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J.F. Groote, J. van Wamel

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CWI
P.O. Box 94079
1090 GB Amsterdam
The Netherlands

Basic Theorems for Parallel Processes in Timed μCRL

Jan Friso Groote[1,2] & Jos van Wamel[1]

1. *CWI*

P.O. Box 94079, 1090 GB Amsterdam, The Netherlands

2. *Eindhoven University of Technology*

Department of Mathematics and Computing Science

P.O. Box 513, 5600 MB Eindhoven, The Netherlands

E-mail: `{jfg,jos}@cwi.nl`

ABSTRACT

Timed μCRL is a process algebra-based formalism for the specification and verification of parallel, communicating systems with explicit time [5]. In this paper various basic results are derived, such as theorems for *basic forms*, the expansion of terms with operators for parallelism, elimination of parallelism, and commutativity and associativity of the merge and communication merge (the operators $\|$ and $|$).
The interpretation of the operators, in particular the left merge, is far from trivial, and more in general, it has to be stated that working with a time-based formalism such as timed μCRL can be fairly complicated. Therefore we pay a lot of attention to all kinds of proof details that could enhance the understanding – and thus facilitate the use – of the formalism.
Many basic lemmas are included, and examples are used to illustrate the intuition behind the various results.

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1 Introduction

The relevance of timing aspects in many modern, parallel communicating systems does not need much explanation. For instance, the correct behaviour of systems based on digitalised audio and video, such as multimedia systems, strongly depends on time parameters, but the same also holds for more classical time-critical systems, such as railway control systems, automatic manufacturing systems, etc.

For this reason, about ten years ago the study of explicit time in formal specification languages started to become an increasingly important topic of research. Also in the field of process algebra, where we are primarily interested in ACP (Algebra of Communicating Processes, see for example [3]) and its derivatives, such as abstract μCRL (*micro* Common Representation Language, see for example [7]), serious efforts were made.

In [1] the first axiomatisations of *real-time process algebra* appeared and in [9] and [4] different versions were proposed. [9] is a state of the art work for real-time process algebra, with many semantic results. In the realm of *discrete-time process algebra* new developments started via [2]. More recently [10] appeared. It contains various semantic results for discrete-time process algebras without abstraction. It may also serve as a good survey of the field.

A common conclusion that could be drawn from all this research is that it is possible to effectively axiomatise process algebras with time, but that there are usually several hard matters to be dealt with. We mention the most important issues:

1. A framework is needed for reasoning with time parameters, including a calculus with conditionals, (in)equalities and variable binding constructs;

2. Axiomatisation of the *left merge* operator $\mathbin{\|\!\!\lfloor}$ turned out to be a complicated matter. All axiomatisations use additional machinery in the form of predicates and operators in order to define $\mathbin{\|\!\!\lfloor}$ properly;

3. Verifications turn out to get complex, even for simple systems. This may partly have been due to a lack of systematic study and experience, but also to the relative complexity of the axiom systems;

4. Definition and use of abstraction and bisimulation equivalences appear to be complicated, and therefore unpractical when applied in a setting with time;

Timed μCRL [5], or $\mu\mathrm{CRL}_t$, is a new formalism for the algebraic specification and verification of processes with explicit time. Although not all of the above problems have been tackled to satisfaction yet, we have reasons to believe that timed μCRL has certain definite advantages over the existing formalisms, and that various important results from our predecessors will carry over to timed μCRL.

In the first place, abstract μCRL provides a variable binding construct, conditionals, and all facilities for reasoning with processes parameterised with data terms [6]. Therefore, not much additional theory was needed and time could be incorporated in μCRL as an abstract data type.

In the second place, some concession was made in the axiomatisation of the left merge operator. In particular, two actions are allowed to happen at the same time and yet after each other: $a^{\backprime}2\, b^{\backprime}2$ ($a^{\backprime}2$ stands for action a at time 2) is a terminating process in $\mu\mathrm{CRL}_t$, whereas in other formalisms it leads to an a at time 2 followed by a deadlock at time 2.

In the third place, many verifications have been made in abstract μCRL, so that much experience and techniques are already available. Much of this may easily be generalised to the timed variant. We are confident this will be the case, because $\mu\mathrm{CRL}_t$ was designed in such a way that when the time parameters are removed from a specification, a correct abstract μCRL specification results. Actually, the main results in this paper also point in this direction.

For specification and verification practice, many basic results are needed in order to get smoothly through the usually complex calculations. This paper contains various such results. Besides a large number of elementary lemmas, we found that all *process-closed* $\mu\mathrm{CRL}_t$ processes (i.e. terms without process variables) can be proven equal to terms in some basic form (Theorem 2.6), and we derived an expansion theorem (Theorem 3.11) for calculating with terms that contain parallel operators. We moreover have a theorem for the elimination of parallel operators from $\mu\mathrm{CRL}_t$-terms (Theorem 4.2), and associativity and commutativity of $\|$ and $|$ (Section 4).

A consequence of the design of the $\mathbin{\|\!\!\lfloor}$-operator is the phenomenon of $\mathbin{\|\!\!\lfloor}$*-leaking*: A process $p \mathbin{\|\!\!\lfloor} \delta^{\backprime}\mathbf{0}$ is not simply equal to $\delta^{\backprime}\mathbf{0}$ but it can perform that part of p that is enabled at time $\mathbf{0}$. It turns out that in all proofs involving $\mathbin{\|\!\!\lfloor}$, and thus $\|$, this $\mathbin{\|\!\!\lfloor}$-leaking has to be studied as a separate case.

At the moment, a different paper with three case studies in $\mu\mathrm{CRL}_t$ is in progress [8]. It appears that reasoning with our basic forms is quite natural, and moreover, that not much additional calculations are needed once the Expansion Theorem can be applied.

Also the *empty process* ϵ with the property $\epsilon\, x = x\, \epsilon = x$ is studied. Notations using ϵ could possibly simplify our basic forms, and therefore the various proofs, considerably. However, addition of ϵ to the language will probably not be a trivial exercise.

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Contents

2 Timed μCRL

The axiom system $p\text{CRL}_t$ for *pico* CRL with time is presented. It serves as the basic framework for our studies. The following step is to incorporate operators for parallelism and introduce μCRL_t. We work in a setting without the silent step τ, and without abstraction or general operators for renaming. We also define a notion of *basic forms* and prove that all terms over the signature $\Sigma(p\text{CRL}_t)$ without process variables are provably equal to basic forms.

2.1 Axioms for pCRL with time

Atomic actions are the building blocks of processes. Therefore, axiom systems in process algebra have a set of atomic actions A as a parameter. The actions are parameterised with data, and w.l.o.g. we may assume that all actions have exactly one such parameter. For process variables we use $x, y, z, \ldots$, and for process terms we use $p, q, r, \ldots$. Choice or alternative composition is modelled by $+$, and sequential composition by $\cdot$, which is often omitted from expressions. We write $\cdot$ only in the tables of axioms. Deadlock is modelled by δ. We use $a, b, c, \ldots$ to denote elements from either A or $A \cup \{\delta\}$ (A_δ).

Table 1 lists the 'core' axioms of abstract pCRL, with A6 replaced by AT6. Axioms A1–A5 and A7 are well known from process algebra, axiom AT6 expresses that a deadlock at time $\mathbf{0}$ may always be eliminated from an alternative composition. The $\sum$-operator will be explained below.

Data types in μCRL are algebraically specified in the standard way using sorts, functions and axioms (see e.g. [11]). For data sorts we use $D, E, \ldots$, and for data variables of the respective sorts we use $d, e, \ldots$. Two special sorts are assumed in μCRL_t: **Bool** and **Time**.

Sort **Bool** contains the constants t ("true") and f ("false"). Typical boolean variables are $b, c, \alpha, \beta, \ldots$, and the use of booleans in process expressions may become clear from the axioms C1 and C2 for the conditional construct $\_ \lhd \_ \rhd \_$. For sort **Bool** we assume connectives $\neg, \wedge, \vee, \rightarrow$ with straightforward interpretations, and for the construction of proofs we (implicitly) use the proof theory for μCRL [6], which also provides a rule for structural induction on data terms. For booleans, this implies that we may use the principle of *case distinction* in proofs, i.e., if a formula ϕ holds for both $b = \mathsf{t}$ and $b = \mathsf{f}$

A1	$x+y=y+x$	SUM1	$\sum_{d:D} x = x$
A2	$x+(y+z)=(x+y)+z$	SUM3	$\sum X = \sum X + Xd$
A3	$x+x=x$	SUM4	$\sum_{d:D}(Xd+Yd) = \sum X + \sum Y$
A4	$(x+y)\cdot z = x\cdot z + y\cdot z$	SUM5	$(\sum X)\cdot x = \sum_{d:D}(Xd\cdot x)$
A5	$(x\cdot y)\cdot z = x\cdot(y\cdot z)$	SUM11	$(\forall d{\in}D\ Xd = Yd) \to \sum X = \sum Y$
AT6	$x + \delta{}^{\backprime}\mathbf{0} = x$		
A7	$\delta\cdot x = \delta$	C1	$x \lhd \mathsf{t} \rhd y = x$
		C2	$x \lhd \mathsf{f} \rhd y = y$

Table 1: Core axioms of $p\mathrm{CRL}_t$

then ϕ holds in general. As a consequence, we have to require that for the data specifications only *minimal* models are considered.

Sort **Time** contains a constant $\mathbf{0}$ ("zero"), which serves as a minimal element for the total ordering $\leq$. Axioms for $\leq$, *eq* (equality), *min* (minimum), and *if* (if-then-else) are listed in Table 2. A function $<$ is used to abbreviate terms $t \leq u \wedge \neg eq(t,u)$ to $t < u$, and $u \leq t \leq v$ abbreviates $u \leq t \wedge t \leq u$. Typical elements of sort **Time** are $t, u, v, \ldots$, and unless stated explicitly, such as in axioms with $\sum_{t:\mathbf{Time}}$, **Time** is treated as a normal μCRL data type.

An expression of the form $p[d_0/d]$ denotes process p with data term d_0 substituted for variable d. We will always try to be clear when we want to distinguish between data variables and data terms. *Process-closed terms* are terms without process variables, but possibly with bound and free data variables.

The *at* operator adds time parameters to processes: $p{}^{\backprime}t$ should be interpreted as p at time t. Table 2 contains the axioms for the *at* operator. In $p\mathrm{CRL}_t$, we have by axiom ATA1 that $\delta = \sum_{t:\mathbf{Time}} \delta{}^{\backprime}t$, so δ models the process that will never do a step, terminate or block. Processes $\delta{}^{\backprime}t$ do model deadlocks at time t. Therefore we call them *time deadlocks*.

We see that if a deadlock $\delta{}^{\backprime}t$ occurs in a process term with δ as an alternative, it vanishes:

$$\delta + \delta{}^{\backprime}u \stackrel{\mathrm{ATA1}}{=} \sum_{t:\mathbf{Time}} \delta{}^{\backprime}t + \delta{}^{\backprime}u \stackrel{\mathrm{SUM3}}{=} \sum_{t:\mathbf{Time}} \delta{}^{\backprime}t \stackrel{\mathrm{ATA1}}{=} \delta.$$

In general, for $n > 0$ finite sums $p_1 + \ldots + p_n$ are abbreviated by $\sum_{i\in I} p_i$, where $I = \{1,\ldots,n\}$. In μCRL, a summation construct of the form $\sum_{d:D} p$ is a binder of variable d of data sort D in p. D may be infinite. We use the convention that $\sum_{i\in\emptyset} p \equiv \delta{}^{\backprime}\mathbf{0}$.

In axioms SUMx distinction is made between *sum operators* $\sum$ and *sum constructs* $\sum_{d:D} p$. The X in $\sum X$ may be instantiated with functions from some data sort to the sort of processes, such as $\lambda d{:}D.p$, where variable d in p may not become bound by $\sum$. We also have expressions $\sum_{d:D} x$, where some term p that is substituted for x may not contain free variable d. Data terms are considered modulo *α-conversion*, e.g., the terms $\sum_{d:D} p(d)$ and $\sum_{e:E} p(e)$ are equal.

$FV(p)$ and $FV(b)$ are used to denote the sets of free variables in process p and boolean b, respectively. We do not formalise these concepts further.

In our calculations we work modulo associativity and commutativity of $+$, and we do not mention the use of simple properties of $\neg \vee \wedge, \rightarrow, \leq, min, eq$ and *if*. So axioms Time$x$ are used implicitly, and so are C1 and C2.

Throughout this paper the following proof principle will be of great use. The notation $x \subseteq y$ stands for $x + y = y$ ($y \supseteq x$ stands for the same).

Lemma 2.1 (*Summand Inclusion*). *If $x \subseteq y$ and $y \subseteq x$ then $x = y$.*

Proof. By definition of $\subseteq$ $x + y = y$ and $y + x = x$. So $x = y + x \stackrel{\mathrm{A1}}{=} x + y = y$. □

Time1	$(t_1 \leq t_2 \wedge t_2 \leq t_3 = \mathsf{t}) \rightarrow t_1 \leq t_3 = \mathsf{t}$
Time2	$\mathbf{0} \leq t = \mathsf{t}$
Time3	$t_1 \leq t_2 \vee t_2 \leq t_1 = \mathsf{t}$
Time4	$(t_1 \leq t_2 \wedge t_2 \leq t_1 = \mathsf{t}) \rightarrow t_1 = t_2$
Time5	$eq(t_1, t_2) = t_1 \leq t_2 \wedge t_2 \leq t_1$
Time6	$min(t_1, t_2) = if(t_1 \leq t_2, t_1, t_2)$
Time7	$if(\mathsf{t}, t_1, t_2) = t_1$
Time8	$if(\mathsf{f}, t_1, t_2) = t_2$
ATA1	$x = \sum_{t:\mathbf{Time}} x^{\varsigma} t$
ATA2	$\delta^{\varsigma} t \triangleleft t \leq u \triangleright \delta^{\varsigma}\mathbf{0} + a^{\varsigma} u = a^{\varsigma} u$
ATA3	$a^{\varsigma} t \cdot x = a^{\varsigma} t \cdot (\sum_{u:\mathbf{Time}} x^{\varsigma} u \triangleleft t \leq u \triangleright \delta^{\varsigma}\mathbf{0} + \delta^{\varsigma} t)$
ATB1	$\delta^{\varsigma} t^{\varsigma} u = \delta^{\varsigma} min(t, u)$
ATB2	$a^{\varsigma} t^{\varsigma} u = a^{\varsigma} t \triangleleft eq(t, u) \triangleright \delta^{\varsigma} min(t, u)$
ATB3	$(x + y)^{\varsigma} t = x^{\varsigma} t + y^{\varsigma} t$
ATB4	$(x \cdot y)^{\varsigma} t = x^{\varsigma} t \cdot y$
ATB5	$(\sum_{d:D} Xd)^{\varsigma} t = \sum_{d:D} Xd^{\varsigma} t$

Table 2: Time related axioms of pCRL$_t$, where $a \in A_\delta$

Finally, axiom ATA3 implies that successive actions have non-decreasing time parameters. A remarkable observation is that, in this paper, we never have to apply it.

2.2 Addition of time and operators for parallelism

The axioms of μCRL$_t$ are the axioms of pCRL$_t$, combined with the axioms in the tables 3 and 4. The signature $\Sigma(\mu\mathrm{CRL}_t)$ is as $\Sigma(p\mathrm{CRL}_t)$, extended with the operators for parallelism and the $\ll$ operator.

For communication we have a binary function γ, which is only defined on *action labels*. In order for a communication to occur between actions $c, c' \in A$, $\gamma(c, c')$ should be defined, and the data parameters of the actions should match according to axiom CF. By definition, the function γ is commutative and associative.

Concurrency is basically described by three operators: the *merge* $\parallel$, the *left merge* $\lfloor\!\lfloor$ and the *communication merge* $|$. The process $p \parallel q$ symbolises the parallel execution of p and q. It 'starts' with an action of either p or q, or with a communication, or *synchronisation*, between p and q. $p \lfloor\!\lfloor q$ is as $p \parallel q$, but the first action that is performed comes from p.

For the axiomatisation of the left merge $\lfloor\!\lfloor$ the auxiliary *before* operator is defined; $p \ll q$ should be interpreted as the process that behaves like p, provided that p can do a step before or at the moment t_0 after which q gets definitively disabled. Otherwise $p \ll q$ becomes a time deadlock at time t_0. A small example may facilitate the understanding of the $\ll$ operator. We will use various identities from Appendix A, which is completely devoted to auxiliary lemmas.

Example 2.2. Let $a, b, c \in A$ and t_1, t_2, t_3 be closed terms of sort **Time**. We analyse the following term:

$$\begin{array}{rcl} a^{\varsigma} t_1 \ll (b^{\varsigma} t_2 + c^{\varsigma} t_3) & \stackrel{\ll 2}{=} & a^{\varsigma} t_1 \ll b^{\varsigma} t_2 + a^{\varsigma} t_1 \ll c^{\varsigma} t_3 \\ & \stackrel{\ll 5, \ll 1}{=} & \sum_{u:\mathbf{Time}} a^{\varsigma} t_1{}^{\varsigma} u \triangleleft u \leq t_2 \triangleright \delta^{\varsigma}\mathbf{0} + \sum_{u:\mathbf{Time}} a^{\varsigma} t_1{}^{\varsigma} u \triangleleft u \leq t_3 \triangleright \delta^{\varsigma}\mathbf{0} \end{array}$$

$$\begin{array}{cl}
\overset{\mathrm{SUM4,A.1.7}}{=} & \sum_{u:\mathbf{Time}} a\text{`}t_1\text{`}u \lhd u \leq t_2 \vee u \leq t_3 \rhd \delta\text{`}\mathbf{0} \\
\overset{\mathrm{A.2.2}}{=} & \sum_{u:\mathbf{Time}} a\text{`}u\text{`}t_1 \lhd u \leq max(t_2,t_3) \rhd \delta\text{`}\mathbf{0} \\
\overset{\mathrm{A.5.4}}{=} & a\text{`}t_1 \lhd t_1 \leq max(t_2,t_3) \rhd \delta\text{`}\mathbf{0} + \delta\text{`}t_1\text{`}max(t_2,t_3).
\end{array}$$

If $t_1 \leq max(t_2,t_3)$ then using axiom ATB1 we find $a\text{`}t_1 + \delta\text{`}t_1 \overset{\mathrm{ATA2}}{=} a\text{`}t_1$, otherwise the above process equals $\delta\text{`}max(t_2,t_3)$. □

Process $p \mid q$ is as $p \parallel q$, but the first action is a communication between p and q. Encapsulation operators ∂_H block atomic actions in H by renaming them to δ. They are used to enforce communication between processes.

The various operators of $\Sigma(\mu\mathrm{CRL}_t)$ are listed in order of decreasing binding strength:

$$\text{`} \quad \cdot \quad \ll \quad \{\lhd\,\rhd, \parallel, \mathbin{\lfloor\!\lfloor}, \mid\} \quad \textstyle\sum_{d:D} \quad +.$$

Brackets are omitted from expressions according to this convention.

ATB6	$(x \mathbin{\lfloor\!\lfloor} y)\text{`}t = x\text{`}t \mathbin{\lfloor\!\lfloor} y$
ATB7	$(x \mid y)\text{`}t = x\text{`}t \mid y$
ATB8	$(x \mid y)\text{`}t = x \mid y\text{`}t$
ATB9	$\partial_H(x\text{`}t) = \partial_H(x)\text{`}t$
$\ll$1	$x \ll a = x$
$\ll$2	$x \ll (y+z) = x \ll y + x \ll z$
$\ll$3	$x \ll y \cdot z = x \ll y$
$\ll$4	$x \ll \sum X = \sum_{d:D} x \ll Xd$
$\ll$5	$x \ll y\text{`}t = \sum_{u:\mathbf{Time}} (x \ll y)\text{`}u \lhd u \leq t \rhd \delta\text{`}\mathbf{0}$

Table 3: Time related axioms of $\mu\mathrm{CRL}_t$, where $a \in A_\delta$

2.3 Basic forms

We work here in a setting without recursion, in order to establish some important, basic results.

Definition 2.3. A *basic form* over $\Sigma(p\mathrm{CRL}_t)$ is a process-closed term of the form

$$\begin{array}{rcl}
r & = & \sum_{i\in I} \sum_{d_1^i:D_1^i} \cdots \sum_{d_{m_i}^i:D_{m_i}^i} \sum_{u:\mathbf{Time}} a_i\text{`}u\, r_i \lhd \alpha_i \rhd \delta\text{`}\mathbf{0} + \\
& & \sum_{j\in J} \sum_{e_1^j:E_1^j} \cdots \sum_{e_{n_j}^j:E_{n_j}^j} \sum_{v:\mathbf{Time}} b_j\text{`}v \lhd \beta_j \rhd \delta\text{`}\mathbf{0}
\end{array}$$

where the $a_i \in A$ and $b_j \in A_\delta$, and the r_i are also basic forms. □

In the sequel, we will often write $\overline{\sum}_{d_1,\ldots,d_m} x$ for $\sum_{d_1:D_1} \ldots \sum_{d_m:D_m} x$, and $\overline{d}_m$ for $d_1,\ldots,d_m$. By convention $\overline{\sum}_{\overline{d}_0} x = x$. We take care that no confusion can arise w.r.t. the sorts of the d_k. For example, if we treat $\sum_{i\in I}$ and $\sum_{j\in J}$ as formal summations we may abbreviate r in the above definition to

$$\overline{\sum}_{i,\overline{d^i}_{m_i},u} a_i\text{`}u\, r_i \lhd \alpha_i \rhd \delta\text{`}\mathbf{0} + \overline{\sum}_{j,\overline{e^j}_{n_j},v} b_j\text{`}v \lhd \beta_j \rhd \delta\text{`}\mathbf{0}.$$

The above form may already be compact, but for the studies to come we need a more general format for representing basic forms.

SUM6	$(\sum X) \mathbin{\|\!\lfloor} x = \sum_{d:D}(Xd \mathbin{\|\!\lfloor} x)$		
SUM7	$(\sum X) \mid x = \sum_{d:D}(Xd \mid x)$	CF	$c(d) \mid c'(e) = \begin{cases} \gamma(c,c')(d) \lhd eq(d,e) \rhd \delta & \text{if sorts of } d \text{ and } e \text{ are equal, and } \gamma(c,c') \text{ defined} \\ \delta & \text{otherwise} \end{cases}$
SUM7′	$x \mid (\sum X) = \sum_{d:D}(x \mid Xd)$		
SUM8	$\partial_H(\sum X) = \sum_{d:D} \partial_H(Xd)$		
CM1	$x \parallel y = x \mathbin{\|\!\lfloor} y + y \mathbin{\|\!\lfloor} x + x \mid y$	CD1	$\delta \mid a = \delta$
CM2	$a \mathbin{\|\!\lfloor} x = (a \ll x) \cdot x$	CD2	$a \mid \delta = \delta$
CM3	$a \cdot x \mathbin{\|\!\lfloor} y = (a \ll y) \cdot (x \parallel y)$		
CM4	$(x + y) \mathbin{\|\!\lfloor} z = x \mathbin{\|\!\lfloor} z + y \mathbin{\|\!\lfloor} z$	DD	$\partial_H(\delta) = \delta$
CM5	$a \cdot x \mid b = (a \mid b) \cdot x$		
CM6	$a \mid b \cdot x = (a \mid b) \cdot x$	D1	$\partial_H(c(d)) = c(d) \quad \text{if } c \notin H$
CM7	$a \cdot x \mid b \cdot y = (a \mid b) \cdot (x \parallel y)$	D2	$\partial_H(c(d)) = \delta \quad \text{if } c \in H$
CM8	$(x + y) \mid z = x \mid z + y \mid z$	D3	$\partial_H(x + y) = \partial_H(x) + \partial_H(y)$
CM9	$x \mid (y + z) = x \mid y + x \mid z$	D4	$\partial_H(x \cdot y) = \partial_H(x) \cdot \partial_H(y)$

Table 4: Axioms for parallelism of μCRL$_t$, where $a, b \in A_\delta$ and $c, c' \in A$

Lemma 2.4 (*Representation*)**.** *Basic form r given in Definition 2.3 can be represented by*

$$\overline{\sum}_{i,\overline{d}_m,u} a_i {}^{\backprime} u\, r_i \lhd \alpha_i \rhd \delta {}^{\backprime}\mathbf{0} + \overline{\sum}_{j,\overline{e}_n,v} b_j {}^{\backprime} v \lhd \beta_j \rhd \delta {}^{\backprime}\mathbf{0},$$

where the sequence $d_1, \ldots, d_m$ contains all data variables from $\bigcup_{i \in I}\{d^i_1, \ldots, d^i_{m_i}\}$, and $e_1, \ldots, e_n$ contains all data variables from $\bigcup_{j \in J}\{e^j_1, \ldots, e^j_{n_j}\}$.

Proof. Consider summand $\overline{\sum}_{\overline{d^i}_{m_i},u} a_i {}^{\backprime} u\, r_i \lhd \alpha_i \rhd \delta {}^{\backprime}\mathbf{0}$ of r, for some $i \in I$.

We may assume that no variable d_k or e_l, where $k = 1, \ldots, m$ and $l = 1, \ldots, n$, occurs free in r, so by axiom SUM1 all summations $\sum_{d_k}$, with d_k not already in the sequence $d^i_1, \ldots, d^i_{m_i}$, may be added in front of this expression. By Lemma A.4 the order of the variables may be changed to $d_1, \ldots, d_m$. As this holds for all summands of r this lemma is proved. □

Lemma 2.5. *If p and q are process-closed terms over $\Sigma(pCRL_t)$ and q is a basic form then there is another basic form r such that $pCRL_t \vdash p\, q = r$.*

Proof. By induction on the structure of p.

1. $p \equiv a$, where $a \in A$:

$$\begin{array}{lcl} a\,q & = & a\,q \lhd \mathsf{t} \rhd \delta {}^{\backprime}\mathbf{0} \\ & \overset{\text{ATA1}}{=} & \sum_{u:\mathbf{Time}}(a\,q \lhd \mathsf{t} \rhd \delta {}^{\backprime}\mathbf{0}) {}^{\backprime} u \\ & \overset{\text{A.1.2,A.2.1}}{=} & \sum_{u:\mathbf{Time}}(a\,q) {}^{\backprime} u \lhd \mathsf{t} \rhd \delta {}^{\backprime}\mathbf{0} \\ & \overset{\text{ATB4}}{=} & \sum_{u:\mathbf{Time}} a {}^{\backprime} u\, q \lhd \mathsf{t} \rhd \delta {}^{\backprime}\mathbf{0}; \end{array}$$

2. $p \equiv \delta$: $\delta\, q \overset{\text{A7}}{=} \delta$. Now proceed as in case 1;

3. $p \equiv p' + p''$: $(p' + p'')\, q \overset{\text{A4}}{=} p'\, q + p''\, q$. By induction, both $p'\, q$ and $p''\, q$ are provably equal to basic forms. The sum of these basic forms is again a basic form;

4. $p \equiv p' \, p''$: $(p' \, p'') \, q \stackrel{\text{A5}}{=} p' \, (p'' \, q)$. By induction, $p'' \, q$ is provably equal to some basic form r. Again by induction, $p' \, r$ equals some other basic form;

5. $p \equiv \sum_{d:D} p'$: $(\sum_{d:D} p') \, q \stackrel{\text{SUM5}}{=} \sum_{d:D} p' \, q$. By induction there is a basic form r such that $p' \, q = r$. Now by axiom SUM4 and Lemma A.4 $\sum_{d:D}$ may be put in the right place in r;

6. $p \equiv p' \triangleleft b \triangleright p''$: $(p' \triangleleft b \triangleright p'') \, q \stackrel{\text{A.1.3,A.1.4}}{=} p' \, q \triangleleft b \triangleright \delta^{\text{‘}}\mathbf{0} + p'' \, q \triangleleft \neg b \triangleright \delta^{\text{‘}}\mathbf{0}$. By induction $p' \, q$ and $p'' \, q$ are provably equal to some basic forms r and r', respectively. It suffices to consider both subterms separately. We only give a proof for the first subterm. The second follows in a similar way.

$$
\begin{aligned}
r \triangleleft b \triangleright \delta^{\text{‘}}\mathbf{0} &\stackrel{2.4,\text{A.1.5}}{=} (\overline{\textstyle\sum}_{i,\overline{d}_m,u} a_i{}^{\text{‘}}u \, r_i \triangleleft \alpha_i \triangleright \delta^{\text{‘}}\mathbf{0}) \triangleleft b \triangleright \delta^{\text{‘}}\mathbf{0} + \\
&\qquad (\overline{\textstyle\sum}_{j,\overline{e}_n,v} b_j{}^{\text{‘}}v \triangleleft \beta_j \triangleright \delta^{\text{‘}}\mathbf{0}) \triangleleft b \triangleright \delta^{\text{‘}}\mathbf{0} \\
&\stackrel{\text{A.3.1,A.1.6}}{=} \overline{\textstyle\sum}_{i,\overline{d}_m,u} a_i{}^{\text{‘}}u \, r_i \triangleleft \alpha_i \wedge b \triangleright \delta^{\text{‘}}\mathbf{0} + \overline{\textstyle\sum}_{j,\overline{e}_n,v} b_j{}^{\text{‘}}v \triangleleft \beta_j \wedge b \triangleright \delta^{\text{‘}}\mathbf{0};
\end{aligned}
$$

7. $p \equiv p'{}^{\text{‘}}t$: $p'{}^{\text{‘}}t \, q \stackrel{\text{ATB4}}{=} (p' \, q)^{\text{‘}}t$. By induction there is a basic form r such that $p' \, q = r$. The proof proceeds as follows:

$$
\begin{aligned}
& r^{\text{‘}}t \\
&\stackrel{2.4,\text{ATB3,ATB5}}{=} \overline{\textstyle\sum}_{i,\overline{d}_m,u} (a_i{}^{\text{‘}}u \, r_i \triangleleft \alpha_i \triangleright \delta^{\text{‘}}\mathbf{0})^{\text{‘}}t + \overline{\textstyle\sum}_{j,\overline{e}_n,v} (b_j{}^{\text{‘}}v \triangleleft \beta_j \triangleright \delta^{\text{‘}}\mathbf{0})^{\text{‘}}t \\
&\stackrel{\text{A.1.2,A.2.1,ATB4}}{=} \overline{\textstyle\sum}_{i,\overline{d}_m,u} a_i{}^{\text{‘}}u^{\text{‘}}t \, r_i \triangleleft \alpha_i \triangleright \delta^{\text{‘}}\mathbf{0} + \overline{\textstyle\sum}_{j,\overline{e}_n,v} b_j{}^{\text{‘}}v^{\text{‘}}t \triangleleft \beta_j \triangleright \delta^{\text{‘}}\mathbf{0} \\
&\stackrel{\text{ATB2,A.1.4}}{=} \overline{\textstyle\sum}_{i,\overline{d}_m,u} \\
&\qquad (a_i{}^{\text{‘}}u \triangleleft eq(u,t) \triangleright \delta^{\text{‘}}\mathbf{0} + \delta^{\text{‘}}min(u,t) \triangleleft \neg eq(u,t) \triangleright \delta^{\text{‘}}\mathbf{0}) \, r_i \triangleleft \alpha_i \triangleright \delta^{\text{‘}}\mathbf{0} + \\
&\qquad \overline{\textstyle\sum}_{j,\overline{e}_n,v} (b_j{}^{\text{‘}}v \triangleleft eq(v,t) \triangleright \delta^{\text{‘}}\mathbf{0} + \delta^{\text{‘}}min(v,t) \triangleleft \neg eq(v,t) \triangleright \delta^{\text{‘}}\mathbf{0}) \triangleleft \beta_j \triangleright \delta^{\text{‘}}\mathbf{0} \\
&\stackrel{\text{A4,A.1.3,A.2.3}}{=} \overline{\textstyle\sum}_{i,\overline{d}_m,u} \\
&\qquad (a_i{}^{\text{‘}}u \, r_i \triangleleft eq(u,t) \triangleright \delta^{\text{‘}}\mathbf{0} + \delta^{\text{‘}}min(u,t) \triangleleft \neg eq(u,t) \triangleright \delta^{\text{‘}}\mathbf{0}) \triangleleft \alpha_i \triangleright \delta^{\text{‘}}\mathbf{0} + \\
&\qquad \overline{\textstyle\sum}_{j,\overline{e}_n,v} (b_j{}^{\text{‘}}v \triangleleft eq(v,t) \triangleright \delta^{\text{‘}}\mathbf{0} + \delta^{\text{‘}}min(v,t) \triangleleft \neg eq(v,t) \triangleright \delta^{\text{‘}}\mathbf{0}) \triangleleft \beta_j \triangleright \delta^{\text{‘}}\mathbf{0} \\
&\stackrel{\text{A.1.5,A.1.6,ATB1}}{=} \overline{\textstyle\sum}_{i,\overline{d}_m,u} \\
&\qquad (a_i{}^{\text{‘}}u \, r_i \triangleleft eq(u,t) \wedge \alpha_i \triangleright \delta^{\text{‘}}\mathbf{0} + \delta^{\text{‘}}t^{\text{‘}}u \triangleleft \neg eq(u,t) \wedge \alpha_i \triangleright \delta^{\text{‘}}\mathbf{0}) + \\
&\qquad \overline{\textstyle\sum}_{j,\overline{e}_n,v} (b_j{}^{\text{‘}}v \triangleleft eq(v,t) \wedge \beta_j \triangleright \delta^{\text{‘}}\mathbf{0} + \delta^{\text{‘}}t^{\text{‘}}v \triangleleft \neg eq(v,t) \wedge \beta_j \triangleright \delta^{\text{‘}}\mathbf{0}) \\
&\stackrel{\text{A.1.12,SUM4}}{=} \overline{\textstyle\sum}_{i,\overline{d}_m,u} a_i{}^{\text{‘}}u \, r_i \triangleleft eq(u,t) \wedge \alpha_i \triangleright \delta^{\text{‘}}\mathbf{0} + \\
&\qquad \overline{\textstyle\sum}_{i,\overline{d}_m,u} \delta^{\text{‘}}t \triangleleft t \leq u \wedge \neg eq(u,t) \wedge \alpha_i \triangleright \delta^{\text{‘}}\mathbf{0} + \\
&\qquad \overline{\textstyle\sum}_{i,\overline{d}_m,u} \delta^{\text{‘}}u \triangleleft u \leq t \wedge \neg eq(u,t) \wedge \alpha_i \triangleright \delta^{\text{‘}}\mathbf{0} + \\
&\qquad \overline{\textstyle\sum}_{j,\overline{e}_n,v} b_j{}^{\text{‘}}v \triangleleft eq(v,t) \wedge \beta_j \triangleright \delta^{\text{‘}}\mathbf{0} + \\
&\qquad \overline{\textstyle\sum}_{j,\overline{e}_n,v} \delta^{\text{‘}}t \triangleleft t \leq v \wedge \neg eq(v,t) \wedge \beta_j \triangleright \delta^{\text{‘}}\mathbf{0} + \\
&\qquad \overline{\textstyle\sum}_{j,\overline{e}_n,v} \delta^{\text{‘}}v \triangleleft v \leq u \wedge \neg eq(v,t) \wedge \beta_j \triangleright \delta^{\text{‘}}\mathbf{0}.
\end{aligned}
$$

The second and fifth summand still have to be put in the right form. We show how this is done for the second. By Lemma A.5.1 it is equal to

$$
\begin{aligned}
& \overline{\textstyle\sum}_{i,\overline{d}_m,u} (\textstyle\sum_v (\delta^{\text{‘}}v \triangleleft v \leq u \wedge \neg eq(u,v) \wedge \alpha_i \triangleright \delta^{\text{‘}}\mathbf{0}) \triangleleft eq(v,t) \triangleright \delta^{\text{‘}}\mathbf{0}) \\
& \stackrel{\text{A.1.6}}{=} \overline{\textstyle\sum}_{i,\overline{d}_m,u,v} \delta^{\text{‘}}v \triangleleft v \leq u \wedge \neg eq(u,v) \wedge \alpha_i \wedge eq(v,t) \triangleright \delta^{\text{‘}}\mathbf{0},
\end{aligned}
$$

which is a basic form.

$\square$

Theorem 2.6 (*Basic Forms*)**.** *If q is a process-closed term over $\Sigma(pCRL_t)$ then there is a basic form p such that $\mu CRL_t \vdash p = q$.*

Proof. Easy; by induction on the structure of q. The case where $q \equiv r\, s$ uses Lemma 2.5. □

We introduce a criterion that will be used in inductive proofs.

Definition 2.7. Let r be a process-closed term over $\Sigma(p\mathrm{CRL}_t)$ as in Definition 2.3. The depth $|r|$ of r is defined as 1 if $I = \emptyset$, and $1 + max_{i \in I}|r_i|$ otherwise. □

3 A theorem for expanding the merge operator

In this section we derive various results for the application of parallel operators to $\mu\mathrm{CRL}_t$ terms. All results can be used for calculating with basic forms, but also for recursively specified processes, provided that they are in the prescribed format.

3.1 Preparatory steps

We consider arbitrary $\mu\mathrm{CRL}_t$ processes in the following syntactical format:

$$P \stackrel{\mathrm{def}}{=} \overline{\sum}_{i,\overline{d}_m,u} a_i{}^{\varsigma}u\, P_i \lhd \alpha_i \rhd \delta^{\varsigma}\mathbf{0} + \overline{\sum}_{i',\overline{d'}_{m'},u} a_{i'}{}^{\varsigma}u \lhd \alpha_{i'} \rhd \delta^{\varsigma}\mathbf{0} \qquad = P' + P'';$$

$$Q \stackrel{\mathrm{def}}{=} \overline{\sum}_{j,\overline{e}_n,v} b_j{}^{\varsigma}v\, Q_j \lhd \beta_j \rhd \delta^{\varsigma}\mathbf{0} + \overline{\sum}_{j',\overline{e'}_{n'},v} b_{j'}{}^{\varsigma}v \lhd \beta_{j'} \rhd \delta^{\varsigma}\mathbf{0} \qquad = Q' + Q''.$$

The terms are split in subterms x' and x'' in order to make the coming proofs a little more manageable. Let $i \in I, i' \in I'$ where $I \cap I' = \emptyset$, and $j \in J, j' \in J'$ where $J \cap J' = \emptyset$. The sets I^*, J^* stand for $I \cup I', J \cup J'$, respectively. The vectors $\overline{d^*}_{m^*}$ and $\overline{e^*}_{n^*}$ stand for $\overline{d}_m, \overline{d'}_{m'}$ and $\overline{e}_n, \overline{e'}_{n'}$, respectively.

Lemma 3.1. *It holds that:*

$$P \,\|\, Q = P' \,⌊\!⌊\, Q + P'' \,⌊\!⌊\, Q + Q' \,⌊\!⌊\, P + Q'' \,⌊\!⌊\, P + P' \,|\, Q' + P' \,|\, Q'' + P'' \,|\, Q' + P'' \,|\, Q''.$$

Proof. Straightforward. By the axioms CM1, CM4, CM8 and CM9. □

The term $a {\ll} Q$ denotes the process that may do an a-step as long as Q is enabled. The following lemma expresses this more formally.

Lemma 3.2. *It holds that:*

1. *If $\exists_{v:\mathbf{Time},j^* \in J^*}.\beta_{j^*} = \mathsf{t}$ then*

$$a {\ll} Q = \overline{\sum}_{\overline{e^*}_{n^*},v,t} a^{\varsigma}t \lhd t \leq v \wedge \bigvee_{j^* \in J^*} \beta_{j^*} \rhd \delta^{\varsigma}\mathbf{0};$$

2. *If $\forall_{v:\mathbf{Time},j^* \in J^*}.\beta_{j^*} = \mathsf{f}$ then*

$$a {\ll} Q = a^{\varsigma}\mathbf{0}.$$

Proof. $a {\ll} Q = a {\ll} (Q' + Q'') \stackrel{\ll 2}{=} a {\ll} Q' + a {\ll} Q''$. We show how $a {\ll} Q'$ can be expanded.

$$a {\ll} Q' \;= a {\ll} \overline{\sum}_{j,\overline{e}_n,v} b_j{}^{\varsigma}v\, Q_j \lhd \beta_j \rhd \delta^{\varsigma}\mathbf{0}$$

$$
\begin{array}{cl}
\stackrel{\ll 4, \mathrm{A.1.13}}{=} & \overline{\sum}_{j,\overline{e}_n,v} a \ll b_j {}^{\varsigma} v\, Q_j \lhd \beta_j \rhd a \ll \delta {}^{\varsigma}\mathbf{0} \\
\stackrel{\ll 3, \mathrm{A.2.6}}{=} & \overline{\sum}_{j,\overline{e}_n,v} a \ll b_j {}^{\varsigma} v \lhd \beta_j \rhd a {}^{\varsigma}\mathbf{0} \\
\stackrel{\mathrm{A.1.4}}{=} & \overline{\sum}_{j,\overline{e}_n,v} (a \ll b_j {}^{\varsigma} v \lhd \beta_j \rhd \delta {}^{\varsigma}\mathbf{0} + a {}^{\varsigma}\mathbf{0} \lhd \neg\beta_j \rhd \delta {}^{\varsigma}\mathbf{0}) \\
\stackrel{\mathrm{A.4, A.1.7}}{=} & \overline{\sum}_{\overline{e}_n,v} (a \ll b_j {}^{\varsigma} v \lhd \bigvee_{j\in J} \beta_j \rhd \delta {}^{\varsigma}\mathbf{0} + a {}^{\varsigma}\mathbf{0} \lhd \neg \bigwedge_{j\in J} \beta_j \rhd \delta {}^{\varsigma}\mathbf{0}) \\
\stackrel{\ll 5, \ll 1}{=} & \overline{\sum}_{\overline{e}_n,v} ((\sum_t a {}^{\varsigma} t \lhd t \leq v \rhd \delta {}^{\varsigma}\mathbf{0}) \lhd \bigvee_{j\in J} \beta_j \rhd \delta {}^{\varsigma}\mathbf{0} + a {}^{\varsigma}\mathbf{0} \lhd \neg \bigwedge_{j\in J} \beta_j \rhd \delta {}^{\varsigma}\mathbf{0}) \\
\stackrel{\mathrm{SUM1}}{=} & \overline{\sum}_{\overline{e^*}_{n^*},v} ((\sum_t a {}^{\varsigma} t \lhd t \leq v \rhd \delta {}^{\varsigma}\mathbf{0}) \lhd \bigvee_{j\in J} \beta_j \rhd \delta {}^{\varsigma}\mathbf{0} + a {}^{\varsigma}\mathbf{0} \lhd \neg \bigwedge_{j\in J} \beta_j \rhd \delta {}^{\varsigma}\mathbf{0}).
\end{array}
$$

In a similar way $a \ll Q''$ can be expanded. It follows that

$$
a \ll Q'' = \overline{\sum}_{\overline{e^*}_{n^*},v} ((\sum_t a {}^{\varsigma} t \lhd t \leq v \rhd \delta {}^{\varsigma}\mathbf{0}) \lhd \bigvee_{j'\in J'} \beta_{j'} \rhd \delta {}^{\varsigma}\mathbf{0} + a {}^{\varsigma}\mathbf{0} \lhd \neg \bigwedge_{j'\in J'} \beta_{j'} \rhd \delta {}^{\varsigma}\mathbf{0}),
$$

so that

$$
\begin{array}{l}
a \ll Q' + a \ll Q'' \stackrel{\mathrm{SUM4, A.1.7}}{=} \\
\overline{\sum}_{\overline{e^*}_{n^*},v} ((\sum_t a {}^{\varsigma} t \lhd t \leq v \rhd \delta {}^{\varsigma}\mathbf{0}) \lhd \bigvee_{j^*\in J^*} \beta_{j^*} \rhd \delta {}^{\varsigma}\mathbf{0} + a {}^{\varsigma}\mathbf{0} \lhd \neg \bigwedge_{j^*\in J^*} \beta_{j^*} \rhd \delta {}^{\varsigma}\mathbf{0}).
\end{array}
$$

The following step is to distinguish various cases for the conditions in Q. Let $j_0, j_1 \in J^*$.

1. (a) Assume $\beta_{j_0}[t_0/v] = \mathsf{t}$ and $\beta_{j_1}[t_1/v] = \mathsf{f}$. (Note that these assumptions are allowed by minimality of the model.) We continue our calculations. By SUM4 and Lemma A.5.2 the above term equals

$$
\begin{array}{cl}
 & \overline{\sum}_{\overline{e^*}_{n^*}} (\sum_v (\sum_t a {}^{\varsigma} t \lhd t \leq v \rhd \delta {}^{\varsigma}\mathbf{0}) \lhd \bigvee_{j^*\in J^*} \beta_{j^*} \rhd \delta {}^{\varsigma}\mathbf{0} + a {}^{\varsigma}\mathbf{0}) \\
\stackrel{\mathrm{SUM3}}{=} & \overline{\sum}_{\overline{e^*}_{n^*}} (\sum_v (\sum_t a {}^{\varsigma} t \lhd t \leq v \rhd \delta {}^{\varsigma}\mathbf{0}) \lhd \bigvee_{j^*\in J^*} \beta_{j^*} \rhd \delta {}^{\varsigma}\mathbf{0} + \\
 & \quad \sum_t a {}^{\varsigma} t \lhd t \leq t_0 \rhd \delta {}^{\varsigma}\mathbf{0} + a {}^{\varsigma}\mathbf{0}).
\end{array}
$$

   Again we may use SUM3. At a first application the third inner summand ($a {}^{\varsigma}\mathbf{0}$) is cancelled by the second. At a second application the second inner summand vanishes again. Finally, $\bigvee_{j^*\in J^*} \beta_{j^*}$ is pushed into the innermost summation using Lemma A.3.3;

   (b) Assume $\beta_{j_0}[t_0/v] = \mathsf{t}$ and that there are no $j^* \in J^*$ and v for which $\beta_{j^*} = \mathsf{f}$. Now $a {}^{\varsigma}\mathbf{0} \lhd \neg \bigwedge_{j^*\in J^*} \beta_{j^*} \rhd \delta {}^{\varsigma}\mathbf{0}$ reduces to $\delta {}^{\varsigma}\mathbf{0}$, which is cancelled by axiom AT6. Again, $\bigvee_{j^*\in J^*} \beta_{j^*}$ is pushed into the innermost summation using Lemma A.3.3;

2. Assume there are no $j^* \in J^*, v$ such that $\beta_{j^*} = \mathsf{t}$. Consequently, $\bigvee_{j^*\in J^*} \beta_{j^*} = \mathsf{f}$ and $\neg \bigwedge_{j^*\in J^*} \beta_{j^*} = \mathsf{t}$. It follows easily that $a \ll Q = a {}^{\varsigma}\mathbf{0}$.

□

3.2 Expansion of the left merge

The left merge $P \mathbin{|\!\!\lfloor} Q$ denotes the parallel composition of P and Q, where the first step originates from P. This first step may take place as long as Q is enabled, i.e., Q can do some b_j-step. A time deadlock occurs exactly at moment that either P or Q gets definitively disabled. Later we will show that such time deadlocks can often be cancelled.

We see that process $P \mathbin{|\!\!\lfloor} \delta {}^{\varsigma}\mathbf{0}$ is not simply equal to $\delta {}^{\varsigma}\mathbf{0}$ but that it can perform that part of P that is enabled at time $\mathbf{0}$. This must be regarded a consequence of the design of the $\mathbin{|\!\!\lfloor}$-operator, and we refer to this phenomenon as $\mathbin{|\!\!\lfloor}$*-leaking*. In general, $\delta {}^{\varsigma}\mathbf{0}$'s as arguments of the parallel operators are not desirable, so in practical applications of $\mu\mathrm{CRL}_t$ such cases will probably only model pathological processes. However, in this paper $\mathbin{|\!\!\lfloor}$-leaking imposes extra proof obligations for formulas involving $\mathbin{|\!\!\lfloor}$, and thus $\|$.

Below, we derive these results algebraically.

Lemma 3.3. *It holds that:*

$$\delta\text{‹}\mathbf{0} \mathbin{\lfloor\!\lfloor} Q = \delta\text{‹}\mathbf{0}.$$

Proof. By Lemma 3.2 the term $\delta{\ll}Q$ has to be considered for two cases. We only consider the first case; the second also has a simple proof.

$$\begin{array}{lcl}
\delta\text{‹}\mathbf{0} \mathbin{\lfloor\!\lfloor} Q & \stackrel{\text{ATB6,CM2,ATB4}}{=} & (\delta{\ll}Q)\text{‹}\mathbf{0}\, Q \\
& \stackrel{3.2.1}{=} & (\overline{\sum}_{\overline{e^*}_{n^*},v,t}\delta\text{‹}t \triangleleft t \leq v \wedge \bigvee_{j^* \in J^*} \beta_{j^*} \triangleright \delta\text{‹}\mathbf{0})\text{‹}\mathbf{0}\, Q \\
& \stackrel{\text{ATB5,A.1.2,A.2.2,A.2.1}}{=} & (\overline{\sum}_{\overline{e^*}_{n^*},v,t}\delta\text{‹}\mathbf{0} \triangleleft t \leq v \wedge \bigvee_{j^* \in J^*} \beta_{j^*} \triangleright \delta\text{‹}\mathbf{0})\, Q \\
& \stackrel{\text{A.1.1,SUM1,A.2.3}}{=} & \delta\text{‹}\mathbf{0}.
\end{array}$$

□

Lemma 3.4 (*Left Merge*)**.** *It holds that:*

1. *If* $\exists_{v:\mathbf{Time},j^* \in J^*}.\beta_{j^*} = \mathsf{t}$ *then*

$$\begin{array}{rl}
P \mathbin{\lfloor\!\lfloor} Q = & \overline{\sum}_{i,\overline{d}_m,\overline{e^*}_{n^*},v,u} a_i\text{‹}u\,(P_i \parallel Q) \triangleleft u \leq v \wedge \alpha_i \wedge \bigvee_{j^* \in J^*} \beta_{j^*} \triangleright \delta\text{‹}\mathbf{0} + \\
& \overline{\sum}_{i',\overline{d'}_{m'},\overline{e^*}_{n^*},v,u} a_{i'}\text{‹}u\, Q \triangleleft u \leq v \wedge \alpha_{i'} \wedge \bigvee_{j^* \in J^*} \beta_{j^*} \triangleright \delta\text{‹}\mathbf{0} + \\
& \overline{\sum}_{i^*,j^*,\overline{d^*}_{m^*},\overline{e^*}_{n^*},u,v} \delta\text{‹}u\text{‹}v \triangleleft \alpha_{i^*} \wedge \beta_{j^*} \triangleright \delta\text{‹}\mathbf{0};
\end{array}$$

2. *If* $\forall_{v:\mathbf{Time},j^* \in J^*}.\beta_{j^*} = \mathsf{f}$ *then* $Q = \delta\text{‹}\mathbf{0}$ *and*

$$P \mathbin{\lfloor\!\lfloor} Q = \overline{\sum}_{i,\overline{d}_m} a_i\text{‹}\mathbf{0}\,(P_i[\mathbf{0}/u] \parallel \delta\text{‹}\mathbf{0}) \triangleleft \alpha_i[\mathbf{0}/u] \triangleright \delta\text{‹}\mathbf{0} + \sum_{i',\overline{d'}_{m'}} a_{i'}\text{‹}\mathbf{0}\,\delta\text{‹}\mathbf{0} \triangleleft \alpha_{i'}[\mathbf{0}/u] \triangleright \delta\text{‹}\mathbf{0}.$$

Proof. $P \mathbin{\lfloor\!\lfloor} Q = (P' + P'') \mathbin{\lfloor\!\lfloor} Q = P' \mathbin{\lfloor\!\lfloor} Q + P'' \mathbin{\lfloor\!\lfloor} Q$. We consider $P' \mathbin{\lfloor\!\lfloor} Q$:

$$\begin{array}{lcll}
P' \mathbin{\lfloor\!\lfloor} Q & = & (\overline{\sum}_{i,\overline{d}_m,u} a_i\text{‹}u\, P_i \triangleleft \alpha_i \triangleright \delta\text{‹}\mathbf{0}) \mathbin{\lfloor\!\lfloor} Q & \\
& \stackrel{\text{SUM6}}{=} & \overline{\sum}_{i,\overline{d}_m,u} (a_i\text{‹}u\, P_i \triangleleft \alpha_i \triangleright \delta\text{‹}\mathbf{0}) \mathbin{\lfloor\!\lfloor} Q & \\
& \stackrel{\text{A.1.14}}{=} & \overline{\sum}_{i,\overline{d}_m,u} (a_i\text{‹}u\, P_i \mathbin{\lfloor\!\lfloor} Q) \triangleleft \alpha_i \triangleright (\delta\text{‹}\mathbf{0} \mathbin{\lfloor\!\lfloor} Q) & \\
& \stackrel{\text{A.2.7,3.3}}{=} & \overline{\sum}_{i,\overline{d}_m,u} (a_i{\ll}Q)\text{‹}u\,(P_i \parallel Q) \triangleleft \alpha_i \triangleright \delta\text{‹}\mathbf{0} & (*)
\end{array}$$

Now we may use Lemma 3.2.

1. If $\exists_{v:\mathbf{Time},j^* \in J^*}.\beta_{j^*} = \mathsf{t}$ then

$$\begin{array}{l}
P' \mathbin{\lfloor\!\lfloor} Q \quad = \\
\overline{\sum}_{i,\overline{d}_m,u} (\overline{\sum}_{\overline{e^*}_{n^*},v,t} a_i\text{‹}t \triangleleft t \leq v \wedge \bigvee_{j^* \in J^*} \beta_{j^*} \triangleright \delta\text{‹}\mathbf{0})\text{‹}u\,(P_i \parallel Q) \triangleleft \alpha_i \triangleright \delta\text{‹}\mathbf{0} \\
\stackrel{\text{ATB5,A.3.2}}{=} \\
\overline{\sum}_{i,\overline{d}_m,u} (\overline{\sum}_{\overline{e^*}_{n^*},v,t} a_i\text{‹}t\text{‹}u \triangleleft t \leq v \wedge \bigvee_{j^* \in J^*} \beta_{j^*} \triangleright \delta\text{‹}\mathbf{0})\,(P_i \parallel Q) \triangleleft \alpha_i \triangleright \delta\text{‹}\mathbf{0} \\
\stackrel{\text{A.3.3}}{=} \\
\overline{\sum}_{i,\overline{d}_m,u} (\overline{\sum}_{\overline{e^*}_{n^*},v} (\sum_t a_i\text{‹}t\text{‹}u \triangleleft t \leq v \triangleright \delta\text{‹}\mathbf{0}) \triangleleft \bigvee_{j^* \in J^*} \beta_{j^*} \triangleright \delta\text{‹}\mathbf{0})\,(P_i \parallel Q) \triangleleft \alpha_i \triangleright \delta\text{‹}\mathbf{0} \\
\stackrel{\text{A.5.4}}{=} \\
\overline{\sum}_{i,\overline{d}_m,u} (\overline{\sum}_{\overline{e^*}_{n^*},v} (a_i\text{‹}u \triangleleft u \leq v \triangleright \delta\text{‹}\mathbf{0} + \delta\text{‹}u\text{‹}v) \triangleleft \bigvee_{j^* \in J^*} \beta_{j^*} \triangleright \delta\text{‹}\mathbf{0})\,(P_i \parallel Q) \triangleleft \alpha_i \triangleright \delta\text{‹}\mathbf{0} \\
\stackrel{\text{A.1.5,A.1.6}}{=} \\
\overline{\sum}_{i,\overline{d}_m,u} \\
(\overline{\sum}_{\overline{e^*}_{n^*},v} (a_i\text{‹}u \triangleleft u \leq v \wedge \bigvee_{j^* \in J^*} \beta_{j^*} \triangleright \delta\text{‹}\mathbf{0} + \delta\text{‹}u\text{‹}v \triangleleft \bigvee_{j^* \in J^*} \beta_{j^*} \triangleright \delta\text{‹}\mathbf{0}))\,(P_i \parallel Q) \triangleleft \alpha_i \triangleright \delta\text{‹}\mathbf{0}
\end{array}$$

$$\stackrel{\text{SUM5,A4}}{=}$$
$$\overline{\sum}_{i,\overline{d}_m,u}(\overline{\sum}_{\overline{e^*}_{n^*},v}((a_i ` u \lhd u \leq v \wedge \bigvee_{j^* \in J^*} \beta_{j^*} \rhd \delta ` \mathbf{0})\,(P_i \parallel Q) + (\delta ` u ` v \lhd \bigvee_{j^* \in J^*} \beta_{j^*} \rhd \delta ` \mathbf{0})\,(P_i \parallel Q))) \lhd \alpha_i \rhd \delta ` \mathbf{0}$$
$$\stackrel{\text{A.1.3,ATB4,A.2.3}}{=}$$
$$\overline{\sum}_{i,\overline{d}_m,u} (\overline{\sum}_{\overline{e^*}_{n^*},v}(a_i ` u\,(P_i \parallel Q) \lhd u \leq v \wedge \bigvee_{j^* \in J^*} \beta_{j^*} \rhd \delta ` \mathbf{0} + \delta ` u ` v \lhd \bigvee_{j^* \in J^*} \beta_{j^*} \rhd \delta ` \mathbf{0})) \lhd \alpha_i \rhd \delta ` \mathbf{0}$$
$$\stackrel{\text{SUM4,A.1.5}}{=}$$
$$\overline{\sum}_{i,\overline{d}_m,u}((\overline{\sum}_{\overline{e^*}_{n^*},v} a_i ` u\,(P_i \parallel Q) \lhd u \leq v \wedge \bigvee_{j^* \in J^*} \beta_{j^*} \rhd \delta ` \mathbf{0}) \lhd \alpha_i \rhd \delta ` \mathbf{0} + (\overline{\sum}_{\overline{e^*}_{n^*},v} \delta ` u ` v \lhd \bigvee_{j^* \in J^*} \beta_{j^*} \rhd \delta ` \mathbf{0}) \lhd \alpha_i \rhd \delta ` \mathbf{0})$$
$$\stackrel{\text{A.3.1,A.1.6}}{=}$$
$$\overline{\sum}_{i,\overline{d}_m,u}(\overline{\sum}_{\overline{e^*}_{n^*},v} a_i ` u\,(P_i \parallel Q) \lhd u \leq v \wedge \alpha_i \wedge \bigvee_{j^* \in J^*} \beta_{j^*} \rhd \delta ` \mathbf{0} + \overline{\sum}_{\overline{e^*}_{n^*},v} \delta ` u ` v \lhd \alpha_i \wedge \bigvee_{j^* \in J^*} \beta_{j^*} \rhd \delta ` \mathbf{0})$$
$$\stackrel{\text{SUM4,A.1.7,A.4}}{=}$$
$$\overline{\sum}_{i,\overline{d}_m,\overline{e^*}_{n^*},v,u} a_i ` u\,(P_i \parallel Q) \lhd u \leq v \wedge \alpha_i \wedge \bigvee_{j^* \in J^*} \beta_{j^*} \rhd \delta ` \mathbf{0} + \overline{\sum}_{i,j^*,\overline{d}_m,\overline{e^*}_{n^*},u,v} \delta ` u ` v \lhd \alpha_i \wedge \beta_{j^*} \rhd \delta ` \mathbf{0}.$$

In a similar way, $P'' \mathbin{⫻\!\!\!\!\_} Q$ can be derived. It follows that

$$P'' \,\underline{\|}\, Q \;=\; \overline{\sum}_{i',\overline{d'}_{m'},\overline{e^*}_{n^*},v,u} a_{i'} ` u\, Q \lhd u \leq v \wedge \alpha_{i'} \wedge \bigvee_{j^* \in J^*} \beta_{j^*} \rhd \delta ` \mathbf{0} + \overline{\sum}_{i',j^*,\overline{d'}_{m'},\overline{e^*}_{n^*},u,v} \delta ` u ` v \lhd \alpha_{i'} \wedge \beta_{j^*} \rhd \delta ` \mathbf{0}.$$

We see that both $P' \,\underline{\|}\, Q$ and $P'' \,\underline{\|}\, Q$ have δ-summands. By axiom SUM1 we may extend the corresponding vectors $\overline{d}_m$ and $\overline{d'}_{m'}$ to $\overline{d^*}_{m^*}$. Now in the sum $P' \,\underline{\|}\, Q + P'' \,\underline{\|}\, Q$ the δ-summands add up to

$$\overline{\sum}_{i^*,j^*,\overline{d^*}_{m^*},\overline{e^*}_{n^*},u,v} \delta ` u ` v \lhd \alpha_{i^*} \wedge \beta_{j^*} \rhd \delta ` \mathbf{0},$$

which finishes this case;

2. If $\forall_{v:\mathbf{Time},j^* \in J^*}.\beta_{j^*} = \mathsf{f}$ then clearly $Q = \delta ` \mathbf{0}$, and

$$\begin{aligned} P' \,\underline{\|}\, \delta ` \mathbf{0} &\stackrel{(*),3.2.2}{=} \overline{\sum}_{i,\overline{d}_m,u} a_i ` \mathbf{0} ` u\,(P_i \parallel \delta ` \mathbf{0}) \lhd \alpha_i \rhd \delta ` \mathbf{0} \\ &\stackrel{\text{ATB2}}{=} \overline{\sum}_{i,\overline{d}_m,u} (a_i ` \mathbf{0} \lhd eq(u,\mathbf{0}) \rhd \delta ` \mathbf{0})\,(P_i \parallel \delta ` \mathbf{0}) \lhd \alpha_i \rhd \delta ` \mathbf{0} \\ &\stackrel{\text{A.1.3,A.2.3}}{=} \overline{\sum}_{i,\overline{d}_m,u} (a_i ` \mathbf{0}\,(P_i \parallel \delta ` \mathbf{0}) \lhd eq(u,\mathbf{0}) \rhd \delta ` \mathbf{0}) \lhd \alpha_i \rhd \delta ` \mathbf{0} \\ &\stackrel{\text{A.1.6,A.5.1}}{=} \overline{\sum}_{i,\overline{d}_m} a_i ` \mathbf{0}\,(P_i[\mathbf{0}/u] \parallel \delta ` \mathbf{0}) \lhd \alpha_i[\mathbf{0}/u] \rhd \delta ` \mathbf{0}. \end{aligned}$$

$P'' \,\underline{\|}\, \delta ` \mathbf{0}$ follows in a similar way.

□

We have now studied the first two terms of $P \parallel Q$ according to Lemma 3.1. By symmetry, results for $Q' \,\underline{\|}\, P$ and $Q'' \,\underline{\|}\, P$ follow in a similar way.

The following proposition is not needed for proving further results, but it may provide the reader with some good intuition about the behaviour of the left merge. We state it without proof.

Proposition 3.5. *Consider Lemma 3.4.1. Whenever there exist terms* Δ, v_0:**Time** *such that one of the conditions*

1. $\bigvee_{j^* \in J^*} \beta_{j^*}[v_0/v] = \mathsf{t}$ *and* $u > v_0 \wedge \bigvee_{i^* \in I^*} \alpha_{i^*} = \mathsf{f}$*, or*
2. $v - \Delta \leq v \wedge v < v_0 \wedge \bigvee_{j^* \in J^*} \beta_{j^*} = \mathsf{t}$ *and* $u \geq v_0 \wedge \bigvee_{i^* \in I^*} \alpha_{i^*} = \mathsf{f}$

is satisfied, the third summand (the δ*-summand) of* $P \mathbin{|\!\lfloor} Q$ *is cancelled.*

In other words, the above proposition implies that if Q can perform some action at least until (but not necessarily including) the moment that P gets definitively disabled, the δ-summand is cancelled.
We give two examples to further illustrate the behaviour of $\mathbin{|\!\lfloor}$. The first example shows that the conditions in Proposition 3.5 do not have to be satisfied in order to get rid of the δ's.

Example 3.6. Let p be the process that can perform an a-action before or at time 2, and q be the process that can do a b-action at time 1. (Note that the conditions in Proposition 3.5 are not satisfied.) For $p \mathbin{|\!\lfloor} q$ it easily follows that:

$$
\begin{aligned}
p \mathbin{|\!\lfloor} q \quad & = && (\textstyle\sum_u a^{\varsigma}u \lhd u \leq 2 \rhd \delta^{\varsigma}\mathbf{0}) \mathbin{|\!\lfloor} \textstyle\sum_v b^{\varsigma}v \lhd eq(v,1) \rhd \delta^{\varsigma}\mathbf{0} \\
& \overset{3.4.1,A.5.1}{=} && \overline{\textstyle\sum}_{v,u} a^{\varsigma}u\, b^{\varsigma}1 \lhd u \leq v \wedge u \leq 2 \wedge eq(v,1) \rhd \delta^{\varsigma}\mathbf{0} + \\
& && \overline{\textstyle\sum}_{u,v} \delta^{\varsigma}u^{\varsigma}v \lhd u \leq 2 \wedge eq(v,1) \rhd \delta^{\varsigma}\mathbf{0} \\
& \overset{A.4,A.3.3}{=} && \textstyle\sum_u (\sum_v a^{\varsigma}u\, b^{\varsigma}1 \lhd eq(v,1) \rhd \delta^{\varsigma}\mathbf{0}) \lhd u \leq 1 \rhd \delta^{\varsigma}\mathbf{0} + \\
& && \textstyle\sum_u (\sum_v \delta^{\varsigma}u^{\varsigma}v \lhd eq(v,1) \rhd \delta^{\varsigma}\mathbf{0}) \lhd u \leq 2 \rhd \delta^{\varsigma}\mathbf{0} \\
& \overset{A.5.1,A.5.2}{=} && \textstyle\sum_u a^{\varsigma}u\, b^{\varsigma}1 \lhd u \leq 1 \rhd \delta^{\varsigma}\mathbf{0} + \sum_u \delta^{\varsigma}u^{\varsigma}1 \lhd u \leq 2 \rhd \delta^{\varsigma}\mathbf{0} \\
& \overset{A.3.2}{=} && \textstyle\sum_u a^{\varsigma}u\, b^{\varsigma}1 \lhd u \leq 1 \rhd \delta^{\varsigma}\mathbf{0} + (\sum_u \delta^{\varsigma}u \lhd u \leq 2 \rhd \delta^{\varsigma}\mathbf{0})^{\varsigma}1 \\
& \overset{A.5.3,ATB1}{=} && \textstyle\sum_u a^{\varsigma}u\, b^{\varsigma}1 \lhd u \leq 1 \rhd \delta^{\varsigma}\mathbf{0} + \delta^{\varsigma}1.
\end{aligned}
$$

By axiom SUM3 $a^{\varsigma}1\, b^{\varsigma}1$ is a summand of $p \mathbin{|\!\lfloor} q$, so by Lemma A.2.5 $\delta^{\varsigma}1$ is cancelled. $\square$

The following example shows a left merge that induces a 'hard' deadlock.

Example 3.7. Let

$$
\begin{aligned}
& p \overset{\text{def}}{=} \textstyle\sum_u a^{\varsigma}u \lhd 1 \leq u \leq 2 \vee 4 \leq u \leq 5 \rhd \delta^{\varsigma}\mathbf{0}; \\
& q \overset{\text{def}}{=} \textstyle\sum_v b^{\varsigma}v \lhd v \leq 3 \rhd \delta^{\varsigma}\mathbf{0}.
\end{aligned}
$$

We have that:

$$
\begin{aligned}
& p \mathbin{|\!\lfloor} q \\
& \overset{3.4.1}{=} && \overline{\textstyle\sum}_{v,u} a^{\varsigma}u\, q \lhd u \leq v \wedge (1 \leq u \leq 2 \vee 4 \leq u \leq 5) \wedge v \leq 3 \rhd \delta^{\varsigma}\mathbf{0} + \\
& && \overline{\textstyle\sum}_{u,v} \delta^{\varsigma}u^{\varsigma}v \lhd (1 \leq u \leq 2 \vee 4 \leq u \leq 5) \wedge v \leq 3 \rhd \delta^{\varsigma}\mathbf{0} \\
& \overset{A.3.3,A.2.2}{=} && \overline{\textstyle\sum}_{v,u} a^{\varsigma}u\, q \lhd u \leq v \wedge 1 \leq u \leq 2 \wedge v \leq 3 \rhd \delta^{\varsigma}\mathbf{0} + \\
& && \textstyle\sum_u (\sum_v \delta^{\varsigma}v^{\varsigma}u \lhd v \leq 3 \rhd \delta^{\varsigma}\mathbf{0}) \lhd 1 \leq u \leq 2 \vee 4 \leq u \leq 5 \rhd \delta^{\varsigma}\mathbf{0} \\
& \overset{A.3.2}{=} && \overline{\textstyle\sum}_{v,u} a^{\varsigma}u\, q \lhd u \leq v \wedge 1 \leq u \leq 2 \wedge v \leq 3 \rhd \delta^{\varsigma}\mathbf{0} + \\
& && \textstyle\sum_u (\sum_v \delta^{\varsigma}v \lhd v \leq 3 \rhd \delta^{\varsigma}\mathbf{0})^{\varsigma}u \lhd 1 \leq u \leq 2 \vee 4 \leq u \leq 5 \rhd \delta^{\varsigma}\mathbf{0} \\
& \overset{A.5.3,A.2.2}{=} && \overline{\textstyle\sum}_{v,u} a^{\varsigma}u\, q \lhd u \leq v \wedge 1 \leq u \leq 2 \wedge v \leq 3 \rhd \delta^{\varsigma}\mathbf{0} + \\
& && \textstyle\sum_u \delta^{\varsigma}u^{\varsigma}3 \lhd 1 \leq u \leq 2 \vee 4 \leq u \leq 5 \rhd \delta^{\varsigma}\mathbf{0}
\end{aligned}
$$

$$\begin{aligned}
&\overset{\text{A.1.7,SUM4}}{=} && \overline{\sum}_{v,u} a^{\varsigma}u\, q \lhd u \leq v \wedge 1 \leq u \leq 2 \wedge v \leq 3 \rhd \delta^{\varsigma}\mathbf{0} + \\
& && \textstyle\sum_u \delta^{\varsigma}u^{\varsigma}3 \lhd 1 \leq u \leq 2 \rhd \delta^{\varsigma}\mathbf{0} + \\
& && \textstyle\sum_u \delta^{\varsigma}u^{\varsigma}3 \lhd 4 \leq u \leq 5 \rhd \delta^{\varsigma}\mathbf{0} \\
&\overset{\text{A.3.2}}{=} && \overline{\sum}_{v,u} a^{\varsigma}u\, q \lhd u \leq v \wedge 1 \leq u \leq 2 \wedge v \leq 3 \rhd \delta^{\varsigma}\mathbf{0} + \\
& && (\textstyle\sum_u \delta^{\varsigma}u \lhd 1 \leq u \leq 2 \rhd \delta^{\varsigma}\mathbf{0})^{\varsigma}3 + \\
& && (\textstyle\sum_u \delta^{\varsigma}u \lhd 4 \leq u \leq 5 \rhd \delta^{\varsigma}\mathbf{0})^{\varsigma}3 \\
&\overset{\text{A.5.3,ATB1,ATA2}}{=} && \overline{\sum}_{v,u} a^{\varsigma}u\, q \lhd u \leq v \wedge 1 \leq u \leq 2 \wedge v \leq 3 \rhd \delta^{\varsigma}\mathbf{0} + \delta^{\varsigma}3.
\end{aligned}$$

The intuition behind this expression is that a should happen between 1 and 2, and then a b before or at time 3. Since $3 > 2$ the deadlock at 3 cannot be cancelled. Actually, $\delta^{\varsigma}3$ models the alternative that p would 'wait' to do an a-step between 4 and 5, while process q can only wait until time 3. □

3.3 Expansion of the communication merge

We continue with the communications between P and Q. Communication between any two actions of P and Q may take place as long as both actions are enabled. In general, time deadlocks will occur, most of which can be eliminated right away.

Lemma 3.8 (*Communication Merge*). *It holds that:*

$$\begin{aligned}
P \mid Q = \; & \overline{\sum}_{i,j,\overline{d}_m,\overline{e}_n,u} (a_i \mid b_j)^{\varsigma}u\, (\, P_i \parallel Q_j[u/v]) \lhd \alpha_i \wedge \beta_j[u/v] \rhd \delta^{\varsigma}\mathbf{0} + \\
& \overline{\sum}_{i',j,\overline{d'}_{m'},\overline{e}_n,u} (a_{i'} \mid b_j)^{\varsigma}u\, Q_j[u/v] \lhd \alpha_{i'} \wedge \beta_j[u/v] \rhd \delta^{\varsigma}\mathbf{0} + \\
& \overline{\sum}_{i,j',\overline{d}_m,\overline{e'}_{n'},u} (a_i \mid b_{j'})^{\varsigma}u\, P_i \lhd \alpha_i \wedge \beta_{j'}[u/v] \rhd \delta^{\varsigma}\mathbf{0} + \\
& \overline{\sum}_{i',j',\overline{d'}_{m'},\overline{e'}_{n'},u} (a_{i'} \mid b_{j'})^{\varsigma}u \lhd \alpha_{i'} \wedge \beta_{j'}[u/v] \rhd \delta^{\varsigma}\mathbf{0} + \\
& \overline{\sum}_{i^*,j^*,\overline{d^*}_{m^*},\overline{e^*}_{n^*},u,v} \delta^{\varsigma}u^{\varsigma}v \lhd \alpha_{i^*} \wedge \beta_{j^*} \rhd \delta^{\varsigma}\mathbf{0}.
\end{aligned}$$

Proof. We give the full expansion of $P' \mid Q'$:

$$\begin{aligned}
P' \mid Q' \quad &= && (\overline{\sum}_{i,\overline{d}_m,u} a_i{}^{\varsigma}u\, P_i \lhd \alpha_i \rhd \delta^{\varsigma}\mathbf{0}) \mid \overline{\sum}_{j,\overline{e}_n,v} b_j{}^{\varsigma}v\, Q_j \lhd \beta_j \rhd \delta^{\varsigma}\mathbf{0} \\
&\overset{\text{SUM7}}{=} && \overline{\sum}_{i,\overline{d}_m,u} (a_i{}^{\varsigma}u\, P_i \lhd \alpha_i \rhd \delta^{\varsigma}\mathbf{0} \mid \overline{\sum}_{j,\overline{e}_n,v} b_j{}^{\varsigma}v\, Q_j \lhd \beta_j \rhd \delta^{\varsigma}\mathbf{0}) \\
&\overset{\text{SUM7}',\text{A.4}}{=} && \overline{\sum}_{i,j,\overline{d}_m,\overline{e}_n,u,v} (a_i{}^{\varsigma}u\, P_i \lhd \alpha_i \rhd \delta^{\varsigma}\mathbf{0} \mid b_j{}^{\varsigma}v\, Q_j \lhd \beta_j \rhd \delta^{\varsigma}\mathbf{0}) \\
&\overset{\text{A.1.15,A.1.16}}{=} && \overline{\sum}_{i,j,\overline{d}_m,\overline{e}_n,u,v} \\
& && \quad ((a_i{}^{\varsigma}u\, P_i \mid b_j{}^{\varsigma}v\, Q_j) \lhd \beta_j \rhd (a_i{}^{\varsigma}u\, P_i \mid \delta^{\varsigma}\mathbf{0})) \\
& && \quad \lhd \alpha_i \rhd ((\delta^{\varsigma}\mathbf{0} \mid b_j{}^{\varsigma}v\, Q_j) \lhd \beta_j \rhd (\delta^{\varsigma}\mathbf{0} \mid \delta^{\varsigma}\mathbf{0})) \\
&\overset{\text{A.2.\{8,9,10\}}}{=} && \overline{\sum}_{i,j,\overline{d}_m,\overline{e}_n,u,v} \\
& && \quad ((a_i{}^{\varsigma}u\, P_i \mid b_j{}^{\varsigma}v\, Q_j) \lhd \beta_j \rhd \delta^{\varsigma}\mathbf{0}) \lhd \alpha_i \rhd (\delta^{\varsigma}\mathbf{0} \lhd \beta_j \rhd \delta^{\varsigma}\mathbf{0}) \\
&\overset{\text{A.1.1,A.1.6}}{=} && \overline{\sum}_{i,j,\overline{d}_m,\overline{e}_n,u,v} (a_i{}^{\varsigma}u\, P_i \mid b_j{}^{\varsigma}v\, Q_j) \lhd \alpha_i \wedge \beta_j \rhd \delta^{\varsigma}\mathbf{0} \\
&\overset{\text{ATB\{4,8,7\}}}{=} && \overline{\sum}_{i,j,\overline{d}_m,\overline{e}_n,u,v} (a_i\, P_i \parallel b_j\, Q_j)^{\varsigma}u^{\varsigma}v \lhd \alpha_i \wedge \beta_j \rhd \delta^{\varsigma}\mathbf{0} \\
&\overset{\text{CM7,ATB4}}{=} && \overline{\sum}_{i,j,\overline{d}_m,\overline{e}_n,u,v} (a_i \mid b_j)^{\varsigma}u^{\varsigma}v\, (\, P_i \parallel Q_j) \lhd \alpha_i \wedge \beta_j \rhd \delta^{\varsigma}\mathbf{0} \\
&\overset{\text{ATB2,A.1.3}}{=} && \overline{\sum}_{i,j,\overline{d}_m,\overline{e}_n,u,v} \\
& && \quad ((a_i \mid b_j)^{\varsigma}u\, (\, P_i \parallel Q_j) \lhd eq(u,v) \rhd \delta^{\varsigma}min(u,v)\, (\, P_i \parallel Q_j)) \lhd \alpha_i \wedge \beta_j \rhd \delta^{\varsigma}\mathbf{0}
\end{aligned}$$

$$
\begin{array}{ll}
\stackrel{\text{A.2.3,A.1.8}}{=} & \overline{\sum}_{i,j,\overline{d}_m,\overline{e}_n,u,v}(\\
& \quad (a_i \mid b_j)^{\text{c}}u\,(\,P_i \parallel Q_j) \triangleleft eq(u,v) \wedge \alpha_i \wedge \beta_j \triangleright \delta^{\text{c}}\mathbf{0} + \\
& \quad \delta^{\text{c}}min(u,v) \triangleleft \neg eq(u,v) \wedge \alpha_i \wedge \beta_j \triangleright \delta^{\text{c}}\mathbf{0}) \\
\stackrel{\text{A.2.5,A.1.5}}{=} & \overline{\sum}_{i,j,\overline{d}_m,\overline{e}_n,u,v}(\\
& \quad (a_i \mid b_j)^{\text{c}}u\,(\,P_i \parallel Q_j) \triangleleft eq(u,v) \wedge \alpha_i \wedge \beta_j \triangleright \delta^{\text{c}}\mathbf{0} + \\
& \quad \delta^{\text{c}}u \triangleleft eq(u,v) \wedge \alpha_i \wedge \beta_j \triangleright \delta^{\text{c}}\mathbf{0} + \\
& \quad \delta^{\text{c}}min(u,v) \triangleleft \neg eq(u,v) \wedge \alpha_i \wedge \beta_j \triangleright \delta^{\text{c}}\mathbf{0}) \\
\stackrel{\text{A.1.17,ATB1,A.1.9}}{=} & \overline{\sum}_{i,j,\overline{d}_m,\overline{e}_n,u,v}(\\
& \quad (a_i \mid b_j)^{\text{c}}u\,(\,P_i \parallel Q_j) \triangleleft eq(u,v) \wedge \alpha_i \wedge \beta_j \triangleright \delta^{\text{c}}\mathbf{0} + \\
& \quad \delta^{\text{c}}u^{\text{c}}v \triangleleft \alpha_i \wedge \beta_j \triangleright \delta^{\text{c}}\mathbf{0}) \\
\stackrel{\text{SUM4,A.1.6}}{=} & \overline{\sum}_{i,j,\overline{d}_m,\overline{e}_n}(\\
& \quad \sum_v(\sum_u (a_i \mid b_j)^{\text{c}}u\,(\,P_i \parallel Q_j) \triangleleft \alpha_i \wedge \beta_j \triangleright \delta^{\text{c}}\mathbf{0}) \triangleleft eq(u,v) \triangleright \delta^{\text{c}}\mathbf{0} + \\
& \quad \sum_{u,v} \delta^{\text{c}}u^{\text{c}}v \triangleleft \alpha_i \wedge \beta_j \triangleright \delta^{\text{c}}\mathbf{0}) \\
\stackrel{\text{A.5.1,ATB1,SUM4}}{=} & \overline{\sum}_{i,j,\overline{d}_m,\overline{e}_n,u}(a_i \mid b_j)^{\text{c}}u\,(\,P_i \parallel Q_j[u/v]) \triangleleft \alpha_i \wedge \beta_j[u/v] \triangleright \delta^{\text{c}}\mathbf{0} + \\
& \overline{\sum}_{i,j,\overline{d}_m,\overline{e}_n,u,v} \delta^{\text{c}}u^{\text{c}}v \triangleleft \alpha_i \wedge \beta_j \triangleright \delta^{\text{c}}\mathbf{0}.
\end{array}
$$

The processes $P' \mid Q''$, $P'' \mid Q'$ and $P'' \mid Q''$ are expanded in a similar way. The four terms with $\mid$ each lead to different δ-summands. Some elementary calculations show that these add up to

$$\overline{\sum}_{i^*,j^*,\overline{d^*}_{m^*},\overline{e^*}_{n^*},u,v} \delta^{\text{c}}u^{\text{c}}v \triangleleft \alpha_{i^*} \wedge \beta_{j^*} \triangleright \delta^{\text{c}}\mathbf{0},$$

which finishes this proof. □

In the following example we show that the term with the time deadlocks is essential.

Example 3.9. Consider the process $a^{\text{c}}1 \mid b^{\text{c}}2$. Straightforward application of the axioms of μCRL_t leads to:

$$a^{\text{c}}1 \mid b^{\text{c}}2 \stackrel{\text{ATB8,ATB7}}{=} (a \mid b)^{\text{c}}1^{\text{c}}2 \stackrel{\text{ATB2}}{=} \delta^{\text{c}}1.$$

We can also use the above lemma for communication, and then we need

(i) $a^{\text{c}}1 \stackrel{\text{A.5.1}}{=} \sum_u a^{\text{c}}u \triangleleft eq(u,1) \triangleright \delta^{\text{c}}\mathbf{0}$;
(ii) $b^{\text{c}}2 \stackrel{\text{A.5.1}}{=} \sum_v b^{\text{c}}v \triangleleft eq(v,2) \triangleright \delta^{\text{c}}\mathbf{0}$.

Application of Lemma 3.8 yields:

$$
\begin{array}{lll}
a^{\text{c}}1 \mid b^{\text{c}}2 & \stackrel{\text{i,ii,3.8}}{=} & \sum_u (a \mid b)^{\text{c}}u \triangleleft eq(u,1) \wedge eq(u,2) \triangleright \delta^{\text{c}}\mathbf{0} + \\
& & \sum_{u,v} \delta^{\text{c}}u^{\text{c}}v \triangleleft eq(u,1) \wedge eq(v,2) \triangleright \delta^{\text{c}}\mathbf{0} \\
& \stackrel{\text{A.2.2,A.3.3}}{=} & \sum_u \delta^{\text{c}}\mathbf{0} + \sum_u(\sum_v \delta^{\text{c}}v^{\text{c}}u \triangleleft eq(v,2) \triangleright \delta^{\text{c}}\mathbf{0}) \triangleleft eq(u,1) \triangleright \delta^{\text{c}}\mathbf{0} \\
& \stackrel{\text{SUM1,A.3.2}}{=} & \delta^{\text{c}}\mathbf{0} + \sum_u(\sum_v \delta^{\text{c}}v \triangleleft eq(v,2) \triangleright \delta^{\text{c}}\mathbf{0})^{\text{c}}u \triangleleft eq(u,1) \triangleright \delta^{\text{c}}\mathbf{0} \\
& \stackrel{\text{AT6,A.5.1}}{=} & \delta^{\text{c}}2^{\text{c}}1 \\
& \stackrel{\text{ATB1}}{=} & \delta^{\text{c}}1.
\end{array}
$$

Note that this $\delta^{\text{c}}1$ was generated by the δ-summand in $P \mid Q$. □

3.4 Expansion Theorem

Before we can present our main result we have to derive a lemma for splitting the central δ-summand.

Lemma 3.10. *It holds that:*

$$
\begin{array}{lll}
\multicolumn{2}{l}{\overline{\sum}_{i^*,j^*,\overline{d^*}_{m^*},\overline{e^*}_{n^*},u,v}\delta\text{‘}u\text{‘}v \lhd \alpha_{i^*}\wedge\beta_{j^*}\rhd\delta\text{‘}\mathbf{0}} & \\
= & \overline{\sum}_{i,\overline{d}_m,\overline{e^*}_{n^*},v,u}\delta\text{‘}u \lhd u\leq v\wedge\alpha_i\wedge\bigvee_{j^*\in J^*}\beta_{j^*}\rhd\delta\text{‘}\mathbf{0}\;+ & (D1)\\
 & \overline{\sum}_{i',\overline{d'}_{m'},\overline{e^*}_{n^*},v,u}\delta\text{‘}u \lhd u\leq v\wedge\alpha_{i'}\wedge\bigvee_{j^*\in J^*}\beta_{j^*}\rhd\delta\text{‘}\mathbf{0}+ & (D2)\\
 & \overline{\sum}_{j,\overline{e}_n,\overline{d^*}_{m^*},u,v}\delta\text{‘}v \lhd v\leq u\wedge\beta_j\wedge\bigvee_{i^*\in I^*}\alpha_{i^*}\rhd\delta\text{‘}\mathbf{0}+ & (D3)\\
 & \overline{\sum}_{j',\overline{e'}_{n'},\overline{d^*}_{m^*},u,v}\delta\text{‘}v \lhd v\leq u\wedge\beta_{j'}\wedge\bigvee_{i^*\in I^*}\alpha_{i^*}\rhd\delta\text{‘}\mathbf{0} & (D4)
\end{array}
$$

Proof. We derive

$$
\begin{array}{ll}
\multicolumn{2}{l}{\overline{\sum}_{i^*,j^*,\overline{d^*}_{m^*},\overline{e^*}_{n^*},u,v}\delta\text{‘}u\text{‘}v \lhd \alpha_{i^*}\wedge\beta_{j^*}\rhd\delta\text{‘}\mathbf{0}}\\
\overset{\text{A.1.10,A.2.2}}{=} & \overline{\sum}_{i^*,j^*,\overline{d^*}_{m^*},\overline{e^*}_{n^*},u,v}\\
 & (\delta\text{‘}u\text{‘}v \lhd u\leq v\wedge\alpha_{i^*}\wedge\beta_{j^*}\rhd\delta\text{‘}\mathbf{0}+\delta\text{‘}v\text{‘}u \lhd v\leq u\wedge\alpha_{i^*}\wedge\beta_{j^*}\rhd\delta\text{‘}\mathbf{0})\\
\overset{\text{A.1.11,SUM4}}{=} & \overline{\sum}_{i^*,j^*,\overline{d^*}_{m^*},\overline{e^*}_{n^*},u,v}\delta\text{‘}u \lhd u\leq v\wedge\alpha_{i^*}\wedge\beta_{j^*}\rhd\delta\text{‘}\mathbf{0}+\\
 & \overline{\sum}_{i^*,j^*,\overline{d^*}_{m^*},\overline{e^*}_{n^*},u,v}\delta\text{‘}v \lhd v\leq u\wedge\alpha_{i^*}\wedge\beta_{j^*}\rhd\delta\text{‘}\mathbf{0}\\
\overset{\text{A.4,A.1.7}}{=} & \overline{\sum}_{i^*,\overline{d^*}_{m^*},\overline{e^*}_{n^*},v,u}\delta\text{‘}u \lhd u\leq v\wedge\alpha_{i^*}\wedge\bigvee_{j^*\in J^*}\beta_{j^*}\rhd\delta\text{‘}\mathbf{0}+\\
 & \overline{\sum}_{j^*,\overline{e^*}_{n^*},\overline{d^*}_{m^*},u,v}\delta\text{‘}v \lhd v\leq u\wedge\beta_{j^*}\wedge\bigvee_{i^*\in I^*}\alpha_{i^*}\rhd\delta\text{‘}\mathbf{0}\\
= & \overline{\sum}_{i,\overline{d^*}_{m^*},\overline{e^*}_{n^*},v,u}\delta\text{‘}u \lhd u\leq v\wedge\alpha_i\wedge\bigvee_{j^*\in J^*}\beta_{j^*}\rhd\delta\text{‘}\mathbf{0}+\\
 & \overline{\sum}_{i',\overline{d^*}_{m^*},\overline{e^*}_{n^*},v,u}\delta\text{‘}u \lhd u\leq v\wedge\alpha_{i'}\wedge\bigvee_{j^*\in J^*}\beta_{j^*}\rhd\delta\text{‘}\mathbf{0}+\\
 & \overline{\sum}_{j,\overline{e^*}_{n^*},\overline{d^*}_{m^*},u,v}\delta\text{‘}v \lhd v\leq u\wedge\beta_j\wedge\bigvee_{i^*\in I^*}\alpha_{i^*}\rhd\delta\text{‘}\mathbf{0}+\\
 & \overline{\sum}_{j',\overline{e^*}_{n^*},\overline{d^*}_{m^*},u,v}\delta\text{‘}v \lhd v\leq u\wedge\beta_{j'}\wedge\bigvee_{i^*\in I^*}\alpha_{i^*}\rhd\delta\text{‘}\mathbf{0}.
\end{array}
$$

By axiom SUM1 we may omit all sum operators that do not bind any variables in their arguments. This finishes the proof. □

Now we can derive an expansion theorem for timed μCRL. Lemma 3.4 was derived for two cases, the second of which should be regarded as theoretical; A sound specification will generally contain no time deadlocks. Consequently, an expansion theorem can be derived for four cases, two of which are derived below. The other two cases follow trivially.

Theorem 3.11 (*Expansion*)**.** *It holds that:*

1. *If* $\exists_{u:\mathbf{Time},i^*\in I^*}.\alpha_{i^*}=\mathtt{t}$ *and* $\exists_{v:\mathbf{Time},j^*\in J^*}.\beta_{j^*}=\mathtt{t}$ *then*

$$
\begin{array}{llll}
P\parallel Q & = & \overline{\sum}_{i,\overline{d}_m,\overline{e^*}_{n^*},v,u}a_i\text{‘}u\,(P_i\parallel Q)\lhd u\leq v\wedge\alpha_i\wedge\bigvee_{j^*\in J^*}\beta_{j^*}\rhd\delta\text{‘}\mathbf{0}+ & (M1)\\
 & & \overline{\sum}_{i',\overline{d'}_{m'},\overline{e^*}_{n^*},v,u}a_{i'}\text{‘}u\,Q\lhd u\leq v\wedge\alpha_{i'}\wedge\bigvee_{j^*\in J^*}\beta_{j^*}\rhd\delta\text{‘}\mathbf{0}+ & (M2)\\
 & & \overline{\sum}_{j,\overline{e}_n,\overline{d^*}_{m^*},u,v}b_j\text{‘}v\,(Q_j\parallel P)\lhd v\leq u\wedge\beta_j\wedge\bigvee_{i^*\in I^*}\alpha_{i^*}\rhd\delta\text{‘}\mathbf{0}+ & (M3)\\
 & & \overline{\sum}_{j',\overline{e'}_{n'},\overline{d^*}_{m^*},u,v}b_{j'}\text{‘}v\,P\lhd v\leq u\wedge\beta_{j'}\wedge\bigvee_{i^*\in I^*}\alpha_{i^*}\rhd\delta\text{‘}\mathbf{0}+ & (M4)\\
 & & \overline{\sum}_{i,j,\overline{d}_m,\overline{e}_n,u}(a_i\mid b_j)\text{‘}u\,(P_i\parallel Q_j[u/v])\lhd\alpha_i\wedge\beta_j[u/v]\rhd\delta\text{‘}\mathbf{0}+ & (M5)\\
 & & \overline{\sum}_{i',j,\overline{d'}_{m'},\overline{e}_n,u}(a_{i'}\mid b_j)\text{‘}u\,Q_j[u/v]\lhd\alpha_{i'}\wedge\beta_j[u/v]\rhd\delta\text{‘}\mathbf{0}+ & (M6)\\
 & & \overline{\sum}_{i,j',\overline{d}_m,\overline{e'}_{n'},u}(a_i\mid b_{j'})\text{‘}u\,P_i\lhd\alpha_i\wedge\beta_{j'}[u/v]\rhd\delta\text{‘}\mathbf{0}+ & (M7)\\
 & & \overline{\sum}_{i',j',\overline{d'}_{m'},\overline{e'}_{n'},u}(a_{i'}\mid b_{j'})\text{‘}u\lhd\alpha_{i'}\wedge\beta_{j'}[u/v]\rhd\delta\text{‘}\mathbf{0} & (M8)
\end{array}
$$

2. *If* $\exists_{u:\mathbf{Time},i^*\in I^*}.\alpha_{i^*} = \mathsf{t}$ *and* $\forall_{v:\mathbf{Time},j^*\in J^*}.\beta_{j^*} = \mathsf{f}$ *then*

$$P \,\|\, Q = \overline{\sum}_{i,\overline{d}_m} a_i{}^{\varsigma}\mathbf{0}\,(P_i[\mathbf{0}/u] \parallel \delta{}^{\varsigma}\mathbf{0}) \lhd \alpha_i[\mathbf{0}/u] \rhd \delta{}^{\varsigma}\mathbf{0} + \overline{\sum}_{i',\overline{d'}_{m'}} a_{i'}{}^{\varsigma}\mathbf{0}\,\delta{}^{\varsigma}\mathbf{0} \lhd \alpha_{i'}[\mathbf{0}/u] \rhd \delta{}^{\varsigma}\mathbf{0}.$$

Proof. We only prove the first case. A proof for the second is straightforward.

We may combine the results from the lemmas 3.4.1 and 3.8. In total, two different terms with time deadlocks are involved, which are easily proven equal using the lemmas A.2.2 and A.4. The resulting expansion is as follows:

$$\begin{array}{rcl} P \parallel Q & = & M1 + \ldots + M8 + \\ & & \overline{\sum}_{i^*,j^*,\overline{d^*}_{m^*},\overline{e^*}_{n^*},u,v} \delta{}^{\varsigma}u{}^{\varsigma}v \lhd \alpha_{i^*} \wedge \beta_{j^*} \rhd \delta{}^{\varsigma}\mathbf{0} \quad (M9) \end{array}$$

By Lemma 3.10 the time deadlocks in summand $M9$ may be distributed over the four summands $D1$–$D4$. We consider summand $M1$ and show that it cancels $D1$. We derive

$$\begin{array}{rcl} M1 + D1 & \overset{\text{SUM4}}{=} & \overline{\sum}_{i,\overline{d}_m,\overline{e^*}_{n^*},v,u}(\\ & & \quad a_i{}^{\varsigma}u\,(P_i \parallel Q) \lhd u \leq v \wedge \alpha_i \wedge \bigvee_{j^*\in J^*} \beta_{j^*} \rhd \delta{}^{\varsigma}\mathbf{0} + \\ & & \quad \delta{}^{\varsigma}u \lhd u \leq v \wedge \alpha_i \wedge \bigvee_{j^*\in J^*} \beta_{j^*} \rhd \delta{}^{\varsigma}\mathbf{0}) \\ & \overset{\text{A.1.5,A.2.5}}{=} & \overline{\sum}_{i,\overline{d}_m,\overline{e^*}_{n^*},v,u} a_i{}^{\varsigma}u\,(P_i \parallel Q) \lhd u \leq v \wedge \alpha_i \wedge \bigvee_{j^*\in J^*} \beta_{j^*} \rhd \delta{}^{\varsigma}\mathbf{0}, \end{array}$$

which is again $M1$. In a similar way it is proved that summand $M2$ cancels $D2$. By symmetry, $D3$ is cancelled by $M3$ and $D4$ by $M4$; □

4 Properties of the parallel operators

We prove some basic properties of the parallel operators. First we prove that they may be eliminated from process-closed μCRL$_t$-terms, then we show that $\parallel$ and $\mid$ are commutative and associative for process-closed terms.

4.1 Elimination

In this section we consider processes with parallel operators, and show that all process-closed terms are provably equal to $\Sigma(p\mathrm{CRL}_t)$-terms.

Lemma 4.1. *Let p and q be process-closed terms over $\Sigma(pCRL_t)$. The terms $p \ll q$ and $\partial_H(p)$ are provably equal to basic forms.*

Proof. Easy; by induction on the structure of q and p, respectively. □

Theorem 4.2 (*Elimination***).** *For any process-closed term q over $\Sigma(\mu CRL_t)$ there is a basic form p such that $\mu CRL_t \vdash p = q$.*

Proof. (Sketch.) Let r and s be process-closed terms over $\Sigma(p\mathrm{CRL}_t)$. We take three steps:

1. By Theorem 2.6 r and s are provably equal to some basic forms $\tilde{r}$ and $\tilde{s}$, respectively. Now using the Expansion Theorem it is easily proved by induction on the sum of depths $|\tilde{r}| + |\tilde{s}|$ (see Definition 2.7) that the term $\tilde{r} \parallel \tilde{s}$ reduces to some basic form;

2. By step 1 and the lemmas 3.4 and 3.8 $r \mathbin{\|\!\_} s$ and $r \mid s$ also reduce to basic forms;

3. By Lemma 4.1 and the steps 1 and 2 any process-closed term q over $\Sigma(\mu\mathrm{CRL}_t)$ with at most one operator from $\{\ll, \partial_H, \|, \lfloor\!\lfloor, \mid\}$ is provably equal to some basic form. Obviously, any term with $n+1$ operators from this set contains subterms with only one such operator, such that after elimination of one of these, a term with n parallel operators results. By induction on n we conclude that all these operators can be eliminated, so that q is provably equal to some basic form p.

□

4.2 Associativity and commutativity

In this section we consider basic forms p, q and r over $p\mathrm{CRL}_t$ of the following form:

$$\begin{array}{lll} p \stackrel{\text{def}}{=} \overline{\sum}_{i,\overline{d}_l,t} a_i \text{‹} t\, p_i \lhd \alpha_i \rhd \delta \text{‹} \mathbf{0} + \overline{\sum}_{i',\overline{d'}_{l'},t} a_{i'} \text{‹} t \lhd \alpha_{i'} \rhd \delta \text{‹} \mathbf{0} & = p' + p''; \\ q \stackrel{\text{def}}{=} \overline{\sum}_{j,\overline{e}_m,u} b_j \text{‹} u\, q_j \lhd \beta_j \rhd \delta \text{‹} \mathbf{0} + \overline{\sum}_{j',\overline{e'}_{m'},u} b_{j'} \text{‹} u \lhd \beta_{j'} \rhd \delta \text{‹} \mathbf{0} & = q' + q''; \\ r \stackrel{\text{def}}{=} \overline{\sum}_{k,\overline{f}_n,v} c_k \text{‹} v\, r_k \lhd \nu_k \rhd \delta \text{‹} \mathbf{0} + \overline{\sum}_{k',\overline{f'}_{n'},v} c_{k'} \text{‹} v \lhd \nu_{k'} \rhd \delta \text{‹} \mathbf{0} & = r' + r''. \end{array}$$

Theorem 4.3 (*Commutativity*). *It holds that:*

1. $p \mid q = q \mid p$;
2. $p \parallel q = q \parallel p$.

Proof. We prove the identities 1 and 2 simultaneously by induction on the sum of depths $|p| + |q|$. The induction hypothesis is $\tilde{p} \| \tilde{q} = \tilde{q} \| \tilde{p}$, where $|\tilde{p}| + |\tilde{q}| < |p| + |q|$. The base case of the proof, where $|p| = |q| = 1$ is easy, and therefore omitted. For the general case, we first prove $p \mid q = q \mid p$:

$$p \mid q = (p' + p'') \mid (q' + q'') \stackrel{\text{CM8,CM9}}{=} p' \mid q' + p' \mid q'' + p'' \mid q' + p'' \mid q'' \qquad (*)$$

$$\begin{array}{lcl} p' \mid q' & = & (\overline{\sum}_{i,\overline{d}_l,t} a_i \text{‹} t\, p_i \lhd \alpha_i \rhd \delta \text{‹} \mathbf{0}) \mid \overline{\sum}_{j,\overline{e}_m,u} b_j \text{‹} u\, q_j \lhd \beta_j \rhd \delta \text{‹} \mathbf{0} \\ & \stackrel{3.8}{=} & \overline{\sum}_{i,j,\overline{d}_l,\overline{e}_m,t} (a_i \mid b_j) \text{‹} t\, (p_i \| q_j[t/u]) \lhd \alpha_i \wedge \beta_j[t/u] \rhd \delta \text{‹} \mathbf{0} + \\ & & \overline{\sum}_{i,j,\overline{d}_l,\overline{e}_m,t,u} \delta \text{‹} t \text{‹} u \lhd \alpha_i \wedge \beta_j \rhd \delta \text{‹} \mathbf{0} \\ & \stackrel{\text{A.4,i.h.,A.2.2}}{=} & \overline{\sum}_{j,i,\overline{e}_m,\overline{d}_l,t} (a_i \mid b_j) \text{‹} t\, (q_j[t/u] \| p_i) \lhd \beta_j[t/u] \wedge \alpha_i \rhd \delta \text{‹} \mathbf{0} + \\ & & \overline{\sum}_{j,i,\overline{e}_m,\overline{d}_l,u,t} \delta \text{‹} u \text{‹} t \lhd \beta_j \wedge \alpha_i \rhd \delta \text{‹} \mathbf{0}. \end{array}$$

Consider the first summand. By axiom CF and the commutativity of the communication function γ $a_i \mid b_j = b_j \mid a_i$. By α-conversion t may be renamed to u and u to t. Now $p' \mid q' = q' \mid p'$ follows readily, and the identities $p' \mid q'' = q'' \mid p'$, $p'' \mid q' = q' \mid p''$ and $p'' \mid q'' = q'' \mid p''$ follow in a similar way. (No application of the induction hypothesis is needed there.) Finally, using some simple manipulations, we may derive from $(*)$ that $p \mid q = q \mid p$. Commutativity of $\|$ now follows easily:

$$p \| q \stackrel{\text{CM1}}{=} p \lfloor\!\lfloor q + q \lfloor\!\lfloor p + p \mid q \stackrel{4.3.1}{=} q \lfloor\!\lfloor p + p \lfloor\!\lfloor q + q \mid p \stackrel{\text{CM1}}{=} q \| p.$$

□

Theorem 4.4 (*Associativity*). *It holds that:*

1. $(p \| q) \| r = p \| (q \| r)$;
2. $(p \mid q) \mid r = p \mid (q \mid r)$.

Proof.

1. See Appendix B;
2. (Sketch.) We distinguish two cases:
   (a) None of the processes p, q, r equals $\delta\text{`}\mathbf{0}$. By the axioms CM8 and CM9 we have that

$$\begin{aligned}(p \mid q) \mid r \quad = \quad & (p' \mid q') \mid r' + (p' \mid q') \mid r'' + (p' \mid q'') \mid r' + (p' \mid q'') \mid r'' + \\ & (p'' \mid q') \mid r' + (p'' \mid q') \mid r'' + (p'' \mid q'') \mid r' + (p'' \mid q'') \mid r''.\end{aligned}$$

   We consider $(p' \mid q') \mid r'$ and compress a few steps in the analysis. First Theorem 3.8 may be applied to $p' \mid q'$ (see the proof of Theorem 4.3), and then the δ-summand may be split in two subterms using Lemma A.1.12 and axiom SUM4. Next, according to axiom CM8 the communication with r' may be distributed over the three summands. (Where necessary, Lemma A.4 is used to modify the order of the summands.) Finally, Theorem 3.8 may be applied again. This results in:

$$\overline{\sum}_{i,j,k,\overline{d}_l,\overline{e}_m,\overline{f}_n,t}((a_i \mid b_j) \mid c_k)\text{`}t\,((p_i \,\|\, q_j[t/u]) \,\|\, r_k[t/v]) \lhd \alpha_i \wedge \beta_j[t/u] \wedge \nu_k[t/v] \rhd \delta\text{`}\mathbf{0} \; + \quad (R1)$$

$$\overline{\sum}_{i,j,k,\overline{d}_l,\overline{e}_m,\overline{f}_n,t,v}\delta\text{`}t\text{`}v \lhd \alpha_i \wedge \beta_j[t/u] \wedge \nu_k \rhd \delta\text{`}\mathbf{0} + \quad (R2)$$

$$\overline{\sum}_{i,j,k,\overline{d}_l,\overline{e}_m,\overline{f}_n,u,t}\;(\delta \mid c_k)\text{`}t\, r_k[t/v] \lhd t \leq u \wedge \alpha_i \wedge \beta_j \wedge \nu_k[t/v] \rhd \delta\text{`}\mathbf{0} + \quad (R3)$$

$$\overline{\sum}_{i,j,k,\overline{d}_l,\overline{e}_m,\overline{f}_n,u,t,v}\delta\text{`}t\text{`}v \lhd t \leq u \wedge \alpha_i \wedge \beta_j \wedge \nu_k \rhd \delta\text{`}\mathbf{0} + \quad (R4)$$

$$\overline{\sum}_{i,j,k,\overline{d}_l,\overline{e}_m,\overline{f}_n,t,u}(\delta \mid c_k)\text{`}u\, r_k[u/v] \lhd u \leq t \wedge \alpha_i \wedge \beta_j \wedge \nu_k[u/v] \rhd \delta\text{`}\mathbf{0} + \quad (R5)$$

$$\overline{\sum}_{i,j,k,\overline{d}_l,\overline{e}_m,\overline{f}_n,t,u,v}\delta\text{`}u\text{`}v \lhd u \leq t \wedge \alpha_i \wedge \beta_j \wedge \nu_k \rhd \delta\text{`}\mathbf{0} \quad (R6)$$

   By axiom CD1 and Lemma A.2.3 $(\delta \mid c_k)\text{`}t\, r_k[t/v] = \delta\text{`}t$ and $(\delta \mid c_k)\text{`}u\, r_k[u/v] = \delta\text{`}u$.
   By the associativity of the communication function γ, axiom CF and Theorem 4.4.1 $R1 = R1'$. A number of routine calculations give us the identities $R2 = R2', \ldots, R6 = R6'$.

$$\overline{\sum}_{i,j,k,\overline{d}_l,\overline{e}_m,\overline{f}_n,t}(a_i \mid (b_j \mid c_k))\text{`}t\,(p_i \,\|\, (q_j[t/u] \,\|\, r_k[t/v])) \lhd \alpha_i \wedge \beta_j[t/u] \wedge \nu_k[t/v] \rhd \delta\text{`}\mathbf{0} \; + \quad (R1')$$

$$\overline{\sum}_{i,j,k,\overline{d}_l,\overline{e}_m,\overline{f}_n,t,u,v}\delta\text{`}t\text{`}u\text{`}v \lhd eq(t,u) \wedge \alpha_i \wedge \beta_j \wedge \nu_k \rhd \delta\text{`}\mathbf{0} + \quad (R2')$$

$$\overline{\sum}_{i,j,k,\overline{d}_l,\overline{e}_m,\overline{f}_n,t,u,v}\delta\text{`}t\text{`}u\text{`}v \lhd t \leq u \wedge eq(t,v) \wedge \alpha_i \wedge \beta_j \wedge \nu_k \rhd \delta\text{`}\mathbf{0} + \quad (R3')$$

$$\overline{\sum}_{i,j,k,\overline{d}_l,\overline{e}_m,\overline{f}_n,t,u,v}\delta\text{`}t\text{`}u\text{`}v \lhd t \leq u \wedge \alpha_i \wedge \beta_j \wedge \nu_k \rhd \delta\text{`}\mathbf{0} + \quad (R4')$$

$$\overline{\sum}_{i,j,k,\overline{d}_l,\overline{e}_m,\overline{f}_n,t,u,v}\delta\text{`}t\text{`}u\text{`}v \lhd u \leq t \wedge eq(u,v) \wedge \alpha_i \wedge \beta_j \wedge \nu_k \rhd \delta\text{`}\mathbf{0} + \quad (R5')$$

$$\overline{\sum}_{i,j,k,\overline{d}_l,\overline{e}_m,\overline{f}_n,t,u,v}\delta\text{`}t\text{`}u\text{`}v \lhd u \leq t \wedge \alpha_i \wedge \beta_j \wedge \nu_k \rhd \delta\text{`}\mathbf{0} \quad (R6')$$

   We see that $R3' \subseteq R4'$ and $R5' \subseteq R6'$, so $R3' + R4' + R5' + R6' = R4' + R6' = R7$:

$$\overline{\sum}_{i,j,k,\overline{d}_l,\overline{e}_m,\overline{f}_n,t,u,v}\delta\text{`}t\text{`}u\text{`}v \lhd \alpha_i \wedge \beta_j \wedge \nu_k \rhd \delta\text{`}\mathbf{0} \quad (R7)$$

   Since $R2' \subseteq R7$ we have that $(p' \mid q') \mid r' = R1' + R7$.
   By commutativity of $\mid$ we have that $p' \mid (q' \mid r') = (q' \mid r') \mid p'$, so by symmetry it must be that the δ-summands of $p' \mid (q' \mid r')$ also add up to $R7$. Thus, after application of α-conversion to $R1'$ we may conclude that $(p' \mid q') \mid r' = p' \mid (q' \mid r')$.
   Associativity of the other subterms of $(p \mid q) \mid r$ follows in a similar way, and identity with $p \mid (q \mid r)$ follows readily;

   (b) For the various possibilities where one or more of the arguments p, q, r equals $\delta\text{`}\mathbf{0}$ it is easily proved that $(p \mid q) \mid r = \delta\text{`}\mathbf{0} = p \mid (q \mid r)$.

□

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A Elementary lemmas

In this appendix we provide a large number of elementary lemmas needed for proving our main results.

Lemma A.1 (*Laws for Conditional Expressions*)**.** *It holds that:*

1. $x \lhd b \rhd x = x$;
2. $(x \lhd b \rhd y)^{\varsigma}t = x^{\varsigma}t \lhd b \rhd y^{\varsigma}t$;
3. $(x \lhd b \rhd y)\, z = x\, z \lhd b \rhd y\, z$;
4. $x \lhd b \rhd y = x \lhd b \rhd \delta^{\varsigma}\mathbf{0} + y \lhd \neg b \rhd \delta^{\varsigma}\mathbf{0}$;
5. $(x + y) \lhd b \rhd z = x \lhd b \rhd z + y \lhd b \rhd z$;
6. $(x \lhd b \rhd \delta^{\varsigma}\mathbf{0}) \lhd c \rhd \delta^{\varsigma}\mathbf{0} = x \lhd b \wedge c \rhd \delta^{\varsigma}\mathbf{0}$;
7. $x \lhd b \vee c \rhd \delta^{\varsigma}\mathbf{0} = x \lhd b \rhd \delta^{\varsigma}\mathbf{0} + x \lhd c \rhd \delta^{\varsigma}\mathbf{0}$;
8. $(x \lhd b \rhd y) \lhd c \rhd \delta^{\varsigma}\mathbf{0} = x \lhd b \wedge c \rhd \delta^{\varsigma}\mathbf{0} + y \lhd \neg b \wedge c \rhd \delta^{\varsigma}\mathbf{0}$;
9. $x \lhd c \rhd \delta^{\varsigma}\mathbf{0} = x \lhd b \wedge c \rhd \delta^{\varsigma}\mathbf{0} + x \lhd \neg b \wedge c \rhd \delta^{\varsigma}\mathbf{0}$;
10. $x \lhd b \rhd \delta^{\varsigma}\mathbf{0} = x \lhd t \leq u \wedge b \rhd \delta^{\varsigma}\mathbf{0} + x \lhd u \leq t \wedge b \rhd \delta^{\varsigma}\mathbf{0}$;
11. $\delta^{\varsigma}t^{\varsigma}u \lhd t \leq u \wedge b \rhd \delta^{\varsigma}\mathbf{0} = \delta^{\varsigma}t \lhd t \leq u \wedge b \rhd \delta^{\varsigma}\mathbf{0}$;
12. $\delta^{\varsigma}t^{\varsigma}u \lhd b \rhd \delta^{\varsigma}\mathbf{0} = \delta^{\varsigma}t \lhd t \leq u \wedge b \rhd \delta^{\varsigma}\mathbf{0} + \delta^{\varsigma}u \lhd u \leq t \wedge b \rhd \delta^{\varsigma}\mathbf{0}$;
13. $x \ll (y \lhd b \rhd z) = x \ll y \lhd b \rhd x \ll z$;
14. $(x \lhd b \rhd y) \mathbin{\|\!\lfloor} z = (x \mathbin{\|\!\lfloor} z) \lhd b \rhd (y \mathbin{\|\!\lfloor} z)$;
15. $(x \lhd b \rhd y) \mid z = (x \mid z) \lhd b \rhd (y \mid z)$;
16. $x \mid (y \lhd b \rhd z) = (x \mid y) \lhd b \rhd (x \mid z)$;
17. $\delta^{\varsigma}t^{\varsigma}u \lhd eq(t,u) \wedge b \rhd \delta^{\varsigma}\mathbf{0} = \delta^{\varsigma}t \lhd eq(t,u) \wedge b \rhd \delta^{\varsigma}\mathbf{0}$.

Proof. By case distinction on the booleans. □

Lemma A.2 (*Simple Process Identities*)**.** *It holds that:*

1. $\delta^{\varsigma}\mathbf{0}^{\varsigma}t = \delta^{\varsigma}\mathbf{0}$;
2. $a^{\varsigma}t^{\varsigma}u = a^{\varsigma}u^{\varsigma}t$;
3. $\delta^{\varsigma}t\, x = \delta^{\varsigma}t$;
4. $a^{\varsigma}t^{\varsigma}t = a^{\varsigma}t$;
5. $a^{\varsigma}t\, x = a^{\varsigma}t\, x + \delta^{\varsigma}t$;
6. $a \ll \delta^{\varsigma}\mathbf{0} = a^{\varsigma}\mathbf{0}$;
7. $a^{\varsigma}t\, x \mathbin{\|\!\lfloor} y = (a \ll y)^{\varsigma}t\, (x \parallel y)$;
8. $\delta^{\varsigma}t \mid a^{\varsigma}u\, x = \delta^{\varsigma}min(t,u)$;

9. $a\text{‘}t\, x \mid \delta\text{‘}u = \delta\text{‘}min(t,u)$;

10. $\delta\text{‘}t \mid \delta\text{‘}u = \delta\text{‘}min(t,u)$.

Proof. Easy. $\square$

Lemma A.3 (*Simple Laws with Sum Operators*). *It holds that:*

1. $(\sum X) \triangleleft b \triangleright \delta\text{‘}\mathbf{0} = \sum X \triangleleft b \triangleright \delta\text{‘}\mathbf{0}$;
2. $(\sum X \triangleleft b \triangleright \delta\text{‘}\mathbf{0})\text{‘}t = \sum X\text{‘}t \triangleleft b \triangleright \delta\text{‘}\mathbf{0}$;
3. $(\sum X \triangleleft b \triangleright \delta\text{‘}\mathbf{0}) \triangleleft c \triangleright \delta\text{‘}\mathbf{0} = \sum X \triangleleft b \wedge c \triangleright \delta\text{‘}\mathbf{0}$.

Proof.

1. By case distinction on b;
2. By axiom ATB5 and the lemmas A.1.2 and A.2.1;
3. By the lemmas A.3.1 and A.1.6.

$\square$

Lemma A.4 (*Sum Permutation*). *Let $\overline{\mathcal{P}(d)}_n$ be some permutation of a finite sequence of data variables $\overline{d}_n$. It holds that:*

$$\overline{\sum}_{\overline{\mathcal{P}(d)}_n} r = \overline{\sum}_{\overline{d}_n} r.$$

Proof. For $n = 0$ or 1 the proof is trivial. For $n \geq 2$ we first show that two adjacent summations may interchanged. We prove

$$\sum_{d_k:D_k} \sum_{d_{k+1}:D_{k+1}} p = \sum_{d_{k+1}:D_{k+1}} \sum_{d_k:D_k} p \quad (*)$$

where $k = 1, \ldots, n-1$, by Summand Inclusion.
$\subseteq$: Consider the r.h.s. of the identity to be proved. Let $e \notin FV(p)$.

$$\sum_{d_{k+1}:D_{k+1}} \sum_{d_k:D_k} p \overset{\text{SUM3,SUM4}}{=} \sum_{d_{k+1}:D_{k+1}} \sum_{d_k:D_k} p + \sum_{d_{k+1}:D_{k+1}} p[e/d_k].$$

Next we apply axiom SUM11, which adds $\sum_{e:D_k}$ to both the l.h.s. and r.h.s. of the above equation. Application of SUM4 and SUM1 yields

$$\sum_{d_{k+1}:D_{k+1}} \sum_{d_k:D_k} p = \sum_{d_{k+1}:D_{k+1}} \sum_{d_k:D_k} p + \sum_{e:D_k} \sum_{d_{k+1}:D_{k+1}} p[e/d_k].$$

By α-conversion e may be renamed to d_k, which proves this case.
$\supseteq$: By symmetry.

It is easily understood that any two adjacent summations may change place in expressions with $n \geq 3$: Any term $\overline{\sum}_{\overline{d}_n} r$ may be written in the form $\overline{\sum}_{\overline{d}_k} \sum_{d_{k+1}:D_{k+1}} \sum_{d_{k+2}:D_{k+2}} p$ $(k = 0, \ldots, n-2)$, so that (i) can be applied to the subterm $\sum_{d_{k+1}:D_{k+1}} \sum_{d_{k+2}:D_{k+2}} p$ in order to change the two outermost summands.

Finally, if any two adjacent elements in a finite sequence $\overline{d}_n$ may be interchanged, any permutation $\overline{\mathcal{P}(d)}_n$ of $\overline{d}_n$ can be constructed in a finite number of swaps. $\square$

Lemma A.5 (*Sum Elimination*). *It holds that:*

1. $\sum_{d:D} p \lhd eq(d,e) \rhd \delta^{\varsigma}\mathbf{0} = p[e/d]$;
2. *Let* $d \notin FV(p)$. *If* $b[e/d] = \mathtt{t}$ *then* $\sum_{d:D} p \lhd b \rhd \delta^{\varsigma}\mathbf{0} = p$;
3. *If* $u \leq v = \mathtt{t}$ *then* $\sum_{t:\mathbf{Time}} \delta^{\varsigma}t \lhd u \leq t \leq v \rhd \delta^{\varsigma}\mathbf{0} = \delta^{\varsigma}v$;
   (note that always $0 \leq v = \mathtt{t}$.*)*
4. $\sum_{t:\mathbf{Time}} a^{\varsigma}t^{\varsigma}u \lhd t \leq v \rhd \delta^{\varsigma}\mathbf{0} = a^{\varsigma}u \lhd u \leq v \rhd \delta^{\varsigma}\mathbf{0} + \delta^{\varsigma}u^{\varsigma}v$.

Proof.

1. By Summand Inclusion, see [5];
2. By Summand Inclusion.
   $\subseteq$: $p \overset{\text{SUM1}}{=} \sum_{d:D} p = \sum_{d:D} p \lhd b \vee \neg b \rhd \delta^{\varsigma}\mathbf{0} \overset{\text{A.1.7,SUM4}}{=} \sum_{d:D} p \lhd b \rhd \delta^{\varsigma}\mathbf{0} + \sum_{d:D} p \lhd \neg b \rhd \delta^{\varsigma}\mathbf{0}$.
   $\supseteq$: $\sum_{d:D} p \lhd b \rhd \delta^{\varsigma}\mathbf{0} \overset{\text{SUM3}}{=} \sum_{d:D} p \lhd b \rhd \delta^{\varsigma}\mathbf{0} + p[e/d] \lhd b[e/d] \rhd \delta^{\varsigma}\mathbf{0}$.
   By assumption $p[e/d] = p$ and $b[e/d] = \mathtt{t}$, which proves the case;
3. By Summand Inclusion. First we prove $\subseteq$:

$$\begin{array}{lll} \delta^{\varsigma}v & \overset{\text{ATA2}}{=} & \delta^{\varsigma}v + \delta^{\varsigma}t \lhd (u \leq t \vee t \leq u) \wedge t \leq v \rhd \delta^{\varsigma}\mathbf{0} \\ & \overset{\text{A.1.7}}{=} & \delta^{\varsigma}v + \delta^{\varsigma}t \lhd u \leq t \leq v \rhd \delta^{\varsigma}\mathbf{0} + \delta^{\varsigma}t \lhd t \leq u \wedge t \leq v \rhd \delta^{\varsigma}\mathbf{0}. \end{array}$$

   After applying $^{\varsigma}t$ to both sides of the above equation, and by SUM11 we find that

$$\textstyle\sum_{t:\mathbf{Time}} \delta^{\varsigma}v^{\varsigma}t = \sum_{t:\mathbf{Time}} (\delta^{\varsigma}v + \delta^{\varsigma}t \lhd u \leq t \leq v \rhd \delta^{\varsigma}\mathbf{0} + \delta^{\varsigma}t \lhd t \leq u \wedge t \leq v \rhd \delta^{\varsigma}\mathbf{0})^{\varsigma}t.$$

   Application of ATB3, SUM4 and ATA1 leads to:

$$\begin{array}{lll} \delta^{\varsigma}v & = & \delta^{\varsigma}v + \sum_{t:\mathbf{Time}} (\delta^{\varsigma}t \lhd u \leq t \leq v \rhd \delta^{\varsigma}\mathbf{0})^{\varsigma}t + \\ & & \sum_{t:\mathbf{Time}} (\delta^{\varsigma}t \lhd t \leq u \wedge t \leq v \rhd \delta^{\varsigma}\mathbf{0})^{\varsigma}t \\ & \overset{\text{A.1.2,A.2.4,A.2.1}}{=} & \delta^{\varsigma}v + \sum_{t:\mathbf{Time}} \delta^{\varsigma}t \lhd u \leq t \leq v \rhd \delta^{\varsigma}\mathbf{0} + \\ & & \sum_{t:\mathbf{Time}} \delta^{\varsigma}t \lhd t \leq u \wedge t \leq v \rhd \delta^{\varsigma}\mathbf{0}. \end{array}$$

   $\supseteq$: By SUM3 $\sum_{t:\mathbf{Time}} \delta^{\varsigma}t \lhd u \leq t \leq v \rhd \delta^{\varsigma}\mathbf{0} \supseteq (\delta^{\varsigma}t \lhd u \leq t \leq v \rhd \delta^{\varsigma}\mathbf{0})[v/t]$. By assumption $u \leq v = \mathtt{t}$, which proves the case;
4. We have that:

$$\begin{array}{ll} \multicolumn{2}{l}{\sum_{t:\mathbf{Time}} a^{\varsigma}t^{\varsigma}u \lhd t \leq v \rhd \delta^{\varsigma}\mathbf{0}} \\ \overset{\text{ATB2}}{=} & \sum_{t:\mathbf{Time}} (a^{\varsigma}t \lhd eq(t,u) \rhd \delta^{\varsigma}min(t,u)) \lhd t \leq v \rhd \delta^{\varsigma}\mathbf{0} \\ \overset{\text{A.1.8}}{=} & \sum_{t:\mathbf{Time}} (a^{\varsigma}t \lhd eq(t,u) \wedge t \leq v \rhd \delta^{\varsigma}\mathbf{0} + \\ & \quad \delta^{\varsigma}min(t,u) \lhd \neg eq(t,u) \wedge t \leq v \rhd \delta^{\varsigma}\mathbf{0}) \\ \overset{\text{ATA2,A.1.5}}{=} & \sum_{t:\mathbf{Time}} (a^{\varsigma}t \lhd eq(t,u) \wedge t \leq v \rhd \delta^{\varsigma}\mathbf{0} + \\ & \quad \delta^{\varsigma}t \lhd eq(t,u) \wedge t \leq v \rhd \delta^{\varsigma}\mathbf{0} + \\ & \quad \delta^{\varsigma}min(t,u) \lhd \neg eq(t,u) \wedge t \leq v \rhd \delta^{\varsigma}\mathbf{0}) \\ \overset{\text{A.1.17,ATB1}}{=} & \sum_{t:\mathbf{Time}} (a^{\varsigma}t \lhd eq(t,u) \wedge t \leq v \rhd \delta^{\varsigma}\mathbf{0} + \\ & \quad \delta^{\varsigma}min(t,u) \lhd eq(t,u) \wedge t \leq v \rhd \delta^{\varsigma}\mathbf{0} + \\ & \quad \delta^{\varsigma}min(t,u) \lhd \neg eq(t,u) \wedge t \leq v \rhd \delta^{\varsigma}\mathbf{0}) \end{array}$$

$$
\begin{array}{cl}
\overset{\text{A.1.9,SUM4,ATB1}}{=} & \sum_{t:\mathbf{Time}} a\cdot t \lhd eq(t,u) \wedge t \leq v \rhd \delta\cdot\mathbf{0} \;+ \sum_{t:\mathbf{Time}} \delta\cdot t\cdot u \lhd t \leq v \rhd \delta\cdot\mathbf{0} \\
\overset{\text{A.1.6,A.5.1}}{=} & a\cdot u \lhd u \leq v \rhd \delta\cdot\mathbf{0} \;+ \sum_{t:\mathbf{Time}} \delta\cdot t\cdot u \lhd t \leq v \rhd \delta\cdot\mathbf{0} \\
\overset{\text{A.3.2}}{=} & a\cdot u \lhd u \leq v \rhd \delta\cdot\mathbf{0} + (\sum_{t:\mathbf{Time}} \delta\cdot t \lhd t \leq v \rhd \delta\cdot\mathbf{0})\cdot u \\
\overset{\text{A.5.3,A.2.2}}{=} & a\cdot u \lhd u \leq v \rhd \delta\cdot\mathbf{0} + \delta\cdot u\cdot v.
\end{array}
$$

□

B A proof of the associativity of the merge operator

In the parts A, B and C we prove the associativity of $\|$ for arguments p, q, r, all not equal to $\delta\cdot\mathbf{0}$. In this case we may restrict ourselves to the normal formulation of the Expansion Theorem (case 3.11.1). In part D we consider the various $\delta\cdot\mathbf{0}$-cases.

The first part of this proof is carried out by induction on the depth $|p| + |q| + |r|$, where we assume that $|p|, |q|$ and $|r|$ are at least equal to 1. As induction hypothesis we assume $(\tilde{p} \,\|\, \tilde{q}) \,\|\, \tilde{r} = \tilde{p} \,\|\, (\tilde{q} \,\|\, \tilde{r})$ for all $|\tilde{p}| + |\tilde{q}| + |\tilde{r}| < |p| + |q| + |r|$.

First the base case, where $|p| = |q| = |r| = 1$, has to be proved. In essence, this part of the proof is similar to the proof given below, but with the sets I, J, K put to $\emptyset$, so that no application of the i.h. would be necessary. (No term with three parallel processes occurs at the r.h.s. of the the expansions.) We omit this part of the proof. The general case, where $|p| + |q| + |r| > 3$ is proven below in detail.

A. Expansion of $(p \,\|\, q) \,\|\, r$. By the Expansion Theorem $p \,\|\, q$ follows easily. By Lemma A.1.7 the generalised disjunctions in the conditions may be converted to summations, and by axiom SUM1 various $\sum$-operators may be added. By Lemma A.4 the sums may be rearranged arbitrarily, so that

$$
\begin{array}{rcll}
p \,\|\, q & = & \overline{\sum}_{i,j^*,\overline{d^*}_{l^*},\overline{e^*}_{m^*},u,t} a_i\cdot t\,(p_i \,\|\, q) \lhd t \leq u \wedge \alpha_i \wedge \beta_{j^*} \rhd \delta\cdot\mathbf{0} + & (M1) \\
& & \overline{\sum}_{i',j^*,\overline{d^*}_{l^*},\overline{e^*}_{m^*},u,t} a_{i'}\cdot t\, q \lhd t \leq u \wedge \alpha_{i'} \wedge \beta_{j^*} \rhd \delta\cdot\mathbf{0} + & (M2) \\
& & \overline{\sum}_{i^*,j,\overline{d^*}_{l^*},\overline{e^*}_{m^*},t,u} b_j\cdot u\,(q_j \,\|\, p) \lhd u \leq t \wedge \alpha_{i^*} \wedge \beta_j \rhd \delta\cdot\mathbf{0} + & (M3) \\
& & \overline{\sum}_{i^*,j',\overline{d^*}_{l^*},\overline{e^*}_{m^*},t,u} b_{j'}\cdot u\, p \lhd u \leq t \wedge \alpha_{i^*} \wedge \beta_{j'} \rhd \delta\cdot\mathbf{0} + & (M4) \\
& & \overline{\sum}_{i,j,\overline{d^*}_{l^*},\overline{e^*}_{m^*},t} (a_i \mid b_j)\cdot t\,(p_i \,\|\, q_j[t/u]) \lhd \alpha_i \wedge \beta_j[t/u] \rhd \delta\cdot\mathbf{0} + & (M5) \\
& & \overline{\sum}_{i',j,\overline{d^*}_{l^*},\overline{e^*}_{m^*},t} (a_{i'} \mid b_j)\cdot t\, q_j[t/u] \lhd \alpha_{i'} \wedge \beta_j[t/u] \rhd \delta\cdot\mathbf{0} + & (M6) \\
& & \overline{\sum}_{i,j',\overline{d^*}_{l^*},\overline{e^*}_{m^*},t} (a_i \mid b_{j'})\cdot t\, p_i \lhd \alpha_i \wedge \beta_{j'}[t/u] \rhd \delta\cdot\mathbf{0} + & (M7) \\
& & \overline{\sum}_{i',j',\overline{d^*}_{l^*},\overline{e^*}_{m^*},t} (a_{i'} \mid b_{j'})\cdot t \lhd \alpha_{i'} \wedge \beta_{j'}[t/u] \rhd \delta\cdot\mathbf{0} & (M8)
\end{array}
$$

Each main summand Mn has indices i, i', i^*, j, j', j^*, which may be combined into a single index $\xi_n \in \Xi_n$, where $n = 1, \ldots, 8$. Let $\Xi \overset{\text{def}}{=} \bigcup_{n=1,\ldots,7} \Xi_n$, and $\Xi' \overset{\text{def}}{=} \Xi_8$. By convention, $\Xi^* = \Xi \cup \Xi'$. To the summands $M3$ and $M4$ α-conversion may be applied, so that we finally obtain the following simple abbreviation:

$$
p \,\|\, q \overset{\text{abbr.}}{=} \overline{\sum}_{\xi,\overline{d^*}_{l^*},\overline{e^*}_{m^*},u,t} \tilde{a}_\xi\cdot t\, \tilde{p}_\xi \lhd \tilde{\alpha}_\xi \rhd \delta\cdot\mathbf{0} + \overline{\sum}_{\xi',\overline{d^*}_{l^*},\overline{e^*}_{m^*},u,t} \tilde{a}_{\xi'}\cdot t \lhd \tilde{\alpha}_{\xi'} \rhd \delta\cdot\mathbf{0}.
$$

Again the Expansion Theorem may be applied. Let $\overline{d}_\chi \overset{\text{def}}{=} \overline{d^*}_{l^*}, \overline{e^*}_{m^*}, \overline{f^*}_{n^*}$.

$$
\begin{array}{rcll}
(p \,\|\, q) \,\|\, r & = & \overline{\sum}_{\xi,k^*,\overline{d}_\chi,t,u,v} \tilde{a}_\xi\cdot t\,(\tilde{p}_\xi \,\|\, r) \lhd t \leq v \wedge \tilde{\alpha}_\xi \wedge \nu_{k^*} \rhd \delta\cdot\mathbf{0} + & (N1) \\
& & \overline{\sum}_{\xi',k^*,\overline{d}_\chi,t,u,v} \tilde{a}_{\xi'}\cdot t\, r \lhd t \leq v \wedge \tilde{\alpha}_{\xi'} \wedge \nu_{k^*} \rhd \delta\cdot\mathbf{0} + & (N2) \\
& & \overline{\sum}_{\xi^*,k,\overline{d}_\chi,t,u,v} c_k\cdot v\,(r_k \,\|\, (p \,\|\, q)) \lhd v \leq t \wedge \tilde{\alpha}_{\xi^*} \wedge \nu_k \rhd \delta\cdot\mathbf{0} + & (N3) \\
& & \overline{\sum}_{\xi^*,k',\overline{d}_\chi,t,u,v} c_{k'}\cdot v\,(p \,\|\, q) \lhd v \leq t \wedge \tilde{\alpha}_{\xi^*} \wedge \nu_{k'} \rhd \delta\cdot\mathbf{0} + & (N4)
\end{array}
$$

$$
\begin{array}{ll}
\overline{\sum}_{\xi,k,\overline{d}_\chi,t,u}(\tilde{a}_\xi \mid c_k)\text{‹}t\,(\tilde{p}_\xi \,\|\, r_k[t/v]) \lhd \tilde{\alpha}_\xi \wedge \nu_k[t/v] \rhd \delta\text{‹}\mathbf{0}\,+ & (N5)\\
\overline{\sum}_{\xi',k,\overline{d}_\chi,t,u}(\tilde{a}_{\xi'} \mid c_k)\text{‹}t\, r_k[t/v] \lhd \tilde{\alpha}_{\xi'} \wedge \nu_k[t/v] \rhd \delta\text{‹}\mathbf{0}\,+ & (N6)\\
\overline{\sum}_{\xi,k',\overline{d}_\chi,t,u}(\tilde{a}_\xi \mid c_{k'})\text{‹}t\, \tilde{p}_\xi \lhd \tilde{\alpha}_\xi \wedge \nu_{k'}[t/v] \rhd \delta\text{‹}\mathbf{0}\,+ & (N7)\\
\overline{\sum}_{\xi',k',\overline{d}_\chi,t,u}(\tilde{a}_{\xi'} \mid c_{k'})\text{‹}t \lhd \tilde{\alpha}_{\xi'} \wedge \nu_{k'}[t/v] \rhd \delta\text{‹}\mathbf{0} & (N8)
\end{array}
$$

First we examine the summand $N1 + N2$:

$$
\begin{array}{ll}
\overline{\sum}_{i,j^*,k^*,\overline{d}_\chi,t,u,v} a_i\text{‹}t\,((p_i \,\|\, q) \,\|\, r) \lhd t \leq v \wedge t \leq u \wedge \alpha_i \wedge \beta_{j^*} \wedge \nu_{k^*} \rhd \delta\text{‹}\mathbf{0}\,+ & (T1)\\
\overline{\sum}_{i',j^*,k^*,\overline{d}_\chi,t,u,v} a_{i'}\text{‹}t\,(q \,\|\, r) \lhd t \leq v \wedge t \leq u \wedge \alpha_{i'} \wedge \beta_{j^*} \wedge \nu_{k^*} \rhd \delta\text{‹}\mathbf{0}\,+ & (T2)\\
\overline{\sum}_{i^*,j,k^*,\overline{d}_\chi,t,u,v} b_j\text{‹}u\,((q_j \,\|\, p) \,\|\, r) \lhd u \leq v \wedge u \leq t \wedge \alpha_{i^*} \wedge \beta_j \wedge \nu_{k^*} \rhd \delta\text{‹}\mathbf{0}\,+ & (T3)\\
\overline{\sum}_{i^*,j',k^*,\overline{d}_\chi,t,u,v} b_{j'}\text{‹}u\,(p \,\|\, r) \lhd u \leq v \wedge u \leq t \wedge \alpha_{i^*} \wedge \beta_{j'} \wedge \nu_{k^*} \rhd \delta\text{‹}\mathbf{0}\,+ & (T4)\\
\overline{\sum}_{i,j,k^*,\overline{d}_\chi,t,v}(a_i \mid b_j)\text{‹}t\,((p_i \,\|\, q_j[t/u]) \,\|\, r) \lhd t \leq v \wedge \alpha_i \wedge \beta_j[t/u] \wedge \nu_{k^*} \rhd \delta\text{‹}\mathbf{0}\,+ & (T5)\\
\overline{\sum}_{i',j,k^*,\overline{d}_\chi,t,v}(a_{i'} \mid b_j)\text{‹}t\,(q_j[t/u] \,\|\, r) \lhd t \leq v \wedge \alpha_{i'} \wedge \beta_j[t/u] \wedge \nu_{k^*} \rhd \delta\text{‹}\mathbf{0}\,+ & (T6)\\
\overline{\sum}_{i,j',k^*,\overline{d}_\chi,t,v}(a_i \mid b_{j'})\text{‹}t\,(p_i \,\|\, r) \lhd t \leq v \wedge \alpha_i \wedge \beta_{j'}[t/u] \wedge \nu_{k^*} \rhd \delta\text{‹}\mathbf{0}\,+ & (T7)\\
\overline{\sum}_{i',j',k^*,\overline{d}_\chi,t,v}(a_{i'} \mid b_{j'})\text{‹}t\, r \lhd t \leq v \wedge \alpha_{i'} \wedge \beta_{j'}[t/u] \wedge \nu_{k^*} \rhd \delta\text{‹}\mathbf{0} & (T8)
\end{array}
$$

Note that in the summands $T3$ and $T4$ α-conversion is applied in order to undo the α-conversion we applied earlier to $M3$ and $M4$.

We continue with $N3$:

$$
\begin{array}{ll}
\overline{\sum}_{i,j^*,k,\overline{d}_\chi,t,u,v} c_k\text{‹}v\,(r_k \,\|\, (p \,\|\, q)) \lhd v \leq t \wedge t \leq u \wedge \alpha_i \wedge \beta_{j^*} \wedge \nu_k \rhd \delta\text{‹}\mathbf{0}\,+ & (T9)\\
\overline{\sum}_{i',j^*,k,\overline{d}_\chi,t,u,v} c_k\text{‹}v\,(r_k \,\|\, (p \,\|\, q)) \lhd v \leq t \wedge t \leq u \wedge \alpha_{i'} \wedge \beta_{j^*} \wedge \nu_k \rhd \delta\text{‹}\mathbf{0}\,+ & (T10)\\
\overline{\sum}_{i^*,j,k,\overline{d}_\chi,t,u,v} c_k\text{‹}v\,(r_k \,\|\, (p \,\|\, q)) \lhd v \leq u \wedge u \leq t \wedge \alpha_{i^*} \wedge \beta_j \wedge \nu_k \rhd \delta\text{‹}\mathbf{0}\,+ & (T11)\\
\overline{\sum}_{i^*,j',k,\overline{d}_\chi,t,u,v} c_k\text{‹}v\,(r_k \,\|\, (p \,\|\, q)) \lhd v \leq u \wedge u \leq t \wedge \alpha_{i^*} \wedge \beta_{j'} \wedge \nu_k \rhd \delta\text{‹}\mathbf{0}\,+ & (T12)\\
\overline{\sum}_{i,j,k,\overline{d}_\chi,t,v} c_k\text{‹}v\,(r_k \,\|\, (p \,\|\, q)) \lhd v \leq t \wedge \alpha_i \wedge \beta_j[t/u] \wedge \nu_k \rhd \delta\text{‹}\mathbf{0}\,+ & (T13)\\
\overline{\sum}_{i',j,k,\overline{d}_\chi,t,v} c_k\text{‹}v\,(r_k \,\|\, (p \,\|\, q)) \lhd v \leq t \wedge \alpha_{i'} \wedge \beta_j[t/u] \wedge \nu_k \rhd \delta\text{‹}\mathbf{0}\,+ & (T14)\\
\overline{\sum}_{i,j',k,\overline{d}_\chi,t,v} c_k\text{‹}v\,(r_k \,\|\, (p \,\|\, q)) \lhd v \leq t \wedge \alpha_i \wedge \beta_{j'}[t/u] \wedge \nu_k \rhd \delta\text{‹}\mathbf{0}\,+ & (T15)\\
\overline{\sum}_{i',j',k,\overline{d}_\chi,t,v} c_k\text{‹}v\,(r_k \,\|\, (p \,\|\, q)) \lhd v \leq t \wedge \alpha_{i'} \wedge \beta_{j'}[t/u] \wedge \nu_k \rhd \delta\text{‹}\mathbf{0} & (T16)
\end{array}
$$

α-conversion was applied to $T11$ and $T12$.

$N4$:

$$
\begin{array}{ll}
\overline{\sum}_{i,j^*,k',\overline{d}_\chi,t,u,v} c_{k'}\text{‹}v\,(p \,\|\, q) \lhd v \leq t \wedge t \leq u \wedge \alpha_i \wedge \beta_{j^*} \wedge \nu_{k'} \rhd \delta\text{‹}\mathbf{0}\,+ & (T17)\\
\overline{\sum}_{i',j^*,k',\overline{d}_\chi,t,u,v} c_{k'}\text{‹}v\,(p \,\|\, q) \lhd v \leq t \wedge t \leq u \wedge \alpha_{i'} \wedge \beta_{j^*} \wedge \nu_{k'} \rhd \delta\text{‹}\mathbf{0}\,+ & (T18)\\
\overline{\sum}_{i^*,j,k',\overline{d}_\chi,t,u,v} c_{k'}\text{‹}v\,(p \,\|\, q) \lhd v \leq u \wedge u \leq t \wedge \alpha_{i^*} \wedge \beta_j \wedge \nu_{k'} \rhd \delta\text{‹}\mathbf{0}\,+ & (T19)\\
\overline{\sum}_{i^*,j',k',\overline{d}_\chi,t,u,v} c_{k'}\text{‹}v\,(p \,\|\, q) \lhd v \leq u \wedge u \leq t \wedge \alpha_{i^*} \wedge \beta_{j'} \wedge \nu_{k'} \rhd \delta\text{‹}\mathbf{0}\,+ & (T20)\\
\overline{\sum}_{i,j,k',\overline{d}_\chi,t,v} c_{k'}\text{‹}v\,(p \,\|\, q) \lhd v \leq t \wedge \alpha_i \wedge \beta_j[t/u] \wedge \nu_{k'} \rhd \delta\text{‹}\mathbf{0}\,+ & (T21)\\
\overline{\sum}_{i',j,k',\overline{d}_\chi,t,v} c_{k'}\text{‹}v\,(p \,\|\, q) \lhd v \leq t \wedge \alpha_{i'} \wedge \beta_j[t/u] \wedge \nu_{k'} \rhd \delta\text{‹}\mathbf{0}\,+ & (T22)\\
\overline{\sum}_{i,j',k',\overline{d}_\chi,t,v} c_{k'}\text{‹}v\,(p \,\|\, q) \lhd v \leq t \wedge \alpha_i \wedge \beta_{j'}[t/u] \wedge \nu_{k'} \rhd \delta\text{‹}\mathbf{0}\,+ & (T23)\\
\overline{\sum}_{i',j',k',\overline{d}_\chi,t,v} c_{k'}\text{‹}v\,(p \,\|\, q) \lhd v \leq t \wedge \alpha_{i'} \wedge \beta_{j'}[t/u] \wedge \nu_{k'} \rhd \delta\text{‹}\mathbf{0} & (T24)
\end{array}
$$

α-conversion was applied to $T19$ and $T20$.

$N5 + N6$:

$$
\begin{aligned}
&\overline{\sum}_{i,j^*,k,\overline{d}_\chi,t,u}(a_i \mid c_k){}^{\backprime}t\,((p_i \,\|\, q) \,\|\, r_k[t/v]) \lhd t \leq u \wedge \alpha_i \wedge \beta_{j^*} \wedge \nu_k[t/v] \rhd \delta{}^{\backprime}\mathbf{0} + && (T25)\\
&\overline{\sum}_{i',j^*,k,\overline{d}_\chi,t,u}(a_{i'} \mid c_k){}^{\backprime}t\,(q \,\|\, r_k[t/v]) \lhd t \leq u \wedge \alpha_{i'} \wedge \beta_{j^*} \wedge \nu_k[t/v] \rhd \delta{}^{\backprime}\mathbf{0} + && (T26)\\
&\overline{\sum}_{i^*,j,k,\overline{d}_\chi,t,u}(b_j \mid c_k){}^{\backprime}u\,((q_j \,\|\, p) \,\|\, r_k[u/v]) \lhd u \leq t \wedge \alpha_{i^*} \wedge \beta_j \wedge \nu_k[u/v] \rhd \delta{}^{\backprime}\mathbf{0} + && (T27)\\
&\overline{\sum}_{i^*,j',k,\overline{d}_\chi,t,u}(b_{j'} \mid c_k){}^{\backprime}u\,(p \,\|\, r_k[u/v]) \lhd u \leq t \wedge \alpha_{i^*} \wedge \beta_{j'} \wedge \nu_k[u/v] \rhd \delta{}^{\backprime}\mathbf{0} + && (T28)\\
&\overline{\sum}_{i,j,k,\overline{d}_\chi,t}((a_i \mid b_j) \mid c_k){}^{\backprime}t\,((p_i \,\|\, q_j[t/u]) \,\|\, r_k[t/v]) \lhd \alpha_i \wedge \beta_j[t/u] \wedge \nu_k[t/v] \rhd \delta{}^{\backprime}\mathbf{0} + && (T29)\\
&\overline{\sum}_{i',j,k,\overline{d}_\chi,t}((a_{i'} \mid b_j) \mid c_k){}^{\backprime}t\,(q_j[t/u] \,\|\, r_k[t/v]) \lhd \alpha_{i'} \wedge \beta_j[t/u] \wedge \nu_k[t/v] \rhd \delta{}^{\backprime}\mathbf{0} + && (T30)\\
&\overline{\sum}_{i,j',k,\overline{d}_\chi,t}((a_i \mid b_{j'}) \mid c_k){}^{\backprime}t\,(p_i \,\|\, r_k[t/v]) \lhd \alpha_i \wedge \beta_{j'}[t/u] \wedge \nu_k[t/v] \rhd \delta{}^{\backprime}\mathbf{0} + && (T31)\\
&\overline{\sum}_{i',j',k,\overline{d}_\chi,t}((a_{i'} \mid b_{j'}) \mid c_k){}^{\backprime}t\,r_k[t/v] \lhd \alpha_{i'} \wedge \beta_{j'}[t/u] \wedge \nu_k[t/v] \rhd \delta{}^{\backprime}\mathbf{0} && (T32)
\end{aligned}
$$

α-conversion was applied to $T27$ and $T28$.

$N7 + N8$:

$$
\begin{aligned}
&\overline{\sum}_{i,j^*,k',\overline{d}_\chi,t,u}(a_i \mid c_{k'}){}^{\backprime}t\,(p_i \,\|\, q) \lhd t \leq u \wedge \alpha_i \wedge \beta_{j^*} \wedge \nu_{k'}[t/v] \rhd \delta{}^{\backprime}\mathbf{0} + && (T33)\\
&\overline{\sum}_{i',j^*,k',\overline{d}_\chi,t,u}(a_{i'} \mid c_{k'}){}^{\backprime}t\,q \lhd t \leq u \wedge \alpha_{i'} \wedge \beta_{j^*} \wedge \nu_{k'}[t/v] \rhd \delta{}^{\backprime}\mathbf{0} + && (T34)\\
&\overline{\sum}_{i^*,j,k',\overline{d}_\chi,t,u}(b_j \mid c_{k'}){}^{\backprime}u\,(q_j \,\|\, p) \lhd u \leq t \wedge \alpha_{i^*} \wedge \beta_j \wedge \nu_{k'}[u/v] \rhd \delta{}^{\backprime}\mathbf{0} + && (T35)\\
&\overline{\sum}_{i^*,j',k',\overline{d}_\chi,t,u}(b_{j'} \mid c_{k'}){}^{\backprime}u\,p \lhd u \leq t \wedge \alpha_{i^*} \wedge \beta_{j'} \wedge \nu_{k'}[u/v] \rhd \delta{}^{\backprime}\mathbf{0} + && (T36)\\
&\overline{\sum}_{i,j,k',\overline{d}_\chi,t}((a_i \mid b_j) \mid c_{k'}){}^{\backprime}t\,(p_i \,\|\, q_j[t/u]) \lhd \alpha_i \wedge \beta_j[t/u] \wedge \nu_{k'}[t/v] \rhd \delta{}^{\backprime}\mathbf{0} + && (T37)\\
&\overline{\sum}_{i',j,k',\overline{d}_\chi,t}((a_{i'} \mid b_j) \mid c_{k'}){}^{\backprime}t\,q_j[t/u] \lhd \alpha_{i'} \wedge \beta_j[t/u] \wedge \nu_{k'}[t/v] \rhd \delta{}^{\backprime}\mathbf{0} + && (T38)\\
&\overline{\sum}_{i,j',k',\overline{d}_\chi,t}((a_i \mid b_{j'}) \mid c_{k'}){}^{\backprime}t\,p_i \lhd \alpha_i \wedge \beta_{j'}[t/u] \wedge \nu_{k'}[t/v] \rhd \delta{}^{\backprime}\mathbf{0} + && (T39)\\
&\overline{\sum}_{i',j',k',\overline{d}_\chi,t}((a_{i'} \mid b_{j'}) \mid c_{k'}){}^{\backprime}t \lhd \alpha_{i'} \wedge \beta_{j'}[t/u] \wedge \nu_{k'}[t/v] \rhd \delta{}^{\backprime}\mathbf{0} && (T40)
\end{aligned}
$$

α-conversion was applied to $T35$ and $T36$.

B. Expansion of $p \,\|\, (q \,\|\, r)$. Now we are going to derive a large number of summands from the other term.

$$
\begin{aligned}
q \,\|\, r \quad = \quad &\overline{\sum}_{j,k^*,\overline{e^*}_{m^*},\overline{f^*}_{n^*},v,u} b_j{}^{\backprime}u\,(q_j \,\|\, r) \lhd u \leq v \wedge \beta_j \wedge \nu_{k^*} \rhd \delta{}^{\backprime}\mathbf{0} + && (U1)\\
&\overline{\sum}_{j',k^*,\overline{e^*}_{m^*},\overline{f^*}_{n^*},v,u} b_{j'}{}^{\backprime}u\,r \lhd u \leq v \wedge \beta_{j'} \wedge \nu_{k^*} \rhd \delta{}^{\backprime}\mathbf{0} + && (U2)\\
&\overline{\sum}_{j^*,k,\overline{e^*}_{m^*},\overline{f^*}_{n^*},u,v} c_k{}^{\backprime}v\,(r_k \,\|\, q) \lhd v \leq u \wedge \beta_{j^*} \wedge \nu_k \rhd \delta{}^{\backprime}\mathbf{0} + && (U3)\\
&\overline{\sum}_{j^*,k',\overline{e^*}_{m^*},\overline{f^*}_{n^*},u,v} c_{k'}{}^{\backprime}v\,q \lhd v \leq u \wedge \beta_{j^*} \wedge \nu_{k'} \rhd \delta{}^{\backprime}\mathbf{0} + && (U4)\\
&\overline{\sum}_{j,k,\overline{e^*}_{m^*},\overline{f^*}_{n^*},u} (b_j \mid c_k){}^{\backprime}u\,(\,q_j \,\|\, r_k[u/v]) \lhd \beta_j \wedge \nu_k[u/v] \rhd \delta{}^{\backprime}\mathbf{0} + && (U5)\\
&\overline{\sum}_{j',k,\overline{e^*}_{m^*},\overline{f^*}_{n^*},u} (b_{j'} \mid c_k){}^{\backprime}u\,r_k[u/v] \lhd \beta_{j'} \wedge \nu_k[u/v] \rhd \delta{}^{\backprime}\mathbf{0} + && (U6)\\
&\overline{\sum}_{j,k',\overline{e^*}_{m^*},\overline{f^*}_{n^*},u} (b_j \mid c_{k'}){}^{\backprime}u\,q_j \lhd \beta_j \wedge \nu_{k'}[u/v] \rhd \delta{}^{\backprime}\mathbf{0} + && (U7)\\
&\overline{\sum}_{j',k',\overline{e^*}_{m^*},\overline{f^*}_{n^*},u} (b_{j'} \mid c_{k'}){}^{\backprime}u \lhd \beta_{j'} \wedge \nu_{k'}[u/v] \rhd \delta{}^{\backprime}\mathbf{0} && (U8)
\end{aligned}
$$

Each summand Un has indices j, j', j^*, k, k', k^*, which may be combined into a single index $\phi_n \in \Phi_n$, where $n = 1, \ldots, 8$. Let $\Phi \stackrel{\text{def}}{=} \bigcup_{n=1,\ldots,7} \Phi_n$, and $\Phi' \stackrel{\text{def}}{=} \Phi_8$. By convention, $\Phi^* = \Phi \cup \Phi'$. To $U3$ and $U4$ α-conversion may be applied, so that we have the following abbreviation:

$$
q \,\|\, r \quad \stackrel{\text{abbr.}}{=} \quad \overline{\sum}_{\phi,\overline{e^*}_{m^*},\overline{f^*}_{n^*},v,u} \tilde{b}_\phi{}^{\backprime}u\,\tilde{q}_\phi \lhd \tilde{\beta}_\phi \rhd \delta{}^{\backprime}\mathbf{0} + \overline{\sum}_{\phi',\overline{e^*}_{m^*},\overline{f^*}_{n^*},v,u} \tilde{b}_{\phi'}{}^{\backprime}u \lhd \tilde{\beta}_{\phi'} \rhd \delta{}^{\backprime}\mathbf{0}.
$$

Again the Expansion Theorem may be applied. We obtain

$$\begin{array}{rcll}
p \,\|\, (q \,\|\, r) & = & \overline{\sum}_{i,\phi^*,\overline{d}_\chi,t,u,v} a_i\text{‘}t\,(p_i \,\|\, (q \,\|\, r)) \lhd t \le u \wedge \alpha_i \wedge \tilde{\beta}_{\phi^*} \rhd \delta\text{‘}\mathbf{0} + & (V1)\\
& & \overline{\sum}_{i',\phi^*,\overline{d}_\chi,t,u,v} a_{i'}\text{‘}t\,(q \,\|\, r) \lhd t \le u \wedge \alpha_{i'} \wedge \tilde{\beta}_{\phi^*} \rhd \delta\text{‘}\mathbf{0} + & (V2)\\
& & \overline{\sum}_{i^*,\phi,\overline{d}_\chi,t,u,v} \tilde{b}_{\phi}\text{‘}u\,(\tilde{q}_\phi \,\|\, p) \lhd u \le t \wedge \alpha_{i^*} \wedge \tilde{\beta}_{\phi} \rhd \delta\text{‘}\mathbf{0} + & (V3)\\
& & \overline{\sum}_{i^*,\phi',\overline{d}_\chi,t,u,v} \tilde{b}_{\phi'}\text{‘}u\, p \lhd u \le t \wedge \alpha_{i^*} \wedge \tilde{\beta}_{\phi'} \rhd \delta\text{‘}\mathbf{0} + & (V4)\\
& & \overline{\sum}_{i,\phi,\overline{d}_\chi,t,v} (a_i \mid \tilde{b}_\phi)\text{‘}t\,(p_i \,\|\, \tilde{q}_\phi[t/u]) \lhd \alpha_i \wedge \tilde{\beta}_\phi[t/u] \rhd \delta\text{‘}\mathbf{0} + & (V5)\\
& & \overline{\sum}_{i',\phi,\overline{d}_\chi,t,v} (a_{i'} \mid \tilde{b}_\phi)\text{‘}t\,\tilde{q}_\phi[t/u] \lhd \alpha_{i'} \wedge \tilde{\beta}_\phi[t/u] \rhd \delta\text{‘}\mathbf{0} + & (V6)\\
& & \overline{\sum}_{i,\phi',\overline{d}_\chi,t,v} (a_i \mid \tilde{b}_{\phi'})\text{‘}t\,p_i \lhd \alpha_i \wedge \tilde{\beta}_{\phi'}[t/u] \rhd \delta\text{‘}\mathbf{0} + & (V7)\\
& & \overline{\sum}_{i',\phi',\overline{d}_\chi,t,v} (a_{i'} \mid \tilde{b}_{\phi'})\text{‘}t \lhd \alpha_{i'} \wedge \tilde{\beta}_{\phi'}[t/u] \rhd \delta\text{‘}\mathbf{0} & (V8)
\end{array}$$

$V1$:

$$\begin{array}{ll}
\overline{\sum}_{i,j,k^*,\overline{d}_\chi,t,u,v} a_i\text{‘}t\,(p_i \,\|\, (q \,\|\, r)) \lhd t \le u \wedge u \le v \wedge \alpha_i \wedge \beta_j \wedge \nu_{k^*} \rhd \delta\text{‘}\mathbf{0} + & (W1)\\
\overline{\sum}_{i,j',k^*,\overline{d}_\chi,t,u,v} a_i\text{‘}t\,(p_i \,\|\, (q \,\|\, r)) \lhd t \le u \wedge u \le v \wedge \alpha_i \wedge \beta_{j'} \wedge \nu_{k^*} \rhd \delta\text{‘}\mathbf{0} + & (W2)\\
\overline{\sum}_{i,j^*,k,\overline{d}_\chi,t,u,v} a_i\text{‘}t\,(p_i \,\|\, (q \,\|\, r)) \lhd t \le v \wedge v \le u \wedge \alpha_i \wedge \beta_{j^*} \wedge \nu_{k} \rhd \delta\text{‘}\mathbf{0} + & (W3)\\
\overline{\sum}_{i,j^*,k',\overline{d}_\chi,t,u,v} a_i\text{‘}t\,(p_i \,\|\, (q \,\|\, r)) \lhd t \le v \wedge v \le u \wedge \alpha_i \wedge \beta_{j^*} \wedge \nu_{k'} \rhd \delta\text{‘}\mathbf{0} + & (W4)\\
\overline{\sum}_{i,j,k,\overline{d}_\chi,t,u} a_i\text{‘}t\,(p_i \,\|\, (q \,\|\, r)) \lhd t \le u \wedge \alpha_i \wedge \beta_j \wedge \nu_{k}[u/v] \rhd \delta\text{‘}\mathbf{0} + & (W5)\\
\overline{\sum}_{i,j',k,\overline{d}_\chi,t,u} a_i\text{‘}t\,(p_i \,\|\, (q \,\|\, r)) \lhd t \le u \wedge \alpha_i \wedge \beta_{j'} \wedge \nu_{k}[u/v] \rhd \delta\text{‘}\mathbf{0} + & (W6)\\
\overline{\sum}_{i,j,k',\overline{d}_\chi,t,u} a_i\text{‘}t\,(p_i \,\|\, (q \,\|\, r)) \lhd t \le u \wedge \alpha_i \wedge \beta_j \wedge \nu_{k'}[u/v] \rhd \delta\text{‘}\mathbf{0} + & (W7)\\
\overline{\sum}_{i,j',k',\overline{d}_\chi,t,u} a_i\text{‘}t\,(p_i \,\|\, (q \,\|\, r)) \lhd t \le u \wedge \alpha_i \wedge \beta_{j'} \wedge \nu_{k'}[u/v] \rhd \delta\text{‘}\mathbf{0} & (W8)
\end{array}$$

α-conversion was applied to $W3$ and $W4$.
$V2$:

$$\begin{array}{ll}
\overline{\sum}_{i',j,k^*,\overline{d}_\chi,t,u,v} a_{i'}\text{‘}t\,(q \,\|\, r) \lhd t \le u \wedge u \le v \wedge \alpha_{i'} \wedge \beta_j \wedge \nu_{k^*} \rhd \delta\text{‘}\mathbf{0} + & (W9)\\
\overline{\sum}_{i',j',k^*,\overline{d}_\chi,t,u,v} a_{i'}\text{‘}t\,(q \,\|\, r) \lhd t \le u \wedge u \le v \wedge \alpha_{i'} \wedge \beta_{j'} \wedge \nu_{k^*} \rhd \delta\text{‘}\mathbf{0} + & (W10)\\
\overline{\sum}_{i',j^*,k,\overline{d}_\chi,t,u,v} a_{i'}\text{‘}t\,(q \,\|\, r) \lhd t \le v \wedge v \le u \wedge \alpha_{i'} \wedge \beta_{j^*} \wedge \nu_{k} \rhd \delta\text{‘}\mathbf{0} + & (W11)\\
\overline{\sum}_{i',j^*,k',\overline{d}_\chi,t,u,v} a_{i'}\text{‘}t\,(q \,\|\, r) \lhd t \le v \wedge v \le u \wedge \alpha_{i'} \wedge \beta_{j^*} \wedge \nu_{k'} \rhd \delta\text{‘}\mathbf{0} + & (W12)\\
\overline{\sum}_{i',j,k,\overline{d}_\chi,t,u} a_{i'}\text{‘}t\,(q \,\|\, r) \lhd t \le u \wedge \alpha_{i'} \wedge \beta_j \wedge \nu_{k}[u/v] \rhd \delta\text{‘}\mathbf{0} + & (W13)\\
\overline{\sum}_{i',j',k,\overline{d}_\chi,t,u} a_{i'}\text{‘}t\,(q \,\|\, r) \lhd t \le u \wedge \alpha_{i'} \wedge \beta_{j'} \wedge \nu_{k}[u/v] \rhd \delta\text{‘}\mathbf{0} + & (W14)\\
\overline{\sum}_{i',j,k',\overline{d}_\chi,t,u} a_{i'}\text{‘}t\,(q \,\|\, r) \lhd t \le u \wedge \alpha_{i'} \wedge \beta_j \wedge \nu_{k'}[u/v] \rhd \delta\text{‘}\mathbf{0} + & (W15)\\
\overline{\sum}_{i',j',k',\overline{d}_\chi,t,u} a_{i'}\text{‘}t\,(q \,\|\, r) \lhd t \le u \wedge \alpha_{i'} \wedge \beta_{j'} \wedge \nu_{k'}[u/v] \rhd \delta\text{‘}\mathbf{0} & (W16)
\end{array}$$

α-conversion was applied to $W11$ and $W12$.
$V3 + V4$:

$$\begin{array}{ll}
\overline{\sum}_{i^*,j,k^*,\overline{d}_\chi,t,u,v} b_j\text{‘}u\,((q_j \,\|\, r) \,\|\, p) \lhd u \le t \wedge u \le v \wedge \alpha_{i^*} \wedge \beta_j \wedge \nu_{k^*} \rhd \delta\text{‘}\mathbf{0} + & (W17)\\
\overline{\sum}_{i^*,j',k^*,\overline{d}_\chi,t,u,v} b_{j'}\text{‘}u\,(r \,\|\, p) \lhd u \le t \wedge u \le v \wedge \alpha_{i^*} \wedge \beta_{j'} \wedge \nu_{k^*} \rhd \delta\text{‘}\mathbf{0} + & (W18)\\
\overline{\sum}_{i^*,j^*,k,\overline{d}_\chi,t,u,v} c_k\text{‘}v\,((r_k \,\|\, q) \,\|\, p) \lhd v \le t \wedge v \le u \wedge \alpha_{i^*} \wedge \beta_{j^*} \wedge \nu_{k} \rhd \delta\text{‘}\mathbf{0} + & (W19)\\
\overline{\sum}_{i^*,j^*,k',\overline{d}_\chi,t,u,v} c_{k'}\text{‘}v\,(q \,\|\, p) \lhd v \le t \wedge v \le u \wedge \alpha_{i^*} \wedge \beta_{j^*} \wedge \nu_{k'} \rhd \delta\text{‘}\mathbf{0} + & (W20)\\
\overline{\sum}_{i^*,j,k,\overline{d}_\chi,t,u} (b_j \mid c_k)\text{‘}u\,((\,q_j \,\|\, r_k[u/v]) \,\|\, p) \lhd u \le t \wedge \alpha_{i^*} \wedge \beta_j \wedge \nu_{k}[u/v] \rhd \delta\text{‘}\mathbf{0} + & (W21)\\
\overline{\sum}_{i^*,j',k,\overline{d}_\chi,t,u} (b_{j'} \mid c_k)\text{‘}u\,(r_k[u/v] \,\|\, p) \lhd u \le t \wedge \alpha_{i^*} \wedge \beta_{j'} \wedge \nu_{k}[u/v] \rhd \delta\text{‘}\mathbf{0} + & (W22)\\
\overline{\sum}_{i^*,j,k',\overline{d}_\chi,t,u} (b_j \mid c_{k'})\text{‘}u\,(q_j \,\|\, p) \lhd u \le t \wedge \alpha_{i^*} \wedge \beta_j \wedge \nu_{k'}[u/v] \rhd \delta\text{‘}\mathbf{0} + & (W23)\\
\overline{\sum}_{i^*,j',k',\overline{d}_\chi,t,u} (b_{j'} \mid c_{k'})\text{‘}u\, p \lhd u \le t \wedge \alpha_{i^*} \wedge \beta_{j'} \wedge \nu_{k'}[u/v] \rhd \delta\text{‘}\mathbf{0} & (W24)
\end{array}$$

α-conversion was applied to $W19$ and $W20$.
$V5 + V7$:

$$\overline{\sum}_{i,j,k^*,\overline{d}_\chi,t,v}(a_i \mid b_j) \cdot t\,(p_i \,\|\, (q_j[t/u] \,\|\, r)) \lhd t \leq v \wedge \alpha_i \wedge \beta_j[t/u] \wedge \nu_{k^*} \rhd \delta \cdot \mathbf{0} + \qquad (W25)$$
$$\overline{\sum}_{i,j',k^*,\overline{d}_\chi,t,v}(a_i \mid b_{j'}) \cdot t\,(p_i \,\|\, r) \lhd t \leq v \wedge \alpha_i \wedge \beta_{j'}[t/u] \wedge \nu_{k^*} \rhd \delta \cdot \mathbf{0} + \qquad (W26)$$
$$\overline{\sum}_{i,j^*,k,\overline{d}_\chi,t,u}(a_i \mid c_k) \cdot t\,(p_i \,\|\, (r_k[t/v] \,\|\, q)) \lhd t \leq u \wedge \alpha_i \wedge \beta_{j^*} \wedge \nu_k[t/v] \rhd \delta \cdot \mathbf{0} + \qquad (W27)$$
$$\overline{\sum}_{i,j^*,k',\overline{d}_\chi,t,u}(a_i \mid c_{k'}) \cdot t\,(p_i \,\|\, q) \lhd t \leq u \wedge \alpha_i \wedge \beta_{j^*} \wedge \nu_{k'}[t/v] \rhd \delta \cdot \mathbf{0} + \qquad (W28)$$
$$\overline{\sum}_{i,j,k,\overline{d}_\chi,t}(a_i \mid (b_j \mid c_k)) \cdot t\,(p_i \,\|\, (q_j[t/u] \,\|\, r_k[t/v])) \lhd \alpha_i \wedge \beta_j[t/u] \wedge \nu_k[t/v] \rhd \delta \cdot \mathbf{0} + \qquad (W29)$$
$$\overline{\sum}_{i,j',k,\overline{d}_\chi,t}(a_i \mid (b_{j'} \mid c_k)) \cdot t\,(p_i \,\|\, r_k[t/v]) \lhd \alpha_i \wedge \beta_{j'}[t/u] \wedge \nu_k[t/v] \rhd \delta \cdot \mathbf{0} + \qquad (W30)$$
$$\overline{\sum}_{i,j,k',\overline{d}_\chi,t}(a_i \mid (b_j \mid c_{k'})) \cdot t\,(p_i \,\|\, q_j[t/u]) \lhd \alpha_i \wedge \beta_j[t/u] \wedge \nu_{k'}[t/v] \rhd \delta \cdot \mathbf{0} + \qquad (W31)$$
$$\overline{\sum}_{i,j',k',\overline{d}_\chi,t}(a_i \mid (b_{j'} \mid c_{k'})) \cdot t\, p_i \lhd \alpha_i \wedge \beta_{j'}[t/u] \wedge \nu_{k'}[t/v] \rhd \delta \cdot \mathbf{0} \qquad (W32)$$

α-conversion was applied to $W27$ and $W28$.
$V6 + V8$:

$$\overline{\sum}_{i',j,k^*,\overline{d}_\chi,t,v}(a_{i'} \mid b_j) \cdot t\,(q_j[t/u] \,\|\, r) \lhd t \leq v \wedge \alpha_{i'} \wedge \beta_j[t/u] \wedge \nu_{k^*} \rhd \delta \cdot \mathbf{0} + \qquad (W33)$$
$$\overline{\sum}_{i',j',k^*,\overline{d}_\chi,t,v}(a_{i'} \mid b_{j'}) \cdot t\, r \lhd t \leq v \wedge \alpha_{i'} \wedge \beta_{j'}[t/u] \wedge \nu_{k^*} \rhd \delta \cdot \mathbf{0} + \qquad (W34)$$
$$\overline{\sum}_{i',j^*,k,\overline{d}_\chi,t,u}(a_{i'} \mid c_k) \cdot t\,(r_k[t/v] \,\|\, q) \lhd t \leq u \wedge \alpha_{i'} \wedge \beta_{j^*} \wedge \nu_k[t/v] \rhd \delta \cdot \mathbf{0} + \qquad (W35)$$
$$\overline{\sum}_{i',j^*,k',\overline{d}_\chi,t,u}(a_{i'} \mid c_{k'}) \cdot t\, q \lhd t \leq u \wedge \alpha_{i'} \wedge \beta_{j^*} \wedge \nu_{k'}[t/v] \rhd \delta \cdot \mathbf{0} + \qquad (W36)$$
$$\overline{\sum}_{i',j,k,\overline{d}_\chi,t}(a_{i'} \mid (b_j \mid c_k)) \cdot t\,(q_j[t/u] \,\|\, r_k[t/v]) \lhd \alpha_{i'} \wedge \beta_j[t/u] \wedge \nu_k[t/v] \rhd \delta \cdot \mathbf{0} + \qquad (W37)$$
$$\overline{\sum}_{i',j',k,\overline{d}_\chi,t}(a_{i'} \mid (b_{j'} \mid c_k)) \cdot t\, r_k[t/v] \lhd \alpha_{i'} \wedge \beta_{j'}[t/u] \wedge \nu_k[t/v] \rhd \delta \cdot \mathbf{0} + \qquad (W38)$$
$$\overline{\sum}_{i',j,k',\overline{d}_\chi,t}(a_{i'} \mid (b_j \mid c_{k'})) \cdot t\, q_j[t/u] \lhd \alpha_{i'} \wedge \beta_j[t/u] \wedge \nu_{k'}[t/v] \rhd \delta \cdot \mathbf{0} + \qquad (W39)$$
$$\overline{\sum}_{i',j',k',\overline{d}_\chi,t}(a_{i'} \mid (b_{j'} \mid c_{k'})) \cdot t \lhd \alpha_{i'} \wedge \beta_{j'}[t/u] \wedge \nu_{k'}[t/v] \rhd \delta \cdot \mathbf{0} \qquad (W40)$$

α-conversion was applied to $W35$ and $W36$.

C. Proving identity of $(p \,\|\, q) \,\|\, r$ and $p \,\|\, (q \,\|\, r)$. Finally it has to be proved that $T1 + \ldots + T40 = W1 + \ldots + W40$.

Here we tacitly apply the i.h., commutativity of $\|$, and the properties $a \mid b = b \mid a$ and $(a \mid b) \mid c = b \mid (a \mid b)$ of $\mid$, which follow from the definition of the communication function γ and axiom CF.

Fortunately, a large number of identities follows right away by inspection of the equations:

$T3 = W17$	$T7 = W26$	$T27 = W21$	$T31 = W30$	$T35 = W23$	$T39 = W32$
$T4 = W18$	$T8 = W34$	$T28 = W22$	$T32 = W38$	$T36 = W24$	$T40 = W40$
$T5 = W25$	$T25 = W27$	$T29 = W29$	$T33 = W28$	$T37 = W31$	
$T6 = W33$	$T26 = W35$	$T30 = W37$	$T34 = W36$	$T38 = W39$	

Four more identities have to be proven:

(*i*) $T1 = W1 + \ldots + W8$;
(*ii*) $T2 = W9 + \ldots + W16$;
(*iii*) $T9 + \ldots + T16 = W19$;
(*iv*) $T17 + \ldots + T24 = W20$.

We prove (*i*):

$$W1 + W2$$
$$\stackrel{\text{SUM4}}{=} \quad \overline{\sum}_{i,j^*,k^*,\overline{d}_\chi,t,u,v} a_i \cdot t\,(p_i \,\|\, (q \,\|\, r)) \lhd t \leq u \wedge u \leq v \wedge \alpha_i \wedge \beta_{j^*} \wedge \nu_{k^*} \rhd \delta \cdot \mathbf{0};$$

$$W3+W4$$
$$\stackrel{\text{SUM4}}{=} \quad \overline{\sum}_{i,j^*,k^*,\overline{d}_\chi,t,u,v} a_i \text{‘}t\,(p_i \,\|\, (q \,\|\, r)) \lhd t \leq v \wedge v \leq u \wedge \alpha_i \wedge \beta_{j^*} \wedge \nu_{k^*} \rhd \delta\text{‘}\mathbf{0};$$
$$W5+W6+W7+W8$$
$$\stackrel{\text{SUM4}}{=} \quad \overline{\sum}_{i,j^*,k^*,\overline{d}_\chi,t,u} a_i \text{‘}t\,(p_i \,\|\, (q \,\|\, r)) \lhd t \leq u \wedge \alpha_i \wedge \beta_{j^*} \wedge \nu_{k^*}[u/v] \rhd \delta\text{‘}\mathbf{0}$$
$$\stackrel{\text{A.5.1,A.1.6}}{=} \quad \overline{\sum}_{i,j^*,k^*,\overline{d}_\chi,t,u,v} a_i \text{‘}t\,(p_i \,\|\, (q \,\|\, r)) \lhd t \leq u \wedge eq(u,v) \wedge \alpha_i \wedge \beta_{j^*} \wedge \nu_{k^*} \rhd \delta\text{‘}\mathbf{0}.$$

By axiom SUM4 and axiom A.1.7 $W1 + \ldots + W8 =$

$$\overline{\sum}_{i,j^*,k^*,\overline{d}_\chi,t,u,v} a_i \text{‘}t\,(p_i \,\|\, (q \,\|\, r))$$
$$\lhd ((t \leq u \wedge u \leq v) \vee (t \leq v \wedge v \leq u) \vee (t \leq u \wedge eq(u,v))) \wedge \alpha_i \wedge \beta_{j^*} \wedge \nu_{k^*} \rhd \delta\text{‘}\mathbf{0}.$$

Application of the i.h. and some elementary calculations for the condition show that this term is equal to $T1$. The identities (ii), (iii) and (iv) follow in a similar way. □

D. Cases with $\delta\text{‘}\mathbf{0}$. We finally have to study the cases where at least one of the processes p, q, r equals $\delta\text{‘}\mathbf{0}$. We may distinguish the following cases (C stands for commutativity of $\|$):

$D.1.$ $(\delta\text{‘}\mathbf{0} \,\|\, p) \,\|\, q = \delta\text{‘}\mathbf{0} \,\|\, (p \,\|\, q)$. See below;

$D.2.$ $(p \,\|\, \delta\text{‘}\mathbf{0}) \,\|\, q \stackrel{\text{C}}{=} (\delta\text{‘}\mathbf{0} \,\|\, p) \,\|\, q \stackrel{\text{D.1}}{=} \delta\text{‘}\mathbf{0} \,\|\, (p \,\|\, q) \stackrel{\text{C}}{=} \delta\text{‘}\mathbf{0} \,\|\, (q \,\|\, p) \stackrel{\text{D.1}}{=} (\delta\text{‘}\mathbf{0} \,\|\, q) \,\|\, p \stackrel{\text{C}}{=} p \,\|\, (\delta\text{‘}\mathbf{0} \,\|\, q)$;

$D.3.$ $(p \,\|\, q) \,\|\, \delta\text{‘}\mathbf{0} \stackrel{\text{C}}{=} \delta\text{‘}\mathbf{0} \,\|\, (p \,\|\, q) \stackrel{\text{C}}{=} \delta\text{‘}\mathbf{0} \,\|\, (q \,\|\, p) \stackrel{\text{D.1}}{=} (\delta\text{‘}\mathbf{0} \,\|\, q) \,\|\, p \stackrel{\text{C}}{=} (q \,\|\, \delta\text{‘}\mathbf{0}) \,\|\, p \stackrel{\text{C}}{=} p \,\|\, (q \,\|\, \delta\text{‘}\mathbf{0})$;

$D.4.$ $(p \,\|\, \delta\text{‘}\mathbf{0}) \,\|\, \delta\text{‘}\mathbf{0} = p \,\|\, (\delta\text{‘}\mathbf{0} \,\|\, \delta\text{‘}\mathbf{0})$. Easy; by induction on $|p|$;

$D.5.$ $(\delta\text{‘}\mathbf{0} \,\|\, p) \,\|\, \delta\text{‘}\mathbf{0} \stackrel{\text{C}}{=} \delta\text{‘}\mathbf{0} \,\|\, (\delta\text{‘}\mathbf{0} \,\|\, p) \stackrel{\text{C}}{=} \delta\text{‘}\mathbf{0} \,\|\, (p \,\|\, \delta\text{‘}\mathbf{0})$;

$D.6.$ $(\delta\text{‘}\mathbf{0} \,\|\, \delta\text{‘}\mathbf{0}) \,\|\, p \stackrel{\text{C}}{=} p \,\|\, (\delta\text{‘}\mathbf{0} \,\|\, \delta\text{‘}\mathbf{0}) \stackrel{\text{D.4}}{=} (p \,\|\, \delta\text{‘}\mathbf{0}) \,\|\, \delta\text{‘}\mathbf{0} \stackrel{\text{C}}{=} \delta\text{‘}\mathbf{0} \,\|\, (p \,\|\, \delta\text{‘}\mathbf{0}) \stackrel{\text{C}}{=} \delta\text{‘}\mathbf{0} \,\|\, (\delta\text{‘}\mathbf{0} \,\|\, p)$;

$D.7.$ $(\delta\text{‘}\mathbf{0} \,\|\, \delta\text{‘}\mathbf{0}) \,\|\, \delta\text{‘}\mathbf{0} \stackrel{\text{C}}{=} \delta\text{‘}\mathbf{0} \,\|\, (\delta\text{‘}\mathbf{0} \,\|\, \delta\text{‘}\mathbf{0})$.

Proof of identity D.1. By induction on $|p| + |q|$. As in part A of the proof we tacitly apply the lemmas A.1.7 and A.4, and axiom SUM1.

$$(\delta\text{‘}\mathbf{0} \,\|\, p) \,\|\, q \quad \stackrel{4.3.2,3.11.2,\text{A.5.1,A.1.6}}{=} \quad \begin{array}{l}(\overline{\sum}_{i,\overline{d}_l,t} a_i \text{‘}t\,(p_i \,\|\, \delta\text{‘}\mathbf{0}) \lhd \alpha_i \wedge eq(t,\mathbf{0}) \rhd \delta\text{‘}\mathbf{0} + \\ \overline{\sum}_{i',\overline{d'}_{l'},t} a_{i'} \text{‘}t\,\delta\text{‘}\mathbf{0} \lhd \alpha_{i'} \wedge eq(t,\mathbf{0}) \rhd \delta\text{‘}\mathbf{0}) \,\|\, q \end{array} \qquad (*)$$

Here we have a tricky part of the proof; The term just obtained also has to be considered for two cases, namely:

1. If $\exists_{i^* \in I^*}.\alpha_{i^*}[\mathbf{0}/t] = \mathsf{t}$ then by Theorem 3.11.1 the above term equals

$$\begin{array}{l}
\overline{\sum}_{i,j^*,\overline{d^*}_{l^*},\overline{e^*}_{m^*},u,t} a_i \text{‘}t\,((p_i \,\|\, \delta\text{‘}\mathbf{0}) \,\|\, q) \lhd t \leq u \wedge eq(t,\mathbf{0}) \wedge \alpha_i \wedge \beta_{j^*} \rhd \delta\text{‘}\mathbf{0} + \\
\overline{\sum}_{i',j^*,\overline{d^*}_{l^*},\overline{e^*}_{m^*},u,t} a_{i'} \text{‘}t\,(\delta\text{‘}\mathbf{0} \,\|\, q) \lhd t \leq u \wedge eq(t,\mathbf{0}) \wedge \alpha_{i'} \wedge \beta_{j^*} \rhd \delta\text{‘}\mathbf{0} + \\
\overline{\sum}_{i^*,j,\overline{d^*}_{l^*},\overline{e^*}_{m^*},t,u} b_j \text{‘}u\,(q_j \,\|\, (\delta\text{‘}\mathbf{0} \,\|\, p)) \lhd u \leq t \wedge eq(t,\mathbf{0}) \wedge \alpha_{i^*} \wedge \beta_j \rhd \delta\text{‘}\mathbf{0} + \\
\overline{\sum}_{i^*,j',\overline{d^*}_{l^*},\overline{e^*}_{m^*},t,u} b_{j'} \text{‘}u\,(\delta\text{‘}\mathbf{0} \,\|\, p) \lhd u \leq t \wedge eq(t,\mathbf{0}) \wedge \alpha_{i^*} \wedge \beta_{j'} \rhd \delta\text{‘}\mathbf{0} + \\
\overline{\sum}_{i,j,\overline{d^*}_{l^*},\overline{e^*}_{m^*},t} (a_i \mid b_j) \text{‘}t\,((p_i \,\|\, \delta\text{‘}\mathbf{0}) \,\|\, q_j[t/u]) \lhd eq(t,\mathbf{0}) \wedge \alpha_i \wedge \beta_j[t/u] \rhd \delta\text{‘}\mathbf{0} + \\
\overline{\sum}_{i',j,\overline{d^*}_{l^*},\overline{e^*}_{m^*},t} (a_{i'} \mid b_j) \text{‘}t\,(\delta\text{‘}\mathbf{0} \,\|\, q_j[t/u]) \lhd eq(t,\mathbf{0}) \wedge \alpha_{i'} \wedge \beta_j[t/u] \rhd \delta\text{‘}\mathbf{0} + \\
\overline{\sum}_{i,j',\overline{d^*}_{l^*},\overline{e^*}_{m^*},t} (a_i \mid b_{j'}) \text{‘}t\,(p_i \,\|\, \delta\text{‘}\mathbf{0}) \lhd eq(t,\mathbf{0}) \wedge \alpha_i \wedge \beta_{j'}[t/u] \rhd \delta\text{‘}\mathbf{0} + \\
\overline{\sum}_{i',j',\overline{d^*}_{l^*},\overline{e^*}_{m^*},t} (a_{i'} \mid b_{j'}) \text{‘}t\,\delta\text{‘}\mathbf{0} \lhd eq(t,\mathbf{0}) \wedge \alpha_{i'} \wedge \beta_{j'}[t/u] \rhd \delta\text{‘}\mathbf{0}.
\end{array}$$

By the lemmas A.1.6 and A.5.1 this may be simplified to

$$\begin{array}{lr}
\overline{\sum}_{i,j^*,\overline{d^*}_{l^*},\overline{e^*}_{m^*},u} a_i\text{‹}\mathbf{0}\,((p_i[\mathbf{0}/t] \parallel \delta\text{‹}\mathbf{0}) \parallel q) \lhd \alpha_i[\mathbf{0}/t] \wedge \beta_{j^*} \rhd \delta\text{‹}\mathbf{0} + & (X1)\\
\overline{\sum}_{i',j^*,\overline{d^*}_{l^*},\overline{e^*}_{m^*},u} a_{i'}\text{‹}\mathbf{0}\,(\delta\text{‹}\mathbf{0} \parallel q) \lhd \alpha_{i'}[\mathbf{0}/t] \wedge \beta_{j^*} \rhd \delta\text{‹}\mathbf{0} + & (X2)\\
\overline{\sum}_{j,\overline{d^*}_{l^*},\overline{e^*}_{m^*}} b_j\text{‹}\mathbf{0}\,(q_j[\mathbf{0}/u] \parallel (\delta\text{‹}\mathbf{0} \parallel p)) \lhd \bigvee_{i^*\in I^*} \alpha_{i^*}[\mathbf{0}/t] \wedge \beta_j[\mathbf{0}/u] \rhd \delta\text{‹}\mathbf{0} + & (X3)\\
\overline{\sum}_{j',\overline{d^*}_{l^*},\overline{e^*}_{m^*}} b_{j'}\text{‹}\mathbf{0}\,(\delta\text{‹}\mathbf{0} \parallel p) \lhd \bigvee_{i^*\in I^*} \alpha_{i^*}[\mathbf{0}/t] \wedge \beta_{j'}[\mathbf{0}/u] \rhd \delta\text{‹}\mathbf{0} + & (X4)\\
\overline{\sum}_{i,j,\overline{d^*}_{l^*},\overline{e^*}_{m^*}} (a_i \mid b_j)\text{‹}\mathbf{0}\,((p_i[\mathbf{0}/t] \parallel \delta\text{‹}\mathbf{0}) \parallel q_j[\mathbf{0}/u]) \lhd \alpha_i[\mathbf{0}/t] \wedge \beta_j[\mathbf{0}/u] \rhd \delta\text{‹}\mathbf{0} + & (X5)\\
\overline{\sum}_{i',j,\overline{d^*}_{l^*},\overline{e^*}_{m^*}} (a_{i'} \mid b_j)\text{‹}\mathbf{0}\,(\delta\text{‹}\mathbf{0} \parallel q_j[\mathbf{0}/u]) \lhd \alpha_{i'}[\mathbf{0}/t] \wedge \beta_j[\mathbf{0}/u] \rhd \delta\text{‹}\mathbf{0} + & (X6)\\
\overline{\sum}_{i,j',\overline{d^*}_{l^*},\overline{e^*}_{m^*}} (a_i \mid b_{j'})\text{‹}\mathbf{0}\,(p_i[\mathbf{0}/t] \parallel \delta\text{‹}\mathbf{0}) \lhd \alpha_i[\mathbf{0}/t] \wedge \beta_{j'}[\mathbf{0}/u] \rhd \delta\text{‹}\mathbf{0} + & (X7)\\
\overline{\sum}_{i',j',\overline{d^*}_{l^*},\overline{e^*}_{m^*}} (a_{i'} \mid b_{j'})\text{‹}\mathbf{0}\,\delta\text{‹}\mathbf{0} \lhd \alpha_{i'}[\mathbf{0}/t] \wedge \beta_{j'}[\mathbf{0}/u] \rhd \delta\text{‹}\mathbf{0} & (X8)
\end{array}$$

By assumption at least one of the $\alpha_{i^*}[\mathbf{0}/t]$ equals t, so $\bigvee_{i^*\in I^*} \alpha_{i^*}[\mathbf{0}/t] = \mathsf{t}$. By Lemma A.5.2 we may replace $X3 + X4$ by

$$\begin{array}{lr}
\overline{\sum}_{j,\overline{d^*}_{l^*},\overline{e^*}_{m^*},t} (b_j\text{‹}\mathbf{0}\,(q_j[\mathbf{0}/u] \parallel (\delta\text{‹}\mathbf{0} \parallel p)) \lhd \beta_j[\mathbf{0}/u] \rhd \delta\text{‹}\mathbf{0}) \lhd \bigvee_{i^*\in I^*} \alpha_{i^*} \rhd \delta\text{‹}\mathbf{0} + & \\
\overline{\sum}_{j',\overline{d^*}_{l^*},\overline{e^*}_{m^*},t} (b_{j'}\text{‹}\mathbf{0}\,(\delta\text{‹}\mathbf{0} \parallel p) \lhd \beta_{j'}[\mathbf{0}/u] \rhd \delta\text{‹}\mathbf{0}) \lhd \bigvee_{i^*\in I^*} \alpha_{i^*} \rhd \delta\text{‹}\mathbf{0} & \\
\overset{\text{A.1.6}}{=} \quad \overline{\sum}_{i^*,j,\overline{d^*}_{l^*},\overline{e^*}_{m^*},t} b_j\text{‹}\mathbf{0}\,(q_j[\mathbf{0}/u] \parallel (\delta\text{‹}\mathbf{0} \parallel p)) \lhd \alpha_{i^*} \wedge \beta_j[\mathbf{0}/u] \rhd \delta\text{‹}\mathbf{0} + & (X3')\\
\qquad\quad \overline{\sum}_{i^*,j',\overline{d^*}_{l^*},\overline{e^*}_{m^*},t} b_{j'}\text{‹}\mathbf{0}\,(\delta\text{‹}\mathbf{0} \parallel p) \lhd \alpha_{i^*} \wedge \beta_{j'}[\mathbf{0}/u] \rhd \delta\text{‹}\mathbf{0} & (X4')
\end{array}$$

The term $\delta\text{‹}\mathbf{0} \parallel (p \parallel q)$ may be expanded according to $M1 + \ldots + M8$ given under part A of the proof, and Theorem 3.11.2. It turns out that, by the i.h., the resulting system is identical to $X1 + X2 + X3' + X4' + X5 + X6 + X7$, which finishes this case;

2. If $\forall_{i^*\in I^*}.\alpha_{i^*}[\mathbf{0}/t] = \mathsf{f}$ then by commutativity of $\parallel$ and Theorem 3.11.2 $\delta\text{‹}\mathbf{0} \parallel p = \delta\text{‹}\mathbf{0}$, so expression $(*)$ equals $\delta\text{‹}\mathbf{0} \parallel q$. It is left to prove that $\delta\text{‹}\mathbf{0} \parallel (p \parallel q) = \delta\text{‹}\mathbf{0} \parallel q$, which can be done by induction on $|q|$. Again we may use $M1 + \ldots + M8$ for $p \parallel q$, and by the theorems 4.3.2 and 3.11.2 we find that

$$\begin{array}{rcl}
\delta\text{‹}\mathbf{0} \parallel (p \parallel q) & = & \overline{\sum}_{j,\overline{d^*}_{l^*},\overline{e^*}_{m^*},t} b_j\text{‹}\mathbf{0}\,((q_j[\mathbf{0}/u] \parallel p) \parallel \delta\text{‹}\mathbf{0}) \lhd \bigvee_{i^*\in I^*} \alpha_{i^*} \wedge \beta_j[\mathbf{0}/u] \rhd \delta\text{‹}\mathbf{0} + \\
& & \overline{\sum}_{j',\overline{d^*}_{l^*},\overline{e^*}_{m^*},t} b_{j'}\text{‹}\mathbf{0}\,(p \parallel \delta\text{‹}\mathbf{0}) \lhd \bigvee_{i^*\in I^*} \alpha_{i^*} \wedge \beta_{j'}[\mathbf{0}/u] \rhd \delta\text{‹}\mathbf{0}.
\end{array}$$

The base case of the proof, where $|q| = 1$, is included in the general case, where $|q| \geq 1$, which we prove here. By assumption at least one of the α_{i^*} must be true at some time, so by Lemma A.5.2, commutativity of $\parallel$, and the i.h. we get

$$\begin{array}{rcl}
\delta\text{‹}\mathbf{0} \parallel (p \parallel q) & = & \overline{\sum}_{j,\overline{d^*}_{l^*},\overline{e^*}_{m^*}} b_j\text{‹}\mathbf{0}\,(q_j[\mathbf{0}/u] \parallel (\delta\text{‹}\mathbf{0} \parallel p)) \lhd \beta_j[\mathbf{0}/u] \rhd \delta\text{‹}\mathbf{0} + \\
& & \overline{\sum}_{j',\overline{d^*}_{l^*},\overline{e^*}_{m^*}} b_{j'}\text{‹}\mathbf{0}\,(\delta\text{‹}\mathbf{0} \parallel p) \lhd \beta_{j'}[\mathbf{0}/u] \rhd \delta\text{‹}\mathbf{0}.
\end{array}$$

Now, $\delta\text{‹}\mathbf{0} \parallel p = \delta\text{‹}\mathbf{0}$, redundant sums $\overline{\sum}_{\overline{d^*}_{l^*}}$ may be removed, and finally, by Lemma 3.11.2 and commutativity of $\parallel$, we may conclude that this expression equals $\delta\text{‹}\mathbf{0} \parallel q$. This finishes the proof of identity D.1.

□