A TUTORIAL INTRODUCTION TO DIFFERENTIABLE MANIFOLDS AND VECTOR FIELDS

Michiel Hazewinkel

Dept. Math., Erasmus Univ. Rotterdam

In this tutorial I try by means of several examples to illustrate the basic definitions and concepts of differentiable manifolds. There are few proofs (not that there are ever many at this level of the theory). This material should be sufficient to understand the use made of these concepts in the other contributions in this volume, or, at least, it should help in explaining the terminology employed.

1. INTRODUCTION AND A FEW MOTIVATIONAL REMARKS

Roughly an n-dimensional differentiable manifold is a gadget which locally looks like $\mathbb{R}^n$ but globally perhaps not; A precise definition is given below in section 2. Examples are the sphere and the torus, which are both locally like $\mathbb{R}^2$ but differ globally from $\mathbb{R}^2$ and from each other.

Such objects often arise naturally when discussing problems in analysis (e.g. differential equations) and elsewhere in mathematics and its applications. A few advantages which may come about by doing analysis on manifolds rather than just on $\mathbb{R}^n$ are briefly discussed below.

M. Hazewinkel and J. C. Willems (eds.), Stochastic Systems: The Mathematics of Filtering and Identification and Applications, 77–93.

1.1 Coordinate freeness ("Diffeomorphisms"). A differentiable manifold can be viewed as consisting of pieces of $\mathbb{R}^n$ which are glued together in a smooth (= differentiable) manner. And it is on the basis of such a picture that the analysis (e.g. the study of differential equations) often proceeds. This brings more than a mere extension of analysis on $\mathbb{R}^n$ to analysis on spheres, tori, projective spaces and the like; it stresses the "coordinate free approach", i.e. the formulation of problems and concepts in terms which are invariant under (nonlinear) smooth coordinate transformations and thus also helps to bring about a better understanding even of analysis on $\mathbb{R}^n$. The more important results, concepts and definitions tend to be "coordinate free".

1.2 Analytic continuation. A convergent power series in one complex variable is a rather simple object. It is considerably more difficult to obtain an understanding of the collection of all analytic continuations of a given power series, especially because analytic continuation along a full circle may yield a different function value than the initial one. The fact that the various continuations fit together to form a Riemann surface (a certain kind of 2-dimensional manifold usually different from $\mathbb{R}^2$) was a major and most enlightening discovery which contributes a great deal to our understanding.

1.3 Submanifolds. Consider an equation $\dot{x} = f(x)$ in $\mathbb{R}^n$. Then it often happens, especially in problems coming from mechanics, that the equation is such that it evolves in such a way that certain quantities (e.g. energy, angular momentum) are conserved. Thus the equation really evolves on a subset $\{x \in \mathbb{R}^n | E(x) = c\}$ which is often a differentiable submanifold. Thus it could happen that $\dot{x} = f(x)$, f smooth, is constrained to move on a 2-sphere which then immediately tells us that there is an equilibrium point.

Also one might meet 2 seemingly different equations, say, one in $\mathbb{R}^4$ and one in $\mathbb{R}^3$ (perhaps both intended as a description of the same process) of which the first has two conserved quantities and the second one. It will then be important to decide whether the surfaces on which the equations evolve are diffeomorphic, i.e. the same after a suitable invertible transformation and whether the equations on these submanifolds correspond under these transformations.

1.4 Behaviour at infinity. Consider a differential equation in the plane $\dot{x} = P(x,y)$, $\dot{y} = Q(x,y)$ where P and Q are relatively prime polynomials. To study the behavior of the paths far out in the plane and such things as solutions escaping to infinity and coming back, Poincaré already completed the plane to real projective 2-space (an example of a differential manifold). Also the projective plane is by no means the only smooth manifold compatifying $\mathbb{R}^2$ and it will be of some importance for the behaviour of the equation near infinity whether the "right" compactification to which the equation can be extended will be a projective 2-space, a sphere or a torus, or, ..., or, whether no such compactification exists at all. A good example of a set of equations which are practically impossible to analyse completely without bringing in manifolds are the matrix Riccati equations (which naturally live on Grassmann manifolds (which also gives in this case a very considerable saving in the number of dimensions needed)).

1.5 Avoiding confusion between different kinds of objects. Consider an ordinary differential equation $\dot{x} = f(x)$ on $\mathbb{R}^n$, where $f(x)$ is a function $\mathbb{R}^n \to \mathbb{R}^n$. When one now tries to generalize this idea of a differential equation to a differential equation on a manifold one discovers that $\dot{x}$ and hence $f(x)$ are a different kind of object; they are not functions, but as we shall see, they are vectorfields; in other words under a nonlinear change of coordinates they transform in a different way than functions do.

2. DIFFERENTIABLE MANIFOLDS

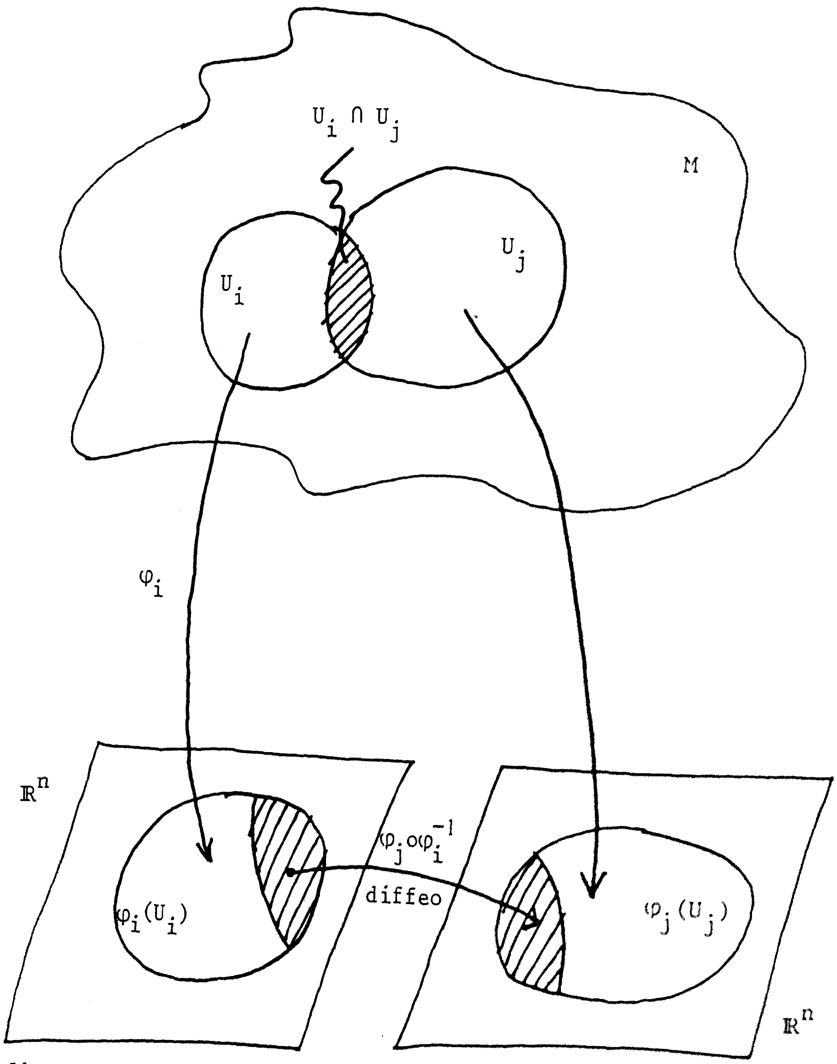

fig.1. Pictorial definition of a differentiable manifold.

Let U be an open subset of $\mathbb{R}^n$, e.g. an open ball. A function $f: U \to \mathbb{R}$ is said to be C^∞ or smooth if all partial derivatives (any order) exist at all $x \in U$. A mapping $\mathbb{R}^n \supset U \to \mathbb{R}^m$ is smooth if all components are smooth; $\varphi: U \to V$, $U \subset \mathbb{R}^n, V \subset \mathbb{R}^n$ is called a diffeomorphism if φ is $1-1$, onto, and both φ and φ^{-1} are smooth.

As indicated above a smooth n-dimensional manifold is a gadget consisting of open pieces of $\mathbb{R}^n$ smoothly glued together. This gives the above pictorial definition of a smooth n-dimensional manifold M (fig.1).

2.1 Example. The circle $S^1 = \{(x_1,x_2)\,|\,x_1^2 + x_2^2 = 1\} \subset \mathbb{R}^2$

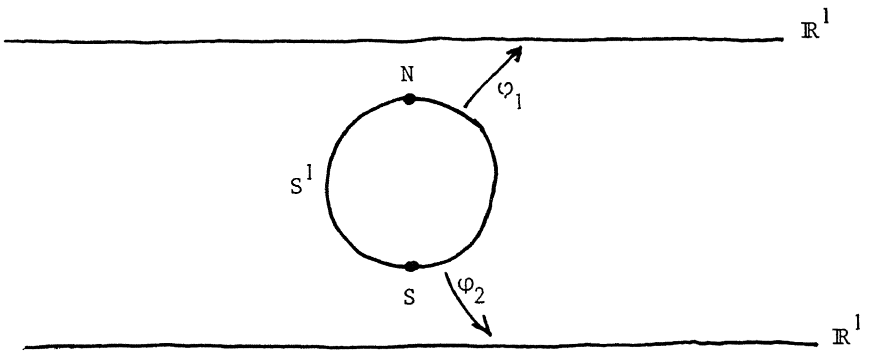

fig.2. Example: the circle

$U_1 = S^1 \setminus \{S\}$, $U_2 = S^1 \setminus \{N\}$ so $U_1 \cup U_2 = S^1$. The "coordinate charts" φ_1 and φ_2 are given by

$$\varphi_1(x_1,x_2) = \frac{x_1}{1+x_2} \quad , \quad \varphi_2(x_1,x_2) = \frac{x_1}{1-x_2}$$

Thus $\varphi_1(U_1 \cap U_2) = \mathbb{R} \setminus \{0\}$, $\varphi_2(U_1 \cap U_2) = \mathbb{R} \setminus \{0\}$ and the map

$\varphi_2 \circ \varphi_1^{-1}: \mathbb{R}\setminus\{0\} \to \mathbb{R}\setminus\{0\}$ is given by $x \mapsto x^{-1}$ which is a diffeomorphism.

2.2 Formal definition of a differentiable manifold.

The data are

- M, a Hausdorff topological space
- A covering $\{U_i\}_{i\in I}$ by open subsets of M
- Coordinate maps $\varphi_i: U_i \to \varphi_i(U_i) \subset \mathbb{R}^n$, $\varphi_i(U_i)$ open in $\mathbb{R}^n$.

These data are subject to the following condition

- $\varphi_i \circ \varphi_j^{-1}: \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ is a diffeomorphism

Often one also adds the requirement that M be paracompact. We shall however disregard these finer points; nor shall we need them in this volume.

2.3 Constructing differentiable manifolds 1: embedded manifolds. Let M be a subset of $\mathbb{R}^N$. Suppose for every $x \in M$ there exists an open neighbourhood $U \subset \mathbb{R}^n$ and a smooth function ψ: $U \to \mathbb{R}^N$ mapping U homeomorphically onto an open neighbourhood V of x in M. Suppose moreover that the Jacobian matrix of ψ has rank n at all $u \in U$. Then M is a smooth manifold of dimension n. (Exercise: the coordinate neighbourhoods are the V's and the coordinate maps are the ψ^{-1}; use the implicit function theorem). Virtually the same arguments show that if $\varphi: U \to \mathbb{R}^k$, $U \subset \mathbb{R}^{n+k}$, is a smooth map and the rank of the Jacobian matrix $J(f)(x)$ is k for all $x \in \varphi^{-1}(0)$, then $\varphi^{-1}(0)$ is a smooth n-dimensional manifold. We shall not pursue this approach but concentrate instead on:

2.4 Constructing differentiable manifolds 2: gluing. Here the data are as follows

- an index set I
- for every $i \in I$ an open subset $U_i \subset \mathbb{R}^n$
- for every ordered pair (i,j) an open subset $U_{ij} \subset U_i$

- diffeomorphisms $\varphi_{ij}: U_{ij} \to U_{ji}$ for all $i,j \in I$

These data are supposed to satisfy the following compatibility conditions

- $U_{ii} = U_i$, $\varphi_{ii} = \mathrm{id}$
- $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ (where appropriate)

(where the last identity is supposed to imply also that $\varphi_{ij}(U_{ij} \cap U_{ik}) \subset U_{jk}$ so that $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{jk} \cap U_{ji}$).

These are not all conditions but the present lecturer e.g. has often found it advantageous to stop right here so to speak, and to view a manifold simply as a collection of open subsets of $\mathbb{R}^n$ together with gluing data (coordinate transformation rules).

From the data given above one now defines an abstract topological space M by taking the disjoint union of the U_i and then identifying $x \in U_i$ and $y \in U_j$ iff $x \in U_{ij}$, $y \in U_{ji}$, $\varphi_{ij}(x) = y$. This gives a natural injection $U_i \to M$ with image U_i' say. Let $\varphi_i: U_i' \to U_i$ be the inverse map. Then this gives us a differentiable manifold M in the sense of definition 2.2 provided that M is Hausdorff and paracompact, and these are the conditions which must be added to the gluing compatibility conditions above.

2.5 Functions on a "glued manifold". Let M be a differentiable manifold obtained by the gluing process described in 2.4 above. Then a differentiable function $f: M \to \mathbb{R}$ consist simply of a collection of functions $f_i: U_i \to \mathbb{R}$ such that $f_j \circ \varphi_{ij} = f_i$ on U_{ij}, as illustrated in fig. 3.

Thus for example a function on the circle S^1, cf. figure 2, can be described either as a function of two variables restricted to S^1 or as two functions f_1, f_2 of one variable on U_1 and U_2 such that $f_1(x) = f_2(x^{-1})$. Obviously the latter approach can have considerable advantages.

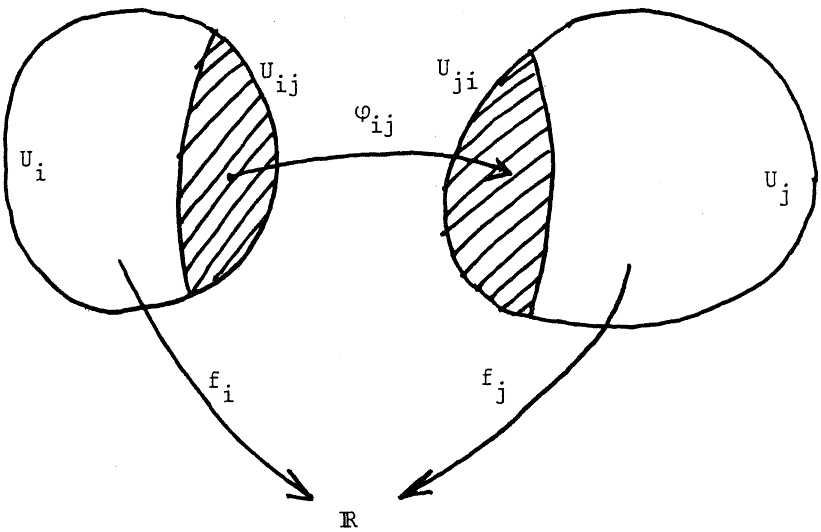

fig.3. Functions on a glued manifold

2.6 Example of a 2 dimensional manifold: the Möbius band.
The (open) Möbius band is obtained by taking a strip in $\mathbb{R}^2$ as indicated below in fig. 4 without its upper and lower edges and identifying the left hand and right hand edges as indicated

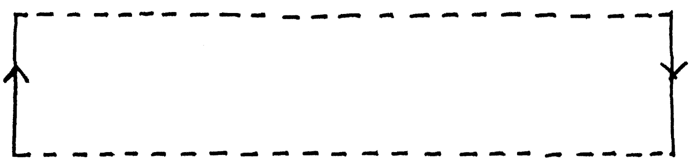

fig.4. Construction of the Möbius band

The resulting manifold (as a submanifold of $\mathbb{R}^3$) looks something like the following figure 5.

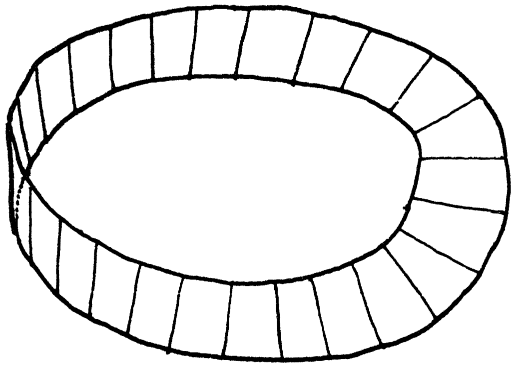

fig.5. The Möbius band

It is left as an exercise to the reader to cast this description in the form required by the gluing description of 2.4 above. The following pictorial description (fig. 6) will suffice.

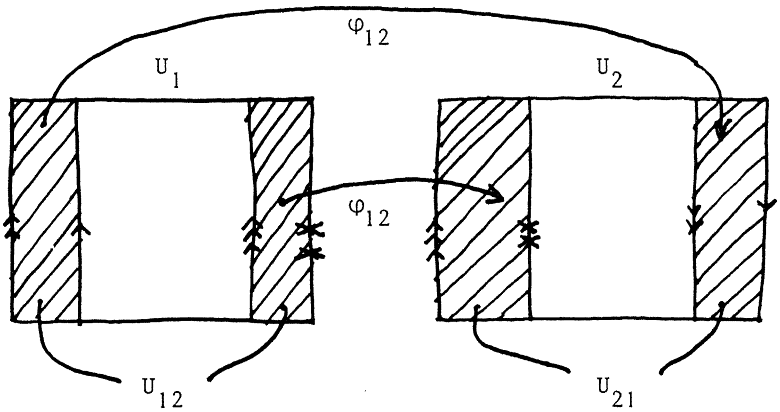

fig.6. Gluing description of the Möbius band

2.7 Example: the 2-dimensional sphere. The picture in fig. 7 below shows how the 2-sphere $S^2 = \{(x_1,x_2,x_3) \mid x_1^2 + x_2^2 + x_3^2 = 1\}$ can be obtained by gluing two disks together. If the surface of the earth is viewed as a model for S^2 the first disk covers everything north of Capricorn and the second everything south of Cancer.

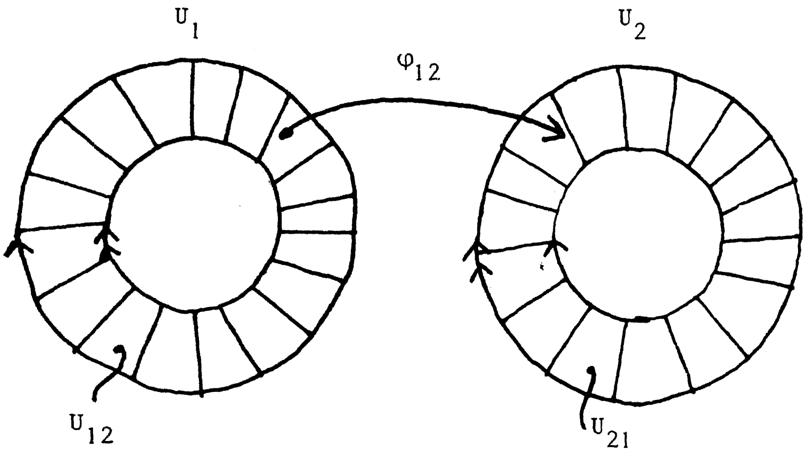

fig.7. Gluing description of the 2-sphere S^2

2.8 Morphisms of differentiable manifolds. Let M and N be differentiable manifolds obtained by the gluing process of section 2.4 above. Say M is obtained by gluing together open subsets U_i of $\mathbb{R}^n$ and N by gluing together open subsets V_j of $\mathbb{R}^m$. Then a smooth map $f: M \to N$ (a morphism) is given by specifying for all i,j an open subset $U_{ij} \subset U_i$ and a smooth map $f_{ij}: U_{ij} \to V_j$ such that $\bigcup_j U_{ij} = U_i$ and the f_{ij} are compatible under the identifications $\varphi_{ii'}: U_{ii'} \to U_{i'i}$, $\varphi_{jj'}: V_{jj'} \to V_{j'j}$, i.e. $f_{i'j'} \circ \varphi_{ii'} = \varphi_{jj'} \circ f_{ij}$ whenever appropriate. (Here the φ's are the gluing diffeomorphisms for M and the ψ's are the gluing diffeomorphisms for N).

2.9 Exercise: Show that the description of the circle S^1 as in 2.1 above gives an injective morphism $S^1 \to \mathbb{R}^2$.

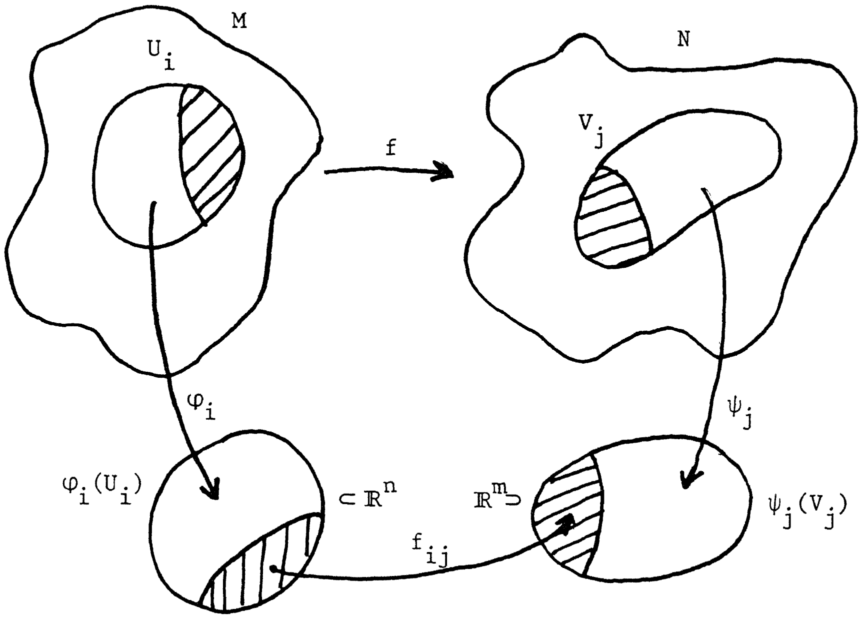

fig.8. Morphisms

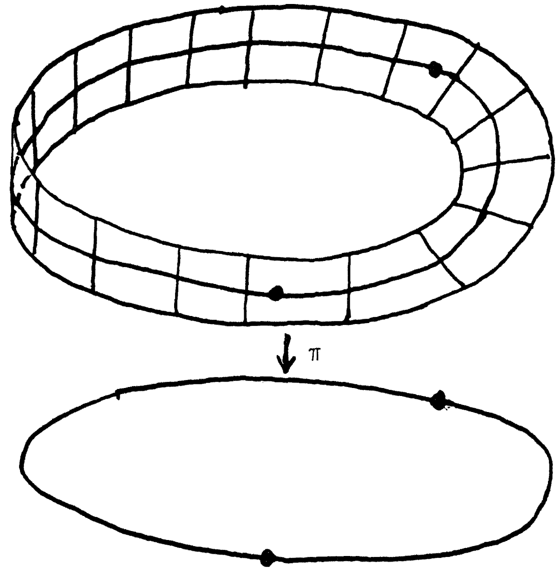

fig.9. The Möbius band as vectorbundle over the circle

3. DIFFERENTIABLE VECTORBUNDLES

Intuitively a vectorbundle over a space S is a family of vectorspaces parametrized by S. Thus for example the Möbius band of example 2.6 can be viewed as a family of open intervals in $\mathbb{R}$ parametrized by the circle, cf. fig. 9 above, and if we are willing to identify the open intervals with $\mathbb{R}$ this gives us a family of one dimensional vectorspaces parametrized by S^1 which locally (i.e. over small neighbourhoods in the base space S^1) looks like a product but globally is not equal to a product.

3.1 Formal definition of a differentiable vectorbundle. A differentiable vectorbundle of dimension m over a differentiable manifold M consists of a surjective morphism $\pi: E \to M$ of differentiable manifolds and a structure of an m-dimensional real vectorspace on $\pi^{-1}(x)$ for all $x \in M$ such that moreover there is for all $x \in M$ an open neighbourhood $U \subset M$ containing x and a diffeomorphism $\varphi_U: U \times \mathbb{R}^m \to \pi^{-1}(U)$ such that the following diagram commutes

$$\begin{tikzcd} U \times \mathbb{R}^m \arrow[rr, "\varphi_U"] \arrow[dr] & & \pi^{-1}(U) \arrow[dl, "\pi"] \\ & U & \end{tikzcd}$$

where the lefthand arrow is the projection on the first factor, and such that φ_U induces for every $y \in U$ an isomorphism $\{y\} \times \mathbb{R}^m \to \pi^{-1}(y)$ of real vectorspaces.

3.2 Constructing vectorbundles. The definition given above is not always particularly easy to assimilate. It simply means that a vectorbundle over M is obtained by taking an open covering $\{U_i\}$ of M and gluing together products $U_i \times \mathbb{R}^m$ by means of dif-

feomorphisms which are linear (i.e. vectorspace structure preserving) in the second coordinate. Thus an m-dimensional vectorbundle over M is given by the following data
- an open covering $\{U_i\}_{i\in I}$ of M.
- for every i,j a smooth map $\varphi_{ij}: U_i \cap U_j \to GL_m(\mathbb{R})$ where $GL_m(\mathbb{R})$ is the space of all invertible real m × m matrices considered as an open subset of $\mathbb{R}^{m^2}$. These data are subject to the following compatibility conditions
- $\varphi_{ii}(x) = I_m$, the identity matrix, for all $x \in U_i$
- $\varphi_{jk}(x)\, \varphi_{ij}(x) = \varphi_{ik}(x)$ for all $x \in U_i \cap U_j \cap U_k$

From these data E is constructed by taking the disjoint union of the $U_i \times \mathbb{R}^m$, $i \in I$ and identifying $(x,v) \in U_i \times \mathbb{R}^m$ with $(y,w) \in U_j \times \mathbb{R}^m$ if and only if $x = y$ and $\varphi_{ij}(x)v = w$. The morphism π is induced by the first coordinate projections $U_i \times \mathbb{R}^m \to U_i$.

3.3 Constructing vectorbundles 2. If the base manifold M is itself viewed as a smoothly glued together collection of open sets in $\mathbb{R}^n$ we can descripe the gluing for M and a vectorbundle E over M all at once. The combined data are than as follows
- open sets $U_i \times \mathbb{R}^m$, $U_i \subset \mathbb{R}^n$ for all $i \in I$
- open subsets $U_{ij} \subset U_i$ for all $i,j \in I$
- diffeomorphisms $\varphi_{ij}: U_{ij} \to U_{ji}$
- diffeomorphisms $\tilde{\varphi}_{ij}: U_{ij} \times \mathbb{R}^m \to U_{ji} \times \mathbb{R}^m$ of the form $(x,v) \mapsto (\varphi_{ij}(x), A_{ij}(x)v)$ where $A_{ij}(x)$ is an m × m invertible real matrix depending smoothly on x.

These data are then subject to the same compatibility conditions for the $\tilde{\varphi}_{ij}$'s (and hence the φ_{ij}) as described in 2.4 above.

3.4 Example: the tangent vectorbundle of a smooth manifold. Let the smooth manifold M be given by the data U_i, U_{ij}, φ_{ij} as in 2.4. Then the tangent bundle TM is given by the data

- $U_i \times \mathbb{R}^n$, $U_{ij} \times \mathbb{R}^n \subset U_i \times \mathbb{R}^n$
- $\tilde{\varphi}_{ij}: U_{ij} \times \mathbb{R}^n \to U_{ji} \times \mathbb{R}^n$, $\tilde{\varphi}_{ij}(x,v) = (\varphi_{ij}(x), J(\varphi_{ij})(x)v)$

where $J(\varphi_{ij})(x)$ is the Jacobian matrix of φ_{ij} at $x \in U_{ij}$.

Exercise: check that these gluing morphisms do indeed define a vectorbundle; i.e. check the compatibility (chain rule !)

3.5 Morphisms of vectorbundles. A morphism of vectorbundles from the vectorbundle $\pi: E \to M$ to the vectorbundle $\pi': E' \to M'$ is a pair of smooth maps $\tilde{f}: E \to E'$, $f: M \to M'$ such that $\pi' \circ \tilde{f} = f \circ \pi$ and such that the induced map $\tilde{f}_x: \pi^{-1}(x) \to \pi^{-1}(f(x))$ is a homomorphism of vectorspaces for all $x \in M$. We leave it to the reader to translate this into a local pieces and gluing data description.

As an example consider two manifolds M, N both described in terms of local pieces and gluing data. Let $f: M \to N$ be given in these terms by the $f_{ij}: U_{ij} \to V_j$ (cf. 2.8 above). Then the maps $\tilde{f}_{ij}: U_{ij} \times \mathbb{R}^n \to V_j \times \mathbb{R}^m$ defined by $\tilde{f}_{ij}(x,v) = (f_{ij}(x), J(f_{ij})(x)v)$ combine to define a morphism of vectorbundles $\tilde{f} = Tf: TM \to TN$.

4. VECTORFIELDS

A vectorfield on a manifold M assigns in a differentiable manner to every $x \in M$ a tangent vector at x, i.e. an element of the fibre $T_xM = \pi^{-1}(x)$ of the tangent bundle TM. Slightly more precisely this gives the

4.1 Definitions. Let $\pi: E \to M$ be a vectorbundle. Then a section of E is a smooth map $s: M \to E$ such that $\pi \circ s = \mathrm{id}$. A section of the tangent vectorbundle $TM \to M$ is called a vectorfield.

Suppose that M is given by a local pieces and gluing data description as in 2.4 above. Then a vectorfield s is given by "local sections" $s_i^!: U_i \to U_i \times \mathbb{R}^n$ of the form $s_i^!(x) = (x, s_i(x))$, i.e. by a collection of functions $s_i: U_i \to \mathbb{R}^n$ such that $J(\varphi_{ij})(x)(s_i(x)) = s_j(x)$ for all $x \in U_{ij}$.

4.2 Derivations. Let A be an algebra over $\mathbb{R}$. Then a derivation is an $\mathbb{R}$-linear map $D: A \to A$ such that $D(fg) = (Df)g + f(Dg)$ for all $f,g \in A$.

4.3 Derivations and vectorfields. Now let M be a differentiable manifold and let S(M) be the $\mathbb{R}$-algebra of smooth functions $M \to \mathbb{R}$. Then every vectorfield s on M defines a derivation of S(M), (which assigns to a function f its derivative along s), which can be described as follows. Let M be given in terms of local pieces U_i and gluing data U_{ij}, φ_{ij}. Let $f: M \to \mathbb{R}$ and the section $s: M \to TM$ be given by the local functions $f_i: U_i \to \mathbb{R}$, $s_i: U_i \to \mathbb{R}^n$. Now define $g_i: U_i \to \mathbb{R}$ by the formula

$$g_i(x) = \sum_k s_i(x)_k \frac{\partial f_i}{\partial x_k}(x) \tag{4.4}$$

where $s_i(x)_k$ is the k-th component of the n-vector $s_i(x)$. It is now an easy exercise to check that $g_j(\varphi_{ij}(x)) = g_i(x)$ for all $x \in U_{ij}$ (because $(\varphi_{ij})(x)s_i(x) = s_j(x)$ for these x) so that the $g_i(x)$ combine to define a function $g = D_s(f): M \to \mathbb{R}$. This defines a map $D: S(M) \to S(M)$ which is seen to be a derivation. Inversely every derivation of S(M) arises in this way.

4.5 The Lie bracket of derivations and vectorfields. Let D_1, D_2 be derivations of an $\mathbb{R}$-algebra A. Then, as is easily checked, so is

$$[D_1,D_2] = D_1D_2 - D_2D_1$$

So if s_1, s_2 are vectorfields on M, then there is a vectorfield $[s_1,s_2]$ on M corresponding to the derivation $[D_{s_1},D_{s_2}]$. This vectorfield is called the Lie bracket of s_1 and s_2 and $(s_1,s_2) \mapsto [s_1,s_2]$ defines a Lie algebra structure on the vectorspace V(M) of all vectorfields on M.

If M is given in terms of local pieces U_i and gluing data U_{ij},φ_{ij} then the Lie bracket operation can be described as follows. Let the vectorfields s and t be given by the local functions $s_i,t_i\colon U_i \to \mathbb{R}^n$. Then [s,t] is given by the local functions

$$\sum_j s_j \frac{\partial t_i}{\partial x_j} - \sum_j t_j \frac{\partial s_i}{\partial x_j}$$

4.6 <u>The $\frac{\partial}{\partial x}$ notation.</u> Let the vectorfield s: M → TM be given by the functions $s_i\colon U_i \to \mathbb{R}^n$. Then using the symbols $\frac{\partial}{\partial x_k}$ in first instance simply as labels for the coordinates in $\mathbb{R}^n$ we can write

$$s_i = \sum s_i(x)_k \frac{\partial}{\partial x_k} \tag{4.7}$$

This is a most convenient notation because as can be seen from (4.4) this gives precisely the local description of the differential operator (derivation) D_s associated to s.

4.7 <u>Differential equations on a manifold.</u> A differential equation on a manifold M is given by an equation

$$\dot{x} = s(x) \tag{4.8}$$

where s: M → TM is a vectorfield, i.e. a section of the tangentbundle. At every moment t, equation (4.8) tells us in which direction and how fast x(t) will evolve by specifying a tangent vector s(x(t)) at x(t).

Again it is often useful to take a local pieces and gluing data point of view. Then the differential equation (4.8) is given by a collection of differential equations $\dot{x} = s_i(x)$ in the usual sense of the word on U_i where the functions $s_i(x)$ satisfy $J(\varphi_{ij})(x)s_i(x) = s_j(x)$ for all $x \in U_{ij}$.

In these terms a solution of the differential equation is simply a collection of solutions of the local equations, i.e. a collection of maps $f_i: V_i \to U_i$, $V_i \subset \mathbb{R}(\geqq 0)$ such that $\cup V_i = \mathbb{R}(\geqq 0)$, $\frac{d}{dt} f_i(t) = s_i(f_i(t))$ which fit together to define a morphism $\mathbb{R}(\geqq 0) \to M$, i.e. such that $\varphi_{ij}(f_i(t)) = f_j(t)$ if $t \in V_i \cap V_j$.

In more global terms a solution of (4.8) which passes through x_0 at time 0 is a morphism of smooth manifolds $f: \mathbb{R} \to M$ such that $Tf: T\mathbb{R} \to TM$ satisfies $Tf(t,1) = s(f(t))$ for all $t \in \mathbb{R}$ (or a suitable subset of $\mathbb{R}$), i.e. Tf takes the vectorfield (section) $1: \mathbb{R} \to T\mathbb{R} = \mathbb{R} \times \mathbb{R}$, $t \mapsto (t,1)$ into the vectorfield (section) $s: M \to TM$.

4.9 Conclusion. Here, where it starts to get interesting, is, according to a developing tradition in textbook writing, a good place to stop.