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ABSTRACT
A drift-diffusion model for semiconductors with nonlinear diffusion is considered. The model consists of two quasilinear degenerated parabolic equations for the carrier densities and the Poisson equation for the electric potential. We also assume Lipschitz continuous non linearities in the drift and $generation\text{-}recombination$ terms.

Existence of weak solutions is proven by using a regularization technique. Uniqueness of solutions is proven when either the diffusion term φ is strictly increasing and solutions have spatial derivatives in $L^1(Q_T)$ or when φ is non decreasing and a suitable entropy condition is fullfilled by the electric potential.

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1. Introduction

In solid state physics, the drift-diffusion equations are today the most widely used model to describe semiconductor devices. The drift-diffusion models describe the flow of the electrons in the conduction band of the semiconductor material and of the holes (or defect electrons) in the valence band of the crystal, influenced by the electric field. Mathematically, they form a system of parabolic equations for the electron density u, the hole density v and the Poisson equation for the electric potential w that together with physically motivated auxiliary conditions form the problem

$$\begin{cases} u_t - div\,(\nabla\varphi(u) - b(u)\nabla w) = F(u,v) & \text{in } Q_T := \Omega\times(0,T), \\ v_t - div\,(\nabla\varphi(v) + b(v)\nabla w) = F(u,v) & \text{in } Q_T, \\ -\Delta w = v - u + C & \text{in } Q_T, \\ \nabla\varphi(u)\cdot\nu = 0,\ \nabla\varphi(v)\cdot\nu = 0,\ \nabla w\cdot\nu = 0, & \text{on } \Sigma_{NT} := \Gamma_N\times(0,T), \\ \varphi(u) = \varphi(u_D),\ \varphi(v) = \varphi(v_D),\ w = w_D, & \text{on } \Sigma_{DT} := \Gamma_D\times(0,T), \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & \text{in } \Omega. \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(1 \le N \le 3)$ is the (bounded) domain occupied by the semiconductor crystal. Here, $C = C(x)$ denotes the doping profile (fixed charged background ions) characterizing the semiconductor under consideration, $\varphi(s)$ is the pressure function, $b(s)/s$ the mobility of the particles, and $F(s,\sigma)$ the recombination-generation rate. The boundary $\partial\Omega$ splits into two disjoint subsets Γ_D and Γ_N. The carrier densities and the potential are fixed at Γ_D (Ohmic contacts), whereas Γ_N models the union of insulating boundary segments.

The standard drift-diffusion model corresponds to $\varphi(s) = s$, $b(s) = s$ and $F(u,v) = q(u,v)(u_i^2(x) - uv)$, where $q(u,v)$ is a positive function and $u_i(x) > 0$ is the so-called intrinsic density. The standard model can be derived from Boltzmann's equation under the assumption that the semiconductor device is in the low injection regime (i.e. for small absolute values of the applied voltage). It is shown in [24] that in the high injection regime the diffusion terms are no longer linear. An useful choice of function $\varphi(s)$ is the one given by $\varphi(s) = s^\alpha$ for any $s \ge 0$ and with $\alpha = \frac{5}{3}$. With this pressure function, the equations in (1.1) become of degenerate type, and the existence of solutions does not follow from standard theory. In this paper we present results including both the low injection case as well as the high injection case.

The function φ can be interpreted in the language of gas dynamics. We assume that the particles behave (thermodynamically spoken) as an ideal gas such that the gas law $\varphi = u\theta$ holds (φ denotes the pressure, θ the particle temperature). In the isothermal case θ =const. the pressure turns out to be linear: $\varphi(u) = u$. In the isentropic case, however, the temperature (only) depends on the concentrations. Then $\theta(u) = u^{2/3}$ holds for particles without spin in adiabatic and hence for isentropic states [10], which reads $\varphi(u) = u^{5/3}$. And an analogous expression holds for the holes.

In the isentropic (or high injection) case, the functions $b(s) = s$ and $F(u,v) = -uv(u^\beta + v^\beta)$ with $\beta = 2/3$ are used in [24]. In the present paper we consider functions $b(s)$ and $F(u,v)$ under some general assumptions which are fulfilled in all the above cases.

The standard (low injection) model has been mathematically and numerically investigated in many papers (see [33], [34] and references therein). The existence and uniqueness of weak solutions have been shown. The isentropic (high injection) model for $b(s) = s$ and a monotonic function F (including the non Lipschitz continuous case) was analyzed in [22], [23], [25] and [24]. There, the existence of weak solutions has been proved. However, there is a lack on results concerning the uniqueness of solutions when the system actually degenerates. Furthermore, there are no results for general mobility functions. This paper is devoted to the proof of the existence and the uniqueness of weak solutions for monotone pressure functions φ satisfying $\varphi'(0) = 0$ and for general smooth functions $b(s)$ and $F(s,\sigma)$ (see next section for the precise assumptions).

The outline of the paper is as follows. In Section 2 we present the assumptions on the data of the problem and prove the existence of weak solutions by means of a regularization technique that involves the consideration of a non degenerate problem for which existence of solutions is proven by a fixed point argument. In Section 3 we study the uniqueness of solutions and present three results depending, mainly, on the behaviour of function φ (strictly increasing or non decreasing) and on the regularity of solutions.

2. Existence of solutions

In this section we prove the existence of weak solutions of problem (1.1). The main result is Theorem 2.1 where we prove the existence of such solutions in the most interesting case: when the parabolic equations of (1.1) are of degenerate type. The transport terms $div(b(u)\nabla w)$ and $div(b(v)\nabla w)$ are the main difficulty in the proof due to the fact that natural a priori estimates of the problem are obtained in terms of $\varphi(u)$ (with $\varphi'(0)=0$) and their spatial derivatives meanwhile transport terms contain $b(u)$ and $b(v)$ that, in general, are not bounded by the former.

This difficulty leads us to consider an auxiliar non degenerate problem for which we obtain existence of weak solutions (Theorem 2.2) and that allows us, by means of techniques of regularization and passing to the limit, to prove the result for the general formulation.

Before stating the first result we introduce a set of assumptions on the data as well as the definition of weak solution of (1.1) and some consequences of the Sobolev's embedding theorems that we shall use.

Assumptions on the data.

$\mathbf{H}_1$. $\Omega \subset I\!\!R^N$, $N \leq 3$, is an open, bounded and connected set. The boundary of Ω, $\partial\Omega$, is of class $\mathcal{C}^{1,1}$ and its $(N-1)-$dimensional Haussdorf measure is finite; $\partial\Omega$ splits in two disjoint components Γ_D (with positive measure) and Γ_N (open in $\partial\Omega$). We assume that for any function ψ satisfying

$$\begin{cases} \Delta\psi \in L^s(\Omega) \\ \psi = 0 \text{ on } \Gamma_D \\ \nabla\cdot\psi = 0 \text{ on } \Gamma_N \end{cases} \tag{2.1}$$

the regularity $\psi \in W^{2,s}(\Omega)$, for $s \in [1,\infty)$ holds. Finally, we suppose that $T>0$ is fixed arbitrarily.

Definition. *A function $f : I\!\!R^n \to I\!\!R$ is* sublinear *if there exists a positive constant c such that*

$$|f(s_1,...,s_n)| \leq c\left(1+\sum_{i=1}^{n}|s_i|\right), \quad \forall\,(s_1,...,s_n) \in I\!\!R^n.$$

$\mathbf{H}_2$. We assume that

$$\varphi \in \mathcal{C}([0,\infty)) \cap \mathcal{C}^1((0,\infty)), \quad \varphi(0)=0, \quad \varphi \text{ non decreasing},$$

$$F \in \mathcal{C}^{0,1}_{loc}([0,\infty)^2\,; I\!\!R), \tag{2.2}$$

$b \in \mathcal{C}^1_{loc}([0,\infty))$ is sublinear and satisfies

$$|b'(s)| \leq c\,(1+\varphi'(s))\,, \quad \forall\, s \in [0,\infty)\,, \tag{2.3}$$

for some constant $c>0$.

$\mathbf{H}_3$. The auxiliary data satisfy

$$\begin{aligned} &u_0, v_0 \in L^\infty(\Omega), \quad u_0 \geq 0, \quad v_0 \geq 0 \quad \text{in} \quad \Omega, \\ &\varphi(u_D), \varphi(v_D) \in L^\infty(0,T;H^1(\Omega)) \cap L^\infty(Q_T) \cap H^1(0,T;L^2(\Omega)), \\ &w_D \in L^\infty(0,T;W^{2,\infty}(\Omega)). \end{aligned}$$

We also assume that $C \in L^\infty(Q_T)$.

$\mathbf{H}_4$. If both F and b are nonlinear then we assume that $\varphi^{-1} \in \mathcal{C}^{0,\alpha}([0,\infty))$, for some $\alpha \in (0,1)$.

We remark that the property assumed for (2.1) actually represents a condition on the contact angles of the boundary segments Γ_D and Γ_N (see, e.g., [32]). In particular, if both components of the

boundary are open and closed (so they do not meet) then the assumption is a well known result (see, e.g., [41]). As stated in (2.2), in this article we shall consider a Lipschitz continuous *recombination-generation* term F. The case of a monotone F was already treated in [24] obtaining similar results on the existence of weak solutions under somehow stronger conditions on φ and b. As shown in [12], a monotone non Lipschitz continuous recombination-generation term may imply the formation of *dead cores* (sets where the components u, v of the solution vanish even when the initial data are strictly positive) and play an important role in applications through the phenomenon known as *vacuum solutions* (see [12]). We consider a notion of weak solution similar to that introduced in [1]:
Definition of weak solution. *Set*

$$\mathcal{V} := \left\{ z \in H^1(\Omega) : z = 0 \quad \text{on} \quad \Gamma_D \right\},$$

and assume $\mathbf{H}_3$. *Then* (u, v, w) *is a* weak solution of (1.1) *if the following properties hold:*
(i) $u, v \in L^\infty(Q_T)$, $\varphi(u) \in \varphi(u_D) + L^2(0,T;\mathcal{V})$, $\varphi(v) \in \varphi(v_D) + L^2(0,T;\mathcal{V})$ and $w \in w_D + L^2(0,T;\mathcal{V}) \cap L^\infty(Q_T)$.
(ii)

$$\begin{aligned} &\int_0^T \langle u_t, \zeta \rangle + \int_0^T \int_\Omega (\nabla\varphi(u) - b(u)\nabla w) \cdot \nabla\zeta = \int_0^T \int_\Omega F(u,v)\zeta, \\ &\int_0^T \langle v_t, \zeta \rangle + \int_0^T \int_\Omega (\nabla\varphi(v) - b(v)\nabla w) \cdot \nabla\zeta = \int_0^T \int_\Omega F(u,v)\zeta, \\ &\int_0^T \int_\Omega \nabla w \cdot \nabla\zeta = \int_0^T \int_\Omega (v - u - C)\,\zeta \end{aligned} \tag{2.4}$$

for any test function $\zeta \in L^2(0,T;\mathcal{V})$ *(notice that due to* (2.2) $F(u,v) \in L^2(Q_T)$*).*
(iii) $u_t, v_t \in L^2(0,T;\mathcal{V}')$ *and the initial data are verified in the following sense:*

$$\begin{aligned} &\int_0^T \langle u_t, \zeta \rangle + \int_0^T \int_\Omega (u - u_0)\,\zeta_t = 0, \\ &\int_0^T \langle v_t, \zeta \rangle + \int_0^T \int_\Omega (v - v_0)\,\zeta_t = 0, \end{aligned} \tag{2.5}$$

for any test function $\zeta \in L^2(0,T;\mathcal{V}) \cap W^{1,1}(0,T;L^2(\Omega))$ *with* $\zeta(T) = 0$.
We shall use the notation:

$$|||f||| := \|f\|_{L^\infty(0,T;L^2(\Omega))} + \|f\|_{L^2(0,T;\mathcal{V})}, \qquad 2^* := \begin{cases} 6 & \text{if} \quad N = 3, \\ s \in [1,\infty) & \text{if} \quad N = 2, \\ \infty & \text{if} \quad N = 1, \end{cases}$$

$\|\cdot\|_{L^p} := \|\cdot\|_{L^p(Q_T)}$ and $\|\cdot\|_{L^p(L^q)} := \|\cdot\|_{L^p(0,T;L^q(\Omega))}$.

The following lemma is a consequence of Sobolev's Theorem and standard inequalities (see [20]):

Lemma 2.1 *Suppose that* $f \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;\mathcal{V})$ *and let*

$$1 \le r < 4(1 - \frac{1}{2^*}). \tag{2.6}$$

Then there exist a positive constant $C(\Omega)$ *such that*

$$\|f\|_{L^r(Q_T)} \le C(\Omega)\, |||f|||.$$

Next we state the main result of this section:

Theorem 2.1 *Assume* $\mathbf{H}_1$-$\mathbf{H}_4$ *and that the auxiliary data satisfy*

$$\begin{aligned} k \geq u_0, v_0 \geq m \geq 0 \quad &\text{a.e. in } \Omega \qquad \text{and} \\ \varphi(ke^{\lambda_0 t}) \geq \varphi(u_D), \varphi(v_D) \geq \varphi(me^{-\lambda_1 t}) \geq 0 \quad &\text{a.e. in } \Sigma_{DT}, \end{aligned}$$

for some non negative constants $k, m, \lambda_0, \lambda_1$. *Then there exists a* $\lambda \geq 0$ *independent of* φ *such that the problem* (1.1) *has, at least, one weak solution verifying*

$$\begin{aligned} & ke^{\lambda t} \geq u, v \geq me^{-\lambda t} \geq 0 \quad \text{a.e. in } Q_T, \\ & u, v \in \mathcal{C}([0,T]; \mathcal{V}'), \\ & w \in L^\infty(0,T; W^{2,s}(\Omega)) \quad \text{for all } s \in [1, \infty). \end{aligned}$$

Moreover, if $\varphi \in \mathcal{C}^1([0,\infty))$ *then*

$$\sqrt{\varphi'(u)}\nabla u, \sqrt{\varphi'(v)}\nabla v \in L^2(Q_T).$$

The proof of this theorem is based on the following previous result for the non degenerate problem:

Theorem 2.2 *Assume* $\mathbf{H}_1$-$\mathbf{H}_3$ *and let* φ *be a sublinear strictly increasing function. Suppose that* φ^{-1} *is Lipschitz continuous and that* F *is sublinear. Assume that*

$$\begin{aligned} k \geq u_0, v_0 \geq m \geq 0 \quad &\text{a.e. in } \Omega \qquad \text{and} \\ \varphi(ke^{\lambda_0 t}) \geq \varphi(u_D), \varphi(v_D) \geq \varphi(me^{-\lambda_1 t}) \geq 0 \quad &\text{a.e. in } \Sigma_{DT}, \end{aligned}$$

for some non negative constants $k, m, \lambda_0, \lambda_1$. *Then there exists a* $\lambda \geq 0$ *independent of* φ *such that problem* (1.1) *has, at least, a weak solution verifying*

$$\begin{aligned} & ke^{\lambda t} \geq u, v \geq me^{-\lambda t} \geq 0 \quad \text{a.e. in } Q_T \qquad \text{and} \\ & u, v \in \mathcal{C}([0,T]; L^1(\Omega)), \\ & w \in L^\infty(0,T; W^{2,s}(\Omega)) \quad \text{for all } s \in [1, \infty). \end{aligned} \tag{2.7}$$

The proof of Theorem 2.2 is based on a fixed point technique. To define the fixed point operator in L^p spaces we need, due to the lack of regularity of the term $\nabla b(u) \cdot \nabla w$, to uncouple problem (1.1) and to consider two auxiliary problems (see (2.10) and (2.11)). First we apply a fixed point argument to obtain the existence of solutions, (u, v), of (2.10) and we also show that this solution satisfies (2.7). Then, we solve problem (2.11) and use again a fixed point argument to couple the system, obtaining in this way a weak solution of (1.1) with the property (2.7). The additional regularity is obtained by applying general results on L^p spaces (see [39]).

Proof of Theorem 2.2.

Step **1.** Let $T > 0$, $0 < \rho < c_\rho$, with c_ρ a positive constant to be fixed and p an exponent satisfying the following restriction:

$$\frac{r}{r-2} < 3 < p < r, \tag{2.8}$$

with r given in (2.6). Consider the set

$$K := \left\{ h \in L^p(0,T; W^{2,p}(\Omega)) : \Delta h \in L^\infty(Q_T),\ h = 0 \text{ on } \Sigma_{DT}, \|\Delta h\|_{L^p} + \|\nabla h\|_{L^2} \leq \rho \right\}. \tag{2.9}$$

Clearly, K is convex. Moreover, as $2 \leq \frac{Np}{N-p}$ (due to the choice of p) it follows that $\|\Delta w\|_{L^p} + \|\nabla w\|_{L^2}$ is a norm in $L^p(0,T; W^{2,p}(\Omega))$ and therefore K is weakly compact in $L^p(0,T; W^{2,p}(\Omega))$. These properties of K will be used later to apply a fixed point argument. Given $h \in K$ we introduce the problems

$$\begin{cases} u_t - div\,(\nabla\varphi(u) - b(u)\nabla h) = F(u,v) & \text{in } Q_T, \\ v_t - div\,(\nabla\varphi(v) + b(v)\nabla h) = F(u,v) & \text{in } Q_T, \\ \nabla\varphi(u)\cdot\nu = 0, \quad \nabla\varphi(v)\cdot\nu = 0, & \text{on } \Sigma_{NT}, \\ \varphi(u) = \varphi(u_D), \quad \varphi(v) = \varphi(v_D), & \text{on } \Sigma_{DT}, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & \text{in } \Omega \end{cases} \tag{2.10}$$

and

$$\begin{cases} -\Delta w = v - u - C & \text{in } Q_T, \\ w = w_D & \text{on } \Sigma_{DT}, \\ \nabla w \cdot \nu = 0 & \text{on } \Sigma_{NT}, \end{cases} \tag{2.11}$$

with similar notions of weak solutions as for problem (1.1).

Step **2.** *Definition of the fixed point operator for* (2.10). Consider the problems

$$\begin{cases} u_t - \Delta\varphi(u) = f \quad \text{in } Q_T, \\ \varphi(u) = \varphi(u_D) \text{ on } \Gamma_D, \ \nabla\varphi(u)\cdot\nu = 0 \text{ on } \Gamma_N, \\ u(x,0) = u_0(x) \quad \text{in } \Omega \end{cases} \tag{2.12}$$

and

$$\begin{cases} v_t - \Delta\varphi(v) = g \quad \text{in } Q_T, \\ \varphi(v) = \varphi(v_D) \text{ on } \Gamma_D, \ \nabla\varphi(v)\cdot\nu = 0 \text{ on } \Gamma_N, \\ v(x,0) = v_0(x), \quad \text{in } \Omega, \end{cases} \tag{2.13}$$

with $f, g \in L^2(Q_T)$. Since these problems are uniformly parabolic, from well known results (see, e.g., [1], [5], [30]) we have that (2.12) and (2.13) have a unique weak solution $u, v \in L^r(Q_T) \cap \mathcal{C}([0,T]; L^1(\Omega))$, $\varphi(u) \in \varphi(u_D) + L^2(0,T;\mathcal{V})$, $\varphi(v) \in \varphi(v_D) + L^2(0,T;\mathcal{V})$, where r was given in (2.6). We introduce the set

$$K^* := \left\{(f,g) \in L^2(Q_{T^*}) \times L^2(Q_{T^*}) : \|f\|_{L^2}, \|g\|_{L^2} < R\right\}, \quad 0 < T^* \leq T,$$

which is convex and weakly compact in $L^2(Q_{T^*}) \times L^2(Q_{T^*})$, and the mapping $Q : K^* \to L^2(Q_{T^*}) \times L^2(Q_{T^*})$ given by

$$Q(f,g) := (F(u,v) - div(b(u)\nabla h), F(u,v) + div(b(v)\nabla h)),$$

where u, v are the solutions of (2.12), (2.13). It can be shown that, as a consequence of (2.3) and the sublinearity of F and b, the operator Q is well defined. Notice also that a fixed point of Q is a weak solution of (2.10). To prove the existence of such a point we search for R and T^* such that
(i) $Q(K^*) \subset K^*$, and
(ii) Q is weakly-weakly sequentially continuous in $L^2(Q_{T^*}) \times L^2(Q_{T^*})$,
that will allow us to apply the Arino, Gauthier and Penot's fixed point theorem [4] to conclude the result. Since problems (2.12) and (2.13) share the same structure we shall only work out the properties that solutions of (2.12) satisfy, being those of (2.13) obtained in an identical manner.

Step **3.** *A priori estimates for problems* (2.12) *and* (2.13) *and proof of* $Q(K^*) \subset K^*$. This last condition reads as

$$I_u := \|F(u,v) - \nabla b(u)\cdot\nabla h - b(u)\Delta h\|_{L^2} \leq R, \tag{2.14}$$

and similarly for v. Taking $\zeta = \varphi(u) - \varphi(u_D)$ as a test function for problem (2.12) we get

$$\begin{aligned} \int_\Omega \Phi(u(t)) + \int_{Q_T} |\nabla\varphi(u)|^2 \ &= \int_\Omega \Phi(u_0) - \int_{Q_T} (u - u_0)\,\varphi(u_D)_t + \int_\Omega (u(t) - u_0)\,\varphi(u_D) \\ &+ \int_{Q_T} \nabla\varphi(u)\cdot\nabla\varphi(u_D) + \int_{Q_T} f\,(\varphi(u) - \varphi(u_D)), \end{aligned} \tag{2.15}$$

with $\Phi(s) := \displaystyle\int_0^s \varphi(\sigma)d\sigma$. Using that φ^{-1} is Lipschitz continuous and standard inequalities we get from (2.15)

$$\|\Phi(u)\|_{L^\infty(L^1)} + \|\nabla\varphi(u)\|_{L^2}^2 \leq \Lambda + \|f\|_{L^2}^2, \tag{2.16}$$

where Λ is a positive constant depending only on the initial and boundary data. Then, from Lemma 2.1 and $f \in K^*$ we get

$$\|u\|_{L^r} \leq c\, |||u||| \leq c\,(\Lambda + R)\,. \tag{2.17}$$

Since $r > 2$, we have

$$\|u\|_{L^2} \leq A_0(T^*)\, \|u\|_{L^r} \leq cA_0(T^*)\,(\Lambda + R)\,, \tag{2.18}$$

with $A_0(T^*) := |Q_{T^*}|^{\frac{r-2}{2r}}$ and, since φ is sublinear, we have that there exists a continuous non decreasing function $\eta : (0,\infty) \to (0,\infty)$ such that

$$\|\nabla\varphi(u)\|^2_{L^2} \leq \Lambda + \eta(T^*), \tag{2.19}$$

with $\eta(T^*) \to 0$ as $T^* \to 0$ (see [16], Lemma 6). We are now ready to estimate the terms in (2.14): F sublinear and (2.18) imply

$$\|F(u,v)\|_{L^2} \leq cA_1(T^*), \tag{2.20}$$

with $A_1(T^*) := A_0(T^*)\,(\Lambda + 2R) + |Q_{T^*}|^{1/2}$ and c a positive constant that shall vary along the proof. Now, from (2.3), (2.19) and the regularity of h we get

$$\|\nabla b(u) \cdot \nabla h\|_{L^2} \leq c\, \|\nabla\varphi(u)\|_{L^2}\, \|\nabla h\|_{L^\infty} \leq c\,(\Lambda + \eta(T^*))\, \|\nabla h\|_{L^\infty}\,, \tag{2.21}$$

and since b is sublinear

$$\|b(u)\Delta h\|_{L^2} \leq \|b(u)\|_{L^2}\, \|\Delta h\|_{L^\infty} \leq cA_2(T^*)\, \|\Delta h\|_{L^\infty}\,, \tag{2.22}$$

with $A_2(T^*) := |Q_{T^*}|^{\frac{1}{2}} + |Q_{T^*}|^{\frac{r-2}{2r}}\,(\Lambda + R)$. Gathering (2.20), (2.21) and (2.22) we obtain

$$I_u \leq cA_1(T^*) + c\,(\Lambda + \eta(T^*))\, \|\nabla h\|_{L^\infty} + cA_2(T^*)\, \|\Delta h\|_{L^\infty}\,,$$

and as we want $I_u \leq R$, it is sufficient to make

$$cA_1(T^*) + c\,(\Lambda + \eta(T^*))\, \|\nabla h\|_{L^\infty} + cA_2(T^*)\, \|\Delta h\|_{L^\infty} \leq R. \tag{2.23}$$

Since A_1, A_2, η are non decreasing continuous functions in $\mathbb{R}_+$, we have that, fixing R such that

$$R > cA_1(T) + c\,(\Lambda(u_0, \varphi(u_D)) + \eta(T))\, \|\nabla h\|_{L^\infty(Q_T)} + cA_2(T)\, \|\Delta h\|_{L^\infty(Q_T)}\,,$$

inequality (2.23) is satisfied for all $T^* \in [0,T]$. An identical argument allows us to get $I_v \leq R$. Therefore, we have proven the existence of a radius R and an instant T^* (that can be taken as $T^* = T$) such that $Q(K^*) \subset K^*$.

Step **4.** *Continuity of* Q. Consider any sequence $(f_j, g_j) \subset K^* \rightharpoonup (f,g)$ weakly in $L^2(Q_T) \times L^2(Q_T)$, and let us show that

$$\begin{array}{ll} div(b(u_j)\nabla h) \rightharpoonup div(b(u)\nabla h) & \text{in } L^2(Q_T) \quad \text{and} \\ F(u_j, v_j) \rightharpoonup F(u,v) & \text{in } L^2(Q_T), \end{array}$$

where u_j, v_j, u, v are the solutions of (2.12), (2.13) associated to f_j, g_j, u, v, respectively. By (2.17) we have that there exists a subsequence u_j such that $\|u_j\|_{L^r} \leq c\, |||u_j||| \leq$ const. and therefore also holds $\|u_{jt}\|_{L^2(0,T;\mathcal{V}')} \leq c$. Passing to a subsequence, if necessary, we obtain

$$\begin{array}{ll} u_j \rightharpoonup u & \text{weakly in } L^r(Q_T), \\ u_j \to u & \text{strongly in } L^2(Q_T) \text{ and a.e. in } Q_T, \\ \nabla u_j \rightharpoonup \nabla u & \text{weakly in } L^2(Q_T), \\ u_{jt} \rightharpoonup u_t & \text{weakly in } L^2(0,T;\mathcal{V}'). \end{array}$$

Since F is sublinear, $\|(F(u_j,v_j)\|_{L^r} \leq$const., because u_j, v_j are bounded in $L^r(Q_T)$, so $F(u_j,v_j) \rightharpoonup \tilde{F}$, for some $\tilde{F} \in L^r(Q_T)$. Moreover, the continuity of F together with the a.e. convergence of u_j, v_j implies that $F(u_j,v_j) \to F(u,v)$ a.e. in Q_{T^*}, and therefore $\tilde{F} \equiv F(u,v)$, i.e. $F(u_j,v_j) \to F(u,v)$ in $L^r(Q_T)$. A similar argument shows that $b(u_j) \to b(u)$ in $L^r(Q_T)$. Finally $div(b(u_j)\nabla h) = \nabla b(u_j)\cdot\nabla h + b(u_j)\Delta h$ and since $\nabla b(u_j) \rightharpoonup \nabla b(u)$ and $b(u_j) \to b(u)$ in $L^r(Q_T)$ and $\nabla h, \Delta h \in L^\infty$ it follows that $div(b(u_j)\nabla h) \rightharpoonup div(b(u)\nabla h)$ in $L^2(Q_T)$. Hence, Q is weakly-weakly sequentially continuous. By the fixed point theorem [4] we have that there exists a weak solution (u,v) of (2.10) with the same regularity obtained for the solutions of (2.12) and (2.13) when $f, g \in L^2(Q_T)$ is assumed.

Step **5.** *Lower bound and L^∞ regularity of u, v.* We introduce the change of unknowns $U := ue^{-\beta t}$ and $V := ve^{-\beta t}$ with $\beta > 0$ in problem (2.10) so (U,V) satisfies

$$\begin{cases} U_t + \beta U - e^{-\beta t} div\left(\nabla\varphi(e^{\beta t}U) - b(e^{\beta t}U)\nabla h\right) = \hat{F} & \text{in } Q_T, \\ V_t + \beta V - e^{-\beta t} div\left(\nabla\varphi(e^{\beta t}V) + b(e^{\beta t}V)\nabla h\right) = \hat{F} & \text{in } Q_T, \\ \nabla\varphi(e^{\beta t}U)\cdot\nu = 0, \quad \nabla\varphi(e^{\beta t}V)\cdot\nu = 0, & \text{on } \Sigma_{NT}, \\ \varphi(e^{\beta t}U) = \varphi(u_D), \quad \varphi(e^{\beta t}V) = \varphi(v_D), & \text{on } \Sigma_{DT}, \\ U(x,0) = u_0(x), \quad V(x,0) = v_0(x), & \text{in } \Omega, \end{cases}$$

with $\hat{F} := e^{-\beta t}F(e^{\beta t}U, e^{\beta t}V)$. To get the lower bound we compare U and V with the function $z := me^{-(\lambda+\beta)t}$ for a suitable $\lambda > \lambda_1$. By assumption, $u_D \geq me^{-\lambda_1 t} \geq me^{-\lambda t}$ and then we can take $Z_u := \min\{U - z, 0\}$ as test function obtaining

$$\begin{aligned} \int_\Omega Z_u (U-z)_t \quad & -\lambda\int_\Omega zZ_u + \beta\int_\Omega Z_u^2 + e^{-\beta t}\int_\Omega \nabla\varphi(e^{\beta t}U)\cdot\nabla Z_u = \\ & = -e^{-\beta t}\int_\Omega Z_u\left[\nabla b(e^{\beta t}U)\cdot\nabla h + b(e^{\beta t}U)\Delta h\right] + \\ & + e^{-\beta t}\int_\Omega Z_u F(e^{\beta t}U, e^{\beta t}V). \end{aligned}$$

Since b is Lipschitz continuous (with constant M_b), by estimating

$$\int_\Omega Z_u b'(e^{\beta t}U)\nabla Z_u\cdot\nabla h \leq M_b\int_\Omega |\nabla Z_u|^2 + M_b\|\nabla h\|^2_{L^\infty}\int_\Omega Z_u^2$$

and

$$\begin{aligned} \int_\Omega Z_u b(e^{\beta t}U)\Delta h &= \int_\Omega Z_u\left(b(e^{\beta t}U) - b(e^{\beta t}z) + b(e^{\beta t}z)\right)\Delta h \leq \\ &\leq ce^{\beta t}M_b\|\Delta h\|_{L^\infty}\left(\int_\Omega Z_u^2 + \int_\Omega z|Z_u|\right) \end{aligned}$$

we obtain

$$\frac{d}{dt}\int_\Omega Z_u^2 + \lambda'\int_\Omega z|Z_u| + \beta'\int_\Omega Z_u^2 \leq e^{-\beta t}\int_\Omega Z_u F(e^{\beta t}U, e^{\beta t}V), \tag{2.24}$$

with $\lambda' := \lambda - cM_b\|\Delta h\|_{L^\infty}$, $\beta' := \beta - M_b\|\nabla h\|^2_{L^\infty} - cM_b\|\Delta h\|_{L^\infty}$, $c > 0$ and where we have used that $-\lambda Z_u = \lambda|Z_u|$. Since F is Lipschitz continuous we can use a similar argument to show that

$$Z_u F(e^{\beta t}U, e^{\beta t}V) \leq c_1 Z_u\left(Z_u + Z_v + F(e^{\beta t}z, e^{\beta t}z)\right), \tag{2.25}$$

with $Z_v := (V-z)_-$ and $c_1 > 0$. Adding to (2.24) the analogous inequality for V we get in the right hand side of the resulting inequality the term

$$e^{-\beta t}\int_\Omega F(e^{\beta t}U, e^{\beta t}V)\,(Z_u + Z_v),$$

which using (2.25) and the analogous estimate for $Z_v F(e^{\beta t}U, e^{\beta t}V)$ may be estimated as

$$e^{-\beta t} \int_\Omega F(e^{\beta t}U, e^{\beta t}V)\,(Z_u + Z_v) \le c_1 \int_\Omega \left(Z_u^2 + Z_v^2 + z\,|Z_u| + z\,|Z_v|\right).$$

Then, for $\beta \ge M_b \|\nabla h\|_{L^\infty}^2 + cM_b \|\Delta h\|_{L^\infty} + c_1$ and $\lambda \ge cM_b \|\Delta h\|_{L^\infty} + c_1$ (notice that neither β nor λ depend on φ) we obtain

$$\frac{d}{dt} \int_\Omega \left(Z_u^2 + Z_v^2\right) \le 0,$$

from where the result follows. The estimate $u, v \in L^\infty(Q_T)$ is obtained in a similar way and we omit therefore the proof (see, e.g., [20] for details).

Step **6.** *End of the proof of existence of local solutions of* (1.1). Let $\tilde{T} \in (0, T]$, to be fixed, $h \in \tilde{K}$ (defined in (2.9) with T changed by $\tilde{T}$) and u, v be solutions of (2.12), (2.13) associated to h. Consider the problem (2.11) in $Q_{\tilde{T}}$. Since $u, v, C \in L^\infty(Q_{\tilde{T}})$ (2.11) has a unique solution $w \in L^\infty(0, \tilde{T}, W^{2,s}(\Omega)) \subset L^p(0, \tilde{T}, W^{2,p}(\Omega))$, for all $s \in (p, \infty)$. Define $P : \tilde{K} \to L^p(0, \tilde{T}, W^{2,p}(\Omega))$ by $P(h) = w$, being w such solution. Notice that if w is a fixed point for P then (u, v, w) is a solution of (1.1). To prove the existence of a fixed point we use the same technique than before, showing that
(i) $P(\tilde{K}) \subset \tilde{K}$, i.e., $\Delta w \in L^\infty(Q_T)$, and $\|\Delta w\|_{L^p} < \rho$ and
(ii) P is weakly-weakly sequentially continuous in $L^p(0, \tilde{T}, W^{2,p}(\Omega))$.
From (2.11):

$$\|\Delta w\|_{L^s} \le \|u\|_{L^s} + \|v\|_{L^s} + \|C\|_{L^s}, \quad \text{for all } s \in [1, \infty]. \tag{2.26}$$

Multiplying the equation in (2.11) by $w - w_D$ and using Hölder and Poincaré's inequalities we obtain

$$\|\nabla w\|_{L^2} \le c\left(\|u\|_{L^2} + \|v\|_{L^2} + \|C\|_{L^2} + \|w_D\|_{L^2} + \|\nabla w_D\|_{L^2}\right). \tag{2.27}$$

From (2.26), (2.27) and $p > 2$ we get

$$\|\Delta w\|_{L^p} + \|\nabla w\|_{L^2} \le c\left(\|u\|_{L^p} + \|v\|_{L^p} + \|C\|_{L^p} + \|w_D\|_{L^2} + \|\nabla w_D\|_{L^2}\right). \tag{2.28}$$

By (2.8) we have $p < r$ and therefore

$$\|u\|_{L^p} \le A(\tilde{T})\,\|u\|_{L^r} \le cA(\tilde{T})\,|||u|||, \tag{2.29}$$

with $A(\tilde{T}) := |Q_{\tilde{T}}|^{\frac{rp}{r-p}}$. Assume that the estimate

$$|||u||| \le G(\rho, \tilde{T}) \tag{2.30}$$

holds, where G is continuous, bounded as a function of $\tilde{T}$ and increasing with respect to ρ in an interval $(0, c_\rho)$ with c_ρ small enough. This estimate will be proven later on (see step **7**). Then, from (2.29) $\|u\|_{L^p} \le cA(\tilde{T})G(\rho, \tilde{T})$. A similar estimate holds for v. Since $C \in L^\infty(Q_{\tilde{T}})$ and $w_D \in L^\infty(0, T; H^1(\Omega))$ we have $\|C\|_{L^p} + \|w_D\|_{L^2} + \|\nabla w_D\|_{L^2} = B_0(\tilde{T})$ for a non decreasing continuous function B_0 satisfying $B_0(0) = 0$. From (2.28) we deduce

$$\|\Delta w\|_{L^p} + \|\nabla w\|_{L^2} \le A(\tilde{T})G(\rho, \tilde{T}) + B_0(\tilde{T}),$$

and since we want to make $\|\Delta w\|_{L^p} \le \rho$, it suffices to find a $\tilde{T} > 0$ such that

$$A(\tilde{T})G(\rho, \tilde{T}) + B_0(\tilde{T}) = \rho.$$

Since G is bounded as a function of ρ and $A(\tilde{T}), B_0(\tilde{T}) \downarrow 0$ as $\tilde{T} \downarrow 0$ it is straightforward to see that such $\tilde{T}$ exists, so (i) is satisfied.

To prove the continuity we consider a sequence h_n in $\tilde{K}$ such that $h_n \rightharpoonup h$ in $L^p(0,\tilde{T};W^{2,p}(\Omega))$ and we show that $w_n \rightharpoonup w$ in $L^p(0,\tilde{T};W^{2,p}(\Omega))$, being w_n, w solutions of (2.11) associated to h_n, h. Since $h_n \in \tilde{K}$, $\|\Delta h_n\|_{L^p} \le \rho < c_\rho$, and then from (2.30) we get $|||u_n|||\,, |||v_n||| \le G(c_\rho,\tilde{T}) \le$const., and therefore, $\|u_n\|_{L^r}, \|v_n\|_{L^r} \le$const. Then $u_n \rightharpoonup u$ in $L^r(Q_{\tilde{T}})$, and analogously for v_n, and therefore $\Delta w_n \rightharpoonup \Delta w$ in $L^r(Q_{\tilde{T}}) \subset L^p(Q_{\tilde{T}})$, because $p < r$. We deduce from the fixed point theorem [4] that P has a fixed point, (u,v,w), which is a weak solution of (1.1) in $Q_{\tilde{T}}$ with the regularity inherited from problems (2.10) and (2.11). Moreover, since the estimates are continuous with respect to $\tilde{T}$ we can take $\tilde{T} = T$, so the solution is global in time. Let us, finally, prove estimate (2.30):

Step **7**. *Estimating* $|||u|||^2 + |||v|||^2$ *of problem* (2.10). Taking $\varphi(u) - \varphi(u_D)$ as test function for (2.10) and reasoning as in (2.15) (with $f := F(u,v) - div(b(u)\nabla w)$) we get

$$\begin{aligned} \|\Phi(u)\|_{L^\infty(L^1)} + \|\nabla\varphi(u)\|_{L^2}^2 \quad &\le \quad \Lambda + \int_{Q_T} F(u,v)\,(\varphi(u) - \varphi(u_D)) + \\ &+ \int_{Q_T} b(u)\nabla h \cdot \nabla\,(\varphi(u) - \varphi(u_D))\,, \end{aligned} \tag{2.31}$$

with Λ depending only on the auxiliary data. Since F is sublinear we again get (2.20). Defining $B(s) := b(s)\varphi(s) - \int_0^s b'(\sigma)\varphi(\sigma)d\sigma$ and using the assumptions on sublinearity on φ, b and (2.3) we get $|B(s)| \le c(1 + |s| + s^2)$. We have

$$\begin{aligned} \int_{Q_T} b(u)\nabla h \cdot (\nabla\varphi(u) - \nabla\varphi(u_D)) \quad &= \quad -\int_{Q_T} (B(u) - B(u_D))\,\Delta h - \\ &\quad -\int_{Q_T} (b(u) - b(u_D))\,\nabla h \cdot \nabla\varphi(u_D). \end{aligned}$$

The first term is estimated as

$$\begin{aligned} \int_{Q_T} (B(u) - B(u_D))\,\Delta h \quad &\le \quad \|B(u) - B(u_D)\|_{L^{p'}} \|\Delta h\|_{L^p} \le \\ &\le \quad c\left(|Q_T|^{1/p'} + \|u\|_{L^{2p'}}^2 + \|u_D\|_{L^{2p'}}^2\right)\|\Delta h\|_{L^p}\,, \end{aligned}$$

and since $h \in K$, and $2p' \le r$ due to the choice of p (see (2.8)), we deduce $\|u\|_{L^{2p'}} \le c\,\|u\|_{L^r} \le c\,|||u|||$ and therefore

$$\int_{Q_T} (B(u) - B(u_D))\,\Delta h \le c\left(|Q_T|^{1/p'} + \|u_D\|_{L^{2p'}}^2 + |||u|||^2\right)\rho.$$

Second term is estimated as follows:

$$\int_{Q_T} (b(u) - b(u_D))\,\nabla h \cdot \nabla\varphi(u_D) \le \|b(u) - b(u_D)\|_{L^r(L^2)}\,\|\nabla h\|_{L^p(L^\infty)}\,\|\nabla\varphi(u_D)\|_{L^2}\,,$$

and since b is sublinear and $2 < r$ we have

$$\begin{aligned} \int_{Q_T} (b(u) - b(u_D))\,\nabla h \cdot \nabla\varphi(u_D) \quad &\le \quad c\,\|\nabla\varphi(u_D)\|_{L^q(L^2)}\left(|Q_T|^{1/p'} + \|u_D\|_{L^r} + |||u|||\right)\rho \\ &\le \quad \rho^2\,|||u|||^2 + c\,\|\nabla\varphi(u_D)\|_{L^2}^2 + \\ &\qquad + c\rho\,\|\nabla\varphi(u_D)\|_{L^q(L^2)}\left(|Q_T|^{1/p'} + \|u_D\|_{L^r}\right). \end{aligned}$$

Proceeding in a similar way for the v equation we get from (2.31) that

$$\left(|||u|||^2 + |||v|||^2\right) \le \Lambda_1(\rho) + c\left(|||u|||^2 + |||v|||^2\right)(\rho + \rho^2)$$

with $\Lambda_1(\rho) := c_1 + c_2\rho$, and c_1, c_2 depending on the norms of the auxiliary conditions and on T (continuous and non decreasingly). Hence, defining $G(\rho, T) := \dfrac{\Lambda_1(\rho)}{\sqrt{1 - c(\rho + \rho^2)}}$ with $\rho \in (0, c_\rho)$ and $c_\rho := \min\left\{1, \frac{1}{2c}\right\}$ we finish. □

Now we can afford the

Proof of Theorem 2.1. The proof uses a regularization technique and the Theorem 2.2. In view of the constructive method that we shall use in one of the uniqueness results, we consider two different regularizations of problem (1.1) depending on whether φ is a strictly increasing or a non decreasing function. In the first case we consider the following perturbation of the auxiliary data

$$\left\{\begin{array}{ll} \varphi_\varepsilon(u_D) = \varphi(u_D) + \varphi(\varepsilon e^{-\lambda_1 t}) & \text{on } \Sigma_{DT}, \\ \varphi_\varepsilon(v_D) = \varphi(v_D) + \varphi(\varepsilon e^{-\lambda_1 t}) & \text{on } \Sigma_{DT}, \\ u_{0\varepsilon} = u_0 + \varepsilon, \quad v_{0\varepsilon} = v_0 + \varepsilon & \text{in } \Omega, \end{array}\right. \tag{2.32}$$

for some $\lambda_1 > 0$, remaining the other auxiliary conditions the same, and we consider the function

$$\varphi_\varepsilon(s) := \left\{\begin{array}{ll} \varphi(\varepsilon e^{-\lambda T}) \exp\left\{\mu(s - \varepsilon e^{-\lambda T})\right\} & \text{si } s < \varepsilon e^{-\lambda T}, \\ \varphi(s) & \text{si } s \in [\varepsilon e^{-\lambda T}, k], \\ \varphi'(k)s + \varphi(k) - k\varphi'(k) & \text{si } s \geq k, \end{array}\right. \tag{2.33}$$

where k is an L^∞ bound of the auxiliary data and $\mu := \frac{\varphi'(\varepsilon e^{-\lambda T})}{\varphi(\varepsilon e^{-\lambda T})}$, so the matching is $\mathcal{C}^1$, $\varphi_\varepsilon(0) > 0$ and $\varphi_\varepsilon'(s) > 0$ in $s \geq 0$. It is straightforward to check that the sequence of problems $(1.1)_\varepsilon$ associated to the data (2.32) and the function (2.33) satisfy the conditions of Theorem 2.2. Finally, since φ and φ_ε coincides in the range of $u_\varepsilon, v_\varepsilon$ we may assume that $\varphi \equiv \varphi_\varepsilon$.

In the second case, in which φ is non decreasing, we consider for each $\varepsilon > 0$ the regularization given by $\varphi_\varepsilon(s) := \varphi(s) + \varepsilon s$ (without any change in the auxiliary conditions) and proceed in a similar way than above to show that the requirements of Theorem 2.2 are satisfied, obtaining therefore the existence of a sequence of solutions of $(1.1)_\varepsilon$ with the regularity and properties stated in that proposition.

A priori estimates. In both cases we proceed in a similar way: we use $\varphi_\varepsilon(u_\varepsilon) - \varphi(u_{D\varepsilon})$ as a test function for the first equation in (1.1) and as in the step **7** of the proof of Theorem 2.2 we obtain

$$\sup_{0 \leq t \leq T} \int_\Omega \Phi(u_\varepsilon(t)) + \int_{Q_T} |\nabla\varphi(u_\varepsilon)|^2 + \varepsilon \int_{Q_T} |\nabla u_\varepsilon|^2 \leq C, \tag{2.34}$$

with C independent of ε (because the L^∞ bounds of $u_\varepsilon, v_\varepsilon$ are independent of φ_ε). Using now $\xi \in L^2(0, T; \mathcal{V})$ as a test function we get

$$\begin{aligned} \left| \int_0^T \langle u_{\varepsilon t}, \xi \rangle_{\mathcal{V}', \mathcal{V}} \right| \quad \leq \quad & \|\nabla\varphi_\varepsilon(u_\varepsilon)\|_{L^2} \|\nabla\xi\|_{L^2} + \|b(u_\varepsilon)\|_{L^\infty} \|\nabla w_\varepsilon\|_{L^2} \|\nabla\xi\|_{L^2} + \\ & + \|F(u_\varepsilon, v_\varepsilon)\|_{L^2} \|\xi\|_{L^2}, \end{aligned}$$

from where we deduce

$$\|u_{\varepsilon t}\|_{L^2(0,T;\mathcal{V}')} \leq c, \tag{2.35}$$

with c independent of ε. A similar estimate holds for v_ε. From the third equation of (1.1) we get

$$\|\Delta w_\varepsilon\|_{L^\infty(Q_T)} \leq \|v_\varepsilon - u_\varepsilon + C\|_{L^\infty(Q_T)} \leq \text{const.} \tag{2.36}$$

Therefore, by (2.34)-(2.36) and standard compactness results we can extract subsequences (labeled again by ε) such that

$$\begin{array}{ll} u_\varepsilon \rightharpoonup u & \text{weakly} * \text{ in } L^\infty(Q_T), \\ \varphi(u_\varepsilon) \rightharpoonup \xi & \text{weakly in } \varphi(u_D) + L^2(0,T;\mathcal{V}), \\ u_{\varepsilon t} \rightharpoonup u_t & \text{weakly in } L^2(0,T;\mathcal{V}'), \\ \varepsilon^{1/2} u_\varepsilon \rightharpoonup 0 & \text{weakly in } L^2(0,T;\mathcal{V}), \\ w_\varepsilon \rightharpoonup w & \text{weakly } * - \text{ weakly in } L^\infty(0,T;W^{2,s}(\Omega)), \text{ for all } s<\infty. \end{array} \tag{2.37}$$

From the compact imbedding $L^\infty(\Omega) \subset H^{-1}(\Omega)$ and Corollary 4 (p. 85) of [39] we also have that

$$u_\varepsilon \to u \quad \text{in} \quad \mathcal{C}([0,T],\mathcal{V}'),$$

and since φ is continuous and non decreasing we have that $-\Delta\varphi(\cdot)$ is a maximal monotone graph in $L^2(0,T;\mathcal{V}')$, and therefore, it is strongly weakly closed in such space (see, e.g., [8]), from where we deduce that $\xi = \varphi(u)$.

Assume, now, that $\mathbf{H}_4$ holds. In order to pass to the limit on $b(u_\varepsilon)$ and $F(u_\varepsilon, v_\varepsilon)$ we shall prove that $u_\varepsilon \to u$ in $L^q(Q_T)$ for all $q<\infty$. To do that we use a modification of the arguments given in [17], [31] (see also [19]). Defining the space

$$\mathcal{H} = \{u \in L^{2/\alpha}(0,T;W^{\alpha,2/\alpha}(\Omega)), u_t \in L^2(0,T;\mathcal{V}')\},$$

it is easy to see that u_n is uniformly bounded in $\mathcal{H}$. Then, from the compact imbedding $\mathcal{H} \subset L^{2/\alpha}(Q_T)$ we conclude that there exists a subsequence of u_n such that

$$u_n \to u \quad \text{strongly in} \quad L^{2/\alpha}(Q_T) \quad \text{and} \quad \text{a.e. in} \quad Q_T.$$

This fact together with the weak $*$ convergence of u_ε to u in $L^\infty(Q_T)$ implies that $u_\varepsilon \to u$ in $L^q(Q_T)$ for all $q<\infty$. And similarly for v.

Identification of the limit. With the above convergences we are ready to identify the limit (u,v,w) as a solution of (1.1). Let $\zeta \in L^2(0,T;\mathcal{V})$ be a test function. By (2.37) it is clear that

$$\int_0^T \langle u_{\varepsilon t}, \zeta\rangle \to \int_0^T \langle u_t, \zeta\rangle \quad \text{and} \quad \int_0^T\int_\Omega \nabla\varphi(u_\varepsilon)\cdot\nabla\zeta \to \int_0^T\int_\Omega \nabla\varphi(u)\cdot\nabla\zeta.$$

From the convergence a.e. in Q_T of $u_\varepsilon, v_\varepsilon$ to u, v we get $F(u_\varepsilon, v_\varepsilon) \to F(u,v)$ a.e. in Q_T, and since F is Lipschitz continuous we obtain

$$\|F(u_\varepsilon, v_\varepsilon)\|_{L^2} \le c\,(\|u_\varepsilon\|_{L^2} + \|v_\varepsilon\|_{L^2} + 1) \le \text{const.},$$

so $F(u_\varepsilon, v_\varepsilon) \rightharpoonup \tilde F \in L^2(Q_T)$. Lebesgue's theorem implies $\tilde F \equiv F(u,v)$, and therefore

$$\int_0^T\int_\Omega F(u_\varepsilon, v_\varepsilon)\zeta \to \int_0^T\int_\Omega F(u,v)\zeta.$$

We also have that since $b(u_\varepsilon)$ is bounded in $L^\infty(Q_T)$ (b is continuous in $[0,\infty)$) and since $b(u_\varepsilon)$ is bounded in $L^\infty(Q_T)$ and $w_\varepsilon \to w$ in $L^2(0,T;\mathcal{V})$ (due to the compact imbedding $L^2(0,T;W^{2,2}(\Omega)) \subset L^2(0,T;\mathcal{V})$) then we get

$$\int_{Q_T} b(u_\varepsilon)\nabla w_\varepsilon\cdot\nabla\zeta \to \int_{Q_T} b(u)\nabla w\cdot\nabla\zeta. \tag{2.38}$$

It also holds $\nabla w \in L^\infty(Q_T)$ because $w \in L^\infty(0,T;W^{2,s}(\Omega))$ for all $s<\infty$ and $b(u_\varepsilon) \rightharpoonup b(u)$ in $L^2(Q_T)$ because of the continuity of b and a.e. in Q_T convergences of u_ε to u. Hence

$$\int_{Q_T} b(u_\varepsilon)\nabla w\cdot\nabla\zeta \to \int_{Q_T} b(u)\nabla w\cdot\nabla\zeta. \tag{2.39}$$

We deduce from (2.38) and (2.39) that

$$\int_{Q_T} (b(u_\varepsilon)\nabla w_\varepsilon - b(u)\nabla w) \cdot \nabla\zeta \to 0,$$

from where we have identified u as the first component of a solution of (1.1). The other components are handled in a very similar way and we skip therefore the proof. In the case in which $\mathbf{H}_4$ does not hold, i.e., when both b and F are linear functions, the passing to the limit is easier since we do not need to enssure the a.e. convergence of $u_\varepsilon, v_\varepsilon$ to u, v. In this situation the identification of the limit is just a consequence of the weak convergences in (2.37). Finally, from [3], Theorem 2.2, we obtain the additional regularity

$$\sqrt{\varphi'(u)}\nabla u, \sqrt{\varphi'(v)}\nabla v \in L^2(Q_T).$$

To finish, notice that since for all $\varepsilon > 0$ we have, due to Theorem 2.2, that (2.7) holds for any $\varepsilon > 0$ we deduce that this property also holds in the limit $\varepsilon \to 0$. □

Remark. The technique we have used is also applicable when $F(u, v)$ is a maximal monotone graph (see [15] for a likely system but without transport terms). We also point out that functions $\varphi(u)$ and $\varphi(v)$ (as well as $b(u)$ and $b(v)$) may be different as long as they fulfill the assumptions given on the data.

3. Uniqueness of solutions

As in the question of existence, the main difficulty to prove uniqueness of solutions relies in the simultaneous presence of a transport term and a non linear (degenerate) diffusion term. This kind of difficulty has already received the attention of many authors and has been solved for scalar equations of the type

$$u_t - div\,(\nabla\varphi(u) + b(u)\mathbf{e}) = F(u), \tag{3.1}$$

where $\mathbf{e}$ is a prescribed vector field. The most successful technique developed to prove uniqueness of solutions of this problem is based on the use of the test function $sign_+(u_1 - u_2)$ in (3.1), where u_1, u_2 are solutions *in some sense*. The core of this technique is to show that the solution has enough regularity to define the sign function as an admissible test function. This justification has been carry out by different means. One of them, introduced by Kruzhkov in [28] to prove an L^1 contraction property of entropy solutions of hyperbolic equations, is based in doubling the time variable and performing a passing to the limit in which these variables collapse. This technique has been also applied to parabolic scalar equations (see, e.g., [29], [9], [18], [19] and [35]) and, recently, in [36], also to certain systems of parabolic equations coupled through reaction terms (but not through transport terms). However, systems coupled through transport terms in which transport is not dominated by diffusion (say $b' \leq c\varphi'$ does not hold), have not been, as far as we know, solved by using this technique, so other means have to be applied.

We present in this section three theorems on the uniqueness question for problem (1.1) that share the *duality technique* in their proofs, i.e., the searching of suitable test functions that allow, by means of different arguments in each theorem, to conclude the uniqueness property.

The first result has been obtained by using a technique introduced by Antontsev, Díaz and Domansky [2] for a system of two-phase filtration in porous medium. Here we assume $(b'(s))^2 \leq c\varphi'(s)$, that holds in the important case when diffusion and transport are both linear and also in the case in which they are degenerate in a suitable way. It is worth noting that this type of condition also arises as sufficient condition to ensure the existence of strong solutions of (3.1) (see [6]).

The second result uses a technique introduced by Rulla [37] for a scalar equation in the Stefan problem with prescribed convection. In this case we only assume that φ is non decreasing, but an entropy type condition for the electric field on the Dirichlet boundary must be introduced: $\nabla w \cdot \nu = 0$

on Σ_{DT}. Conditions of this type are already classical in the literature of hyperbolic equations (see [28]) and they arise as natural conditions that allow to select a unique solution (the so called entropy solution) when uniqueness fails for weak solutions (see also [9]).

Our last theorem applies to the case in which problem (1.1) has strong solutions in the following sense: $u, v \in L^1(0,T;\mathcal{W})$, with

$$\mathcal{W} := \left\{ h \in W^{1,p}(\Omega) : h = 0 \text{ on } \Gamma_D \right\}. \tag{3.2}$$

with $p > N$ if $N \geq 2$ and $p = 1$ if $N = 1$. To obtain the result we used a method due to Kalashnikov [26] which consists of making a comparaison between an arbitrary weak solution of (1.1) and the weak solution constructed as the limit of a sequence of solutions of regularized problems (see Theorem 2.1). Our result is strongly based on the technique introduced by Díaz and Kersner [13] to study a one dimensional scalar equation (this technique was later used in [7] for an $N-$dimensional scalar equation) and, to achieve it, we generalized a comparaison argument introduced in [13] to handle some singular boundary integrals.

In the sequel we shall assume that the component w of solutions is non trivial in the sense that $\|\nabla w\|_{L^2(Q_T)} \neq 0$. On the contrary, the system reduces to the equation $u_t - \Delta\varphi(u) = F(u, u - C)$, in fact simpler than (3.1) that, as we already mentioned, is well understood.

THEOREM 3.1 *Suppose that assumptions* $\mathbf{H}_1$-$\mathbf{H}_3$ *hold and that there exists a constant* $M > 0$ *such that*

$$\left(b'(s)\right)^2 \leq M\varphi'(s) \quad \textit{for any } s > 0 \tag{3.3}$$

and

$$\left(\frac{\partial}{\partial s_i} F(s_1, s_2)\right)^2 \leq M\varphi'(s_i), \quad \textit{for any } s_i > 0, \quad i = 1, 2. \tag{3.4}$$

Then problem (1.1) *has a unique solution in the class of weak solutions such that*

$$\begin{aligned} &\sqrt{\varphi'(u)}\nabla u, \sqrt{\varphi'(v)}\nabla v \in L^2(Q_T), \\ &w \in L^\infty(0,T;W^{1,\infty}(\Omega)). \end{aligned}$$

Proof. Suppose that (u_1, v_1, w_1) and (u_2, v_2, w_2) are two weak solutions of (1.1) and define $u := u_1 - u_2$, $v := v_1 - v_2$, $w := w_1 - w_2$, $F_i := F(u_i, v_i)$, $i = 1, 2$ and $\hat{F} := F_1 - F_2$. Then (u, v, w) satisfies

$$\left\{ \begin{array}{l} u_t - \Delta\left(\varphi(u_1) - \varphi(u_2)\right) + div\left(b(u_1)\nabla w + \left(b(u_1) - b(u_2)\right)\nabla w_2\right) = \hat{F}, \\ v_t - \Delta\left(\varphi(v_1) - \varphi(v_2)\right) - div\left(b(v_1)\nabla w + \left(b(v_1) - b(v_2)\right)\nabla w_2\right) = \hat{F}, \\ -\Delta w + u - v = 0, \end{array} \right. \tag{3.5}$$

in Q_T with the auxiliary conditions

$$\left\{ \begin{array}{lll} \nabla\varphi(u_i)\cdot\nu = 0, \quad \nabla\varphi(v_i)\cdot\nu = 0, \quad \nabla w\cdot\nu = 0 & \text{on } \Sigma_{NT}, \\ \varphi(u_i) = \varphi(u_D), \quad \varphi(v_i) = \varphi(v_D), \quad w = 0 & \text{on } \Sigma_{DT}, \\ u(x,0) = 0, \quad v(x,0) = 0 & \text{in } \Omega, \end{array} \right.$$

$i = 1, 2$. Taking ψ, ξ, η as smooth test functions for each of the three equations in (3.5), integrating by parts and adding the resulting integral identities we obtain

$$\begin{aligned} \int_\Omega \psi(T)u(T) + \xi(T)v(T) &= \int_{Q_T} u\left(\psi_t + A_u\Delta\psi + \mathbf{B}_u\cdot\nabla\psi + \eta + F_u\left(\psi + \xi\right)\right) + \\ &+ \int_{Q_T} v\left(\xi_t + A_v\Delta\xi - \mathbf{B}_v\nabla\xi - \eta + F_v\left(\psi + \xi\right)\right) + \\ &- \int_{Q_T} w div\left(b(u_1)\nabla\psi - b(v_1)\nabla\xi + \nabla\eta\right), \end{aligned} \tag{3.6}$$

where $A_u := \dfrac{\varphi(u_1)-\varphi(u_2)}{u}$, $\mathbf{B}_u := \dfrac{b(u_1)-b(u_2)}{u}\nabla w_2$ and $F_u := \dfrac{F(u_1,v_1)-F(u_2,v_1)}{u}$ whenever $u \neq 0$ and $A_u = \mathbf{B}_u = F_u = 0$ if $u = 0$, and similar definitions for $A_v, \mathbf{B}_v$ and F_v. Notice that since $b \in \mathcal{C}^1([0,\infty))$, $u_i, v_i \in L^\infty(Q_T)$, F is Lipschitz continuous and $\nabla w_2 \in L^\infty(Q_T)$ we have that $\mathbf{B}_u, F_u$ and $\mathbf{B}_v, F_v$ are bounded in $L^\infty(Q_T)$. We define the differential operators

$$\begin{aligned}
\mathcal{L}_1(\psi,\xi,\eta) &:= \psi_t + A_u^\varepsilon \Delta\psi + \mathbf{B}_u \cdot \nabla\psi + \eta + F_u\,(\psi+\xi)\,,\\
\mathcal{L}_2(\psi,\xi,\eta) &:= \xi_t + A_v^\varepsilon \Delta\xi - \mathbf{B}_v \cdot \nabla\xi - \eta + F_v\,(\psi+\xi)\,,\\
\mathcal{L}_3(\psi,\xi,\eta) &:= div\,(b(u_1)\nabla\psi - b(v_1)\nabla\xi + \nabla\eta)\,,
\end{aligned}$$

with $A_u^\varepsilon := A_u + \varepsilon$ and $\varepsilon > 0$, (and a similar definition for A_v^ε) and set the following problem to choose the test functions:

$$\left\{\begin{array}{ll}
\mathcal{L}_1(\psi,\xi,\eta) = u & \text{in } Q_T,\\
\mathcal{L}_2(\psi,\xi,\eta) = v & \text{in } Q_T,\\
\mathcal{L}_3(\psi,\xi,\eta) = 0 & \text{in } Q_T,\\
\nabla\psi\cdot\nu = \nabla\xi\cdot\nu = \nabla\eta\cdot\nu = 0 & \text{on } \Sigma_{NT},\\
\psi = \xi = \eta = 0 & \text{on } \Sigma_{DT},\\
\psi(T) = \xi(T) = 0 & \text{in } \Omega.
\end{array}\right. \tag{3.7}$$

Lemma 3.1 *Problem* (3.7) *has a unique solution with the regularity of the test functions of* (1.1) *(see* (2.4) *and* (2.5)*). Moreover,*

$$\psi,\xi,\eta \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)),$$

and there exists a positive constant $C(T)$ independent of ε such that

$$\varepsilon \int_{Q_T} \left(|\Delta\psi|^2 + |\Delta\xi|^2\right) \leq C(T). \tag{3.8}$$

Continuation of the proof of Theorem 3.1**.** Introducing in (3.6) these test functions we get

$$\varepsilon \int_{Q_T} (u\Delta\psi + v\Delta\xi) = \int_{Q_T} \left(u^2+v^2\right).$$

From Young's inequality (with parameter $\sqrt{\varepsilon}$) and (3.8) we obtain

$$\int_{Q_T} \left(u^2+v^2\right) \leq \sqrt{\varepsilon}\int_{Q_T} \left(u^2+v^2\right) + \sqrt{\varepsilon}C(T).$$

Hence, taking the limit $\varepsilon \to 0$, we conclude that $u \equiv v \equiv 0$ a.e. in Q_T, that also implies $w \equiv 0$ a.e. in Q_T. □

Proof of Lemma 3.1.

Step **1.** *A prori estimates.* Due to (3.3) we can estimate

$$\int_\Omega (\mathbf{B}_u\cdot\nabla\psi)\,\Delta\psi \leq \delta\int_\Omega A_u^\varepsilon\,|\Delta\psi|^2 + \frac{M}{\delta}\int_\Omega |\nabla\psi|^2\,, \tag{3.9}$$

for $\delta > 0$. A similar estimate holds, thanks to (3.4), for $\displaystyle\int_\Omega F_u\,(\psi+\xi)\,\Delta\psi$. Multiplying the third equation of (3.7) by η and using the regularity $u_i, v_i \in L^\infty(Q_T)$ and the continuity of b we get

$$\int_\Omega |\nabla\eta|^2 \leq c_0(T)\int_\Omega \left(|\nabla\psi|^2 + |\nabla\xi|^2\right) \tag{3.10}$$

with $c_0(T) \geq 0$. Finally, multiplying the first equation of (3.7) by $\Delta\psi$ and using (3.9), the analogous expression for the F term and (3.10) we obtain

$$-\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\psi|^2 + \int_\Omega A_u^\varepsilon |\Delta\psi|^2 \leq c\left(\int_\Omega |\nabla\psi|^2 + \int_\Omega |\nabla\xi|^2 + \int_{Q_T} u^2\right), \tag{3.11}$$

for a suitable δ. From the second equation of (3.7) we obtain a similar inequality for ξ which, being added to (3.11) and taking into account that $A_u^\varepsilon, A_v^\varepsilon > \varepsilon$ allows us to obtain

$$\begin{aligned} -\frac{1}{2}\frac{d}{dt}\int_\Omega \left(|\nabla\psi|^2 + |\nabla\xi|^2\right) + \frac{\varepsilon}{2}\int_\Omega \left(|\Delta\psi|^2 + |\Delta\xi|^2\right) \quad &\leq \quad c\left(\int_\Omega \left(|\nabla\psi|^2 + |\nabla\xi|^2\right) + \right. \\ &\left. + \int_\Omega \left(u^2 + v^2\right)\right), \end{aligned} \tag{3.12}$$

where c is independent of ε. On one hand, we deduce from Gronwall's Lemma that

$$\int_\Omega \left(|\nabla\psi(t)|^2 + |\nabla\xi(t)|^2\right) \leq c_1(T)e^{cT}, \tag{3.13}$$

with $c_1(T)$ independent of ε, and, on the other hand, integrating (3.12) in $(0,T)$ and using (3.13) we obtain

$$\frac{\varepsilon}{2}\int_{Q_T} \left(|\Delta\psi|^2 + |\Delta\xi|^2\right) \leq c_2(T)e^{cT}, \tag{3.14}$$

with $c_2(T)$ independent of ε. So we deduced (3.8). Finally, from the third equation of (3.7) we have that

$$\Delta\eta = \nabla b(v_1)\cdot\nabla\xi + b(v_1)\Delta\xi - \nabla b(u_1)\cdot\nabla\psi - b(u_1)\Delta\psi,$$

and from (3.3) and the regularity $\sqrt{\varphi'(v_1)}\nabla v_1 \in L^2(Q_T)$ (see Theorem 2.1) we obtain

$$\int_\Omega |\nabla b(v_1)|^2 = \int_\Omega b'(v_1)^2 |\nabla v_1|^2 \leq c\int_\Omega \varphi'(v_1)|\nabla v_1|^2 \leq \text{const.} \tag{3.15}$$

Hence, using Hölder and Young's inequalities and estimates (3.10) and (3.15) we obtain the $L^2(0,T;H^2(\Omega)$ regularity of η.

Step **2**. *Existence of solutions of (3.7).* We proceed by a fixed point argument. First we consider the set

$$K := \left\{h \in L^2(0,T^*;\mathcal{V}) : \|h\|_{L^2(0,T^*;\mathcal{V})} \leq R\right\},$$

where T^* and R will be suitably chosen. It is clear that K is convex and weakly compact in $L^2(0,T^*;\mathcal{V})$. In this set we define the mapping $Q : K \subset L^2(0,T^*;\mathcal{V}) \to L^2(0,T^*;\mathcal{V})$ by $Q(\hat\eta) := \eta$, where η is the unique solution of the problem $\mathcal{L}_3(\hat\psi,\hat\xi,\eta) = 0$, $(\hat\psi,\hat\xi)$ being the unique solution of

$$\left\{\begin{array}{l} \mathcal{L}_1(\hat\psi,\hat\xi,\hat\eta) = u, \\ \mathcal{L}_2(\hat\psi,\hat\xi,\hat\eta) = v, \end{array}\right. \tag{3.16}$$

with the same auxiliary conditions as in (3.7). Since $u, v, \hat\eta \in L^2(Q_{T^*})$ we have that, thanks to the a priori estimates in Step **1** of this proof, any weak solution of (3.16) has the regularity

$$\hat\psi,\hat\xi \in H^1(0,T^*;L^2(\Omega)) \cap L^\infty(0,T^*;H^1(\Omega)) \cap L^2(0,T^*;H^2(\Omega)). \tag{3.17}$$

From this regularity and the linearity of the differential operators it follows the uniqueness of solutions of (3.16). The existence of solutions of (3.16) is proven by uncoupling the problem and applying again a fixed point technique. Assume for the moment that such a solution exists and, therefore, it is unique and satisfies (3.17). We have then that the solution of $\mathcal{L}_3(\hat{\psi}, \hat{\xi}, \eta) = 0$ satisfies $\eta \in L^2(0, T^*; H^2(\Omega))$, because it is a linear elliptic problem with smooth coefficients and right hand side term in $L^2(Q_{T^*})$. Notice that if Q has a fixed point, $\hat{\eta}$, then $(\hat{\psi}, \hat{\xi}, \hat{\eta})$ is a solution of (3.7). To prove the existence of such a fixed point we shall prove that
(i) $Q(K) \subset K$, for suitable $R, T^* > 0$,
(ii) Q is weakly-weakly sequentially continuous in $L^2(0, T^*; \mathcal{V})$,
and apply the fixed point theorem [4]. The first point is deduced from the previous a priori estimates, which will be justified thanks to the regularity of $\hat{\psi}, \hat{\xi}$ and η. Taking $T = T^*$, from (3.10) we get that

$$\|\eta\|_{L^2(0,T^*;\mathcal{V})} \leq c_0(T^*) \left(\left\|\hat{\psi}\right\|_{L^2(0,T^*;\mathcal{V})} + \left\|\hat{\xi}\right\|_{L^2(0,T^*;\mathcal{V})} \right),$$

and from (3.13) (and the corresponding estimate for ξ) we obtain

$$\left\|\hat{\psi}\right\|_{L^2(0,T^*;\mathcal{V})} + \left\|\hat{\xi}\right\|_{L^2(0,T^*;\mathcal{V})} \leq c_1(T^*) \|\hat{\eta}\|_{L^2(0,T^*;\mathcal{V})} e^{cT^*}.$$

It follows that

$$\|Q(\hat{\eta})\|_{L^2(0,T^*;\mathcal{V})} \leq c_3(T^*) e^{cT^*} R.$$

Notice that the functions $c_i(T^*)$ are continuous non decreasing with $c_i(0) = 0$ (they depend on the norms of the data in Q_T) and therefore we can take T^* small enough to obtain $c_3(T^*)e^{cT^*} \leq 1$, deducing $Q(K) \subset K$. The second point is a direct consequence of the linearity and a priori estimates and we omit the proof (see [20]). This finishes the proof of the existence of a fixed point and, therefore, of a local solution of (3.7). Notice that the continuity of the estimates with respect to the time implies that $T^* = T$ for any $T > 0$, i.e., the solution is global in time. Finally, the uniqueness of solutions is again a consequence of the linearity of the problem and the regularity of the solution. To finish, notice that the proof of existence of solutions of (3.16) may be performed in a similar way. □

Now we state the second result on uniqueness of solutions of (1.1). The main feature of this theorem is that it allows to consider a non linear diffusion, φ, not necessarily strictly increasing. However, we need to assume that an entropy type condition on the electric field holds on the Dirichlet boundary.

THEOREM 3.2 *Assume that* $\mathbf{H}_1$-$\mathbf{H}_3$ *hold and that* $b(s) = s$. *If*

$$\nabla w \cdot \nu = 0 \quad \text{on} \quad \Gamma_D \times (0, T), \tag{3.18}$$

and

$$|F(s_1, \sigma_1) - F(s_2, \sigma_2)| \leq c_1 \left[(\varphi(s_1) - \varphi(s_2)) + (\varphi(\sigma_1) - \varphi(\sigma_2)) \right], \tag{3.19}$$

then problem (1.1) *has a unique solution in the class of weak solutions such that*

$$w \in L^\infty(0, T; W^{2,\infty}(\Omega)).$$

Remark. The equality in (3.18) is a consequence of the different sign that the transport terms have in the equations satisfied by u and v. Indeed, suppose that there exist two solutions $(u_1, \tilde{v}, w_1)$ and $(u_2, \tilde{v}, w_2)$. Then, under the conditions of the above theorem, with the equality sign in (3.18) substituted by $\geq$, it is possible to show that $u_1 \equiv u_2$ and $w_1 \equiv w_2$ a.e. in Q_T.

Proof of Theorem 3.2. Following the proof of Theorem 3.1 with $b(s) := s$, we deduce, from (3.6), the following identity

$$\begin{aligned}\int_{Q_T} u_t\psi + v_t\xi &= \int_{Q_T} (\varphi(u_1) - \varphi(u_2))\,\Delta\psi + u\nabla w_2\cdot\nabla\psi + u\eta \\ &+ \int_{Q_T} (\varphi(v_1) - \varphi(v_2))\,\Delta\xi - v\nabla w_2\cdot\nabla\xi - v\eta \\ &- \int_{Q_T} w div\,(u_1\nabla\psi - v_1\nabla\xi + \nabla\eta) - \int_{Q_T} (F_1 - F_2)\,(\psi+\xi)\,.\end{aligned}$$

We choose the test functions, for each $t \in (0,T)$, as solutions of the problem

$$\begin{cases} -\Delta\psi(t) = u(t) & \text{in } \Omega, \\ -\Delta\xi(t) = v(t) & \text{in } \Omega, \\ -\Delta\eta(t) = div\,(v_1(t)\nabla\xi(t) - u_1(t)\nabla\psi(t)) & \text{in } \Omega, \\ \nabla\psi\cdot\nu = \nabla\xi\cdot\nu = \nabla\eta\cdot\nu = 0, & \text{on } \Sigma_N, \\ \psi = \xi = \eta = 0, & \text{on } \Sigma_D. \end{cases} \tag{3.20}$$

The existence, uniqueness and regularity of solutions is a consequence of the theory of linear elliptic equations. Notice that since $u,v \in \mathcal{C}([0,T];\mathcal{V}')$ we conclude that $\psi(t),\xi(t) \in \mathcal{V}$ so, in particular, $v_1(t)\nabla\xi(t) \in L^2(\Omega)$ and therefore $\eta(t) \in H^1(\Omega)$. Using these test functions we get

$$\begin{aligned}\int_\Omega \left(|\nabla\psi(T)|^2 + |\nabla\xi(T)|^2\right) &+ \int_{Q_T} [u\,(\varphi(u_1)-\varphi(u_2)) + v\,(\varphi(v_1)-\varphi(v_2))] = \\ &= \int_{Q_T} u\nabla w_2\cdot\nabla\psi - v\nabla w_2\cdot\nabla\psi + \\ &+ \int_{Q_T} [\nabla\psi\cdot\nabla\eta - \nabla\xi\cdot\nabla\eta + |F_1 - F_2|\,|\psi+\xi|]\,.\end{aligned} \tag{3.21}$$

Now we perform the arguments to handle the terms involving u. The terms involving v are similarly hundled (with a change of sign). Due to the choice of the test functions

$$\int_{Q_T} u\nabla w_2\cdot\nabla\psi = \int_{Q_T} -\Delta\psi\nabla w_2\cdot\nabla\psi.$$

As in [37], let us show that (3.18) implies

$$\int_{Q_T} -\Delta\psi\nabla w_2\cdot\nabla\psi \le \frac{1}{2}\,\|w_2\|_{L^\infty(W^{2,\infty})}\int_{Q_T} |\nabla\psi|^2\,. \tag{3.22}$$

Integrating formally by parts the left hand side of (3.22) we get

$$\int_\Omega -\Delta\psi\,(\nabla w_2\cdot\nabla\psi) = \int_\Omega \nabla\psi\cdot\nabla\,(\nabla w_2\cdot\nabla\psi) - \int_{\partial\Omega} (\nabla w_2\cdot\nabla\psi)\,(\nabla\psi\cdot\nu)\,. \tag{3.23}$$

See [37] for a rigorous derivation of this identity. Using the boundary conditions and $\psi = 0$ on Γ_D imply that $\nabla\psi$ has the same direction as ν we obtain

$$\int_{\partial\Omega} (\nabla w_2\cdot\nabla\psi)\,(\nabla\psi\cdot\nu) = \int_{\Gamma_D} |\nabla\psi|^2\,\nabla w_2\cdot\nu. \tag{3.24}$$

Denoting by $H(\cdot)$ the Hessian matrix we get after integrating by parts

$$\begin{aligned}\int_\Omega \nabla\psi\cdot\nabla\,(\nabla w_2\cdot\nabla\psi) &= \int_\Omega \nabla\psi : H(\nabla w_2) : \nabla\psi - \frac{1}{2}\int_\Omega \Delta w_2\,|\nabla\psi|^2 + \\ &+ \frac{1}{2}\int_{\Gamma_D} |\nabla\psi|^2\,\nabla w_2\cdot\nu.\end{aligned} \tag{3.25}$$

Substituting (3.24) and (3.25) in (3.23) leads to

$$\begin{aligned}\int_\Omega -\Delta\psi\,(\nabla w_2\cdot\nabla\psi) &= \int_\Omega \nabla\psi : H(w_2):\nabla\psi - \frac{1}{2}\int_\Omega \Delta w_2\,|\nabla\psi|^2 - \\ &\quad -\frac{1}{2}\int_{\Gamma_D} |\nabla\psi|^2\,\nabla w_2\cdot\nu,\end{aligned}$$

and using $\nabla w_2\cdot\nu\geq 0$ on Γ_D (due to (3.18)) and the regularity assumed on w_2 we obtain (3.22). For problem (3.20) the estimate (3.10) also holds and then we have

$$\int_{Q_T} (\nabla\psi\cdot\nabla\eta - \nabla\xi\cdot\nabla\eta) \leq \tilde{c}\int_{Q_T}\left(|\nabla\psi|^2+|\nabla\xi|^2\right), \tag{3.26}$$

with $\tilde{c}>0$. Finally, Hölder's, Young's and Poincaré's inequalities together with (3.19) give us

$$\begin{aligned}\int_{Q_T} |F_1-F_2|\,|\psi+\xi| &\leq \varepsilon c_1\int_{Q_T}\left[(\varphi(u_1)-\varphi(u_2))^2+(\varphi(v_1)-\varphi(v_2))^2\right] + \\ &\quad + C_\varepsilon c_2\int_{Q_T}\left(|\nabla\psi|^2+|\nabla\xi|^2\right). \end{aligned}\tag{3.27}$$

Then, using that φ is Lipschitz continuous and non decreasing, and substituting estimates (3.22) (and the corresponding for v), (3.26) and (3.27) in (3.21) and choosing ε,δ small enough we obtain

$$\int_\Omega\left(|\nabla\psi(T)|^2+|\nabla\xi(T)|^2\right)\leq C\int_{Q_T}\left(|\nabla\psi|^2+|\nabla\xi|^2\right),$$

with $C>0$. We conclude, by Gronwall's inequality that $\nabla\psi\equiv\nabla\xi\equiv 0$ a.e. in Q_T, from where the assertion follows. □

We finally present our third result. Here we shall assume a condition on the Dirichlet boundary to perform some estimates of a singular boundary integral.

Theorem 3.3 *Assume that* $\mathbf{H}_1$–$\mathbf{H}_3$ *hold and suppose that there exists an open set* $\tilde{B}\subset\Gamma_D$ *such that the* $(N-1)-$*dimensional Haussdorf measure of* $\tilde{B}$ *and* Γ_D *coincides. Suppose that*

$$\varphi\in C^2((0,\infty)),\quad \textit{with}\quad \varphi'(0)=0$$

and that there exist a positive constant C *and a convex function* $\mu\in C^0([0,\infty))\cap C^2((0,\infty))$ *such that* $\mu(0)=0$,

$$0<\mu'(r)\leq\varphi'(r)\quad\textit{and}\quad\varphi(r)\leq C\mu(r)\quad\textit{for}\quad r>0. \tag{3.28}$$

Then problem (1.1) *has a unique solution in the class of weak solutions satisfying*

$$\begin{aligned}&u,v\in L^1(0,T;\mathcal{W}),\\ &w\in L^\infty(0,T;W^{1,\infty}(\Omega)),\end{aligned}$$

with $\mathcal{W}$ *given by* (3.2).

Proof. Consider, as in the proof of Theorem 2.1, the sequence of regularized problems $(1.1)_\varepsilon$ in which we approximate solutions of the degenerate problem (1.1) by taking the auxiliary conditions given by (2.32) where the remaining conditions are unchanged. We know from Theorem 2.2 that for each $\varepsilon>0$ problem $(1.1)_\varepsilon$ has, at least, a weak solution $(u_\varepsilon,v_\varepsilon,w_\varepsilon)$ with the same regularity as stated in the mentioned theorem and converging to a weak solution of (1.1) (Theorem 2.1). Moreover, there exist positive constants λ and c, independent of φ and ε, such that

$$c\geq u_\varepsilon,v_\varepsilon\geq\varepsilon e^{-\lambda t}\quad\text{a.e. in } Q_T \tag{3.29}$$

and

$$\|\nabla w_\varepsilon\|_{L^\infty(Q_T)} \le c. \tag{3.30}$$

Suppose that there exists another weak solution, (u_2, v_2, w_2), of (1.1) and define $(U_\varepsilon, V_\varepsilon, W_\varepsilon) := (u_\varepsilon - u_2, v_\varepsilon - v_2, w_\varepsilon - w_2)$. Then $(U_\varepsilon, V_\varepsilon, W_\varepsilon)$ satisfy

$$\begin{cases} U_{\varepsilon t} - \Delta\left(\varphi(u_\varepsilon) - \varphi(u_2)\right) + div\left(b(u_2)\nabla W_\varepsilon + \left(b(u_\varepsilon) - b(u_2)\right)\nabla w_\varepsilon\right) = \\ \qquad = F(u_\varepsilon, v_\varepsilon) - F(u_2, v_2), \\ V_{\varepsilon t} - \Delta\left(\varphi(v_\varepsilon) - \varphi(v_2)\right) - div\left(b(v_2)\nabla W_\varepsilon + \left(b(v_\varepsilon) - b(v_2)\right)\nabla w_\varepsilon\right) = \\ \qquad F(u_\varepsilon, v_\varepsilon) - F(u_2, v_2), \\ -\Delta W_\varepsilon + U_\varepsilon - V_\varepsilon = 0, \end{cases} \tag{3.31}$$

in Q_T and the auxiliary conditions

$$\begin{cases} \varphi(u_{D\varepsilon}) = \varphi(u_D + \varepsilon e^{-\lambda_1 t}), \quad \varphi(v_{D\varepsilon}) = \varphi(v_D + \varepsilon e^{-\lambda_1 t}) & \text{on } \Sigma_{DT}, \\ \varphi(u_{D2}) = \varphi(u_D), \quad \varphi(v_{D2}) = \varphi(v_D), \quad W_{D\varepsilon} = 0 & \text{on } \Sigma_{DT}, \\ \nabla\varphi(u_{N\varepsilon})\cdot\nu = \nabla\varphi(v_{N\varepsilon})\cdot\nu = \nabla W_\varepsilon\cdot\nu = \nabla\varphi(u_2)\cdot\nu = \nabla\varphi(v_2)\cdot\nu = 0 & \text{on } \Sigma_{NT}, \\ U_\varepsilon(x,0) = V_\varepsilon(x,0) = \varepsilon & \text{in } \Omega. \end{cases}$$

Taking for (3.31) smooth test functions ψ, ξ, η with homogeneous mixed boundary conditions we get

$$\begin{aligned} \int_\Omega \psi(T)U_\varepsilon(T) + \xi(T)V_\varepsilon(T) \quad = \quad & \int_\Omega \psi(0)U_\varepsilon(0) + \xi(0)V_\varepsilon(0) + \\ & + \int_{Q_T} U_\varepsilon\left(\psi_t + A_u^\varepsilon\Delta\psi + \mathbf{B}_u^\varepsilon\cdot\nabla\psi + F_u^\varepsilon\left(\psi + \xi\right) + \eta\right) + \\ & + \int_{Q_T} V_\varepsilon\left(\xi_t + A_v^\varepsilon\Delta\xi - \mathbf{B}_v^\varepsilon\nabla\xi + F_v^\varepsilon\left(\psi + \xi\right) - \eta\right) - \\ & - \int_{Q_T} W_\varepsilon\left(\Delta\eta + div\left(b(u_2)\nabla\psi - b(v_2)\nabla\xi\right)\right) - \\ & - \int_{\Sigma_{DT}} \left[\left(\varphi(u_\varepsilon) - \varphi(u_2)\right)\nabla\psi + \left(\varphi(v_\varepsilon) - \varphi(v_2)\right)\nabla\xi\right]\cdot\nu, \end{aligned} \tag{3.32}$$

with $A_u^\varepsilon := \dfrac{\varphi(u_\varepsilon) - \varphi(u_2)}{U_\varepsilon}$, $\mathbf{B}_u^\varepsilon := \dfrac{b(u_\varepsilon) - b(u_2)}{U_\varepsilon}\nabla w_\varepsilon$ and $F_u^\varepsilon := \dfrac{F(u_\varepsilon, v_\varepsilon) - F(u_2, v_\varepsilon)}{U_\varepsilon}$ whenever $U_\varepsilon \neq 0$ and $A_u^\varepsilon = \mathbf{B}_u^\varepsilon = F_u^\varepsilon = 0$ if $U_\varepsilon = 0$ and a similar definitions for A_v^ε, $\mathbf{B}_v^\varepsilon$ and F_v^ε. Due to (3.29) and (3.30) and thanks to the condition (3.28) and to the Lipschitz continuity of b and F there exist constants k_0 and

$$k(\varepsilon) := \varepsilon^{-1} e^{\lambda T}\mu(\varepsilon e^{-\lambda T}) \tag{3.33}$$

such that

$$0 < k(\varepsilon) \le A_z^\varepsilon(x,t) \le k_0, \tag{3.34}$$

and

$$\max\left\{\left|\mathbf{B}_z^\varepsilon(x,t)\right|, \left|F_z^\varepsilon(x,t)\right|\right\} \le k_0. \tag{3.35}$$

We consider sequences of $\mathcal{C}^\infty(Q_T)$ functions such that

$$A_z^{\varepsilon,n} \to A_z^\varepsilon, \quad \mathbf{B}_z^{\varepsilon,n} \to \mathbf{B}_z^\varepsilon, \quad F_z^{\varepsilon,n} \to F_z^\varepsilon, \quad \text{y} \quad b_z^n \to b(z_2),$$

strongly in $L^2(Q_T)$ when $n \to \infty$, for $z = u, v$, where $A_z^{\varepsilon,n}$ is taken monotone decreasing on n and $\mathbf{B}_z^{\varepsilon,n}$, $F_z^{\varepsilon,n}$ and b_z^n monotone increasing on n. Because of (3.34) and (3.35) and the $L^\infty(Q_T)$ regularity of solutions of (1.1) we have

$$0 < k(\varepsilon) \le A_z^{\varepsilon,n} \le k_0, \quad \text{and} \quad \max\{|\mathbf{B}_z^{\varepsilon,n}|, |F_z^{\varepsilon,n}|, |b_z^{\varepsilon,n}|\} \le k_0 \tag{3.36}$$

in Q_T. We write identity (3.32) as

$$\begin{aligned}
\int_\Omega \psi(T)U_\varepsilon(T) + \xi(T)V_\varepsilon(T) \quad = \quad & \int_\Omega \psi(0)U_\varepsilon(0) + \xi(0)V_\varepsilon(0) + \\
& + \int_{\Sigma_{DT}} [(\varphi(u_\varepsilon) - \varphi(u_2))\nabla\psi + (\varphi(v_\varepsilon) - \varphi(v_2))\nabla\xi] \cdot \nu + \\
& + \int_{Q_T} U_\varepsilon [(A_u^\varepsilon - A_u^{\varepsilon,n})\Delta\psi + (\mathbf{B}_u^\varepsilon - \mathbf{B}_u^{\varepsilon,n}) \cdot \nabla\psi] + \\
& + \int_{Q_T} V_\varepsilon [(A_v^\varepsilon - A_v^{\varepsilon,n})\Delta\xi - (\mathbf{B}_v^\varepsilon - \mathbf{B}_v^{\varepsilon,n})\nabla\xi] + \\
& + \int_{Q_T} U_\varepsilon (F_u^\varepsilon - F_u^{\varepsilon,n})(\psi + \xi) + V_\varepsilon (F_v^\varepsilon - F_v^{\varepsilon,n})(\psi + \xi) + \\
& + \int_{Q_T} \nabla W_\varepsilon \cdot ((b(u_2) - b_u^n)\nabla\psi - (b(v_2) - b_v^n)\nabla\xi) - \\
& - \int_{Q_T} W_\varepsilon (\Delta\eta + div\,(b_u^n \nabla\psi - b_v^n \nabla\xi)) + \\
& + \int_{Q_T} U_\varepsilon (\psi_t + A_u^{\varepsilon,n}\Delta\psi + \mathbf{B}_u^{\varepsilon,n} \cdot \nabla\psi + F_u^{\varepsilon,n}(\psi + \xi) + \eta) + \\
& + \int_{Q_T} V_\varepsilon (\xi_t + A_v^{\varepsilon,n}\Delta\xi - \mathbf{B}_v^{\varepsilon,n} \cdot \nabla\xi + F_v^{\varepsilon,n}(\psi + \xi) - \eta) \\
=: \quad & I_1 + \ldots + I_9
\end{aligned} \tag{3.37}$$

and set the following problem to choose the test functions:

$$\begin{cases}
\psi_t + A_u^{\varepsilon,n}\Delta\psi + \mathbf{B}_u^{\varepsilon,n} \cdot \nabla\psi + F_u^{\varepsilon,n}(\psi + \xi) + \eta = 0 & \text{in } Q_T, \\
\xi_t + A_v^{\varepsilon,n}\Delta\xi - \mathbf{B}_v^{\varepsilon,n} \cdot \nabla\xi + F_v^{\varepsilon,n}(\psi + \xi) - \eta = 0 & \text{in } Q_T, \\
\Delta\eta + div\,(b_u^n \nabla\psi - b_v^n \nabla\xi) = 0 & \text{in } Q_T, \\
\psi = \xi = \eta = 0 & \text{on } \Sigma_{DT}, \\
\nabla\psi \cdot \nu = \nabla\xi \cdot \nu = \nabla\eta \cdot \nu = 0 & \text{on } \Sigma_{NT}, \\
\psi(T,x) = \chi_u^\delta(x), \quad \xi(T,x) = \chi_v^\delta(x) & \text{in } \Omega,
\end{cases} \tag{3.38}$$

with $\chi_z^\delta \in \mathcal{C}_0^\infty(\Omega)$, $dist(\Sigma_D, supp(\chi_z^\delta)) \ge \delta$ and $\chi_z^\delta \rightharpoonup sign\,\{U(T)\}$ in $L^1(\Omega)$ as $\delta \to 0$.

LEMMA 3.2 *Problem* (3.38) *has a unique solution with the regularity of the test functions of* (1.1) *(see* (2.4) *and* (2.5)*). Moreover,*

$$\psi, \xi \in H^1(0,T; L^2(\Omega)) \cap L^\infty(0,T; H^1(\Omega)) \cap L^2(0,T; H^2(\Omega)), \tag{3.39}$$

$$\eta \in H^1(0,T; L^2(\Omega)) \cap L^\infty(0,T; H^1(\Omega)),$$

and their norms in these spaces are uniformily bounded with respect to n. *Finally, there exists a positive constant* $C(T)$ *independent of* ε *such that*

$$\|\psi\|_{L^\infty(Q_T)}, \|\xi\|_{L^\infty(Q_T)} \le C(T). \tag{3.40}$$

Proof of Lemma 3.2. We can follow step by step the proof of Lemma 3.1 to get the existence, uniqueness and regularity of solutions of problem (3.38). The new point we must show is that uniform estimates hold in the norms of the spaces stated in Lemma 3.2. Notice that in the estimates (3.10), (3.13) and (3.14) the dependence with respect to n of the coefficients may be avoided thanks to (3.36) and, therefore, we easily obtain that the norms of the solution in the spaces of (3.39) are independent of n. However, the proof of (3.40) requires more work: Suppose that $\hat{\psi}, \hat{\xi} \in L^\infty(Q_T)$ and define

$$z := \eta + b_u^n \hat{\psi} - b_v^n \hat{\xi}. \tag{3.41}$$

From the third equation of (3.38) we conclude that z satisfies

$$\begin{cases} \Delta z = div\left(\hat{\psi}\nabla b_u^n - \hat{\xi}\nabla b_v^n\right) & \text{in } Q_T, \\ z = 0 & \text{on } \Sigma_{DT}, \\ \nabla z \cdot \nu = 0 & \text{on } \Sigma_{NT}. \end{cases}$$

By well known results (see [40]) we have the following estimate

$$\|z\|_{L^\infty(\Omega)} \leq c\left(\left\|\hat{\xi}\nabla b_v^n\right\|_{L^p(\Omega)} + \left\|\hat{\psi}\nabla b_u^n\right\|_{L^p(\Omega)}\right),$$

from where we get

$$\|z\|_{L^\infty(\Omega)} \leq c\left(\left\|\hat{\xi}\right\|_{L^\infty(\Omega)} \|\nabla b_v^n\|_{L^p(\Omega)} + \left\|\hat{\psi}\ \right\|_{L^\infty(\Omega)} \|\nabla b_u^n\|_{L^p(\Omega)}\right).$$

By assumptions $u_2, v_2 \in L^1(0,T;\mathcal{W})$ and $b \in \mathcal{C}([0,\infty))$ and recalling the definition of z we obtain

$$\|\eta\|_{L^1(0,T;L^\infty(\Omega))} \leq c\left(\left\|\hat{\psi}\right\|_{L^\infty(Q_T)} + \left\|\hat{\xi}\right\|_{L^\infty(Q_T)}\right). \tag{3.42}$$

On the other hand, given $\hat{\eta} \in L^1(0,T;L^\infty(\Omega))$ we have that, thanks to the maximum principle of Alexandrov (see [27]) the solutions of the two first equations of (3.38) (with η substituted by $\hat{\eta}$) are bounded in $L^\infty(Q_T)$ uniformily in ε. Therefore, this fact together with (3.42) and the fixed point argument used to get existence of solutions imply that also the solution of the coupled problem (3.38) satisfy these L^∞ estimates uniform in ε.□

Continuation of the proof of Theorem 3.3. With the test functions of Lemma 3.2 we have in (3.37) that $I_7 = I_8 = I_9 = 0$. Now we shall take limits in the resulting identity, first when $n \to \infty$ and then when $\varepsilon, \delta \to 0$. Since we have uniform estimates of $\|\nabla\psi\|_{L^2(Q_T)}$, $\|\nabla\xi\|_{L^2(Q_T)}$, $\|\Delta\psi\|_{L^2(Q_T)}$ and $\|\Delta\xi\|_{L^2(Q_T)}$ with respect to n we deduce that I_3, I_4, I_5 and I_6 tend to zero when $n \to \infty$. Therefore, identity (3.37) is reduced to

$$\begin{aligned} \int_\Omega \left(\chi_u^\delta U_\varepsilon(T) + \chi_v^\delta V_\varepsilon(T)\right) &= \varepsilon\int_\Omega (\psi(0) + \xi(0)) - \\ &- \int_{\Sigma_{DT}} \left[(\varphi(u_\varepsilon) - \varphi(u_2))\nabla\psi + (\varphi(v_\varepsilon) - \varphi(v_2))\nabla\xi\right]\cdot\nu. \end{aligned} \tag{3.43}$$

Since, by Lemma 3.2, the estimates of $\|\psi\|_{L^\infty(Q_T)}$, $\|\xi\|_{L^\infty(Q_T)}$ are uniform with respect to ε we get

$$\varepsilon\int_\Omega (\psi(0) + \xi(0)) \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{3.44}$$

The following Lemma will allow us to estimate the integral over the Dirichlet boundary:

Lemma 3.3 *Let* A_ε, $\mathbf{B}_\varepsilon$, $g_\varepsilon \in L^\infty(Q_T)$ *with*

$$k(\varepsilon) < A_\varepsilon, \tag{3.45}$$

where $k(\varepsilon)$ *is given by* (3.33). *Consider the problem*

$$\begin{cases} \psi_t + A_\varepsilon \Delta\psi + \mathbf{B}_\varepsilon \cdot \nabla\psi + g_\varepsilon = 0 & \text{in } Q_T, \\ \psi = 0 & \text{on } \Sigma_{DT}, \\ \nabla\psi \cdot \nu = 0 & \text{on } \Sigma_{NT}, \\ \psi(T,x) = \chi_\delta(x) & \text{in } \Omega, \end{cases}$$

with $\delta > 0$. *Then, there exist a* $\delta(\varepsilon) > 0$ *and a positive constant* c*, independent of* ε*, such that if* $\delta < \delta(\varepsilon)$ *then*

$$\nabla\psi \cdot \nu \geq -c \frac{\|\mathbf{B}_\varepsilon\|_{L^\infty(Q_T)} \|\psi\|_{L^\infty(Q_T)}}{k(\varepsilon)} \quad \text{a.e. in} \quad \Sigma_{DT}. \tag{3.46}$$

Continuation of the proof of Theorem 3.3. Applying this lemma to the problem (3.38) we have that, evaluating u_ε and u_2 on Σ_{DT} and using (3.46) and that $\|\mathbf{B}_\varepsilon\|_{L^\infty(Q_T)}$ and $\|\psi\|_{L^\infty(Q_T)}$ have bounds which are independent of ε we get

$$-\int_{\Sigma_{DT}} (\varphi(u_\varepsilon) - \varphi(u_2)) \nabla\psi \cdot \nu = -\int_{\Sigma_{DT}} \varphi(\varepsilon e^{-\lambda_1 t}) \nabla\psi \cdot \nu \leq c \frac{\varphi(\varepsilon)}{k(\varepsilon)},$$

where we have used that φ is increasing. By (3.28) and (3.33) we get $\varphi(\varepsilon) \leq \hat{c}\varepsilon k(\varepsilon)$, for a certain $\hat{c} > 0$, and therefore

$$\int_{\Sigma_{DT}} (\varphi(u_\varepsilon) - \varphi(u_2)) \nabla\psi \cdot \nu \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

A similar argument may be applied to the term involving v_ε. Choosing χ_u^δ as

$$\chi_u^\delta(x,t) := \begin{cases} sign\,\{U(x,T)\} & \text{if } x \in \Omega_\delta, \\ 0 & \text{if } x \in \Omega \backslash \bar{\Omega}_\delta, \end{cases}$$

where $\Omega_\delta := \{x \in \Omega : dist\,[\partial\Omega, supp\,(U_\varepsilon(x,T))] > \delta\}$ (similar for χ_v^δ) we get, when $\varepsilon, \delta \to 0$ that

$$\int_\Omega \chi_u^\delta U_\varepsilon(T) \to \int_\Omega |U(T)|,$$

(and analogously for the term involving v). We then deduce from (3.43) that

$$\int_\Omega |U(T)| + |V(T)| \leq 0,$$

and, therefore, the desired result. □

Proof of Lemma 3.3. Since $\partial\Omega$ is regular, Ω satisfies the exterior sphere condition, i.e., for all $x_0 \in \partial\Omega$ there exists a $R_1 > 0$ and a $x_1 \in I\!R^N \backslash \bar{\Omega}$ such that

$$B(x_1, R_1) \cap \bar{\Omega} = \{x_0\},$$

where $B(x_1, R_1) := \left\{x \in I\!R^N : |x - x_1| < R_1\right\}$. Let us fix $x_0 \in Interior(\Gamma_D)$. It is clear that this set is non empty because, by hypothesis, there exists an open set $\tilde{B}$ such that $\tilde{B} \subset \Gamma_D$. Therefore, there exists a small enough $\delta > 0$ such that, by defining $R_2 := \delta + R_1$, it holds $B(x_1, R_2) \cap \partial\Omega \subset \Gamma_D$. Moreover, since $dist(\partial\Omega, supp(\chi_\delta)) \geq \delta$, we also have that $\chi_\delta \equiv 0 \quad \text{in} \quad \omega := \Omega \cap B(x_1, R_2)$. We shall use the notation $k_0(\varepsilon) := \|g\|_{L^\infty(Q_T)}$, $k_1(\varepsilon) := \left(\frac{N-1}{R_1}+1\right)\|\mathbf{B}\|_{L^\infty(Q_T)}$ and $k_2(\varepsilon) := \|\psi\|_{L^\infty(Q_T)}$. We define

$$\mathcal{L}(\psi) := \psi_t + A_\varepsilon \Delta\psi + \mathbf{B}\cdot\nabla\psi \quad \text{and} \quad w(x,t) := \psi(x,t) + \sigma(r),$$

where $(x,t) \in \omega \times (0,t)$, $r := |x - x_0|$ and $\sigma \in C^2([R_1, R_2])$ will be chosen such that the maximum of w in $\bar\omega \times [0,T]$ is attained in $\{x_0\} \times [0,T]$, and such that $\sigma''(r) \geq 0$ and $\sigma'(r) \leq 0$. Assuming these properties we get, due to (3.45), that w satisfies

$$\mathcal{L}(w) = -g + A_\varepsilon\Delta\sigma + \mathbf{B}\cdot\nabla\sigma \geq k(\varepsilon)\sigma''(r) + k_1(\varepsilon)\sigma'(r) - k_0(\varepsilon).$$

Choosing $\sigma(r) := \frac{k_0(\varepsilon)}{k_1(\varepsilon)} r + C_2 e^{-\frac{k_1(\varepsilon)}{k(\varepsilon)} r}$, with C_2 an arbitrary constant, we obtain

$$k(\varepsilon)\sigma''(r) + k_1(\varepsilon)\sigma'(r) - k_0(\varepsilon) = 0, \quad \sigma''(r) \geq 0 \quad \text{and}$$

$$\text{if} \quad C_2 \geq k(\varepsilon)\frac{k_0(\varepsilon)}{k_1^2(\varepsilon)} e^{\frac{k_1(\varepsilon)}{k(\varepsilon)}R_2} \quad \text{then} \quad \sigma'(r) \leq 0. \tag{3.47}$$

Taking C_2 with this restriction we have that $\mathcal{L}(w) \geq 0$ in $\bar\omega \times [0,T]$ and therefore, by the Maximum Principle we deduce that w attains its maximum on the parabolic boundary of $\omega \times [0,T]$. On this boundary the values of w may be estimated as follows:

$$\begin{cases} w(x,t) = \sigma(r) \leq \sigma(R_1) & \text{on } (\Gamma_D \cap \partial\omega) \times [0,T], \\ w(x,t) = \psi(x,t) + \sigma(r) \leq k_2(\varepsilon) + \sigma(R_2) & \text{on } (\partial B(x_1, R_2) \cap \partial\omega) \times [0,T], \\ w(x_0,t) = \sigma(R_1) & \text{on } [0,T], \\ w(x,T) = \sigma(r) + \chi_\delta(x) \leq \sigma(R_1) & \text{in } \omega, \end{cases}$$

where we have used that $\chi_\delta \equiv 0$ in ω. It is a straightforward computation to see that we can choose C_2 (by making δ small enough) such that (3.47) and $\sigma(R_1) = k_2(\varepsilon) + \sigma(R_2)$ hold. As a consequence we obtain that $\nabla w(x_0,t)\cdot\nu \geq 0$ and by the definition of w and taking δ suitably we obtain

$$\nabla\psi(x_0,t)\cdot\nu \geq -c\frac{k_1(\varepsilon)k_2(\varepsilon)}{k(\varepsilon)} \quad \text{in } [0,T].$$

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