Edge-disjoint Paths in Graphs on Surfaces

Maurits de Graaf
Department of Mathematics
University of Amsterdam
Plantage Muidergracht 24, 1018 TV Amsterdam
email: maurits@fwi.uva.nl

We present a survey of results on the edge-disjoint paths problem and relate this problem to the edge-disjoint homotopic path and the edge-disjoint homotopic cycle problem. The latter problem is: given a graph $G = (V, E)$ embedded on a surface S and closed curves $C_1, \dots, C_k$ on S, find necessary and sufficient conditions for the existence of pairwise edge-disjoint cycles $\widetilde{C}_1, \dots, \widetilde{C}_k$ in G so that $\widetilde{C}_i$ is homotopic to C_i for $i = 1, \dots, k$. We explain that a certain cut condition, which is easily seen to be necessary, is also sufficient for a fractional solution of the edge-disjoint homotopic cycle problem. To this end we use a theorem stating that any system of closed curves can be made 'minimally crossing' by 'Reidemeister moves'. This establishes a relation between the edge-disjoint homotopic cycle problem and the theory of knots.

1. Introduction

In this article we present a survey of recent results on the *edge-disjoint paths problem* and the *edge-disjoint cycle problem*. We consider the following version of the edge-disjoint paths problem:

$$\begin{array}{ll} \textbf{given}: & \text{a planar graph } G = (V, E) \text{ embedded in the plane and} \\ & \text{pairs } \{r_1, s_1\}, \dots, \{r_k, s_k\} \text{ of vertices of } G, \\ \textbf{find}: & \text{pairwise edge-disjoint paths } P_1, \dots, P_k \text{ in } G \text{ where each } P_i \\ & \text{is a path with endpoints } r_i \text{ and } s_i \quad (i = 1, \dots, k). \end{array} \tag{1}$$

It is assumed throughout that $r_i \neq s_i$ for $i = 1, \dots, k$.

We are particularly interested in finding necessary and sufficient conditions to guarantee the *existence* of edge-disjoint paths $P_1, \dots, P_k$ as in (1). The question of actually *finding* the paths if they exist will not be considered here, although polynomial-time algorithms are known for the specific problems discussed in this article.

Let us proceed by defining some notation and terminology.

Paths and cycles in a graph are not allowed to use an edge more than once. They may, however, have repeated vertices. The pairs $\{r_i, s_i\}$ are called *commodities* and the set of commodities is denoted by R, that is $R := \{\{r_1, s_1\}, \ldots, \{r_k, s_k\}\}$. Given a graph $G = (V, E)$ and $W \subset V$, $\delta(W)$ denotes the cardinality of the set of edges with one end in W and the other in $V \backslash W$. We define $\rho(W) := |\{\{v, w\} \in R | v \in W, w \notin W\}|$. A graph is called *eulerian* if all its vertices have even degree.

The following *cut condition* is clearly a necessary condition for (1):

$$\text{(cut condition)} \qquad \text{for each } W \subset V, \delta(W) \geq \rho(W). \tag{2}$$

The content of Menger's theorem is that in the special case $|R| = 1$ the cut condition (2) is *sufficient* for the existence of edge-disjoint paths $P_1, \ldots, P_k$ as in (1). One may derive from this that condition (2) is also sufficient in case $\cap_{i=1}^{k} \{r_i, s_i\} \neq \emptyset$.

One aim of this article is to show how the edge-disjoint path problem (1) can be interpreted as an edge-disjoint homotopic path problem in graphs embedded on a surface. We will study two special cases of the edge-disjoint homotopic path problem for which sufficient conditions for the existence of the edge-disjoint homotopic paths are known. After that, we will see how the edge-disjoint homotopic path problem can be regarded as an edge-disjoint homotopic cycle problem. For this problem we will discuss the fact that a cut condition which is easily seen to be necessary, is sufficient for the existence of 'fractionally' edge-disjoint cycles. In order to point out the connection of the edge-disjoint homotopic cycle problem with problems in algebraic topology, some steps towards a proof of the latter result are sketched.

In order to interpret the cut condition and the edge-disjoint path problem in a different way, we introduce some more notation and definitions in the next section.

2. Homotopy and crossings of curves

Let S denote a triangulizable surface. This means that S is homeomorphic to a 2-sphere S^2, with a finite number of open disks removed and either a finite number of handles or a finite number of cross-caps adjoined.

By an *open disk* on S we mean a subset of S which is homeomorphic to the open unit disk in $\mathbb{R}^2$. For $D \subset S$, we let $\mathrm{bd}(D)$ denote the boundary of D with respect to S.

What a *handle* is, should be clear from Figure 1. Adjoining a *cross-cap* means deleting an open disk D on S and identifying diametrically opposite points on $\mathrm{bd}(D)$.

Some examples of surfaces are: the *sphere*, the *plane*, the *torus* and the *projective plane*. The plane can be considered as a sphere with one open disk removed, the torus as a sphere with one handle adjoined, the projective plane as a sphere with one cross-cap adjoined.

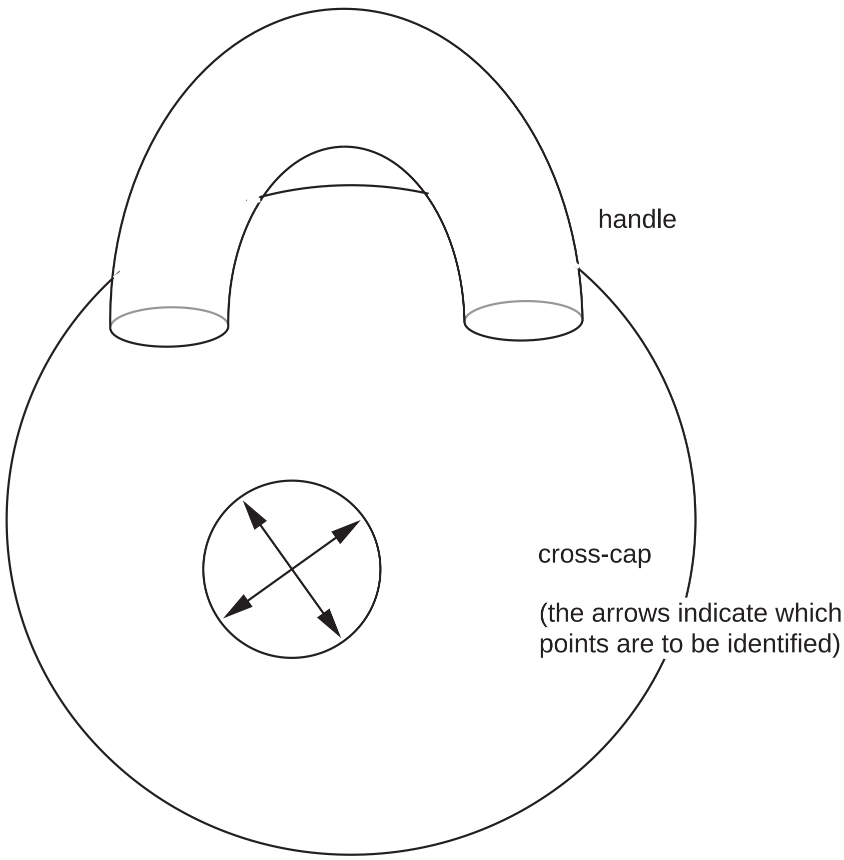


FIGURE 1. Handles and cross-caps.

A *curve* on S is a continuous function $C : [0,1] \to S$; a *closed curve* on S is a curve C with $C(0) = C(1)$. For a curve C, we call $C(0)$ and $C(1)$ the *endpoints* of C. If no confusion arises, we identify (closed) curves with their images on S.

Two curves C and C' are *homotopic*, in notation $C \sim C'$, if there exists a continuous function $\Phi : [0,1] \times [0,1] \to S$ such that $\Phi(0,z) = C(z)$ and $\Phi(1,z) = C'(z)$ for all $z \in [0,1]$ and $\Phi(t,0) = C(0) = C'(0)$ and $\Phi(t,1) = C(1) = C'(1)$ for all $t \in [0,1]$.

Two closed curves C and C' are *freely homotopic*, in notation $C \sim_f C'$, if there exists a continuous function $\Phi : [0,1] \times [0,1] \to S$ such that $\Phi(0,z) = C(z)$ and $\Phi(1,z) = C'(z)$ for all $z \in [0,1]$ (so the endpoint need not be fixed). A closed curve is called *nullhomotopic* if it is freely homotopic to a point.

For convencience, we let 'C is homotopic to C'' be the shorthand notation for: 'C is freely homotopic to C' if C is a closed curve' and 'C is homotopic to

C' if C is not a closed curve'. In notation: '$C \sim C'$' means $C \sim_f C'$ if C is a closed curve and $C \sim C'$ if C is not a closed curve.

Loosely speaking, two curves are homotopic if they both go in the 'same' way around (or over) the removed open disks and adjoined handles and crosscaps. For example, on the sphere and plane (where *no* open disks are removed or handles or crosscaps adjoined) all closed curves are nullhomotopic and all curves with the same endpoints are homotopic.

For any curve C on S, the number of self-crossings (counting multiplicities) of C is denoted by $\mathrm{cr}(C)$. That is,

$$\mathrm{cr}(C) = \tfrac{1}{2}|\{(w,z) \in [0,1] \times [0,1] | C(w) = C(z), w \neq z\}|. \tag{3}$$

A curve C is called *simple* if $\mathrm{cr}(C) = 0$. Moreover, $\mathrm{mincr}(C)$ denotes the minimum of $\mathrm{cr}(C')$ where C' ranges over all curves homotopic to C. So,

$$\mathrm{mincr}(C) = \min\{\mathrm{cr}(C') | C' \sim C\}. \tag{4}$$

For any pair of curves C, D on S, the number of crossings of C and D (counting multiplicities) is denoted by $\mathrm{cr}(C, D)$. That is,

$$\mathrm{cr}(C,D) = |\{(w,z) \in [0,1] \times [0,1] | C(w) = D(z)\}|. \tag{5}$$

Moreover, $\mathrm{mincr}(C, D)$ denotes the minimum of $\mathrm{cr}(C', D')$ where C' and D' range over all curves homotopic to C and D, respectively. So,

$$\mathrm{mincr}(C,D) = \min\{\mathrm{cr}(C',D') | C' \sim C, D' \sim D\}. \tag{6}$$

For a graph G and a curve C we denote by $\mathrm{cr}(G, C)$ the number of intersections of G with C. That is,

$$\mathrm{cr}(G,C) = |\{z \in S^1 | C(z) \in G\}|. \tag{7}$$

3. Homotopic edge-disjoint path problem

Let us reconsider the edge-disjoint paths problem (1). By letting $C_1, \ldots, C_k$ be curves on the sphere S^2 so that the endpoints of C_i are r_i and s_i for $i = 1, \ldots, k$, we see that (1) is equivalent to the following:

$$\begin{array}{ll} \textbf{given}: & \text{a planar graph } G = (V,E) \text{ embedded on the sphere } S^2 \\ & \text{and curves } C_1, \ldots, C_k \text{ on } S^2, \\ \textbf{find}: & \text{pairwise edge-disjoint paths } P_1, \ldots, P_k \text{ in } G \text{ so that} \\ & P_i \sim C_i, \text{ for } i = 1, \ldots, k. \end{array} \tag{8}$$

A natural generalization of problem (8) is to allow the graph G to be embedded on any surface S:

given : a surface S and a graph $G = (V, E)$ embedded on S and curves $C_1, \ldots, C_k$ on S,
find : pairwise edge-disjoint paths $P_1, \ldots, P_k$ in G so that $P_i \sim C_i$ on S, for $i = 1, \ldots, k$. (9)

Problem (9) will be called the *edge-disjoint homotopic paths problem.* This problem is 'easier' in the sense that in (9) it is prescribed how the paths we are searching for 'globally look'. Note that, even though these homotopic paths may not exist in the graph, there might exist edge-disjoint paths $P_1, \ldots, P_k$ in G so that P_i has the same endpoints as C_i $(i = 1, \ldots, k)$. On the other hand, the problem of finding edge-disjoint paths $P_1, \ldots, P_k$ in G so that P_i has the same endpoints as C_i $(i = 1, \ldots, k)$ can be solved by enumerating homotopy types and solving several edge-disjoint homotopic path problems. Thus, problem (9) is interesting for further study.

The cut condition (2) can be restated as: for each closed curve D on S, intersecting G only a finite number of times and not intersecting V, one has

$$\mathrm{cr}(G, D) \geq \textstyle\sum_{i=1}^{k} \min_{C' \sim C_i} \mathrm{cr}(C', D). \tag{10}$$

Let us consider the edge-disjoint homotopic path problem (9) in the case where G is embedded on S^2. Let $I_1, \ldots, I_p$ denote the interiors of some fixed faces of G and let the curves $C_1, \ldots, C_k$ have their endpoints on $\mathrm{bd}(\cup_{j=1}^{p} I_j)$. Let S denote a sphere with the p open disks removed: $S := S^2 \backslash (\cup_{j=1}^{p} I_j)$, and consider the edge-disjoint homotopic path problem (9).

Note that for $p = 1$ the problems (1) and (9) are equivalent. For $p = 1$, the cut condition (10) is *not* sufficient for the existence of edge-disjoint paths. In Figure 2 a counterexample is given. This example does not satisfy the following *Euler condition:*

$$\text{(Euler condition)} \qquad G \cup C_1 \cup \cdots \cup C_k \text{ is eulerian.} \tag{11}$$

Here $G \cup C_1 \cup \cdots \cup C_k$ denotes the graph formed by adding the curves $C_1, \ldots, C_k$ to G as edges. Okamura and Seymour [7] show that if a graph G together with a set of curves $C_1, \ldots, C_k$ satisfies the Euler condition then the cut condition is necessary and sufficient.

Theorem 1 [7]. *Let G be a graph embedded on the sphere S^2. Let I be the interior of some fixed face and $C_1, \ldots, C_k$ be curves on $S^2 \backslash I$ each with endpoints on bd(I) so that the Euler condition (11) is satisfied. Then there exist edge-disjoint paths $P_1, \ldots, P_k$ in G so that $P_i \sim C_i$ on $S^2 \backslash I$ for $i = 1, \ldots, k$ if and only if the cut condition (10) is satisfied.*

A generalization of this theorem to the case where the curves $C_1, \ldots, C_k$ have their endpoints on the boundary of *two* fixed faces ($p = 2$), is shown by Van Hoesel and Schrijver [4].

Theorem 2 [4]. *Let G be a graph embedded on the sphere S^2. Let I_1 and I_2 be the interiors of two fixed faces of G and $C_1, \ldots, C_k$ be curves on $S^2 \backslash (I_1 \cup I_2)$,*

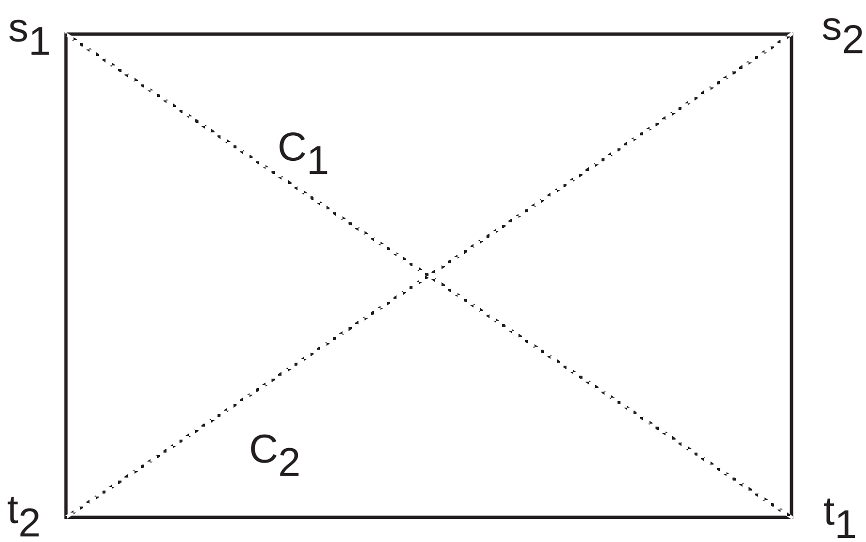


FIGURE 2. The cut condition is not sufficient.

each with endpoints on bd$(I_1 \cup I_2)$ *so that the Euler condition (11) is satisfied. Then there exist edge-disjoint paths* $P_1, \ldots, P_k$ *in* G *so that* $P_i \sim C_i$ *on* $S^2 \backslash (I_1 \cup I_2)$ *for* $i = 1, \ldots, k$ *if and only if the cut condition (10) is satisfied.*

If G is a planar graph embedded on the sphere S^2 and the curves $C_1, \ldots, C_k$ have their endpoints on the boundary of *three* faces ($p = 3$), conditions (10) and (11) are not sufficient for the existence of edge disjoint paths $P_i \sim C_i$ ($i = 1, \ldots, k$) as is shown in Figure 3.

4. PROJECTIVE PLANE

Let us go back to the edge-disjoint paths problem (9) where S is the sphere S^2 and the endpoints of $C_1, \ldots, C_k$ are all on the boundary of one face I. Denote the endpoints of C_i by s_i and t_i for $i = 1, \ldots, k$. Suppose that the commodities $s_1, t_1, \ldots, s_k, t_k$ occur on bd(I) in the order $s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k$.

In that case, the general picture (see for example Figure 4) looks like a picture of the projective plane. (The projective plane can be regarded as a closed unit disk D where diametrically opposite points on bd(D) are identified.) There are only two free homotopy classes for closed curves: a closed curve is either nullhomotopic or homotopic to a closed curve connecting two diametrically opposite points on bd(D). Let B denote some non-nullhomotopic closed curve.

Now instead of asking for edge-disjoint paths $P_1, \ldots, P_k$ in G homotopic to the given curves $C_1, \ldots, C_k$ on the disk, we can ask for (many) edge-disjoint *closed* curves freely homotopic to B on the projective plane. For this surface the cut condition reads: for each closed curve D on S, intersecting G only a finite number of times and not intersecting V, one has

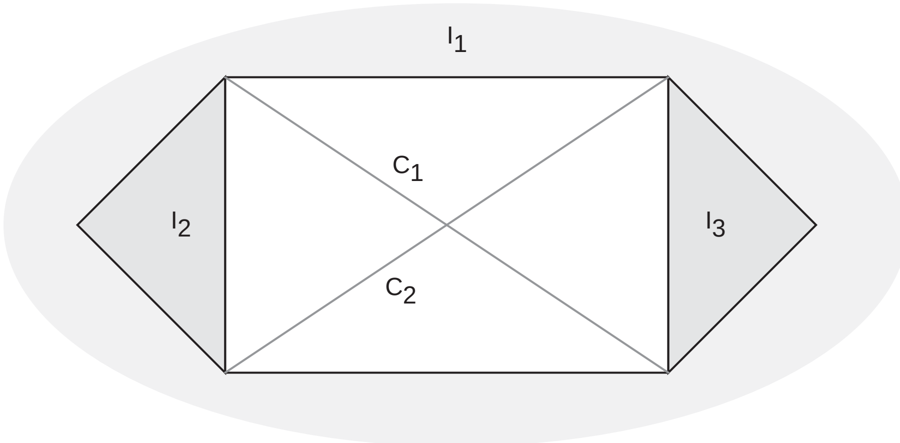


FIGURE 3. Endpoints of curves on the boundary of three faces

$$\mathrm{cr}(G,D) \geq \textstyle\sum_{i=1}^{k} \mathrm{mincr}(D,B). \tag{12}$$

Note that $\mathrm{mincr}(D,B) = 1$ if $D \sim B$ and $\mathrm{mincr}(D,B) = 0$ otherwise. The cut condition (12) is not sufficient for the existence of edge-disjoint closed curves homotopic to B, as the example of Figure 4 shows. However, for eulerian graphs embedded on the projective plane, the following theorem was proved by LINS [6].

THEOREM 3 [6]. *Let S be the projective plane and G be an eulerian graph embedded on S. Then G contains k nontrivial edge-disjoint closed curves if and only if $cr(G,D) \geq k$ for each nontrivial curve D on $S \backslash V$.*

Indeed Lins' theorem can be seen to imply Theorem 1 by an easy construction.

5. SURFACES

The problem of finding k edge-disjoint non-nullhomotopic closed curves in a graph embedded on the projective plane is an example of the *edge-disjoint homotopic cycle problem*:

$$\begin{array}{ll} \textbf{given}: & \text{a graph } G=(V,E) \text{ embedded on a surface } S \text{ and closed} \\ & \text{curves } C_1,\ldots,C_k \text{ on } S, \\ \textbf{find}: & \text{pairwise edge-disjoint cycles } \widetilde{C}_1,\ldots,\widetilde{C}_k \text{ in } G \text{ so that} \\ & \widetilde{C}_i \sim C_i \text{ for } i=1,\ldots,k. \end{array} \tag{13}$$

Problem (13) has the following connection with the edge-disjoint homotopic path problem (9). Given an instance of (9) we make a small hole at the endpoints of each C_i and attach a handle H_i connecting the two holes, add an edge e_i connecting the two endpoints of C_i in such a way that e_i goes over handle H_i and let C_i' be the closed curve $C_i \cup e_i$ for $i = 1,\ldots,k$. Furthermore we

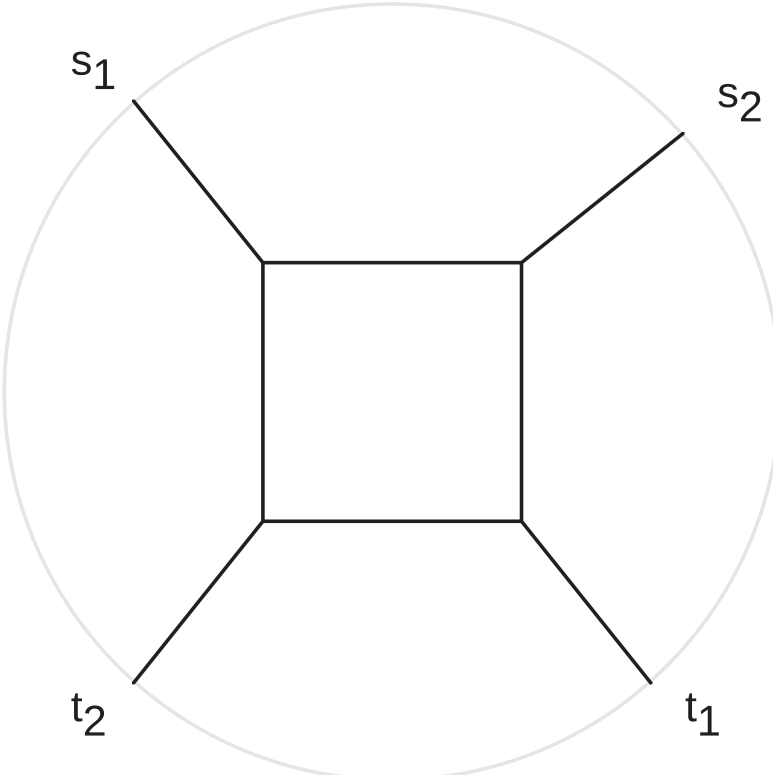


FIGURE 4. The cut condition is not sufficient on the projective plane.

let $\hat{G}$, $\hat{S}$ denote the extended graph, respectively surface thus obtained. There exist pairwise edge-disjoint cycles $\widetilde{C}'_1, \ldots, \widetilde{C}'_k$ in $\hat{G}$ on $\hat{S}$ so that $\widetilde{C}'_i \sim C'_i$ for $i = 1, \ldots, k$ if and only if there exist pairwise edge-disjoint paths $P_1, \ldots, P_k$ in the original graph G so that $P_i \sim C_i$ for $i = 1, \ldots, k$ on surface S.

The formulation (13) has the advantage that the endpoints of the cycles C_i ($i = 1, \ldots, k$) lose their specific role. In a closed curve every point is equal. Furthermore we can apply tools from algebraic topology, where closed curves correspond (in a sense that will not be specified here) to distance-preserving functions on the 'universal covering' surface.

The cut condition for (13) becomes: for each closed curve D on S, intersecting G only a finite number of times and not intersecting V, one has:

$$\text{(cut condition)} \qquad \mathrm{cr}(G, D) \geq \sum_{i=1}^{k} \mathrm{mincr}(C_i, D). \tag{14}$$

In general, however, this cut condition is not sufficient as was already mentioned for the projective plane. For the case where the surface S is the torus T, FRANK and SCHRIJVER [1] consider the following *parity condition*:

$$\begin{array}{l}\text{(parity condition) for each closed curve } D \text{ on } T, \text{ not intersecting} \\ \text{vertices of } G, \text{ the number of crossings of } D \text{ with edges of } G, \text{ plus} \\ \text{the number of crossings with } C_1, \ldots, C_k, \text{ is an even number.}\end{array} \quad (15)$$

THEOREM 4 [1]. *Let $G = (V, E)$ be a graph embedded on the torus T, and let $C_1, \ldots, C_k$ be simple closed curves on T, such that the parity condition* (15) *holds. Then there exist pairwise edge-disjoint closed cycles $\widetilde{C}_1, \ldots, \widetilde{C}_k$ in G so that $\widetilde{C}_i \sim C_i$ $(i = 1, \ldots, k)$, if and only if the cut condition* (14) *holds.*

Currently we are investigating if the cut condition and the parity condition are sufficient for graphs embedded on the Klein bottle as well.

We say that $\widetilde{C}_1, \ldots, \widetilde{C}_u$ is a *fractional packing of cycles homotopic to* $C_1, \ldots, C_k$ if there exist $\lambda_1, \ldots, \lambda_u \geq 0$ so that the following is satisfied:

(i) $\sum_{j=1, \widetilde{C}_j \sim C_i}^{u} \lambda_j = 1 \quad (i = 1, \ldots, k)$,

(ii) $\sum_{j=1}^{u} \lambda_j \chi^{\widetilde{C}_j}(e) \leq 1 \quad (e \in E)$.

Here $\chi^C(e)$, where C is a cycle in G, denotes the number of times that C traverses e. A *fractional path packing* is defined in a similar way.

As the cut condition is not sufficient for the existence of edge-disjoint cycles in general, it is an interesting question to decide if the cut condition is sufficient for the existence of a fractional packing of cycles homotopic to $C_1, \ldots, C_k$. This gives the following problem.

$$\begin{array}{ll}\textbf{given}: & \text{a graph } G = (V, E) \text{ embedded on } S \text{ and closed curves} \\ & C_1, \ldots, C_k \text{ on } S, \\ \textbf{find}: & \text{cycles } \widetilde{C}_1, \ldots, \widetilde{C}_u \text{ in } G \text{ such that } \widetilde{C}_1, \ldots, \widetilde{C}_u \text{ is a} \\ & \text{fractional packing of cycles homotopic to } C_1, \ldots, C_k.\end{array} \quad (16)$$

Note that the cut condition is not sufficient for the existence of a fractional path packing for problem (1). A counterexample is given in [7]. But for problem (16) the cut condition is sufficient. This is shown in [9] for the case where G is embedded on an orientable compact surface and in [3] for the case where G is embedded on any compact surface.

THEOREM 5 [9, 3]. *Let $G = (V, E)$ be a graph embedded on a compact surface S and $C_1, \ldots, C_k$ be cycles in G. Then there exists a fractional packing of cycles homotopic to $C_1, \ldots, C_k$ if and only the cut condition* (14) *is satisfied.*

In this theorem, the necessity of the cut condition is straightforward. The essence of the theorem is that the cut condition is sufficient for the existence of fractional edge-disjoint cycles. In order to illustrate the connection of problem (16) with problems in topology, we will sketch the proof of Theorem 5. (For details see [3] or [9].)

First, the problem is formulated as a polyhedral problem. To that aim let K be the convex cone in $\mathbb{R}^k \times \mathbb{R}^E$ generated by the vectors

$$(\epsilon_i, \chi^\Gamma) \quad (i = 1, \ldots, k,\ \Gamma \text{ cycle in } G \text{ with } \ \Gamma \sim C_i); \tag{17}$$

$$(0, \epsilon_e) \qquad\qquad (e \in E). \tag{18}$$

Here ϵ_i denotes the i-th unit basis vector in $\mathbb{R}^k$. Similarly, ϵ_e denotes the e-th unit basis vector in $\mathbb{R}^E$, 0 denotes the origin in $\mathbb{R}^k$.

The question now is:

Does condition (14) imply that the vector $\mathbf{1}$ belongs to K ?

Here $\mathbf{1}$ denotes the all-one vector in $\mathbb{R}^k \times \mathbb{R}^E$.

By Farkas' lemma, $\mathbf{1} \in K$ if and only if each vector $(p, b) \in \mathbb{Q}^k \times \mathbb{Q}^E$ with nonnegative inner product with each of the vectors (17), (18) also has nonnegative inner product with the vector $\mathbf{1}$. By an easy argument it is sufficient to restrict the vectors (p, b) to those vectors (p, b) where each entry is an even integer and $b > 0$. Let G' be the graph arising from G by replacing each edge e by a path of length $b(e)$. Now G' is a bipartite graph. The surface dual of a bipartite graph is a eulerian graph. From this, one can show that it suffices to prove the following result on eulerian graphs embedded on a surface.

THEOREM 6 [9, 3]. *Let G be an eulerian graph embedded on a compact surface S. Then the edges of G can be decomposed into cycles $C_1, \ldots, C_t$ in such a way that for each closed curve D on $S \setminus V$:*

$$\operatorname{mincr}(G, D) = \textstyle\sum_{i=1}^t \operatorname{mincr}(C_i, D). \tag{19}$$

Here $\operatorname{mincr}(G, D)$ denotes the minimum of $\operatorname{cr}(G, D')$ where D' ranges over all curves on $S \setminus V$ that are homotopic (on S) to D. That is,

$$\begin{aligned}\operatorname{mincr}(G, D) := \min\{&\operatorname{cr}(G, D') | D' \sim D,\ D' \text{ does not traverse} \\ &\text{vertices of } G\ \}.\end{aligned} \tag{20}$$

Note that for *any* decomposition of the edges of G into cycles $C_1, \ldots, C_t$ we have

$$\operatorname{mincr}(G, D) \geq \textstyle\sum_{i=1}^t \operatorname{mincr}(C_i, D). \tag{21}$$

The content of the theorem is that there exists a decomposition having equality. In Section 6 we present an outline of the proof of Theorem 6.

6. MAKING CURVE SYSTEMS MINIMALLY CROSSING BY REIDEMEISTER MOVES

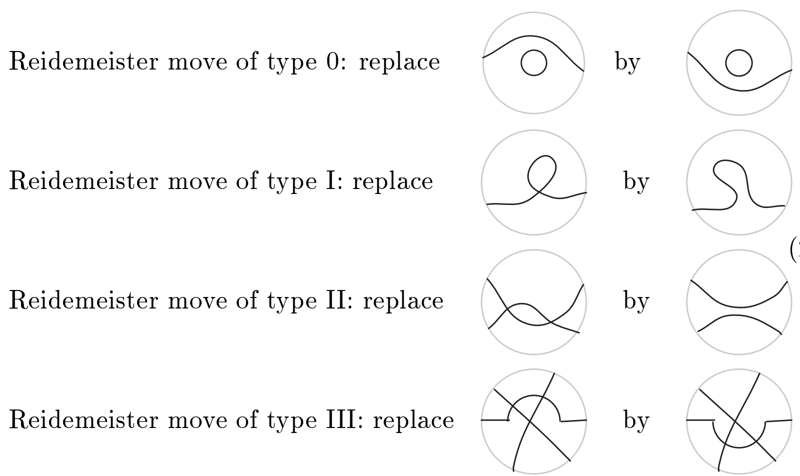
At last, we show that Theorem 6 is implied by a theorem which in itself is interesting from a topological point of view. In order to formulate this we need a few more definitions.

Let $C_1, \ldots, C_k$ be a system of closed curves on S. We call $C_1, \ldots, C_k$ *minimally crossing* if

$$\text{(i)} \qquad \operatorname{cr}(C_i) = \operatorname{mincr}(C_i) \qquad \text{for each } \ i = 1, \ldots, k \tag{22}$$

$$\text{(ii)} \quad \operatorname{cr}(C_i, C_j) = \operatorname{mincr}(C_i, C_j) \quad \text{for all } i, j = 1, \ldots, k \text{ with } i \neq j. \tag{23}$$

We call $C_1, \ldots, C_k$ a *regular* system of curves if $C_1, \ldots, C_k$ have only a finite number of (self-)intersections, each being a crossing of only two curve parts. That is, no point on S is traversed more than twice by $C_1, \ldots, C_k$ and each point of S traversed twice has a disk-neighborhood on which the curve parts are topologically two crossing straight lines. On such a system of curves we define four operations called *Reidemeister moves*, depicted below in (24).



(24)

These moves were introduced by Reidemeister in the study of knots. In particular he showed that any two knots are 'equivalent' if and only if their diagrams can be moved to one another by a series of moves similar to those in (24) (see [5] and [8]).

The pictures in (24) represent the intersection of the union of $C_1, \ldots, C_k$ with a closed disk on S. So no other curve parts than the ones shown intersect such a disk.

The main result of [2] is:

THEOREM 7 [2]. *Any regular system of closed curves on S can be transformed to a minimally crossing system on S by a series of Reidemeister moves.*

It is important to note that the main content of Theorem 7 is that we do not need to apply any of the Reidemeister moves in the reverse direction—otherwise the result would follow quite straightforwardly with the techniques of simplicial approximation. This is also the reason why we have to include the Reidemeister move of type 0. If we were allowed to apply the Reidemeister moves in both directions it could have been replaced by two Reidemeister moves of type II.

The Reidemeister moves can be applied on graphs as well. It is easy to see that:

$$\text{if } G' \text{ rises from } G \text{ by one Reidemeister move of type III then } \mathrm{mincr}(G', D) = \mathrm{mincr}(G, D) \text{ for each curve } D \text{ on } S\backslash V. \tag{25}$$

We show the following proposition:

Theorem 7 implies Theorem 6. (26)

PROOF.

I. We may assume that each vertex v of G has degree at most 4. If v would have a degree larger than 4, we can replace G in a neighborhood of v as in Figure 5. This modification does not change the value of $\mathrm{mincr}(G, D)$ for any D

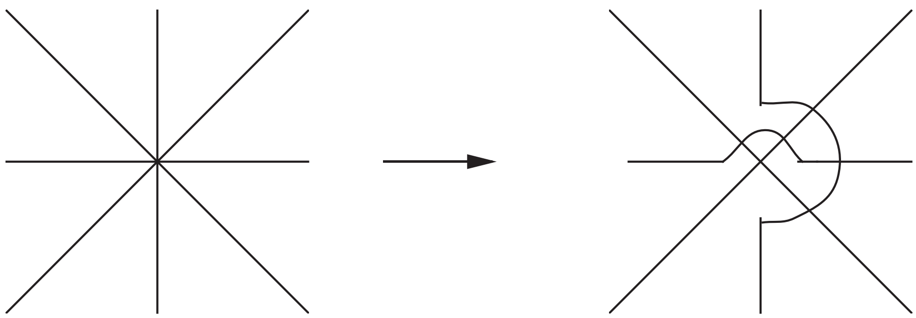


FIGURE 5. Modification of a vertex with degree 8.

on $S\backslash V$. Moreover, closed curves decomposing the edges of the modified graph satisfying (19), directly yield a decomposition of the edges of the original graph satisfying (19).

II. Let $G = (V, E)$ be a counterexample to Theorem 6 with each vertex having degree at most 4 and with $|V| + |E|$ minimal. We will call such a graph a *minimal counterexample*. By (25) it is clear that:

$$\text{if } G' \text{ arises from } G \text{ by one Reidemeister move of type III, then } G' \text{ is also a minimal counterexample.} \tag{27}$$

Let $C_1, \ldots, C_t$ be the *straight decomposition* of G. That is, $C_1, \ldots, C_t$ form a system of closed curves such that each edge of G is traversed exactly once by $C_1, \ldots, C_t$ and such that each vertex of degree 4 represents a (self-)crossing of $C_1, \ldots, C_t$. By the minimality of G, no C_i $(i = 1, \ldots, t)$ is a nullhomotopic curve without intersections with the other curves (otherwise we could delete C_i). Moreover, we have:

$$\text{no series of Reidemeister moves of type III when applied to } C_1, \ldots, C_t \text{ creates a situation where a Reidemeister move of type I or of type II can be applied.} \tag{28}$$

To see (28), we may assume by (27) that G is a minimal counterexample where a Reidemeister move of type I or of type II can be applied. We will lead this to a contradiction. Consider the situation just before the move of type II as in Figure 6(a). Let H denote the graph obtained from G by replacing this

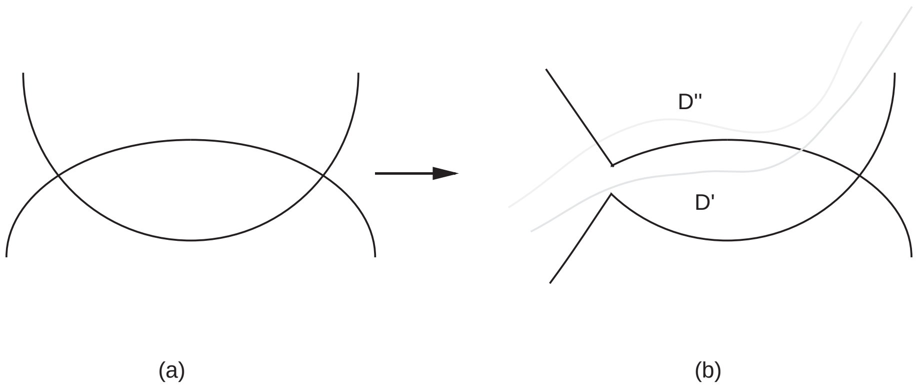


FIGURE 6. configuration where a Reidemeister move of type II can be applied.

configuration by the configuration in Figure 6(b).

It is clear that $\mathrm{mincr}(H, D) \leq \mathrm{mincr}(G, D)$. We claim that $\mathrm{mincr}(H, D) = \mathrm{mincr}(G, D)$ for each closed curve D not intersecting vertices of G. For suppose there is a curve D with $\mathrm{mincr}(H, D) < \mathrm{mincr}(G, D)$. Then any curve $D' \sim D$ with $\mathrm{cr}(H, D') = \mathrm{mincr}(H, D)$ should pass through the opening at v. However, it is possible to reroute the curve D' to obtain a curve D'' (as indicated in Figure 6(b)) which does not pass through the opening at v. Hence we obtain the contradiction:

$$\mathrm{mincr}(G, D) > \mathrm{mincr}(H, D) = \mathrm{cr}(H, D') = \mathrm{cr}(H, D'') = \mathrm{cr}(G, D'').$$

Similar reasoning shows that in a minimal counterexample we cannot apply a Reidemeister move of type I. This gives (28).

III. We complete the proof by showing that $C_1, \ldots, C_t$ satisfy (19). Choose a closed curve D. We may assume that $D, C_1, \ldots, C_t$ form a regular system. By Theorem 7 we can apply Reidemeister moves so as to obtain a minimally crossing system $D', C_1', \ldots, C_t'$. By the arguments above we did not apply Reidemeister moves of type 0, type I or of type II to $C_1, \ldots, C_t$ so for the graph

G' obtained from the final $C'_1, \ldots, C'_t$ we have by (25) that $\mathrm{mincr}(G', D) = \mathrm{mincr}(G, D)$. This yields,

$$\mathrm{mincr}(G,D) = \mathrm{mincr}(G',D) \leq \mathrm{cr}(G',D') = \sum_{i=1}^{t} \mathrm{cr}(C'_i, D') = \\ = \sum_{i=1}^{t} \mathrm{mincr}(C_i, D).$$

Since the converse inequality holds by (21), we have (19). This completes the proof of (26). □

REFERENCES

1. A. FRANK and A. SCHRIJVER (1992). Edge-disjoint circuits in graphs on the torus, *J. Combin. Theory Ser. B* **55**, pp. 9-17.
2. M. DE GRAAF and A. SCHRIJVER (1993). *Making curve systems minimally crossing by Reidemeister moves*, preprint.
3. M. DE GRAAF and A. SCHRIJVER (1993). *Decomposition of graphs on surfaces*, preprint.
4. C. VAN HOESEL and A. SCHRIJVER (1990). Edge-disjoint homotopic paths in a planar graph with one hole, *J. Combin. Theory Ser. B* **48**, pp. 77-91.
5. L.H. KAUFFMAN (1987). *On knots*, Annals of Mathematics Studies (number 115), Princeton University Press, Princeton.
6. S. LINS (1981). A minimax theorem on circuits in projective graphs, *J. Combin. Theory Ser. B* **30**, pp. 253-262.
7. H. OKAMURA and P. D. SEYMOUR (1981). Multicommodity flows in planar graphs, *J. Combin. Theory Ser. B* **31**, pp. 75 - 81.
8. K. REIDEMEISTER (1926). Elementare Begründung der Knotentheorie, *Abh. math. Semin. Hamburg. Univ.* **Bd. 5**, pp. 24 - 32.
9. A. SCHRIJVER (1981). Decompositions of graphs on surfaces and a homotopic circulation theorem, *J. Combin. Theory Ser. B* **51**, pp. 161-210.