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Uniform in ε Convergence of Finite Element Method for Convection-Diffusion Equations Using a priori Chosen Meshes

Mariana Nikolova and Owe Axelsson
Faculty of Mathematics and Informatics
University of Nijmegen, P.O. box 9010, 6500 GL Nijmegen The Netherlands

In this paper we consider the standard Galerkin finite element method applied to singularly perturbed convection-diffusion problems in 2D. Their discretizations are based on piecewise bilinears on a Shishkin mesh with total number of points $O(N^2)$. Discretization error estimates of order $O(N^{-1})$ in L_2 norm, which hold uniformly in ε, are provided. The theoretical results are supported by numerical results for both, the standard Galerkin and the streamline upwind diffusion, methods on a Shishkin mesh. This *a priori* adapted mesh gives good results for exponential and parabolic layers. Nevertheless, the use of Shishkin meshes to solve problems with interior and more complicated layers, the location of which is unknown *a priori*, is still questionable. In this case we use the black box tool - *a posteriori* adapted meshes obtained by some adaptive refinement procedure combined with a special discretization method to get a high order of accuracy. The advantages and disadvantages of *a priori* – and *a posteriori* – adaptive refinement techniques are illustrated by numerical experiments in 2D.

1. Introduction

We consider the convection-diffusion problem (1.1) with boundary conditions (1.2),

$$\mathcal{L}_\varepsilon u = -\varepsilon \Delta u + \mathbf{b} \cdot \nabla u + cu = f, \qquad \mathbf{x} \in \Omega \subset \mathbb{R}^2, \tag{1.1}$$

$$u = 0,\ \mathbf{x} \in \Gamma_-, \qquad \frac{\partial u}{\partial \mathbf{n}} = g,\ \mathbf{x} \in \Gamma_1 = \partial\Omega/\Gamma_- = \Gamma_+ \cup \Gamma_0. \tag{1.2}$$

The solution is driven by the vector field $\mathbf{b}$, and $\Gamma_- = \{\mathbf{x} \in \Gamma, \mathbf{b} \cdot \mathbf{n} < 0\}$, $\Gamma_+ = \{\mathbf{x} \in \Gamma, \mathbf{b} \cdot \mathbf{n} > 0\}$, $\Gamma_0 = \{\mathbf{x} \in \Gamma, \mathbf{b} \cdot \mathbf{n} = 0\}$ are the corresponding inflow, outflow and characteristic parts of Γ, the boundary of Ω.

Here Ω is a bounded convex domain of polygonal type and $\mathbf{b}$, f are given sufficiently smooth functions in Ω. The singular perturbation parameter ε, $0 \le \varepsilon \le 1$ is used to measure the relative amount of diffusion to convection. The function g is given on Γ_1 and $\mathbf{n}$ is the outward normal vector on this part of the boundary. Further, we assume

$$\min_{\mathbf{x} \in \Omega} (c - \tfrac{1}{2} \mathrm{div}\, \mathbf{b}) \ge c_0 > 0. \tag{1.3}$$

Under this condition it can be shown that (1.1), (1.2) has a unique solution in $H_0^1(\Omega)$.
In practice, (1.3) can always be satisfied by a proper variable transformation of u. The assumption on homogeneous Dirichlet boundary conditions are made just to simplify the presentation, because a linear transformation $\widetilde{u} = u_- + u$, where u_- represents any given inhomogeneous boundary condition on Γ_-, leads to homogeneous boundary conditions for u.

The major problem in the numerical solution of (1.1) is to find a numerical approximation scheme which is uniformly accurate in ε and with a solution cost which does not grow with ε. The standard Galerkin finite element scheme on a uniform mesh does not belong to this class. Moreover, it is numerically unstable and gives an oscillating approximate solution unless the finite element mesh is extremely fine. Furthermore, the pointwise error is not necessarily reduced by successive uniform refinement of the mesh in contrast to solving nonsingularly perturbed problems. Recently, three books [9], [10], [14] appeared about the numerical solution of singularly perturbed problems.

The introduction of specially adapted (*a priori* or *a posteriori*) meshes in the layer region(s) overcomes these difficulties. Bakhvalov was the first using a uniform mesh outside the layer(s) and specially graded mesh at the layer(s), see [5]. Then, Shishkin [16] introduced *piecewise equidistant meshes* and showed that one can also obtain uniform convergence in the layer. These Shishkin meshes combined with standard Galerkin (SG) or streamline upwind diffusion (SUPD) finite elements discretization are the focus of our paper. It is organized as follows. In Section 2, *the uniform convergence in* ε of the SG method applied to problems with exponential layers is analyzed. In Section 3, we do similar analysis of problems with parabolic layer. The uniform L_2-norm error estimates, obtained in these sections are of optimal order $O(N^{-1})$. The SUPD method is introduced in Section 4. Several numerical experiments on Shishkin meshes as well as some illustrations concerning the necessity of using *a posteriori* adapted meshes for more general problems are given in Section 5.

NOTATION 1. *Throughout the paper we denote with C a generic constant independent of ε and the mesh parameters.*

NOTATION 2. *We denote by $(.,.)$ and $\|.\|$ the usual $L_2(\Omega)$ product and L_2-norm.*

Definition 1.1. *The weighted energy norm is defined by*

$$|||u|||^2 = \varepsilon\|\nabla u\|^2 + \|u\|^2.$$

2. Convergence analysis of problems with exponential layers. L_2- norm error estimates

We consider the problem (1.1) in $\Omega = (0,1)^2$ with homogeneous boundary conditions on Γ and the additional assumption that

$$\mathbf{b} = (b_1, b_2) \geq (\beta_1, \beta_2) > (0,0). \tag{2.1}$$

Exponential boundary layers occur then at the outflow boundary $\Gamma_+ = \{x = 1 \cup y = 1\}$. Condition (2.1) excludes the occurrence of internal and parabolic boundary layer(s). To avoid also layers caused by data incompatibility at the corners of the unit square the zero-order compatibility conditions should be imposed,

$$f(0,0) = f(0,1) = f(1,0) = f(1,1) = 0, \tag{2.2}$$

which ensure that $u(x,y) \in \mathbf{C}^2(\overline{\Omega})$, for more details see [8]. Roos [13] assumes in addition the first-order compatibility conditions at the corner (0,0) of the unit square

$$f_x(0,0) = f_y(0,0) = 0, \tag{2.3}$$

and deduces Theorem 2.1, which plays important role in the estimate of certain derivatives of $u(x,y)$. Similar estimates are also given in [2].

Theorem 2.1. *Let* $\mathbf{b}$ *and* f *be sufficiently smooth and assume that* f *satisfies (2.2), (2.3). Then, the solution* u *of (1.1) has the representation* $u = u_0 + v$, *where* u_0 *is the smooth part and*

$$\left|\frac{\partial^{i+j}u_0(x,y)}{\partial x^i \partial y^j}\right| \leq C,$$

and $v = v_1 + v_2 + v_{12}$, *where*

$$\left|\frac{\partial^{i+j}v_1(x,y)}{\partial x^i \partial y^j}\right| \leq C\varepsilon^{-i}exp(-\beta_1(1-x)/\varepsilon),$$

$$\left|\frac{\partial^{i+j}v_2(x,y)}{\partial x^i \partial y^j}\right| \leq C\varepsilon^{-j}exp(-\beta_2(1-y)/\varepsilon),$$

$$\left|\frac{\partial^{i+j}v_{12}(x,y)}{\partial x^i \partial y^j}\right| \leq C\varepsilon^{-(i+j)}exp(-\beta_1(1-x)/\varepsilon)exp(-\beta_2(1-y)/\varepsilon),$$

for all $(x,y) \in \Omega$ *and* $0 \leq i+j \leq 2$.

The variational formulation of (1.1) is: find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = (f,v), \quad \text{for all } v \in H_0^1(\Omega), \tag{2.4}$$

where $H_0^1(\Omega)$ is the usual Sobolev space of functions satisfying homogeneous Dirichlet boundary conditions and $a(u,v)$ is the corresponding bilinear form

$$a(u,v) = \varepsilon(\nabla u, \nabla v) + (\mathbf{b} \cdot \nabla u, v) + (cu, v).$$

Based on Green's formula and (1.3) we obtain

$$a(u,u) \geq \varepsilon(\nabla u, \nabla u) + c_0(u,u) \geq min\{1, c_0\}|||u|||^2,$$

thus

$$a(u,u) \geq C|||u|||^2. \tag{2.5}$$

We use the standard Galerkin finite element method on a Shishkin mesh to discretize (2.4). Let V^N be a finite element subspace of $H_0^1(\Omega)$ consisting of piecewise bilinear functions $\{\varphi_{i,j}(x,y)\}_{i,j=1}^N$ on Ω that vanish on $\partial\Omega$. The finite element approximation $u^N \in V^N$ of u satisfies

$$a(u^N, \varphi) = (f, \varphi), \quad \text{for all } \varphi \in V^N. \tag{2.6}$$

The uniqueness of u^N is guaranteed by (2.5). Subtracting (2.6) from (2.4), we obtain the orthogonal relation,

$$a(u - u^N, v^N) = 0, \quad \text{for all } v^N \in V^N. \tag{2.7}$$

The Shishkin mesh is defined by a tensor product of two one-dimensional piecewise equidistant meshes. Let N_x and N_y be the points in $x-$ and $y-$direction, correspondingly. Then, we set

$$\tau_x = min\{\frac{1}{2}, \frac{2}{\beta_1}\varepsilon \ln N_x\}, \qquad \tau_y = min\{\frac{1}{2}, \frac{2}{\beta_2}\varepsilon \ln N_y\},$$

and call $1-\tau_x$ and $1-\tau_y$ the transition points from the coarse to the fine mesh in the corresponding directions. The coarse and fine meshsizes are defined by

$$\begin{array}{ll} H_x = (1-\tau_x)/(N_x/2), & H_y = (1-\tau_y)/(N_y/2), \\ h_x = \tau_x/(N_x/2), & h_y = \tau_y/(N_y/2), \end{array}$$

and written formally

$$\begin{aligned} &\Omega_x = \Omega_{c,x} \cup \Omega_{f,x}, \qquad \text{where} \\ &\Omega_{c,x} = \{x_i = iH_x, i = 0, ..., N_x/2\}, \\ &\Omega_{f,x} = \{x_i = 1 - \tau_x + (i - N_x/2)h_x, i = N_x/2 + 1, ..., N_x\}. \end{aligned} \tag{2.8}$$

Analogously, Ω_y is defined. Then, the piecewise equidistant Shishkin mesh in Ω is $\Omega_{xy} = \Omega_x \times \Omega_y$. It is coarse on $[0, 1-\tau_x] \times [0, 1-\tau_y]$ and much finer near Γ_+. Observe that very long stretched elements are used in the layer regions.

For notational simplicity we assume that $N_x = N_y = N$ in the rest of the section. Further, we derive an estimate for $\|u - u^N\|$.

NOTATION 3. *u^I denotes the bilinear interpolant to u on our mesh.*

DEFINITION 2.1. $\Omega_c = [0, 1-\tau_x] \times [0, 1-\tau_y]$ *and* $\Omega_f = \Omega_{xy}/\Omega_c$.

THEOREM 2.2. *Let u be the solution of (2.4) and u^N be the standard Galerkin solution of (2.6). Then, we have*

$$\|u - u^N\| \leq CN^{-1}.$$

PROOF. The coercivity (2.5) of the bilinear form and the relation (2.7) give,

$$C|||u^I - u^N|||^2 \leq a(u^I - u^N, u^I - u^N) = a(u^I - u, u^I - u^N) + a(u - u^N, u^I - u^N)$$
$$= a(u^I - u, u^I - u^N).$$

Using the Green's formula we obtain,

$$a(u^I - u, u^I - u^N) = \varepsilon(\nabla(u^I - u), \nabla(u^I - u^N)) + (\mathbf{b} \cdot \nabla(u^I - u), u^I - u^N)$$
$$+ c(u^I - u, u^I - u^N) =$$
$$\varepsilon(\nabla(u^I - u), \nabla(u^I - u^N)) - (u^I - u, \mathbf{b} \cdot \nabla(u^I - u^N)) + (c - \operatorname{div} \mathbf{b})(u^I - u, u^I - u^N).$$

Let us consider each term separately. In the estimate of the first term we shall use that u^N and u^I are piecewise bilinears and then on each rectangle $\tau \in \Omega_{xy}$, $(u^I - u^N)_x$ and $(u^I - u^N)_y$ do not depend on x and y, respectively. Therefore, $(u^I - u^N)_{xx}$ and $(u^I - u^N)_{yy}$ are zero on each τ. Then,

$$|\varepsilon((u^I - u)_x, (u^I - u^N)_x)| = \left| \int_0^1 \Big(\sum_{1 \leq i \leq N} \int_{x_{i-1}}^{x_i} \varepsilon(u^I - u)_x (u^I - u^N)_x dx \Big) dy \right| \leq$$

$$\left| \int_0^1 \Big(\sum_{1 \leq i \leq N} \varepsilon(u^I - u)|_{x_{i-1}}^{x_i} (u^I - u^N)_x \Big) dy \right| \leq$$

$$C\varepsilon \|u^I - u\|_{\infty,\Omega} \left| \sum_{1 \leq i \leq N} \int_0^1 (u^I - u^N)_x dy \right| \leq$$

$$C\varepsilon N \|u^I - u\|_{\infty,\Omega} \left| \int_0^1 \int_0^1 (u^I - u^N)_x dy dx \right| \leq$$

$$C\varepsilon^{1/2}N\|u^I-u\|_{\infty,\Omega}\varepsilon^{1/2}\|(u^I-u^N)_x\|.$$

Similarly, we have

$$|\varepsilon((u^I-u)_y,(u^I-u^N)_y)|\le C\varepsilon^{1/2}N\|u^I-u\|_{\infty,\Omega}\varepsilon^{1/2}\|(u^I-u^N)_y\|.$$

For the third term we obtain,

$$|(c-\operatorname{div}\mathbf{b})(u^I-u,u^I-u^N)|\le\|c-\operatorname{div}\mathbf{b}\|_{\infty,\Omega}\|u^I-u\|\|u^I-u^N\|$$
$$\le C\|u^I-u\|_{\infty,\Omega}\|u^I-u^N\|.$$

M. Stynes and E. O'Riordan prove in [17] (Theorem 4.2), that

$$\|u^I-u\|_{\infty,\Omega}\le CN^{-2}\ln^2 N. \tag{2.9}$$

They essentially use Theorem 2.1 and the facts that $e^{-\beta_1\tau_x/\varepsilon}=N^{-2}$ and $e^{-\beta_2\tau_y/\varepsilon}=N^{-2}$. Hence,

$$|\varepsilon(\nabla(u^I-u),\nabla(u^I-u^N))+(c-div\mathbf{b})(u^I-u,u^I-u^N)|\le$$

$$\begin{aligned}&C(\varepsilon^{1/2}(\|(u^I-u^N)_x\|+\|(u^I-u^N)_y\|)+\|u^I-u^N\|)\|u^I-u\|_{\infty,\Omega}\le\\&CN^{-2}\ln^2 N|||u^I-u^N|||.\end{aligned}\tag{2.10}$$

Finally, we consider the second term of $a(u^I-u,u^I-u^N)$.

$$|(u^I-u,\mathbf{b}\cdot\nabla(u^I-u^N))|\le\|u^I-u\|_{\Omega_c}\|\mathbf{b}\cdot\nabla(u^I-u^N)\|_{\Omega_c}$$
$$+\|u^I-u\|_{\infty,\Omega_f}\int_{\Omega_f}|\mathbf{b}\cdot\nabla(u^I-u^N)|d\Omega_f.$$

In Ω_c we have an equidistant coarse mesh and the standard inverse inequality holds, i.e.,

$$\|\mathbf{b}\cdot\nabla(u^I-u^N)\|_{\Omega_c}\le CN\|u^I-u^N\|_{\Omega_c},$$

therefore,

$$\|u^I-u\|_{\Omega_c}\|\mathbf{b}\cdot\nabla(u^I-u^N)\|_{\Omega_c}\le CN^{-2}N\|u^I-u^N\|_{\Omega_c}\le CN^{-1}|||u^I-u^N|||.$$

Using the Cauchy-Schwarz inequality and (2.9),

$$\|u^I-u\|_{\infty,\Omega_f}\int_{\Omega_f}|\mathbf{b}\cdot\nabla(u^I-u^N)|d\Omega_f$$

$$\le C\|u^I-u\|_{\infty,\Omega_f}(\operatorname{area}\Omega_f)^{1/2}\|\nabla(u^I-u^N))\|_{\Omega_f}\le$$
$$C(N^{-2}\ln^2 N)(\varepsilon\ln N)^{1/2}\|\nabla(u^I-u^N)\|_{\Omega_f}\le CN^{-1}|||u^I-u^N|||.$$

The factor $N^{-1}(\ln N)^{5/2}$ above is less than 1 for all N. Summing up,

$$|(u^I - u, \mathbf{b} \cdot \nabla(u^I - u^N))| \leq CN^{-1} |||u^I - u^N|||. \tag{2.11}$$

Based on (2.10) and (2.11) we obtain,

$$|||u^I - u^N|||^2 \leq CN^{-2} \ln^2 N |||u^I - u^N||| + CN^{-1} |||u^I - u^N|||,$$

thus,

$$|||u^I - u^N||| \leq CN^{-1}.$$

Then,

$$\|u - u^N\| \leq \|u - u^I\| + \|u^I - u^N\| \leq C\|u - u^I\|_\infty + |||u^I - u^N||| \leq$$

$$CN^{-2} \ln^2 N + CN^{-1},$$

which completes the proof. □

REMARK 2.1. *The uniform in ε error estimate in L_2-norm is the highest order possible for the standard Galerkin method and it cannot be improved using the Aubin-Nitsche trick.*

3. Convergence analysis of problems with parabolic layers. L_2-norm error estimates

We consider the problem (1.1) in $\Omega = (0,1)^2$ with velocity vector $\mathbf{b} = (0, b)$, where $b > 0$ and boundary conditions

$$\begin{aligned} u\big|_{x=0} = 0, \quad u\big|_{x=1} = g_1, \\ u\big|_{y=0} = 0, \quad u\big|_{y=1} = g_2, \end{aligned} \tag{3.1}$$

where g_1 and g_2 are smooth and continuous at the corner points.
Let u_0 be the solution of the reduced problem

$$b(u_0)_y + cu_0 = f(x, y), \qquad \text{on } \Omega, \qquad u_0\big|_{y=0} = 0,$$

which is called Cauchy problem and if b is a constant it has the solution

$$u_0 = \frac{1}{b} exp(\frac{c}{b} y) \int_0^y f(x, z) exp(-\frac{c}{b} z) dz,$$

which is smooth without any additional assumption on the data at the corner (0,0) in contrast with the solution of the reduced equation in the case of exponential layer.
In general, u has a boundary layer along $x = 0$. Introducing $\eta = \frac{x}{\sqrt{\varepsilon}} \in [0, 1/\sqrt{\varepsilon}]$ the layer correction $w(\eta, y)$ satisfies,

$$-w_{\eta\eta} + bw_y + cw = 0, \qquad \text{on } \Omega, \tag{3.2}$$

$$w|_{y=0} = 0,\ w|_{\eta=0} = -u_0(0, y) = \gamma(y). \tag{3.3}$$

The equation (3.2) with the boundary conditions (3.3) form an initial boundary value problem of parabolic type, so at $x = 0$ we say that we have a parabolic layer.
The exact solution of (3.2)&(3.3) has the integral representation

$$w(\eta, y) = \frac{1}{\sqrt{2\pi}} \int\limits_{\eta/\sqrt{\frac{2y}{b}}}^{\infty} exp(-\frac{t^2}{2})\gamma(y - \frac{b\eta^2}{2t^2})exp(-\frac{c\eta^2}{2t^2})dt. \tag{3.4}$$

If $\gamma(0) = \gamma'(0) = 0$ then $\frac{\partial^2 w}{\partial y^2}$ is uniformly bounded in $\Omega \cup \partial\Omega$. In the case $\gamma'(0) \neq 0$, $\frac{\partial^2 w}{\partial y^2}$ has a singularity at the origin.
We define $g_1 = (u_0 + w)|_{x=1}$ and $g_2 = (u_0 + w)|_{y=1}$ in order to exclude the development of a second parabolic layer at $x = 1$ and an exponential layer at $y = 1$. In this way we avoid an overlap of parabolic and exponential layers at the outflow corners (0,1) and (1,1) and make the asymptotic structure $u_{as} = u_0 + w$ much more simple.
For the sake of completeness we mention here the well known lemma presented in papers concerning the asymptotic expansion of solution of problem with parabolic layer.

LEMMA 3.1. *There exists a positive constants C independent of ε such that*

$$|u(x, y) - u_{as}(x, y)| \leq C\varepsilon, \qquad \textit{for all } (x, y) \in \Omega \cup \partial\Omega.$$

S. Shih and R.Kellogg prove in [15] the following corollary.

COROLLARY 3.1. *(Corollary of Theorem 3.8 in [15]) There exist two positive constants C and m independent of ε such that*

$$\left| \frac{\partial^i w(\eta, y)}{\partial y^i} \right| \leq Ce^{-m\eta}, \qquad i{=}0,1,2, \tag{3.5}$$

$$\left| \frac{\partial^{2i} w(\eta, y)}{\partial \eta^{2i}} \right| \leq Ce^{-m\eta}, \qquad i{=}0,1,$$

for all $(\eta, y) \in [0, 1/\sqrt{\varepsilon}] \times [0, 1]$.

Since $\frac{\partial x}{\partial \eta} = \sqrt{\varepsilon}$ we obtain the estimate

$$\left| \frac{\partial^2 w(x, y)}{\partial x^2} \right| \leq C\varepsilon^{-1} e^{-mx/\sqrt{\varepsilon}}, \quad \text{for all } (x, y) \in \Omega, \tag{3.6}$$

which is the key to the correct choice of the transition point between the coarse and the fine mesh. Because of the boundary conditions (3.1) the solution u belongs to

$$H_g^1(\Omega) = \{u \in H^1(\Omega); u = g \text{ on } \Gamma\},$$

where g is defined by (3.1). $H_g^1(\Omega)$ is not a linear space but for any fixed $u^* \in H_g^1(\Omega)$ we can write

$$H_g^1(\Omega) = \{\widetilde{u} \in H^1(\Omega); \widetilde{u} = u + u^*, u \in H_0^1(\Omega)\},$$

and define

$$(f^*, v) = (f, v) - a(u^*, v).$$

Then the variational formulation is: find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = (f^*, v), \quad \text{for all } v \in H_0^1(\Omega), \tag{3.7}$$

where

$$a(u, v) = \varepsilon(\nabla u, \nabla v) + (bu_y, v) + (cu, v) = \varepsilon(\nabla u, \nabla v) - (u, bv_y) + (cu, v). \tag{3.8}$$

We use the standard Galerkin finite element method on a Shishkin mesh to discretize (3.7). The finite element approximation $u^N \in V^N$ of u satisfies

$$a(u^N, \varphi) = (f^*, \varphi), \quad \text{for all } \varphi \in V^N. \tag{3.9}$$

Similarly to the previous section we have a coercivity bound of type (2.5) and an orthogonal relation as (2.7).
The Shishkin mesh is defined as follows. Let we have N_x and N_y points in $x-$ and $y-$direction, correspondingly. Then, we set

$$\tau_x = min\{\frac{1}{2}, \frac{2}{m}\sqrt{\varepsilon}\ln N_x\}$$

and call τ_x the transition point from the fine to the coarse mesh in the $x-$direction.

REMARK 3.1. *The constant m in the definition of τ_x does not have a fixed value since the e stimate (3.6) is used, where m is an arbitrary positive constant.*

The coarse and fine meshsizes are defined by

$$H_x = (1 - \tau_x)/(N_x/2), \qquad H_y = 1/N_y,$$
$$h_x = \tau_x/(N_x/2),$$

and written formally

$$\Omega_y = \{y_j = jH_y, j = 0, ..., N_y\}, \qquad \Omega_x = \Omega_{c,x} \cup \Omega_{f,x}, \qquad \text{where}$$
$$\Omega_{f,x} = \{x_i = ih_x, i = 0, ..., N_x/2\}, \tag{3.10}$$
$$\Omega_{c,x} = \{x_i = \tau_x + (i - N_x/2)H_x, i = N_x/2 + 1, ..., N_x\}.$$

Then, the Shishkin mesh in Ω is $\Omega_{xy} = \Omega_x \times \Omega_y$. It is coarse on $[\tau_x, 1] \times [0, 1]$ and much finer near $x = 0$ in the $x-$ direction.

For notational simplicity we assume that $N_x = N_y = N$ in the rest of the section. Further, we derive an estimate for $\|u - u^N\|$.

DEFINITION 3.1. $\Omega_c = [\tau_x, 1] \times [0, 1]$ *and* $\Omega_f = \Omega_{xy}/\Omega_c$.

THEOREM 3.1. *Let u be the solution of (3.7) and u^N be the standard Galerkin solution of (3.9). Then, we have*

$$\|u - u^N\| \leq CN^{-1}.$$

PROOF. The coercivity of the bilinear form $a(.,.)$ and the orthogonal equality give,

$$C|||u^I - u^N|||^2 \leq \varepsilon(\nabla(u^I - u), \nabla(u^I - u^N)) - (u^I - u, b(u^I - u^N)_y) + c(u^I - u, u^I - u^N).$$

Following the proof of Theorem 2.2 we get an estimate similar to (2.10), i.e.,

$$|\varepsilon(\nabla(u^I - u), \nabla(u^I - u^N)) + c(u^I - u, u^I - u^N)| \leq CN^{-2} \ln^2 N |||u^I - u^N|||.$$

Here we have used that $\|u - u^I\|_\infty \leq CN^{-2} \ln^2 N$, the proof of which is based on estimates (3.5) &(3.6), the choice of h_x and the fact that $e^{-\frac{mx}{\sqrt{\varepsilon}}} \leq e^{-\frac{m\tau_x}{\sqrt{\varepsilon}}} = N^{-2}$ for $x \in [\tau_x, 1]$.
The second term of the bilinear form $a(u^I - u, u^I - u^N)$ is bounded by

$$|(u^I - u, b(u^I - u^N)_y| \leq \|u^I - u\|_{\Omega_c} \|b(u^I - u^N)_y\|_{\Omega_c}$$
$$+ \|u^I - u\|_{\infty, \Omega_f} \int\limits_{\Omega_f} |b(u^I - u^N)_y| d\Omega_f.$$

In the $y-$direction we have an equidistant coarse mesh with meshsize $H_y = 1/N$ and the standard inverse inequality holds, i.e.,

$$\|(u^I - u^N)_y\|_{\begin{Bmatrix}\Omega_c \\ \Omega_f\end{Bmatrix}} \leq CN \|u^I - u^N\|_{\begin{Bmatrix}\Omega_c \\ \Omega_f\end{Bmatrix}}.$$

Therefore,

$$\|u^I - u\|_{\Omega_c} \|b(u^I - u^N)_y\|_{\Omega_c} \leq CN^{-1} |||u^I - u^N|||$$

and

$$\|u^I - u\|_{\infty, \Omega_f} \int\limits_{\Omega_f} |b(u^I - u^N)_y| d\Omega_f \leq C\|u^I - u\|_{\infty, \Omega_f} (\text{area}\, \Omega_f)^{1/2} \|(u^I - u^N)_y\|_{\Omega_f} \leq$$

$$C(N^{-2} \ln^2 N)(\sqrt{\varepsilon} \ln N)^{1/2} N \|u^I - u^N\|_{\Omega_f} \leq CN^{-1} |||u^I - u^N|||.$$

The factor $\varepsilon^{1/4} (\ln N)^{5/2}$ above is a small constant when $\varepsilon \to 0$. Summing up,

$$|(u^I - u, b(u^I - u^N)_y)| \leq CN^{-1} |||u^I - u^N|||.$$

Analogously to the end of the proof of Theorem 2.2 we obtain

$$\|u - u^N\| \leq CN^{-1}.$$

□

4. Streamline upwind diffusion method

The streamline upwind diffusion (SUPD) method was introduced by Hughes and Brooks [6] and its behavior on uniform meshes is studied by many authors, but on nonuniform meshes, in particular Shishkin meshes, it is unknown. Recently, the paper [18] by M. Stynes and L. Tobiska gave the first analysis of the error dependence on the user-chosen parameter τ_0 specifying the Shishkin mesh.

The SUPD method for solving (1.1)&(1.2) can be formulated as a Petrov-Galerkin method with special test functions or as a standard Galerkin method for the third order problem,

$$\mathcal{L}_\varepsilon u - \nabla \cdot ((\underline{\delta} \cdot \mathbf{b})\mathcal{L}_\varepsilon u) = f - \nabla \cdot ((\underline{\delta} \cdot \mathbf{b}) f) \tag{4.1}$$

with properly chosen boundary conditions. Here $\underline{\delta} = (\delta_1, \delta_2)$ is the streamline diffusion method parameter and $(\underline{\delta} \cdot \mathbf{b})$ is a pointwise vector multiplication, i.e, $(\underline{\delta} \cdot \mathbf{b}) = (\delta_1 b_1, \delta_2 b_2)$. The bilinear form corresponding to (4.1) is

$$a_{SD}(u,v) = a(u,v) + b_{SD}(u,v) = (f,v) + (f, (\underline{\delta} \cdot \mathbf{b}) \cdot \nabla v),$$

where

$$a(u,v) = \varepsilon(\nabla u, \nabla v) + (\mathbf{b} \cdot \nabla u, v) + (cu, v), \tag{4.2}$$

$$b_{SD}(u,v) = (\mathcal{L}_\varepsilon u, (\underline{\delta} \cdot \mathbf{b}) \cdot \nabla v). \tag{4.3}$$

Let σ be an arbitrary rectangle from our Shishkin mesh. We rewrite $b_{SD}(.,.)$ in the form

$$\begin{aligned} \widetilde{b}_{SD}(u,v) \; &= \sum_{\sigma \in \Omega_{xy}} \Bigg(\int_\sigma \nabla \cdot (-\varepsilon \nabla u)((\underline{\delta} \cdot \mathbf{b}) \cdot \nabla v) d\sigma + \\ &\int_\sigma (\mathbf{b} \cdot \nabla u + c)((\underline{\delta} \cdot \mathbf{b}) \cdot \nabla v) d\sigma \Bigg). \end{aligned} \tag{4.4}$$

Note that $u \in H^2(\Omega)$ implies that $\nabla \cdot (\varepsilon \nabla u) \in L_2(\Omega)$, however, if we consider the finite element space V^N and $u^N \in V^N$ we have in general $\nabla \cdot (\varepsilon \nabla u^N) \notin L_2(\Omega)$ as $V^N \not\subset H^2(\Omega)$. Hence, the Galerkin finite element formulation must be based on (4.2)&(4.4). Thus, the finite element approximation u^N of u satisfies

$$a(u^N, v) + \widetilde{b}_{SD}(u^N, v) = (f,v) + (f, (\underline{\delta} \cdot \mathbf{b}) \cdot \nabla v), \quad \text{for all } v \in V^N. \tag{4.5}$$

A standard way of stabilizing (4.5) is to choose

$$\delta_1 = \text{width}(\sigma)/(2b_1(P_1)), \qquad \delta_2 = \text{height}(\sigma)/(2b_2(P_2)), \tag{4.6}$$

where P_1 and P_2 are the middle points in the $x-$ and the $y-$direction of σ. Now we note that if u^N is piecewise bilinear then $\nabla \cdot (\nabla u^N)$ is zero on each σ and the first term of $\widetilde{b}_{SD}(.,.)$ in (4.4) vanishes. The second term of $\widetilde{b}_{SD}(.,.)$ corresponds to a diffusion term along the streamline direction of $\mathbf{b}$. This term is

not derived by an artificial streamline diffusion method, i.e., by a perturbation of the equation (1.1). Instead, we have embedded (1.1) into (4.1) and the righthand side has been changed accordingly. An analysis of this method on a uniform mesh can be found in [1], [7], [11]. Estimates on arbitrary and Shishkin meshes in one dimensional case are given in [18].
The SUPD method has the following advantages compared to the classical Galerkin method:

(i) There are no or only minor oscillations due to the stabilization effect of the streamline upwind diffusion term;

(ii) It gives a coercive form uniformly bounded in ε and the resulting matrix is positive definite, which is a valuable property for an iterative solver;

(iii) The order of convergence is $O(N^{-3/2})$ on a uniform mesh;

The last property is not theoretically proven yet on a Shishkin mesh in 2D but the numerical calculations of $\|.\|_\infty$ and $\|.\|$ norms of the error, presented in the next section, are good evidence of that.

5. Numerical results

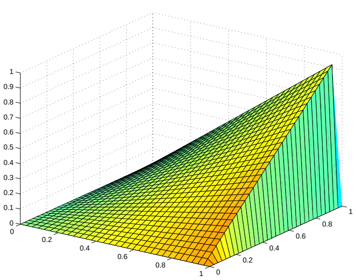


Figure 1. Exact solution (5.2), $\varepsilon = 10^{-5}$.

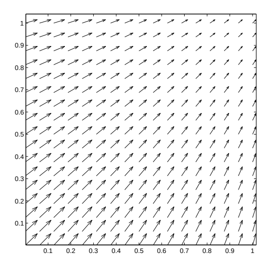
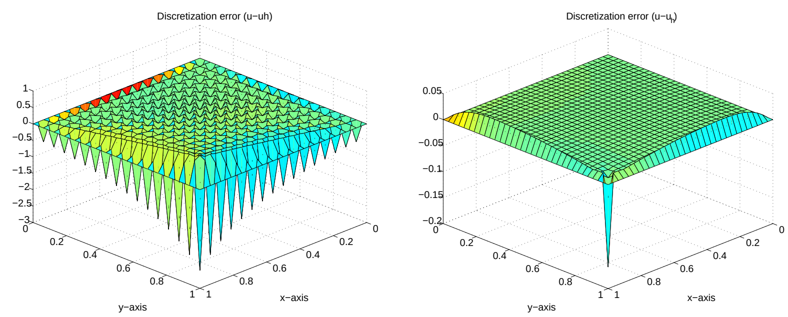


Figure 2. Velocity field of (5.1).

In this section we shall consider three numerical examples. The first one has two exponential boundary layers while the second one has only one parabolic layer. The third example illustrates a case when the Shishkin mesh fails and a posteriori adapted mesh is needed.
Since our problems are nonsymmetric and ill-conditioned, see Tables 2 and 6, we use GCG-MR solver preconditioned by (M)ILU factorization, as presented in [4].



(a) Uniform mesh, $h = 2^{-5}$, $e = 10^{-5}$, $\| u - u_h \|_\infty = 2.5782$

(b) Uniform mesh, $h = 2^{-5}$, $e = 10^{-5}$, $\| u - u_h \|_\infty = 0.166112$

FIGURE 3. Discretization error of: (a) Standard Galerkin, (b) SUPD method.

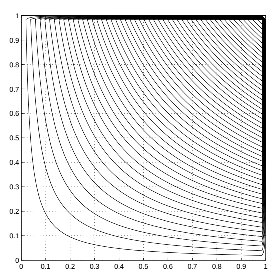


FIGURE 4. Contour lines of (5.2).

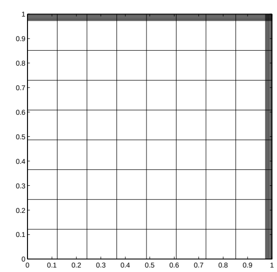


FIGURE 5. Shishkin mesh

5.1. Exponential layer

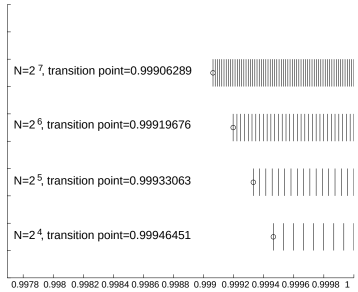
We consider the problem

$$-\varepsilon \Delta u + \mathbf{b} \nabla u + cu = f, \qquad \mathbf{x} \in \Omega = (0,1)^2, \tag{5.1}$$

$$u\big|_{\partial\Omega} = 0,$$

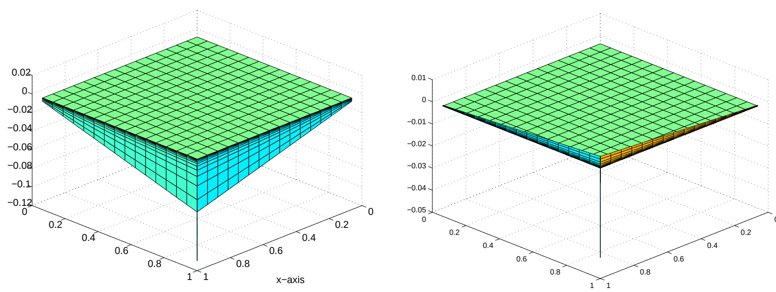
where $\varepsilon \in [10^{-8}, 10^{-4}]$, $c = 1$, $\mathbf{b} = [\frac{\sqrt{2}}{2}(1 - \frac{\sqrt{2}}{2}x), \frac{\sqrt{2}}{2}(1 - \frac{\sqrt{2}}{2}y)]$, see Figure 2, and $f(x)$ is such that the exact solution is:

$$u(x,y) = xy(1 - exp(-\frac{1-x}{\varepsilon}))(1 - exp(-\frac{1-y}{\varepsilon})). \tag{5.2}$$



aspect ratio	
N	$\frac{H}{h} = \frac{1-\tau}{\tau}$
2^7	≈ 2133
2^6	≈ 2489
2^5	≈ 2987
2^4	≈ 3734

FIGURE 6. The positions of the transition points for $\varepsilon = 10^{-5}$ and different N and the corresponding aspect ratios of the rectangular elements used in the layer regions.



(a) Shishkin mesh, $N = 32$, $e = 10^{-5}$, $\| u - u_h \|_\infty = 0.109488$

(b) Shishkin mesh, $N = 32$, $e = 10^{-5}$, $\| u - u_h \|_\infty = 0.037212$

FIGURE 7. Discretization error of: (a) Standard Galerkin, (b) SUPD method.

The righthand side satisfies the compatibility conditions (2.2), (2.3).
Problem (5.1) is characterized by the existence of exponential boundary layers at $x = 1$ and $y = 1$, see Figure 1. These layers cause serious instabilities in the standard Galerkin (SG) finite element scheme on a uniform mesh resulting in oscillatory numerical solution, see Figure 3(a). The amplitude of the oscillations turns out to depend on the height and width of the layers. The streamline upwind diffusion method (SUPD) is much more stable. There are no oscillations, see Figure 3(b) but is does not converge on a sequence of refined uniform meshes as long as $N^{-1} \gg \varepsilon$ in contrast with nonsingular problems.
In order to overcome these difficulties we use the Shishkin mesh defined by (2.8). The numbers of points used in both directions are equal to N. In prac-

tice, this *a priori* refined mesh follows the contour lines of the exact solution, compare Figure 5 with Figure 4. The positions of the transition point for fixed $\varepsilon = 10^{-5}$ and different N are illustrated in Figure 6. It slowly moves to the smooth part of the solution when the number of points doubles, which is an advantage of the Shishkin mesh from an approximation point of view but a disadvantage when a solver such as a multi-grid method will be applied. The parameters of the Shishkin mesh - coarse- (H), fine- (h) meshsizes and their ratio are given in Table 1.

Our matrices obtained by standard Galerkin FE discretization on a Shishkin mesh are very ill-conditioned and a powerful solver like GCG-MR preconditioned by ILU is required. From Table 2 we see that their condition numbers are consistent with theoretical condition number estimate $O(\varepsilon^{-1}(N/\ln N)^2)$, provided by Roos [12]. This large order is significantly improved by (M)ILU preconditioner.

All numerical results described below are obtained by either SG or SUPD method using a Shishkin mesh. The pointwise errors of both methods for $\varepsilon = 10^{-5}$ and $N = 32$ are plotted in Figure 7(a)&(b). We see that there are now no oscillations and both methods perform much better than on a uniform mesh as far as the error amplitude is concerned. The largest error still originates in the layer regions despite of the fact that a very fine mesh is introduced there.

The computed $L_{2,discrete}-$ and $\infty-$ norms of discretization error $(u - u_h)$ are provided in Tables 3 and 4 for $\varepsilon \in [10^{-9}, 10^{-4}]$ and number of points in one direction $N \in [2^4, 2^5, 2^6, 2^7]$. The first 4 columns of each table concerns the SG method while the next 4 columns the SUPD method. The diffusion parameter δ is defined by (4.6). In practice, it means that extra diffusion is added mainly outside the layer regions and the amount introduced inside them is negligible since $h \ll H$. The numerical results in Tables 3 show very clearly, first, the convergence uniformly in ε, and second, the consistency with the theoretical rate of convergence estimates $O(N^{-1})$ and close to $O(N^{-3/2})$, respectively for SG and SUPD method. Based on the results in Table 4 we can say that these two important properties of the L_2-norm of the discretization error hold even for its $\infty-$ norm, a fact which is not theoretically proven yet. In practice, we get even better results than the theoretical analysis shows.

5.2. Parabolic layers

We consider the problem

$$-\varepsilon \Delta u + u_y = f, \qquad \mathbf{x} \in \Omega = (0,1)^2, \tag{5.3}$$

$$\begin{aligned} u\big|_{x=0} &= 0, \quad u\big|_{x=1} = g_1, \\ u\big|_{y=0} &= 0, \quad u\big|_{y=1} = g_2, \end{aligned} \tag{5.4}$$

The parameters of the Shishkin mesh						
	$N = 2^4$			$N = 2^5$		
	H	h	H/h	H	h	H/h
$\varepsilon = 10^{-3}$	1.21653e-01	3.34681e-03	36	6.04082e-02	2.09176e-03	29
$\varepsilon = 10^{-4}$	1.24665e-01	3.34681e-04	372	6.22908e-02	2.09176e-04	298
$\varepsilon = 10^{-5}$	1.24967e-01	3.34681e-05	3734	6.24791e-02	2.09176e-05	2987
$\varepsilon = 10^{-6}$	1.24997e-01	3.34681e-06	37348	6.24979e-02	2.09176e-06	29878
$\varepsilon = 10^{-7}$	1.25000e-01	3.34681e-07	373489	6.24998e-02	2.09176e-07	298791
$\varepsilon = 10^{-8}$	1.25000e-01	3.34681e-08	3734898	6.25000e-02	2.09176e-08	2987918
	$N = 2^6$			$N = 2^7$		
	H	h	H/h	H	h	H/h
$\varepsilon = 10^{-3}$	2.99949e-02	1.25505e-03	24	1.48929e-02	7.32115e-04	20
$\varepsilon = 10^{-4}$	3.11245e-02	1.25505e-04	248	1.55518e-02	7.32115e-05	212
$\varepsilon = 10^{-5}$	3.12374e-02	1.25505e-05	2489	1.56177e-02	7.32115e-06	2133
$\varepsilon = 10^{-6}$	3.12487e-02	1.25505e-06	24898	1.56243e-02	7.32115e-07	21341
$\varepsilon = 10^{-7}$	3.12499e-02	1.25505e-07	248992	1.56249e-02	7.32115e-08	213422
$\varepsilon = 10^{-8}$	3.12500e-02	1.25505e-08	2489932	1.56250e-02	7.32115e-09	2134227

TABLE 1. H - coarse meshsize, h - fine meshsize, H/h - aspect ratio of the rectangular elements used in the layer regions

	N=32		N=64	
ε	cond(A)	$\varepsilon^{-1}(N/\ln N)^2$	cond(A)	$\varepsilon^{-1}(N/\ln N)^2$
10^{-3}	2.085073e+03	8.525287e+04	6.225454e+03	2.368135e+05
10^{-4}	1.735536e+04	8.525287e+05	5.448806e+04	2.368135e+06
10^{-5}	1.726450e+05	8.525287e+06	5.322959e+05	2.368135e+07
10^{-6}	1.725283e+06	8.525287e+07	5.311899e+06	2.368135e+08
10^{-7}	1.725163e+07	8.525287e+08	5.310761e+07	2.368135e+09
10^{-8}	1.725151e+08	8.525287e+09	5.310647e+08	2.368135e+10

TABLE 2. Exponential layer, Condition numbers of SG FE matrices on a Shishkin mesh.

where $\varepsilon \in [10^{-9}, 10^{-4}]$, g_1 and g_2 will be specified later and $f(x)$ is such that the exact solution is (see Figure 8):

$$u(x,y) = \frac{1}{\sqrt{\pi}}(1 - exp(-\sqrt{\frac{\pi}{\varepsilon}}x))exp(\pi y),$$

The solution u has a representation $u = u_0 + w$, where $u_0 = \frac{1}{\sqrt{\pi}}exp(\pi y)$ is the smooth part satisfying the reduced problem

$$(u_0)_y = \sqrt{\pi}exp(\pi y), \qquad \text{on } \Omega, \qquad u_0|_{y=0} = \frac{1}{\sqrt{\pi}},$$

and $w = -\frac{1}{\sqrt{\pi}}exp(-\sqrt{\frac{\pi}{\varepsilon}}x)exp(\pi y)$ is the solution of initial boundary value problem of parabolic type

$$-\varepsilon w_{xx} + w_y = 0, \qquad \text{on } \Omega,$$

SG, Shishkin mesh				
$\varepsilon \downarrow, N \rightarrow$	16	32	64	128
10^{-4}	0.086148	0.037776	0.016026	0.006740
10^{-5}	0.072972	0.030865	0.012703	0.005202
10^{-6}	0.068727	0.028657	0.011652	0.004715
10^{-7}	0.067379	0.027958	0.011319	0.004562
10^{-8}	0.066953	0.027737	0.011214	0.004513
10^{-9}	0.066818	0.027667	0.011180	0.004498

SUPD, Shishkin mesh				
$\varepsilon \downarrow, N \rightarrow$	16	32	64	128
10^{-4}	0.015093	0.005532	0.001971	0.000931
10^{-5}	0.012536	0.004568	0.001567	0.000720
10^{-6}	0.011736	0.004251	0.001437	0.000653
10^{-7}	0.011484	0.004150	0.001396	0.000631
10^{-8}	0.011405	0.004118	0.001383	0.000625
10^{-9}	0.011380	0.004108	0.001378	0.000623

TABLE 3. Exponential layer, $\| u - u_h \|_{L^2,\, discrete}$

SG, Shishkin mesh				
$\varepsilon \downarrow, N \rightarrow$	16	32	64	128
10^{-4}	0.281740	0.109467	0.034321	0.010757
10^{-5}	0.281955	0.109488	0.034318	0.010758
10^{-6}	0.281977	0.109490	0.034318	0.010758
10^{-7}	0.281979	0.109490	0.034318	0.010758
10^{-8}	0.281979	0.109491	0.034318	0.010758
10^{-9}	0.281979	0.109491	0.034318	0.010758

SUPD, Shishkin mesh				
$\varepsilon \downarrow, N \rightarrow$	16	32	64	128
10^{-4}	0.103861	0.037295	0.016829	0.005995
10^{-5}	0.103709	0.037212	0.016804	0.005989
10^{-6}	0.103694	0.037203	0.016802	0.005989
10^{-7}	0.103693	0.037203	0.016802	0.005989
10^{-8}	0.103693	0.037202	0.016802	0.005989
10^{-9}	0.103693	0.037202	0.016802	0.005989

TABLE 4. Exponential layer, $\| u - u_h \|_\infty$

$$w|_{y=0} = 0,\ w|_{x=0} = -u_0(0, y).$$

Along the line $x = 0$ there is a parabolic layer. We define $g_1 = (u_0 + w)|_{x=1}$ and $g_2 = (u_0 + w)|_{y=1}$ in order to exclude the development of a second parabolic layer at $x = 1$ and an exponential layer at $y = 1$.

We discretize (5.3) using SG and SUPD methods on Shishkin mesh (3.10). The numbers of points used in both directions are equal to N. The distance τ of the transition point from the parabolic layer is presented in Table 5. The condition numbers of the SG matrices are given in Table 6. They are proportional to the number $\sqrt{\varepsilon^{-1}}(N/\ln N)^2$ computed in 2nd and 4th columns of the same table.

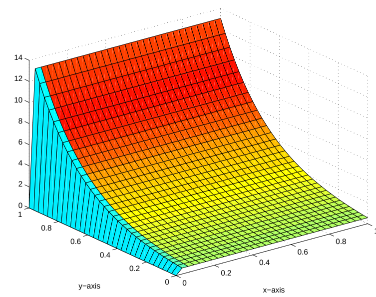


FIGURE 8. Exact solution, $\varepsilon = 10^{-6}$.

$\tau = \min\{1/2, \sqrt{\varepsilon}\ln N\}$						
$\sqrt{\varepsilon}\ln N \searrow$	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$
$\varepsilon = 10^{-3}$	0.087677	0.109596	0.131515	0.153435	0.175354	0.197273
$\varepsilon = 10^{-4}$	0.027726	0.034657	0.041589	0.048520	0.055452	0.062383
$\varepsilon = 10^{-5}$	0.008768	0.010960	0.013152	0.015343	0.017535	0.019727
$\varepsilon = 10^{-6}$	0.002773	0.003466	0.004159	0.004852	0.005545	0.006238
$\varepsilon = 10^{-7}$	0.000877	0.001096	0.001315	0.001534	0.001754	0.001973

TABLE 5. The transition point is of distance τ from the parabolic layer

From the arising graphs of the SG and SUPD pointwise discretization errors we can draw the conclusion that the Shishkin mesh is very efficient as far as the reduction of the error is concerned but it is not optimal with respect to the number of points introduced. The parabolic layer has a different nature compared with the exponential layer and its width varies from 0 to $O(\sqrt{\varepsilon})$, thus the layer region has a curved boundary. In practice, the Shishkin mesh covers the rectangle circumscribing the layer region and remains insensitive to the variation of the width because of its limited adaptivity, which follows by its construction.

The computed $L_{2,discrete}-$ and $\infty-$ norms of discretization error $(u - u_h)$ are provided in Tables 7 and 8. The numerical results clearly show a uniform

	N=16		N=32	
ε	cond(A)	$\sqrt{\varepsilon^{-1}}(N/\ln N)^2$	cond(A)	$\sqrt{\varepsilon^{-1}}(N/\ln N)^2$
10^{-4}	6.160691e+02	3.330190e+03	1.053108e+03	8.525287e+03
10^{-5}	1.904352e+03	1.053099e+04	3.330218e+03	2.695933e+04
10^{-6}	5.928196e+03	3.330190e+04	1.053108e+04	8.525287e+04
10^{-7}	1.863741e+04	1.053099e+05	3.330218e+04	2.695933e+05
10^{-8}	5.882288e+04	3.330190e+05	1.053108e+05	8.525287e+05
10^{-9}	1.858991e+05	1.053099e+06	3.330218e+05	2.695933e+06

TABLE 6. Parabolic layer, Condition numbers of SG FE matrices on a Shishkin mesh.

convergence. Moreover, we see that the standard Galerkin solution u^N approximates u to almost $N^{-3/2}$ order in L_2- (even $\infty-$) norm, which is better than our theoretical analysis shows. However, see the results in [1], illustrating instances when this order arises.

In case of parabolic layer, the SUPD scheme introduces extra diffusion only in the direction of the nonzero velocity component, i.e. in y-direction if $\mathbf{b} = [0, Const]$. The diffusion parameter is defined by (4.6) and in our case is $\delta = H_y/2$, where $H_y = 1/N$. Observe that we do not have coarse and fine meshsizes in $y-$ direction. Since the SUPD scheme does not add stabilization term in $x-$direction, where the parabolic layer is located, we cannot expect an essential improvement of the error amplitude in contrast with the exponential layer. The numerical results in Tables 7, 8 are good illustrations of this fact. The SUPD methods gives smaller errors than SG method but the difference is not substantial.

SG, Shishkin mesh				
$\varepsilon \downarrow, N \rightarrow$	16	32	64	128
10^{-4}	0.178645	0.054223	0.011846	0.003165
10^{-5}	0.191508	0.068764	0.023046	0.006324
10^{-6}	0.189778	0.069273	0.024923	0.008714
10^{-7}	0.187336	0.068199	0.024590	0.008822
10^{-8}	0.185700	0.067390	0.024208	0.008666
10^{-9}	0.184723	0.066899	0.023964	0.008551

SUPD, Shishkin mesh				
$\varepsilon \downarrow, N \rightarrow$	16	32	64	128
10^{-4}	0.109573	0.030462	0.013946	0.005230
10^{-5}	0.113318	0.031845	0.013546	0.004965
10^{-6}	0.112597	0.031451	0.013228	0.004791
10^{-7}	0.111731	0.031037	0.013012	0.004675
10^{-8}	0.111160	0.030776	0.012879	0.004604
10^{-9}	0.110819	0.030624	0.012800	0.004562

TABLE 7. Parabolic layer, $\| u - u_h \|_{L^2,\, discrete}$

5.3. Special layer

We consider the problem

$$-\varepsilon \Delta u + \mathbf{b} \nabla u + cu = f, \qquad \mathbf{x} \in \Omega = (0,1)^2, \tag{5.5}$$

$$u|_{x=0} = 0, \quad u|_{x=1} = 1,$$
$$u|_{y=0} = 0, \quad u|_{y=1} = 1,$$

where $\varepsilon \in [10^{-7}, 10^{-3}]$, $c = 1$, $\mathbf{b} = [(1-x)^2, (1-y)^2]$ and $f(x)$ is such that the exact solution is:

SG, Shishkin mesh				
$\varepsilon \downarrow, N \rightarrow$	16	32	64	128
10^{-4}	0.125372	0.051399	0.020237	0.007148
10^{-5}	0.131183	0.052356	0.020753	0.007381
10^{-6}	0.131818	0.052629	0.020899	0.007470
10^{-7}	0.132329	0.052705	0.020934	0.007488
10^{-8}	0.133818	0.052728	0.020944	0.007492
10^{-9}	0.134290	0.052735	0.020946	0.007493

SUPD, Shishkin mesh				
$\varepsilon \downarrow, N \rightarrow$	16	32	64	128
10^{-4}	0.124525	0.038127	0.013630	0.005560
10^{-5}	0.126846	0.038409	0.013590	0.005625
10^{-6}	0.127162	0.038631	0.013544	0.005619
10^{-7}	0.127218	0.038678	0.013527	0.005614
10^{-8}	0.127232	0.038691	0.013521	0.005612
10^{-9}	0.127236	0.038694	0.013519	0.005611

TABLE 8. Parabolic layer, $\| u - u_h \|_\infty$

$$u(x,y) = exp(-\frac{(1-x)^3(1-y)^3}{\varepsilon}), \qquad \text{(Figure 9)}. \tag{5.6}$$

The contour lines of (5.6) illustrated in Figure 11&12 show that the width of the layer as well as its position in Ω varies when ε decreases. For $\varepsilon \in [10^{-4}, 10^{-1}]$ we have a typical interior layer which gradually localizes as boundary layer along the lines $x = 1, y = 1$ when $\varepsilon \rightarrow 0$. First, the motion of the layer in the domain makes the construction of the proper Shishkin mesh problematic. Second, the width of the layer around the corner (1,1) is greater than the one at the corners (1,0), (0,1) due to dispersion. This is in contrast with its steepness. That is why our trial to use Shishkin mesh constructed in the same way as for exponential/parabolic layers of width $O(\varepsilon)/O(\sqrt{\varepsilon})$ ended up with an easily explainable result, illustrated in Figure 13 - the largest error occurs in the subdomain wider than $O(\sqrt{\varepsilon})$, which is not covered by the Shishkin mesh. On the other hand, a construction of a Shishkin mesh covering this subdomain leads to waste of computational effort and memory resources due to introduction of too many unnecessary points around the corners (1,0) and (0,1).

In an attempt to obtain a mesh which corresponds to the behavior of the solution we use *a posteriori* adaptive refinement based on a defect-correction technique. In Figure 14 , 15, 16 the corresponding discretization error, the final graded and patched mesh, which follows the contour lines of the exact solution and a zoom of the adaptive mesh are shown. The maximal values of $|e_h|$, at each step of adaptive refinement are given in column 4 of Table 9. More details about the adaptive refinement procedure used here are given in [3].

Adaptive refinement			
#level	#points	min(h)	$\| u - u_h \|_\infty$
0	289	1/16	—
1	708	1/32	0.025979
2	2007	1/64	0.013073
3	4391	1/128	0.004451
4	10007	1/256	0.001294
5	15957	1/512	0.000351

TABLE 9. Discretization error after 5 levels of refinement, $\varepsilon = 10^{-5}$.

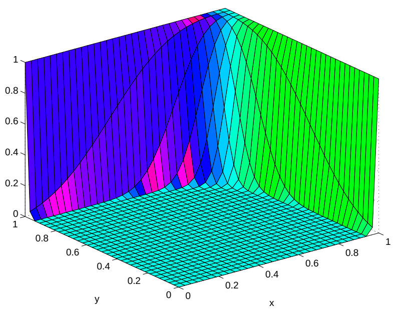


FIGURE 9. Exact solution (5.6), $\varepsilon = 10^{-5}$.

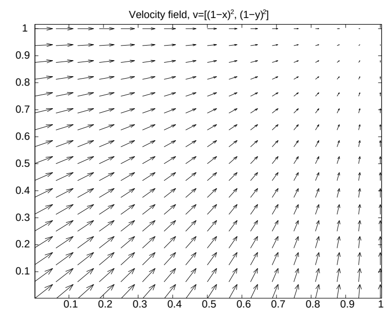


FIGURE 10. Velocity field of (5.5).

6. CONCLUSIONS

It has been demonstrated that the Shishkin meshes are very efficient in reduction of the error for both exponential and parabolic layers but they are not optimal with respect to the number of points in the latter case. We showed that the standard Galerkin finite element method converges uniformly in the perturbation parameter ε, of optimal order $O(N^{-1})$ in L_2-norm, where the total number of points is $O(N^2)$. The numerical results clearly illustrate this fact. The experiments with streamline upwind diffusion method on Shishkin mesh show that it gives higher order of approximation, better stability and matrix properties, so that the iterative solver converges faster than in the case of the standard Galerkin method. The necessity of using adaptive refinement techniques based on stable methods as a defect-correction method and carefully chosen *a posteriori* error estimators was well demonstrated by the last numerical example.

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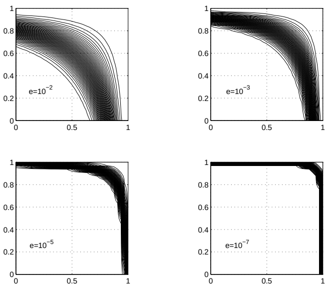

FIGURE 11. Contour lines of (5.6) .

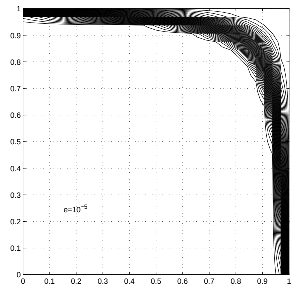

FIGURE 12. Zoom of Figure 11 for $\varepsilon = 10^{-5}$.

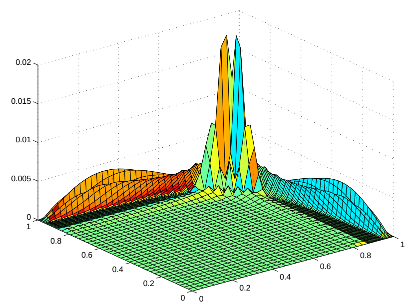

FIGURE 13. $\varepsilon = 10^{-5}$, Discretization error of SG on Shishkin mesh, $N = 64$, $\| u - u_h \|_\infty = 0.019508$

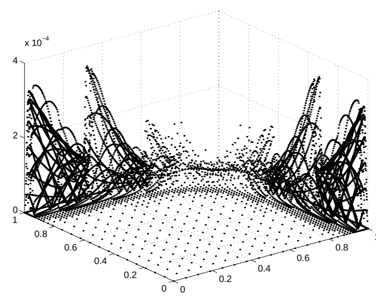

FIGURE 14. Discretization error $\| u - u_h \|_\infty = 0.000351$ after 5 levels of adaptive refinement.

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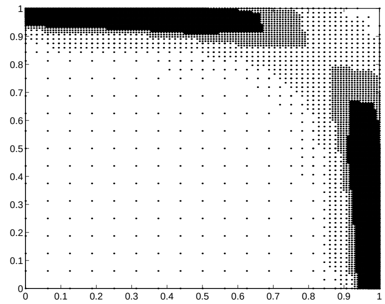

FIGURE 15. Final mesh

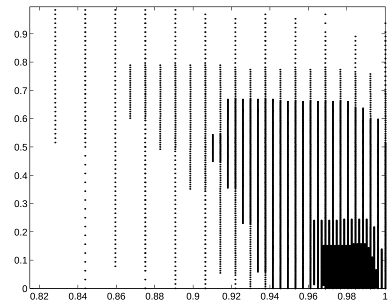

FIGURE 16. Zoom of the mesh

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