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THE SOLUTION OF THE ORDER EQUATIONS OF A FOUR-POINT, FOURTH ORDER, TWO-STEP RUNGE-KUTTA METHOD

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The solution of the order equations of a four-point, fourth order, two-step Runge-Kutta method

by

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ABSTRACT

In this report we present solutions of the order equations of a class of four-point, fourth order, two-step Runge-Kutta method. The solutions include possible variation of integration steps.

KEY WORDS & PHRASES: *Differential equations, explicit Runge-Kutta methods.*

1. STATEMENT OF THE PROBLEM

In [1] VAN DER HOUWEN presents the following multipoint two-step Runge-Kutta method

$$(1.1)\qquad \vec{y}_{n+1} = (a-1)\vec{y}_n + b\vec{y}_{n-1} + ch_n\vec{f}(\vec{y}_{n-1}) + \vec{y}^{(RK)}_{n+1},$$

where $\vec{y}^{(RK)}_{n+1}$ is the result of a single-step Runge-Kutta formula which can be characterized by the array of Runge-Kutta parameters

$$\Lambda = \begin{pmatrix} \lambda_{10} & & & 0 \\ \vdots & \ddots & & \\ \lambda_{m-1\,0} & \cdots & \lambda_{m-1\,m-2} & \\ \theta_0 & \cdots & \theta_{m-2} & \theta_{m-1} \end{pmatrix}.$$

Fourth order consistency of (1.1) leads to the equations

$$(1.2)\qquad a + b = 1,$$

$$(1.3)\qquad \sum_{i=0}^{m-1} \theta_i = \alpha_0,$$

$$(1.4)\qquad \sum_{i=1}^{m-1} \theta_i \mu_i = \frac{1}{2}\alpha_1,$$

$$(1.5)\qquad \sum_{i=2}^{m-1} \theta_i \sum_{j=1}^{i-1} \lambda_{ij} \mu_j = \frac{1}{6}\alpha_2,$$

$$(1.6)\qquad \sum_{i=1}^{m-1} \theta_i \mu_i^2 = \frac{1}{3}\alpha_2,$$

$$(1.7)\qquad \sum_{i=3}^{m-1} \theta_i \sum_{j=2}^{i-1} \lambda_{ij} \sum_{k=1}^{j-1} \lambda_{jk} \mu_k = \frac{1}{24}\alpha_3,$$

$$(1.8)\qquad \sum_{i=2}^{m-1} \theta_i \sum_{j=1}^{i-1} \lambda_{ij} \mu_j^2 = \frac{1}{12}\alpha_3,$$

$$(1.9)\qquad \sum_{i=2}^{m-1} \theta_i \mu_i \sum_{j=1}^{i-1} \lambda_{ij} \mu_j = \frac{1}{8}\alpha_3,$$

$$(1.10)\qquad \sum_{i=1}^{m-1} \theta_i \mu_i^3 = \frac{1}{4}\alpha_3,$$

where

$$\alpha_i = 1 - q^i(bq+(i+1)c),\ i = 0,1,2,3,\quad (q = -h_{n-1}/h_n),$$

and

$$\mu_i = \sum_{j=0}^{i-1} \lambda_{ij},\ i = 1(1)m-1.$$

Shanks has already given the complete solution of fourth order, four-point, single-step Runge-Kutta formulas (see [2]); in section 2 we will generalize these results for a four-point, fourth order formula using two steps in a similar way.

2. SOLUTION OF THE PROBLEM

We will consider three different types of solutions.

Case I. μ_1, μ_2 and μ_3 are distinct.

From (1.4), (1.6) and (1.10) follow

$$\theta_1 = \frac{\frac{1}{2}\alpha_1\mu_2\mu_3 - \frac{1}{3}\alpha_2(\mu_2+\mu_3) + \frac{1}{4}\alpha_3}{\mu_1(\mu_2-\mu_1)(\mu_3-\mu_1)}$$

and similar expressions for θ_2 and θ_3.

(1.5) and (1.9) lead to

$$\lambda_{21} = \frac{\frac{1}{6}\alpha_2\mu_3 - \frac{1}{8}\alpha_3}{\theta_2\mu_1(\mu_3-\mu_2)},$$

$$(2.1)\qquad \lambda_{31}\mu_1 + \lambda_{32}\mu_2 = \frac{\frac{1}{6}\alpha_2\mu_2 - \frac{1}{8}\alpha_3}{\theta_3(\mu_2-\mu_3)}.$$

Next, from (1.8) and (2.1), we get

$$\lambda_{32} = \frac{\frac{1}{6}\alpha_2\mu_1 - \frac{1}{12}\alpha_3}{\theta_3\mu_2(\mu_1-\mu_2)} .$$

Now only (1.7) is not yet fulfilled. A straightforward calculation, using the expressions above, results in

$$\theta_3\lambda_{32}\lambda_{21}\mu_1 = \frac{(\frac{1}{6}\alpha_2\mu_1 - \frac{1}{12}\alpha_3)(\frac{1}{6}\alpha_2\mu_3 - \frac{1}{8}\alpha_3)}{\frac{1}{2}\alpha_1\mu_1\mu_3 - \frac{1}{3}\alpha_2(\mu_1+\mu_3) + \frac{1}{4}\alpha_3} = \frac{1}{24}\alpha_3,$$

and gives the following condition

$$\mu_3 = \frac{\alpha_2\alpha_3}{4\alpha_2^2 - 3\alpha_1\alpha_3} .$$

Thus, case I gives a two-parameter (μ_1,μ_2) family of solutions.

Case II. $\mu_1 = \mu_2$.

Defining $\tilde{\theta}_2 = \theta_1 + \theta_2$ it follows from (1.4), (1.6) and (1.10)

$$\tilde{\theta}_2 = \frac{\frac{1}{2}\alpha_1\mu_3 - \frac{1}{3}\alpha_2}{\mu_2(\mu_3-\mu_2)} \quad \text{(similar expression for } \theta_3\text{)}$$

and

$$\mu_3 = \frac{\frac{1}{3}\alpha_2\mu_2 - \frac{1}{4}\alpha_3}{\frac{1}{2}\alpha_1\mu_2 - \frac{1}{3}\alpha_2} .$$

From (1.5) and (1.8) it is easily verified that

$$\mu_2 = \frac{\alpha_3}{2\alpha_2} ,$$

thus, giving as in case I,

$$\mu_3 = \frac{\alpha_2\alpha_3}{4\alpha_2^2 - 3\alpha_1\alpha_3} .$$

To obtain expressions for the remaining parameters we proceed as in case I. From (1.5) and (1.9) follow

$$\lambda_{31} + \lambda_{32} = \frac{\frac{1}{24}\alpha_3}{\theta_3\mu_2(\mu_3-\mu_2)}$$

and

$$\lambda_{21} = \frac{\frac{1}{6}\alpha_2\mu_3 - \frac{1}{8}\alpha_3}{\theta_2\mu_2(\mu_3-\mu_2)} .$$

Finally we get from (1.7)

$$\lambda_{32} = \frac{\frac{1}{24}\alpha_3}{\theta_3\lambda_{21}\mu_2} .$$

Thus, case II results in a one-parameter (e.g. θ_2) family of solutions.

Case III. $\mu_1 = \mu_3$.

Proceeding as in case I and case II, we find

$$\theta_3 = \tilde{\theta}_3 - \theta_1, \quad \tilde{\theta}_3 = \frac{\frac{1}{2}\alpha_1\mu_2 - \frac{1}{3}\alpha_2}{\mu_3(\mu_2-\mu_3)} ,$$

$$\theta_2 = \frac{\frac{1}{2}\alpha_1\mu_3 - \frac{1}{3}\alpha_2}{\mu_2(\mu_3-\mu_2)} ,$$

$$\lambda_{21} = \frac{\mu_2^2}{2\mu_3} ,$$

$$\lambda_{32} = \frac{\frac{1}{6}\alpha_2\mu_3 - \frac{1}{12}\alpha_3}{\theta_3\mu_2(\mu_3-\mu_2)} , \quad \lambda_{31} = \frac{\frac{1}{6}\alpha_2 - \theta_2\lambda_{21}\mu_3 - \theta_3\lambda_{32}\mu_2}{\theta_3\mu_3} ,$$

$$\mu_2 = \frac{\alpha_3}{2\alpha_2} ,$$

$$\mu_3 = \frac{\alpha_2\alpha_3}{4\alpha_2^2 - 3\alpha_1\alpha_3} .$$

Case III also gives a one-parameter (e.g. θ_1) family of solutions.

We now show that the choice $\mu_2 = \mu_3$ leads to a contradiction. (1.4), (1.6) and (1.10) give the condition

$$(2.2) \qquad \mu_1 = \frac{\frac{1}{3}\alpha_2\mu_2 - \frac{1}{4}\alpha_3}{\frac{1}{2}\alpha_1\mu_2 - \frac{1}{3}\alpha_2} .$$

(1.5) and (1.9) lead to $\mu_2 = \frac{3\alpha_3}{4\alpha_2}$, which, substituted in (2.2) delivers $\mu_1 = 0$. But $\mu_1 = 0$ contradicts with (1.7).

3. EXAMPLES

Below, we present two schemes, both representing case II of section 2.

(i) <u>$b=1$, $c = \frac{1}{3}$, $q = -1$.</u>

These values lead to the following formula

$$\vec{y}_{n+1} = -\vec{y}_n + \vec{y}_{n-1} + \tfrac{1}{3}h\vec{f}(\vec{y}_{n-1}) + \vec{y}_n^{(RK)},$$

with

$$\Lambda = \begin{bmatrix} \frac{2}{3} & & & 0 \\ \frac{1}{3} & \frac{1}{3} & & \\ \frac{1}{4} & 0 & \frac{3}{4} & \\ \frac{4}{3} & 0 & 0 & \frac{1}{3} \end{bmatrix} .$$

(ii) <u>$b = .3$, $c = .1$, $q = -1$.</u>

The scheme is given by

$$\vec{y}_{n+1} = .3(\vec{y}_{n-1} - \vec{y}_n) + .1h\vec{f}(\vec{y}_{n-1}) + \vec{y}_n^{(RK)}$$

and

$$\Lambda = \begin{pmatrix} \frac{11}{20} & & & 0 \\ 0 & \frac{11}{20} & & \\ \frac{330}{10609} & \frac{66000}{1092727} & \frac{1067000}{1092727} & \\ \frac{3717913}{7042200} & \frac{9100}{35211} & \frac{9100}{35211} & \frac{1092727}{7042200} \end{pmatrix}.$$

Note, that the choice $q = -1$ refers to constant step integration. Furthermore, we remark that the term $-\vec{y}_n$ (of scheme (i)) and the term $\vec{y}_n$ appearing in $\vec{y}_n^{(RK)}$, cancel out.

REFERENCES

[1] HOUWEN, P.J. van der (1976): *Construction of integration formulas for initial value problems*, North-Holland Publishing Co., Amsterdam.

[2] SHANKS, E.B. (1966): *Solution of differential equations by evaluation of functions*, Math. Comp., 20, pp.21-38.