by refinement maps).



(3.1.3) (Compatibility property). If $k \leq n$, $K \in \Gamma$, $\mathcal{V}_1, \mathcal{V}_2 \in \mathrm{cov}^f(\gamma K)$, $\mathcal{V}_1 \leq \mathcal{V}_2 \leq \mathcal{U}'$ than the following diagram commutes



(3.1.4) (Norm condition). For every $\tau_{\mathcal{V}}$ and for every $\sigma \in \check{C}^{n+1}(K, \mathcal{U}')$ there is a $C \in \mathcal{C}$ such that $\mathrm{rsupp}(\sigma) \subset C \cap K$ and $\tau_{\mathcal{V}}(\sigma)$ is on $C \cap \gamma K$; i.e. $\mathrm{rsupp}(\sigma') \cap C \cap \gamma K \neq \phi$ for all σ' occurring in $\tau_{\mathcal{V}}(\sigma)$.

In the sequel we shall use n-geometric as an abbreviation for geometric in dimensions $\leq n$ with respect to compactly generated Čech homology.

A finite cover $\mathcal{C}$ is geometric if there exists a finite open refinement $\mathcal{U}' \in \mathrm{cov}^f(X)$ of $\mathcal{C}$ such that there are for every $n \in N$ a cofinal collection of compact sets Γ_n, a map $\gamma_n : \Gamma_n \to \Gamma_n$, and for every $K_n \in \Gamma_n$ and $\mathcal{V} \in \mathrm{cov}^f(\gamma_n K_n)$, $\mathcal{V} \leq \mathcal{U}'$ chain maps $\tau_{\mathcal{V}}^{(n)} : \Delta\check{C}^{n+1}(K_n, \mathcal{U}') \to \Delta\check{C}(\gamma_n K_n, \mathcal{V})$ such that (3.1.1) - (3.1.4) hold. Note that $\mathcal{U}'$ is not allowed to depend on n (but that everything else may depend on n).

Examples of geometric covers are all finite open covers of compact spaces which admit a weak semicomplex structure, cf. 3.2 and 3.3 below.

If X is a not necessarily compact (normal) space and $\mathcal{C}$ is a finite closed convexoid cover which has a finite open refinement then $\mathcal{C}$ is geometric, cf. 8.4.

As in the case of weak semicomplexes the existence of geometric covers is closely related to various forms of local acyclicity (or local connectedness) of the space X, Cf. 4.5 and sections 7 and 8.

Let $\mathcal{C}$ be an n-geometric (or geometric) cover of a space X. Then $\mathcal{O}(\mathcal{C})$ denotes the collection of all covers $\mathcal{U}'$ of X refining $\mathcal{C}$ such that there exist corresponding Γ, γ, $\tau_{\mathcal{V}}$ satisfying the conditions listed above.

Note that $\mathcal{U}' \in \mathcal{O}(\mathcal{C})$ and $\mathcal{U}'' \in \mathrm{cov}^f(X)$, $\mathcal{U}'' \leq \mathcal{U}' \Rightarrow \mathcal{U}'' \in \mathcal{O}(\mathcal{C})$ and $\mathcal{U}' \in \mathcal{O}(\mathcal{C})$, $\mathcal{C} \leq \mathcal{C}' \Rightarrow \mathcal{U}' \in \mathcal{O}(\mathcal{C}')$, i.e. every cover of X refined by an n-geometric one is itself n-geometric.

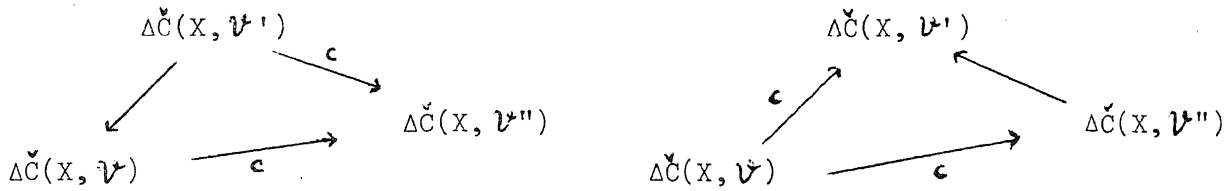
3.2. Weak Semicomplex Structures (Thompson [10])

Let X be a compact space. A weak semicomplex structure (WSC) on X consists of

(i) for every $\mathcal{U} \in \mathrm{cov}^f(X)$ a cofinal subset $\Omega(\mathcal{U}) \subset \mathrm{cov}^f(X)$ with a coarsest element $\alpha(\mathcal{U}) \in \Omega(\mathcal{U})$, $\alpha(\mathcal{U}) \leq \mathcal{U}$

(ii) for every $\mathcal{V}, \mathcal{W} \in \Omega(\mathcal{U})$, $\mathcal{V} \leq \mathcal{W}$ an augmentation preserving chain map $c_{\mathcal{V}}^{\mathcal{W}} : \Delta\check{C}(X, \mathcal{W}) \to \Delta\check{C}(X, \mathcal{V})$ such that the following conditions are satisfied

(3.2.1) If $\mathcal{V}'' \leq \mathcal{V}' \leq \mathcal{V}$ in $\Omega(\mathcal{U})$ then the following diagrams are commutative up to homotopy (where the unlabelled arrows are refinement maps)



(3.2.2) For each $\mathcal{V} \in \Omega(\mathcal{U})$, $c_{\mathcal{V}}^{\mathcal{V}}$ induces an idempotent homomorphism $H(\mathcal{V}) \to H(\mathcal{V})$ of which the image coincides with the image of $\check{H}(X)$ in $H(\mathcal{V})$ (under the natural map)

(3.2.3) If $\mathcal{V} \leq \mathcal{W}$ in $\Omega(\mathcal{U})$ then the chain map $c_{\mathcal{V}}^{\mathcal{W}}$ satisfies the following norm condition: For every $c_{\mathcal{V}}^{\mathcal{W}} : \Delta\check{C}(X, \mathcal{W}) \to \Delta\check{C}(X, \mathcal{V})$ and every simplex $\sigma \in \check{C}(X, \mathcal{W})$ there exists a $U \in \mathcal{U}$ such that $\mathrm{supp}(\sigma) \subset U$ and $\mathrm{supp}(c_{\mathcal{V}}^{\mathcal{W}}(\sigma)) \subset U$.

3.3. Theorem.

Let X be a compact space which admits a WSC structure. Then every finite open cover of X is geometric.

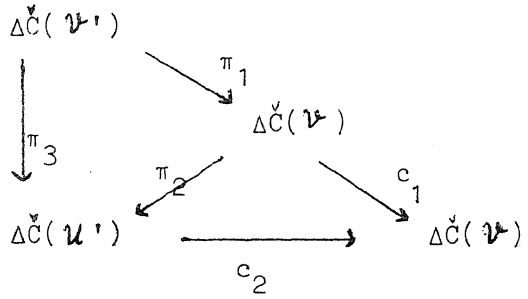
Proof. Let $\mathcal{U} \in \mathrm{cov}^f(X)$. Take $\mathcal{U}' = \alpha(\mathcal{U})$, cf 3.2 above. We take $\Gamma = \{X\}$, and define $\tau_{\mathcal{V}} : \check{C}(X, \mathcal{U}') \to \Delta\check{C}(X, \mathcal{V})$ for every $\mathcal{V} \in \mathrm{cov}^f(X)$ refining $\mathcal{U}'$ as follows:

if $\mathcal{V} \in \Omega(\mathcal{U})$ take $\tau_{\mathcal{V}} = c_{\mathcal{V}}^{\mathcal{U}'}$

if $\mathcal{V} \notin \Omega(\mathcal{U})$ let $\mathcal{V}' \in \Omega(\mathcal{U})$ be a refinement of $\mathcal{V}$ and define $\tau_{\mathcal{V}}$ as the composite of $c_{\mathcal{V}'}^{\mathcal{U}'}$ with the refinement map $\Delta\check{C}(X, \mathcal{V}') \to \Delta\check{C}(X, \mathcal{V})$

This definition does not depend on $\mathcal{V}'$ (up to homotopy) because of (3.2.1) (second diagram).

We check the various axioms. (3.1.1) is automatic and (3.1.3) and (3.1.4) follow from respectively (3.2.1) (second diagram and (3.2.3). It remains to prove the factorization condition. Again it suffices to do this for $\mathcal{V} \in \Omega(\mathcal{U})$. Let $\mathcal{V}' \leq \mathcal{V}$ be such that the image of $H(\mathcal{V}')$ in $H(\mathcal{V})$ is equal to the image of $\check{H}(X)$ in $H(\mathcal{V})$. Such a $\mathcal{V}' \in \mathrm{cov}^f(X)$ exists by the Lefshetz theorem 2.5. We have a diagram



and this diagram is commutative up to homotopy. Now according to (3.2.2) c_{1*} is idempotent with as image the image of $H(X)$ in $H(\mathcal{V})$.

It follows that $c_{1*}\,\pi_{1*} = \pi_{1*}$. And hence we have

$\pi_{1*} = c_{1*}\pi_{1*} = c_{2*}\pi_{2*}\pi_{1*} = c_{2*}\pi_{3*}$ which proves the factorization.

Inversely we have

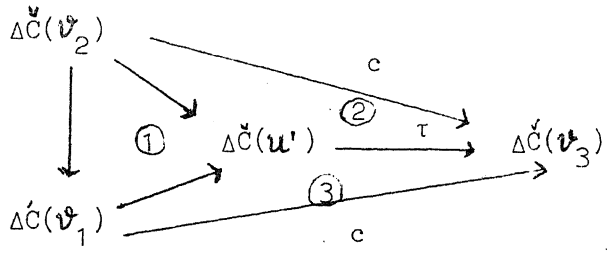
3.4. Theorem.

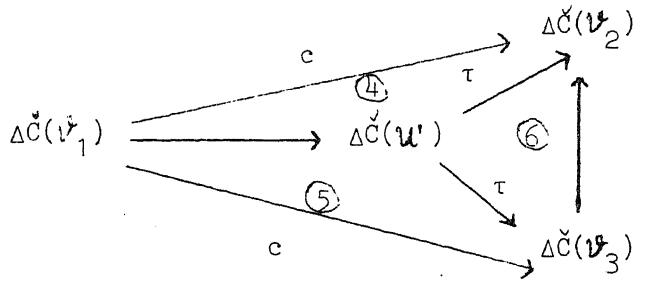
Let X be a compact space and suppose that every open covering of X is geometric. Then X admits a WSC structure.

Proof. Let $\mathcal{U}_o \in \mathrm{cov}^f(X)$, let $\mathcal{U}$ be a starrefinement of $\mathcal{U}_o$. The cover $\mathcal{U}$ is geometric by hypothesis; let $\mathcal{U}' \in \mathcal{A}(\mathcal{U})$. We take $\alpha(\mathcal{U}_o) = \mathcal{U}'$ and $\Omega(\mathcal{U}_o) = \{\, \mathcal{V} \in \mathrm{cov}^f(X) \mid \mathcal{V} \leq \mathcal{U}' \}$. Let Γ, γ, $\tau_{\mathcal{V}}$ be the other structure elements which make $\mathcal{U}$ a geometric cover. Taking $X = K \in \Gamma$ we have augmentation preserving chain maps

$$\tau_{\mathcal{V}} : \check{C}(\mathcal{U}') \to \Delta\check{C}(\mathcal{V})$$

for all $\mathcal{V} \in \Omega(\mathcal{U}_o)$ satisfying (3.1.2) - (3.1.4). For $\mathcal{V} \leq \mathcal{W} \leq \mathcal{U}'$ we define the inverse projections $c_{\mathcal{V}}^{\mathcal{W}} : \check{C}(\mathcal{W}) \to \Delta\check{C}(\mathcal{V})$ as the composite of a refinement map $\check{C}(\mathcal{W}) \to \check{C}(\mathcal{U}')$ with $\tau_{\mathcal{V}} : \check{C}(\mathcal{U}') \to \Delta\check{C}(\mathcal{V})$. It is now not difficult to check the commutativity up to homotopy of the diagrams (3.2.1). Indeed if $\mathcal{V}_3 \leq \mathcal{V}_2 \leq \mathcal{V}_1 \leq \mathcal{U}'$ we have the diagrams





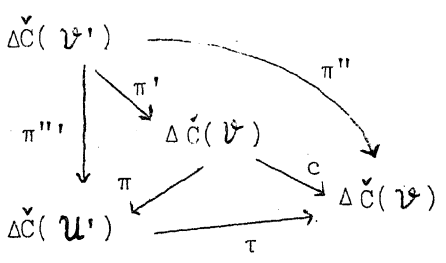
where the unlabelled arrows are refinement maps. The triangles 1, 2, 3, 4, 5, 6 are commutative up to homotopy because of respectively: triangle of refinement maps, definition c, definition c, definition c, definition c, compatibility. The outer triangles are therefore commutative which is what we needed to prove.

We now check the norm condition. Let $\mathcal{V} \leq \mathcal{W}$, c: $\Delta\check{C}(\mathcal{W}) \to \Delta\check{C}(\mathcal{V})$ the inverse projection. The map c is defined as the composite of a refinement map $\pi : \Delta\check{C}(\mathcal{W}) \to \Delta C(\mathcal{U}')$ and $\tau_{\mathcal{V}} : \Delta\check{C}(\mathcal{U}') \to \Delta\check{C}(\mathcal{V})$. Let σ be a simplex of $\Delta\check{C}(\mathcal{W})$, then $\pi\sigma$ is a simplex of $\Delta\check{C}(\mathcal{U}')$. Hence because of the norm condition (3.1.4) there is a $U \in \mathcal{U}$ such that

$$\text{rsupp}(\pi\sigma) \subset U \quad \text{and} \quad \tau(\pi\sigma) \text{ is on } U$$

Because $\mathcal{U}$ is a starrefinement of $\mathcal{U}_o$ this implies that there is $U_o \in \mathcal{U}_o$ such that $\text{supp}(\pi\sigma) \subset U_o$ and $\text{supp}(\tau\pi\sigma) \subset U_o$ and because π is a refinement map $\text{supp}(\pi\sigma) \subset U_o$ implies $\text{supp}(\sigma) \subset U_o$.

The last condition we have to check is (3.2.2). The chain homomorphism c: $\Delta\check{C}(\mathcal{V}) \to \Delta\check{C}(\mathcal{V})$ is defined as $\tau\pi$. According to the factorization axiom there is a $\mathcal{V}' \leq \mathcal{V}$ such that the outer edge triangle of the following diagram commutes up to homotopy. It follows that $c\pi' \simeq \pi''$. This holds for all fine enough $\mathcal{V}'$. Hence H(c) maps the image of H(X) in $H(\mathcal{V})$ identically onto itself.



Further the various τ define (because of (3.1.3) a map of projective systems $H\check{C}(\mathcal{U}') \to H\check{C}(X,-)$ and hence a homomorphism $\tilde{\tau}_* : H\check{C}(\mathcal{U}') \to \check{H}(X)$.

We then have $q_{\mathcal{V}}\tilde{\tau}_* = \tau_*$. Now $c_* = \tau_*\pi_* = q_{\mathcal{V}}\tilde{\tau}_*\pi_*$ which proves that c_* maps $H(\mathcal{V})$ into the image of $H(X)$ in $H(\mathcal{V})$. This concludes the proof of the theorem.

3.5. Remark.

Theorems 3.3 and 3.4 show that the compact spaces which admit a WSC structure are precisely the compact spaces for which every finite open covering is geometric. This also shows, we feel, that the property "admits a WSC structure" is rather more natural than is maybe apparent from Thompson's original definition. Especially if we notice (cf. 4.4) that conditions (3.1.2) and (3.1.3) really say that the $H(\tau_{\mathcal{V}})$ define a homomorphism of projective systems $H(\mathcal{U}') \to H\check{C}(X,-)$ such that the composed map $H\check{C}(X,-) \to H(\mathcal{U}') \to H\check{C}(X,-)$ is the identity homomorphism (between projective systems), where the projective systems are indexed by the set of open coverings finer than $\mathcal{U}'$.

Spaces which admit WSC structures include compact polyhedra or more generally Lefshetz's HLC^* spaces (cf. Lefshetz [14]) and finite unions of compact convex subsets of locally convex topological vector spaces (cf. Thompson [15]).

3.6. Compactly Generated Lebesque Covering Dimension.

Let K be a compact space. We say that K has Lebesque covering dimension $\leq n$ if for every $\mathcal{U} \in \mathrm{cov}^f(K)$ there is a $\mathcal{V} \in \mathrm{cov}^f(K)$, $\mathcal{V} \leq \mathcal{U}$ such that $\dim(\check{C}(K,\mathcal{V})) \leq n$.

Now let X be a not necessarily compact space. Then we say that X has compactly generated Lebesque covering dimension $\leq n$ if every compact subspace K of X has Lebesque covering dimension $\leq n$. We simply write $\dim X \leq n$.

3.7. Proposition.

If $\dim X \leq n$, then every n-geometric cover is geometric.

Proof. Let $\mathcal{C}$ be a geometric cover and $\mathcal{U}' \in \mathrm{cl}(\mathcal{C})$. (Note that $\dim\check{C}(X,\mathcal{U}')$ may well be larger than n and it may not be possible to repair this by taking a refinement of $\mathcal{U}'$). Let Γ, γ, $\tau_{\mathcal{V}}$ be the corresponding structure elements.

$$\tau_{\mathcal{V}} : \check{C}^{n+1}(K,\mathcal{U}') \to \Delta\check{C}(\gamma K,\mathcal{V})$$

If $\mathcal{V} \in \mathrm{cov}^f(\gamma K)$ refines $\mathcal{U}'$ and $\dim \mathcal{V} \leq n$, then the chain map $\tau_{\mathcal{V}}$ can be extended to a chain map $\tilde{\tau}_{\mathcal{V}} : \Delta C(K, \mathcal{U}') \to \Delta C(\gamma K, \mathcal{V})$ by taking $\tilde{\tau}_{\mathcal{V}}(\sigma) = 0$ if $\dim \sigma > n+1$. If $\mathcal{V} \in \mathrm{cov}^f(\gamma K)$ is any covering refining $\mathcal{U}'$ choose a $\mathcal{V}'$ refining such that $\dim \mathcal{V}' \leq n$, and define $\tilde{\tau}_{\mathcal{V}}$ as the composite of $\tilde{\tau}_{\mathcal{V}'}$ and a refinement map. One easily checks that the $\tilde{\tau}_{\mathcal{V}}$ satisfy (3.1.1) - (3.1.4).

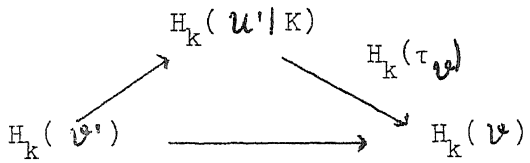
4. CONSEQUENCES OF THE EXISTENCE OF GEOMETRIC COVERS

The existence of geometric covers has strong consequences for the homology and local acyclicity of a space.

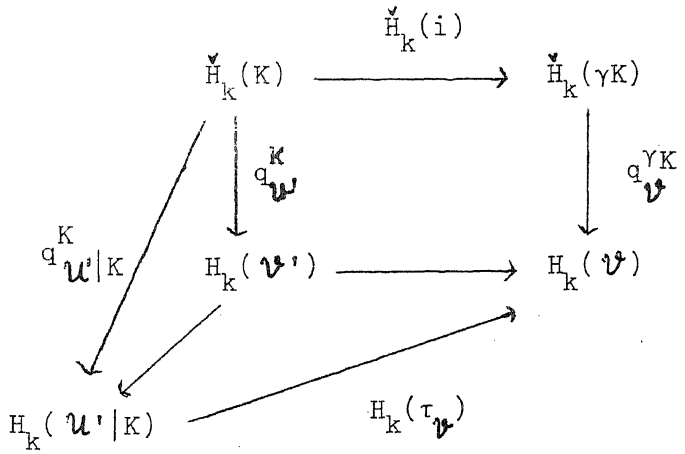
4.1. Theorem.

Let $\mathcal{C}$ be an n-geometric cover of the space X. Then the natural map $\check{H}_k(X) \to H_k(\mathcal{U}')$ is monomorphic for all $k \leq n$ and $\mathcal{U}' \in \mathcal{O}(\mathcal{C})$. Consequently $\check{H}_k(X)$ is finitely generated for all $k \leq n$.

Proof. Let $\mathcal{U}' \in \mathcal{O}(\mathcal{C})$ and let τ, Γ, γ be such that the conditions (3.1.1) and (3.1.2) of 3.1 are satisfied. Let $K \in \Gamma$, $\mathcal{V} \leq \mathcal{U}'$. Then according to (3.1.2) there exists a $\mathcal{V}' \in \mathrm{cov}^f(K)$, $\mathcal{V}' \leq \mathcal{V}$ such that the following diagram commutes for all $k \leq n$.



Now consider the following diagram (where the unlabelled arrows are induced by refinement maps and $i: K \to \gamma K$ is the natural inclusion



The leftmost triangle and the square are commutative by the definition of Čech homology groups, cf. 2.2; the lower triangle is commutative because of (3.1.2). It follows that the whole diagram is commutative.

We now have for all $z \in H_k(K)$

$$(4.1.1) \qquad q^K_{\mathcal{U}'|K}(z) = 0 \quad \Rightarrow \check{H}_k(i)(z) = 0$$

Indeed if $q^K_{\mathcal{U}'|K}(z) = 0$ then $q^{\gamma K}_{\mathcal{V}} H_k(i)(z) = 0$ for all $\mathcal{V} \in \mathrm{cov}^f(\gamma K)$ refining $\mathcal{U}'|\gamma K$. It follows that $H_k(i)(z) = 0$.

Now let $z \in \check{H}_k(X)$ and suppose that $q_{\mathcal{U}'}(z) = 0$. Since Γ is cofinal there is a $K \in \Gamma$ such that z comes from K, i.e. K is such that $z \in \mathrm{Im}(\check{H}_k(K) \to \check{H}_k(X))$. Taking a larger $K \in \Gamma$ if necessary we can assume that $\check{C}(K, \mathcal{U}') = \check{C}(X, \mathcal{U}')$ (i.e. if $U_1 \cap \ldots \cap U_r \neq \phi$, $U_i \in \mathcal{U}'$, then $U_1 \cap \ldots \cap U_r \cap K \neq \phi$). It follows (cf. 2.3) that $q^K_{K|\mathcal{U}'}(\bar{z}) = 0$ for all $\bar{z} \in \check{H}_k(K)$ mapping onto $z \in \check{H}_k(X)$ and hence that $z = 0$ because of (4.1.1).

4.2. Corollary.

If $\mathcal{C}$ is a geometric cover of a space X then $\check{H}_*(X)$ is finitely generated. In particular $\check{H}_k(X) = 0$ for k large enough.

4.3. Remarks.

The "uniformity" of $\mathcal{U}'$ with respect to $K \in \Gamma$ and dimension n is essential for these results.

Note that properties (3.1.3) and (3.1.4) of a geometric cover have not been used.

4.4. Remark.

Property (3.1.3) says that the maps $H_k(\tau_{\mathcal{V}})$ define a morphism of proobjects

$$H_k(\mathcal{U}'|K) \to H_k\check{C}(\gamma K, -)$$

and property (3.1.2) then says that the composition

$$H_k\check{C}(K,-) \to H_k(\mathcal{U}'|K) \to H_k\check{C}(\gamma K,-)$$

is the natural homomorphism of proobjects induced by the inclusion $K \hookrightarrow \gamma K$.

It follows that the composed homomorphism

$$\check{H}_k(X) \to H_k(\mathcal{U}') \xrightarrow{\tau_*} H_k(\gamma K) \xrightarrow{i_{\gamma K}} \check{H}_k(X)$$

is the identity for K large enough. (The last map is induced by the inclusion $\gamma K \to X$). Indeed, because $H_k(X)$ is finitely generated there is a compact set K such that the natural map $i_K : \check{H}_k(K) \to \check{H}_k(X)$ is surjective.

Let $x \in \check{H}_k(X)$, $x' \in \check{H}_k(K)$ such that $i_K(x') = x$. Then because $q(x) = q^K_{\mathcal{U}'}(x')$ (cf. 2.3) and (4.4.1) above we have

$$i_{\gamma K}\tau_* q(x) = i_{\gamma K}\tau_* q^K_{\mathcal{U}'}(x') = x.$$

NB The homomorphism $i_{\gamma K} \circ \tau_*$ may depend on K.

4.5. Proposition.

Let $\mathcal{C}$ be an n-geometric cover of a space X and let $\mathcal{U}' \in \sigma(\mathcal{C})$. Then $\check{H}^{\sim}_k(U') \to \check{H}^{\sim}_k(X)$ is the zero map for all $U' \in \mathcal{U}'$, $k = 0, 1, \ldots, n$.

Proof. Let $U' \in \mathcal{U}'$ and let $K \subset U'$ be compact. We have to show that there exists a $K' \subset X$ such that the inclusion $K \to K'$ induces the zero map on reduced homology. Take $K' = \gamma K$.

Because $K \subset U'$ we have $\mathcal{U}'|K \leq \{K\} \leq \mathcal{U}'|K$ and hence $\check{H}(\mathcal{U}'|K) = 0$. But from the proof of theorem (4.1) we have

$$(4.5.1) \qquad q^K_{\mathcal{U}'|K}(z) = 0 \Rightarrow \check{H}_k(i)(z) = 0 \quad \text{for } z \in \check{H}_k(K)$$

The same holds for reduced homology. (By using the Lefshetz theorem for reduced homology). This proves the proposition.

5. GEOMETRIC COVERS AND ALMOST FIXED POINTS

We are now in a position to state and prove a Lefshetz type almost fixed point theorem

5.1. Lefshetz Number.

If X is a space such that $\check{H}_*(X)$ is finitely generated and $f: X \to X$ is continuous we define $L(f) = \Sigma(-1)^k \mathrm{Tr}(\check{H}_k(f))$ where $\mathrm{Tr}(g)$ denotes the trace of a linear map g between (finite dimensional) vector spaces

5.2. Lemma.

Let $\mathcal{C}$ be a finite closed covering of a space X and let $\mathcal{U}$ be a finite open covering of X such that $\mathcal{C} \leq \mathcal{U}$. For each $C \in \mathcal{C}$ choose $U_C \in \mathcal{U}$ such that $C \subset U_C$. Then there exists a finite open covering $\mathcal{U}'$ of X such that $U' \in \mathcal{U}'$ and $U' \cap C \neq \phi$ imply $U' \subset U_C$.

Proof. For each partition $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$ of $\mathcal{C}$ into two disjoint parts we define the open set

$$U'_{\mathcal{A},\mathcal{B}} = \left(\bigcap_{C \in \mathcal{A}} X \setminus C \right) \cap \left(\bigcap_{C \in \mathcal{B}} U_C \right)$$

Take for $\mathcal{U}'$ the covering consisting of the nonempty $U'_{\mathcal{A},\mathcal{B}}$.
($\mathcal{U}'$ is a covering because

$x \in U'_{\mathcal{A}_x, \mathcal{B}_x}$ with $\mathcal{B}_x = \{C \in \mathcal{C} \mid x \in C\}$, $\mathcal{A}_x = \{C \in \mathcal{C} \mid x \notin C\}$).

5.3. Theorem.

Let X be a space with a geometric covering $\mathcal{C}$, and let f: X → X be continuous. Then for every finite open covering $\mathcal{U} \geq \mathcal{C}$ we have $L(f) \neq 0 \Rightarrow \exists U \in \mathcal{U}$ such that $U \cap f(U) \neq \phi$.

Proof. $\mathcal{C}$ is a geometric cover. Let $\mathcal{U} \geq \mathcal{C}$. For each $C \in \mathcal{C}$ choose $U_C \in \mathcal{U}$ such that $C \subset U_C$. Now choose $\mathcal{U}' \in \mathcal{O}(\mathcal{C})$ such that $U' \in \mathcal{U}'$ and $U' \cap C \neq \phi$ imply $U' \subset U_C$. Such a $\mathcal{U}'$ can be found by 5.2, and because any refinement of a cover in $\mathcal{O}(\mathcal{C})$ is also in $\mathcal{O}(\mathcal{C})$. Cf. 3.1. Let $n = \dim \check{C}(X, \mathcal{U}')$. The cover $\mathcal{C}$ is n-geometric. Let Γ, γ, $\tau_{\mathcal{V}}$ be the other structure elements corresponding to $\mathcal{U}'$ which go into the definition of an n-geometric covering.

Assume that $U \cap f(U) = \phi$ for all $U \in \mathcal{U}$. We are going to prove that $L(f) = 0$. Let K be compact such that $\check{C}(K, \mathcal{U}') = \check{C}(\mathcal{U}')$ and such that $\check{H}(K) \to \check{H}(X)$ is surjective. Let $K' \supset \gamma K$ be such that $f(\gamma K) \subset K'$. Let $\mathcal{V}$ be a finite open covering of γK such that $\mathcal{V} \leq \mathcal{U}'$ and such that for each $V \in \mathcal{V}$ there is a $U' \in \mathcal{U}'$ such that $f(V) \subset U'$. (This can be done because γK is compact). Then we have an induced chain map

$$f_\bullet : \check{C}(\mathcal{V}) \to \check{C}(\mathcal{U}') = \check{C}(\mathcal{U}', K')$$

and composing this with $\tau_{\mathcal{V}} : \check{C}(\mathcal{U}') \to \Delta\check{C}(\mathcal{V})$ we obtain an induced chain map

$$f_{\bullet\bullet} : \Delta\check{C}(\mathcal{U}') \to \Delta\check{C}(\mathcal{U}')$$

On the other hand we have a map of proobjects (cf. 4.4)

$\tau^K_{*k} : H_k(\mathcal{U}' | K) \to H_k(\gamma K, -)$ and a by f induced homomorphism

$f^{\gamma K}_{*k} : H_k(\gamma K, -) \to H_k(K, -)$. Composing this and taking the projective limit gives a homomorphism

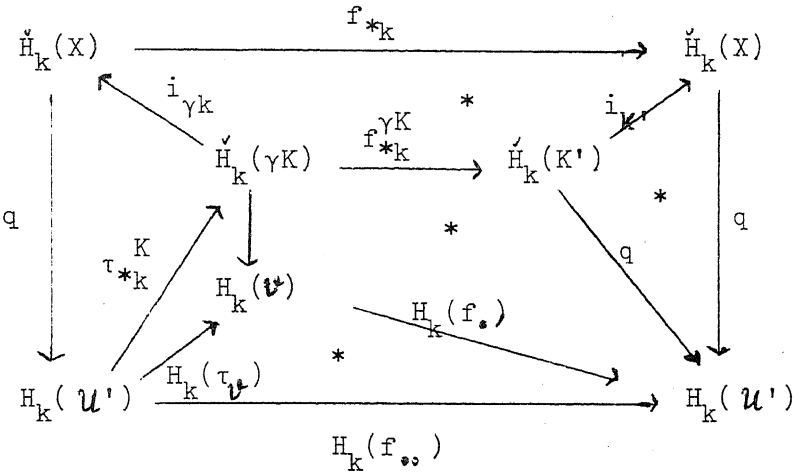
$$f^{\gamma K}_{*k} \circ \tau^{K}_{*k} : H_k(\mathcal{U}') = H_k(\mathcal{U}'|K) \to \check{H}_k(K')$$

Composing this with the natural map $\check{H}_k(K') \to \check{H}_k(X)$ gives us a map

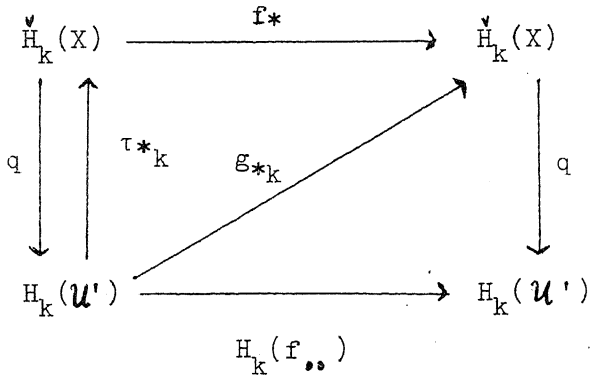
$$g_{*k} : H_k(\mathcal{U}') \to \check{H}_k(X)$$

Now consider the following diagram

(5.3.1)



The starred triangles and quadrangles are commutative and by (4.4) we have that $i_{\gamma k}\tau^{K}_{*k}$ q = id (left most triangle). Retaining only what we need, we find a diagram



We have $q\, g_{*k} = H_k(f_{\bullet\bullet})$ and $f_*\tau_{*k} = g_{*k}$. Hence $g_{*k}q = f_*\tau_{*k}q = f_*$ and $qf_* = H_k(f_{\bullet\bullet})q$.

It follows that $H_k(f_{\bullet\bullet})(H_k(\mathcal{U}')) \subset \check{H}_k(X) \subset H_k(\mathcal{U}')$, where $\check{H}_k(X)$ is seen as a subvectorspace of $H_k(\mathcal{U}')$ by means of the injection $q_{\mathcal{U}'}$ (with inverse projection τ_{*k}, cf 4.4). And from this it follows that

$$\text{(5.3.2)} \qquad \mathrm{Tr}(H_k(f_{\bullet\bullet})) = \mathrm{Tr}(f_{*k}) \quad \text{for all } k$$

Now by (3.1.4) there is for every $\sigma \in \check{C}(\mathcal{U}'|K)$ an element $C \in \mathcal{C}$ such that

$$\mathrm{rsupp}(\sigma) \subset C \text{ and } \mathrm{rsupp}(\sigma') \cap C \neq \phi \text{ for all } \sigma' \text{ in } \tau_{\mathcal{V}}(\sigma)$$

It follows that (because $U' \in \mathcal{U}'$ and $U' \cap C \neq \phi$ imply $U' \subset U_C$)

$$\mathrm{supp}(\sigma) \subset U_C \text{ and } \mathrm{supp}(\sigma') \subset U_C \text{ for all } \sigma' \text{ in } \tau_{\mathcal{V}}(\sigma)$$

But $U_C \cap f(U_C) = \phi$. Hence σ does not occur with nonzero coefficient in $f_*\tau_{\mathcal{V}}(\sigma) = f_{\bullet\bullet}(\sigma)$. Hence $\mathrm{Tr}(f_{\bullet\bullet})_k = 0$.

And by the Hopf theorem and (5.3.2) we then have

$$L(f) = \Sigma(-1)^k \mathrm{Tr}(f_{*k}) = \Sigma(-1)^k \mathrm{Tr}(H_k(f_{\bullet\bullet})) = \Sigma(-1)^k \mathrm{Tr}((f_{\bullet\bullet})_k) = 0$$

5.4. Remark. This proof is quite similar in spirit to the proofs of various other Lefshetz type fixed point theorem. Cf. eg. Thompson [10].

5.5. Addendum.

It is possible to extend theorem (5.2). A closed continuous surjective map $f: Y \to X$ is called a Vietoris map if the subspace $f^{-1}(x)$ is homologically trivial (with respect to $\check{H}$) for all $x \in X$. The extended version of theorem (5.2) then reads

Theorem. Let X be a normal space and $\mathcal{C}$ a closed geometric cover of X. Let Y be a topological space and f,g : Y → X two continuous maps of which f is a Vietoris map, then if L(f,g) ≠ 0 then for every finite open cover $\mathcal{U} \geq \mathcal{C}$ there is an y ∈ Y and U ∈ $\mathcal{U}$ such that f(y) ∈ U, g(y) ∈ U.

Here L(f,g) is defined as $L(f,g) = \Sigma(-1)^k Tr(\check{H}_k(g)\check{H}_k(f)^{-1})$ which makes sense because the Vietoris map f induces isomorphisms on the homology groups.

This theorem allows one to deal with multifunctions F: X → X and gives as a corollary an Eilenberg-Montgomery type (cf. [4]) fixed point theorem by taking X compact (Cf. the introduction).

The chief technical difficulty in proving this theorem (as compared to theorem 5.2) lies in the obtaining of $\check{H}_k(f)^{-1}$ as H_k of a suitably controlled chain map.

These things are to appear in Van der Vel [12].

6. LOCAL CONNECTEDNESS

As in the case of e.g. WSC structures (cf. Thompson [10] and [11]) some kind of local n-connectedness (with respect to $\check{H}$) is related to the existence of n-geometric covers. This and the following sections are concerned with this connection.

6.1. (Partial) Realizations.

Let $S_1 \subset S$ be a pair of simplicial complexes; S_1 is said to be dense in S if S_1^0, the zero skeleton of S_1, is equal to S^0.

Let $\mathcal{W}$ be a finite (open) cover of a space X. A partial realization of S in $\mathcal{W}$ is an augmentation preserving chain map

$$\tau_1 : S_1 \to \Delta\check{C}(X,\mathcal{W})$$

where S_1 is dense subcomplex of S. If $S_1 = S$ we speak of a (full) realization.

If σ is a simplex of S, then S(σ) denotes the subcomplex of S consisting of all faces of σ (including σ itself).

The partial realization $\tau_1 : S_1 \to \Delta C(X,\mathcal{W})$ is said to be of norm $\leq \mathcal{C}$, where $\mathcal{C}$ is another covering of X, if for every σ ∈ S there is a C ∈ $\mathcal{C}$ such that $\tau_1(\mu)$ is on C for all μ ∈ $S_1 \cap S(\sigma)$.

Let $X_2 \subset X_1$ be a pair of topological spaces; let τ_1, $\mathcal{U}_1$ and τ_2, $\mathcal{U}_2$ be finite covers of X_1, X_2 respectively with $\mathcal{U}_1$ and $\mathcal{U}_2$ open covers and $\tau_2 \leq \tau_1$, $\mathcal{U}_2 \leq \mathcal{U}_1$. Then we say that the pair $(\mathcal{U}_1, \mathcal{U}_2)$ has enough controlled realizations for dimensions $\leq n$ ($(\mathcal{U}_1, \mathcal{U}_2)$ has ECR(n)) with respect to (τ_1, τ_2) if for every $S_1 \subset S$ every partial realization $\tau_1 : S_1 \to \Delta\check{C}(X_2, \mathcal{U}_2)$ of norm $\leq \tau_2$ extends to a partial realization $\tau : S^n \cup S_1 \to \Delta\check{C}(X_1, \mathcal{U}_1)$ of norm $\leq \tau_1$. I.e. we have a commutative diagram

$$\begin{array}{ccc} S_1 & \xrightarrow{\tau_1} & \Delta\check{C}(X_2, \mathcal{U}_2) \\ \downarrow & & \downarrow i \\ S^n \cup S_1 & \xrightarrow{\tau} & \Delta\check{C}(X_1, \mathcal{U}_1) \end{array}$$

for some suitable refining homomorphism i.

6.2. ℓc^n and c-ℓc^n Refinements.

Let $X_2 \subset X_1$ be a pair of topological spaces and let τ_2, τ_1 be covers of X_2, X_1 respectively such that $\tau_2 \leq \tau_1$. Then τ_2 is an ℓc^n refinement of τ_1 if for every $\mathcal{U}_1 \in \mathrm{cov}^f(X_1)$ there exists a $\mathcal{U}_2 \in \mathrm{cov}^f(X_2)$ such that $\mathcal{U}_2 \leq \mathcal{U}_1$ and $(\mathcal{U}_1, \mathcal{U}_2)$ has ECR(n+1) with respect to (τ_1, τ_2). If τ_2 is an ℓc^n refinement of τ_1 for every n, τ_2 is said to be an ℓc^ω refinement of τ_1

NB. ℓc^n corresponds to ECR(n+1).

Let X be a space and let $\tau_2 \leq \tau_1$ be covers of X. We say that τ_2 is a c-ℓc^n refinement of τ_1 if for every compact set $K_2 \subset X$ there is larger compact set K_1 in X such that $\tau_2|K_2$ is an ℓc^n refinement of $\tau_1|K_1$. If τ_2 is an c-ℓc^n refinement of τ_1 for every n, τ_2 is said to be an c-ℓc^ω refinement of τ_1.

6.3. Elementary properties.

Let $X_3 \subset X_2 \subset X_1$ and let τ_3, τ_2, τ_1 be covers of X_3, X_2, X_1 respectively such that $\tau_3 \leq \tau_2 \leq \tau_1$. Then

(i) If τ_2 is an ℓc^n(ℓc^ω) refinement of τ_1 then so is τ_3

(ii) If τ_3 is an ℓc^n(ℓc^ω) refinement of τ_2 then it is also an ℓc^n(ℓc^ω) refinement of τ_1.

Let $\mathcal{T}_2 \leq \mathcal{T}_1$ be two covers of a compact space X. Then

(iii) $\mathcal{T}_2$ is an c-ℓc^n(resp. c-ℓc^ω) refinement of $\mathcal{T}_1$ iff $\mathcal{T}_2$ is an ℓc^n(resp. ℓc^ω) refinement of $\mathcal{T}_1$.

6.4. ℓc^n Spaces and ℓc^n and c-ℓc^n Covers.

A compact space is said to be ℓc^n (resp. ℓc^ω) if every finite open cover has an ℓc^n (resp. ℓc^ω) refinement.

A covering $\mathcal{T}$ of a space X is c-ℓc^n (resp. ℓc^n) if it is a c-ℓc^n (resp. ℓc^n) refinement of the trivial cover.

One could perfectly well define what a c-ℓc^n (resp. c-ℓc^ω) space would be. But there seem to be very few examples, which are noncompact; we know none. The property ℓc^ω seems somewhat weaker than ℓc^*. (cf. e.g. Begle [1] or Thompson [10] for a definition of ℓc^*).

7. ACYCLICITY AND c-ℓc^n REFINEMENTS

We have seen (cf. 4.5) that some kind of local acyclicity is implied by the existence of n-geometric covers. On the other kind given acyclicity properties of a suitable kind one can go a fair way towards the construction of n-geometric covers as we shall attempt to show in this and the next section. The first step is to show that given suitable acyclicity conditions, partial realizations can be extended. One has even better control over the supports than is needed for c-ℓc^n refinements and this results in some "uniqueness up to homotopy" statements and these in turn will permit us to construct n-geometric covers in section 8.

7.1. Lemma.

Let $K_2 \subset K_1$ be compact spaces and suppose that for a certain n the induced homomorphism $\check{H}_n(K_2) \to \check{H}_n(K_1)$ is trivial. Then for every finite open cover $\mathcal{U}_1$ of K_1 there is a finite open cover $\mathcal{U}_2$ of K_2 such that $\mathcal{U}_2 \leq \mathcal{U}_1$ and

$$\hat{H}_n(K_2, \mathcal{U}_2) \to \hat{H}_n(K_1, \mathcal{U}_1)$$

is the trivial map.

Proof. We have a commutative diagram

$$\begin{array}{ccc} \check{H}_n^{\sim}(K_2) & \xrightarrow{i_*} & \check{H}_n^{\sim}(K_1) \\ \downarrow q^2_{\mathcal{U}_1} & & \downarrow q^1_{\mathcal{U}_1} \\ \hat{H}_n(K_2, \mathcal{U}_1) & \xrightarrow{i_*} & \hat{H}_n(K_1, \mathcal{U}_1) \end{array}$$

where $i : K_2 \to K_1$ is the inclusion. By the Lefshetz theorem (cf. (2.5)) there is a finite open cover $\mathcal{U}_2$ of K_2 refining $\mathcal{U}_1$ such that

$$\mathrm{Im}(\hat{H}_n(K_1, \mathcal{U}_2) \to \hat{H}_n(K_2, \mathcal{U}_1)) = \mathrm{Im}(\check{H}_n^{\sim}(K_2) \to \hat{H}_n(K_2, \mathcal{U}_1))$$

It follows that the natural map $\hat{H}_n(K_2, \mathcal{U}_2) \to \hat{H}_n(K_1, \mathcal{U}_1)$ is trivial. Note that any refinement $\mathcal{U}_2'$ of $\mathcal{U}_2$ also works.

7.2. Definition and Construction.

Let $X_2 \subset X_1$ be topological spaces and let τ_2, τ_1 be finite covers of X_2 and X_1 respectively such that $\tau_2 \leq \tau_1$. Let $j: \tau_2 \to \tau_1$ be a refinement map. We say that j is acyclic in dimension n if for all subsets $\mathcal{A}_2 \subset \tau_2$ and all compact subsets $K_2 \subset \cap \mathcal{A}_2$ there is a compact subset $K_1 \subset \cap j(\mathcal{A}_2)$ such that $K_2 \subset K_1$ and such that

(7.2.1) $\qquad \check{H}_n(K_2) \to \check{H}_n(K_1)$ is the zero map

Now let n_o and n be nonnegative integers. Suppose we have a sequence of finite closed covers $\tau_o \leq \tau_1 \leq \ldots \leq \tau_{n+1}$ of a space X with refinement maps $j_k : \tau_k \to \tau_{k+1}$, k = 0,1, ..., n such that j_k is acyclic in dimension $n_o + k$.

Let K be any compact subset of X. Then there exists a sequence of compact subsets

(7.2.2) $\qquad K = K_o \subset K_1 \subset \ldots \subset K_{n+1}$

such that for all $\mathcal{A}_k \subset \tau_k$ such that $K_k \cap (\cap \mathcal{A}_k) \neq \phi$,

(7.2.3) $\quad \check{H}^{\sim}_{n_o+k}(K_k \cap (\cap \mathcal{A}_k)) \to \check{H}^{\sim}_{n_o+k}(K_{k+1} \cap (\cap j_k(\mathcal{A}_k)))$ is the zero map

and consequently for every finite open cover $\mathcal{V}_{n+1}$ of K_{n+1} there exists a sequence of open covers $\mathcal{V}_k$, k = 0, 1, ..., n,

(7.2.4) $$\mathcal{V}_o \leq \mathcal{V}_1 \leq \mathcal{V}_2 \leq \ldots \leq \mathcal{V}_{n+1}$$

such that for all $\mathcal{A}_k \subset \mathcal{C}_k$ such that $K_k \cap (\cap \mathcal{A}_k) \neq \phi$

(7.2.5) $$\check{H}_{n_o+k}(K_k \cap (\cap \mathcal{A}_k), \mathcal{V}_k) \to \check{H}_{n_o+k}(K_{k+1} \cap (\cap\, j_k \mathcal{A}_k), \mathcal{V}_{k+1})$$

is the zero map

The sequence (7.2.2) is constructed as follows. Suppose we have found K_k, $k \geq 0$ for every $\mathcal{A}_k \subset \mathcal{C}_k$ such that $K_k \cap (\cap \mathcal{A}_k) \neq \phi$ let $K_k'(\mathcal{A}_k)$ be a compact set containing $K_k \cap (\cap \mathcal{A}_k)$ and contained in $\cap\, j_k(\mathcal{A}_k)$ such that (7.2.1) is satisfied.

Now let K_{k+1} be the union of all the $K_k'(\mathcal{A}_k)$. Then of course $K_{k+1} \cap (\cap\, j_k \mathcal{A}_k) \supset K_k'(\mathcal{A}_k)$ so that (7.2.3) is satisfied. To find the sequence (7.2.4) such that (7.2.5) is satisfied, apply 7.1 repeatedly.

7.3. Proposition.

Let $\mathcal{C}_o \leq \ldots \leq \mathcal{C}_{n+1}$ be a sequence of finite closed covers of a space X with corresponding refinement maps $j_k: \mathcal{C}_k \to \mathcal{C}_{k+1}$ such that j_k is acyclic in dimension $n_o + k$. Let $\mathcal{C}_o' \leq \mathcal{C}_o$ have a refinement map $i: \mathcal{C}_o' \to \mathcal{C}_o$ such that $C_o' \subset \text{interior}(i(C_o'))$ for all $C_o' \in \mathcal{C}_o'$. Let $S_1 \subset S$ be a pair of (finite) simplicial complexes.

Let $K = K_o \subset K_1 \subset \ldots \subset K_{n+1}$ be as in 7.2. Then for every $\mathcal{V}_{n+1} \in \text{cov}^f(K_{n+1})$ there exists a sequence $\mathcal{V}_o \leq \mathcal{V}_1 \leq \ldots \leq \mathcal{V}_{n+1}$, $\mathcal{V}_k \in \text{cov}^f(K_k)$ such that for every partial realization

$$\tau_o : S^{n_o} \cup S_1 \to \Delta\check{C}(K_o, \mathcal{V}_o)$$

of norm $\leq \mathcal{C}_o'$ there exists a sequence of partial realizations

$$\tau_k : S^{n_o+k} \cup S_1 \to \Delta\check{C}(K_k, \mathcal{V}_k), \quad k = 0, \ldots, n+1$$

such that

(7.3.1) τ_{k+1} extends τ_k, $k = 0, 1, \ldots, n$

(7.3.2) if σ is a simplex of $S^{n_o+k} \cup S_1$ then $\tau_k(\sigma)$ is on $\cap\, \mathcal{A}_k(\sigma) \cap K_k$, $k = 0, 1, \ldots, n+1$

where $\mathcal{A}_k(\sigma) = j_{k-1} \ldots j_o i \mathcal{A}(\sigma)$, and $\mathcal{A}(\sigma)$ is defined as follows. Let $\sigma_1, \ldots, \sigma_s$ be the maximal simplices of S. For each σ_r choose a $C'_o(r)$ such that $\tau_o((S^{n_o} \cup S_1) \cap S(\sigma_r))$ is on $C'_o(r)$. (Such $C'_o(r)$ exist because τ_o is of norm $\leq \mathfrak{r}'_o$). Now define $\mathcal{A}(\sigma) = \{C'_o(r) | \sigma$ is a face of $\sigma_r\}$.

Moreover in the case $n_o = 0$ there is the following homotopy property: If $\mathcal{V}_o \leq \mathcal{V}'_1 \leq \ldots \leq \mathcal{V}'_n \leq \mathcal{V}_{n+1}$ is a second series of refinements and $\tau'_o, \tau'_1, \ldots, \tau'_{n+1}$ a second series of extensions such that (7.3.1) and (7.3.2) hold and if $\tau'_o = \tau_o$ on S_1 then $H_k(\tau'_{n+1}) = H_k(\tau_{n+1})$ for $k = 0, 1, \ldots, n$.

Remarks. 1. If $n_o > 0$ and $\tau_o = \tau'_o$ then also $H_k(\tau'_{n+1}) = H_k(\tau_{n+1})$ for $k = 0, \ldots, n_o+n$.

2. In general there are several different choices for the $C'_o(r)$ and correspondingly one finds different $\mathcal{A}(\sigma)$ and different τ_k.

The proof of proposition 7.3 is in several steps: subsections 7.4 - 7.7. The first step is to choose $\mathcal{V}_o \leq \ldots \leq \mathcal{V}_{n+1}$ such that (7.3.2) holds for $k = 0$

7.4. Remarks on the $\mathcal{A}(\sigma)$ and the Sequence $\mathcal{V}_0 \leq \ldots \leq \mathcal{V}_{n+1}$.

Choose a sequence of covers $\mathcal{V}_0 \leq \mathcal{V}_1 \leq \ldots \leq \mathcal{V}_{n+1}$, $\mathcal{V}_k \in \mathrm{cov}^f(K_k)$ such that (7.2.5) is satisfied. Refining $\mathcal{V}_0$ if necessary we can also assume that $V_0 \in \mathcal{V}_0$, $V_0 \cap C_0' \neq \phi \Rightarrow V_0 \subset i(C_0') \in \mathcal{L}_0$.

Let σ be a simplex of $S^{n_0} \cup S_1$. Let σ be a face of σ_r. Then $\tau_0(\mu)$ is on $C_0'(r)$ for all vertices μ of σ. Hence $V_0 \cap C_0'(r) \neq \phi$ for all V_0 occurring in $\tau_0(\mu)$, hence $V_0 \subset i(C_0'(r))$ for all vertices V_0 occurring in $\tau_0(\mu)$, μ a vertex of σ. Hence $\mathrm{supp}(\tau_0(\sigma)) \subset i(C_0'(r))$ for all r such that $\sigma \subset \sigma_r$ hence $\mathrm{supp}(\tau_0(\sigma)) \subset (\cap \mathcal{A}_0(\sigma)) \cap K_0$ which certainly implies (7.3.2) which says that $\mathrm{rsupp}(\sigma') \cap (\cap \mathcal{A}_0(\sigma)) \cap K_0 \neq \phi$ for every simplex σ' occurring in $\tau_0(\sigma)$.

Note that

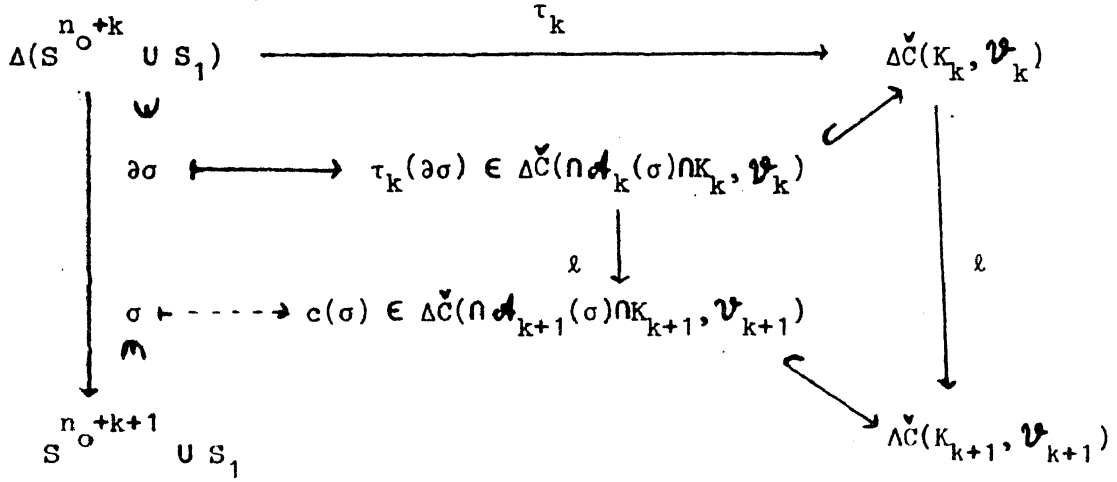
(7.4.1) $\qquad \mu$ a face of $\sigma \Rightarrow \mathcal{A}(\mu) \supset \mathcal{A}(\sigma)$

(For if σ is a face of σ_r then so is μ).

7.5. Existence of the Sequence of Extensions $\tau_0, \tau_1, \ldots, \tau_{n+1}$.

Let $\mathcal{V}_0 \leq \ldots \leq \mathcal{V}_{n+1}$ be the sequence of refinements of 7.4 above. We have just seen that τ_0 satisfies 7.3.2. By induction we can suppose that τ_k has been constructed such that (7.3.1) and (7.3.2) hold.

Consider the following diagram



Let $\sigma \in S^{n_o+k+1} \setminus S^{n_o+k} \setminus S_1$. Let μ be an $n_o + k$ face of σ. Then $\tau_k(\mu)$ is on $\cap \mathcal{A}_k(\mu) \cap K_k$ hence it is certainly on $\cap \mathcal{A}_k(\sigma) \cap K_k$ by (7.4.1). Therefore $\tau_k(\partial\sigma)$ is on $\cap \mathcal{A}_k(\sigma) \cap K_k$. The image of $\tau_k(\partial\sigma)$ under ℓ is homologous to zero because (7.2.5) holds (NB $\tau_k(\partial\sigma)$ is a cycle because τ_k is an augmentation preserving chainmap). Therefore there exists a $c(\sigma)$ such that $\partial c(\sigma) = \ell\tau_k(\partial\sigma)$. Now define $\tau_{k+1}(\sigma) = c(\sigma)$. Do this for every $\sigma \in S^{n_o+k+1} \setminus S^{n_o+k} \setminus S_1$, and define $\tau_{k+1}(\mu) = \ell\tau_k(\mu)$ for $\mu \in S^{n_o+k} \cup S_1$. Note that τ_{k+1} satisfies (7.3.1) and (7.3.2).

7.6. The Homotopy Property for Equal Refining Sequences.

Consider the simplicial complex S x I and let S_2 be the subcomplex $S_2 = S_1 \times I$. Now define

$$T_o : (S\times I)^o \cup S_2 \to \Delta\check{C}(K_o, \mathcal{V}_o) \tag{7.6.1}$$

as follows. Let $e_1, \ldots, e_s$ be an ordening of the vertices of S. Then the simplices of S x I are all sets of the form $\{e^o_{i(1)},\ldots,e^o_{i(r)},e^1_{i(r+1)},\ldots,e^1_{i(t)}\}$ such that $i(1) <\ldots< i(r) \leq i(r+1) <\ldots< i(t)$ and $\{e_{i(1)},\ldots,e_{i(t)}\}$ is a simplex of S. We now define T_o on the vertices of S x I by $T_o(e^o_i) = \tau_o(e_i)$, $T_o(e^1_i) = \tau'_o(e_i)$ and on $S_1 \times I = S_2$ we define T_o by $T_o(\{e^o_{i(1)},\ldots,e^o_{i(r)},e^1_{i(r+1)},\ldots,e^1_{i(t)}\}) = \tau_o\{e_{i(1)},\ldots,e_{i(t)}\} = \tau'_o\{e_{i(1)},\ldots,e_{i(t)}\}$. Then T_o satisfies (7.3.2). Now extend T_o to T_{n+1} exactly as we extended τ_o to τ_{n+1} in 7.5, taking care to define $T_k(\sigma) = \tau_k(\sigma)$ if σ is a k-simplex of the form $\{e^o_{i(1)},..,e^o_{i(k+1)}\}$ and $T_k(\sigma) = \tau'_k(\sigma)$ if σ is a k-simplex of the form $\{e^1_{i(1)}, \ldots, e^1_{i(k+1)}\}$. We then have a chain map

$$T_{n+1} : (S+I)^{n+1} \cup S_2 \to \Delta\check{C}(K_{n+1}, \mathcal{V}_{n+1})$$

which restricts to τ_{n+1} on $S^{n+1} \times \{0\} \cup S_1 \times \{0\}$ and to τ'_{n+1} on $S^{n+1} \times \{1\} \cup S_1 \times \{0\}$.

This proves that $H_k(\tau'_{n+1}) = H_k(\tau_{n+1})$ for $k = 0,1, \ldots, n$.

7.7. The Homotopy Property for Different Refining Sequences.

Now let $\mathcal{V}_o \leq \mathcal{V}_1 \leq \ldots \leq \mathcal{V}_{n+1}$, $\mathcal{V}_o \leq \mathcal{V}'_1 \leq \ldots \leq \mathcal{V}'_n \leq \mathcal{V}_{n+1}$ and $\tau_o, \tau_1, \ldots, \tau_{n+1}$; $\tau'_o, \tau'_1, \ldots, \tau'_{n+1}$ be two different sequences of refinements with corresponding τ_k and τ'_k such that (7.3.1) and (7.3.2) are satisfied. It suffices to prove that $H_m(\tau_{n+1}) = H_m(\tau'_{n+1})$, $m = 0, \ldots, n$ in case $\mathcal{V}'_k \leq \mathcal{V}_k$ for all $k = 0, \ldots, n+1$. (Take a common refinement of the two refinement sequences such that (7.2.5) holds for this common refinement sequence). Define $\tau''_k = \ell_k \circ \tau'_k$ where ℓ_k is induced by a refinement map; we can take $\ell_o = \ell_{n+1} =$ identity. Then $\tau_o, \ldots, \tau_{n+1}$; $\tau''_o, \ldots, \tau''_{n+1}$ are two sequences of extensions corresponding to the same refinement sequence satisfying (7.3.1) and (7.3.2). Therefore $H_m(\tau_{n+1}) = H_m(\tau'_{n+1})$ for $m = 0,1, \ldots, n$.

7.8. Corollary.

Let $\mathcal{C}'_o \leq \mathcal{C}_o \leq \ldots \leq \mathcal{C}_{n+1}$ be a sequence of closed covers as in 7.2, 7.3 and let $\mathcal{C}''_o \leq \mathcal{C}'_o$ be a c-ℓc^{n_o-1} extension. Then $\mathcal{C}''_o \leq \mathcal{C}_{n+1}$ is a c-ℓc^{n_o+n} extension. In particular if $n_o = 0$ then $\mathcal{C}'_o \leq \mathcal{C}_{n+1}$ is a c-ℓc^n extension.

Proof. Let K be compact, choose K_o such that for every $\mathcal{V}_o \in \mathrm{cov}^f(K_o)$ there is a $\mathcal{V}'_o \in \mathrm{cov}^f(K)$ such that $(\mathcal{V}_o, \mathcal{V}'_o)$ has ECR(n_o) with respect to $(\mathcal{C}'_o, \mathcal{C}''_o)$. Now let $K_o \subset K_1 \subset \ldots \subset K_{n+1}$ be as in 7.2. For every $\mathcal{V}_{n+1} \in \mathrm{cov}^f(K_{n+1})$ let $\mathcal{V}_o \leq \mathcal{V}_1 \leq \ldots \leq \mathcal{V}_{n+1}$ be a sequence of refinements as in 7.2. Now let $\mathcal{V}'_o$ be such that $(\mathcal{V}_o, \mathcal{V}'_o)$ has ECR(n_o) with respect to $(\mathcal{C}'_o, \mathcal{C}''_o)$ and let τ be a partial realization of S in $\Delta\check{C}(K, \mathcal{V}'_o)$, $\tau : S_1 \to \Delta C(K, \mathcal{V}'_o)$ of norm $\leq \mathcal{C}''_o$.

Then there exists an extension $\tau_o : S_1 \cup S^{n_o} \to \Delta\check{C}(K_o, \mathcal{V}_o)$ of norm $\leq \mathcal{C}'_o$ which in turn can be extended to $\tau_{n+1}: S_1 \cup S^{n_o+n+1} \longrightarrow \Delta\check{C}(K_{n+1}, \mathcal{V}_{n+1})$ by 7.3. This extension τ_{n+1} satisfies (7.3.2) and therefore is of norm $\leq \mathcal{C}_{n+1}$.

7.9. Examples of c-ℓc^n Covers and Refinements.

Let τ'_o be any cover of $\mathbb{R}^m$ and $\tau_o = \ldots = \tau_n = \{\mathbb{R}^m\}$. Applying the corollary and 6.3 we see that any cover of $\mathbb{R}^m$ is a c-ℓc^n cover for all n, i.e. a c-ℓc^ω cover.

Let τ be a closed convex cover of $\mathbb{R}^m$ and $\tau' \leq \tau$ such that for every $C' \in \tau'$ there is a $C \in \tau$ with $C' \subset$ interior(C) then $\tau' \leq \tau$ is an c-ℓc^ω refinement.

7.10. Corollary.

Let dim$X \leq n$ and let $\tau' \leq \tau$ be a c-ℓc^n refinement where τ is a closed cover. Let $\tau \leq \tau''$ such that for every $C \in \tau$ there is a $C'' \in \tau''$ with $C \subset$ interior(C''). Then $\tau' \leq \tau''$ is an c-ℓc^ω refinement.

Proof. Take $\tau_o = \ldots = \tau_{n+1} = \tau''$ in 7.8. The acyclicity conditions now follow from the finite dimension assumption (cf.3.4).

8. ACYCLICITY AND n-GEOMETRIC COVERINGS

We can now construct n-geometric coverings given suitable acyclicity assumptions.

8.1. Theorem.

Let $\mathcal{U}' \leq \tau'_o \leq \tau_o \leq \ldots \leq \tau_{n+1}$ be a sequence of covers of a space X such that

(i) $\mathcal{U}'$ is a finite open cover; $\tau'_o, \tau_o, \ldots, \tau_{n+1}$ are finite closed covers.

(ii) for every $C'_o \in \tau'_o$ there is a $C_o \in \tau_o$ such that $C'_o \subset$ interior(C_o)

(iii) $\mathcal{U}'$ is a starrefinement of τ'_o

(iv) $\tau_k \leq \tau_{k+1}$ is acyclic in dimension k.

Then τ_{n+1} is n-geometric and $\mathcal{U}' \in \mathcal{O}(\tau_{n+1})$.

Let Γ be any cofinal collection of compact subsets of X. For every $K \in \Gamma$ choose a sequence $K = K_o \subset K_1 \subset \ldots \subset K_{n+1}$ such that (7.2.3) is satisfied. Enlarging K_{n+1} if necessary we can assume that also $K_{n+1} \in \Gamma$ Define $\gamma K = K_{n+1}$. For every $\mathcal{V} = \mathcal{V}_{n+1} \in \text{cov}(K_{n+1})$ refining $\mathcal{U}'$ choose a sequence

$$\mathcal{V}_o \leq \mathcal{V}_1 \leq \ldots \leq \mathcal{V}_{n+1} \qquad \mathcal{V}_i \in \text{cov}^f(K_i)$$

such that (7.2.5) is satisfied. For each $C_o'(i) \in \mathcal{C}_o'$ choose $C_o(i) \in \mathcal{C}_o$ such that $C_o'(i) \subset \text{interior}(C_o(i))$. Refining $\mathcal{V}_o$ if necessary we can assume that

(8.1.1) $\quad V_o \in \mathcal{V}_o,\ V_o \cap C_o'(i) \neq \phi \Rightarrow V_o \subset C_o(i)$

Let $S = \check{C}(K, \mathcal{U}')$, $S_1 = S^o$. We now define $\tau_o: S^o \to \check{C}(K_o, \mathcal{V}_o)$ as follows. For each $U'(i) \in \mathcal{U}'$ choose a $V_o(i)$ such that $V_o(i) \cap U'(i) \neq \phi$, and define $\tau_o(U'(i)) = V_o(i)$. I.e. we have

(8.1.2) $\quad \tau_o(U'(i)) \cap U'(i) \cap K_o \neq \phi$

For each $U'(i)$ choose $C_o'(i)$ such that star $(U'(i)) \subset C_o'(i)$. I.e.

(8.1.3) $\quad U'(j) \cap U'(i) \neq \phi \Rightarrow U'(j) \subset C_o'(i)$

Now let $\sigma = \{U'(i_o),\ldots,U'(i_m)\}$ be a maximal simplex of $S = \check{C}(K, \mathcal{U}')$, i.e. $U'(i_o) \cap \ldots \cap U'(i_m) \cap K \neq \phi$ and hence

(8.1.4) $\quad U'(i_r) \subset C_o'(i_s) \qquad r,s = 0,1, \ldots, m$

It follows that

(8.1.5) $\quad \text{supp}(\sigma) \subset C_o'(i_s) \cap K \qquad s = 0,1, \ldots, m$

Now $\tau_o(U'(i_r)) \cap U'(i_r) \cap K_o \neq \phi$, $K_o \cap U'(i_r) \subset C_o'(i_s) \cap K_o$. Hence

(8.1.6) $\quad \tau_o(U'(i_r))$ is on $C_o'(i_s) \cap K_o \qquad r = 0, \ldots, m;\ s = 0, \ldots, m$

which by (8.1.1) implies

(8.1.7) $\quad \tau_o(U'(i_r)) \subset C_o(i_s) \cap K_o \qquad r, s = 0, \ldots, m$

We can now choose the $\mathcal{A}(\sigma)$ as follows. For each maximal simplex σ_t choose a vertex $U'(t)$ of σ_t and let $C'_o(\sigma_t) = C'_o(t)$. Then (cf. 7.3) $\mathcal{A}(\sigma) = \{C'_o(\sigma_t) | \sigma \subset \sigma_t, \sigma_t \text{ maximal}\}$. In view of (8.1.5), (8.1.7) we have for all $\sigma \in \check{C}(K, \mathcal{U}')$ and $\mu \in \check{C}^o(K, \mathcal{U}')$

$$(8.1.8) \qquad \operatorname{supp}(\sigma) \subset (\cap \mathcal{A}(\sigma)) \cap K, \quad \operatorname{supp}(\tau_o(\mu)) \subset (\cap \mathcal{A}(\mu)) \cap K_o$$

Now, using these $\mathcal{A}(\sigma)$, construct a sequence of extensions $\tau_o, \tau_1, \ldots, \tau_{n+1}$ as in (7.3). This gives chain maps

$$\tau_{\mathcal{V}} = \tau_{n+1} : C^{n+1}(K, \mathcal{U}') \to \Delta C(\gamma K, \mathcal{V})$$

We now check the various axiomes which the $\tau_{\mathcal{V}}$ have to satisfy.

(a) The norm condition (cf. (3.1.4). This is satisfied because $\tau_{n+1} = \tau_{\mathcal{V}}$ satisfies (7.3.2) and because we have (8.1.8) (first part).

(b) The factorization property (cf. 3.1.2). Let $i : \check{C}(K, \mathcal{V}_o) \to \check{C}(K, \mathcal{U}')$ be any refining map; let $i_k : \check{C}(K_k, \mathcal{V}_k) \to C(K_{k+1}, \mathcal{V}_{k+1})$ be refining maps for $k = 0, 1, \ldots, n$. Define $\bar{\tau}_k = \tau_k \circ i$, $k = 0, \ldots, n+1$ and $\tau'_o = \mathrm{id}$, $\tau'_{k+1} = i_k \circ \ldots \circ i_o$, $k = 0, \ldots, n$. We then have two sequences of maps

$$\bar{\tau}_k, \tau'_k : \check{C}(K, \mathcal{V}_o) \to C(K_k, \mathcal{V}_k)$$

satisfying (7.3.1) and (7.3.2). Hence $H_m(\tau_{n+1}) \circ H_m(i) = H_m(\tau'_{n+1})$, $m = 0, \ldots, n$ which proves the factorization property with $\mathcal{V}' = \mathcal{V}_o$.

(c) The comptability property (cf. 3.1.3)

Let $\mathcal{V}_1, \mathcal{V}_2$ be two open covers of $\gamma K = K_{n+1}$ such that $\mathcal{V}_1 \leq \mathcal{V}_2 \leq \mathcal{U}'$ let $\mathcal{V}_o(1), \ldots, \mathcal{V}_{n+1}(1) = \mathcal{V}_1$; $\mathcal{V}_o(2), \ldots, \mathcal{V}_{n+1}(2) = \mathcal{V}_2$ and let $\tau_o(1), \ldots, \tau_{n+1}(1) = \tau_{\mathcal{V}_1}$; $\tau_o(2), \ldots, \tau_{n+1}(2) = \tau_{\mathcal{V}_2}$ be the corresponding

sequences of chain maps. Choose a common refinement $\mathcal{V}_o$ of $\mathcal{V}_o(1)$ and $\mathcal{V}_o(2)$. Define $\tau_o: \check{C}^o(K, \mathcal{U}') \to \Delta C(\check{K}_o, \mathcal{V}_o)$ by assigning to each U' a V_o such that $U' \cap V_o \neq \phi$.

Let $i_1: \mathcal{V}_o \to \mathcal{V}_o(1)$ (resp. $i_2: \mathcal{V}_o \to \mathcal{V}_o(2)$)be any refinement map. Define $\tau_o'(1) = i_1\tau_o$, $\tau_o'(2) = i_2\tau_o$ and let $\tau_o'(1), \ldots, \tau_{n+1}'(1)$ (resp. $\tau_o'(2), \ldots, \tau_{n+1}'(2)$) be the sequences of extensions obtained by using $\mathcal{V}_o(1) \leq \ldots \leq \mathcal{V}_{n+1}(1)$ (resp. $\mathcal{V}_o(2) \leq \ldots \leq \mathcal{V}_{n+1}(2)$). Then $H_m(\tau_{n+1}'(1)) = H_m(\tau_{n+1}(1))$ and $H_m(\tau_{n+1}'(2)) = H_m(\tau_{n+1}(2))$ for $m = 0, \ldots, n$ The sequences

$$\tau_o, \tau_1'(1), \ldots, \tau_n'(1), i\circ\tau_{n+1}'(1); \tau_o, \tau_1'(2), \ldots, \tau_{n+1}'(2)$$

where i is any refinement map $C(\gamma K, \mathcal{V}_1) \to C(\gamma K, \mathcal{V}_2)$, are sequences of extensions corresponding to the sequences of refinements

$$\mathcal{V}_o, \mathcal{V}_1(1), \ldots, \mathcal{V}_n(1), \mathcal{V}_2; \quad \mathcal{V}_o, \mathcal{V}_1(2), \ldots, \mathcal{V}_n(2), \mathcal{V}_2$$

and therefore we have that $H_m(i\tau_{n+1}'(1)) = H_m(\tau_{n+1}'(2))$, $m = 0, \ldots, n$ and hence $H_m(i\tau_{n+1}(1)) = H_m(\tau_{n+1}(2)$ for $m = 0,1, \ldots, n$, which proves the compatability.

This theorem is especially useful in the case of convex or more generally convexoid covers.

8.2. Convexoid Covers (Definition)

A finite closed cover $\mathcal{C}$ of a space X is called convexoid in dimensions $\leq n$, if $\mathcal{C} \leq \mathcal{C}$ is acylic in dimension k for all $k = 0, 1, \ldots, n$ I.e. for every $k = 0, 1, \ldots, n$ and for every subset $\mathcal{A} \subset \mathcal{C}$ and every compact set $K \subset \cap\mathcal{A}$ there is a larger compact set $K' \subset \cap\mathcal{A}$, $K \subset K'$ such that $\check{H}_k^{\sim}(K) \to \check{H}_k^{\sim}(K')$ is the zero map. $\mathcal{C}$ is called convexoid if is convexoid in dimensions $\leq n$ for all $n \in \mathbb{N}$.

8.3. Corollary.

Let $\mathcal{U}' \leq \mathcal{C}' \leq \mathcal{C}$ be a sequence of covers such that

(i) $\mathcal{U}'$ is finite open, $\mathcal{C}', \mathcal{C}$ are finite closed covers

(ii) $\mathcal{U}'$ is a starrefinement of $\mathcal{C}'$

(iii) for every $C' \in \mathcal{C}'$ there is a $C \in \mathcal{C}$ such that $C' \subset \text{interior}(C)$

(iv) $\mathcal{C}$ is n-convexoid

Then $\mathcal{C}$ is n-geometric and $\mathcal{U}' \in \mathcal{O}(\mathcal{C})$.

Proof. Apply 8.1 with $\mathcal{C}'_o = \mathcal{C}'$, $\mathcal{C}_o = \mathcal{C}_1 = \ldots = \mathcal{C}_{n+1} = \mathcal{C}$.

8.4. Theorem.

Let $\mathcal{C}$ be a finite closed convexoid covering of a normal space X which admits a finite open refinement. Then $\mathcal{C}$ is geometric

Proof. Let $\mathcal{U}$ be a finite open refinement of $\mathcal{C}$. Because X is normal there exists a finite open starrefinement $\mathcal{V}$ of $\mathcal{U}$. Let $\mathcal{C}'$ be the covering consisting of the closures of elements of $\mathcal{V}$. Finally let $\mathcal{U}'$ be a finite open starrefinement of $\mathcal{V}$. The chain of coverings

$$\mathcal{U}' \leq \mathcal{C}' \leq \mathcal{C}_o \leq \ldots \leq \mathcal{C}_{n+1}$$

with $\mathcal{C}=\mathcal{C}_o = \mathcal{C}_1 = \ldots = \mathcal{C}_{n+1}$ then satisfies the conditions of theorem 8.1.

q.e.d.

8.6. Further Remark.

Instead of relying on acyclicity conditions to construct geometric covers one can also rely on c-ℓc^n refinements in order to be able to construct the necessary chain maps. In fact for the application to the construction of n-geometric covers a somewhat weaker notion: weak c-ℓc^n refinement is sufficient. This is defined as follows.

Let $K \subset K'$, $\mathcal{V} \in \mathrm{cov}^f(K)$, $\mathcal{V}' \in \mathrm{cov}^f(K')$. The pair $(\mathcal{V}', \mathcal{V})$ is said to have weak ECR(n) with respect to $(\mathcal{C}', \mathcal{C})$ if for every complex S of dimension $\leq n$ and partial realization $\tau : S_1 \to \Delta C(K, \mathcal{V})$ of norm $\leq \mathcal{C}$ there exists a realization $\tau' : S \to \Delta C(K', \mathcal{V}')$ of norm $\leq \mathcal{C}'$ extending τ.

A pair of covers $\mathcal{C}_1 \leq \mathcal{C}_2$ of a space X is then a weak c-ℓc^n refinement if for every compact $K_1 \subset X$ there is a larger compact K_2 such that for every $\mathcal{V}_2 \in \mathrm{cov}^f(K_2)$ there is a refinement $\mathcal{V}_1 \in \mathrm{cov}^f(K_1)$ such that $(\mathcal{V}_2, \mathcal{V}_1)$ has weak ECR(n+1) with respect to $(\mathcal{C}_2, \mathcal{C}_1)$.

One now has e.g. the following theorem

Theorem. Suppose we have a sequence of covers $\mathcal{U}' \leq \mathcal{C}' \leq \mathcal{C}'' \leq \mathcal{C}$ of a space X such that

(i) $\mathcal{U}'$ is a finite open cover; $\mathcal{C}'$, $\mathcal{C}''$, $\mathcal{C}$ are finite closed covers

(ii) $\mathcal{U}'$ is a starrefinement of $\mathcal{C}'$; $\mathcal{C}''$ is a star refinement of $\mathcal{C}$.

(iii) $\mathcal{C}$ is a weak c-ℓc^n covering of X (i.e. $\mathcal{C} \leq \{X\}$ is a weak c-ℓc^n refinement.

And suppose that in addition one of the following conditions is satisfied

(iv) $\mathcal{C}'$ is a weak c-ℓc^n refinement of $\mathcal{C}''$

(v) dim $\mathcal{U}' \leq n$ and $\mathcal{C}'$ is a weak c-ℓc^{n-1} refinement of $\mathcal{C}''$

(vi) dim$X \leq n$ and $\mathcal{C}'$ is a c-ℓc^{n+1} refinement of $\mathcal{C}''$

Then $\mathcal{C}$ is n-geometric and $\mathcal{U}' \in \sigma(\mathcal{C})$.

The proof of this theorem is very similar to the proof of theorem 8.1 However, one has slightly weaker control of the supports and it is to overcome this that one needs the extra starrefinement $\mathcal{C}'' \leq \mathcal{C}$ and condition (iii). For details cf. [12].

9. CONVEX COVERS OF EUCLIDEAN SPACES.

In order to prove that euclidean spaces have the almost fixed point property with respect to finite open convex covers and continuous maps we need the following refinement-of-convex-coverings result.

9.1. Theorem.

Let A be a finite intersection of closed halfspaces in $\mathbb{R}^n$ or $A = \mathbb{R}^n$. Then for every finite open convex covering $\mathcal{U}$ of A there exists a finite closed convex covering $\mathcal{C}$ such that

(i) $\mathcal{C} \leq \mathcal{U}$, i.e. $\mathcal{C}$ refines $\mathcal{U}$

(ii) The interiors of the elements of $\mathcal{C}$ still cover A.

To prove this we use some lemmas. Let $U \subset \mathbb{R}^n$, $U \neq \mathbb{R}^n$ be a convex set, $\bar{U}$ its closure. We define a function $r_U : \bar{U} \to \mathbb{R}$ by $r_U(x) = \sup\{r \mid B(x,r) \subset U\}$ where $B(x,r)$ is the open ball of radius r and center x.

9.2. Lemma.

$r_U : \bar{U} \to \mathbb{R}$ is a concave continuous function. I.e.

$r_U(\lambda x+(1-\lambda)y) \geq \lambda\, r_U(x) + (1-\lambda)r_U(y)$ for $x, y \in \bar{U}$.

9.3. Lemma.

Let C be a closed convex subset of an open convex set $U \subset \mathbb{R}^n$. Then there is a closed convex subset C' such that $C \subset \mathrm{int}(C') \subset C' \subset U$.

Proof. Define

$$C'' = \{x \in \bar{U} \mid \exists y \in C \text{ such that } \| x-y\| \leq \tfrac{1}{2} r_U(y)\}$$

We check that C'' is convex. Let $\| x_1-y_1\| \leq \frac{1}{2}r_U(y_1)$, $\| x_2-y_2\| \leq \frac{1}{2}r_U(y_2)$. Let $0 \leq \lambda \leq 1$. Then

$$\| (\lambda x_1+(1-\lambda)x_2) - (\lambda y_1+(1-\lambda)y_2)\| \leq \lambda\|x_1-y_1\| + (1-\lambda)\| x_2-y_2\| \leq$$

$\leq \frac{1}{2}\lambda r_U(y_1) + \frac{1}{2}(1-\lambda)r_U(y_2) \leq \frac{1}{2}\, r_U(\lambda y_1+(1-\lambda)y_2)$ because r_U is concave. Now let x be a point in the closure of C''. We show that $x \in U$.

This will prove the lemma. (Take $C' = \bar{C}''$). Let (x_i), $x_i \in C''$ be a sequence of points converging to x. Let y_i be such that $\| x_i - y_i \| \leq \frac{1}{2} r_U(y_i)$. Note that $r_U(x_i) \geq \frac{1}{2} r_U(y_i)$. Consider the sequence of real positive numbers $r_U(y_i)$. If $\liminf r_U(y_i) > 0$ then $\liminf r_U(x_i) > 0$ and hence $r_U(x) > 0 \Rightarrow x \in U$. If $\liminf r_U(y_i) = 0$ we can assume by taking a subsequence that $\lim r_U(y_i) = 0$ because $\| x_i - y_i \| \leq \frac{1}{2} r_U(y_i)$ it follows that y_i converges to x. But $y_i \in C$ and C is closed hence $x \in C \subset U$.

q.e.d.

9.4. Lemma.

Let $A \subset \mathbb{R}^n$ be a convex set and let x be a point in the interior of A. Let ℓ be a ray starting in x and suppose $\ell \subset A$. Then

(i) $\ell \subset \text{int}(A)$

(ii) $\exists \varepsilon > 0$ such that $d(y,\ell) \leq \varepsilon \Rightarrow y \in \text{int}(A)$, where $d(y,\ell)$ is the distance of y to ℓ.

Proof. Let $y \in \ell$, and let y' be a point on ℓ twice as far from x as y. Let B be small open ball around x such that $B \subset A$. The linear combinations $\frac{1}{2}x + \frac{1}{2}y'$, $y' \in B$ then constitute an open ball around y, which proves that $y \in \text{int}(A)$. This proves (i). To prove (ii) consider the function $r_A : \ell \to \mathbb{R}$ this function is concave and continuous and $r_A(y) > 0$ for all $y \in \ell$. It follows that there is an $\varepsilon > 0$ such that $r_A(y) \geq \varepsilon$ for all $y \in \ell$. This proves (ii).

9.5. Proof of the theorem.

If $\mathcal{U}$ is a finite open covering of A we denote with $s(\mathcal{U})$ the total number of simplices in $\check{C}(\mathcal{U})$. By induction we can assume that the theorem has been proved for dim A < n and dim A = n and $s(\mathcal{U}) < s$. (The cases dim A = 1 and $s(\mathcal{U}) = 1$ being trivial)
Let $\mathcal{U}$ be a convex open covering of A, dim A = n and $s(\mathcal{U}) = s$.

There are two cases to consider

(a) $\cap \mathcal{U} = \phi$

(b) $\cap \mathcal{U} \neq \phi$

In case (a) let $\mathcal{U}' \subset \mathcal{U}$ be a maximal subset such that $\cap \mathcal{U}' \neq \phi$. Choose $U \in \mathcal{U} \setminus \mathcal{U}'$. By the separating hyperplane theorem there is a hyperplane H such that U is on one side of H and $\cap \mathcal{U}'$ on the other. Let A^+ and A^- be the intersections of A with the closed halfspaces determined by H. Then

$$s(\mathcal{U}|A^+) < s(\mathcal{U}) \text{ and } s(\mathcal{U}|A^-) < s(\mathcal{U})$$

and by induction we are done with this case.
Suppose we are in case (b). We can assume that $A \subset \mathbb{R}^n$, dim A = n hence int(A) $\neq \phi$, dim U = n for all $U \in \mathcal{U}$. Let $x \in \cap \mathcal{U}$. We can see to it that also $x \in$ int(A). Let S be a sphere with center x. Each point $s \in S$ corresponds uniquely to a ray ℓ_s starting in x. For every ray ℓ_s there is a $U \in \mathcal{U}$ such that $\ell_s \subset U$. This is seen as follows. If $\ell_s \cap A \neq \ell_s$ then there is a unique point $y_s \in \ell_s \cap A$ such that $\ell_s \cap A$ = segment joining x and y_s; y_s is in the boundary of A. If $\ell_s \cap A = \ell_s$ choose points $y_1, y_2, y_3, \ldots$ on ℓ_s at distance 1,2,3,... from x. At least one $U \in \mathcal{U}$ contains infinitely many of these points. Then $\ell_s \subset U$.
For every $U \in \mathcal{U}$ we now define a set C_U as follows

$$C_U = \{y \in A | \exists s \in S \text{ such that } y \in \ell_s \subset U\}$$

Concerning these C_U we have

(i) C_U is convex, $C_U \subset U$

(ii) C_U is closed

Claim (i) is a triviality. To prove (ii), consider $C_U \cap S$.

Let (s_i), $s_i \in C_U \cap S$ be a sequence of points converging to $s \in S$. Let z be a point of ℓ_s. Then z is the limit of a sequence of points $z_i \in \ell_{s_i}$ (take $z_i \in \ell_{s_i}$ such that $\| z_i - x \| = \| z - x \|$. Hence $z \in \bar{U}$. Hence $\ell_s \subset \bar{U}$ and hence $\ell_s \subset \text{int}(\bar{U})$ by lemma 9.4. This proves (ii).

To deal with the rays ℓ_s such that $\ell_s \cap A \neq \ell_s$ we use the following construction. The boundary bd(A) of A is a finite union $A = \bigcup_{t=1}^{k} A_t$, dim $A_t = n-1$, A_t a finite intersection of closed halfspaces. For each t let $D_1(t), \ldots, D_{n_t}(t)$ be a finite closed convex covering of A_t which refines $\mathcal{U} | A_t$. (Induction!)

For each t and $i \in \{1, \ldots, n_t\}$ we define

$$C_{t,i} = \{y \in A \mid \exists z \in D_i(t) \text{ with } y \text{ on the sequent joining } x \text{ and } z\}$$

For each $C_{t,i}$ choose $U \in \mathcal{U}$ such that $D_i(t) \subset U$. Then we have

(iii) $C_{t,i} \subset U$ and $C_{t,i}$ is convex

(iv) $\overline{C_{t,i}} \subset U$

Claim (iii) is a triviality. To see (iv) let $T \subset S$ be the subset of $s \in S$ corresponding to rays in $C_{t,i}$. Let (s_i), $s_i \in T$ be a sequence of points converging to $s \in S$. There are two possibilities. First $\ell_s \cap A \neq \ell_s$. The sequence y_{s_i} of endpoints then converges to y_s and because $y_{s_i} \in D_i(t)$, $y_s \in D_i(t)$ so that $\ell_s \cap A \subset C_{t,i} \subset U$. Secondly suppose that $\ell_s \cap A = \ell_s$ (i.e. the points y_{s_i} run off to infinity). Let $z \in \ell_s$. The distance $\| x - y_{s_i} \|$ goes to infinity as $i \to \infty$ (Follows from lemma 9.4 (ii). Hence z is the limit of a sequence of points $z_i \in \ell_{s_i} \cap A \subset U$. Hence $z \in \bar{U}$. Thus $\ell_s \subset \bar{U}$ hence $\ell_s \subset U$ by lemma 9.4. (i). We have now found a closed convex finite covering consisting of

$$C_U, \; U \in \mathcal{U} \; ; \; \overline{C}_{t,i}, \; t = 1, \ldots, k; \; i = 1, \ldots, n_t$$

of A which refines $\mathcal{U}$. Thickening each C_U and $\bar{C}_{t,i}$ as in lemma 9.3 then gives a finite closed convex covering $\mathcal{C}$ which refines $\mathcal{U}$ and such that their interiors still cover A. This concludes the proof of the theorem.

We can now prove the following almost fixed point theorem for euclidean spaces which was conjectured by De Groot, cf. [2].

9.6. Almost Fixed Point Theorem for Euclidean Spaces.

Let $\mathcal{U}$ be a finite open convex covering of $\mathbb{R}^n$, and $f: \mathbb{R}^n \to \mathbb{R}^n$ a continuous map. Then there is a $U \in \mathcal{U}$ such that $U \cap f(U) \neq \phi$.

Proof. Let $\mathcal{C}$ be a finite convex closed refinement of $\mathcal{U}$ such that the interiors of the sets in $\mathcal{C}$ still cover $\mathbb{R}^n$. Such a $\mathcal{C}$ exists by theorem 9.1. The covering $\mathcal{C}$ is geometric by theorem 8.4. It now suffices to apply theorem 5.2.

Addendum to Report 7501 (On almost-fixed point theory, by M. Hazewinkel and M. van de Vel)

1. Very recently it was pointed out to us that in [16], a paper which deserves to be much more widely known than apparently it is, Dugundji also gives a proof of de Groot's conjecture. The methods are different.

2. Using practically the same arguments as in report 7501 one can prove Theorem. Let $\underline{U}$ be a finite convex open covering of a locally convex space X and let $f: X \to X$ be a continuous map. Then there is a $U \in \underline{U}$ such that $U \cap f(U) \neq \phi$. This result is not covered by Dugundji's theorem.

April 30, 1975.

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