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R.A. Zuidwijk

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CWI
P.O. Box 94079
1090 GB Amsterdam
The Netherlands

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Rob Zuidwijk

CWI

P.O. Box 94079, 1090 GB Amsterdam, The Netherlands

e-mail: R.A.Zuidwijk@cwi.nl

ABSTRACT

Combined use of the X-ray (Radon) transform and the wavelet transform has proved to be useful in application areas such as diagnostic medicine and seismology. In the present paper, the wavelet X-ray transform is introduced. This transform performs one-dimensional wavelet transforms along lines in $\mathbb{R}^n$, which are parameterized in the same fashion as for the X-ray transform. It is shown that the transform has the same convenient inversion properties as the wavelet transform. The reconstruction formula receives further attention in order to obtain usable discretizations of the transform. Finally, a connection between the wavelet X-ray transform and the filtered backprojection formula is discussed.

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1. Introduction

Both the X-ray transform (or Radon transform) and the wavelet transform have received considerable attention in the literature and are of great mathematical interest in their own right. Moreover, these transforms play a significant role in a large range of application areas. In some of these application areas, such as diagnostic medicine and seismology, the two transforms are applied in combination.

Combined use of the Radon transform and wavelet transform relevant to diagnostic medicine can be found in [BW] and [OD]. In order to reduce the amount of data required for proper reconstruction of the density of the object under consideration, localized inversion methods of the Radon transform using wavelets are proposed there. The filtered backprojection formula plays a decisive role in these approaches. This important formula receives attention in Section 5 of the present paper.

The results mentioned above are also relevant to cross-borehole tomography in seismic exploration; see for example [DL].

On the other hand, the Radon transform has proved to be useful in reflection seismology. If one models the earth's subsurface as a stratified medium, then the Radon transform can be used to transform seismic data in such a way that arriving wavefronts with distinct propagating velocities are separated. In this context, the Radon transform is referred to as a *slant stack* [Rob].

We shall now shortly describe the aforementioned integral transforms. The *X-ray transform* [Nat]

$$Pf(\theta, x) = \int_{\mathbb{R}} f(x + t\theta)\, dt$$

integrates a function f on $\mathbb{R}^n$ along an affine line $x + \mathbb{R}\theta$, where $x \in \mathbb{R}^n$ is perpendicular to the direction θ. Observe that (θ, x), where θ is a unit vector and x a vector orthogonal to θ, parameterize all lines in $\mathbb{R}^n$. In particular, the distance of the line $x + \mathbb{R}\theta$ to the origin is given by $\|x\|$. The relevance of the X-ray transform to diagnostic medicine can be understood as follows: The attenuation of X-ray beams (along lines in $\mathbb{R}^2$ or $\mathbb{R}^3$) passing through a medium with density f is modelled by the integral of the density function along these lines. It is the aim of computerized tomography to reconstruct the density function from these attenuation data, i.e., from the X-ray transformed function [SSW, Nat].

The X-ray transform belongs to the same family of transformations as the *Radon transform*

$$Rf(\theta, s) = \int_{\theta^\perp} f(x + s\theta)\, dx,$$

which integrates f along the affine hyperplane $\theta^\perp + s\theta$. As a matter of fact, the Radon transform is also referred to as the $(n-1)$-dimensional X-ray transform [Sol]. Note that for $n = 2$, the Radon transform and the (one-dimensional) X-ray transform actually coincide. In the present paper, the Radon transform appears in Section 5 in relation to the filtered backprojection formula.

The *wavelet transform*

$$W_g f(b, a) = \int_{\mathbb{R}} f(t) \frac{1}{\sqrt{a}} \overline{g\left(\frac{t-b}{a}\right)}\, dt, \quad b \in \mathbb{R}, \quad a > 0,$$

which puts a function f to its wavelet coefficients $W_g f(b, a)$, is often considered as an alternative for the windowed Fourier transform in the time-frequency analysis of nonstationary signals; e.g., see [Mey, RV]. The transform actually computes inner products of f with respect to translated and dilated versions of one and the same function g, which is referred to as the wavelet. Usually, the function g satisfies an admissibility condition to ensure that the function f can be reconstructed from its wavelet coefficients $W_g f$; details are given in Section 2.

In [FKV, FKV2], it has been argued that the Radon transform (as a slant stack) and the wavelet transform (as a time-frequency analysis tool) have complementary useful features to remove noise from seismic reflection data. For this reason, the two transforms are applied in a cascaded fashion. This work motivates the definition of a transformation which combines the properties of the wavelet and the X-ray transform. Indeed, we consider the *wavelet X-ray transform*

$$P_g f(\theta, x, b, a) = \int_{\mathbb{R}} f(x + t\theta) \frac{1}{\sqrt{a}} \overline{g\left(\frac{t-b}{a}\right)}\, dt.$$

This transform computes one-dimensional wavelet transforms along lines in $\mathbb{R}^n$ which are parameterized in the same fashion as for the X-ray transform. In Section 3, the wavelet X-ray transform will be discussed in detail. It is shown there that the wavelet X-ray transform has the same convenient reconstruction properties (Theorem 3.3) as the usual wavelet transform (Theorem 2.2). In Section 4, it is shown that the reconstruction formula of the wavelet X-ray transform is an integral over *elementary orthogonal projections*. It is expected that these projections will play a role in usable discretizations of the wavelet X-ray transform. In this spirit, it is shown that certain products of these elementary projections give rise to *separable wavelets*. Such wavelets were also proposed for seismic data processing in [CC]. An effective use of the wavelet X-ray transform for seismic data processing in a more advanced fashion is an aim for further research.

The wavelet X-ray transform is closely related to *the windowed X-ray transform* [KS, Tak], which is given by

$$X_g f(y,v) = \int_{\mathbb{R}} f(y+tv)\overline{g(t)}\, dt,$$

where $x \in \mathbb{R}^n$ and $0 \neq v \in \mathbb{R}^n$. If we set

$$\theta = \frac{v}{\|v\|}, \quad x = y - \langle y, \theta\rangle\theta, \quad b = \langle y, \theta\rangle, \quad a = \|v\|,$$

then $X_g f(y,v) = P_g f(\theta, x, b, a)$. The reconstruction formulas in Section 3 are simpler than the corresponding ones in [KS, Tak]. Moreover, in the present paper, discrete versions of the transform are discussed.

2. The continuous wavelet transform

The continuous wavelet transform has been studied thoroughly by several authors. The transform was introduced in [GMP1, GMP2] and several books on wavelets [Dau, Hol2, Koo] provide an introduction to the subject. For completeness, we shall state and prove some important results from the literature concerning the continuous wavelet transform acting on $L^2(\mathbb{R})$, the space of square integrable functions on $\mathbb{R}$. For $f, g \in L^2(\mathbb{R})$, consider the expression

$$W_g f(b,a) = \int_{\mathbb{R}} f(t) \frac{1}{\sqrt{a}} \overline{g\left(\frac{t-b}{a}\right)}\, dt, \quad (b,a) \in \mathbb{H},$$

where $\mathbb{H} = \{(b,a) \in \mathbb{R}^2 \mid a > 0\}$ denotes the open upper half plane. We shall refer to W_g as the *continuous wavelet transform*. The function g plays the role of the *wavelet* and will normally satisfy a so-called admissibility condition which will be specified later on. In most signal analysis applications, the wavelet under consideration should also be (essentially) localized in a compact interval; see [Mey, RV] for an explanation of these matters. If we introduce the shorthand notation

$$g_{a,b}(t) = \frac{1}{\sqrt{a}} g\left(\frac{t-b}{a}\right), \quad g_a(t) = g_{a,0}(t), \quad t \in \mathbb{R}, \quad (b,a) \in \mathbb{H},$$

we get $W_g f(b,a) = \langle f, g_{a,b}\rangle_{L^2(\mathbb{R})}$. Observe that $g_{a,b}$ represents a dilated and translated version of the function g, which is normalized in such a way that $\|g_{a,b}\|_{L^2(\mathbb{R})} = \|g\|_{L^2(\mathbb{R})}$. To fix notation, we remark that the Fourier transform $f \mapsto \widehat{f}$, given by

$$\widehat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i\omega x}\, dx,$$

is a unitary operator on $L^2(\mathbb{R})$. The wavelet transform $W_g f$ of f can also be described in terms of the convolution product. The convolution product of $f_1, f_2 \in L^2(\mathbb{R})$ is given by

$$(f_1 * f_2)(x) = \int_{\mathbb{R}} f_1(x-t) f_2(t)\, dt.$$

Observe that this formula is defined for almost all $x \in \mathbb{R}$, but that $f_1 * f_2$ needs not be in $L^2(\mathbb{R})$. Using the notation $\widetilde{g}(t) = \overline{g(-t)}$, we get $W_g f(b,a) = (f * \widetilde{g}_a)(b)$. This fact plays a role in the proof of the following theorem, which states Plancherel's formula for the continuous wavelet transform. In the theorem, the wavelet g should satisfy the following *admissibility condition*: the expression

$$c_g = \int_0^\infty \frac{|\widehat{g}(a\omega)|^2}{a}\, da \tag{2.1}$$

must assume a strictly positive value and should be constant for almost all $\omega \in \mathbb{R}$. A function $g \in L^2(\mathbb{R})$ which satisfies the admissibility condition is called an *admissible wavelet*. At the end of this section, a short exposition on the collection of admissible wavelets will be given.

Theorem 2.1 (Plancherel's formula) *Let $g \in L^2(\mathbb{R})$ be an admissible wavelet, then for $f \in L^2(\mathbb{R})$, one gets*

$$\int_{\mathbb{R}} |f(t)|^2\, dt = \frac{1}{c_g} \int_{\mathbb{H}} |W_g f(b,a)|^2\, db\, \frac{da}{a^2}.$$

Proof First, we write

$$\int_{\mathbb{H}} |W_g f(b,a)|^2\, db\, \frac{da}{a^2} = \int_0^\infty \int_{\mathbb{R}} |(f * \widetilde{g}_a(b)|^2\, db\, \frac{da}{a^2}.$$

It is straightforward to verify (c.f. [Koo]) that

$$\int_0^\infty \int_{\mathbb{R}} |(f * \widetilde{g}_a(b)|^2\, db\, \frac{da}{a^2} = \int_0^\infty \int_{\mathbb{R}} |\widehat{f}(\omega)|^2 |\widehat{g}_a(\omega)|^2\, d\omega\, \frac{da}{a^2},$$

where both sides may be positive infinite. We remark that $\widehat{g}_{a,b}(\omega) = \sqrt{a}\ \widehat{g}(a\omega) e^{-i\omega b}$. This leads to

$$\int_{\mathbb{H}} |W_g f(b,a)|^2\, db\, \frac{da}{a^2} = \int_0^\infty \int_{\mathbb{R}} |\widehat{f}(\omega)|^2 |\widehat{g}(a\omega)|^2\, d\omega\, \frac{da}{a} =$$

$$\int_{\mathbb{R}} \int_0^\infty |\widehat{f}(\omega)|^2 |\widehat{g}(a\omega)|^2\, \frac{da}{a}\, d\omega = c_g \int_{\mathbb{R}} |\widehat{f}(\omega)|^2\, d\omega = c_g \|f\|^2_{L^2(\mathbb{R})}.$$

Here, we have used Fubini's theorem and Plancherel's formula for the Fourier transform. This proves the theorem. □

We remark that, by polarization, we may derive from Theorem 2.1 Parseval's formula: if $f, k \in L^2(\mathbb{R})$, then

$$\langle f, k \rangle_{L^2(\mathbb{R})} = \frac{1}{c_g} \int_{\mathbb{H}} W_g f(b,a) \overline{W_g k(b,a)}\, db\, \frac{da}{a^2} = \frac{1}{c_g} \langle W_g f, W_g k \rangle_{L^2(\mathbb{H})}.$$

We now proceed with a reconstruction formula for the wavelet transform.

Theorem 2.2 (Reconstruction formula) *Let $g \in L^2(\mathbb{R})$ be an admissible wavelet, and let $f \in L^2(\mathbb{R})$. Then*

$$f = \frac{1}{c_g} \int_{\mathbb{H}} W_g f(b,a) g_{a,b}(\cdot)\, db\, \frac{da}{a^2}. \tag{2.2}$$

The integral converges in the norm of $L^2(\mathbb{R})$.

Proof We first prove that the right-hand side of (2.2) is in $L^2(\mathbb{R})$. In the main body of the proof, we will replace $W_g f$ by an arbitrary $\sigma \in L^2(\mathbb{H})$. Let $(K_m)_{m=1}^{\infty}$ be a non-decreasing sequence of compact and measurable subsets of the open upper half plane $\mathbb{H}$, such that $\bigcup_{m=1}^{\infty} K_m = \mathbb{H}$. Define

$$\mathcal{I}_m = \frac{1}{c_g} \int_{K_m} \sigma(b,a) g_{a,b}(\cdot)\, db\, \frac{da}{a^2}, \quad m \in \mathbb{Z}^+.$$

Then for any $k \in L^2(\mathbb{R})$,

$$\langle \mathcal{I}_m, k\rangle_{L^2(\mathbb{R})} = \frac{1}{c_g} \int_{\mathbb{R}} \int_{K_m} \sigma(b,a) g_{a,b}(t) \overline{k(t)}\, db\, \frac{da}{a^2}\, dt.$$

Note that the integrand is in $L^1(\mathbb{R} \times K_m)$. Indeed,

$$\int_{K_m} \int_{\mathbb{R}} |\sigma(b,a) g_{a,b}(t) k(t)|\, dt\, db\, \frac{da}{a^2} \leq \int_{K_m} |\sigma(b,a)| \cdot \|g\|_{L^2(\mathbb{R})} \|k\|_{L^2(\mathbb{R})}\, db\, \frac{da}{a^2} \leq$$

$$\|g\|_{L^2(\mathbb{R})} \|k\|_{L^2(\mathbb{R})} \|\sigma\|_{L^2(\mathbb{H})} \sqrt{\mu(K_m)},$$

where $\mu(K_m)$ denotes the measure of K_m with respect to $a^{-2} db da$. Therefore, we may apply Fubini's theorem and arrive at

$$\langle \mathcal{I}_m, k\rangle_{L^2(\mathbb{R})} = \frac{1}{c_g} \int_{K_m} \sigma(b,a) \overline{\int_{\mathbb{R}} k(t) \overline{g_{a,b}(t)}\, dt}\, db\, \frac{da}{a^2} = \frac{1}{c_g} \langle \sigma, W_g k\rangle_{L^2(K_m)}.$$

This implies

$$|\langle \mathcal{I}_m, k\rangle_{L^2(\mathbb{R})}| \leq \frac{1}{\sqrt{c_g}} \|\sigma\|_{L^2(\mathbb{H})} \|k\|_{L^2(\mathbb{R})},$$

and we get $\mathcal{I}_m \in L^2(\mathbb{R})$. Next, we prove the convergence in norm of the right-hand side of (2.2). It suffices to prove that the sequence $(\mathcal{I}_m)_{m=1}^{\infty}$ is a Cauchy sequence in $L^2(\mathbb{R})$. Let $p < q$ be positive integers and consider for $k \in L^2(\mathbb{R})$ the expression

$$\langle \mathcal{I}_q - \mathcal{I}_p, k\rangle_{L^2(\mathbb{R})} = \int_{K_q/K_p} \sigma(b,a) \overline{W_g k(b,a)}\, db\, \frac{da}{a^2} \leq$$

$$\sqrt{\int_{K_q/K_p} |\sigma(b,a)|^2\, db\, \frac{da}{a^2}} \cdot \sqrt{\int_{K_q/K_p} |W_g k(b,a)|^2\, db\, \frac{da}{a^2}} \leq$$

$$\sqrt{\int_{K_q/K_p} |\sigma(b,a)|^2 \, db \, \frac{da}{a^2}} \cdot \|W_g k\|_{L^2(\mathbb{H})} = \sqrt{c_g} \cdot \|k\|_{L^2(\mathbb{R})} \cdot \sqrt{\int_{K_q/K_p} |\sigma(b,a)|^2 \, db \, \frac{da}{a^2}}.$$

This implies

$$\|\mathcal{I}_q - \mathcal{I}_p\|_{L^2(\mathbb{R})} \leq \sqrt{c_g} \cdot \sqrt{\int_{K_q/K_p} |\sigma(b,a)|^2 \, db \, \frac{da}{a^2}}.$$

As $p, q \to \infty$, the integral tends to zero, since the integrand is in $L^1(\mathbb{H})$. To prove the equality in (2.2), let $\sigma = W_g f$ to obtain

$$\lim_{m\to\infty} \langle \mathcal{I}_m, k\rangle_{L^2(\mathbb{R})} = \lim_{m\to\infty} \frac{1}{c_g} \langle W_g f, W_g k\rangle_{L^2(K_m)} = \langle f, k\rangle_{L^2(\mathbb{R})}.$$

This proves the theorem. □

Define the adjoint operator $W_g^* : L^2(\mathbb{H}) \to L^2(\mathbb{R})$ by duality:

$$\langle f, W_g^* \sigma\rangle_{L^2(\mathbb{R})} = \langle W_g f, \sigma\rangle_{L^2(\mathbb{H})}, \quad f \in L^2(\mathbb{R}), \quad \sigma \in L^2(\mathbb{H}).$$

Then W_g^* is a bounded operator with norm $\|W_g^*\| = \|W_g\| = \sqrt{c_g}$. We may write

$$W_g^* \sigma = \int_{\mathbb{H}} \sigma(b,a) g_{a,b}(\cdot) \, db \, \frac{da}{a^2}.$$

By the proof of the preceding theorem, the integral converges in $L^2(\mathbb{R})$. The following theorem states that W_g is left-invertible, and the orthogonal projection onto its range is given explicitly.

Theorem 2.3 *If $g \in L^2(\mathbb{R})$ is an admissible wavelet, then*

$$f = \frac{1}{c_g} W_g^* W_g f$$

for all $f \in L^2(\mathbb{R})$. Moreover, the orthoprojector onto ran $W_g \subseteq L^2(\mathbb{H})$ *is given by*

$$\frac{1}{c_g} W_g W_g^* : L^2(\mathbb{H}) \to L^2(\mathbb{H}).$$

Proof The first part of the theorem follows immediately from Parseval's formula. The bounded operator $\Pi_g = c_g^{-1} W_g W_g^*$ obviously satisfies $\Pi_g^2 = \Pi_g$ and $\Pi_g^* = \Pi_g$. Remains to prove that ran $\Pi_g =$ ran W_g. Clearly, ran $\Pi_g \subseteq$ ran W_g and the converse inclusion follows from $\Pi_g W_g f = c_g^{-1} W_g W_g^* W_g f = W_g f$. □

A few remarks concerning the admissibility condition (2.1) are in order. Given $g \in L^2(\mathbb{R})$, the admissibility condition (2.1) comes down to

$$c_g = \int_0^\infty \frac{|\hat{g}(a)|^2}{a} \, da = \int_0^\infty \frac{|\hat{g}(-a)|^2}{a} \, da \in (0, \infty).$$

Indeed, the cases $\omega < 0$ and $\omega > 0$ in (2.1) are reduced to the cases $\omega = -1$ and $\omega = 1$, respectivily, by change of the variable of integration.

Moreover, if $\hat{g}$ is continuous at 0 (which is the case when $g \in L^1(\mathbb{R})$), then the existence of the integrals above implies that $\hat{g}(0) = \int_{\mathbb{R}} g(x)\,dx = 0$. The following lemma shows that for certain classes of functions, admissible wavelets are actually characterized by a vanishing mean value.

Lemma 2.4 *Let $g \in L^2(\mathbb{R})$ be a non-zero real-valued function.*

(a) If there exists $\alpha > 0$ such that

$$\hat{g}(\omega) - \hat{g}(0) = O(|\omega|^\alpha), \quad \omega \to 0,$$

then: g is admissible if and only if $\hat{g}(0) = 0$.

(b) If there exists $\alpha > 0$ such that $g(\cdot)\ (1 + |\cdot|)^\alpha \in L^1(\mathbb{R})$, then: g is admissible if and only if $\int_{\mathbb{R}} g(x)dx = 0$.

(c) If g is compactly supported, then: g is admissible if and only if $\int_{\mathbb{R}} g(x)\,dx = 0$.

Proof In all three cases, g is real-valued, so $\hat{g}$ is symmetric. Together with the fact that g is not identically zero, g is an admissible wavelet if and only if

$$\int_0^\infty \frac{|\hat{g}(\omega)|^2}{\omega}\,d\omega < \infty.$$

In addition, we have assumed that $g \in L^2(\mathbb{R})$, hence $\hat{g} \in L^2(\mathbb{R})$. It follows that the admissibility condition is equivalent to

$$\int_0^1 \frac{|\hat{g}(\omega)|^2}{\omega}\,d\omega < \infty.$$

In all three cases, the only if part follows from the discussion before the lemma. Observe that for $g \in L^1(\mathbb{R})$, $\hat{g}(0) = \int_{\mathbb{R}} g(x)dx$. We now focus on the if parts, where we may and do assume that $0 < \alpha < 1$.

(a) If $\hat{g}(0) = 0$, then $\hat{g}(\omega) = O(|\omega|^\alpha)$ for $\omega \to 0$, so

$$\int_0^1 \frac{|\hat{g}(\omega)|^2}{\omega}\,d\omega \le C \cdot \int_0^1 \omega^{2\alpha-1}\,d\omega = \frac{C}{2\alpha} < \infty.$$

(b) If $g(\cdot)\ (1 + |\cdot|)^\alpha \in L^1(\mathbb{R})$, then

$$|\hat{g}(\omega) - \hat{g}(0)| \le \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |g(x)||e^{-i\omega x} - 1|\,dx =$$

$$\frac{|\omega|}{\sqrt{2\pi}} \int_{|x|\le 1} |g(x)| \left| \frac{e^{-i\omega x} - 1}{\omega x} \right| dx \;+\; \frac{|\omega|^\alpha}{\sqrt{2\pi}} \int_{|x|\ge 1} |g(x)| \cdot |x|^\alpha \left| \frac{e^{-i\omega x} - 1}{(\omega x)^\alpha} \right| dx$$

can be estimated from above by $C \cdot |\omega|^\alpha$ for $|\omega| < 1$, say. We are now back in situation (a).

(c) If g is compactly supported, then $g(\cdot)\ (1+|\cdot|)^\alpha \in L^1(\mathbb{R})$ and we are in situation (b).

This proves the lemma. □

An obvious example of an admissible wavelet which is not compactly supported is the *mexican hat* given by $t \mapsto (1-t^2)e^{-t^2/2}$.

3. The continuous wavelet X-ray transform

We shall now consider a transform acting on square integrable functions on $\mathbb{R}^n$. This transform actually performs one-dimensional wavelet transforms (see the preceding section) along lines in $\mathbb{R}^n$. These lines are parameterized in the same fashion as for the usual X-ray transform (see [Nat, Sol]), i.e., by means of the vector bundle on the unit sphere

$$\mathcal{T} = \{(\theta, x) \mid \theta \in S^{n-1}, x \in \theta^\perp\}.$$

Here $\theta^\perp$ denotes the orthoplement of $\theta \in S^{n-1}$ in $\mathbb{R}^n$. Let $g \in L^2(\mathbb{R}),\ f \in L^2(\mathbb{R}^n)$, and define

$$P_g f(\theta, x, b, a) = \int_{\mathbb{R}} f(x+t\theta)\overline{g_{a,b}(t)}\, dt, \quad (\theta, x) \in \mathcal{T}, \quad (b, a) \in \mathbb{H}.$$

The transform P_g will be called the *continuous wavelet X-ray transform*. If we fix $\theta \in S^{n-1}$, we shall write

$$P_{g,\theta} f(x, b, a) = P_g(\theta, x, b, a), \quad x \in \theta^\perp,\ (b, a) \in \mathbb{H}.$$

We shall also formulate results in terms of this transform, i.e., for the wavelet X-ray transform with fixed direction $\theta \in S^{n-1}$. In the next theorem, we derive Parseval's formula for the wavelet X-ray transform.

Theorem 3.1 (Parseval's formulas) *Let $g \in L^2(\mathbb{R})$ be an admissible wavelet, then*

$$P_g : L^2(\mathbb{R}^n) \to L^2(\mathcal{T} \times \mathbb{H})$$

is a multiple of an isometry. In fact,

$$\langle P_g f, P_g k\rangle_{L^2(\mathcal{T}\times\mathbb{H})} = c_g \cdot |S^{n-1}| \cdot \langle f, k\rangle_{L^2(\mathbb{R}^n)}.$$

Moreover, for fixed $\theta \in S^{n-1}$, the transformation

$$P_{g,\theta} : L^2(\mathbb{R}^n) \to L^2(\theta^\perp \times \mathbb{H})$$

is also a multiple of an isometry with

$$\langle P_{g,\theta} f, P_{g,\theta} k\rangle_{L^2(\theta^\perp\times\mathbb{H})} = c_g \cdot \langle f, k\rangle_{L^2(\mathbb{R}^n)}.$$

We state and prove a lemma which has Theorem 3.1 as an immediate corollary. Note that $P_g = W_g \Gamma$ can be written as a cascade of two operators, where $\Gamma : L^2(\mathbb{R}^n) \to L^2(\mathcal{T} \times \mathbb{R})$ is defined by

$$\Gamma f(\theta, x, t) = f(x + t\theta), \quad (\theta, x, t) \in \mathcal{T} \times \mathbb{R},$$

and where $W_g : L^2(\mathcal{T} \times \mathbb{R}) \to L^2(\mathcal{T} \times \mathbb{H})$ is a lifted version of the one-dimensional wavelet transform, given by

$$W_g \sigma(\theta, x, b, a) = \int_{\mathbb{R}} \sigma(\theta, x, t) \overline{g_{a,b}(t)}\, dt, \quad (\theta, x) \in \mathcal{T}, \quad (b, a) \in \mathbb{H}.$$

In the same fashion, for fixed $\theta \in S^{n-1}$, we get $P_{g,\theta} = W_g \Gamma_\theta$, where $\Gamma_\theta : L^2(\mathbb{R}^n) \to L^2(\theta^\perp \times \mathbb{R})$ is given by $\Gamma_\theta f(x, t) = f(x + t\theta)$ for $x \in \theta^\perp$ and $\theta \in S^{n-1}$. Here $W_g : L^2(\theta^\perp \times \mathbb{R}) \to L^2(\theta^\perp \times \mathbb{H})$ is another lifted version of the one-dimensional wavelet transform.

Lemma 3.2 *The mapping*

$$\Gamma : L^2(\mathbb{R}^n) \to L^2(\mathcal{T} \times \mathbb{R})$$

is a multiple of an isometry. For fixed $\theta \in S^{n-1}$, the mapping

$$\Gamma_\theta : L^2(\mathbb{R}^n) \to L^2(\theta^\perp \times \mathbb{R}), \quad \theta \in S^{n-1},$$

is an isometry, and if $g \in L^2(\mathbb{R})$ is an admissible wavelet, then

$$W_g : L^2(\mathcal{T} \times \mathbb{R}) \to L^2(\mathcal{T} \times \mathbb{H}), \quad W_g : L^2(\theta^\perp \times \mathbb{R}) \to L^2(\theta^\perp \times \mathbb{H})$$

are multiples of an isometry.

Proof First, for any $f \in L^2(\mathbb{R}^n)$, we get

$$\|\Gamma_\theta f\|^2_{L^2(\theta^\perp \times \mathbb{R})} = \int_{\theta^\perp} \int_{\mathbb{R}} |\Gamma_\theta f(x, t)|^2\, dt\, dx =$$

$$\int_{\theta^\perp} \int_{\mathbb{R}} |f(x + t\theta)|^2\, dt\, dx = \|f\|^2_{L^2(\mathbb{R}^n)}.$$

With a slight modification of the preceding argument, one obtains

$$\|\Gamma f\|^2_{L^2(\mathcal{T} \times \mathbb{R})} = |S^{n-1}| \cdot \|f\|^2_{L^2(\mathbb{R}^n)}.$$

Next, if $\sigma, \tau \in L^2(\mathcal{T} \times \mathbb{R})$, then

$$\langle W_g \sigma, W_g \tau \rangle_{L^2(\mathcal{T} \times \mathbb{H})} = \int_{\mathcal{T}} \int_{\mathbb{H}} W_g \sigma(\theta, x, b, a) \overline{W_g \tau(\theta, x, b, a)}\, db\, \frac{da}{a^2}\, d(\theta, x) =$$

$$\int_{\mathcal{T}} \langle W_g \sigma(\theta, x, \cdot, \cdot), W_g \tau(\theta, x, \cdot, \cdot) \rangle_{L^2(\mathbb{H})}\, d(\theta, x) = \int_{\mathcal{T}} c_g \langle \sigma(\theta, x, \cdot), \tau(\theta, x, \cdot) \rangle_{L^2(\mathbb{R})}\, d(\theta, x) =$$

$$c_g \int_{\mathcal{T}} \int_{\mathbb{R}} \sigma(\theta, x, t) \overline{\tau(\theta, x, t)}\, dt\, d(\theta, x) = c_g \langle \sigma, \tau \rangle_{L^2(\mathcal{T} \times \mathbb{R})}.$$

The fact that $W_g : L^2(\theta^\perp \times \mathbb{R}) \to L^2(\theta^\perp \times \mathbb{H})$ is a multiple of an isometry is proved in the same fashion. $\square$

Theorem 3.3 (Reconstruction formulas) *Let $g \in L^2(\mathbb{R})$ be an admissible wavelet, then for any $f \in L^2(\mathbb{R}^n)$, we get*

$$f = \frac{1}{c_g \cdot |S^{n-1}|} \int_{S^{n-1}} \int_{\mathbb{H}} P_g f(\theta, E_\theta \cdot, b, a) g_{a,b}(\langle \cdot, \theta \rangle)\, db\, \frac{da}{a^2}\, d\theta, \tag{3.1}$$

where $E_\theta = I - \langle \cdot, \theta \rangle$ denotes the orthoprojector onto $\theta^\perp \subseteq \mathbb{R}^n$. Moreover, for $\theta \in S^{n-1}$ fixed,

$$f = \frac{1}{c_g} \int_{\mathbb{H}} P_{g,\theta} f(E_\theta \cdot, b, a) g_{a,b}(\langle \cdot, \theta \rangle)\, db\, \frac{da}{a^2}. \tag{3.2}$$

Both integrals converge in $L^2(\mathbb{R}^n)$.

Proof We first prove that the right-hand side of (3.1) is in $L^2(\mathbb{R}^n)$. In the main body of the proof, we will replace $P_g f$ by an arbitrary $\Sigma \in L^2(\mathcal{T} \times \mathbb{H})$. Let $(K_m)_{m=1}^\infty$ be a non-decreasing sequence of compact and measurable subsets of the open upper half plane $\mathbb{H}$, such that $\bigcup_{m=1}^\infty K_m = \mathbb{H}$. For $m \in \mathbb{Z}^+$, define

$$\mathcal{I}_m = \frac{1}{c_g |S^{n-1}|} \int_{S^{n-1}} \int_{K_m} \Sigma(\theta, E_\theta \cdot, b, a) g_{a,b}(\langle \cdot, \theta \rangle)\, db\, \frac{da}{a^2}\, d\theta.$$

Then, for any $k \in L^2(\mathbb{R}^n)$,

$$\langle \mathcal{I}_m, k \rangle_{L^2(\mathbb{R}^n)} = \frac{1}{c_g |S^{n-1}|} \int_{\mathbb{R}^n} \int_{S^{n-1}} \int_{K_m} \Sigma(\theta, E_\theta y, b, a) g_{a,b}(\langle y, \theta \rangle) \overline{k(y)}\, db\, \frac{da}{a^2}\, d\theta\, dy.$$

The integrand is in $L^1(\mathbb{R}^n \times S^{n-1} \times K_m)$. Indeed,

$$\int_{K_m} \int_{S^{n-1}} \int_{\theta^\perp} \int_{\mathbb{R}} |\Sigma(\theta, x, b, a) g_{a,b}(t) k(x + t\theta)|\, dt\, dx\, d\theta\, db\, \frac{da}{a^2} \le$$

$$\int_{K_m} \int_{S^{n-1}} \int_{\theta^\perp} |\Sigma(\theta, x, b, a)| \cdot \|g\|_{L^2(\mathbb{R})} \sqrt{\int_{\mathbb{R}} |k(x + t\theta)|^2\, dt}\ dx\, d\theta\, db\, \frac{da}{a^2}\, dx \le$$

$$\|g\|_{L^2(\mathbb{R})} \|k\|_{L^2(\mathbb{R}^n)} \int_{K_m} \int_{S^{n-1}} \sqrt{\int_{\theta^\perp} |\Sigma(\theta, x, b, a)|^2\, dx}\ d\theta\, db\, \frac{da}{a^2} \le$$

$$\|g\|_{L^2(\mathbb{R})} \|k\|_{L^2(\mathbb{R})} \sqrt{\mu(K_m) \cdot |S^{n-1}|} \cdot \|\Sigma\|_{L^2(\mathcal{T} \times \mathbb{H})}.$$

Therefore, by Fubini's theorem,

$$\langle \mathcal{I}_m, k \rangle_{L^2(\mathbb{R}^n)} = \frac{1}{c_g |S^{n-1}|} \int_{K_m} \int_{S^{n-1}} \int_{\theta^\perp} \int_{\mathbb{R}} \Sigma(\theta, x, b, a) g_{a,b}(t) \overline{k(x + t\theta)}\, dt\, dx\, d\theta\, db\, \frac{da}{a^2} =$$

$$\frac{1}{c_g |S^{n-1}|} \int_{K_m} \int_{S^{n-1}} \int_{\theta^\perp} \Sigma(\theta, x, b, a) \overline{P_g k(\theta, x, b, a)}\, dx\, db\, \frac{da}{a^2} = \frac{1}{c_g |S^{n-1}|} \langle \Sigma, P_g k \rangle_{L^2(\mathcal{T} \times K_m)}.$$

This implies

$$|\langle \mathcal{I}_m, k\rangle_{L^2(\mathbb{R}^n)}| \le \frac{1}{\sqrt{c_g|S^{n-1}|}}\|\Sigma\|_{L^2(\mathcal{T}\times\mathbb{H})}\|k\|_{L^2(\mathbb{R}^n)},$$

and henceforth $\mathcal{I}_m \in L^2(\mathbb{R}^n)$.

Next, we prove the convergence in norm of the right-hand side of (3.1). It suffices to prove that the sequence $(\mathcal{I}_m)_{m=1}^{\infty}$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$. Let $p < q$ be positive integers and consider for $k \in L^2(\mathbb{R}^n)$ the expression

$$\langle \mathcal{I}_q - \mathcal{I}_p, k\rangle_{L^2(\mathbb{R}^n)} =$$

$$\frac{1}{c_g|S^{n-1}|}\int_{K_q/K_p}\int_{S^{n-1}}\int_{\theta^\perp}\int_{\mathbb{R}}\Sigma(\theta,x,b,a)\overline{P_g k(\theta,x,b,a)}\,dt\,dx\,d\theta\,db\,\frac{da}{a^2} \le$$

$$\frac{1}{c_g|S^{n-1}|}\sqrt{\int_{K_q/K_p}\int_{S^{n-1}}\int_{\theta^\perp}|\Sigma(\theta,x,b,a)|^2\,dx\,d\theta\,db\,\frac{da}{a^2}}\times$$

$$\times\sqrt{\int_{K_q/K_p}\int_{S^{n-1}}\int_{\theta^\perp}|P_g k(\theta,x,b,a)|^2\,dx\,d\theta\,db\,\frac{da}{a^2}} \le$$

$$\frac{1}{\sqrt{c_g|S^{n-1}|}}\cdot\sqrt{\int_{K_q/K_p}\int_{S^{n-1}}\int_{\theta^\perp}|\Sigma(\theta,x,b,a)|^2\,dx\,d\theta\,db\,\frac{da}{a^2}}\cdot\|k\|_{L^2(\mathbb{R}^n)}.$$

Therefore,

$$\|\mathcal{I}_q - \mathcal{I}_p\|_{L^2(\mathbb{R}^n)} \le \frac{1}{\sqrt{c_g|S^{n-1}|}}\cdot\sqrt{\int_{K_q/K_p}\int_{S^{n-1}}\int_{\theta^\perp}|\Sigma(\theta,x,b,a)|^2\,dx\,d\theta\,db\,\frac{da}{a^2}}.$$

As $p, q \to \infty$, the integral tends to zero, since the integrand is in $L^1(\mathcal{T}\times\mathbb{H})$.

To prove the equality in (3.1), let $\Sigma = P_g f$. Then

$$\lim_{m\to\infty}\langle \mathcal{I}_m, k\rangle_{L^2(\mathbb{R}^n)} = \lim_{m\to\infty}\frac{1}{c_g|S^{n-1}|}\langle P_g f, P_g k\rangle_{L^2(\mathcal{T}\times K_m)} = \langle f, k\rangle_{L^2(\mathbb{R}^n)}.$$

A slight modification of the preceding part of the proof yields (3.2). This proves the theorem. □

We proceed in the same fashion as for the continuous wavelet transform. Define the dual operator $P_g^* : L^2(\mathcal{T}\times\mathbb{H}) \to L^2(\mathbb{R}^n)$ by

$$\langle P_g^*\Sigma, f\rangle_{L^2(\mathbb{R}^n)} = \langle \Sigma, P_g f\rangle_{L^2(\mathcal{T}\times\mathbb{H})}, \quad \Sigma \in L^2(\mathcal{T}\times\mathbb{H}), \quad f \in L^2(\mathbb{R}^n).$$

Then P_g^* is a bounded operator, with norm $\|P_g^*\| = \|P_g\| = \sqrt{c_g|S^{n-1}|}$, and is given by

$$P_g^*\Sigma = \int_{S^{n-1}} \int_{\mathbb{H}} \Sigma(\theta, E_\theta\cdot, b, a) g_{a,b}(\langle \cdot, \theta\rangle)\, db\, \frac{da}{a^2}\, d\theta, \quad \Sigma \in L^2(\mathcal{T} \times \mathbb{H}).$$

By the proof of Theorem 3.3, the integral converges in $L^2(\mathbb{R}^n)$. Analogously, one may define the dual operator of $P_{g,\theta}$, where $\theta \in S^{n-1}$ is fixed. Indeed, put

$$\langle P_{g,\theta}^*\Sigma, f\rangle_{L^2(\mathbb{R}^n)} = \langle \Sigma, P_{g,\theta} f\rangle_{L^2(\theta^\perp\times\mathbb{H})}, \quad \Sigma \in L^2(\theta^\perp \times \mathbb{H}), \quad f \in L^2(\mathbb{R}^n).$$

Then $P_{g,\theta}^* : L^2(\theta^\perp \times \mathbb{H}) \to L^2(\mathbb{R}^n)$ is given by

$$P_{g,\theta}^*\Sigma = \int_{\mathbb{H}} \Sigma(E_\theta\cdot, b, a) g_{a,b}(\langle \cdot, \theta\rangle)\, db\, \frac{da}{a^2}.$$

This integral also converges in $L^2(\mathbb{R}^n)$. The next theorem identifies the left inverse of P_g and the orthoprojector onto ran P_g. Analogous results can be stated for $P_{g,\theta}$, but we shall omit those.

Theorem 3.4 *If $g \in L^2(\mathbb{R})$ is an admissible wavelet, then*

$$f = \frac{1}{c_g|S^{n-1}|} P_g^* P_g f$$

for all $f \in L^2(\mathbb{R}^n)$. Moreover, the bounded operator

$$\frac{1}{c_g|S^{n-1}|} P_g P_g^* : L^2(\mathcal{T} \times \mathbb{H}) \to L^2(\mathcal{T} \times \mathbb{H})$$

defines the orthoprojector onto ran P_g.

4. Elementary projections

In this section, we may and do assume that $\|g\|_{L^2(\mathbb{R})} = 1$. The reconstruction formula (3.1) can be seen as an expansion of a function along wavelets of distribution type. To make this more precise, assume for the moment that $f \in C(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is a continuous function in $L^2(\mathbb{R}^n)$. Write

$$f(y) = \frac{1}{c_g|S^{n-1}|} \int_{S^{n-1}} \int_{\mathbb{H}} \int_{\mathbb{R}} f(E_\theta y + t\theta) \overline{g_{a,b}(t)}\, dt\, g_{a,b}(\langle y, \theta\rangle)\, db\, \frac{da}{a^2}\, d\theta =$$

$$\frac{1}{c_g|S^{n-1}|} \int_{S^{n-1}} \int_{\mathbb{H}} \int_{\theta^\perp} \int_{\mathbb{R}^n} f(z) \overline{g_{a,b}(\langle z, \theta\rangle)} \delta(E_\theta z - x)\, dz\, g_{a,b}(\langle y, \theta\rangle) \delta(E_\theta y - x)\, dx\, db\, \frac{da}{a^2}\, d\theta.$$

We have introduced here the delta distribution which satisfies

$$k(x) = \int_{\theta^\perp} k(u)\delta(x - u)\, du, \quad x \in \theta^\perp, \quad k \in C(\theta^\perp).$$

If we define the distribution $\psi_{\theta,x,a,b} = \delta(E_\theta \cdot -x) g_{a,b}(\langle \cdot, \theta \rangle)$ and interpret it as a wavelet (see [Hol, Hol2]), i.e., if we write

$$W_\psi f(\theta, x, b, a) = \int_{\mathbb{R}^n} f(z) \overline{\psi_{\theta,x,a,b}(z)} \, dz = P_g(\theta, x, b, a),$$

then we arrive at

$$f(y) = \frac{1}{c_g |S^{n-1}|} \int_{S^{n-1}} \int_{\mathbb{H}} \int_{\theta^\perp} W_\psi f(\theta, x, b, a) \psi_{\theta,x,a,b}(y) \, dx \, db \, \frac{da}{a^2} \, d\theta.$$

Instead of using wavelets of distribution type, we introduce for $\theta \in S^{n-1}$, $(b,a) \in \mathbb{H}$, the *elementary projections*

$$G_{\theta,a,b} f = P_g f(\theta, E_\theta \cdot, b, a) g_{a,b}(\langle \cdot, \theta \rangle), \quad f \in L^2(\mathbb{R}^n),$$

and rewrite the reconstruction formula (3.1) as

$$f = \frac{1}{c_g |S^{n-1}|} \int_{S^{n-1}} \int_{\mathbb{H}} G_{\theta,a,b} f \; db \, \frac{da}{a^2} \, d\theta,$$

and reconstruction formula (3.2) as

$$f = \frac{1}{c_g} \int_{\mathbb{H}} G_{\theta,a,b} f \; db \, \frac{da}{a^2},$$

for fixed $\theta \in S^{n-1}$. As we have seen, the integrals converge in $L^2(\mathbb{R}^n)$. In terms of the aforementioned distribution ψ, we could write for a continuous function f in $L^2(\mathbb{R}^n)$ the formula

$$G_{\theta,a,b} f = \int_{\theta^\perp} W_\psi f(\theta, x, b, a) \psi_{\theta,x,a,b} \, dx,$$

but we will study the elementary projections without further reference to the wavelet of distribution type ψ. Lemma 4.1 shows that the elementary projections indeed are orthogonal projections on $L^2(\mathbb{R}^n)$. In the remainder of this section, first results on discrete versions of the wavelet X-ray transform are presented.

Lemma 4.1 *Let $g \in L^2(\mathbb{R})$ be an admissible wavelet. Then*

$$G_{\theta,a,b} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

is an orthogonal projection onto the subspace

$$M_{\theta,a,b} = \{ f \in L^2(\mathbb{R}^n) \mid f = \varphi(E_\theta \cdot) g_{a,b}(\langle \cdot, \theta \rangle), \text{ for some } \varphi \in L^2(\theta^\perp) \}.$$

Proof We first prove that $G_{\theta,a,b}$ is a bounded operator. Indeed,

$$\int_{\mathbb{R}^n} |G_{\theta,a,b}f(y)|^2\,dy = \int_{\theta^\perp}\int_{\mathbb{R}} |P_g f(\theta,x,b,a)|^2 |g_{a,b}(t)|^2\,dt\,dx =$$

$$\int_{\theta^\perp} |P_g f(\theta,x,b,a)|^2\,dx \le \|f\|^2_{L^2(\mathbb{R}^n)}.$$

Moreover, the operator is self-adjoint:

$$\langle G_{\theta,a,b}f,k\rangle_{L^2(\mathbb{R}^n)} = \int_{\theta^\perp}\int_{\mathbb{R}} P_g f(\theta,x,b,a) g_{a,b}(t)\overline{k(x+t\theta)}\,dt\,dx =$$

$$\int_{\theta^\perp}\int_{\mathbb{R}}\int_{\mathbb{R}} f(x+s\theta)\overline{g_{a,b}(s)}g_{a,b}(t)\overline{k(x+t\theta)}\,ds\,dt\,dx = \langle f, G_{\theta,a,b}k\rangle_{L^2(\mathbb{R}^n)}.$$

Further, the operator is idempotent: for almost all $y \in \mathbb{R}^n$,

$$G^2_{\theta,a,b}f(y) = \int_{\mathbb{R}} G_{\theta,a,b}f(E_\theta y + t\theta)\overline{g_{a,b}(t)}\,dt\,g_{a,b}(\langle y,\theta\rangle) =$$

$$\int_{\mathbb{R}} P_g f(\theta, E_\theta y, b, a) g_{a,b}(t)\overline{g_{a,b}(t)}\,dt\,g_{a,b}(\langle y,\theta\rangle) = G_{\theta,a,b}f(y).$$

Next, we identify the range of $G_{\theta,a,b}$. For each $f \in L^2(\mathbb{R}^n)$, we get

$$G_{\theta,a,b}f = P_g f(\theta, E_\theta\cdot, b, a) g_{a,b}(\langle\cdot,\theta\rangle) \in M_{\theta,a,b}.$$

If $\varphi \in L^2(\theta^\perp)$, then $f = \varphi(E_\theta\cdot)g_{a,b}(\langle\cdot,\theta\rangle)$ satisfies $G_{\theta,a,b}f = f$. □

We also have the following version of Parseval's formula:

Lemma 4.2 *Let $g \in L^2(\mathbb{R})$ be an admissible wavelet. Then for any $f \in L^2(\mathbb{R}^n)$ and $\theta \in S^{n-1}$,*

$$\|f\|^2_{L^2(\mathbb{R}^n)} = \frac{1}{c_g}\int_{\mathbb{H}} \|G_{\theta,a,b}f\|^2_{L^2(\mathbb{R}^n)}\,db\,\frac{da}{a^2}.$$

Proof By Parseval's formula,

$$\int_{\mathbb{R}^n} |f(y)|^2\,dy = \frac{1}{c_g}\int_{\mathbb{H}}\int_{\theta^\perp} |P_g f(\theta,x,a,b)|^2\,dx\,db\,\frac{da}{a^2}.$$

The lemma now follows from

$$\|G_{\theta,a,b}f\|^2_{L_2(\mathbb{R}^n)} = \int_{\theta^\perp}\int_{\mathbb{R}} |P_g(\theta,x,b,a)g_{a,b}(t)|^2\,dt\,dx = \int_{\theta^\perp} |P_g(\theta,x,b,a)|^2\,dx.$$

□

We will now study products of the projections $G_{\theta,a,b}$. If $f \in L^2(\mathbb{R}^n)$, then for arbitrary $(\theta, a, b),\ (\varphi, \alpha, \beta) \in S^{n-1} \times \mathbb{H}$, we get

$$G_{\theta,a,b}G_{\varphi,\alpha,\beta}f(y) = \int_{\mathbb{R}} G_{\varphi,\alpha,\beta}f(E_\theta y + t\theta)\overline{g_{a,b}(t)}\,dt\, g_{a,b}(\langle y,\theta\rangle) =$$

$$\int_{\mathbb{R}}\int_{\mathbb{R}} f(E_\varphi E_\theta y + tE_\varphi\theta + s\varphi)\overline{g_{\alpha,\beta}(s)}\ \overline{g_{a,b}(t)}g_{\alpha,\beta}(\langle y, E_\theta\varphi\rangle + t\langle\theta,\varphi\rangle)\,ds\,dt\ \ g_{a,b}(\langle y,\theta\rangle)$$

for almost all $y \in \mathbb{R}^n$. We will specify two important cases.

CASE 1
Consider the case when $\theta = \varphi$. In this case,

$$G_{\theta,a,b}G_{\theta,\alpha,\beta}f(y) = \int_{\mathbb{R}}\int_{\mathbb{R}} f(E_\theta y + s\theta)\overline{g_{\alpha,\beta}(s)}\ \overline{g_{a,b}(t)}g_{\alpha,\beta}(t)\,ds\,dt\ \ g_{a,b}(\langle y,\theta\rangle) =$$

$$\langle g_{\alpha,\beta}, g_{a,b}\rangle_{L^2(\mathbb{R})}P_g f(\theta, E_\theta y, \beta, \alpha)\ \ g_{a,b}(\langle y,\theta\rangle).$$

CASE 2
In the case when $\theta \perp \varphi$, the product comes down to

$$G_{\theta,a,b}G_{\varphi,\alpha,\beta}f(y) = \int_{\mathbb{R}}\int_{\mathbb{R}} f(E_\varphi E_\theta y + t\theta + s\varphi)\overline{g_{\alpha,\beta}(s)}\ \overline{g_{a,b}(t)}\,ds\,dt\ \ g_{\alpha,\beta}(\langle y,\varphi\rangle)g_{a,b}(\langle y,\theta\rangle).$$

Observe that in this case, $E_\theta E_\varphi = E_\varphi E_\theta$ is the orthoprojector onto the $(n-2)$-dimensional subspace $\{\theta,\varphi\}^\perp$. Moreover, it is immediate that the orthoprojectors $G_{\theta,a,b}$ and $G_{\varphi,\alpha,\beta}$ commute. The following lemma is an immediate consequence of these facts.

Lemma 4.3 *Let $\theta_1, \dots, \theta_n \in S^{n-1}$ be mutually orthogonal unit vectors, and let $(b_j, a_j) \in \mathbb{H}$ for $j = 1, \dots, n$. Write*

$$\underline{\theta} = (\theta_1, \dots, \theta_n), \quad \underline{a,b} = ((a_1, b_1), \dots, (a_n, b_n)).$$

Next, define

$$F_{\underline{\theta},\underline{a,b}} = \prod_{j=1}^{n} g_{a_j,b_j}(\langle\cdot,\theta_j\rangle), \quad G_{\underline{\theta},\underline{a,b}} = \prod_{j=1}^{n} G_{\theta_j,a_j,b_j}.$$

Then $F_{\underline{\theta},\underline{a,b}} \in L^2(\mathbb{R}^n)$, $\|F_{\underline{\theta},\underline{a,b}}\|_{L^2(\mathbb{R}^n)} = 1$, and

$$G_{\underline{\theta},\underline{a,b}}f = \langle f, F_{\underline{\theta},\underline{a,b}}\rangle_{L^2(\mathbb{R}^n)}\ F_{\underline{\theta},\underline{a,b}}.$$

In particular, the operator $G_{\underline{\theta},\underline{a,b}}$ is an orthogonal projector of rank one.

We derive Plancherel's formula concerning the coefficients $\langle f, F_{\underline{\theta},\underline{a},\underline{b}}\rangle_{L^2(\mathbb{R}^n)}$ of f. By Lemma 4.3, we get

$$\|G_{\underline{\theta},\underline{a},\underline{b}} f\|_{L^2(\mathbb{R}^n)} = |\langle f, F_{\underline{\theta},\underline{a},\underline{b}}\rangle_{L^2(\mathbb{R}^n)}|.$$

By repetitive use (n times) of Lemma 4.2, we arrive at

$$\|f\|_{L^2(\mathbb{R}^n)} = \frac{1}{c_g^n |S^{n-1}|^n} \int_{S^{n-1}} \int_{\mathbb{H}} \cdots \int_{S^{n-1}} \int_{\mathbb{H}} |\langle f, F_{\underline{\theta},\underline{a},\underline{b}}\rangle_{L^2(\mathbb{R}^n)}|^2 \, db_n \, \frac{da_n}{a_n^2} \, d\theta_n \cdots db_1 \, \frac{da_1}{a_1^2} \, d\theta_1.$$

It is an aim of future research to study the discrete wavelet X-ray transform. We shall give a preliminary result in this direction. We omit the straightforward proof.

Proposition 4.4 *Let $K \subseteq \mathbb{H}$ be a countable subset and let $g \in L^2(\mathbb{R})$ be a wavelet, such that $\{g_{a,b}\}_{(b,a)\in K}$ is an orthonormal basis of $L^2(\mathbb{R})$. If $\theta_1, \ldots \theta_n \in S^{n-1}$ are mutually orthogonal unit vectors, then $\{F_{\underline{\theta},\underline{a},\underline{b}}\}_{\underline{a},\underline{b}\in K^n}$ is an orthonormal basis of $L^2(\mathbb{R}^n)$. In particular, if $f \in L^2(\mathbb{R}^n)$, then*

$$f = \sum_{\underline{a},\underline{b}\in K^n} \langle f, F_{\underline{\theta},\underline{a},\underline{b}}\rangle_{L^2(\mathbb{R}^n)} F_{\underline{\theta},\underline{a},\underline{b}},$$

and

$$\|f\|^2_{L^2(\mathbb{R}^n)} = \sum_{\underline{a},\underline{b}\in K^n} |\langle f, F_{\underline{\theta},\underline{a},\underline{b}}\rangle_{L^2(\mathbb{R}^n)}|^2.$$

The type of orthonormal wavelet bases in $L^2(\mathbb{R})$ we particularly have in mind in Proposition 4.4 are the ones consisting of compactly supported wavelets [Dau, Dau2]. Wavelets in $L^2(\mathbb{R}^n)$ which are the product of wavelets in $L^2(\mathbb{R})$, such as the ones in Proposition 4.4, are called *separable wavelets*; see for example [Dau].

5. Approximation of the identity

As indicated in the introduction, the reconstruction of a (density) function from its Radon transform is relevant for a range of applications. The backprojection formula, given by Lemma 5.1, enables us to approximately reconstruct an L^1 function from its Radon transform. In fact, we will use the wavelet transform to construct an approximate identity in the same fashion as in [BW]. Proposition 5.3 gives a connection between such an approximate identity and the wavelet X-ray transform.

The *Radon transform* or the *$(n-1)$-dimensional X-ray transform* $R : L^1(\mathbb{R}^n) \to L^1(\mathcal{Z})$ is given by

$$Rf(\theta, s) = \int_{\theta^\perp} f(x + s\theta)\, dx, \quad (\theta, s) \in \mathcal{Z},$$

where $\mathcal{Z} = S^{n-1} \times \mathbb{R}$ is the unit cylinder. It is easily verified that the Radon transform R is a bounded operator on $L^1(\mathbb{R}^n)$. Indeed,

$$\|Rf\|_{L^1(\mathcal{Z})} \leq \int_{S^{n-1}} \int_{\mathbb{R}} \int_{\theta^\perp} |f(x+s\theta)|\, dx\, ds\, d\theta = |S^{n-1}| \cdot \|f\|_{L^1(\mathbb{R}^n)}.$$

We mention that R is *not* a bounded operator on $L^2(\mathbb{R}^n)$; see [Sol]. In this section, convolution products arise of functions in several function spaces. If $h_1, h_2 \in L^1(\mathcal{Z})$, then define

$$(h_1 * h_2)(\theta, s) = \int_{\mathbb{R}} h_1(\theta, s-t) h_2(\theta, t)\, dt, \quad (\theta, s) \in \mathcal{Z}.$$

For $f_1, f_2 \in L^1(\mathbb{R}^n)$, we have the usual convolution product

$$(f_1 * f_2)(y) = \int_{\mathbb{R}^n} f_1(y-z) f_2(z)\, dz, \quad y \in \mathbb{R}^n.$$

We shall consider the dual operator $R^\#$ of R. Identify bounded linear functionals on L^1 with functions in L^∞. The backprojection formula is given by the following lemma. For completeness, we give its proof; see also [Nat].

Lemma 5.1 (Filtered backprojection) *The dual operator $R^\# : L^\infty(\mathcal{Z}) \to L^\infty(\mathbb{R}^n)$ of R is given by*

$$R^\# h = \int_{S^{n-1}} h(\theta, \langle \cdot, \theta \rangle)\, d\theta, \quad h \in L^\infty(\mathcal{Z}).$$

Moreover, if $f \in L^1(\mathbb{R}^n)$ and $h \in L^\infty(\mathcal{Z}) \cap L^1(\mathcal{Z})$, then $h * Rf \in L^\infty(\mathcal{Z}) \cap L^1(\mathcal{Z})$ and

$$R^\#(h * Rf) = R^\# h * f.$$

Proof Observe that, by Fubini's theorem,

$$\int_{\mathcal{Z}} Rf(\theta, s) h(\theta, s)\, d(\theta, s) = \int_{S^{n-1}} \int_{\mathbb{R}} \int_{\theta^\perp} f(x+s\theta) h(\theta, s)\, dx\, ds\, d\theta =$$

$$\int_{\mathbb{R}^n} f(y) \int_{S^{n-1}} h(\theta, \langle y, \theta \rangle)\, d\theta\, dy,$$

and the first part of the lemma is proved. Note that

$$\|h * Rf\|_{L^\infty(\mathcal{Z})} \leq |S^{n-1}| \cdot \|h\|_{L^\infty(\mathcal{Z})} \|f\|_{L^1(\mathbb{R}^n)}$$

and

$$\|h * Rf\|_{L^1(\mathcal{Z})} \leq |S^{n-1}| \cdot \|h\|_{L^1(\mathcal{Z})} \|f\|_{L^1(\mathbb{R}^n)}.$$

Finally, for almost all $y \in \mathbb{R}^n$, we get by Fubini's theorem,

$$R^{\#}(h * Rf)(y) = \int_{S^{n-1}} (h * Rf)(\theta, \langle y, \theta \rangle)\, d\theta = \int_{S^{n-1}} \int_{\mathbb{R}} h(\theta, \langle y, \theta\rangle - s) Rf(\theta, s)\, ds\, d\theta =$$

$$\int_{S^{n-1}} \int_{\mathbb{R}} \int_{\theta^\perp} h(\theta, \langle y, \theta\rangle - s) f(x + s\theta)\, dx\, ds\, d\theta = \int_{\mathbb{R}^n} R^{\#}h(y-z) f(z)\, dz = (R^{\#}h * f)(y).$$

This proves the lemma. □

As we are considering functions in L^1, we shall adapt the normalization of dilated functions to this norm. Indeed, in this section, we shall write for $g \in L^1(\mathbb{R}^k)$

$$g_a(\cdot) = \frac{1}{a^k} g\left(\frac{\cdot}{a}\right),$$

i.e., such that $\|g_a\|_{L^1(\mathbb{R}^k)} = \|g\|_{L^1(\mathbb{R}^k)}$ for $a > 0$. If $h \in L^1(\mathcal{Z})$, then

$$h_a(\theta, \cdot) = \frac{1}{a} h\left(\theta, \frac{\cdot}{a}\right), \quad \theta \in S^{n-1}$$

for $a > 0$. We will need the following classical result (see e.g. [Ru]).

Lemma 5.2 (Approximate identity) *If $h \in L^1(\mathbb{R}^n)$ is positive with norm $\|h\|_{L^1(\mathbb{R}^n)} = 1$, then*

$$\lim_{a \downarrow 0} \|h_a * f - f\|_{L^1(\mathbb{R}^n)} = 0,$$

where $h_a(\cdot) = \frac{1}{a^n} h(\frac{\cdot}{a})$ for $a > 0$ and $f \in L^1(\mathbb{R}^n)$.

Proof Fix $\varepsilon > 0$. Observe that for almost all $y \in \mathbb{R}^n$,

$$(h_a * f)(y) - f(y) = \int_{\mathbb{R}^n} h_a(z) \left[f(y-z) - f(y)\right] dz,$$

hence

$$\|h_a * f - f\|_{L^1(\mathbb{R}^n)} \le \int_{\mathbb{R}^n} h_a(z) \int_{\mathbb{R}^n} |f(y-z) - f(y)|\, dy\, dz.$$

Let φ be a continuous function with compact support contained in a closed cube $[-A, A]^n$, such that $\|f - \varphi\|_{L^1(\mathbb{R}^n)} < \varepsilon/3$. Then φ is uniformly continuous: there exists $0 < \delta < A/2$ such that $\|z\| < \delta$ implies

$$|\varphi(y-z) - \varphi(y)| < \frac{\varepsilon}{3^{n+1} A^n}.$$

We arrive at

$$\|f(\cdot - z) - f\|_{L^1(\mathbb{R}^n)} \le$$

$$\|f(\cdot - z) - \varphi(\cdot - z)\|_{L^1(\mathbb{R}^n)} + \|\varphi(\cdot - z) - \varphi\|_{L^1(\mathbb{R}^n)} + \|\varphi - f\|_{L^1(\mathbb{R}^n)} =$$

$$2\|\varphi - f\|_{L^1(\mathbb{R}^n)} + \|\varphi(\cdot - z) - \varphi\|_{L^1(\mathbb{R}^n)}.$$

Since

$$\|\varphi(\cdot - z) - \varphi\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\varphi(y - z) - \varphi(y)|\, dy =$$

$$\int_{\|y\| \le A+\delta} |\varphi(y - z) - \varphi(y)|\, dy \le (3A)^n \cdot \frac{\varepsilon}{3^{n+1}A^n} = \frac{\varepsilon}{3},$$

we conclude that $\|f(\cdot - z) - f\|_{L^1(\mathbb{R}^n)} < \varepsilon$ if $\|z\| < \delta$. Therefore,

$$\|h_a * f - f\|_{L^1(\mathbb{R}^n)} \le$$

$$\int_{\|z\|<\delta} h_a(z)\|f(\cdot - z) - f\|_{L^1(\mathbb{R}^n)}\, dz + \int_{\|z\|\ge\delta} h_a(z)\|f(\cdot - z) - f\|_{L^1(\mathbb{R}^n)}\, dz \le$$

$$\varepsilon + 2\|f\|_{L^1(\mathbb{R}^n)} \int_{\|z\|\ge\delta} \frac{1}{a^n} h(\frac{z}{a})\, dz = \varepsilon + 2\|f\|_{L^1(\mathbb{R}^n)} \int_{\|z\|\ge\delta/a} h(z)\, dz.$$

Since $h \in L^1(\mathbb{R}^n)$, the latter expression can be made smaller than 2ε by taking a small enough. This proves the lemma. $\square$

We are interested in the following situation. Let $g \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$, $a > 0$, and define $h \in L^\infty(\mathcal{Z}) \cap L^1(\mathcal{Z})$ by $h(\theta, \cdot) = \widetilde{g}$. By Lemma 5.1 and Fubini's theorem, we get for $f \in L^1(\mathbb{R}^n)$,

$$(R^\# h_a * f)(y) = R^\#(h_a * Rf)(y) = \int_{S^{n-1}} \int_{\mathbb{R}} Rf(\theta, s)\overline{g_a(s - \langle y, \theta\rangle)}\, ds\, d\theta =$$

$$\int_{S^{n-1}} \int_{\mathbb{R}} \int_{\theta^\perp} f(x + s\theta)\overline{g_a(s - \langle y, \theta\rangle)}\, dx\, ds\, d\theta =$$

$$\int_{S^{n-1}} \int_{\mathbb{R}} \int_{\theta^\perp} f(x + s\theta)\overline{g_{a,\langle y,\theta\rangle}(s)}\, dx\, ds\, d\theta = \int_{\mathcal{T}} P_g f(\theta, x, \langle y, \theta\rangle, a)\, d(\theta, x).$$

Here P_g maps $L^1(\mathbb{R}^n)$ into $L^1_{\mathrm{loc}}(\mathcal{T} \times \mathbb{H})$, since $P_g \in L^1(\mathcal{T} \times K)$ for all $K \subseteq \mathbb{H}$ compact. Indeed, if $f \in L^1(\mathbb{R}^n)$ and $K \subseteq \mathbb{H}$ compact, then

$$\int_K \int_{\mathcal{T}} |P_g f(\theta, x, b, a)|\, d(\theta, x)\, db\, \frac{da}{a^2} \le$$

$$\int_K \int_{S^{n-1}} \int_{\theta^\perp} \int_{\mathbb{R}} |f(x + t\theta)| \cdot |g_{a,b}(t)|\, dt\, dx\, d\theta\, db\, \frac{da}{a^2} \le$$

$$\int_K \int_{S^{n-1}} \int_{\theta^\perp} \int_{\mathbb{R}} |f(x+t\theta)|\, dt \frac{1}{a} \|g\|_{L^\infty(\mathbb{R})}\, dx\, d\theta\, db\, \frac{da}{a^2} =$$

$$|S^{n-1}| \cdot \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^\infty(\mathbb{R})} \int_K \frac{1}{a}\, db\, \frac{da}{a^2} < \infty.$$

Observe that the function $R^{\#}h$ is radially symmetric:

$$R^{\#}h(y) = \int_{S^{n-1}} \overline{g(\langle y,\theta\rangle)}\, d\theta = |S^{n-1}| \cdot \overline{g(\|y\|)}, \quad y \in \mathbb{R}^n.$$

We will assume in addition that $g \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ is a positive function and that

$$\int_0^\infty g(t) t^{n-1}\, dt = |S^{n-1}|^{-2}.$$

In this case, $R^{\#}h$ is a positive function in $L^1(\mathbb{R}^n)$ with norm $\|R^{\#}h\|_{L^1(\mathbb{R}^n)} = 1$. We may then apply Lemmas 5.1 and 5.2 to obtain the following result.

Proposition 5.3 *If $g \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is a positive function such that $\int_0^\infty g(t)t^{n-1}\, dt = |S^{n-1}|^{-2}$, and $h \in L^1(\mathcal{Z}) \cap L^\infty(\mathcal{Z})$ is given by $h(\theta, \cdot) = \widetilde{g}$, then*

$$|S^{n-1}| \cdot g_a(\|\cdot\|) * f = R^{\#}(h_a * Rf) = \int_{\mathcal{T}} P_g f(\theta, x, \langle \cdot, \theta\rangle, a)\, d(\theta, x)$$

converges in $L^1(\mathbb{R}^n)$ to the function $f \in L^1(\mathbb{R}^n)$ as $a \downarrow 0$.

We remark that in the lemma, $g_a(\|\cdot\|) = \frac{1}{a} g(\frac{\|\cdot\|}{a})$. Note that, by assumption, the wavelet g under consideration is *not* admissible. In some literature (e.g. [Dau]), such a wavelet is called a *father wavelet* (instead of a so-called *mother wavelet*) or a *scaling function*. An example of such a wavelet is given by $g(t) = C \cdot e^{-t^2}$, $t \in \mathbb{R}$, with an appropriate normalization constant C.

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