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**TOPOLOGY AND
ORDER STRUCTURES**

PART 1

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PREFACE

In August, 1980, NATO and Texas Tech University jointly sponsored a workshop on topology and linear orderings in Lubbock, Texas. For a two week period, specialists met to collaborate on problems of mutual interest. This volume includes contributions from most of last year's participants, plus papers by several others who were not able to attend the workshop. Other papers related to the workshop will be included in a second volume, to be published after the workshop's second meeting in August, 1981. We wish to express our gratitude to NATO and to Texas Tech University for their financial support, and to the Mathematical Centre for agreeing to publish this volume. In addition, the editing of this volume was partially supported by research grants from the U.S. National Science Foundation and from the Netherlands Organization for the Advancement of Pure Research (ZWO).

Let us add a preliminary note about terminology in this volume. A topological space (X, T) is *orderable* if there is a linear ordering $<$ of the set X such that T is the usual open interval topology of $<$, and then the triple $(X, <, T)$ is called a *linearly ordered topological space* (LOTS). A less stringent requirement is that there exist some linear ordering $<$ of the set X such that T has a base whose members are order-convex. If, in addition, T is a T_1 -topology, then (X, T) is said to be *suborderable* and the triple $(X, <, T)$ is called a *generalized ordered space* (GO-space). Often the terms "suborderable space", "subordered space", and "GO-space" are used interchangeably, even though this is not quite correct.

H.R. Bennett and D.J. Lutzer

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ORDERABILITY OF CONNECTED GRAPHS
AND NEARNESS SPACES

by

Horst Herrlich

INTRODUCTION

A *nearness space* is a pair (X, μ) , consisting of a set X and a collection μ of (non-empty) covers of X , satisfying the following conditions:

- (N1) $\{X\} \in \mu$;
- (N2) if a cover A of X is refined by some member of μ , then A belongs to μ ;
- (N3) $A \in \mu$ and $B \in \mu$ imply $\{A \cap B \mid A \in A \text{ and } B \in B\} \in \mu$;
- (N4) $A \in \mu$ implies $\{\text{int}_{\mu} A \mid A \in A\} \in \mu$, where $x \in \text{int}_{\mu} A$ iff $\{A, X \setminus \{x\}\} \in \mu$.

For any nearness space (X, μ) there exists a unique topology $\tau(\mu)$ on X - called the *induced topology* - such that int_{μ} is the interior-operator of $(X, \tau(\mu))$.

A nearness space (X, μ) is called a T_1 -*nearness space*, provided $(X, \tau(\mu))$ is a T_1 -space (equivalently: iff $\{X \setminus \{x\}, X \setminus \{y\}\} \in \mu$ for any two different elements x and y of X). (For background on nearness spaces see e.g. [5] and the references therein.)

A subset β of μ is called a *base* for μ , provided every member of μ is refined by some member of β . (For a definition of subbases see WATTEL [9].)

A *subordered* (resp. *ordered*) *nearness space* is a triple (X, \leq, μ) , such that the following conditions hold:

- (ON1) (X, \leq) is a linearly ordered set;
- (ON2) (X, μ) is a T_1 -nearness space;
- (ON3) μ has a base, consisting of covers, whose elements are intervals (resp. open intervals) in (X, \leq) .

If (X, \leq, μ) is a (sub)ordered nearness space, then $(X, \leq, \tau(\mu))$ is a (sub)ordered topological space. A nearness space (X, μ) is called (*sub*-)orderable, provided there exists a linear order \leq on X , such that (X, \leq, μ) is a (sub-)ordered nearness space (HUŠEK [6]).

The problem we are concerned with, is an intrinsic characterization of those nearness spaces which are orderable. Since topological R_0 -spaces (via interior covers), uniform spaces (via uniform covers), and proximity spaces (\cong totally bounded uniform spaces) can be considered as particular nearness spaces, the orderability problem for nearness spaces generalizes simultaneously the orderability problems for proximity spaces (FEDORČUK [3], WATTEL [9]), for uniform spaces (BANASCHEWSKI [1]), and for topological spaces (cf. e.g. EILENBERG [2], KOWALSKY [8], HERRLICH [4], and KOK [7] for the connected case).

The main result of this paper asserts that a connected, regular T_1 -nearness space (X, μ) is orderable if there exists a base for μ , consisting of covers \mathcal{U} , satisfying the following conditions:

- (1) each $U \in \mathcal{U}$, considered as a subspace of (X, μ) , is connected;
- (2) the graph $G(\mathcal{U})$ of \mathcal{U} is orderable.

Because of the latter condition, we start with a section on the orderability of connected graphs.

ORDERABILITY OF CONNECTED GRAPHS

A *graph* is a pair (X, ρ) , consisting of a set X and a reflexive symmetric relation ρ on X . It is called *finite*, provided X is finite. A graph (X, ρ) is called *connected*, provided for any pair (a, b) of elements of X there exists a finite sequence (a_1, a_2, \dots, a_n) in X with $a = a_1$, $b = a_n$, and $a_i \rho a_{i+1}$ for $i = 1, \dots, n-1$. A connected graph (X, ρ) is called *orderable*, provided there exists a convex subset C of the set \mathbb{Z} of integers and a bijection $h: X \rightarrow C$, such that $x \rho y \Leftrightarrow |h(x) - h(y)| \leq 1$ holds.

If (X, ρ) is a graph, Y is a subset of X and σ is the restriction of ρ to Y , then (Y, σ) is called the *subgraph* of (X, ρ) *determined by* Y . An element x of X is called an *endpoint* (resp. *cutpoint*) of a connected graph (X, ρ) , provided the subgraph of (X, ρ) , determined by $X \setminus \{x\}$, is connected (resp. not connected). A graph (X, ρ) contains a *cycle*, provided there exist a subgraph (Y, σ) of (X, ρ) , a natural number $n \geq 3$, and a bijection $h: Y \rightarrow \{1, 2, \dots, n\}$, such that the equivalence $x \rho y \Leftrightarrow |h(x) - h(y)| \leq 1 \pmod{n}$ holds. A graph (X, ρ) contains a *n-star*, provided there exists a subgraph (Y, σ) of (X, ρ) and a bijection $h: Y \rightarrow \{0, 1, \dots, n\}$, such that the equivalence $x \rho y \Leftrightarrow (x=y \text{ or } x=0 \text{ or } y=0)$ holds.

PROPOSITION 1. A finite, connected graph is orderable, iff it has at most two endpoints.

PROPOSITION 2. For a connected graph (X, ρ) the following conditions are equivalent:

- (1) (X, ρ) is orderable;
- (2) (X, ρ) has neither cycles nor 3-stars;
- (3) each connected subgraph of (X, ρ) has at most two endpoints;
- (4) among every three distinct, connected, proper subgraphs of (X, ρ) , there are two, which together do not cover X .

ORDERABILITY OF NEARNESS SPACES

For every cover \mathcal{U} of X , we call $(\mathcal{U} \setminus \{\emptyset\}, \{(U, V) \in \mathcal{U}^2 \mid U \cap V \neq \emptyset\})$ the graph of \mathcal{U} and denote it by $G(\mathcal{U})$. A nearness space (X, μ) is called *connected*, provided $G(\mathcal{U})$ is connected for every $\mathcal{U} \in \mu$. If the induced topological space $(X, \tau(\mu))$ is connected, then so is (X, μ) , but not vice versa. A nearness space (X, μ) is called *regular*, provided for every $U \in \mu$ there exists a *uniform refinement* $V \in \mu$, which means that for every $V \in \mathcal{V}$ there exist $U \in \mathcal{U}$ and $W \in \mu$ with $\text{star}(V, W) \subset U$.

Let (X, μ) be a nearness space and let Y be a subset of X . For each $U \in \mu$ the set $\mathcal{U}_Y = \{U \cap Y \mid U \in \mathcal{U}\}$ is a cover of Y . Moreover $\mu_Y = \{\mathcal{U}_Y \mid \mathcal{U} \in \mu\}$ is a nearness structure on Y . The pair (Y, μ_Y) is called the *nearness subspace* of (X, μ) , determined by Y . A subset Y of X is called *connected* in (X, μ) , provided (Y, μ_Y) is connected.

PROPOSITION 3. A connected nearness space is orderable iff it is suborderable.

THEOREM 1. If a connected, regular T_1 -nearness space (X, μ) has a base β , such that each $U \in \beta$ consists of connected subsets of (X, μ) and has an orderable graph $G(\mathcal{U})$, then (X, μ) is orderable.

PROOF. (0) Convenient assumptions. If X contains at most one element, the result is trivially true. Otherwise let a and b be two different fixed elements of X . Since (X, μ) is regular T_1 , we may assume, without loss of generality, that $b \notin \text{star}(\text{star}(\text{star}(a, U), U)U)$ for every $U \in \beta$. We may further assume $\emptyset \notin U$ for each $U \in \beta$.

(1) Construction of a compatible order. Each $U \in \beta$ can, due to the orderability of $G(U)$, be written in the form $U = \{U_n \mid n \in C_U\}$, where C_U is a convex subset of \mathbb{Z} , such that

$$U_n \cap U_m \neq \emptyset \Leftrightarrow |n-m| \leq 1$$

and

$$\max\{n \mid a \in U_n\} + 2 < \min\{n \mid b \in U_n\}$$

hold.

Next, for each $U \in \beta$, define a relation $\underset{U}{<}$ on X by

$$x \underset{U}{<} y \Leftrightarrow (\max\{n \mid x \in U_n\} + 1 < \min\{n \mid y \in U_n\}).$$

Finally,

$$x \leq y \Leftrightarrow (x = y \text{ or } \exists U \in \beta, x \underset{U}{<} y)$$

defines a relation \leq on X .

(2) \leq is a linear order relation on X . First we show that for elements U and V of β , such that U refines V , the implication $x \underset{V}{<} y \Rightarrow x \underset{U}{<} y$ holds. Since $x \underset{V}{<} y$, we have $y \notin \text{star}(\text{star}(x, V), V)$. Hence $y \notin \text{star}(\text{star}(x, U), U)$, which implies that exactly one of the statements $x \underset{U}{<} y$ or $y \underset{U}{<} x$ holds. Assume the former to be false. Then $y \underset{U}{<} x$ holds. Let $m = \min\{n \mid y \in V_n\}$, $\ell = \min\{n \mid b \in V_n\}$ and $k = \max\{n \mid a \in V_n\}$. Since $b \notin \text{star}(\text{star}(a, V), V)$ we have $k \leq m-2$ or $m+2 \leq \ell$.

Case 1. $k \leq m-2$. This implies $a \underset{V}{<} y$, hence $a \underset{U}{<} y$ or $y \underset{U}{<} a$ as above.

Case 1.1. $k \leq m-2$ and $a \underset{U}{<} y$. This contradicts the connectedness of V_{m-2} in (X, μ) , since we have (with $y \in U_i$):

- (a) $U_i \cap V_{m-2} = \emptyset$;
- (b) there exists $j < i$ with $U_j \cap V_{m-2} \neq \emptyset$, since $a \in U\{V_s \mid s \leq m-2\}$, $y \in V_{m-2}$ and $a \underset{U}{<} y$;
- (c) there exists $j > i$ with $U_j \cap V_{m-2} \neq \emptyset$, since $y \in V_{m-2}$, $x \in U\{V_s \mid s \leq m-2\}$ and $y \underset{U}{<} x$.

Case 1.2. $k \leq m-2$ and $y \underset{U}{<} a$. This contradicts the connectedness of V_{k+2} in (X, μ) , since we have (with $a \in U_i$):

- (a) $U_i \cap V_{k+2} = \emptyset$;

- (b) there exists $j > i$ with $U_j \cap V_{k+2} \neq \emptyset$, since $a \in V_k$, $b \in U\{V_s \mid s \geq k+2\}$ and $a < b$;
 (c) there exists $j < i$ with $U_j \cap V_{k+2} \neq \emptyset$, since $y \in U\{V_s \mid s \geq k+2\}$, $a \in V_k$ and $y < a$.

Case 2. $m+2 \leq \ell$. Then $b \notin \text{star}(y, V)$. Hence $b \notin \text{star}(y, U)$. So, if $y \in U_p$ and $b \in U_q$, we have either $p < q$ or $q < p$.

Case 2.1. $m+2 \leq \ell$ and $q < p$. This contradicts the connectedness of $V_{\ell-2}$ in (X, μ) , since we have:

- (a) $U_q \cap V_{\ell-2} = \emptyset$;
 (b) there exists $i < q$ with $U_i \cap V_{\ell-2} \neq \emptyset$ since $a \in U\{V_s \mid s \leq \ell-2\}$, $b \in V_\ell$ and $a < b$;
 (c) there exists $i > q$ with $U_i \cap V_{\ell-2} \neq \emptyset$, since $b \in V_\ell$, $y \in U\{V_s \mid s \leq \ell-2\}$ and $y \in U\{U_t \mid t > q\}$.

Case 2.2. $m+2 \leq \ell$ and $p < q$. Then $x \underset{U}{<} y$ implies $x \underset{U}{<} b$. Hence $x \underset{U}{<} b$ or $b \underset{U}{<} x$ as above.

Case 2.2.1. $m+2 \leq \ell$ and $x \underset{U}{<} b$. This contradicts the connectedness of V_m in (X, μ) , since we have (with $x \in U_i$):

- (a) $U_i \cap V_m = \emptyset$;
 (b) there exists $j > i$ with $U_j \cap V_m \neq \emptyset$, since $x \in U\{V_s \mid s < m\}$, $b \in U\{V_s \mid m < s\}$ and $x \underset{U}{<} b$;
 (c) there exists $j < i$ with $U_j \cap V_m \neq \emptyset$, since $y \in V_m$ and $y \underset{U}{<} x$.

Case 2.2.2. $m+2 \leq \ell$, $p < q$ and $b \underset{U}{<} x$. This contradicts the connectedness of V_m in (X, μ) , since we have:

- (a) $U_q \cap V_m = \emptyset$;
 (b) there exists $i > q$ with $U_i \cap V_m \neq \emptyset$, since $b \in U\{V_s \mid s > m\}$, $x \in U\{V_s \mid s < m\}$ and $b \underset{U}{<} x$;
 (c) there exists $i < q$ with $U_i \cap V_m \neq \emptyset$, since $p < q$ and $y \in U_p \cap V_m$.

Hence the assumption $y \underset{U}{<} x$ leads to a contradiction. Therefore we have $x \underset{U}{<} y$. Since any two members of β have a common refinement in β , and since $\underset{U}{<}$ is obviously transitive and antisymmetric, the above implies that \leq is an order relation on X . Since (X, μ) is a regular T_1 -nearness space this order relation is linear.

(3) (X, \leq, μ) is an ordered nearness space. According to the above proposition it remains to show that μ has a base consisting of covers, whose elements are intervals in (X, \leq) . In general, the given base β does not have

this property. Denote by \tilde{A} the convex hull of A in (X, \leq) , and define $\tilde{U} = \{\tilde{U} \mid U \in \mathcal{U}\}$ and $\tilde{\beta} = \{\tilde{U} \mid U \in \beta\}$. It remains to show that $\tilde{\beta}$ is a base for μ . Since, by regularity, μ has a base consisting of closed covers (with respect to the topology $\tau = \tau(\mu)$), the latter follows from:

(a) $A \subset \tilde{A} \subset \text{cl}_{(X, \tau)} A$ for each connected set A in (X, μ) .

To show (a), assume it to be wrong. Then there exist a connected set A in (X, μ) and an $x \in \tilde{A} \setminus \text{cl}_{(X, \tau)} A$. Hence there exist $a \in A$, $b \in A$, and $U = \{U_n \mid n \in \mathbb{C}_U\} \in \beta$ with $a \underset{U}{<} x \underset{U}{<} b$ and $\text{star}(x, U) \cap A = \emptyset$. If $a \in U_n$, $x \in U_m$ and $b \in U_k$, then $n < m < k$ and A meets U_n and U_k but not U_m , contradicting the connectedness of A . Consequently, (a) holds and $\tilde{\beta}$ is a base for μ . \square

REMARK. The condition, given in the above theorem for the orderability of connected, regular T_1 -nearness spaces, seems very natural. Nevertheless it is not necessary, as shown by HUSEK [6]. If slightly weakened, it is no longer sufficient, as shown by the following example.

EXAMPLE. Let $X = (\{0\} \times [-1, 1]) \cup \{(x, \sin \frac{1}{x}) \mid x \in [0, 1]\}$ and let μ be the uniform structure induced on X by the Euclidean metric on \mathbb{R}^2 . Then (X, μ) is a connected, regular T_1 -nearness space, such that μ has a base consisting of members with orderable graphs, but (X, μ) is not orderable.

THEOREM 2. For a connected, uniform T_1 -space (X, μ) the following are equivalent:

- (1) (X, μ) is orderable;
- (2) μ has a base, each of whose members U consists of connected subsets of (X, μ) and has an orderable graph $G(U)$.

PROOF. The implication (2) \Rightarrow (1) follows from Theorem 1. The reverse implication (1) \Rightarrow (2) follows immediately from a theorem of HUSEK [6], stating that the large uniform dimension of any orderable uniform space is at most 1. \square

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ORDERABILITY AND SUBORDERABILITY RESULTS
FOR TOTALLY DISCONNECTED SPACES

by

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See [21] in this volume for the basic definitions.

It is shown in [17] that orderability and suborderability theorems for classes for totally disconnected spaces could lead to much more general results. A totally disconnected subset U is chosen from a space X whose components each have at most two boundary points as follows. Suppose K is a component of X . (1) If K is a singleton or open component of X , then choose one point from K to be in U . (2) If K is a nondegenerate nonopen component of X , then choose two points from K , including its boundary points, to be in U . Then X is suborderable iff: (1) Each component of X is orderable, (2) the set of cut points of each component of X is open, (3) each component of X has base of clopen neighbourhoods, and (4) U admits a suborder \leq such that any two points selected from the same component of X are adjacent with respect to \leq . Note condition (1) is topological since there are many good topological characterizations of connected orderable spaces.

Even for some nice fairly narrow classes of totally disconnected spaces there are difficult orderability problems. For example in [15] it was conjectured (reappearing in the problems section of [24]) that orderable is equivalent to monotone normality ([6]) for compact, separable, totally disconnected spaces. The problem is still open.

One should always question the usefulness of a result equating the (sub) orderability of a class of spaces with some other condition. That is, are there spaces for which it is easier to determine whether they satisfy the given condition than to determine their (sub)orderability? Be particularly wary if a (sub)order is transparent from the given condition. The results mentioned in this survey are useful to varying degrees. So in some cases more definitive results are desirable.

Often in (sub)orderability results for a totally disconnected space X a useful condition is found which implies there is a family $\{U_\alpha \mid \alpha \in \kappa\}$, κ

some ordinal, of open partitions of X such that U_β refines U_α for $\alpha \in \beta$ (and often $\{U_\alpha \mid \alpha \in \kappa\}$ is an open base for X). (One might want to allow some U_α to cover only an open subset of X .) Problems usually arise at stages U_α for α a limit ordinal. If such problems can be solved, often a (sub)order is induced on X by induction totally ordering by $<_\alpha$ each U_α such that among other things if $\alpha \in \beta$, $U <_\alpha V$, $U', V' \in U_\beta$, $U' \subseteq U$, and $V' \subseteq V$, then $U' <_\beta V'$. For metric spaces it turns out that one can let $\kappa = \omega_0$ (so the limit stage problem does not arise) and the diameter of each member of U_n is less than $1/n$.

The earliest orderability result of which this author is aware is a 1910 article by L.E.J. BROUWER ([3]) characterizing the Cantor set as a compact, perfect, totally disconnected metric space. The proof employs the techniques described in the above paragraph except no order relation is considered.

SIERPIŃSKI ([25]) in 1920 showed that every countable dense-in-itself metric space is homeomorphic to the rational numbers.

In the same year MAZURKIEWICZ and SIERPIŃSKI ([11]) proved that any compact, countable, metric space is homeomorphic to a well ordered set. Moreover they showed that if $P^{(\alpha)}$ is the last nonempty derived set of P and $|P^{(\alpha)}| = n$, then P is homeomorphic to the ordinal space $(\omega^\alpha \cdot n) + 1$.

A *punctiform* is a space that contains no nondegenerate continua. In 1921 SIERPIŃSKI ([26]) showed that a separable metric punctiform is suborderable iff it is 0-dimensional.

The irrational numbers were characterized in 1928 by ALEXANDROFF and URYSOHN ([1]) as a topologically complete zero-dimensional separable metric space such that no nonempty open set has compact closure.

I.L. LYNN ([8], [9]) in 1961 showed that every zero-dimensional separable metric space is orderable. The following year in his doctoral dissertation H. HERRLICH ([4]) proved that a totally disconnected metric space X is orderable iff $\text{Ind } X = 0$ (also see [5]). Much later the technique of Herrlich's proof was modified in [17] to characterize all suborderable metric spaces utilizing the result mentioned in the second paragraph of this survey.

In 1972 J.W. Baker ([2]) characterized the compact ordinal spaces. If λ is the least ordinal α such that the α^{th} derived set $X^{(\alpha)}$ of a space X is finite and $n = |X^{(\alpha)}|$, then (λ, n) is called the *characteristic* of X . A space is *scattered* if each of its non-empty subspaces has an isolated point. A *linearly ordered base (lob)* of a point $x \in X$ is a neighbourhood base of x which is linearly ordered by reverse inclusion; X satisfies property (D) if each point of X has a lob $\{U_\alpha\}_{\alpha < \tau}$ of clopen sets such that for each limit ordinal $\beta < \tau$ $(\bigcap_{\alpha < \beta} U_\alpha) - U_\beta$ contains at most one point. Baker showed that a

compact scattered space with property (D) and characteristic (λ, n) is homeomorphic to $(\omega^\lambda \cdot n) + 1$. Note compactness is necessary here since the space $\omega_1 \times (\omega_0 + 1)$ is countably compact, scattered, and satisfies property (D), but X is not suborderable (nor is it monotonically normal).

In characterizing all metrizable orderable topological groups M. VENKATARAMAN, M. RAJAGOPALAN, and T. SOUNDARARAJAN ([27]) showed that non-metrizable ones must be totally disconnected. P. NYIKOS and H.-C. REICHEL in 1975 ([14]) showed a nonmetrizable topological group is orderable iff the identity element has a totally ordered local base. Recently M. HUŠEK and REICHEL ([7], [22]) have generalized some of these ideas in their study of *linearly uniformizable* spaces, those spaces whose topology can be derived from a base for a uniformity which is linearly ordered by inclusion. A space is *non-archimedean* if it has a base every pair of elements of which are disjoint or one contains the other. Every nonmetrizable linearly uniformizable space is non-archimedean which in turn is suborderable, hereditarily paracompact, and strongly zerodimensional. A space X is *strongly suborderable* if it admits a suborder such that the pseudogap points are isolated. A set $A \subseteq X$ is a G_κ -set for some cardinal κ iff $A = \bigcap \{U_\alpha \mid \alpha \in \kappa\}$ for some open family $\{U_\alpha \mid \alpha \in \kappa\}$. For X nondiscrete define $\text{ad}(X)$ to be the first ordinal κ such that $\bigcap \{U_\alpha \mid \alpha \in \kappa\}$ is not open for some open family $\{U_\alpha \mid \alpha \in \kappa\}$. The pseudocharacter $\psi(\Delta X)$ of the diagonal of X is the least cardinal κ such that ΔX is a G_κ -set in $X \times X$. Then a non-discrete Hausdorff space X is linearly uniformizable if $\psi(\Delta X) = \text{ad}(X) = \kappa$, X is strongly suborderable, and the set of non-isolated points of X is a G_κ in X . If X is nonmetrizable or $\text{Ind } X = 0$, these conditions are also necessary for a linearly uniformizable space. A linearly uniformizable non-metrizable space X is orderable iff there is a family $\{U_\alpha \mid \alpha \in \kappa\}$ of open partitions of X such that

- (1) $\bigcup \{U_\alpha \mid \alpha \in \kappa\}$ is an open base of X ;
- (2) if $\alpha \in \beta \in \kappa$ then U_β refines U_α , and;
- (3) for β a limit $U_\beta = \bigcap \{U_\alpha \mid \alpha \in \beta\}$ and $S(U_\beta) = \bigcup \{S(U_\alpha) \mid \alpha \in \beta\}$ where $S(U_\alpha) = \bigcup \{K \mid K \text{ is a finite member of } U_\alpha\}$.

A major orderability problem is to determine those suborderable spaces that are orderable. M.E. RUDIN ([23]) satisfactorily solved this for subsets of the real line. A solution of the general problem was also given in [23] but it contained a very complicated last (third) condition. Conditions one and two prevent the obvious counterexamples and allow a reordering of a subordered space that eliminates some of the pseudogap points. To eliminate the remaining "hard core" pseudogap points these points are put into τ subsets

$\{M_\alpha \mid \alpha \in \tau\}$, where τ is a limit ordinal. Then the space is reordered in τ stages eliminating the pseudogap points of M_α at the α^{th} stage. The problem is that although at each stage the new order is an admissible suborder if its predecessor is, the topology could be destroyed when passing to a limit stage. To avoid this problem, reordering about a pseudogap point should be done in a small enough neighbourhood and that can be done if a point - a friend - can be chosen close to the given point. Closeness is in the sense that the cluster points of any set of hard core pseudogap points coincide with those of its set of friends. Condition 3 allows closeness. In [17] it was shown that all suborderable metrizable spaces satisfy closeness.

It was suggested to the author that if a subordered space had enough isolated points then all pseudogap points could be eliminated by throwing sequences of order type ω_0 or ω_0^* at these points. Recalling Rudin's closeness condition the author considered the subset X of the lexicographic product $[0,1] \times \{0,1,2\}$ whose points have second coordinate 0 or 1. The set of isolated points are the pseudogap points and they are the points with second coordinate 1. If the space were orderable, each $(a,1) \in X$ would have as a friend its immediate predecessor or immediate successor with respect to an admissible order. But X does not satisfy closeness. This example helped motivate [21] in this volume.

The results to this point dealt with linearly uniformizable and ordinal spaces. These are lob spaces. The nonlob spaces can cause problems.

The *length* of a scattered space is the least ordinal α such that the α^{th} derived set is empty. In 1976 the author announced ([16], [18]) that a suborderable scattered space of countable length is orderable and hereditarily paracompact. In the announcement it was conjectured that every suborderable scattered space is orderable. For length a countable limit ordinal α , α was mapped onto ω_0 and via this map an order was induced by introducing partial orders in ω_0 stages. This avoided passing through limit stages. For scattered spaces of uncountable length, passing through a limit stage is unavoidable. For a long time this was a stumbling block. In discussions with R. Telgársky in 1980 it became clear that paracompactness is a key to pass through limit stages, since it allowed a decomposition of the space into open subsets of length less than that of the space. But such a decomposition cannot be done on spaces such as ω_1 . A *left gap* in a subordered space is a nonempty clopen convex subset which is cinitial in X and has no maximum. A left gap is a *left Q-gap* if there is a discrete set cofinal in the gap. A left gap A in a space of length α is a *highest level gap* if $A^{(\xi)}$ is cofinal

in A for all $\xi < \alpha$. A left gap is *covered* by a set if the set contains a terminal segment in the gap. Analogous definitions are given for right gaps. The author discovered the desired decomposition could be obtained away from the non-Q-gaps and even at the lower level non-Q-gaps by covering them with sets of length less than α and using a paracompactness-like argument to obtain the desired decomposition. So the problem was at the highest level non-Q-gaps; but considering them as points in the growth of an ordered compactification they (surprisingly) turned out to be discrete. So the space could be decomposed into open sets each of which contains at most one highest level non-Q-gap, and this gap is an endgap. Hence, the space becomes manageable. This is the basis of the proof ([19]) of the conjecture.

A *weak selection* for a space X is a continuous map $s: X^2 \rightarrow X$ such that for all $x, y \in X$, $s(x, y) = s(y, x)$ and $s(x, y) \in \{x, y\}$. Extending a result of E. Michael for continua and an unproved claim by G.S. Young for compact zero-dimensional spaces J. VAN MILL and E. WATTEL ([12]) recently showed that a compact space is orderable iff it has a weak selection.*)

Recently, G. MORAN ([13]) gave a complicated proof that a Hausdorff space is homeomorphic to a compact scattered orderable space iff it is the 2 to 1 continuous image of a compact ordinal.

In a letter Nyikos pointed out that Moran's result can be extended to show that the closed 2 to 1 continuous image of a subspace of a well ordered space is suborderable and hence by [19] is orderable.

After hearing Moran's result, recalling Baker's theorem and taking into account that compact scattered orderable spaces need not be lob spaces, the author proved ([20]) that a compact scattered space X is orderable iff

(1) for each $x \in X$ there is a neighbourhood subbase $\{L_\alpha\}_{\alpha < \tau} \cup \{R_\alpha\}_{\alpha < \gamma}$ consisting of two decreasing nests of clopen sets (these nests may be identical) such that for every limit ordinal β , $\bigcap_{\alpha < \beta} L_\alpha$ has one boundary point if $\beta < \tau$ and $\bigcap_{\alpha < \beta} R_\alpha$ has one boundary point if $\beta < \gamma$; (2) there is no subset Y of X which can be written as $Y = \bigcup \{X_s : s \in S\}$ where the X_s 's are pairwise disjoint, S a stationary set of some uncountable regular ordinal, and for each $s \in S$, X_s is homeomorphic to $(\omega_0 + 1) + \alpha^*$ where α is an uncountable regular ordinal, such that if $x_s \in X_s$ is the point corresponding to ω_0 under the

(Editor's note):

*) More recently, van Mill and Wattel have proved that a Tychonoff space X is suborderable if and only if there is a weak selection $s: X \times X \rightarrow X$ such that if U is open and $x \in U$ then some open V has $x \in V \subset U$ and satisfies $\forall y \in V, \forall z \in X - U, s(y, z) = y \Leftrightarrow s(x, z) = x$.

homeomorphism, then $\{x_s\}_{s \in S}$ is homeomorphic to S . There are obvious ways to strengthen and simplify condition 1 to obtain a sufficient but not necessary condition for X to be orderable. The proof is short, straightforward, and with a little extra effort Moran's result follows. So now there are two characterizations of orderable compact scattered spaces. One is concisely stated but difficult to apply. The other is useful but doesn't look pretty.

A general survey till 1972 of orderability and suborderability results can be found in the historical chapter of [15]. A nice recent survey of ordered spaces appears in [10].

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NON-ORDERABILITY OF SUBORDERABLE
SPACES WITH MANY PSEUDOGAPS

by

Steve Purisch & Evert Wattel

The aim of this note is to use various cardinal functions on particular sets in the suborderable space to show that such a space is not orderable. In particular, the number of pseudogaps should not exceed the density of the space, or the maximum of the density of the derived set, the spread of the space and the number of convexity components of isolated points which have non-compact closures.

This note uses the techniques of [3] and [4] and its results are related to the theorem of HART [1] and the analysis of M.E. RUDIN of orderable subsets in the reals [5]. Our notation is based on HERRLICH's book [2].

This note emerged in the stimulating environment of the NATO workshop on ordered spaces at Lubbock and the authors are especially grateful to Brian M. Scott for his interesting discussions and helpful comments.

1. BASIC DEFINITIONS

DEFINITION 1.1. A subset A of an ordered set (X, \leq) is called *order convex* iff for every two points $a \leq b$ in A we have that $\{c \mid a \leq c \leq b\} \subset A$. A maximal order convex subset C of a set A is called a *convexity component* of A . A Hausdorff topological space (X, \mathcal{T}) with an order relation \leq is called a *subordered space (GO-space)* iff it has an open base for the topology consisting of order-convex sets. Then X is said to be subordered with respect to (w.r.t.) the order \leq . A space which can be supplied with a compatible suborder is called *suborderable*.

DEFINITION 1.2. Let X be a subordered space w.r.t. the order \leq . Then $p \in X$ is called *left isolated* (resp. *right isolated*) if the set $\{x \mid p \leq x\}$ (resp. $\{x \mid x \leq p\}$) is open in X . The collection of left isolated points is called J^L , the collection of right isolated points is called J^R and the members of $J = J^L \cup J^R$ are called *jump points* of X . A pair of adjacent jump points in

X is called a *jump*. A point $p \in J^L$, ($p \in J^R$) is called a *left pseudogap point* (resp. *right pseudogap point*) if $\{x \mid x \leq p; x \neq p\}$ has no maximum (resp. $\{x \mid p \leq x; x \neq p\}$ has no minimum). The collection of left pseudogap points is called P^L , the collection of right pseudogap points is called P^R , and the members of $P = P^L \cup P^R$ are called *pseudogap points*.

2. A SPECIAL CASE

THEOREM 2.1. *Let (X, \leq) be a subordered space and assume that $<$ is also a compatible suborder on X . Let P_{\leq} (resp. $P_{<}$) be the collection of pseudogap points w.r.t. \leq (resp. $<$). Then we have that $|P_{\leq} \setminus P_{<}| \leq d(X)$, in which $d(X)$ denotes the density of X . Therefore if (X, \leq) has more than $d(X)$ pseudogaps then X is not orderable.*

PROOF. Let D be a dense subset of X of cardinality δ . Then D contains all isolated points of X .

First of all we show that almost all jumps in \leq are also jumps in $<$. Define for every $d \in D$

$$F_d^- = \{x \in X \mid x < d\} \quad \text{and} \quad F_d^+ = \{x \in X \mid d < x\};$$

then those sets are clopen in $X \setminus \{d\}$ for every $d \in D$. Now F_d^- and F_d^+ are open and can be partitioned into convexity components w.r.t. \leq . Let C_d^- and C_d^+ denote the collection of \leq -convexity components of F_d^- and F_d^+ respectively. Since the cellularity of a space is not greater than its density we have $|C_d^- \cup C_d^+| \leq \delta$. If we let d run through D we obtain

$$\left| \bigcup_{d \in D} (C_d^- \cup C_d^+) \right| = \delta$$

Moreover, if a pair of points $a, b \in X \setminus D$ do not constitute a jump in $(X, <)$, then there is a point $d \in D$ such that either $a < d < b$ or $b < d < a$. This means that either $a \in F_d^-$ and $b \in F_d^+$ or $a \in F_d^+$ and $b \in F_d^-$. If, in addition a and b constitutes a jump in (X, \leq) , then a and b are extremal points in the members of C_d containing them. Then since

$$\left| \bigcup_{d \in D} (C_d^- \cup C_d^+) \right| = \delta,$$

there are at most δ jumps in (X, \leq) which are not jumps in $(X, <)$. In the same

way there are at most δ jumps in $(X, <)$ which are not jumps in (X, \leq) .

Next we define an equivalence relation \cong on X in the following way:
 $a \cong b$ iff $\{a, b\}$ is a jump in (X, \leq) as well as in $(X, <)$ and neither a nor b is isolated. Now the space $(X, \leq)/\cong$ has at most δ jumps and the same holds for $(X, <)/\cong$. Moreover, if we assume that $(X, <)$ is ordered then $(X, <)/\cong$ also has at most δ jump points, since it has no pseudogaps, so its weight is δ . However, $(X, \leq)/\cong$ has more than δ pseudogaps and its weight is $|P| > \delta$. This is a contradiction, since $(X, \leq)/\cong$ and $(X, <)/\cong$ are two homeomorphic copies of the same space which differ only in their additional order structure.

If $(X, <)$ is only a suborderable space, then we define $X' = X/\cong$. Then X' is sub-orderable w.r.t. \leq and $<$. We define (X'', \leq) to be a subordered space on X' which has a subbase: all convexity components of ordered open sets in (X', \leq) and in $(X', <)$. Then (X'', \leq) has weight δ . If p is a pseudogap point in (X', \leq) but not in $(X', <)$ then either making $(\leftarrow, p]$ or making $[p, \rightarrow)$ open strengthens the topology of $(X'', <)$. This cannot be the case for a $<$ -convexity component of $(\leftarrow, p]$ which does not contain p . This means that it changes the convexity components of $(\leftarrow, p]$ w.r.t. $<$, which means that p has to be a pseudogap point of $(X', <)$. This shows that all but at most δ of the pseudogap points of \leq are also pseudogap points in $<$, which finishes the theorem.

The following example is a suborderable space which fails to be orderable, although the number of pseudogaps is equal to the density of the derived set. The technique of the proof which shows that this example is not orderable will be generalized in the proof of the main Theorem 3.2.

EXAMPLE 2.2. Let $A = [0, 1] \times \{0, 1, 2\}$ lexicographically, and let

$$X = \{(a, b) \in A \mid b = 0 \text{ or } b = 1\}.$$

Then X is not orderable. (Note that X does not have a G_δ diagonal.)

PROOF. Suppose that X were orderable. Let \leq be an admissible order on X . Each $(a, 1) \in X$ which is not an endpoint of (X, \leq) has an immediate predecessor and an immediate successor with respect to \leq . For each $a \in [0, 1]$ where $(a, 1)$ is not an endpoint of (X, \leq) define $(a, 1)'$ to be the immediate successor of $(a, 1)$ if the first coordinate of the immediate successor does not equal a and otherwise define $(a, 1)'$ to be the immediate predecessor of $(a, 1)$. Define a' to be the first coordinate of $(a, 1)'$. For each positive integer n let

$$S_n = \{a \in [0,1] \mid 1/n < |a-a'|\}.$$

Then for some n_0 the set S_{n_0} is uncountable. So there is a strictly increasing sequence $\{a_i\}_{i=1}^{\infty}$ in S_{n_0} with respect to the usual order on $[0,1]$. Then $a_i \rightarrow a^- \in [0,1]$ with the usual topology on $[0,1]$. So $(a_i, 1) \rightarrow (a^-, 0)$ (with the subspace topology induced by the lexicographical order topology on A). But $a_i' \neq a^-$ with the usual topology on $[0,1]$, and so $(a_i, 1)' \neq (a^-, 0)$. However under the order topology on X induced by \leq we have that $(a_i, 1)' \rightarrow (a^-, 0)$ since $(a_i, 1) \rightarrow (a^-, 0)$. So X is not orderable.

3. THE MAIN THEOREM

3.1. Notational conventions

Let (X, \leq) be a subordered space. Then the derived set will be denoted by N , the set of isolated points will be denoted by R and D will be a dense subset of the subspace N with cardinality $\delta = d(N)$. The collection of all closures w.r.t. X of convexity components of R will be denoted by C , and we define

$$C_c = \{C \in C \mid C \text{ is compact}\} \quad \text{and} \quad C_n = C \setminus C_c.$$

The cardinality $|UC_n|$ will be denoted by ν , and $C(p)$ will be the closure w.r.t. X of the convexity component of p in R for every isolated point p . The least upper bound on the cardinalities of closed discrete sets in X will be denoted by κ .

THEOREM 3.2. *Let X be a subordered space with the property that the cardinality of the pseudogaps ψ is larger than:*

- (i) *The density δ of the derived set N ;*
- (ii) *The cardinality ν of the collection of isolated points in convexity components of R with non-compact closure; and*
- (iii) *The least upper bound κ of the cardinalities of closed discrete subsets of X .*

Then X cannot be orderable.

PROOF. To derive a contradiction we subdivide the collection P of pseudogap points into several subcollections. We show that some of those subcollections are small. For the two remaining subcollections we proceed as follows: We

assume that the space admits an order. From that order we construct for almost all pseudogap points in the collection a "friend" which is a close point in that order in the sense that under the topology generated by that order the set of cluster points of any collection of pseudogap points coincides with the set of cluster points of the corresponding collection of "friends". (Compare with condition 3 of the theorem on page 389 of [5].) Finally we construct an open interval in the old order which contains a collection of pseud-gap points clustering to a point in this interval, but the interval is disjoint from the corresponding collection of "friends". This will contradict the concept of friendship.

For $p \in R$ let $C(p)$ be the closure of the convexity component of R containing p .

Define:

$$P_1 = \{p \in N \cap P \mid p \text{ is an isolated point of the subspace } N\};$$

$$P_2 = \{p \in N \cap P \setminus P_1 \mid \exists q \leq p: (q,p) \cap N = \emptyset \text{ or } \exists q \geq p: (p,q) \cap N = \emptyset\};$$

$$P_3 = (P \cap N) \setminus (P_1 \cup P_2);$$

$$P_4 = \{p \in P \cap R \mid C(p) \text{ is not compact}\};$$

$$P_5 = \{p \in P \cap R \mid C(p) \text{ is compact}\}.$$

Clearly,

$$|P_1| \leq \delta.$$

Since for every $p \in P_2$ we have that p is a cluster point of N it follows that p is not a cluster point of the interval (p,q) , (resp. (q,p)), because p is a pseudogap point. So we also have that (p,q) cannot have a minimum (resp. (q,p) cannot have a maximum), and thus p is adjacent to a non-compact convexity component of R . Therefore we have

$$|P_2| \leq \nu.$$

From the definition of ν it is clear that

$$|P_4| \leq \nu.$$

We conclude that at least one of the two collections P_3 and P_5 must have cardinality ψ .

Case 1. $|P_3| = \psi$. Assume that $<$ is an order for X which generates the topology of X , then the subset N is again a suborderable subspace of X . Let P_0 be the collection of all members of P_3 which are still pseudogap points of N w.r.t. the new ordering $<$. According to Theorem 2.1 we obtain that $|P_3 \setminus P_0| \leq \delta$ and therefore $|P_0| = \psi$. Let $p \in P_0$ be a right pseudogap point in N w.r.t. $<$. Then the collection $\{n \in N \mid p < n\}$ has no minimum but it is closed in N w.r.t. the order $<$. Therefore there is an interval (p, q) which is disjoint from N and which starts at p . Choose a point $f(p)$ from this interval; then $f(p) \in R$. We can do a similar thing if p is a left pseudogap point of N w.r.t. $<$ and obtain a mapping from P_3 into R . Note that for each triple of points $p_1 < p_2 < p_3$ in P_0 we have that $p_1 < f(p_2) < p_3$.

Next we return to the order \leq . Let $p \in P_0$, then

$$p = \sup\{n \in N \mid n \leq p, n \neq p\} = \inf\{n \in N \mid p \leq n, p \neq n\}.$$

Let D be dense in N and let I be the collection of all open intervals with endpoints in D . So for every $p \in P_0 \subset P_3$

$$\cap\{I \in I \mid p \in I\} = \{p\}.$$

We assign an interval $I(p)$ to p with endpoints in D such that $p \in I(p)$ and $f(p) \notin I(p)$. We do this for all $p \in P_0$ and we choose ψ times an open interval with endpoints in D . Since $|I| = \delta$, and $\delta \cdot \kappa < \psi$ there is an $I' \in I$ which is assigned to more than κ members of P_0 by the mapping $I(p)$. Let

$$P'_0 = \{p \in P_0 \mid I' \text{ is assigned to } p\},$$

and let $d_0 \leq d_1$ be the endpoints of I' . We consider all closed intervals $[n_0, n_1]$ with endpoints in $I' \cap D$. Since between every pair of points of P_3 there is at least one point of D , we conclude that at most two points of P'_0 do not belong to the union of δ many closed intervals

$$\cup\{[n_0, n_1] \mid n_0, n_1 \in D \cap I'\},$$

namely, one smaller and one larger than all members of the union. Therefore there must be a closed interval I'' and I' which contain more than κ members of $P'_0 \subset P_0$, and we can choose a cluster point q of P'_0 . Clearly $q \in I''$. The collection $\{f(p) \mid p \in P'_0\}$ cannot cluster inside I' since I' is open and $f(p) \notin I'$ for $p \in P'_0$. However if we look in the order $<$, the collection P'_0 clusters to a point q iff there exists a monotonic well ordered sequence p_α which has q as a limit, and in this case the collection $f(p_\alpha)$ has the same limit q which is a contradiction. This finishes Case 1.

Case 2. $|P_5| = \psi$. We again assume that $<$ is an order which generates the topology of X . Let p be a member of P_5 . Now $C(p)$ is compact, and this means that $C(p)$ is either finite or it contains at most one cluster point, which is the limit of an ordinary sequence. We subdivide P_5 according to the possibilities for the closure of the convexity component $C(p)$ of p w.r.t. R and \leq :

- $p \in P_a \Leftrightarrow C(p)$ is finite, not a singleton, and contains only one pseudogap point.
- $p \in P_b \Leftrightarrow C(p)$ is finite and $p \notin P_a$.
- $p \in P_c \Leftrightarrow C(p)$ is infinite and there exists a $q \in P_5$ such that $C(p) \neq \emptyset$ and $C(p)$ and $C(q)$ are adjacent.
- $p \in P_d \Leftrightarrow C(p)$ is infinite and every other $C(q)$ is disjoint from $C(p)$ and not adjacent to $C(p)$ for $q \in P_5$.

If $p \in P_c$ then between p and the point q such that either $C(p) \cap C(q) \neq \emptyset$ or $C(p)$ and $C(q)$ are adjacent, there is a unique limit point ℓ in $C(p)$. This point is isolated in the subspace N and so $|P_c| \leq \delta$.

There is moreover at most one pseudogap point p_0 in P_5 such that

$$\{x \mid x < p_0 \text{ and } x \notin C(p_0)\} = \emptyset$$

and at most one pseudogap point p_1 in P_5 such that

$$\{x \mid p_1 < x \text{ and } x \notin C(p_1)\} = \emptyset.$$

Let P_0 be the intersection of $P_5 \setminus \{p_0, p_1\}$ with a set containing P_a, P_d and precisely one point in P_b from each convexity component intersecting P_b . Clearly, $|P_0| = \psi$.

For every $p \in P_0 \cap P_d$ we define $\lambda(p)$ to be the unique limit point of $C(p)$ and for $p \in P_0 \cap P_a$ we define $\lambda(p)$ to be the unique point of N which is adjacent to $C(p)$ in the order \leq . Next we take a point p of P_0 and consider it in the order $<$. We define:

$$f(p) = \max\{x \in X \mid x < p \text{ and } x \notin C(p)\} \text{ iff this is not } \lambda(p),$$

$$f(p) = \min\{x \in X \mid p < x \text{ and } x \notin C(p)\} \text{ otherwise.}$$

Since we have omitted the two points p_0 and p_1 if they exist, the function f is well defined on P_0 and $f(p) \notin C(p)$ for every $p \in P_5$, but moreover, between p and $f(p)$ we can only have members of $C(p)$. As in Case 1 we have for every three points $p_2 < p_3 < p_4$ that $p_2 < f(p_3) < p_4$ because neither p_2 nor p_4 can be members of $C(p_3)$.

We again return to the order \leq . Let I be again the collection of all open intervals of X with endpoints in the dense set D of N of cardinality δ . Assume that p is a left pseudogap point of P_0 and that I_p is the collection of all $I \in I$ which contain p . We claim that $f(p) \notin \bigcap_p I_p$.

If $p \in P_a$ and $\lambda(p)$ is isolated in N (and hence contained in D) then $\{n \in N \mid n \leq p\}$ has no maximum and so

$$\bigcap\{(d, \lambda(p)) \mid d \in D \text{ and } d \leq p\}$$

is contained in $C(p)$. If $\lambda(p) \notin D$ then

$$\bigcap\{(d_1, d_2) \mid d_1 \leq p \text{ and } p \leq d_2 \text{ and } d_1, d_2 \in D\}$$

contains $C(p)$ and $\lambda(p)$ but nothing more. This means that $f(p) \notin \bigcap_p I_p$.

In the case that $p \in P_d$ a similar argument holds. If $p \in P_b$ then neither the set $\{n \in N \mid n \leq p\}$ has a maximum nor $\{n \in N \mid p \leq n\}$ has a minimum and we obtain that

$$\bigcap\{(d_1, d_2) \mid d_1 \leq p \text{ and } p \leq d_2 \text{ and } d_1, d_2 \in D\}$$

is contained in $C(p)$ which proves that $f(p) \notin \bigcap_p I_p$.

Again we can assign to every $p \in P_0$ an interval $I(p)$ which contains p but not the point $f(p)$ and we can repeat the arguments of Case 1 to show that there should be an open interval I' containing a closed interval I''

with a cluster point of a subset P'_0 of P_0 which is not a cluster point of $\{f(p) \mid p \in P'_0\}$. From there we again derive a contradiction. This proves the theorem. \square

3.3. REMARKS

Clearly the special Case 2.1 follows from the previous theorem since X has at most $d(X)$ isolated points and if $\psi > d(X)$ then of course $\psi > d(N)$. We have included it because the technique is so different.

This theorem admits generalizations of the following type: Require that the cardinality of either the set P_3 or the set P_5 is larger than both κ and δ in the current suborder on the space and then the space cannot be orderable.

Since all pseudogaps of X in Example 2.2 are of type P_5 our theorem implies immediately that this space is not orderable.

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SPACES WITH DENSE ORDERABLE SUBSPACES

by

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A space (X, τ) is *orderable* if there is a linear ordering on X whose induced order topology is τ . Old characterizations of the space Q (of rationals) will show that any first countable separable regular space has a dense subspace embeddable into Q . However, some unexpected classes (e.g. Nyikos' proto-metrizable spaces, see 2.1) or members of other classes also have dense orderable subspaces. The latter is especially true under various set-theoretic hypotheses for normal Moore spaces (3.4), finite products of nowhere separable Souslin lines (4.1), and the Stone- \check{C} ech remainder of a locally compact metric space (6.4).

The initial purpose of this paper was to survey the literature on the class of "spaces with dense orderable subspaces". However, we found the number of gaps in the theory large enough to warrant a research report. What we present is a combination of these two directions. With one exception, we sketch (or indicate) the method of proof of most new and some old results. The exception is in Section 1 where we develop the first characterization for being a space with a dense orderable subspace (1.3).

The paper is sectioned as follows: 0. fundamentals and conventions; 1. the characterization; 2. first countable and other lob spaces; 3. dense metrizable subspaces; 4. product spaces; 5. homeomorphic dense subspaces; 6. Stone- \check{C} ech remainders; 7. examples.

In order to decrease the number of references we have attempted to refer to recent texts and accessible surveys whenever feasible. In particular we make extensive references to the new *Surveys in General Topology* edited by G.M. Reed (Academic Press 1980). Other important surveys are [23] (for orderable spaces), [30] and [43] (for the theory of absolutes), and [27] and [28] (for "blood and guts" base axioms).

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Senior Postdoctoral Fellow during the completion of this article. We acknowledge [12] as our inspiration for considering this topic.

0. FUNDAMENTALS AND CONVENTIONS

In order to simplify our statements and proofs, *all spaces will be assumed infinite, Hausdorff, and completely regular*. However, most of the results can be stated in terms of, and are true for, the class of semi-regular spaces [13]. We use the following notations: "iff" means "if and only if"; \square is used to denote the end of a proof or a theorem not to be proved; ZFC (which we assume) means Zermelo-Frankel set theory with choice; $V=L$ is Gödel's constructible universe; CH is the Continuum Hypothesis; MA is Martin's axiom and C & I means "consistent with and independent of ZFC".

All ordinals and cardinals have the von Neumann definition and will be considered, where applicable, to have the order topology. The symbol $|X|$ is the cardinality of a set X and 2^κ is the cardinality of all subsets of κ . If α is an ordinal and X is a set then ${}^\alpha X$ is the set of functions from α to X . The domain of a function f is denoted $dom(f)$ and the restriction of f to a subset A of its domain is denoted by $f|A$. We use \subseteq (resp. \subset) to mean (proper) subset and $[0,1]$ is the unit interval. For a cardinal κ and space X , $\prod^\kappa X$ is ${}^\kappa X$ with the Tychonov product topology and projections π_α . The Stone- \check{C} ech compactification of X is βX .

0.1. (See [30] or [43]). For a space X and $A \subseteq X$, $int(A)$ and $cl(A)$ denote the interior and closure, respectively, of A in X . A set A is *regular-open* when $A = int(cl(A))$. The collection $\mathcal{R}(X)$, the family of all regular-open sets of X , is a complete Boolean algebra, and thus its Stone space, $S(\mathcal{R}(X))$ is a compact extremally disconnected (all regular-open sets are closed) space. The subspace of $S(\mathcal{R}(X))$ consisting of ultrafilters in $\mathcal{R}(X)$ converging in X is denoted by $\bar{E}(X)$ and is called the *absolute* of X . It is known that $\bar{E}(X)$ is the unique, up to homeomorphism, extremally disconnected pre-image of X under a perfect irreducible surjection.

0.2. A π -base for a space (X, τ) is a cofinal subset of the partially ordered set $(\tau - \{\emptyset\}, \supseteq)$. The π -weight of X is the least cardinal κ for which there exists a π -base of cardinality κ . It is known (see [6] or [43]) that for spaces X and Y , $(\mathcal{R}(X) - \{\emptyset\}, \supseteq)$ and $(\mathcal{R}(Y) - \{\emptyset\}, \supseteq)$ have order-isomorphic cofinal sets iff $\mathcal{R}(X)$ and $\mathcal{R}(Y)$ are isomorphic Boolean algebras iff $S(\mathcal{R}(X))$ and $S(\mathcal{R}(Y))$ are homeomorphic iff $\beta\bar{E}(X)$ ($= \bar{E}(\beta X)$) and $\beta\bar{E}(Y)$ are homeomorphic. When $\bar{E}(X)$ and $\bar{E}(Y)$ are homeomorphic, X and Y are said to be *co-absolute* or X is

said to be *co-absolute* with Y .

0.3. A *suborderable* space is a subspace of an orderable space. A *LOTS* (resp. *GO space*) is an orderable (resp. suborderable) space whose ordering we choose to recognize ([25]). Every GO-space has a π -base which is a *tree* (of regular-open convex sets [39]); i.e., a partially ordered set T in which the induced ordering on the set $t\downarrow$ of predecessors to t is well-ordered for each $t \in T$.

0.4. (See [21] or [28]). Suppose T is a tree. A *branch* of T is any maximal linearly ordered subset of T and $\mathcal{B}r(T)$ is the set of branches of T . For an ordinal α , the α 'th *level* and the α 'th *subtree* are, respectively, the sets

$$lv(T, \alpha) = \{t \in T : t\downarrow \text{ has order type } = \alpha\} \quad \text{and} \quad T\uparrow\alpha = \bigcup_{\beta < \alpha} lv(T, \beta).$$

The *height* of T is $h(T) = \inf\{\alpha : lv(T, \alpha) = \emptyset\}$. Considering the members of a given branch $B \in \mathcal{B}r(T)$ as basic nbhds of B , we find $\mathcal{B}r(T)$ is a space - the *branch space* of T . If each level of T is linearly ordered, $\mathcal{B}r(T)$ is to be given the induced lexicographic ordering. Observe that the order topology on $\mathcal{B}r(T)$ is the branch space topology whenever the level ordering of the immediate successors to each non-maximal $t \in T$ has no first or last element.

1. THE CHARACTERIZATION

In order to characterize " X has a dense orderable subspace" one need only re-formulate global characterizations (see [25] and [22]) of orderability; yet such formulations are, in general, too strong - they obscure properties intrinsic to denseness. One such property is given by

(*) *a dense subspace of a dense subspace is dense.*

Thus, we seek a "near global" property respecting (*) and a local property "undisturbed" by (*).

Towards the "near global" property we may recall the algebraic isomorphism $A \rightarrow \text{int}_X(\text{cl}_X(A))$ between $R(D)$ and $R(X)$ whenever D is a dense subspace of a space X . So an isomorphism invariant property of Boolean algebras is "near global".

THEOREM 1.1. [39]: *For a space X , the following are equivalent:*

- (1) βX is co-absolute with a LOTS;

- (2) X has a π -base T such that (T, \supseteq) is a tree;
 (3) If P is any π -base for X , then (P, \supseteq) has a cofinal tree. \square

Since orderable subspaces of extremally disconnected spaces are discrete, 1.1 alone cannot complete our search. Towards the local property we have S. Davis' generalization of first countability. A *lob space* is a space whose every point has a linearly ordered local base [7]. Now a point in a dense subspace has a linearly ordered local base in the subspace iff it has one in the space. Therefore, "lob space" is undisturbed by (*). However, there are LOTS in which no point has a linearly ordered local base. In order to circumnavigate the latter, we might consider B. Scott's further generalization: the *bi-linearly ordered* local base and the *blob* spaces (see [29] for a definition). Example 7.1 shows the class of blob spaces too large for our purposes.

1.2. A point x in a space X has a *butterflying* local base if there are two collections U_0 and U_1 , of open sets, subject to:

- (1) $U(x) = \{U_0 \cup U_1 \cup \{x\} : U_i \in U_i, i \in 2\}$ is a local base at x ,
 (2) (U_i, \supseteq) is linearly ordered $\forall i \in 2$, and
 (3) for each pair $(U_0, U_1) \in U_0 \times U_1$, $U_0 \cap U_1 = \emptyset$.

The collections U_0 and U_1 will be said to *witness the butterflying at x* , and X is a *butterfly* space when each of its points has a butterflying local base. Obviously a LOTS is a butterfly space.

THEOREM 1.3. *A space X has a dense orderable subspace iff βX is co-absolute with a LOTS and X has a dense butterfly subspace.*

PROOF. As the "only if" is immediate we prove the "if". According to (*) we may assume X is a butterfly space. Fix, for each $x \in X$, the collections $U_0(x)$ and $U_1(x)$ witnessing the butterflying. Let I be the set of isolated points of X , and, from 1.1, let $P \subseteq \mathcal{R}(X)$ be a π -base for X such that (P, \supseteq) is a tree.

Recursively, by its subtrees $T \uparrow \alpha$, we construct a tree T of open sets of X , a function $f: T \rightarrow X$, and a linear ordering \leq on $f[T]$. Let

$$T_1 = (\{\{x\} : x \in I\} \cup \{int(X-I)\}) - \{\emptyset\}.$$

For each $t \in T_1$ arbitrarily choose $f_1(t) \in t$. Let \leq_1 be a discrete ordering

on $f_1[T_1]$ making $f_1(int(X-I))$, if it is defined, the largest element.

Suppose that λ is a given ordinal for which we must construct T_λ , f_λ , and \leq_λ , and suppose that for each $\alpha < \lambda$ we have constructed trees T_α (of open sets ordered by \supseteq), a function $f_\alpha: T_\alpha \rightarrow X$, a linear order \leq_α on $f_\alpha[T_\alpha]$ all subject to the restrictions (i) - (x) below:

- (i) if $\beta < \alpha$, then $T_\alpha \uparrow \beta = T_\beta$, $(f_\alpha|_{T_\beta}) = f_\beta$, and $(\leq_\alpha|_{f_\beta[T_\beta]}) = \leq_\beta$.
- (ii) if $s, t \in T_\alpha$ and if $f_\alpha(s) = f_\alpha(t)$, then $s \cap t \neq \emptyset$.
- (iii) if $\beta < \alpha$, if $r, s \in T_\beta$, and if $t \in T_\alpha - T_\beta$ with $t \subset s$, then $f_\alpha(r) <_\alpha f_\alpha(s) \Rightarrow f_\alpha(r) <_\alpha f_\alpha(t)$, and $f_\alpha(s) <_\alpha f_\alpha(r) \Rightarrow f_\alpha(t) <_\alpha f_\alpha(s)$.
- (iv) if $\beta < \alpha$ and if $int(\cap B)$ is finite $\forall B \in \mathcal{B}(T_\beta)$, then $T_\beta = T_\alpha$.

For the restrictions (v) - (x) we pre-suppose (iv) is vacuous; i.e. for each $\beta < \alpha$ the set

$$J_\beta = \{B \in \mathcal{B}(T_\beta) : |int(\cap B)| \geq \omega\} \text{ is non-empty.}$$

In addition for a fixed $\beta < \alpha$ and $B \in J_\beta$ we set

$$L_B = \{t \in (T_\alpha \uparrow \beta + 1) : t \subset \cap B\}$$

and we designate $\langle B_+ \rangle$ for the statement "there is an $x_B \in X$ such that $f_\alpha(t) = x_B$ for each t in a final segment of (B, \supseteq) ."

- (v) L_B is an infinite collection of pairwise-disjoint open sets whose union is a dense subset of $\cap B$, and $\leq_\alpha|_{L_B}$ is a discrete order with no endpoints.
- (vi) if $\langle B_+ \rangle$, then $\exists U \in \mathcal{U}(x_B)$ such that the set t_B defined by $t_B = U \cap int(\cap B)$ belongs to L_B .
- (vii) if $\langle B_+ \rangle$, then $t_B \cup \{x_B\} \in \mathcal{U}(x_B)$ iff $(\cap B) \cup \{x_B\}$ is a nbhd of x_B iff either $x_B \in t_B$ or $t_B \notin U_0(x_B) \cup U_1(x_B)$.
- (viii) if $\langle B_+ \rangle$, if $t \in L_B - \{t_B\}$, and if $\cap B \subseteq U_0 \cup U_1 \cup \{x_B\} \in \mathcal{U}(x_B)$, then either $t \subset U_0$ and $f_\alpha(t) \leq_\alpha x_B$ or $t \subset U_1$ and $x_B \leq_\alpha f_\alpha(t)$.
- (ix) if $t \in L_B$, then $t \notin P$ iff $\langle B_+ \rangle$ and $t = t_B \notin P$.
- (x) if $t \in L_B$, then $f_\alpha(t) \notin t$ iff $\langle B_+ \rangle$ and $x_B \notin t_B = t$.

Since the above restrictions (i) - (x) precisely describe how the construction, by recursion, of T_λ , f_λ , and \leq_λ takes place, we may assume, for simplicity, the construction proceeds until (iv) is a non-vacuous statement. In this case set $T = T_\lambda$, $f = f_\lambda$, and $(\leq) = (\leq_\lambda)$.

Now (i) and (v) imply that (T, \supseteq) is a tree. Since P is a tree and (iv) is non-vacuous, (v) and (ix) imply T is a π -base. So (i) and (x) imply f is a function and $f[T]$ is dense in X . From (i), (ii) and (iii), and (v) it follows that \leq is linear ordering of $f[T]$.

In order to see that $f[T]$ is orderable, we need only show each $x \in f[T]$ has a local base $W(x) \subseteq U(x)$ such that $W \cap f[T]$ is an open interval of $(f[T], \leq)$. So we suppose $x \in f[T]$ and $U_0 \cup U_1 \cup \{x\} = U \in \mathcal{U}(x)$. If $\exists t \in T$ with $t \subseteq U$ and $t \cup \{x\} \in \mathcal{U}(x)$, then, by (iii) and (viii), we are done. So we suppose no such t exists. From (vi), $\exists \beta < h(T)$, $\exists B \in \mathcal{B}\mathcal{h}(T \uparrow \beta)$ such that $x_B = x$ and $t_B \subseteq U$. For simplicity we may assume (using vii) β is the first such ordinal and $t_B \in \mathcal{U}_0(x)$. Since $U_1(x)$ is linearly ordered, (vii) also implies $\exists s \in B$ with $s \cup \{x\} \in \mathcal{U}(x)$ and $s - V_0 \subseteq U_1$. Thus, we have $t_B \cup (s - V_0) \subseteq U$ and by (iii) and (viii)

$$f[T] \cap (t_B \cup (s - V_0) \cup \{x\})$$

is an open interval of $(f[T], \leq)$. \square

Obviously, every GO space is a butterfly space. Further, the interval topology induced by its underlying linear order is a π -base for the GO-space. Thus, in answer to a question of E. van Douwen and D. Lutzer, we have from 1.3: *Every GO-space has a dense orderable subspace.*

A straight-forward argument shows that each lob space is a butterfly space, and each butterfly space is a blob. Unfortunately (*) is still disturbed by "butterfly local base" since 1.2 (1) implies *each* $U_0 \cup U_1 \cup \{x\}$ *is an open set* of X . On the other hand, we do not know whether there is an "internal" characterization; i.e. one which does not use " X has a dense (blank) subspace". One possibility is to define *wb-spaces* and *weak-butterflying* local bases by replacing 1.2 (1) with the property.

$$w(1): U(x) = \{ \text{int}_X(\text{cl}_X(U_0 \cup U_1)) : U_i \in \mathcal{U}_i, i \in 2 \}$$

is a local base at x .

It is easy to see that x has a *wb* local base in X iff x has a *wb* local base in every extension (dense subspace) of X (in which x is a member).

The term "butterfly space" has been used in a different context in [3]. After receiving a handwritten draft of our paper, D. Lutzer forwarded a copy of [20] where the authors also use the term "butterfly space". Specifically

a space X is a *butterfly space* in the sense of [20] if 1.2 (1) and (2) are satisfied. Generalizing a theorem due to Ponomarev they prove

THEOREM 1.4. [20] (compare this to 2.5): *A space X is butterfly in the sense of [20] iff it is the open continuous image of an orderable space.* \square

2. FIRST COUNTABLE AND OTHER LOB SPACES

The Cantor space $\Pi^{\omega}2$, the space of irrationals Π^{ω} (in fact all $B(\kappa)$, see Section 3), and for $\kappa > \omega$ Hausdorff's κ -metrizable spaces are all examples of suborderable spaces which are *non-archimedean*; i.e. each space has a base in which every pair of elements are either disjoint or related by inclusion. There is a particularly interesting characterization of this property: *X is non-archimedean iff X is ultra-paracompact* (each open cover has a pairwise-disjoint refinement) *and X has an orthobase* (a base \mathcal{B} such that $x \in \cap \mathcal{B}_0$ and $\mathcal{B}_0 \subseteq \mathcal{B}$ imply either \mathcal{B}_0 is a local base at x or $\cap \mathcal{B}_0$ is open) [27]. Since metrizable spaces also have an orthobase, Nyikos responded to the characterization by calling a space *proto-metrizable* if it is paracompact and has an ortho-base (see [27], and [28] for further characterizations).

THEOREM 2.1. *For a space X , the following are equivalent:*

- (1) *X has a dense orderable non-archimedean subspace;*
- (2) *X has a dense proto-metrizable subspace;*
- (3) *X has a dense lob space and βX is co-absolute with a LOTS.*

PROOF. (1) \Rightarrow (2) is obvious. For (2) \Rightarrow (3) observe that every space with an ortho-base is an lob space, and every non-archimedean space is suborderable. To complete the implication we use L. Fuller's nice theorem: a proto-metrizable space is the perfect irreducible image of a non-archimedean space [17]. (3) \Rightarrow (1) follows from the proof of 1.3 and the most useful characterization of non-archimedean spaces: there is a base which is a tree when it is ordered by reverse inclusion. \square

COROLLARY 2.2. *X has a dense orderable non-archimedean subspace if X satisfies any one of the following:*

- (1) [40] *X is a suborderable \check{C} ech-complete space;*
- (2) [39] *X is first countable and βX is co-absolute with a LOTS;*
- (3) *X is \check{C} ech-complete, $|X| < 2^{\omega_1}$, and X is co-absolute with a LOTS.*

(Hint: use the \check{C} ech-Pospišil theorem ([13], 3.12.11).)

There is a multiplicity of first countable spaces without a dense orderable subspace. We shall, in Section 4, see how some first countable spaces with dense orderable subspaces can be used to produce first countable spaces with no dense orderable subspaces.

Hausdorff's κ -metrizable spaces (also known as ω_μ -metrizable spaces) have many characterizations (see [27] and [34]) one of which we use for a definition. If κ is a regular cardinal, a space X is said to be κ -metrizable whenever there is a compatible uniformity for X with a well-ordered base of order type κ . Using this definition Nyikos and Reichel extended the classic result for first countable topological groups by proving that a *topological group is an lob space iff it is a κ -metrizable space for some κ .*

THEOREM 2.3. *A topological group has a dense orderable subspace iff it has a dense butterfly subspace.*

PROOF. We sketch the "if". For the identity e of the group (G, \cdot) fix the families $U_0(e)$ and $U_1(e)$ witnessing the weak-butterflying at e (this is possible by homogeneity and the extension of butterflying local bases in a dense subspace to *wb* local bases in the space). For $x, y \in \bar{E}(G)$ (cf. 0.1) we say $x \sim y$ whenever $\exists a \in G, \exists i \in 2$ such that

$$\text{int}(\text{cl}(U \cdot a)) \in x \cap y, \quad \forall U \in U_i(e).$$

If $E(G)/\sim$ is the resulting quotient space and if q is the quotient map, then we define $f: E(G)/\sim \rightarrow G$ by $f(q(x)) = a$, whenever x converges to a ; f is clearly a perfect irreducible surjection.

If G is an lob space, we use the Nyikos-Reichel result and 2.1 (1). So we suppose G is not an lob space. $E(G)/\sim$ is an lob space. From the definition,

$$G_0 = \{q(x) \in E(X)/\sim: \exists a \in G, \text{int}(\text{cl}(U \cdot a)) \in x, \forall U \in U_0(e)\},$$

is a topological group as a subspace of $E(X)/\sim$. Since G_0 is dense in $E(X)/\sim$, we apply 2.1 (3) to complete the proof. \square

Perhaps 2.3 should be attributed to Nyikos and Reichel since the essentials of their proof for the lob case should be mimicked to prove our theorem. However, our proof has, as a side effect, a corollary reminiscent of Fedorčuk's theory of ordered absolutes (see [30]).

COROLLARY 2.5. *A wb space is the at most 2 to 1 closed continuous irreducible image of an lob space. \square*

Since non-archimedean spaces are zero-dimensional and hereditarily ultra-paracompact [27], l.o.b. GO-spaces (e.g. the Sorgenfrey modification of a LOTS) have a dense subspace possessing those properties. This is no accident.

THEOREM 2.6. *A GO-space has a dense zero-dimensional orderable hereditarily paracompact subspace.*

PROOF. Suppose X is a GO-space. If X is connected, it is the union of compact connected LOTS. From 2.1 (1) the proof is complete. So we suppose WLOG X is a zero-dimensional space. Arbitrarily choose $x(0) \in X$. Suppose λ is an ordinal and for each $\alpha < \lambda$ we have found $x(\alpha) \in X$ to satisfy:

- (i) $X(\alpha) = \{x(\beta) : \beta < \alpha\}$ is hereditarily paracompact;
- (ii) $x(\alpha) \notin cl_X(X(\alpha))$.

If $X(\lambda) = \{x(\alpha) : \alpha < \lambda\}$ is dense, we stop the recursion. Otherwise arbitrarily choose $x(\lambda) \notin cl_X(X(\lambda))$.

If λ is a non-limit ordinal, $X(\lambda)$ is the topological sum of two hereditarily paracompact spaces. So we suppose λ is a limit ordinal and (A, B) is a pseudo-gap (see [25]) of $Y \subseteq X(\lambda)$. If $\exists \beta < \lambda$ with $X(\beta) \cap Y \cap A$ cofinal in A , then from Faber's theorem (see [25]) we may find a closed discrete set $D \subseteq X(\beta) \cap Y \cap A$ cofinal in A . Applying (ii) recursively on $\alpha < \lambda$, we see that D is a closed discrete subset of $X(\lambda)$. If no such β exists for (A, B) , consider the set

$$D = \{x(\alpha_\gamma) : \gamma < cf(\lambda)\}$$

obtained recursively by $\alpha_\gamma = \alpha$, where α is the first ordinal in λ satisfying:

- (iii) $x(\alpha) \in A \cap Y - cl_X\{x \in X : \exists \beta < \alpha, x \leq x(\beta)\}$.

Now (ii) implies $x(\alpha_\gamma)$ is not a limit point of $\{x(\alpha_\delta) : \delta < \gamma\}$ and (iii) implies $x(\alpha_\gamma)$ is not a limit point of $\{x(\alpha_\delta) : \gamma < \delta\}$. So D is a closed discrete subspace of $Y \cap A$, cofinal in A . Similarly, there is such a subset of $Y \cap B$; therefore, Faber's theorem tells us that Y is paracompact. Once again observe that a GO-space has a dense orderable subspace. \square

E. van Douwen has (private communication) extended 2.6 to show every space has a dense subspace which is hereditarily a D-space (see [25]).

3. DENSE METRIZABLE ORDERABLE SUBSPACES

A useful class of completely metrizable spaces are the so-called ([13]) *generalized Baire spaces of weight κ* , $B(\kappa) = \prod^{\omega} D(\kappa)$, where $D(\kappa)$ is the discrete space of infinite cardinality κ . The base of all open sets $\Pi\{G_n : n \in \omega\}$ such that $G_n \neq D(\kappa)$ implies $|G_m| = 1$, $\forall m \leq n$ is a tree (ordered by \supseteq). So each $B(\kappa)$ is non-archimedean and orderable.

FACT 3.1. A metric space X has a dense orderable subspace homeomorphic to a subspace of $B(\kappa)$ where κ is the weight of X .

Hint: Allow the space to have diameter 1. Fix $x \in X$ and find an infinite family D of pairwise-disjoint balls such that $\cup D$ is dense and $B(x, 1/2) \in D$. Now treat each member of D as a space, keeping the center as the fixed point. \square

It is sufficient to determine which spaces have a dense metrizable subspace. The fundamental result on this problem is 3.2 (2) \Rightarrow (1), due to H.E. White, and it surprises several "normal Moore space" enthusiasts (see [16]). An easy proof, paralleling that of 3.1, is straight-forward using the additional equivalence (from [39]) below. For another equivalence see 4.3.

THEOREM 3.2. [38]: For a space X the following are equivalent:

- (1) X has a dense metrizable subspace.
- (2) X has a dense first countable subspace and a σ -disjoint π -base (i.e. a π -base which is the union of countably many families of pairwise-disjoint sets).
- (3) X has a dense first countable subspace and a tree π -base of height at most ω (equivalently, βX and βM are co-absolute for a subspace $M \subseteq B(\kappa)$, where κ is the weight of X). \square

The "first countable" in 3.2 is crucial—just consider $\beta\mathbb{Q} - \mathbb{Q}$ [38]. There is even an lcb LOTS with an σ -disjoint π -base but no dense metrizable subspace (Example 7.3). On the other hand, first countability plus considerable additional structure need not produce dense orderable subspaces. The Pixley-Roy hyperspace of the real line is a ccc Moore space with no dense orderable subspace (see [24]), while the Pixley-Roy hyperspace of a \mathbb{Q} -set (assume $MA + \neg CH$) is all of that, and normal as well (see [8]). Further, we have in Example 7.2 the first "naive" example of a compact connected first countable LOTS with no dense metrizable subspace.

Various classes of "generalized-metrizable spaces" (e.g. M_1 -spaces, p -spaces, stratifiable spaces, etc.) proliferate in topology, and for most of the resulting classes the question "dense orderable subspaces?" is moot - in the sense that there is frequently an axiom with consequence "dense orderable implies dense metrizable". There is a lemma, suggested by known metrizability theorems for G_0 -spaces (see [23] and [25]), illustrating this point.

LEMMA 3.3. [39]: *Suppose G is a countable family of non-empty open sets of a space X , and suppose $\text{int}(\cap G) = \emptyset$. Then*

- (1) X has a σ -disjoint π -base if each $G \in G$ is dense and if βX is co-absolute with a LOTS.
- (2) A point $x \in \cap G$ has a countable local base if x has a weak-butterflying local base. \square

The references [31] and [38] both list and/or prove a number of "dense metrizable subspace" results. As there is not a survey on this topic we include for the reader's convenience a partial list of recent and/or important results. Observe that 3.3 is (implicitly) used (or proved) in each.

THEOREM 3.4. X has a dense metrizable orderable subspace if any one of the following holds:

- (1) [19] X is a Baire p -space with a G_δ -diagonal (and the subspace can be taken to be a G_δ -set);
- (2) [16] X has a σ -locally countable base;
- (3) [39] X is first countable, βX is a co-absolute with a LOTS, and βX is co-absolute with βY for a space Y with a G_δ -diagonal;
- (4) (see 2.2 (1)) X is a suborderable Baire space with a σ -disjoint π -base;
- (5) (FITZPATRICK and FLEISSNER, see [14]). Assume $V = L$, and X is a normal Moore space;
- (6) [1]. every subspace of X is a paracompact p -space. \square

Šanin's 1948 theorem on orderable dyadic spaces (i.e. continuous images of the generalized Cantor set $\Pi^\kappa 2$ for some κ) ultimately motivates our only metrization theorem.

THEOREM 3.5. (Čertanov, see [30]): *A dyadic space has a dense orderable subspace iff it is co-absolute with a LOTS iff it is the continuous image of the Cantor set Π^{ω} (and hence is separable and metrizable).*

The *Sanin number*, $\check{s}(X)$, of a space X is the smallest cardinal κ such that every family of κ^+ many non-empty open sets of X contains a subfamily of κ^+ sets having non-empty intersection. Clearly, \check{s} is not raised by continuous images, or by products of spaces with the same Sanin number. Therefore, if X is dyadic, then $\check{s}(X) = \omega$ [13]. A weak version of $\check{s}(X)$, call it $\check{c}(X)$ (for Čertanov), requires that if the family consists of regular-open sets, then the subfamily has only to satisfy the finite intersection property. If T is a tree, under \supseteq , in $\mathcal{R}(X)$, then $|T| \leq \check{c}(X)$. On the other hand,

$$\omega \leq \check{c}(X) = \check{c}(E(X)) = \check{c}(\beta X) \leq \check{s}(X)$$

for any space X . We have now proved

LEMMA 3.6. (Čertanov): *If $\check{c}(X) = \omega$ and βX is co-absolute with a LOTS Y , then X and Y have countable π -weight. \square*

4. PRODUCT SPACES

Any countable (finite, for fixed κ) product of (κ^-) metrizable spaces is (resp. κ^-) metrizable. The latter gives us easy instances of products with dense orderable subspaces. A few more instances can be gained from a fact we extract from the analysis (see [27]) of productively non-archimedean spaces.

FACT 4.1. *If λ is an ordinal and if, for each $i \in 2$, T_i is a tree for which the height of the tree induced on $\{s \in T_i : t < s\}$ is λ for each $t \in T_i$, then*

$$U\{lv(T_0, \alpha) \times lv(T_1, \alpha) : \alpha \in \lambda\},$$

is a cofinal tree of $T_0 \times T_1$, with the product partial order. \square

This fact is precisely what one uses (along with 2.2 (2)) to prove *each finite product of nowhere separable Souslin lines has a dense orderable subspace*. Barring insulting technicalities on products of butterfly spaces, we know of no other positive results. The rest of the material could be fitted into Section 3 to produce counter-examples.

THEOREM 4.2. *Suppose $X = \prod\{X_\alpha : \alpha \in \kappa\}$ is an infinite product of infinite spaces. If X has a dense orderable subspace, then $|\kappa| = \omega$ and X has a dense metrizable subspace.*

PROOF. Suppose D is a dense orderable subspace of X . For each $d \in D$ we may find $f: \omega \rightarrow \kappa$ and $x \in X$ such that $x(f(n)) \neq d(f(n))$, $n \in \omega$. Apply the lemma 3.3 to

$$G_d = \{\pi_{f(n)}^{-1}(X_{f(n)} - \{x(f(n))\}) : n \in \omega\}. \quad \square$$

THEOREM 4.3. *A space X has a dense metrizable subspace iff $X \times [0,1]$ has a dense orderable subspace.*

PROOF. As 3.1 proves the "only if", we prove the "if". Suppose D is a dense orderable subspace of $X \times [0,1]$. For $d \in D$ set

$$G_d = \{\pi_{[0,1]}^{-1}([0,1] - \{q\}) : q \in Q, \pi_{[0,1]}(d) \neq q\}.$$

Applying 3.3 (2) to G_d shows d has a countable local base. So X has a dense first countable subspace. According to 3.2 we need only find a σ -disjoint π -base for X .

If we apply 3.3 (1) to G_d , we find that $X \times [0,1]$ has a σ -disjoint π -base $\bigcup_{n \in \omega} P_n$. Fix some countable π -base \mathcal{B} for $[0,1]$. For each $B \in \mathcal{B}$ and $n \in \omega$, we may find a (possibly empty) maximal pairwise-disjoint family $\mathcal{U}_{B,n}$ of non-empty open sets of X such that

$$U \in \mathcal{U}_{B,n} \Rightarrow \exists Q \in P_n, \quad U \times B \subseteq Q.$$

So $\bigcup \{\mathcal{U}_{B,n} : (B,n) \in \mathcal{B} \times \omega\}$ is a σ -disjoint π -base for X . \square

We should not ignore the relationship of questions in this paper to the S and L space problems [32]. An immediate corollary to 4.3 shows that for a hereditarily Lindelöf space X , X is separable iff $X \times [0,1]$ has a dense orderable subspace.

We have seen two proofs, 2.3 and 3.5, that the product \mathbb{I}^{\aleph_2} of uncountably many two point spaces fails to have a dense orderable subspace. However, because of its applications (see 6.5 and 7.5), we give yet another proof in 4.5 below. First we generalize a concept from Boolean algebra.

4.4. Suppose X is a space and I is an infinite family of subsets of X . We will call I an *independent family* whenever for every pair J and K of finite non-empty subsets of I we have

(1) $\text{int}(\cap J) \subseteq \text{cl}(\cup K)$ implies $J \cap K \neq \emptyset$.

We say I is *strongly independent* if

(2) $|I| > \sup\{|J|^+ : J \subseteq I, \text{either } \text{int}(\cap J) \neq \emptyset \text{ or } \text{cl}(\cup J) \neq X\}$.

The following sequences of (1) and (2) are routinely proved for an infinite strongly independent family I of a space X :

(3) I is uncountable and $\inf\{|J| : J \text{ is strongly independent subset of } I\}$ is a regular cardinal.

(4) If $\text{int}(I) \subseteq A(I) \subseteq \text{cl}(I)$, $\forall I \in I$, then $\{A(I) : I \in I\}$ is a strongly independent family.

(5) If $J \subseteq I$, then $(I-J) \cup \{X-J : J \in J\}$ is a strongly independent family.

(6) If $f: Y \rightarrow X$ is an open (or closed irreducible) continuous surjection, then $\{f^{-1}(I) : I \in I\}$ is a strongly independent family of Y . \square

Observe that $I = \{\pi_\alpha^{-1}(0) : \alpha < \kappa\}$ is a strongly independent family of $\Pi^\kappa 2$, whenever κ is uncountable. Further, if $\kappa > \omega_1$ and we add all G_δ -sets to the product topology, then I is still strongly independent.

THEOREM 4.5. [42]: *Suppose X is a space with a strongly independent family of clopen sets. Then every orderable subspace of X is nowhere dense.*

PROOF. (sketch): If I is an independent family of a space Y and if $y \in Y$, then $I(y)$ is an independent family, where

$$I(y) = \{\text{int}(I) : y \in \text{int}(I), I \in I\} \cup \{Y - \text{cl}(I) : y \notin \text{cl}(I), I \in I\}.$$

Now if y has a weak butterflying local base, then the Sup Function lemma and the Pressing Down lemma (see [15]) applied to $|I(y)|$ and the character of y , show that $I(y)$ is not a strongly independent family. \square

With respect to the aforementioned applications of 4.5 to Section 6, we note: if one adds, simultaneously, ω_2 random or Cohen reals $\{r_\alpha : \alpha < \omega_2\}$ to any model of set theory, then

$$I = \{\text{cl}_{\beta\omega-\omega}(r_\alpha^{-1}(0)) : \alpha < \omega_2\}$$

is a strongly independent family of clopen sets of $\beta\omega - \omega$ [42].

5. HOMEOMORPHIC DENSE ORDERABLE SUBSPACES

When does a pair of spaces possess homeomorphic dense (not necessarily orderable) subspaces? With the exception of A. Hager's work with the Dedekind-McNeil completion of $C(X)$, [18] and consequences of E. van Douwen's and C. Gates' work on remote points (see [43]), we present all that we know on this question. The first result uses known characterizations of the rationals. The second result combines 2.2 (1) with a kind of "logician's back-and-forth argument".

PROPOSITION 5.1. *Suppose X and Y are first countable separable spaces. Then X and Y have homeomorphic dense orderable countable subspaces iff X and Y have no isolated points. \square*

THEOREM 5.2. [40]: *Suppose X and Y are densely-orderable Čech-complete spaces. Then X and Y have homeomorphic dense orderable subspaces iff βX and βY are co-absolute.*

After one applies 3.4 (1), the next theorem has at least four independent discoverers. Since its first, to my knowledge, appearance was in C. 1977 Ph.D. thesis (University of Kansas), we attribute it to her. The latest appearance of 5.3, and the most general result to date, is as a corollary of 5.2. An elegant proof of 5.3 comes via Lavrentieff's theorem ([13], 4.3.20) and the Dedekind-McNeil completion of $C(X)$, the ordered vector space of real-valued functions on X [18]. However, the most informative proof is a by-product of the lemma 5.4 below.

THEOREM 5.3. (C. Gates): *Suppose X and Y are each Čech-complete spaces with a G_δ -diagonal. Then X and Y have homeomorphic dense (orderable and metrizable) G_δ -sets iff βX and βY are co-absolute. \square*

LEMMA 5.4. [26]: *Suppose M is the class of completely metrizable spaces formed from topological sums of the (various) spaces $B(\kappa)$. If $X, Y \in M$ and if βX and βY are co-absolute, then X and Y are homeomorphic. \square*

Pre-dating the previous three results and the material in Section 3 are their generalizations to various subspaces of the κ -metrizable spaces; for example, parts of 5.5 below are really 3.4 (1) in disguise. Comfort and Negrepontis, in particular, have collected and completely analyzed the η_α -sets (we use the traditional (cf. Sierpinski, Gillman and Jerison) definition

- a linearly ordered set (X, \leq) such that $A, B \subseteq X$, $|A| + |B| < \omega_\alpha$, and $a < b$, $\forall (a, b) \in A \times B$ all imply $\exists c \in X$ with $a < c < b$, $(a, b) \in A \times B$. Chapters 4, 5, 6, 8 and 15 of [6] are an, occasionally hidden, gold mine. But of course once we leave the ease of ω , your set theory prevails.

To aid out study in later sections we collect here some definitions and a theorem. For a space (X, τ) , X_δ is the space generated on the ground set X by the union of all its τ - G_δ sets. The space $(\Pi^{\omega_1 2})_\delta$ is ω_1 -metrizable and homeomorphic to X_δ if X is the LOTS obtained by lexicographically ordering ${}^{\omega_1}2$. A *P-point* of a space X is, by definition, in the interior of every G_δ -set containing it. For a space X , $P_\delta(X)$ is the subspace of P -points of X . The following result extends the Cantor-Hausdorff theorem: all η_1 -sets of cardinality ω_1 are homeomorphic.

THEOREM 5.5. ([6], 6.17 and 15.9): *If X is a compact space of weight ω_1 and if each element of X has character ω_1 , then X_δ is homeomorphic to $(\Pi^{\omega_1 2})_\delta$. Further, if every non-empty G_δ -set of X has non-empty interior, then X_δ and $P_\delta(X)$ are homeomorphic. \square*

6. STONE- \check{C} ECH REMAINDERS

When does $\beta X - X$ have a dense orderable subspace? If we allow pseudo-compact X 's, there is no sane answer to this question (even if we want $\beta X - X$ orderable, see ([4], 4.17). Of course βX is orderable iff X is countably compact and suborderable [36]). So we require X to be real-compact. If we allow nowhere locally compact X 's (such as the rationals, irrationals, or the Sorgenfrey line), we know of no surprising results in this context. So we require X to be locally compact and non-compact.

The first real and surprising response to our question is due to I. Parovičenko (and subsequently improved by Comfort and Negreponis, see 5.5) and said that *whenever X is locally compact non-compact and separable metric, $\beta X - X$ has a dense set homeomorphic to the space $(\Pi^{\omega_1 2})$ - if you assume CH. This result is the basis for this section, and 6.2 below indicates that the question is "reasonable" even if CH is removed. For simplicity the results are stated for the metric case; however, they frequently work with considerably lessened restrictions.*

BASIC FACTS 6.1. (see [37]): *Suppose X is a locally compact non-compact metric space. Then*

- (1) $\beta X - X$ has an open dense set which is the topological sum of spaces each having weight 2^{ω} ;
- (2) $\beta X - X$ is a compact almost P-space (= non-empty G_{δ} sets have non-empty interior) with no isolated points or convergent sequences. \square

LEMMA 6.2. [39]: *If Y is an almost P-space of π -weight at most 2^{ω} , then βY is co-absolute with a LOTS.*

PROOF. (sketch): Assume Y has no isolated points and B is a π -base for Y . Each element of B contains the union of 2^{ω} pairwise-disjoint members of B . Now construct a tree T in (B, \supseteq) so that if $b \in B$ meets 2^{ω} elements of a level of T , then the next level of T contains a member $t \subseteq b$. \square

THEOREM 6.3. [39]: *If X is a locally compact non-compact metric space, then $\beta X - X$ is co-absolute with a LOTS.* \square

With the advent of 6.3, we had hoped that " $\beta X - X$ has a dense orderable subspace, whenever X is a locally compact non-compact metric space" is a theorem of ZFC. However, if Y is a space with no convergent sequences and if $y \in Y$ has a weak-butterflying local base, then y is a P-point of Y . Now recall Shelah's P-point theorem (see [5]).

Since some set-theoretic enhancement of ZFC is necessary for us to achieve our goal, two natural questions arise. How strong, set-theoretically, is Parovičenko's result (mentioned in the second section) or 5.5? What is the least familiar-hypothesis whose assumption yields the dense orderable subspace? The remainder of this section is a response to these two questions.

THEOREM 6.4. *The following are equivalent:*

- (1) CH holds.
- (2) *If K is a compact, zero-dimensional, almost-P, F-space (= co-zero sets are C^* -embedded) of weight 2^{ω} and if K has no isolated point, then K and $\beta\omega - \omega$ are homeomorphic.*
- (3) *If D is a dense orderable subspace of a compact, zero-dimensional, almost-P, F-space of weight 2^{ω} , then D can be embedded into $(\mathbb{N}^{\omega})^2$.*
- (4) *If X is a σ -compact locally compact non-compact space of weight at most 2^{ω} , then $\beta X - X$ has a dense orderable subspace.*

PROOF. Of course (1) \Rightarrow (2) is Parovičenko's famous result (see [37], 3.31); (1) \Rightarrow (3) and (4) follow from 5.5; (2) \Rightarrow (1) is in [11] (also see 7.5); (3) \Rightarrow (1) is in [41] (also see 7.6); (4) \Rightarrow (1) is a consequence of 7.5. \square

In [6] we are told that $MA + 2^\omega = \omega_\alpha$ implies $\beta X - X$ has a dense copy of the canonical η_α -set whenever X is locally compact non-compact metric space. P. SIMON [33] obtained the same conclusion for $\beta\omega - \omega$ with an assumption strictly weaker than MA, namely that $\beta\omega - \omega$ is not the union of 2^ω nowhere dense sets. Of course neither hypothesis of set theory is particularly weak. Ostensibly, one assumes $\beta\omega - \omega$ has a point with a well-ordered base (of order type κ) and one finds that $\beta\omega - \omega$ has a dense non-archimedean subspace. Recently, we discovered [42] that if we assume, in addition, ${}^\omega\omega$ has a κ -scale (see [9]), then $\beta X - X$ has a dense non-archimedean subspace whenever X is locally compact non-compact and metric. Further, there are models of CH where the assumption of κ in the two preceding sentences can be ω_1 [10]; indeed, the non-archimedean space can be the LOTS $(\Pi^{\omega_1}2)_\delta$ even if CH is false. Finally, as an application of (4.5) we have

THEOREM 6.5. [42]: C & I. (*If X is a locally compact non-pseudocompact space, then $\beta X - X$ has a P-point and no dense orderable subspace.*)

PROOF. For any model M of ZFC, let P be the ω_2 -Cohen poset [2], G be a generic filter on P , and for each $\alpha \in \omega_2$ set

$$K(\alpha) = \{n \in \omega : (\alpha, n, 0) \in UG\}.$$

Then $M[G] \models \{(\text{cl}_{\beta\omega}(K(\alpha))) - \omega : \alpha \in \omega_2\}$ is a strongly independent family of $\beta\omega - \omega$. \square

Several readers of a version of this manuscript have complained of "unfairness" in my inclusion of ω_1 in the statement of 6.4 (3). In response to this we note that simple iterated forcing techniques prove [42]: *It is consistent with the axioms of ZFC that $\omega_1 < 2^\omega$ and $\beta X - X$ contains a dense copy of $(\Pi^{\omega_1}2)_\delta$ whenever X is locally compact non-compact metrizable and has weight at most 2^ω .*

7. EXAMPLES

7.1. *Tree π -base is not a sufficient condition, even in the product of LOTS.* Take X to be the first countable compact LOTS obtained from the branch set of special ARONSZAJN tree (see [21]). Every point of $X \times (\Pi^{\omega_1}2)$ (see comments preceding 5.5) has character ω_1 and belongs to a G_δ -set with no interior. From 3.3 (2), $X \times (\Pi^{\omega_1}2)_\delta$ has no dense orderable subspace.

Proposition 5.1 shows that $\beta(X \times (\Pi^{\omega_1} 2)_\delta)$ is co-absolute with a LOTS. \square

7.2. (1) A first countable compact LOTS with no dense metrizable subspace, and; (2) a first countable compact space with no dense orderable subspace [35]. We describe S. Todorčević's absolute examples. Let A be a stationary set (see [15]) in ω_1 . Let T_A be the tree of all countable closed in ω_1 subsets of A , ordered by $s < t$ if s is an initial segment of t . Give A another order \ll so that (A, \ll) is order isomorphic to a subset of $[0, 1]$, and such that the first, induced by ω_1 , successors of each $\alpha \in A$ is order isomorphic, under \ll , to Q . Using the order which \ll induces on the levels of T_A , order $\mathcal{B}\mathcal{h}(T_A)$. If X_A is the Dedekind completion (with end-points) of $\mathcal{B}\mathcal{h}(T_A)$, then X_A is a compact, connected, first countable LOTS with no dense metrizable subspace. From 4.3, $X_A \times [0, 1]$ has no dense orderable subspace. \square

7.3. A non-archimedean LOTS with a σ -disjoint π -base but no dense metrizable subspace. For each $n \in \omega$ let $T_n = \{f \in {}^\omega 2: \omega_n < \alpha < \omega_{n+1}, f \text{ is not constant on a tail of } \omega_n\}$. For $f, g \in T = \bigcup_{n \in \omega} T_n$ define $f < g$ if $f = g \upharpoonright \text{dom}(f)$. The natural order, $0 < 1$, of 2 induces an order, defined recursively, on the levels of T . We use this to order $\mathcal{B}\mathcal{h}(T)$. Now $\bigcup_{n \in \omega} P_n$ is a π -base for $\mathcal{B}\mathcal{h}(T)$, and each P_n is a pairwise-disjoint family, when we let

$$P_n = \{f \in {}^{(\omega_n+1)} 2: f(\omega_n) = 0\} \quad (\text{cf. 0.4}).$$

The space we seek is a dense subspace of $\mathcal{B}\mathcal{h}(T)$, namely

$$\{B \in \mathcal{B}\mathcal{h}(T): n \in \omega, \text{dom}(f) < \omega_n, f \in B\}. \quad \square$$

7.4. A finite or infinite product of spaces may have a dense metrizable subspace even if none of the factors do [35]. Let A and B be disjoint stationary sets in ω_1 , and X_A and X_B be the LOTS defined in 7.2. Todorčević has shown that $X_A \times X_B$ has a dense metrizable subspace. By applying 4.1 and 4.2 to the partial products $X_A \times X_B$, $(X_A \times X_B) \times X_B$, $(X_A \times X_B^2) \times X_B$, etc. we see that $X_A \times \Pi^\omega X_B$ also has a dense metrizable subspace. \square

7.5. A σ -compact locally compact space X such that $\beta X - X$ has no dense orderable subspace. For an infinite cardinal κ , let $X(\kappa) = \omega \times \Pi^\kappa 2$. The space $\beta X(2^\omega) - X(2^\omega)$, and one of its quotients, is used to prove 6.4 (2) \Rightarrow (1) [11]. In [39] a cardinal function argument is used to show $\beta X(\kappa) - X(\kappa)$ is co-absolute with a LOTS iff $\kappa \leq 2^\omega$. Recently, we have shown that

$\beta X(\omega_2) - X(\omega_2)$ has no dense orderable subspace, $X(\omega_2)$ has weight ω_2 , and if $\neg CH$ is assumed, then $X(\omega_2)$ is separable [42]. The last proves 6.4 (3) \Rightarrow (1). \square

7.6. A compact 0-dimensional, almost-P, F-space with no isolated points and with a dense orderable subspace. In [41] a machine is given for producing such objects having a wide range of possible dense orderable subspaces. However, E. van Douwen privately communicated another method we state in the framework of [42]: If X is any compact LOTS, then each ultra-product topology τ_μ on ${}^\omega X$ is orderable (see [5]) and embeds into the remainder $K = \beta(\omega \times X) - (\omega \times X)$; therefore, $cl_K(({}^\omega X, \tau_\mu))$ is the example desired. \square

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SOME GENERAL PROBLEMS ON GENERALIZED METRIZABILITY AND
CARDINAL INVARIANTS IN ORDERED TOPOLOGICAL SPACES

by

M.A. Maurice and K.P. Hart

1. INTRODUCTION

1. In Section 2.1 below we shall formulate some general questions concerning generalized metrizable properties and cardinal invariants in various classes of ordered topological spaces. Next we shall give a short survey of results obtained in answering part of these questions (2.2). It will then be clear what research could still be done in this area.

2. First we want to recapitulate which *ordered topological spaces* (and related spaces) usually are distinguished.

(a) Let $<$ be a *linear order* in a set X .

(i) There is essentially one intrinsic topology, the interval topology, which we denote by $J_{<}$. The triple $(X, <, J_{<})$ is called a LOTS.

(ii) If J is any topology in X , such that $J_{<} \subset J$ and which has a base of $<$ -convex sets, then $(X, <, J)$ is called a GO-space. GO-spaces are of course precisely the subspaces of LOTS's.

(iii) If J is any topology in X , such that $J_{<} \subset J$, then $(X, <, J)$ is called a (weakly, linearly) orderable space.

For (i) and (ii) we refer to [21], and for (iii) we refer to [17]; further references may be found in these papers.

(b) Let $<$ be a *partial order* in X . An important special case is that in which the partial order is derived from a lattice structure in X .

(i) There are several essentially distinct intrinsic topologies. The so-called interval-topology, which we denote by J_{\square} , is the weakest among them.

For linear orderings all these topologies coincide.

(ii) If J is any topology in X , such that $J_{\square} \subset J$, then $(X, <, J)$ is called a POTS. Most often we include in this definition the requirement that the ordering $<$ is J -continuous.

For (i) we refer to [5] and [18], and for (ii) we refer to [23] and [25]; further references may be found in these books and articles.

- (c) Let X be a connected T_2 -space.
- (i) If $p, q \in X$, ($p \neq q$), then $E(p, q)$ denotes the set of those cut points of X each of which separates p and q in X . Also $S(p, q) = E(p, q) \cup \{p, q\}$. There is a well-known natural linear order in $S(p, q)$, the so-called separation order.
 - (ii) X is called tree-like if $E(p, q) \neq \emptyset$ for all $p, q \in X$ such that $p \neq q$. X is a tree if it is tree-like and locally connected. A compact tree-like space (which is automatically a compact tree) is also called a dendron.

See for instance [17], [26] and [27] and the references given there. See also the paper on dendrons by J. van Mill and E. Wattel in these Proceedings.

3. Next we say a few words about *generalized-metrizability properties and cardinal invariants*.

- (a) We use the term "generalized-metrizability property" to indicate an arbitrary topological property which is implied by metrizability. For a survey of the most interesting of these properties and their mutual relations we refer to the appendix of [1] and to [6].
- (b) The term "cardinal invariant" is used for each "function" which is defined on a certain class of topological spaces and which assigns a cardinal number to each space from the class in a topologically invariant way.

A very complete survey may be found in [15], [16]; see also [9].

2. THE GENERAL RESEARCH AREA

1. The questions we are interested in can be formulated in a general form as follows (thereby sub (a) and sub (b) we use the term "ordered space" to indicate any of the spaces mentioned in 1.2).

- (a) Concerning generalized metrizability.
 - (i) Which ordered spaces automatically possess which generalized metrizability properties?
 - (ii) Characterize the various types of generalized metrizability in terms of the order structure.

(iii) Which relations exist between the various types of generalized metrizable spaces in which ordered spaces?

One may ask the same questions for images and pre-images of ordered spaces under certain kinds of mappings.

(b) Concerning cardinal invariants.

(i) Characterize the (values of the) various cardinal invariants in terms of the order structure.

(ii) Which relations exist between the (values of the) various cardinal invariants in which ordered spaces?

Again one may ask the same questions for images and pre-images of ordered spaces under certain kinds of mappings.

(c) Derived questions.

Here, in the first instance, we confine ourselves to LOTS's. There are a number of topological properties which hold for every LOTS. (For instance, monotone normality, strong collectionwise normality, countable paracompactness.)

(i) Which are in general the relations between these properties? (Of course, this concerns only a very limited number of questions.)

(ii) If P_1 and P_2 are any topological properties, such that $P_1 \rightarrow P_2$ for ordered spaces, what then can be said about the implication $P_1 \rightarrow P_2$ for spaces satisfying one or more of the properties mentioned sub (c).

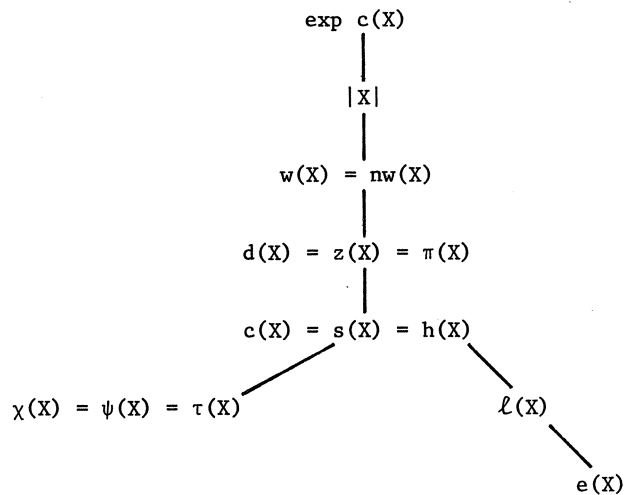
2. We now list several results concerning generalized metrizable spaces in the class of GO-spaces. See also [21], which contains yet other results of this type. Let X be a GO-space.

- (i) X is metrizable $\iff X$ has a σ -discrete dense subset which contains all jumps and pseudo-gaps, [10].
- (ii) X is perfectly normal \iff each relatively discrete subset of X is σ -discrete in X , [10].
- (iii) X is monotonically normal, [14], and hence hereditarily collectionwise normal.
- (iv) X is strongly collectionwise normal (= almost-2-fully normal). In fact, X is \aleph_0 -fully normal, [22].
- (v) X is (hereditarily) countably paracompact, [2].
- (vi) X is paracompact \iff for each gap and each pseudo-gap (A,B) in X , there exist discrete subsets $L \subset A$ and $R \subset B$ which are, respectively, co-final in A and coinital in B , [11], [10].

- (vii) Since X is collectionwise normal, it follows immediately that the following are equivalent: (1) X is paracompact; (2) X is metacompact (= weakly paracompact); (3) X is subparacompact; (4) X is θ -refinable. Moreover, however, these properties are equivalent with: (5) X is hypocompact (= strongly paracompact); (6) X is metalindelöf, [3].
- (viii) X has a G_δ -diagonal $\Rightarrow X$ is hereditarily paracompact [20].
- (ix) X is semi-stratifiable $\Leftrightarrow X$ is metrizable, [20].
- (x) If G is the equivalence relation in X defined by $xGy \Leftrightarrow$ the closed interval $[x,y]$ in X is compact, then the quotient-space X/G has a natural order, with respect to which it is a G_0 -space. Let $g: X \rightarrow X/G$ be the quotient map. Then we have X is a p -space $\Leftrightarrow gX$ is metrizable, [28].
- (xi) X is a strict p -space $\Leftrightarrow X$ is a paracompact p -space, [28].
- (xii) X is p -space $\Rightarrow X$ is an M -space, [28].
- (xiii) X is an M -space $\Leftrightarrow X$ is a $w\Delta$ -space $\Leftrightarrow X$ is quasi-complete, [4], [28].
- (xiv) If C is the equivalence relation in X defined by $xCy \Leftrightarrow$ the closed interval $[x,y]$ in X is countably compact, then the quotient-space X/C has a natural order, with respect to which it is a G_0 -space. Let $c: X \rightarrow X/C$ be the quotient map. Then we have X is an M -space $\Leftrightarrow cX$ is metrizable, [28].
- (xv) The following are equivalent: (1) X is hereditarily a p -space; (2) X is hereditarily an M -space; (3) X is hereditarily a $w\Delta$ -space; (4) X is hereditarily quasi-complete; (5) X is metrizable, [4], [28].
- (xvi) If in particular X is a LOTS, then we also have X is metrizable $\Leftrightarrow X$ has a G_δ -diagonal, [19].

3. Recently, the second author observed that (ii), (iii), (v), (vi), (vii), (viii), (ix), (xi), (xii), (xiii) and (xv) can be generalized to the class of partially ordered sets of finite width, supplied with the interval topology, while (xvi) also holds in the class of lattices of finite width with the interval topology. These facts follow easily by applying a theorem of DILWORTH [7].

4. The class of G_0 -spaces behaves very nicely with respect to cardinal functions. Combining the results from [15] and [9] we get the following diagram:



Moreover, in [15] it is shown that $c(X) \leq d(X) \leq c(X)^+$.

5. Recently the second author showed that the same diagram can be drawn for posets of finite width endowed with the interval-topology. Again, this follows easily by applying a theorem of DILWORTH [7], except for the assertion concerning $z(X)$, which requires a different (and somewhat more complicated) proof. Even more recently, it was shown that for LOTS the following formula holds: $w(X) = \psi w(X) \cdot c(X)$, [12]. As the Sorgenfrey line shows this formula is in general not valid for GO-spaces.

6. As to the relation alluded to sub. 2.c(i) we discuss the following:

- (i) -It is of course very easy to give an example of a countably paracompact, non-normal space: $\omega_1 \times (\omega_1 + 1)$ is not normal but even countably compact.
 - The existence of a normal space which is not countably paracompact (a so-called Dowker-space) has been shown by RUDIN [24].
 - It seems to be not yet known whether or not there exists a monotonically normal Dowker space.
- (ii) E.K. VAN DOUWEN [8] and K.P. HART [13] observed that strong collectionwise normality does not imply and is not implied by monotone normality. In [13] K.P. HART shows moreover that strong collectionwise normality does not imply countable paracompactness. In fact, he proves that M.E. Rudin's Dowker space is strongly collectionwise normal.

7. Finally we give one instance of the type of questions described sub. 2.c(ii): Since for a GO-space we have that $p \rightarrow M$, while any GO-space is monotonically normal, one would like to know whether or not it is true that a monotonically normal p -space is also an M -space. It seems that the answer to this question is not known.

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DENDRONS

by

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1. INTRODUCTION

Let X be a compact connected Hausdorff space. We say that X is a *dendron* provided that for every two distinct points $x, y \in X$ there exists a point $z \in X$ which separates x from y , i.e. $X \setminus \{z\} = U \cup V$ where U and V are disjoint open subsets of X such that $x \in U$ and $y \in V$. Dendrons are natural generalizations of linearly orderable continua. In the last decade several results concerning dendrons have been proved and the aim of this paper is to collect some of these results and to present them in such a way that the underlying ideas which led to these results will be recognized.

2. CONNECTIVITY PROPERTIES

In this section we collect some basic facts which will be important throughout the remaining part of this paper. *The letter D will always denote a given dendron.*

LEMMA 2.1. *Take $x \in D$. If C is a component of $D \setminus \{x\}$, then C is open.*

PROOF. Assume that A and B are disjoint open sets of D and that $A \cup B = D \setminus \{x\}$. We claim that $A \cup \{x\}$ is connected. Suppose not, then there exists a pair of clopen subsets U and V in $A \cup \{x\}$ such that $U \cap V = \emptyset$ and $U \cup V = A \cup \{x\}$. If $x \notin U$, then U is an open subset of the open set A and hence open in D . U is closed in set $A \cup \{x\}$ and hence closed in D . If $x \notin V$ the same arguments hold. This contradicts the connectivity of D and we conclude that $A \cup \{x\}$ is connected.

Next we assume that some quasi-component Q (i.e. the intersection of a maximal collection of clopen subsets) of $D \setminus \{x\}$ is not open. Then Q contains a point q which is in the closure of $D \setminus (Q \cup \{x\})$. Assume that z separates q and x . If $z \notin Q$ then there is a pair of disjoint open subsets A and B such

that $z \in A$ and $q \in B$ and $A \cup B = D \setminus \{x\}$. However, we have seen that $B \cup \{x\}$ is connected and so we conclude that $z \in Q$. From the same argument we find that $C \cup \{x\}$ is connected for every clopen subset $C \subset D \setminus \{x\}$ which misses Q . Therefore

$$U\{C \cup \{x\} \mid C \text{ clopen in } D \setminus \{x\} \text{ and } C \cap Q = \emptyset\} = D \setminus Q$$

is connected. However, q is a member of the closure of $D \setminus Q$ and hence $\{q\} \cup D \setminus Q$ is connected and contains both q and x . Therefore z does not separate q and x . This contradiction shows that Q is open.

Finally, Q is connected, since if Q_1 and Q_2 would be a partition of Q into two clopen parts, then each of those members would be clopen in $D \setminus \{x\}$ and Q would not be a quasi-component. So the collection of quasi-components coincides with the collection of components and the components of $D \setminus \{x\}$ are open. \square

COROLLARY 2.2. *The collection*

$$U(D) = \{U \subset D \mid \exists x \in D \text{ such that } U \text{ is a component of } D \setminus \{x\}\}$$

is an open subbase for the topology of D .

PROOF. If $x, y \in D$ are distinct, then, since D is a dendron there are disjoint $U, V \in U(D)$ with $x \in U$ and $y \in V$. By compactness this easily implies that $U(D)$ is an open subbase. \square

Elements of $U(D)$ are called *cutpoint components*. Define

$$J(D) = \{D \setminus U \mid U \in U(D)\}.$$

Observe that $J(D)$ is a subbase for the closed subsets of D .

LEMMA 2.3. *$J(D)$ consists of connected sets.*

PROOF. Follows directly from the proof of Lemma 2.1. \square

A collection L of subsets of a set X is called *cross-free* provided that for all $L_0, L_1 \in L$ it is true that $L_0 \subset L_1$ or $L_1 \subset L_0$ or $L_0 \cap L_1 = \emptyset$ or $L_0 \cup L_1 = X$.

LEMMA 2.4. *$U(D)$ is cross-free.*

PROOF. Assume that U_1 and U_2 are cutpoint components of $D \setminus \{x_1\}$ (resp. $D \setminus \{x_2\}$). If $x_1 = x_2$ then U_1 and U_2 are clearly either disjoint or equal, and both those possibilities are permitted by the definition of cross-free collections. If $x_1 \neq x_2$ then we distinguish three subcases:

- (a) $x_1 \in U_2$ and $x_2 \in U_1$. Now each cutpoint component C of $D \setminus \{x_1\}$ which does not contain x_2 is a connected subset of D and hence, by connectivity (Lemma 2.1), is contained in U_2 . So $U_2 \cup U_1 = D$.
- (b) $x_1 \notin U_2$. This means that U_2 is a connected subset of $D \setminus \{x_1\}$ and hence either is contained in or disjoint from the cutpoint component U_1 of $D \setminus \{x_1\}$.
- (c) $x_2 \notin U_1$. This case is similar to the previous one. \square

COROLLARY 2.5. $J(D)$ is cross-free. \square

A collection of subsets L of a set X is called *normal* provided that for all disjoint $L_0, L_1 \in L$ there are $S_0, S_1 \in L$ with

$$L_0 \cap S_1 = \emptyset = S_0 \cap L_1 \quad \text{and} \quad S_0 \cup S_1 = X.$$

The sets S_0 and S_1 are called a *screening* of L_0 and L_1 . A collection of subsets L of a set X is called *connected* if there is no partition of X by two non-empty members of L .

LEMMA 2.6. Every cross-free closed subbase J for a connected Hausdorff space X is normal and hence $J(D)$ is normal.

PROOF. Take two disjoint non-empty members T_0 and T_1 from J . Since T_0 is closed and X is connected there exists a point $t_0 \in T_0 \cap (X \setminus T_0)^-$ and similarly we find a point $t_1 \in T_1 \cap (X \setminus T_1)^-$. Since X is Hausdorff we can find two basic closed sets B_0 and B_1 such that $B_0 \cup B_1 = X$, $t_0 \notin B_1$ and $t_1 \notin B_0$. Moreover,

$$B_0 = F_0 \cup F_1 \cup \dots \cup F_m \quad \text{and} \quad B_1 = F_{m+1} \cup F_{m+2} \cup \dots \cup F_n,$$

for a suitably chosen finite subcollection F_0, \dots, F_n of J . Without loss of generality we may assume that no F_i is contained in some F_j . Assume that $t_0 \in F_i \cap F_j$. Then $t_1 \notin F_i \cup F_j$ and since J is cross-free we conclude that either $F_i \subset F_j$ or $F_j \subset F_i$. This means that we can have at most one F , say F_0 , which contains t_0 and one F , say F_n , which contains t_1 .

If some F contains neither t_0 nor t_1 but has an intersection with F_0 then we can choose $t_2 \in F \cap F_0$ and the same argument shows then that $F \subset F_0$ and hence F is superfluous. So we have F_0, F_n , and a collection of F 's disjoint from F_0 and F_n . If there is a point $t_3 \in F$ which is not contained in $F_0 \cup F_n$ then a similar argument shows that $F_0 \cap F_n$ is empty and we have a partition of the space in three disjoint closed subsets, namely F_0, F_n and $\cup\{F_i \mid 0 < i < n\}$. This is a contradiction and we obtain that $F_0 \cup F_n = X$.

Finally we show that $F_n \cap T_0 = \emptyset$. Since $t_0 \in T_0 \setminus F_n$ and $t_1 \in F_n \setminus T_0$, and since t_0 is neither in the interior of T_0 nor in the closure of F_n we obtain that $T_0 \cup F_n \neq X$. We conclude that $T_0 \cap F_n = \emptyset$ and similarly that $T_1 \cap F_0 = \emptyset$ which means that J is normal. \square

A collection of subsets L of a set X is called *binary* provided that for all $M \subset L$ with $\cap M = \emptyset$ there are $M, N \in M$ with $M \cap N = \emptyset$.

LEMMA 2.7. *If X is a compact connected Hausdorff space and its closed subbase J is cross-free then J is binary. Consequently, $J(D)$ is binary.*

PROOF. Suppose not. Assume that M is a subfamily of J in which every two members have a non-empty intersection. We have that X is compact and so $\cap M = \emptyset$ implies that there is a finite subcollection of M containing a minimal number of sets M_1, \dots, M_n which has an empty intersection. Now if $i \neq j$ then $M_i \cap M_j \neq \emptyset$ and M_i is not contained in M_j . So $M_i \cup M_j = X$. In particular, $M_i \cup M_n = X$ for $0 < i < n$ and hence $M_n \cup [\cap_{0 < i < n} M_i] = X$. Moreover, $M_n \cap [\cap_{0 < i < n} M_i] = \emptyset$ which implies that M_n is clopen, contradicting that X is connected. \square

If $x, y \in X$ and if J is a subbase for X then put

$$I_J(x, y) = \cap \{T \in J \mid x, y \in T\}.$$

For notational simplicity, $I_{J(D)}(x, y)$ will be denoted by $I(x, y)$.

LEMMA 2.8. *If $C \subset D$ is an intersection of elements of $J(D)$, then the function $r_C: D \rightarrow C$ defined by*

$$\{r_C(x)\} = \cap_{c \in C} I(x, c) \cap C$$

is a retraction.

PROOF. From the binarity of $J(D)$, Lemma 2.7, it follows that

$$E = \bigcap_{c \in C} I(x, c) \cap C \neq \emptyset.$$

Suppose that there are two distinct points $e_0, e_1 \in E$. Find $T_0, T_1 \in J(D)$ with $e_0 \in T_0 \setminus T_1$, $e_1 \in T_1 \setminus T_0$ and $T_0 \cup T_1 = D$. If $x \in T_0$ then

$$E = \bigcap_{c \in C} I(x, c) \cap C \subset I(x, e_0) \subset T_0,$$

which is impossible since $e_1 \notin T_0$. Similarly we find that $x \notin T_1$. This contradiction shows that r_C is well-defined. Obviously, $r_C(x) = x$ for all $x \in C$.

The only remaining part is to show that r_C is continuous. Let $x \in D$ and suppose that $r_C(x) \notin A \cap C$, for some A in $J(D)$ which intersects C . Since $J(D)$ is binary there is a $c \in C$ such that $I(x, c) \cap A = \emptyset$, and we can find a $B \supset I(x, c)$ such that $B \in J(D)$ and $B \cap A = \emptyset$. Now we can find two sets S_1 and S_2 in $J(D)$ such that $S_1 \cup S_2 = D$, $S_1 \cap A = \emptyset$ and $S_2 \cap D = \emptyset$ (Lemma 2.6). For every point p of the open set $D \setminus S_2$ we obtain that $r_C(p) \notin A$ because $I(p, c) \subset S_1$ which misses A . This proves continuity. \square

The retraction of Lemma 2.8 is called the *canonical retraction* of D onto C .

COROLLARY 2.9. *If $C \subset D$ is an intersection of elements of $J(D)$, then C is connected.* \square

COROLLARY 2.10. *D is locally connected.*

PROOF. Take $x \in D$ and let U be an open neighbourhood of x . Since, by Corollary 2.2, $J(D)$ is a closed subbase for D , we can find finitely many $T_1, T_2, \dots, T_n \in J(D)$ with $x \notin \bigcup_{1 \leq i \leq n} T_i \supset D \setminus U$. Since $J(D)$ is binary (Lemma 2.7) for each $i \leq n$ we can find $T_i' \in J(D)$ with $x \in T_i'$ and $T_i' \cap T_i = \emptyset$ (observe that $\{x\} = \bigcap \{T \in J(D) \mid x \in T\}$). By the normality of $J(D)$, (Lemma 2.6) we can find for each $i \leq n$ an element $T_i'' \in J(D)$ with $T_i' \subset T_i''$, $x \in \text{int}(T_i'')$ and $T_i'' \cap T_i = \emptyset$. Put

$$T = \bigcap_{1 \leq i \leq n} T_i''.$$

Then T is a neighbourhood of x which is contained in U and which, by Corollary 2.9, is connected. \square

For all $x, y \in D$ define

$$S(x, y) = \{p \in D \mid p \text{ separates } x \text{ from } y\} \cup \{x, y\}.$$

We claim that $S(x, y) = I(x, y)$, where $I(x, y)$ is defined as above. We establish that claim in our next two lemmas.

LEMMA 2.11.

$$S(x, y) \subset I(x, y).$$

PROOF. Take $p \in S(x, y) \setminus \{x, y\}$. Then $D \setminus \{p\} = U \cup V$, where U and V are disjoint open subsets of D with $x \in U$ and $y \in V$. Since $I(x, y)$ is connected (Corollary 2.9) and since $x \in I(x, y) \cap U$, $y \in I(x, y) \cap V$, this implies that $p \in I(x, y)$. \square

LEMMA 2.12.

$$I(x, y) \subset S(x, y).$$

PROOF. Let $p \in I(x, y) \setminus S(x, y)$. Suppose that $q \in S(x, y)$ and that $U_x(q)$ (resp. $U_y(q)$) are the cutpoint components of x (resp. y) in $D \setminus \{q\}$. If $p \notin U_x(q) \cup U_y(q)$ then there is a cutpoint component $U_p(q)$ and x and y are both in $D \setminus U_p(q)$, which means that $p \notin I(x, y)$. Therefore every $q \in S(x, y)$ either separates x and p or y and p and $S(x, y) \subset S(x, p) \cup S(y, p)$.

Conversely, if $q \in S(x, p)$ then no cutpoint component of $D \setminus \{q\}$ contains both x and y , since in that case $D \setminus U_p(q)$ contains both x and y in contradiction with $p \in I(x, y)$. So $q \in S(x, y)$ and $S(x, p) \subset S(x, y)$. Similarly $S(p, y) \subset S(x, y)$. Therefore

$$S(x, y) = S(x, p) \cup S(p, y).$$

Define

$$A_x = \bigcup_{q \in S(x, p)} U_x(q) \quad \text{and} \quad A_y = \bigcup_{q \in S(y, p)} U_y(q)$$

Then A_x and A_y are both open. Define

$$A_p = D \setminus (A_x \cup A_y \cup \{p\}).$$

We claim that A_p is open. Let $a \in A_p$ and separate a and p with a point s . Then $s \notin (S(x,p) \cup S(y,p))$. If $U_a(s) \cap A_x \neq \emptyset$ then $\exists r \in S(x,p)$ such that:

$$U_a(s) \cap U_x(r) \neq \emptyset, \quad p \notin U_a(s) \cup U_x(r),$$

$$a \in U_a(s) \setminus U_x(r), \quad x \in U_x(r) \setminus U_a(s),$$

which contradicts Lemma 2.4. Therefore $U_a(s) \cap A_x = \emptyset$, and $U_a(s) \cap A_y = \emptyset$.

$\bigcup_{a \in A_p} U_a(s) = A_p$ so we obtain that A_x , A_y and A_p are a partition of $D \setminus \{p\}$ into open parts, i.e. p is a cutpoint which separates x and y . This contradicts the assumption that $p \notin S(x,y)$ which proves the lemma. \square

COROLLARY 2.13. *If $x, y \in D$, then $I(x,y) = S(x,y)$.* \square

COROLLARY 2.14. *If $C \subset D$ is a subcontinuum, then $C = \bigcap \{T \in J(D) \mid C \subset T\}$.*

PROOF. Take $x \notin C$ and $c \in C$ arbitrarily. Since $I(x,c)$ is connected and $x \notin C$ there has to be a point $y \in I(x,c) \setminus C$ different from x . By Corollary 2.13, y separates c from x . Let U be the component of $D \setminus \{y\}$ containing x . Since C is connected and U is open, $D \setminus (U \cup \{y\})$ is open. Since $y \notin C$ we may conclude that $C \cap U = \emptyset$. Consequently, $T = D \setminus U \in J(D)$ contains C but misses x . \square

COROLLARY 2.15.

- (1) $S(x,y) = \bigcap \{C \subset D \mid x, y \in C \text{ and } C \text{ is a continuum}\}$.
- (2) *Each subcontinuum $C \subset D$ is a retract of D under the retraction $r_C: D \rightarrow C$ defined by*

$$\{r_C(x)\} = \bigcap_{c \in C} S(x,c) \cap C.$$

- (3) *The intersection of an arbitrary family of subcontinua of D is either empty or is a continuum.*

PROOF. Combine Corollary 2.14 and, respectively, Corollary 2.13 and Lemma 2.8. \square

The retraction r_C is called the *canonical retraction of D onto C* .

LEMMA 2.16. *If $a, b, c \in D$ then $S(a,b) \cap S(a,c) \cap S(b,c)$ is a singleton.*

PROOF. By Corollary 2.13 and the binarity of $J(D)$ (Lemma 2.7), we have

$$E = S(a,b) \cap S(b,c) \cap S(a,c) \neq \emptyset.$$

Assume that there are distinct $x, y \in E$. Find $S, T \in J(D)$ with $x \in S \setminus T$, $y \in T \setminus S$ and $T \cup S = D$. At least two points of $\{a, b, c\}$ must be contained in S or T . So, without loss of generality, $a, b \in S$. Then

$$E \subset S(a,b) = I(a,b) \subset S,$$

which is a contradiction since $y \in E \setminus S$. \square

LEMMA 2.17. *If $x, y \in D$ are distinct, $p \in I(x, y)$ and $q \in I(x, y) \setminus I(x, p)$, then $q \in I(p, y)$.*

PROOF. Clearly $q \neq x$ and if $q = y$ then there is nothing to prove. So assume that $q \neq y$. Write $D \setminus \{q\} = U \cup V$ where U and V are disjoint and open, $x \in U$ and $y \in V$. Since $q \notin I(x, p)$ and since $I(x, p)$ is connected (Lemma 2.8) we conclude that $I(x, p) \subset U$. Therefore, by the connectivity of $I(p, y)$ this implies that $q \in I(p, y)$. \square

COROLLARY 2.18. *If $x, y \in D$ are distinct, then $S(x, y)$ is a linearly ordered continuum with order defined by $p \leq q$ iff p separates x from q .*

PROOF. From Corollary 2.13 the relation \leq can also be defined by $p \leq q$ iff $p \in I(x, q)$. If $p \leq q$ and $q \leq p$ then $p \in I(x, q)$, consequently

$$p \in I(x, p) \cap I(p, q) \cap I(x, q).$$

Similarly

$$q \in I(x, p) \cap I(p, q) \cap I(x, q).$$

This implies that $p = q$ (Lemma 2.16). Now we show that \leq is a partial order. If $p \leq q$ and $q \leq r$ then $p \in I(x, q)$ and $q \in I(x, r)$. Therefore $p \in I(x, q) \subset I(x, r)$ or equivalently, $p \leq r$. Let us now show that \leq is linear. Take $p, q \in I(x, y)$ such that $p \not\leq q$ and $q \not\leq p$. Then $p \notin I(x, q)$, hence $p \in I(q, y)$ (Lemma 2.17). Similarly, $q \in I(p, y)$. Therefore

$$p \in I(p, q) \cap I(p, y) \cap I(q, y)$$

and

$$q \in I(p,q) \cap I(p,y) \cap I(q,y),$$

consequently by Lemma 2.16, $p = q$ which is a contradiction.

Let us now show that \leq generates the topology of $I(x,y)$. Clearly

$$\{q \in I(x,y) \mid q \leq p\} = I(x,p)$$

and by Lemma 2.17,

$$\{q \in I(x,y) \mid p \leq q\} = I(p,y).$$

Therefore the initial segments are closed in $I(x,y)$. By the compactness of $I(x,y)$ this implies that \leq generates the topology of $I(x,y)$. \square

NOTES. (for Section 2). Lemma 2.1 (that cutpoint components are open) is due to KOK [9]; see also WARD [23].

The fact that the intersection of an arbitrary family of subcontinua of D is a subcontinuum and that each set of the form $S(x,y)$ is orderable by the order of 2.18 is well-known. See HOCKING & YOUNG [8], MOORE [16], and WHYBURN [27]. The approach developed in this section is implicit in VAN MILL & SCHRIJVER [11], VAN MILL & VAN DE VEL [12] and VAN MILL [10]. The Corollaries 2.10 and 2.14 and some other results are related to the results of GURIN [7], PROIZVOLOV [18], and WARD [23].

3. THE THEOREM OF CORNETTE AND BROUWER

In this section we will show that each dendron is a continuous image of an ordered continuum. We will assume that the reader is familiar with the theory of inverse systems and inverse limits.

Let L and M be ordered continua. A continuous *surjection* $f: L \rightarrow M$ is called *order preserving* if $f(x) \leq f(y)$ for all $x, y \in L$ with $x \leq y$.

LEMMA 3.1. *Let $(L_\alpha, f_{\alpha\beta}, \alpha \in A)$ be an inverse system of ordered continua such that each $f_{\alpha\beta}$ is order preserving. Then $\lim_{\leftarrow} (L_\alpha, f_{\alpha\beta}, \alpha \in A)$ is an ordered continuum.*

PROOF. For each $\alpha \in A$ let $\pi_\alpha: L \rightarrow L_\alpha$ be the projection. Define an order \leq on L by putting

$$x \leq y \text{ iff } \forall \alpha \in A: \pi_\alpha(x) \leq \pi_\alpha(y).$$

It is clear that \leq is a linear order on L which generates the topology of L . It is well-known that the inverse limit of an inverse system consisting of continua is a continuum. Hence L is an ordered continuum. \square

LEMMA 3.2. *Let D be a dendron and let κ be an ordinal. For each $\alpha < \kappa$ let $D_\alpha \subset D$ be a subcontinuum such that $\beta < \alpha$ implies that $D_\beta \subset D_\alpha$. If $r_{\alpha\beta}: D_\alpha \rightarrow D_\beta$ denotes the canonical retraction, then*

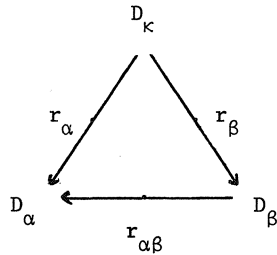
$$\lim_{\leftarrow} (D_\alpha, r_{\alpha\beta}, \alpha < \kappa)$$

is homeomorphic to the closure of $\bigcup_{\alpha < \kappa} D_\alpha$.

PROOF. Let D_κ denote the closure of $\bigcup_{\alpha < \kappa} D_\alpha$ and for each $\alpha < \kappa$ let $r_\alpha: D_\kappa \rightarrow D_\alpha$ be the canonical retraction. It is easy to see that for each $\alpha < \beta < \kappa$ the diagram below commutes, which implies, by compactness, that the function

$$\psi: D_\kappa \rightarrow \lim_{\leftarrow} (D_\alpha, r_{\alpha\beta}, \alpha < \kappa)$$

defined by $\psi(x)_\alpha = r_\alpha(x)$ is a continuous surjection. It therefore suffices



to show that ψ is one to one. To this end, take distinct $x, y \in D_\kappa$. Let V and W be disjoint and connected neighbourhoods of, respectively, x and y (Corollary 2.10). It is clear that for some $\alpha < \kappa$ we have that $V \cap D_\alpha \neq \emptyset \neq D_\alpha \cap W$. Take a point $s \in V \cap D_\alpha$ and a point $t \in W \cap D_\alpha$. Since V is a continuum,

$$I(x, s) \subset V$$

which implies that

$$\{r(x)\} = \bigcap_{d \in D_\alpha} I(x, d) \cap D_\alpha \subset I(x, s) \subset V$$

(Corollary 2.15). We conclude that $r_\alpha(x) \in V$ and, similarly, $r_\alpha(y) \in W$. Consequently, $r_\alpha(x) \neq r_\alpha(y)$. Therefore $\psi(x) \neq \psi(y)$ and ψ is one-to-one. \square

We now come to the main result of this section.

THEOREM 3.3. *Let D be a dendron. Then D is a continuous image of an ordered continuum.*

PROOF. Let $\kappa = |D|$ and let

$$\{d_\alpha \mid \alpha < \kappa \text{ and } \alpha \text{ is a successor}\},$$

enumerate D .

By transfinite induction, for every $\alpha < \kappa$ we will construct a subcontinuum $D_\alpha \subset D$ and an ordered continuum L_α and for each $\beta < \alpha$ an order preserving map $f_{\alpha\beta}: L_\alpha \rightarrow L_\beta$ and a continuous surjection $\pi_\alpha: L_\alpha \rightarrow D_\alpha$ such that for each $\beta < \alpha$ the diagram below commutes. Here $r_{\alpha\beta}$ denotes the canonical retraction.

$$\begin{array}{ccc} L_\beta & \xleftarrow{f_{\alpha\beta}} & L_\alpha \\ \pi_\beta \downarrow & & \downarrow \pi_\alpha \\ D_\beta & \xleftarrow{r_{\alpha\beta}} & D_\alpha \end{array}$$

In addition we will construct the D_α 's in such a way that $d_\alpha \in D_\alpha$ for each successor $\alpha < \kappa$. The construction is a triviality.

Let $D_0 = L_0 = \{d_0\}$ and let π_0 be the identity. Suppose that we have constructed everything for all $\beta < \alpha$. If α is a limit put

$$D_\alpha = \left(\bigcup_{\beta < \alpha} D_\beta \right)^- \quad \text{and} \quad L_\alpha = \lim_{\leftarrow} (L_\beta, f_{\beta\eta}, \beta < \alpha)$$

and define all maps in the obvious way (applying the Lemmas 3.1 and 3.2). If α is a successor and if $d_\alpha \in D_{\alpha-1}$ then we don't do anything, i.e. put $D_\alpha = D_{\alpha-1}$, etc. So suppose that $d_\alpha \notin D_{\alpha-1}$. Let $r: D_\alpha \rightarrow D_{\alpha-1}$ be the canonical retraction and put

$$D_\alpha = D_{\alpha-1} \cup I(d_\alpha, r(d_\alpha)).$$

Observe that $D_{\alpha-1} \cap I(d_\alpha, r(d_\alpha)) = \{r(d_\alpha)\}$. Take a point $y \in L_{\alpha-1}$ with $\pi_{\alpha-1}(y) = r(d_\alpha)$. In $L_{\alpha-1}$ replace $\{y\}$ by an "interval" which maps onto $I(d_\alpha, r(d_\alpha))$ in such a way that the endpoints of this interval are mapped onto $r(d_\alpha)$ (one can take for example two copies of $I(d_\alpha, r(d_\alpha))$ with the points corresponding to d_α identified).

Let L_α be the resulting space and let $\pi: L_\alpha \rightarrow D_\alpha$ be a map with the property that

$$\pi_\alpha(x) = \pi_{\alpha-1}(x) \text{ if } x \in L_{\alpha-1} \setminus \{\text{the endpoints of the added interval}\}.$$

In addition, let $f_{\alpha, \alpha-1}: L_\alpha \rightarrow L_{\alpha-1}$ be the map which collapses the added interval to the point y . It is clear that everything defined in this way is as required. Now put

$$L = \lim_{\leftarrow} (L_\alpha, f_{\alpha\beta}, \alpha < \kappa).$$

By Lemma 3.1, L is an ordered continuum which, by the diagram, maps onto D . \square

COROLLARY 3.4. *Every dendron is hereditarily normal.*

NOTES. (for Section 3). Theorem 3.3 was first shown by CORNETTE [3] and independently, but later, by A.E. BROUWER [1]. Our proof is a simplification of their ideas; see also PEARSON [17] and WARD [26].

A Souslin dendron is a dendron D which satisfies the countable chain condition, is not separable, and which moreover has the property that each countable subset is contained in a metrizable subcontinuum of D . If the above program is carried out with some extra care, it can be shown that each Souslin dendron is a continuous image of a Souslin continuum. In addition, each Souslin continuum can be mapped onto a Souslin dendron. Notice that a Souslin continuum (= a linearly orderable CCC non-separable continuum) is not a Souslin dendron. For details see VAN MILL & WATTEL [13].

Lemma 3.1 is due to CAPEL [2], and Corollary 3.4 is due to GURIN [7], see also PROIZVOLOV [19].

4. THE FIXED POINT PROPERTY

In this section we show that every dendron has the fixed point property.

LEMMA 4.1. *Let L be an ordered continuum. Then L has the fixed point property.*

PROOF. Let $f: L \rightarrow L$ be any self map and put

$$U = \{x \in L \mid x < f(x)\}, \quad \text{and} \quad V = \{x \in L \mid f(x) < x\}$$

respectively. Then U and V are clearly open. Suppose that f has no fixed point. Then $U \cup V = L$ and hence, since $U \cap V = \emptyset$, by connectivity, either $U = \emptyset$ or $V = \emptyset$. If $U = \emptyset$, then $f(\min(L)) < \min(L)$, and if $V = \emptyset$ then $\max(L) < f(\max(L))$, which is impossible. \square

Let D be a dendron. A point $x \in D$ is called an *endpoint* if $D \setminus \{x\}$ is connected. A *finite dendron* is a dendron with only a finite number of endpoints. Note that a finite dendron is nothing but a finite connected acyclic graph.

LEMMA 4.2. *Let D be a finite dendron. Then D has the fixed point property.*

PROOF. Let E denote the set of endpoints of D . We induct on $|E|$. If $|E| \leq 2$ then use Lemma 4.1. So assume that the lemma is true for n and assume that $|E| = n+1$; list E as $\{e_1, \dots, e_{n+1}\}$. Put

$$D' = \cup\{I(e_i, e_j) \mid i, j \in \{1, 2, \dots, n\}\}.$$

Then D' is a subcontinuum of D and hence D' is a dendron (Corollary 2.15(1)). Also D' has precisely n endpoints. Let $r_{D'}: D \rightarrow D'$ be the canonical retraction (Corollary 2.15(2)) and put $x = r_{D'}(e_{n+1})$. Observe that

$$I(e_{n+1}, x) \cap D' = \{x\} \quad \text{and that} \quad I(e_{n+1}, x) \cup D' = D.$$

By Corollary 2.18, $I(e_{n+1}, x)$ is an ordered continuum. Let $f: D \rightarrow D$ be any self-map. Assume that f has no fixed points. If $f(x) \in D'$ then define $g: D' \rightarrow D'$ by

$$\begin{aligned} g(t) &= f(t) & \text{if } f(t) \in D' \\ g(t) &= x & \text{if } f(t) \notin D' \end{aligned}$$

(we just collapse the interval $I(e_{n+1}, x)$ to the point x). By induction hypothesis, g has a fixed point. This point cannot be x and hence must be a fixed point of f . If $f(x) \in I(e_{n+1}, x)$ then we collapse D' to the point x and proceed in the same way. This gives us the required contradiction. \square

We now come to the main result of this section.

THEOREM 4.3. *Let D be a dendron. Then D has the fixed point property.*

PROOF. Let $f: D \rightarrow D$ be any self-map. If f has no fixed point then, by compactness and by the local connectedness of D (Corollary 2.10), there is a finite cover \mathcal{U} of D by non-empty subcontinua such that for every $U \in \mathcal{U}$ we have that

$$U \cap f(U) = \emptyset.$$

Let $F \subset X$ be finite such that for all $U \in \mathcal{U}$ both $F \cap U$ and $F \cap f(U)$ are non-empty. Define

$$D' = \bigcup \{I(x,y) \mid x,y \in F\}.$$

Observe that D' is a finite dendron. Define $g: D' \rightarrow D'$ by

$$g(x) = r_{D'}(f(x)),$$

where $r_{D'}: D \rightarrow D'$ is the canonical retraction (Corollary 2.15(2)). We claim that g has no fixed points which contradicts Lemma 4.2. Take $x \in D'$. There is a $U \in \mathcal{U}$ containing x . Then $f(x) \in f(U)$. Since $f(U)$ is a continuum that intersects D' (observe that $F \subset D'$), by Corollary 2.15(2),

$$r_{D'}(f(x)) \in f(U),$$

consequently, $g(x) \neq x$ since $U \cap f(U) = \emptyset$. \square

NOTES. (for Section 4). Lemma 4.1 is well-known. Theorem 4.3 was first shown by SCHERRER [20] and generalized by WALLACE [22], see also WARD [24], [25].

5. A CHARACTERIZATION OF DENDRONS

In this section we show that a Hausdorff continuum X is a dendron if and only if X possesses a cross-free closed subbase.

LEMMA 5.1. *Let X be a T_1 space and let J be a binary closed subbase for X . Then for any distinct $x, y \in X$ there are disjoint $T_0, T_1 \in J$ with $x \in T_0$ and $y \in T_1$.*

PROOF. Observe that, since X is T_1 and since J is a closed subbase, for every point $z \in X$ it is true that

$$\{z\} = \bigcap \{T \in J \mid z \in T\}.$$

Consequently, the desired result follows directly from the binarity of J . \square

We now come to the main result in this section.

THEOREM 5.2. *Let X be a Hausdorff continuum. Then X is a dendron iff X possesses a cross-free closed subbase.*

PROOF. For the implication "dendron \Rightarrow \exists cross-free closed subbase" see Section 2. So let X be a Hausdorff continuum and let J be a cross-free closed subbase for X . Let $x, y \in X$ such that $x \neq y$. Let $x \in T_0$ and $y \in T_1$ such that $T_0, T_1 \in J$ and $T_0 \cap T_1 = \emptyset$, (cf. 5.1). According to Lemma 2.6 we can find $S_0, S_1 \in J$ such that $S_0 \cup S_1 = X$, and $S_0 \cap T_1 = \emptyset = S_1 \cap T_0$.

Define

$$A = \{T \in J \mid T \cup S_0 = X\}.$$

Since X is connected we have that $A \cup \{S_0\}$ has the property that every two of its elements meet and consequently, by binarity of J (Lemma 2.7), $(\bigcap A) \cap S_0 \neq \emptyset$. We claim that this intersection consists of one point.

Assume to the contrary that $z_0, z_1 \in (\bigcap A) \cap S_0$ such that $z_0 \neq z_1$. In the same way as above there are $R_0, R_1 \in J$ such that $z_0 \in R_0 \setminus R_1$ and $z_1 \in R_1 \setminus R_0$ and $R_0 \cup R_1 = X$. Since $z_0 \notin R_1$ and $z_0 \in \bigcap A$ we have that $R_1 \notin A$ and consequently $R_1 \cup S_0 \neq X$. Hence $S_0 \subset R_1$ or $R_1 \subset S_0$ because $R_1 \cap S_0 = \emptyset$ is impossible since $z_1 \in R_1 \cap S_0$. However, this implies that $R_1 \subset S_0$ since $z_0 \notin R_1$. With the same technique one shows that $R_0 \subset S_0$; but this is a contradiction because $S_0 \neq X$. Let $z_0 = (\bigcap A) \cap S_0$, then z_0 is a separation point of x and y , since S_0 and $\bigcap A$ are closed subsets of X such that $(\bigcap A) \cup S_0 = X$ and $x \in S_0$ and $y \in \bigcap A$. This proves that X is a dendron. \square

NOTES. (for Section 5). Theorem 5.3 is due to VAN MILL & SCHRIJVER [11] and is related to a characterization of ordered spaces in VAN DALEN & WATTEL [4].

6. A CHARACTERIZATION OF SUBSPACES OF DENDRONS

In this section we will use the results of the previous sections to show that a Hausdorff space X can be embedded in a dendron iff X has a cross-free closed subbase. We first show how to modify a given cross-free closed subbase to one with certain additional pleasant properties. Then we use this modified subbase to obtain embeddings into dendrons.

A closed subbase S for a space X is called a T_1 -subbase provided that for all $x \in X$ and $S \in S$ not containing x there exists an element $T \in S$ with $x \in T$ and $T \cap S = \emptyset$.

LEMMA 6.1. *Let X be a Hausdorff space with a cross-free closed subbase S . Then there is a cross-free closed subbase for X which in addition is normal and T_1 .*

PROOF. First of all we extend S to a larger subbase S^t by taking:

$$S^t = S \cup \{\{p\} \mid p \in X\}$$

(i.e. we add all singletons to the subbase). In this case S^t is still cross-free because $\{p\} \cap \{q\} = \emptyset$ for all $p \neq q$ and either $\{p\} \cap S = \emptyset$ or $\{p\} \subset S$ for each $S \in S$. Clearly the subbase S^t is a T_1 collection.

Next we add for each clopen $S \in S^t$ also its complement and obtain

$$S^n = S^t \cup \{X \setminus S \mid S \in S^t \text{ and } S \text{ is clopen}\}.$$

Also S^n is a T_1 collection which is cross-free since if $S, R \in S^t$ then

$$\begin{aligned} S \subset R &\text{ implies } X \setminus S \supset X \setminus R \text{ and } (X \setminus S) \cup R = X, \\ R \subset S &\text{ implies } X \setminus S \subset X \setminus R \text{ and } (X \setminus S) \cap R = \emptyset, \\ R \cap S = \emptyset &\text{ implies } (X \setminus S) \cup (X \setminus R) = X \text{ and } R \subset X \setminus S, \\ R \cup S = X &\text{ implies } (X \setminus S) \cap (X \setminus R) = \emptyset \text{ and } X \setminus S \subset R. \end{aligned}$$

We now show that S^n is not only cross-free but is in addition normal.

Let R and S be two disjoint members of S^n . If S is clopen then also $X \setminus S$ is in S^n and we obtain a screening between S and R by S and $X \setminus S$, and the same holds for R . If neither S nor R is clopen then we can find a point $r \in R$ and a point $s \in S$ such that $r \in \mathcal{C}_X(X \setminus R)$ and $s \in \mathcal{C}_X(X \setminus S)$.

Next we will derive a screening of $\{s\}$ and $\{r\}$ by means of two subbase members. Since X is Hausdorff we can find two basic closed subsets B_s and B_r such that $B_s \cup B_r = X$, $r \notin B_s$ and $s \notin B_r$. B_r is a finite union of subbase members F_{r1}, \dots, F_{rn} , and B_s is a finite union of F_{s1}, \dots, F_{sm} .

Define $F = \{F_{si}\} \cup \{F_{rj}\}$ and $F_s = \{F_{sj} \mid s \in F_{sj}\}$, then for F_{si} and $F_{sj} \in F_s$ we have that

$$s \in F_{si} \cap F_{sj} \quad \text{and} \quad r \notin F_{si} \cup F_{sj}$$

hence either $F_{si} \subset F_{sj}$ or $F_{sj} \subset F_{si}$ and so there exists a largest member $F_s = \cup F_s \in F$. In the same way there is a maximal F_r in F which contains r . We now have two cases. If $F_s \cup F_r = X$ then we have obtained our screening with two members of S .

In the other case we can find a point x in $X \setminus (F_s \cup F_r)$. Let F_x be the maximal member of F containing x . Since

$$r \notin F_x \cup F_s; \quad s \in F_s \setminus F_x \quad \text{and} \quad x \in F_x \setminus F_s$$

we have

$$F_x \cap F_s = \emptyset \quad \text{and similarly} \quad F_x \cap F_r = \emptyset \quad \text{and} \quad F_s \cap F_r = \emptyset.$$

Consequently, we obtain a partition of the space into three disjoint closed parts: F_s , F_r and $U\{F_x \mid x \notin F_s \cup F_r\}$. (The last collection is closed since it is the union of a finite collection because F is finite.) This means that F_s is clopen and $X \setminus F_s$ is in S^n .

Anyway we obtain a screening of s and r by means of two subbase members, call them F'_s and F'_r . Now S does not contain a neighbourhood of s and F'_r is closed and does not contain s and hence $S \cup F'_r \neq X$. Moreover, $s \in S \setminus F'_r$ and $r \in F'_r \setminus S$ and therefore $F'_r \cap S = \emptyset$ and similarly $F'_s \cap R = \emptyset$. Since $F'_s \cup F'_r = X$ we have $R \subset F'_r$ and $S \subset F'_s$ and we obtained a screening of R and S . \square

REMARK 6.2. In the previous lemma the Hausdorff property cannot be omitted since in an infinite space with the cofinite topology the collection of all singletons is a cross-free T_1 subbase, but it cannot have a T_1 normal subbase since a space with a T_1 normal subbase is completely regular (cf. [5]).

A collection S of subsets of a set X is called *strongly connected* provided that X cannot be partitioned into finitely many non-empty elements of S .

LEMMA 6.3. *Let X be a set and let S be cross-free and connected. Then S is strongly connected.*

PROOF. From 6.1 it follows that S is normal and T_1 . Assume that there exists a number n with the property that there is a minimal collection S_1, S_2, \dots, S_n of mutually disjoint sets such that $\bigcup_{1 \leq i \leq n} S_i = X$, but for every number smaller than n there is no such partition of X with members of S . Since S_1 and S_n are disjoint there are two subsets T_1 and T_n in S such that $T_1 \cap S_n = \emptyset$ and $T_n \cap S_1 = \emptyset$ and $T_n \cup T_1 = X$. Let $1 < j < n$ then either $S_j \cap T_1 \neq \emptyset$ or $S_j \cap T_n \neq \emptyset$, say $S_j \cap T_1 \neq \emptyset$. Then $S_j \cup T_1 \neq X$ because S_n is disjoint from both, and therefore $S_j \subset T_1$. Let $J = \{j \mid S_j \subset T_1\}$. Then $\bigcup_{i \notin J} S_i \cup T_1 = X$, is a disjoint cover of X with less than n members. This contradiction shows our lemma. \square

COROLLARY 6.4. *Let X be a compact Hausdorff space and let S be a cross-free connected subbase for X . Then X is connected (and consequently, X is a dendron).*

PROOF. Suppose that X is equal to $G \cup H$ with $G \cap H = \emptyset$ and G and H are closed. Then H is an intersection of a collection of closed base members $\{B_\alpha\}_{\alpha \in A}$ for some index set A . Since $\bigcap_\alpha B_\alpha \cap G = \emptyset$ and since X is compact there is a finite subcollection of B_α 's which misses G and therefore G and H are both finite intersections of finite unions of members of S . We could also write G and H as finite unions of finite intersections of subbasic closed sets. Let m be the minimal number such that there are G_1, \dots, G_m such that:

- (a) G_1, \dots, G_m are non-void intersections of finitely many subbase members;
- (b) $G_1 \cup \dots \cup G_m = X$;
- (c) There is a number $k < m$ such that

$$\bigcup_{1 \leq i \leq k} G_i \neq \emptyset \neq \bigcup_{k < i \leq m} G_i$$

and

$$\left(\bigcup_{1 \leq i \leq k} G_i \right) \cap \left(\bigcup_{k < i \leq m} G_i \right) = \emptyset.$$

We claim that $G_i \cap G_j = \emptyset$ for $i \neq j$, (w.l.o.g. $G_i, G_j \subset G$). Suppose not. Take a point $x \notin G_i \cup G_j$. Then there are subbase members S_i and S_j such that $G_i \subset S_i$ and $G_j \subset S_j$ but $x \notin S_i \cup S_j$. Now $S_i \cap S_j \neq \emptyset$ and $S_i \cup S_j \neq X$, so either $S_i \subset S_j$ or $S_j \subset S_i$ and in both cases the largest of the two contains

$G_i \cup G_j$. Therefore

$$G_i \cup G_j = \cap \{S \in S \mid G_i \cup G_j \subset S\}.$$

But now we can decrease the number m by taking a finite intersection of this collection which misses H , instead of both G_i and G_j . Next we prove that each G_i is a member of S . Suppose that $G_i \notin S$, and let $m \neq i$. Then there is a member $T \in S$ such that $T \cap G_m = \emptyset$ and $G_i \subset T$. The sequence $G_1, \dots, G_{i-1}, T, G_{i+1}, \dots, G_m$ is also a sequence which satisfies (a), (b) and (c) and we conclude that $T \cap G_j = \emptyset$ whenever $1 \leq j \leq m$ with $j \neq i$ and $G_i \subset T$, so $G_i = T$. We found a finite collection of pairwise disjoint members of S which cover X . This contradicts Lemma 6.3. \square

Let S be a subbase for a space X . The superextension $\lambda(X, S)$ has an underlying set, the set of all maximal linked systems in S with topology generated by taking the collection

$$S^+ = \{S^+ \mid S \in S\},$$

where

$$S^+ = \{M \mid M \in \lambda(X, S) \text{ and } S \in M\},$$

as a (closed) subbase. The following facts are well-known and easy to prove:

- S^+ is binary (as a consequence, $\lambda(X, S)$ is compact);
- if S is normal then $\lambda(X, S)$ is Hausdorff;
- if S is a T_1 collection then the function $i: X \rightarrow \lambda(X, S)$ defined by $i(x) = \{S \in S \mid x \in S\}$ is an embedding;
- S is connected iff S^+ is connected.

For details, see [21]. Superextensions were introduced by DE GROOT [6].

LEMMA 6.5. *Let X be a space and let S be a closed subbase of X with the following properties:*

- (a) S is a T_1 collection;
- (b) S is normal;
- (c) S is cross-free.

Then X can be embedded in a dendron T .

PROOF. If S is a connected subbase then $\lambda(X,S)$ is a compact space with a cross-free connected subbase S^+ , and now it follows from 6.4 and 5.2 that $\lambda(X,S)$ is a dendron which contains X .

If S is not connected, then we extend X to a space Y and S to a subbase \tilde{S} in such a way that \tilde{S} is a connected subbase for Y , and since $\lambda(Y,\tilde{S})$ contains X as a subspace we have that X is a subspace of a dendron.

Let $\{\langle H_\alpha, K_\alpha \rangle \mid \alpha \in A\}$ enumerate all the pairs $\langle H,K \rangle \in S * S$ such that $K = X \setminus H$ (in such a way that $\langle H,K \rangle$ and $\langle K,H \rangle$ do not both occur). Let $H = \{H_\alpha \mid \alpha \in A\}$ and $K = \{K_\alpha \mid \alpha \in A\}$. Define

$$Y = X \cup (I \times A), \text{ where } I \text{ is the open unit interval } (0,1).$$

For $\alpha \in A$ we define

$$A_0(\alpha) = \{\beta \in A \setminus \{\alpha\} \mid H_\beta \subset H_\alpha \text{ or } K_\beta \subset H_\alpha\},$$

and

$$A_1(\alpha) = \{\beta \in A \setminus \{\alpha\} \mid H_\beta \supset H_\alpha \text{ or } K_\beta \supset H_\alpha\}.$$

Thus $A = A_0(\alpha) \cup A_1(\alpha) \cup \{\alpha\}$. For $\alpha \in A$ define

$$H_\alpha^\sim = H_\alpha \cup (I \times A_0(\alpha)), \quad K_\alpha^\sim = K_\alpha \cup (I \times A_1(\alpha)).$$

Then for $r \in I$ we define

$$H_{\alpha r}^\sim = H_\alpha^\sim \cup ((0,r) \times \{\alpha\}) \quad \text{and} \quad K_{\alpha r}^\sim = K_\alpha^\sim \cup ([r,1) \times \{\alpha\}).$$

For each $S \in S \setminus (H \cup K)$, let

$$A(S) = \{\alpha \in A \mid H_\alpha \subset S \text{ or } K_\alpha \subset S\};$$

then let

$$S^\sim = S \cup (I \times A(S)).$$

Finally, set

$$S^\sim = \{S^\sim \mid S \in S \setminus (H \cup K)\} \cup \{H_{\alpha r}^\sim \mid \langle r,\alpha \rangle \in I \times A\} \cup \{K_{\alpha r}^\sim \mid \langle r,\alpha \rangle \in I \times A\}.$$

It is easily verified that S^\sim is a connected cross-free subbase satisfying (a) and (b). \square

We now come to the main result of this section.

THEOREM 6.6. *A Hausdorff space X can be embedded in a dendron iff X possesses a cross-free closed subbase.*

PROOF. Corollary 2.5 states that a dendron has a cross-free closed subbase, if we restrict ourselves to a subspace X then the collection of all restrictions of subbase members is still cross-free. Conversely, if X possesses a cross-free closed subbase, then Lemma 6.1 states that X possesses a cross-free closed subbase which is both normal and T_1 . From Lemma 6.5 it follows that X can be embedded in a dendron. \square

NOTES. (for Section 6). Lemma 6.3 and Corollary 6.4 are due to VAN MILL & SCHRIJVER [11]. All other results in this section can be found in VAN MILL & WATTEL [14].

In [15] the authors showed that for compact X the following statements are equivalent:

- (1) X is orderable;
- (2) X has a weak selection;
- (X has a weak selection iff there is a map $s: X^2 \rightarrow X$ such that $s(x,y) = s(y,x) \in \{x,y\}$ for all $x,y \in X$.)

This result suggests the natural question whether for dendrons there is a similar characterization, i.e. is there a natural number $n \in \mathbb{N}$ and algebraic conditions on a map $s: X^n \rightarrow X$ such that a continuum X is a dendron if and only if X has such a map? For this question Ward has given a satisfactory solution in [24], in which he states:

A compact Hausdorff space is a dendron if and only if there exists a continuous function $m: X \times X \rightarrow X$ such that

- (i) m is idempotent, i.e. $m(x,x) = x$;
- (ii) m is associative;
- (iii) m is commutative, i.e. $m(x,y) = m(y,x)$;
- (iv) m is monotone;
- (v) if $m(a,x) = a$ and $m(b,x) = b$, then $m(a,b) \in \{a,b\}$.

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"EXTENDING" MAPS OF ARCS TO
MAPS OF ORDERED CONTINUA

L.B. Treybig

In [1] MARDEŠIĆ and PAPIĆ ask if each locally connected continuum which is the continuous image of a compact ordered space is also the continuous image of an ordered continuum. Continued applications of the techniques of Theorem 2 of [4] suggests that in order to attack the above question it is very desirable to be able to prove:

THEOREM 2. *Let $f: K \rightarrow M$ be a continuous mapping of a compact ordered space K onto a locally connected continuum M such that*

- (1) *no point separates M ; and*
- (2) *M contains an open set U such that:*
 - (a) *$M-U$ is separable; and*
 - (b) *each component of U is homeomorphic to the open interval $(0,1)$.*

Then, M is the continuous image of an ordered continuum.

The general idea of the proof is to find a certain upper semicontinuous decomposition G_2 of M into points and arcs. The resulting Peano continuum M/G_2 is the continuous image of $[0,1]$ under a light map β which is of finite oscillation at local separating points [5]. Since the nondegenerate elements of G_2 are arcs, then certain elements of their inverses under β are also replaced by arcs in order to find an ordered continuum B and a continuous onto map $\alpha: B \rightarrow M$ such that $\phi_2\alpha = \beta k$, where $\phi_2: M \rightarrow M/G_2$ is the natural map.

In [5] we deal with the problem of showing the existence of maps of finite oscillation at local separating points. The references of [6] give an additional guide to the literature. Definitions concerning continuous images of ordered spaces may be found in [3], [4], [6]. The basic definitions and theory of upper semi-continuous collections may be found in [7]. Definitions and basic theory involving local connectivity, irreducible continua and simple closed curves may be found in [2], [7].

A point P of the locally connected metric continuum M is a local separating point of M provided there is a connected open set U containing P so

that $U - P = R \cup S$ mutually separated. If $f: [0,1] \rightarrow M$ is a continuous onto map, then we say f is of finite oscillation at local separating points if for each P, U, R and S as above there is a finite set G of open intervals covering $f^{-1}(R \cup S)$ so that no interval of G intersects both $f^{-1}(R)$ and $f^{-1}(S)$.

THEOREM 1. *Let $f: K \rightarrow M$ be a continuous mapping of a compact ordered space K onto a continuum M such that: (1) no point separates M , and; (2) M contains an open set U such that: (a) $M - U$ is separable; (b) each component u of U is open in M and homeomorphic to $(0,1)$ and \bar{u} is homeomorphic to $[0,1]$; (c) if A and B are mutually exclusive closed subsets of M , then there exist at most finitely many components of U intersecting both A and B ; and (d) if u, v are two components of U , then $\bar{u} \subset M - \bar{v}$. Define a relation R on M so that if $x, y \in M$, then xRy holds if and only if $x = y$ or x and y belong to the closure of a component of U . Then R is an equivalence relation such that the collection G of equivalence classes modulo R is an upper semi-continuous decomposition of M into continua and M/G is a metric continuum.*

PROOF. It is straightforward that R is an equivalence relation and each element of G is a point or arc.

Now suppose the element g_1 of G is a subset of the open set W . There is an open set W_1 so that $g_1 \subset W_1 \subset \bar{W}_1 \subset W$, and use of conditions 2(c), 2(d) of the hypothesis reveals that there is an open set V_1 such that $g_1 \subset V_1 \subset \bar{V}_1 \subset W_1$ and no component of U intersects \bar{V}_1 and $M - W_1$. Thus, if h is an element of G intersecting V_1 , then $h \subset W_1 \subset W$, and G is therefore upper semi-continuous.

Since each element of G contains a point of $M - U$, then $\phi(M - U) = M/G$ is separable, where $\phi: M \rightarrow M/G$ is the natural map. Chapter 7 of [7] reveals that M/G is a continuum, and Theorem 1 of [4] shows that M/G is metric. \square

PROOF OF THEOREM 2. Before we proceed with the main part of the proof we need several lemmas. U' will denote the set of all components of U , and an element u of U' will be denoted by xy , where x and y are the limit points of u in $M - U$. Here $\bar{u} = u \cup \{x, y\}$. For the time being, using the axiom of choice, we will assume that for each such x, y above, U' contains only one such u with $\text{Bd}(u) = \{x, y\}$.

LEMMA 1. *If L is a closed set in M which contains every element of U' which it intersects, then there are at most countably many elements ab of U' so that $a \in L$ and $b \notin L$.*

PROOF. On the contrary, suppose there is an uncountable collection $W = \{a_\alpha b_\alpha, \alpha \in A\}$ of elements of U' such that $a_\alpha \in L, b_\alpha \notin L$, for $\alpha \in A$. Suppose also there is a point c of $M-L$ such that if V is an open set containing c , then V contains b_α for infinitely many α . Let R, S be open sets containing L and c , respectively, such that $\bar{R} \subset M-\bar{S}$, and let $g: M \rightarrow [0,1]$ be a continuous function with $g(\bar{R}) = 0$ and $g(\bar{S}) = 1$. There is a countably infinite subset $\{a_{\alpha_i} b_{\alpha_i}, i = 1, 2, \dots\}$ of W so that $b_{\alpha_i} \in \bar{S}, i = 1, 2, \dots$. For each $i = 1, 2, \dots$ let $d_{\alpha_i} \in a_{\alpha_i} b_{\alpha_i} \cap g^{-1}(1/2)$ and let d be a limit point of $\{d_{\alpha_1}, d_{\alpha_2}, \dots\}$. Since M is locally connected, there is a connected open set W containing d and lying in $M - (\bar{R} \cup \bar{S})$. But for some i , W intersects $a_{\alpha_i} b_{\alpha_i}$ and $M - \{a_{\alpha_i}, b_{\alpha_i}\}$, but not $\{a_{\alpha_i}, b_{\alpha_i}\}$, a contradiction.

Since there is no such c as above, then every open set containing L contains all but finitely many b_α . With the aid of Lemma 2 of [3] we find that every subset of $M-U$ is separable, so let A' be a countable set dense in $\{x: x = a_\alpha \text{ or } b_\alpha, \alpha \in A\}$. Since every open set containing L contains all the a_α and all but finitely many b_α , then each b_α is in A' , a contradiction. \square

DEFINITION. Let H denote the decomposition of M such that the elements of H are the components of $M-U$ and the points of U .

LEMMA 2. H is an upper semi-continuous decomposition of M into continua such that: (1) each subcontinuum B of M/H is locally connected; and (2) there is an ordered continuum A and a continuous onto map $g: A \rightarrow M/H$ such that: (a) if a, b denote the first and last points of A , respectively, then $g(a), g(b) \in \phi(M-U)$, where $\phi: M \rightarrow M/H$ is the natural map; (b) if $x, y \in A, x < y$, and $g(x) = g(y)$, then there exists z in (x, y) with $g(z) \neq g(x)$; and (c) if $u \in U'$ then each component of $g^{-1}(u)$ is mapped onto u .

PROOF. The results at the beginning of Chapter 7 of [7] show why H is upper semi-continuous and why M/H is a locally connected continuum. If B fails to be locally connected, then B fails to be locally connected at each point of some nondegenerate subcontinuum of B . This is impossible since $\phi(M-U)$ is totally disconnected, and B is clearly locally connected at each point of $\phi(U) \cap B$.

By the theorem of [6] there is an ordered continuum A and a continuous onto map $g: A \rightarrow M/H$. By using cut and paste methods we may obtain 2(a). To obtain 2(b) let $x \sim y$ if and only if: (1) $x = y$ or; (2) $g([y, x])$ or $g([x, y]) = g(x)$. The resulting decomposition space results in property 2(b).

Now consider a component (x,y) of $g^{-1}(u)$, $u \in U'$. If $g((x,y)) \neq u$ then $g(x) = g(y)$. If some component (x',y') of $g^{-1}(u)$ maps onto u then for every one, say (r,s) , that does not map onto u let $g[r,s] = g(r) = g(s)$. If no component of $g^{-1}(u)$ maps onto u we let $g[r,s] = g(r) = g(s)$ for all but one such component, say (x',y') . We pick w in (x',y') and modify g (to a continuous map) to let $g[x',w] = \bar{u}$, $g([w,y']) = \bar{u}$, and $g((x',y')) = u$. After the adjustments above we may have to adjust again for 2(b). \square

LEMMA 3. M satisfies the first axiom of countability.

PROOF. Let $x \in M$. If $x \in U$ then the proof is straightforward, so suppose $x \in m$, a component of $M-U$. Since each component of $M-U$ is metrizable, there is a sequence m_1, m_2, \dots of open subsets of m containing x so that: (1) if $x \in n$ and n is open in m , then there exists i so that if $i \leq j$ then $m_j \subset n$; and (2) $m_{i+1} \subset m_i$ for $i = 1, 2, \dots$.

Assume for the moment (**)
M/H satisfies the first axiom of countability.

There is a sequence h_1, h_2, \dots of open sets in M/H such that

- (1) if h is open in M/H , there is a j so that if $i \geq j$, then $m \in h_i \subset h$;
and
(2) $h_{i+1} \subset h_i$ for $i = 1, 2, \dots$.

For each positive integer i let u_i be open in M such that $x \in u_i \subset \bar{u}_i \subset h_i$ and $\bar{u}_i \subset M - (m - m_i)$. Now let $x \in V$, an open subset of M . Let W be an open set so that $x \in W \subset \bar{W} \subset V$.

Let $y \in M - V$. If $y \in m$ there is a positive integer i_y so that if $i \geq i_y$ then $m_i \subset W \cap m$ and thus $(m - W) \cap \bar{u}_i$ is void. Therefore there is an open set R_y so that $y \in R_y \subset M - \bar{u}_{i_y}$. Likewise, if $y \in M - m$ there is a positive integer i_y so that if $i \geq i_y$ then $y \notin \bar{m}_i$. Thus, there is an open set R_y so that $y \in R_y \subset M - \bar{m}_{i_y}$. There is a finite set R_{y_1}, \dots, R_{y_m} which covers $M - V$, so let $N = \sum_{p=1}^m i_{y_p}$. If $i > N$, then $x \in u_i \subset V$. It remains now to show (**).

If x is a point of a component of U , then M/H clearly satisfies the first axiom of countability at x , so suppose x is a component of $M-U$. Let A, g be as in Lemma 2. We now show (***)

- (1) there are only countably many components (u,v) of $A - g^{-1}(x)$; and
(2) for each such component (u,v) there exist $u_1, u_2, \dots, v_1, v_2, \dots$ in (u,v) such that
(i) for each j , $u < u_{j+1} < u_j < v_j < v_{j+1} < v$; and
(ii) u_1, u_2, \dots converges to u and v_1, v_2, \dots converges to v .

First suppose the set $T = \{(u^\alpha, v^\alpha), \alpha \in A'\}$ of components of $A - g^{-1}(x)$ is uncountable. Let S' be a countable set in A such that $a, b \in S'$ and $g(S')$ is dense in $\phi(M-U)$. Let X_1, X_2, \dots be a sequence of finite subsets of A such that

- (1) $a, b \in X$, and $\text{card } X_1 \geq 3$;
- (2) each s' in S' belongs to some X_i ; and
- (3) X_1, X_2, \dots have properties as in paragraph two of the proof of Theorem 2 of [4].

Let G' be the set of all components of $A - \text{Cl}(\bigcup_1^\infty X_i)$. If $g_0 = (r, s) \in G'$ and H' is the set of all elements g' of G such that there is a finite sequence $g_0, g_1, \dots, g_n = g'$ of elements of G such that $g(g_i)$ intersects $g(g_{i+1})$ for $i = 0, \dots, n-1$, then by [4];

- (1) each element (u, v) of H' has the property that $\{g(u), g(v)\} \subset \{g(r), g(s)\}$; and
- (2) if $(t, u) \in H'$ and z_1, z_2 are elements of A so that $z_1 \in \text{Cl}(\bigcup_1^\infty X_i)$, $z_2 \in (t, u)$, and $g(z_1) = g(z_2)$, then $g(z_2) \in \{g(r), g(s)\}$.

Now the collection Q of those (u^α, v^α) containing a point of $\bigcup_1^\infty X_i$ is clearly countable, so suppose $(u^\alpha, v^\alpha) \in T - Q$. Then $(u^\alpha, v^\alpha) \subset (r, s)$, a component of $A - \text{Cl}(\bigcup_1^\infty X_i)$. Thus, if $t \in (r, s)$ and $g(t) \in g(\text{Cl}(\bigcup_1^\infty X_i))$, then $g(t) \in \{g(r), g(s)\}$, so $\{g(u^\alpha), g(v^\alpha)\} \subset \{g(r), g(s)\}$. Therefore there exists a segment (c^α, d^α) of (u^α, v^α) so that $g((c^\alpha, d^\alpha)) \subset U$, (2) $\{g(c^\alpha), g(d^\alpha)\} \subset \phi(M-U)$ and (3) $g(d^\alpha) = g(v^\alpha) = x$. Since $V = \{(c^\alpha, d^\alpha) : (u^\alpha, v^\alpha) \in T - Q\}$ is uncountable, then either (1) there is a component u of U so that $g((c^\alpha, d^\alpha)) = u$ for an uncountable set B of the α 's or (2) there is an uncountable subcollection V' of V so that if $u, v \in V'$ then $g(u) = g(v)$ implies $u = v$.

If (1) holds there is a set of elements $(c^{\alpha i}, d^{\alpha i})$, $i = 1, 2, 3, \dots$, of elements of B , so that $g(c^{\alpha i}) = y$, $g(d^{\alpha i}) = x$, $i = 1, 2, \dots$. If z is a limit point of $\{c^{\alpha 1}, c^{\alpha 2}, \dots\}$ then $g(z) = x$ and y both, a contradiction.

If (2) holds then Lemma 1 implies there is an uncountable subcollection V'' of V' such that if $(c^\alpha, d^\alpha) \in V''$ then $g(c^\alpha) = g(d^\alpha) = x$, which implies that there are uncountably many components of U with endpoints in x , which is metrizable. Thus, if ϕ denotes a metric on x compatible with the relative topology on x . there is an $\epsilon > 0$ and uncountably many components (t, u) of U so that $t, u \in x$ and $\phi(t, u) \geq \epsilon$. We may thus find that condition (c) of Theorem 1 fails to hold, and using the proof of Lemma 1 we find that M is not locally connected, a contradiction. We thus find that the set of all components (u, v) of $A - g^{-1}(x)$ is countable.

If there is such a component (u,v) such that (***) (2) does not hold, then suppose v is not the limit of a countable sequence of elements of (u,v) . Some subinterval (u',v) of (u,v) is a subset of a component (r,s) of $A - \text{Cl}(\bigcup_1^\infty X_i)$. It thus follows that there is an uncountable well ordered sequence $\{t_\alpha, \alpha \in A_1\}$ of points t_α such that

- (1) $g(t_\alpha) \in \phi(M-U)$;
- (2) if $\alpha < \alpha'$ then $u' < t_\alpha < t_{\alpha'} < v$; and
- (3) for each of uncountably many α , $g((t_\alpha, t_{\alpha+1}))$ is a subset of a component u_α of U .

We now obtain contradictions as above.

If for each component (u^α, v^α) of $A - f^{-1}(x)$, $\alpha = 1, 2, 3, \dots$, we let $u_1^\alpha, u_2^\alpha, \dots; v_1^\alpha, v_2^\alpha, \dots$ denote sequences satisfying (***) (2), then U_1, U_2, \dots defined by $U_n = \text{Int}(g(A - \bigcup_1^n (u_n^p, v_n^p)))$ is a countable sequence of open sets satisfying the first axiom of countability at x . This completes the proof of Lemma 3. \square

DEFINITION. Given two points a, b of $M-U$ a subset L of M will be called a J -curve from a to b provided L is the union of two continua g_1, g_2 so that

- (1) $g_1 \cap g_2 = \{a, b\}$; and
- (2) g_i is irreducible from a to b ($i = 1, 2$).

LEMMA 4. *If a and b are distinct points $M-U$, then there is a J -curve L from a to b . Furthermore, given $L = g_1 \cup g_2$ as above, then*

- (1) *if u is a component of U which intersects L then $u \subset g_1$ or $u \subset g_2$; and*
- (2) *if xy and yz are components of U lying in L then $xy \cup \{y\} \cup yz$ is an open subset of L .*

PROOF. Let H_1, H_2, \dots denote a sequence of finite covers of M by connected open sets such that:

- (1) for each positive integer n
 - (i) H_{n+1} is a star refinement of H_n ,
 - (ii) H_n contains elements h_n^a, h_n^b so that h_n^a (resp. h_n^b) is the only element of H_n whose closure contains a (b),
 - (iii) if h is an element of H_n not containing a or b , then \bar{h} does not separate a from b ; and
- (2) $\bigcup_1^\infty h_n^a = \{a\}$ and $\bigcup_1^\infty h_n^b = \{b\}$.

Define a relation T on M so that xTy if and only if $x \in \bigcap_1^\infty \text{st}(y, H_n)$. Clearly xTx holds, so suppose xTy holds. For each positive integer n there is an element h_n of H_n so that $x, y \in h_n$. Therefore $y \in \bigcap_1^\infty \text{st}(x, H_n)$, so yTx

ly and yTz hold. For each positive integer n there exist $x, y \in h_{n+1}$ and $y, z \in g_{n+1}$. But since $st(y, H_{n+1}) \subset k_n$. Therefore $x \in \bigcap_1^\infty st(z, H_n)$, so xTz holds. an equivalence relation, so let T_e denote the set of modulo T .

1 and n is positive integer, then $st(x, H_{n+2}) \subset k_{n+1} \subset \overline{k_{n+1}} \subset \bigcap_1^\infty st(x, H_n)$ for some $k_{n+1} \in H_{n+1}$, $k_n \in H_n$. Therefore $x \in h \in T_e$ and each element of T_e is a continuum. r n such that $Cl(st(x, H_n)) \subset W$. Let $g_i = st(x, H_i)$, $i = n, n+1, g_{n+2} - h$. Now $st(y, H_{n+2})$ is a subset of some element w_{n+1} of w_{n+1} and $st(x, H_{n+1}) \subset W$. Therefore if $y \in k \in T_e$, then $k \subset W$. s upper semi-continuous, so let $\phi_1: M \rightarrow M/T_e$ be the natural

Chapter 7] M/T_e is a locally connected continuum. Since the set of the form $st(x, H_n)$ is countable, the set $X = \{h: h \in T_e \text{ and } \} \text{ for some fixed } x, n\}$ is a countable basis for M/T_e . Therefore metrizable.

se $g \in T_e$ and that $M/T_e - g = R \cup S$ mutually separated where $\{a\} \in R$, but if $x \in g$ then for some n , $st(x, H_{n+1})$ is a subset of some g' in some component Q of $M - g'$ contains a, b . Thus $\phi_1(Q)$ is a connected

$M/T_e - g$ containing $\{a\}$ and $\{b\}$, a contradiction. M/T_e is a locally connected metric continuum such that no point as $\{a\}$ and $\{b\}$, then by Theorem 93, page 138 of MOORE [2] there is a closed curve α in M/T_e , where α contains $\{a\}, \{b\}$. Since α is the of two arcs α_1 and α_2 from $\{a\}$ to $\{b\}$, then we let g_1 be a subcontinuum α_1 which is irreducible from a to b . Clearly $g_1 \cap g_2 = \{a, b\}$.

Suppose u is a component of U intersecting $L = g_1 \cup g_2$. If some arc on a subset of L and meets g_1 and g_2 , then there is a point of $g_1 \cap g_2$ disjoint from a, b , a contradiction. If some component k of $u \cap L$ meets only for example, and $k \neq u$, then k is nondegenerate and some half open interval on k can be removed from g_1 without disconnecting g_1 , a contradiction. Therefore $u \subset g_1$ or $u \subset g_2$.

The last part of the proof follows from standard proofs (concerning connected spaces and simple closed curves) such as those found in Chapter 2 of MOORE [2] and Chapter 2 of WILDER [7]. \square

there is such a component (u, v) such that (***) (2) does not hold. Suppose v is not the limit of a countable sequence of elements of U . Then (u', v) of (u, v) is a subset of a component (r, s) of X_i . It thus follows that there is an uncountable well ordered set $\{t_\alpha, \alpha \in A_1\}$ of points t_α such that $t_\alpha \in \phi(M - U)$; $t_{\alpha'} < t_\alpha < t_{\alpha+1} < v$; and each of uncountably many α , $g((t_\alpha, t_{\alpha+1}))$ is a subset of U .

in contradictions as above. each component (u^α, v^α) of $A - f^{-1}(x)$, $\alpha = 1, 2, 3, \dots$, where $u^\alpha, v^\alpha, \dots$ denote sequences satisfying (***) (2), then $g(A - \bigcup_1^n (u_n^p, v_n^p))$ is a countable sequence of components of $A - f^{-1}(x)$. This completes the proof of the first axiom of countability at x .

Let a, b be two points of $M - U$. A subset L of M will be connected from a to b provided L is the union of two continua g_1, g_2 such that $a \in g_1, b \in g_2$; and g_1 is connected from a to b ($i = 1, 2$).

If a, b are distinct points of $M - U$, then there is a J-continuum $L = g_1 \cup g_2$ as above, then $U \cap L$ is a component of U which intersects L then $u \subset g_1$ or $u \subset g_2$. If u is a component of U lying in L then $xy \cup \{y\} \subset u \cap L$.

Let \mathcal{C} denote a sequence of finite covers of M by continua.

Let n be an integer n . Let \mathcal{C}_n be a star refinement of H_n , where \mathcal{C}_n consists of elements h_n^a, h_n^b so that h_n^a (resp. h_n^b) is a component whose closure contains a (b), and $h_n^a \cap h_n^b = \emptyset$. If h_n^a is a component of H_n not containing a or b , then $h_n^a \cap h_n^b = \emptyset$; and $h_n^a \cap h_n^b = \{b\}$.

Let \mathcal{C}_n be a star refinement of H_n so that xTy if and only if $x \in \bigcap_1^\infty \mathcal{C}_n$ and $y \in \bigcap_1^\infty \mathcal{C}_n$. For each positive integer n , let h_n^a, h_n^b be elements of \mathcal{C}_n such that h_n^a (resp. h_n^b) is a component of H_n not containing a or b , then $h_n^a \cap h_n^b = \emptyset$; and $h_n^a \cap h_n^b = \{b\}$.

LEMMA 5. In M suppose J is a J -curve from a_1 to a_2 . Then, there is a countable subset C of U' such that if ab and cd are distinct elements of U' and $ab \subset J$, then either one of ab, cd belongs to C or $\overline{ab} \subset M - \overline{cd}$.

PROOF. Let M_1 denote the continuum $J \cup (U\{m: m \text{ is a component of } M-U \text{ which intersects } J\})$. Lemma 1 implies that the set C_1 of all elements of U' which have exactly one endpoint in M_1 is countable.

Let P_2 denote a countable set dense in $J \cap (M-U)$ and let C_2 denote the set of all elements xy of U' lying in J such that there is a second element yz of U' lying in J . By Lemma 4, $y \in P_2$ and C_2 is countable.

For each xy in C_2 let $x'y'$ denote an open arc in xy so that the points x, x', y', y lie in the order indicated on \overline{xy} .

We now let M_1 be as above, $K_1 = f^{-1}(M_1)$, $f_1 = f|_{K_1}$, and $U_1 = (U \cap J - UC_2) \cup (U\{x'y': xy \in C_2\})$. Since no two components of U_1 have intersecting closures, Theorem 1 implies that the relation R_1 defined with M_1, U_1 analogous to the way R was defined with M, U defines an upper semi-continuous decomposition G_1 of M_1 into continua so that elements of G_1 are either closures of components of U_1 or points not lying in such closures. Thus M_1/G_1 is a metric continuum.

If the collection C_3 of components of U which have both endpoints in M_1 but do not lie in M_1 is uncountable, then there exist uncountably many such components so that no element of M_1/G_1 contains an endpoint of two such components. Since M_1/G_1 is metric, we obtain a contradiction as in Lemma 3. Therefore C_3 is countable.

Now let $C = C_1 \cup C_2 \cup C_3$ and suppose ab, cd are distinct components of U' and $ab \subset J$. If $\overline{ab} \subset M - \overline{cd}$ we are done, so suppose not. If $cd \subset J$, then $ab, cd \in C_2$. If $cd \notin J$, but has exactly one endpoint in J , then $cd \in C_1$. If $cd \notin J$, but has exactly two endpoints in J , then $cd \in C_3$. This completes the proof of Lemma 5. \square

We return now to the proof of Theorem 2. Let P_1, P_2, \dots be a countable set dense in $M-U$ so that if $i \neq j$, then $P_i \neq P_j$. For each pair of distinct indices i, j where $i < j$ let J_{ij} be a J -curve in M from P_i to P_j , and let C_{ij} be a countable set of components of U satisfying the conclusions of Lemma 5 relative to J_{ij} . Let C denote the countable collection $\cup_{i,j} C_{ij}$.

For each element cd of C let c', d' be points so that c, c', d', d lie on \overline{cd} in the order indicated. Let $M_1 = (M-U) \cup (\cup_{i,j} J_{ij})$ and let $U_1 = M_1 \cap ((U-UC) \cup (U\{c'd': cd \in C\}))$. Now M_1, U_1 satisfy the hypothesis of Theorem 1, so we let R_1 be formed relative to M_1, U_1 as R was formed relative to $M,$

U. Also let G_1 denote the set of equivalence classes modulo R_1 and let $\phi_1: M_1 \rightarrow M_1/G_1$ be the natural map.

Since M_1/G_1 is separable, and thus metric, continuum we use the ideas of the proof of Lemma 3 to show the collection C_1 of all components of U not lying in M_1 is countable. As above for each cd in C_1 let c', d' be points so that c, c', d', d lie on \overline{cd} in the order indicated. Let $M_2 = M$ and let $U_2 = U_1 \cup (U\{c'd': cd \in C_1\})$, and form R_2 relative to M_2, U_2 as R_1 was formed relative to M_1, U_1 . The set G_2 of equivalence classes of M modulo R_2 has elements that are either closures of components of U_2 or points not in such a closure. We let $\phi_2: M \rightarrow M/G_2$ be the natural map, and note that M/G_2 is a locally connected metric continuum such that no point separates it.

By Theorem 3 of [5] there is a continuous onto map $\beta: [0,1] \rightarrow M/G_2$ which is of finite oscillation at local separating points, where each inverse of a point is totally disconnected. (Note Lemma 2.)

Now suppose ab is a typical component of U_2 and $z \in \beta^{-1}(\overline{ab})$. Let S_a, S_b be mutually exclusive connected open sets containing a, b respectively and let $L = \{g: g \in G \text{ and } g \subset ab \cup S_a \cup S_b\}$. Also, let L_a be $\{g: g \in L \text{ and } g \subset S_a\}$ and analogously define $L_b = \{g: g \in L \text{ and } g \subset S_b\}$. Now L, L_a, L_b are open in M/G_2 and $L - \{\overline{ab}\} = L_a \cup L_b$ mutually separated, so \overline{ab} is a local separating point of M/G_2 . Therefore, there is a finite collection G of open intervals (half open at the ends of $[0,1]$) covering $\beta^{-1}(L_a \cup L_b)$ so that no interval of G intersects both $\beta^{-1}(L_a)$ and $\beta^{-1}(L_b)$. There exists $(u,v) \in G$ so that $u < z$ and $z \leq v$, and thus $\beta((u,v))$ does not intersect both L_a and L_b . Also, since $\beta^{-1}(L)$ is open, there is an open interval (r,s) containing z so that $u \leq r < z < s$ and $\beta((r,s)) \subset L$. Therefore, $\beta((r,z) - \beta^{-1}(\overline{ab}))$ is a subset of L_a or L_b . Therefore, we note that $w(z-) = \text{limit of } U\beta(t) \text{ as } t \text{ approaches } z, \text{ where } t < z \text{ and } t \notin \beta^{-1}(\overline{ab}), \text{ exists and is } a \text{ or } b \text{ (i.e. there is a point } w(z-) \text{ of } M \text{ such that if } W \text{ is an open set containing } w(z-), \text{ there is a point } r \text{ of } [0,1] \text{ so that } r < z, \text{ and so that if } t \in (r,z) \text{ and } t \notin \beta^{-1}(\overline{ab}), \text{ then } U\beta(t) \subset W.)$ Correspondingly, the upper limit $w(z+)$ exists and is a or b .

If $w(z-) = a$ and $w(z+) = b$ we replace z by a copy $[0,1]_z$ of $[0,1]$ and define a homeomorphism $f_z: [0,1]_z \rightarrow \overline{ab}$ so that $f_z(0) = a$ and $f_z(1) = b$. Likewise if $w(z-) = b$ and $w(z+) = a$ we define a homeomorphism $f_z: [0,1]_z \rightarrow \overline{ab}$ so that $f_z(0) = b$ and $f_z(1) = a$. If

- (1) $w(z-) = w(z+) = a$; or
- (2) $w(z-) = w(z+) = b$;

we do not replace z unless it is true that for each z in $\beta^{-1}(\overline{ab})$ that (1) or (2) holds. In that case we replace exactly one such point z by a copy $[0,1]_z$

of $[0,1]$ and define a continuous onto map $f_z: [0,1] \rightarrow \overline{ab}$ so that $f_z(0) = f_z(1) = w(z^-)$. This last step is to insure that our desired map α is onto.

We define the ordered continuum B by replacing the various z 's as needed in the description above by copies $[0,1]_z$ of $[0,1]$ and giving B the obvious order. Our map $\alpha: B \rightarrow M$ is defined so that $\alpha(t) = \beta(t)$ if $t \notin [0,1]_z$ for any z , and $\alpha(t) = f_z(t)$ if $t \in [0,1]_z$. Define $k: B \rightarrow [0,1]$ so that

- (1) if z is not replaced then $k(z) = z$; and
- (2) if z is replaced by $[0,1]_z$ then $k([0,1]_z) = z$.

The map k is clearly continuous and the map α is onto. We need only check the continuity of α .

Let $\alpha(t) = t \in S$, where S is open in M . There is an open set W so that

- (1) $y \in W \subset \overline{W} \subset S$ and no closure of a component of U intersects \overline{W} and $M-S$ unless there is a component xy of U_2 whose closure does so; and
- (2) if there is a component xy of U_2 , then $x \in M-\overline{W}$.

CASE 1. There is a component uv of U_2 containing y . Then there is a point z of $[0,1]$ so that $t \in [0,1]_z$. Since f_z is continuous there is an open interval (r,s) containing t so that $f_z(r,s) = \alpha((r,s)) \subset W$.

Case 2. There is a component xy of U_2 . Suppose $t \in [0,1]_z$. If $t \in (0,1)_z$ then we use the ideas of Case 1, so suppose $t = 1_z \in [0,1]_z$, for example. Since $w(z^+) = y$ there is an interval (z,s) of $[0,1]$ so that if $z < u < s$ and $u \notin \beta^{-1}(\overline{xy})$, then $\beta(u) \in W' = \{g \in G_2: g \subset W\}$. Also since only finitely many elements z' of $\beta^{-1}(\overline{xy})$ are replaced by an interval $[0,1]_{z'}$, then s may be chosen so that no such z' lies in (z,s) . Thus if $z < u < s$ and $u \notin \beta^{-1}(\overline{xy})$, then $\beta(u)$ is a single element of M or the closure of a component of U_2 which intersects W and is not xy . Further, if $z < u < s$ and $\beta(u) = \beta(z)$, then u is not replaced and $\alpha(u) = y \in W$.

Let $s_0 = \text{g.l.b. } k^{-1}(s)$. If $1_{z_1} < v < s_0$, then $\alpha(v)$ is either a point of W or an element of the closure of a component of U which intersects W . Thus $\alpha(v) \in S$. By the continuity of f_z there is a point $r_0 \in (0,1)_z$ so that $f_z((r_0, 1_z]) \subset W$. Thus $\alpha((r_0, s_0)) \subset S$.

If $t = 0_z \in [0,1]_z$ or if $t = z$, where z is not replaced, the proof follows analogously.

CASE 3. There is no component uv of U_2 so that $y \in \overline{uv}$. Then $\beta(t) = y$. There is an open interval (r,s) in $[0,1]$ containing t so that if $u \in (r,s)$ then $\beta(u) \in W'$. Let $r_0 = \text{l.u.b. } k^{-1}(r)$ and $s_0 = \text{g.l.b. } k^{-1}(s)$ and suppose

$v \in (r_0, s_0)$. Now $\alpha(v)$ is either a point of W or an element of the closure of a component of U_2 which intersects W . Therefore $\alpha(v) \in S$.

Since β is continuous this completes the proof of Theorem 2 for the case that no two components of U have the same endpoints. We consider now the general case for M .

Define a subcontinuum N of M so that N contains $M-U$ and also contains, for each pair of points x, y which are the endpoints of a component of U , exactly one such component of U . By the proof above there is a continuous map $\beta: B \rightarrow N$ of an ordered continuum B onto N .

Now consider a typical pair of points x, y which are the endpoints of several components of U . Let C_1, \dots, C_n be the set of all such components, where $C_1 \subset N$. We pick one point z of $\beta^{-1}(x)$ and replace z by an interval $[0, 1]_z$ and define a continuous onto map $f_z: [0, 1]_z \rightarrow \bigcup_{p=1}^n C_p$ such that $f_z(0_z) = f_z(1_z) = x$. The proof is now completed much the same as in the special case above. This completes the proof of Theorem 2. \square

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THE HAHN-MAZURKIEWICZ PROBLEM

by

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1. INTRODUCTION

The celebrated Hahn-Mazurkiewicz theorem, which was first proved about 1914 independently by H. HAHN [5] and S. MAZURKIEWICZ [21], characterizes the Hausdorff continuous images of $[0,1]$ (i.e., the *Peano continua*) as the class of locally connected, metrizable continua. It is related in an interesting way to R.L. MOORE's theorem [22] that a Peano continuum is arcwise connected and the theorem of ALEXANDROFF [1] which characterizes the Hausdorff continuous images of the Cantor ternary set as the class of compact metric spaces. The relationship can be illustrated as follows: Given a Peano continuum X and the existence of a mapping f from the Cantor set C on to X , one extends f over the intervals of $[0,1]-C$ to prove the Hahn-Mazurkiewicz theorem. (This is the method of proof used by WILDER [47].) The latter theorem, in turn, can be employed to give a quick and elegant proof of Moore's arc theorem. (See G.T. WHYBURN [46] who attributes this proof to J.L. Kelley.)

It is natural to seek analogues for these results in the category of Hausdorff spaces. For a number of years there seems to have been a sort of folk-conjecture - apparently it never appeared in print - that these three classical theorems might admit straightforward generalizations as indicated below.

Hereafter a *continuum* is a compact connected Hausdorff space. It is helpful to introduce the terminology of A.D. WALLACE [38] and call a subset A of a space an *arc* if A is a continuum with exactly two non-cutpoints. It is well-known (for example, see [9]) that an arc is simply an orderable continuum. A separable arc (i.e., a homeomorph of $[0,1]$) is called a *real arc*. The term *image* will always mean continuous image.

FOLK CONJECTURE 1. *Among Hausdorff spaces, the images of arcs coincide with the locally connected continua.*

FOLK CONJECTURE 2. *Among locally connected, compact Hausdorff spaces, connectedness is equivalent to arcwise connectedness.*

FOLK CONJECTURE 3. *Among Hausdorff spaces, the images of compact ordered spaces coincide with the compact spaces.*

None of these conjectures is true. The first published counter-example was due to MARDEŠIĆ [13] who gave an example of a locally connected continuum which is not arcwise connected, thus exploding Conjecture 2. Mardešić observed that "clearly" the image of an arc is arcwise connected, so that his example also disposed of Conjecture 1. (Proofs of this observation have been given by HARRIS [6] and A.J. WARD [39].) A simple argument disposing of Conjecture 3 was also noted by A.J. WARD [41]: the continuous image of a compact ordered space must be hereditarily normal, and therefore the so-called Tychonoff plank [10] serves as a counterexample. The question remains whether additional hypotheses can be found to provide affirmative solutions to the three conjectures in such a way as to generalize the classical theorems. Mardešić's sequence of papers in the early 1960s, in part in collaboration with his colleague P. Papić, stimulated the current interest in these problems, most notably in the contributions of CORNETTE [3], CORNETTE and LEHMAN [4], PEARSON [23,24], SIMONE [26-29], TYMCHATYN [37], A.J. WARD [39-41] and the authors [30-36] and [42-45].

MARDEŠIĆ [16] has given a survey of the progress on these problems up to 1965. In this paper we review that survey briefly and we describe the work done during the intervening fifteen years.

Several simpler examples of locally connected continua which are not images of arcs have followed Mardešić's original example. For example, see [4] and [14]. In [18] Mardešić gave an example of a locally connected continuum, none of whose nondegenerate proper subcontinua is locally connected. In particular, this continuum contains no arc. The existence of this example depends on the continuum hypothesis, and it is not known whether such an example can be found without assuming the continuum hypothesis.

In 1960 MARDEŠIĆ and PAPIĆ [19] proved the startling result that if a product space $\prod\{X_\alpha\}$ is the image of an arc, then there are at most countably many non-degenerate spaces X_α and each of these is metrizable. G.S. YOUNG [48] used a simple argument to conclude that if L denotes the "long interval"

obtained by inserting copies of $(0,1)$ between consecutive ordinals not greater than ω_1 , then $L \times [0,1]$ is not even the image of some compact ordered space. Of course, this can also be deduced from the fact that $L \times [0,1]$ is not hereditarily normal. Mardešić and Papić also enunciated the following question which remains unsolved and is certainly among the most important in this area.

PROBLEM 1. *If a locally connected continuum X is the image of a compact ordered space, must X also be the image of some arc?*

2. IMAGES OF COMPACT ORDERED SPACES

The results of Mardešić and Papić and of Young alluded to above were substantially improved upon by TREYBIG [30] and A.J. WARD [41] in Theorem 1 below. Alternate proofs of this theorem have been given later by HEATH, LUTZER and ZENOR [8], MARDEŠIĆ in [15], and BULA, DEBSKI and KULPA in [2].

THEOREM 1. *If $f: K \rightarrow X \times Y$ is a continuous map of a compact ordered space K onto a product $X \times Y$, where both X and Y are infinite, then both X and Y are metrizable.*

Sketch of proof (MARDEŠIĆ [15]). We suppose first, since Y contains convergent sequences, that Y is of the form $\{y_1, y_2, \dots, y_\infty\}$, where $y_n \rightarrow y_\infty$, and f is strongly irreducible [30]. For each $n < \infty$, the set $X \times \{y_n\}$ is closed and open in $X \times Y$, so $K_n = f^{-1}(X \times \{y_n\})$ is also closed and open in K , and is thus the union of intervals $I_1^n, \dots, I_{k_n}^n$, which are closed and open. Let $\Pi: X \times Y \rightarrow X$ denote the natural projection. For each $n < \infty$ and subset $\{m_1, \dots, m_s\}$ of $\{1, \dots, k_n\}$ let $U_{m_1}^n, \dots, U_{m_s}^n$ denote $\text{Int } \Pi(\bigcup_{i=1}^s f(I_{m_i}^n))$. The set of all $U_{m_1}^n, \dots, U_{m_s}^n$ can be seen to be a countable basis for X , so X is metrizable. Likewise Y is metrizable. \square

We mention a related result of MARDEŠIĆ and PAPIĆ [20]: a dyadic compactum (i.e., an image of the product of discrete two point spaces) is an image of a compact ordered space if and only if it is metrizable. As with Theorem 1, this demonstrates vividly the great differences between the metric and Hausdorff cases among mapping problems.

The following sequence of theorems on images of compact ordered spaces brings us in chronological fashion up to the present.

THEOREM 2. (TREYBIG [31]). *If the continuum X is the image of a compact ordered space and if X is separated by no subset of fewer than three points, then X is metrizable.*

MARDEŠIĆ [15] has introduced a modification of the large inductive dimension which "neglects metrizable subcontinua" in the category of compact Hausdorff spaces. We set that $\text{Ind}(X, M) = -1$ if $X = \emptyset$ and $\text{Ind}(X, M) \leq 0$ if each component of X is metrizable. Then $\text{Ind}(X, M) \leq n$, ($n > 0$) if for each closed subset F of X and each open set U containing F , there exists an open set V with $F \subset V \subset U$ and $\text{Ind}(\text{Bd } V, M) \leq n - 1$. (Here the symbol M denotes the class of metrizable continua.) It is clear that $\text{Ind}(X, M) \leq \text{Ind } X$ with equality occurring if X contains no metrizable subcontinua.

THEOREM 3. (MARDEŠIĆ [15]). *If X is the Hausdorff image of a compact ordered space, then $\text{Ind}(X, M) \leq 1$.*

Mardešić later used Theorem 3 together with Theorem 1 to prove the following.

THEOREM 4. (MARDEŠIĆ [17]). *If X is the Hausdorff image of a compact ordered space, then X is locally peripherally metrizable.*

If X is a connected space and $x \in X$, we write M_x to denote the set of all $y \in X$ such that x and y lie in a metrizable subcontinuum of X . The sets M_x , called the *metric components* of X , form a partition of the space. Recall that a space is *paraseparable* (*Suslinian*) if each collection of mutually disjoint open sets (non-degenerate subcontinua) is countable. A space is *rim-finite* if each of its elements admits arbitrarily small neighbourhoods with finite boundary.

THEOREM 5. (SIMONE [27]). *If the Suslinian continuum X is the image of some compact ordered space, then the sets M_x are metrizable. Moreover, a paraseparable continuum containing no non-trivial metrizable subcontinuum is the image of some compact ordered space if and only if it is rim-finite.*

THEOREM 6. (SIMONE [26]). *If the continuum X contains no non-trivial metrizable subcontinuum and if X is the image of some compact ordered space, then X is hereditarily locally connected.*

Treybig has obtained the following strengthening of Simone's results.

THEOREM 7. (TREYBIG [32]). *If x and y are distinct elements of the continuum X , if x and y lie in no metrizable subcontinuum of X , and if X is the image of some compact ordered space, then x and y are separated by a finite set.*

SKETCH OF PROOF. Suppose not. If $S(x) = \{p \in X: p \text{ is not separated from } x \text{ in } X \text{ by a finite set}\}$, then $S(x)$ is a continuum containing x and y . If C is a subcontinuum of $S(x)$ which is irreducible from x to y , then by [31], C is the union of proper subcontinua C_1, C_2 , where $x \in C_1 - C_2$ and $y \in C_2 - C_1$. Let U_1, U_2, \dots be open sets containing x so that, for each n , $\bar{U}_{n+1} \subset U_n \subset X - C_2$, and let $Q = \bigcap_{i=1}^{\infty} \bar{U}_i$. If $G = \{p: p = Q \text{ or } p \in X - Q\}$, then X/G is the strongly irreducible image [30] of a compact ordered space K_1 under a map g .

There is a countable subset $\{y_1, y_2, \dots\}$ of K_1 so that for each n , $g^{-1}(Q)$ is covered by a finite set I_n of open intervals in K_1 so that

- (1) each endpoint of each $k \in I_n$ is in $\{y_1, y_2, \dots\}$; and
- (2) $\bigcup\{k: k \in I_n\} \subset g^{-1}(\phi(U_n))$, where $\phi: X \rightarrow X/G$ is the natural map.

Let X_1, X_2, \dots be a sequence of finite subsets of $K_1 = [a', b']$ so that

- (1) $a', b' \in X_1$ and $g(X_1)$ contains three points of C_2 ;
- (2) $\{y_1, \dots, y_n\} \subset X_n \subset X_{n+1}$ for each n ; and
- (3) each X_{n+1} is related to X_n as in Theorem 2 of [31].

It follows that $C_2 \subset g(\text{Cl}(\bigcup_1^{\infty} X_i))$, and Lemma 2 of [30] implies that C_2 is separable. By Theorem 1 of [31], C_2 is metrizable. Likewise C_1 is metrizable, so $C = C_1 \cup C_2$ is also, and this is a contradiction. \square

3. IMAGES OF ARCS

The first affirmative result concerning images of arcs, in a setting more general than the classical Halm-Mazurkiewicz theorem, is due to CORNETTE [3].

THEOREM 8. *The property of being the Hausdorff image of an arc is cyclically extensible and reducible.*

An immediate corollary to this result settled a question raised by PROIZVOLOV [25]. A *tree* is a continuum in which each pair of distinct points can be separated by a third point.

COROLLARY. (CORNETTE [3], PEARSON [23]). *A tree is the image of some arc.*

Subsequently and independently, PEARSON [24] and L.E. WARD, Jr. [43] improved this corollary.

THEOREM 9. (Pearson, Ward). *A rim-finite continuum is the image of some arc.*

In [32] TREYBIG applied Theorem 9 to obtain a partial solution to Problem 1.

THEOREM 10. (Treybig). *If the continuum X contains no non-trivial metrizable subcontinuum and if X is the image of some compact ordered space, then X is the image of some arc.*

In [34] TREYBIG has modified an argument of Mardesić to show that if X is a locally connected continuum which is the image of a compact ordered space, if $P = \{x \in X: \text{every neighbourhood of } x \text{ contains a non-metrizable subcontinuum}\}$ and if G denotes the decomposition of X into components of P and elements of $X - P$, then X/G is the image of an arc.

A *finite tree* is a tree with only finitely many endpoints. A continuum X can be approximated by finite trees if there exists a family J of finite trees such that

- (i) J is directed by inclusion;
- (ii) $\cup J$ is dense in X ; and
- (iii) if U is an open cover of X then there exists $T(U) \in J$ such that if $T(U) \subset T \in J$ and if C is a component of $T - T(U)$, then there exists $U \in U$ such that $C \subset U$.

THEOREM 11. (WARD [44]). *A continuum which can be approximated by finite trees is the image of some arc.*

SKETCH OF PROOF. Let X be a continuum and let J be a family of finite trees which approximates X . If T_1 and T_2 are members of J with $T_1 \subset T_2$ then there is a natural monotone retraction of T_2 onto T_1 ; taking these retractions as bonding maps, the inverse limit T_∞ of J is a tree. Each element (x_α) of T_∞ is a convergent net in X and it follows that the function $g: T_\infty \rightarrow X$ defined by $g((x_\alpha)) = \lim x_\alpha$ is a continuous surjection. By the corollary to Theorem 8, T_∞ is the image of some arc, so the result follows. \square

Among metrizable continua, the property of being approximated by finite trees is actually equivalent to local connectedness. This gives some credence to the possibility of an affirmative answer to this problem:

PROBLEM 2. *Is the converse of Theorem 11 true? I.e., is a continuum the image of an arc if and only if it can be approximated by finite trees?*

A continuum X is *finitely Suslinian* if for each open cover U of X and each infinite family K of disjoint subcontinua, some member of K is contained in a member of U . TYMCHATYN [37] has shown that every finitely Suslinian continuum can be approximated by finite trees and hence is the image of some arc. This generalizes Theorem 9. SIMONE [28] has shown that a continuum which contains no non-trivial metrizable subcontinuum is finitely Suslinian if and only if it is the image of some arc.

4. IRREDUCIBLE HAHN-MAZURKIEWICZ PROBLEMS

A continuous surjection $f: X \rightarrow Y$ is *strongly irreducible* if $f(K) \neq Y$ for each closed proper subset K of X . TREYBIG [30] has observed that every image of a compact ordered space is also the strongly irreducible image of a compact ordered space, but the situation is quite different for arcs and has proven to be surprisingly intractable. Even among metrizable continua the situation remains murky.

PROBLEM 3. *Characterize those continua which are the strongly irreducible images of $[0,1]$.*

The best answer to date was given in 1940 by O.G. HARROLD [7].

THEOREM 12. (Harrold). *If a Peano continuum contains a dense set of non-local separating points, then it is the strongly irreducible image of $[0,1]$.*

A related result is due to L.E. WARD, Jr. [45].

THEOREM 13. (Ward). *A Hausdorff space is a Peano continuum if and only if it is the strongly irreducible image of some dendrite.*

5. ON ARCWISE CONNECTEDNESS

The following question, which may be easier than Problem 1, was posed by MARDEŠIĆ [16].

PROBLEM 4. *If the locally connected continuum X is the continuous image of a compact ordered space, does it follow that X is arcwise connected?*

There are very few results which assert a conclusion of arcwise connectedness in Hausdorff continua. Of course, we have already noted that the

image of an arc is arcwise connected, so Problem 4 has an affirmative answer if Problem 1 has. Perhaps the strongest result on arcwise connectedness in Hausdorff continua is due to R.J. KOCH [12]. (See WARD [42] for another proof.)

THEOREM 14. (Koch). *Let X be a compact Hausdorff space, and suppose X is endowed with a partial order with closed graph. If W is a proper open subset containing no local minima, then each element of W lies in an arc which meets $X - W$.*

COROLLARY. *If X satisfies the hypotheses of Theorem 14, if X contains a zero relative to the partial order, and if $\{y \in X: y \leq x\}$ is a connected set for each $x \in X$, then X is arcwise connected.*

The corollary follows by letting $W = X - \{0\}$. The true strength of this theorem was demonstrated by Virginia Walsh KNIGHT [11] who showed that Peano continua always admit partial orders satisfying the hypotheses of the corollary. Therefore the classical arc theorem of R.L. MOORE [22] follows as a special case of Koch's theorem. It seems possible that Koch's theorem may be applicable to Problem 4.

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GO-SPACES WITH $\delta\theta$ -BASES

by

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In 1966 WORELL and WICKE [9] introduced the concept of a θ -base for a topological space as a generalization of a developable space. In 1967 BENNETT [2] introduced another generalization of developable spaces, namely, quasi-developable spaces. At first glance the notions of a quasi-developable space and a topological space with a θ -base seemed quite different but, in 1971, BENNETT and LUTZER [5] showed that the two concepts are equivalent. In 1974 C.E. AULL [1] introduced topological spaces with $\delta\theta$ -bases, an obvious generalization of topological spaces with θ -bases.

It was shown in [3] that a GO-space with a θ -base also has a point-countable base (the proof is for LOTS but is easily extended to GO-spaces) and it is obvious from the definitions that a point-countable base for a topological space is also a $\delta\theta$ -bases for the space. Hence in the class of GO-spaces we have

$$\theta\text{-base} \rightarrow \text{point-countable base} \rightarrow \delta\theta\text{-base.}$$

In [3] an example is given showing that the first arrow cannot be reversed and, in [4] an example is given showing that the second arrow cannot be reversed.

It is natural to ask when a GO-space with a $\delta\theta$ -base has a point-countable base and in this paper we give an answer to this question.

1. PRELIMINARIES

Let N denote the set of natural numbers, ω_0 the first infinite ordinal and ω_1 the first uncountable ordinal.

DEFINITION 1.1. A base \mathcal{B} for a topological space is a θ -base ($\delta\theta$ -base) if $\mathcal{B} = \cup\{\mathcal{B}_n \mid n \in N\}$ and, given an open set U and a point $x \in X$ such that $x \in U$,

then there exists $n \in \mathbb{N}$ such that x is in finitely (countably) many members of \mathcal{B}_n and there exists $B \in \mathcal{B}_n$ such that $x \in B \subset U$.

It is obvious that topological spaces with $\delta\theta$ -bases are first-countable spaces.

DEFINITION 1.2. A base \mathcal{P} for a topological space X is a *point-countable base* if each $x \in X$ is in at most countably many members of \mathcal{P} .

DEFINITION 1.3. A *linearly ordered topological space* (= LOTS) is a linearly ordered set equipped with the usual open interval topology of the given order. If \leq is the linear order on X , then a subset C of X is *convex* if, whenever a and b are in C such that $a < b$, then $\{x \in X \mid a < x < b\}$ is a subset of C . A *generalized ordered space* (= GO-space) is a linearly ordered set equipped with a T_1 -topology for which there is a base consisting of convex sets. GO-spaces have been studied extensively but the fundamental paper is [7]. All notation and terminology will follow [7].

DEFINITION 1.4. A topological space is *perfect* if closed sets are G_δ -sets.

If A is a set in a topological space X , let $\text{Int}(A)$ denote the interior of the set A , and let $|A|$ denote the cardinality of A . If \mathcal{B} is a collection of sets and p is a point in X , let $\text{ord}(p, \mathcal{B}) = |\{B \in \mathcal{B} \mid p \in B\}|$.

2. GO-SPACES WITH $\delta\theta$ -BASES

The following theorem gives a condition which insures that a GO-space with a $\delta\theta$ -base also has a point-countable base. Since there are Moore spaces (hence, perfect spaces with $\delta\theta$ -bases) that do not have point-countable bases we see that the GO-space structure is needed. Also in [3], [8] it was shown that if there are Souslin lines, then there are Souslin lines with point-countable bases. Since Souslin spaces are perfect the following theorem gives the best conclusion.

THEOREM 2.1. *If X is a perfect GO-space, then X has a point-countable base if and only if X has a $\delta\theta$ -base.*

PROOF. Let $\mathcal{B} = \{B_n \mid n \in \mathbb{N}\}$ be a $\delta\theta$ -base for X with underlying order \leq . No generality is lost if it is assumed that each member of \mathcal{B} is convex.

Let

$$I = \{\{x\} \mid x \in X, \{x\} \text{ open in } X\}.$$

For each $n \in \mathbb{N}$ let $X_n = \{x \in \mathcal{UB}_n \mid 1 \leq \text{ord}(x, \mathcal{B}_n) \leq \omega_0\}$. It follows that X_n is closed in \mathcal{UB}_n . For suppose $p \in \mathcal{UB}_n$ and p is a limit point of X_n . Consider the case where $] \leftarrow, p]$ is open (all other cases follow in a similar fashion). Then there is a monotonic sequence x_1, x_2, \dots of elements of X_n that converges to p . If $\text{ord}(p, \mathcal{B}_n) > \omega_0$ then, there exists $i \in \mathbb{N}$ such that $]x_i, p]$ is contained in uncountably many members of \mathcal{B}_n . Since each $B \in \mathcal{B}$ is convex and $x_i < x_{i+1} < p$, it follows that $\text{ord}(x_{i+1}, \mathcal{B}_n) > \omega_0$. This is a contradiction since $x_{i+1} \in X_n$. Thus $\text{ord}(p, \mathcal{B}_n) \leq \omega_0$ and $p \in X_n$.

Since \mathcal{UB}_n is open in X and X is perfect, $\mathcal{UB}_n = \bigcup \{F(n, i) \mid i \in \mathbb{N}\}$ where each $F(n, i)$ is closed in X . Hence each $F(n, i) \cap X_n$ is closed in X . If $\text{Int}(X_n \cap F(n, i)) \neq \emptyset$, let $A(n, i) = \{B \cap \text{Int}(X_n \cap F(n, i)) \mid B \in \mathcal{B}_n\}$. It is clear that $A(n, i)$ is a point-countable collection of open sets. Let $A = \bigcup \{A(n, i) \mid (n, i) \in \mathbb{N}^2\}$.

Let $G(n, i)$ be the collection of maximal, convex components of $[\mathcal{UB}_n - (X_n \cap F(n, i))] \cup \text{Int}(X_n \cap F(n, i))$. It follows that $UG(n, i)$ is dense in \mathcal{UB}_n and, since $UG(n, i)$ is open, $UG(n, i) = \bigcup \{K(n, i, k) \mid k \in \mathbb{N}\}$ where each $K(n, i, k)$ is closed in X . Let $E(n, i, k)$ be the collection of maximal convex components of $\mathcal{UB}_n - K(n, i, k)$ and let $E = \bigcup \{E(n, i, k) \mid (n, i, k) \in \mathbb{N}^3\}$.

Since $G(n, i)$ is a pairwise disjoint collection of convex open sets in the perfect space X , it follows that $G(n, i)$ is a σ -discrete (in \mathcal{UB}_n) collection [6]. Thus $G(n, i) = \bigcup \{G(n, i, j) \mid j \in \mathbb{N}\}$ such that for each $j \in \mathbb{N}$, $G(n, i, j)$ is a discrete (in \mathcal{UB}_n) collection.

Let $J(n, i, j, k) = \{G \in G(n, i, j) \mid G \cap K(n, i, k) \neq \emptyset\}$. Let $X(n, r) = \{x \in X_n \mid [x, \rightarrow[\text{ is an open set}\}$ and let $\mathcal{B}(n, r) = \{B \in \mathcal{B}_n \mid \text{there exists } x \in X(n, r) \text{ such that } x \text{ is the left endpoint of } B\}$. (Notice that an $x \in X(n, r)$ could be the left endpoint of countably many elements of $\mathcal{B}(n, r)$.)

Since the members of $J(n, i, j, k)$ are convex and $J(n, i, j, k)$ is discrete in \mathcal{UB}_n , if $B \in \mathcal{B}(n, r)$ it makes sense to refer to the first member of $J(n, i, j, k)$ that B intersects. Specifically, G_α is the first member of $J(n, i, j, k)$ that B intersects if $B \cap G_\alpha \neq \emptyset$ and if there does not exist $G_\beta \in J(n, i, j, k)$ such that $G_\beta < G_\alpha$ (i.e. there exists $x_\beta \in G_\beta$, $x_\alpha \in G_\alpha$ such that $x_\beta < x_\alpha$) and $G_\beta \cap B \neq \emptyset$.

Let $J(n, i, j, k) = \{G_\alpha \mid \alpha \in I(n, i, j, k)\}$ where $I(n, i, j, k)$ is some indexing set. For each $\alpha \in I(n, i, j, k)$, let

$$\mathcal{B}(n,i,j,k,\alpha) = \{B \in \mathcal{B}(n,r) \mid B \cap K(n,i,k) \neq \emptyset \text{ and } G_\alpha \text{ is the first member of } J(n,i,j,k) \text{ that } B \text{ intersects}\}.$$

For each $B \in \mathcal{B}(n,i,j,k,\alpha)$, let $C(B)$ be the convex component of $B \cap (X - K(n,i,k))$ that contains the left endpoint of B . Let $\mathcal{C}(n,i,j,k,\alpha) = \{C(B) \mid B \in \mathcal{B}(n,i,j,k,\alpha)\}$. Notice that if $C(B) \in \mathcal{C}(n,i,j,k,\alpha)$, then there does not exist $\beta \in I(n,i,j,k)$, $\beta \neq \alpha$, such that $C(B) \cap G_\beta \neq \emptyset$.

Arbitrarily fix n, i, j and k in N and $\alpha \in I(n,i,j,k)$. Let $G_\alpha \in J(n,i,j,k)$ and consider the following cases:

CASE 1. G_α has a left endpoint a .

- (i) If $a \notin G$, then $a \in X_n$. Thus $|\mathcal{B}(n,i,j,k,\alpha)| \leq \omega_0$ and $|\mathcal{C}(n,i,j,k,\alpha)| \leq \omega_0$.
- (ii) If $a \in G_\alpha$ and $a = a^+$, the right hand point of a jump $[a^-, a^+]$, ($[a^-, a^+] = \emptyset$), then, by maximal convexity of G , it follows that $a^- \in X_n$. Thus $|\mathcal{B}(n,i,j,k,\alpha)| \leq \omega_0$ and $|\mathcal{C}(n,i,j,k,\alpha)| \leq \omega_0$.
- (iii) If $a \in G_\alpha$ and a is the right hand point of a pseudo-gap, then there is a monotonic net $x_1, x_2, \dots, x_\beta, \dots, \beta < \omega_1$, of elements, of X_n such that if $b < a$, then there is an $\alpha < \omega_1$ such that $b < x_\alpha < a$. To obtain this net argue as follows: Since $|\{B \in \mathcal{B}(n,i,j,k,\alpha) \mid a \in B\}| > \omega_0$ choose $x_1 \in X_n$, $x_1 < a$, such that for each $\gamma \in I(n,i,j,k)$, $\gamma \neq \beta$, if $t \in G_\gamma$ and G_γ precedes G_α , then $t < x_1$. Since $|\{B \in \mathcal{B}(n,i,j,k,\alpha) \mid x_1 \in B\}| \leq \omega_0$, choose $x_2 \in X_n$ such that $x_1 < x_2 < a$. Suppose $x_1, x_2, \dots, x_\beta, \dots, \beta < \tau < \omega_0$, have been chosen such that $x_1 < x_2 < \dots < x_\beta < \dots < a$ for each $\beta < \tau$. Since $|\{B \in \mathcal{B}(n,i,j,k,\alpha) \mid x_\beta \in B, \beta < \tau\}| \leq \omega_0$, choose $x_\tau \in X_n$ such that $x_\beta < x_\tau < a$ for each $\beta < \tau$. Thus such a net can be chosen inductively. It is easily seen that x_{ω_1} , cannot be chosen.

CASE 2. G_α does not have an endpoint. Then, in X^+ (= the order completion of X), the left endpoint of G_α represents a gap or a pseudo-gap. In either case if $|\{B \in \mathcal{B}(n,i,j,k,\alpha) \mid B \cap G_\alpha \neq \emptyset\}| > \omega_0$ then construct a monotonic net as in Case 1, part (iii).

For each $\alpha \in I(n,i,j,k)$, if $|\mathcal{C}(n,i,j,k,\alpha)| \leq \omega_0$ let $\mathcal{C}(n,i,j,k,\alpha) = \mathcal{D}(n,i,j,k,\alpha)$. If $|\mathcal{C}(n,i,j,k,\alpha)| \geq \omega_0$ then there is a monotone net $x_1, x_2, \dots, x_\beta, \dots, \beta < \omega_1$, of elements of X_n that converges (in X^+) to $\inf G_\alpha$ (in X^+). If $C(B) \in \mathcal{C}(n,i,j,k,\alpha)$ and y_B is the left endpoint of $C(B)$, let x_B be the first element of the net such that $y_B < x_B$. Let $D(B) = C(B) \cap]x_B, a[$. Let

$\mathcal{D}(n,i,j,k,\alpha,\beta) = \{D(B) \mid C(B) \in \mathcal{C}(n,i,j,k,\alpha), x_B = x_\beta\}$. Notice if $\beta \neq \beta'$, $D \in \mathcal{D}(n,i,j,k,\alpha,\beta)$, $D' \in \mathcal{D}(n,i,j,k,\alpha,\beta')$, then $D \cap D' = \emptyset$. Also notice that $|\mathcal{D}(n,i,j,k,\alpha,\beta)| \leq \omega_0$ since each $C(B)$ meets G_α and thus $x_\beta \in C(B)$. Let $\mathcal{D}(n,i,j,k,\alpha) = \mathcal{U}\{\mathcal{D}(n,i,j,k,\alpha,\beta) \mid \beta < \omega_1\}$. It is clear that $\mathcal{D}(n,i,j,k,\alpha)$ is a point-countable collection. Let $\mathcal{D}(n,i,j,k) = \mathcal{U}\{\mathcal{D}(n,i,j,k,\alpha) \mid \alpha \in I(n,i,j,k)\}$.

Suppose these exist $p \in \mathcal{UB}(n,r)$ such that $\text{ord}(p, \mathcal{D}(n,i,j,k)) > \omega_0$ (i.e. suppose $\mathcal{D}(n,i,j,k)$ is not a point-countable collection). Since each $D \in \mathcal{D}(n,i,j,k)$ is obtained from one $B \in \mathcal{B}(n,r)$, it follows that $p \notin X_n$. Thus $p \in G_\gamma \in \mathcal{G}(n,i,j)$. Suppose $\gamma \in I(n,i,j,k)$. Then, if $p \in D \in \mathcal{D}(n,i,j,k)$, there exists $\beta < \omega_1$ such that $D \in \mathcal{D}(n,i,j,k,\gamma,\beta)$ but $|\mathcal{D}(n,i,j,k,\gamma,\beta)| \leq \omega_0$. Thus $\gamma \notin I(n,i,j,k)$. Hence if $p \in D \in \mathcal{D}(n,i,j,k)$, then there exists $\alpha \in I(n,i,j,k)$ such that $D \in \mathcal{D}(n,i,j,k,\alpha)$. Since $\gamma \notin I(n,i,j,k)$ there exists $x_\beta \in X_n$ such that $p < x_\beta$ and $x_\beta \notin G_\alpha$. Thus if $\text{ord}(p, \mathcal{D}(n,i,j,k,\alpha)) > \omega_0$, then $\text{ord}(p, \mathcal{C}(n,i,j,k,\alpha)) > \omega_0$. Hence, $\text{ord}(p, \mathcal{B}(n,r)) > \omega_0$. Since elements of $\mathcal{B}(n,r)$ are convex it follows that $\text{ord}(x_\beta, \mathcal{B}(n,r)) > \omega_0$. This is a contradiction since $x_\beta \in X_n$. Thus $\mathcal{D}(n,i,j,k)$ is a point-countable collection.

Let

$$\mathcal{D} = \mathcal{U}\{\mathcal{D}(n,i,j,k) \mid (n,i,j,k) \in \mathbb{N}^4\}.$$

In an analogous fashion construct from $X(n,\ell) = \{x \in X_n \mid]\leftarrow, x[\text{ is open}\}$ the point-countable collection

$$\mathcal{H} = \mathcal{U}\{\mathcal{H}(n,i,j,k) \mid (n,i,j,k) \in \mathbb{N}^4\}.$$

Let $\mathcal{P} = \mathcal{I} \cup \mathcal{A} \cup \mathcal{E} \cup \mathcal{D} \cup \mathcal{H}$. It is clear that \mathcal{P} is a point-countable collection of open sets.

To see that \mathcal{P} is a base for X , let $x \in X$ and let U be open in X such that $x \in U$. Consider the following cases:

CASE 1. If $\{x\}$ is open, then $\{x\} \in \mathcal{I} \subset \mathcal{P}$ and $\{x\} \subset U$.

CASE 2. If neither $\{x\},]\leftarrow, x[$ nor $[x, \rightarrow[$ is open, then find $n \in \mathbb{N}$ such that there exists $B \in \mathcal{B}_n$, $x \in B \subset U$ and $1 \leq \text{ord}(x, \mathcal{B}_n) \leq \omega_0$.

(i) if there exists $i \in \mathbb{N}$ such that $x \in \text{Int}(X_n \cap F(n,i))$, then there exists $A \in \mathcal{A}(n,i) \subset \mathcal{A} \subset \mathcal{P}$ such that $x \in A \subset U$.

(ii) If there does not exist $i \in \mathbb{N}$ such that $x \in \text{Int}(X_n \cap F(n,i))$, then arbitrarily choose $i \in \mathbb{N}$ and a and b in x such that $]a, b[\subset U$ and

$a < x < b$. Since $UG(n,i)$ is dense in UB_n , there exists $k \in \mathbb{N}$ such that $K(n,i,k) \cap]a,x[\neq \emptyset$ and $K(n,i,k) \cap]x,b[\neq \emptyset$. Let J_x be the convex component of $UB_n - K(n,i,k)$ that contains x . Hence $x \in J_x \subset]a,b[\subset U$ and $J_x \in E(n,i,k) \subset E \subset \mathcal{P}$.

CASE 3. If $]x, \rightarrow[$ is open and $\{x\}$ is not open, find $n \in \mathbb{N}$ such that $1 \leq \text{ord}(x, B_n) < \omega_0$ and choose $B_x \in B_n$ such that $x \in B_x \subset U$. Then there exists i, j, k in \mathbb{N} and α in $I(n,i,j,k)$ such that $B_x \in B(n,i,j,k,\alpha)$, and $D(B_x) \in \mathcal{D}(n,i,j,k,\alpha)$. Then there exists $D \in \mathcal{D}(n,i,j,k)$ such that $x \in D \subset B \subset U$.

CASE 4. If $\leftarrow, x]$ is open and $\{x\}$ is not open argue, using H , as in Case 3.

Thus \mathcal{P} is a point-countable base for X .

Using techniques similar to [6] the following theorem is obtained.

THEOREM 2.2. *A GO-space with a $\delta\theta$ -base is hereditarily paracompact.*

This theorem is not unexpected since, in the class of GO-spaces, spaces with θ -bases and spaces with point-countable bases are known to hereditarily paracompact [3].

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PRETRANSITIVITY AND PRODUCTS OF
SUBORDERABLE SPACES

by

Ralph Fox

All spaces are T_1 topological spaces.

The classic γ -space conjecture [10,9,11] asserts that all γ -spaces are quasi-metrizable. Recently, Fletcher and Lindgren have introduced the concept of n -pretransitivity of a topological space for non-negative integers n , and pointed out that every n -pretransitive γ -space is quasi-metrizable. The importance of n -pretransitivity is that almost all partial solutions to the γ -space conjecture have used this property: [4,8], [6] and Kofner's proof [7] that suborderable γ -spaces are quasi-metrizable have all shown (even if implicitly) that the spaces concerned are 2- or 3-pretransitive.

In this note we give the first example of a quasi-metrizable space which is not n -pretransitive for any non-negative integer n . The space is the ω^{th} power of the Michael line M , a suborderable quasi-metrizable space [1]. In fact, we show that the n^{th} power M^n of M is not $(n-1)$ -pretransitive. In a forthcoming paper [3] we will show how to construct a counterexample to the γ -space conjecture from a quasi-metrizable space which is not n -pretransitive for any n .

Following [5], a binary relation U on a space X is called a *neighbournet* if for each $x \in X$ the set $U[x]$ is a neighbourhood of x , and a *normal neighbournet* if there exists a sequence $\langle W_k : k \in \mathbb{N} \rangle$ of neighbournets with $W_1 \subseteq V$ and $W_{k+1}^2 \subseteq W_k$ for each $k \in \mathbb{N}$. By U^n we denote the n -fold composite $U \circ U \circ \dots \circ U$ (n times), and by U^0 the diagonal $\{\langle x, x \rangle : x \in X\}$. A space X is called n -pretransitive [2] if whenever U is a neighbournet on X then U^n is a normal neighbournet.

The Michael line M is the space obtained from the real line \mathbb{R} by scattering the irrationals: i.e. rational points have their usual neighbourhoods while irrational points are isolated. Observe that M has a quasi-metric d given by $d(u, v) = 1$ if u is irrational; $d(u, v) = \min\{1, |u-v|\}$ if u is rational.

THEOREM. *The space M^n is not $(n-1)$ -pretransitive.*

PROOF. We will construct a neighbourhood U_n on M^n , and show by induction on n that U_n^{n-1} is not a normal neighbourhood. For any $x \in M^n$, x_i will denote the i^{th} coordinate of x for $1 \leq i \leq n$.

If t is a rational number let $q(t)$ be the smallest positive denominator of t , while if t is irrational let $q(t) = 1$. If $x = \langle x_1, \dots, x_n \rangle \in M^n$ we let $U_n[x]$ be the cartesian product of the following interval neighbourhoods of its coordinates x_i : if x_i is irrational we take as neighbourhood the singleton $\{x_i\}$, while if x_i is rational we take as neighbourhood the largest open interval (r, s) containing x_i such that if t is any rational number in (r, s) other than x_i then $q(t) > \max\{q(x_1), \dots, q(x_n)\}$. The following properties of U_n can be verified.

- (i) If $y \in U_n[a]$ and all coordinates of y are rational, then $U_n[y] \subseteq U_n[a]$.
- (ii) If $y \in U_n[a]$ then $|y_i - a_i| < 1/q(a_n)$.
- (iii) If $y \in U_n[a]$ and y_1, \dots, y_{n-1} are rational, y_n irrational, then $U_n[y] \subseteq U_n[\langle a_1, \dots, a_{n-1}, y_n \rangle]$.

(Properties (i) and (iii) follow from the maximality of the interval (r, s) in the definition of U_n , together with the fact that $q(y_i) \geq q(a_i)$ whenever y_i is rational and $y \in U_n[a]$. Property (ii) follows since if the intervals (r, a_i) or (a_i, s) have length larger than $1/q(a_n)$, they must contain a rational with denominator $q(a_n)$.)

To show that U_n^{n-1} is not normal, we will show that there exists no neighbourhood W on M^n such that $W^n \subseteq U_n^{n-1}$. We will show by induction on n that for any neighbourhood W on M^n there exist $a \in M^n$ with all coordinates rational, and $x \in M^n$ with all coordinates irrational, such that $x \in W^n[a]$ but $x \notin U_n^{n-1}[a]$.

The case $n = 1$ is immediate since $U_1^0[a] = \{a\}$ while every neighbourhood of a rational point in M contains irrational points.

Assume that the inductive hypothesis holds for $n-1$. Since for each irrational x_n and each $x' = \langle x_1, \dots, x_{n-1} \rangle \in M^{n-1}$ we have $U_n[\langle x', x_n \rangle] = U_{n-1}[x'] \times \{x_n\}$, we may apply the inductive hypothesis to the copy $M^{n-1} \times \{x_n\}$ of M^{n-1} to find $a'(x_n) \in M^{n-1}$ with rational coordinates and $x'(x_n) \in M^{n-1}$ with irrational coordinates such that

- (iv) $\langle x'(x_n), x_n \rangle \in W^{n-1}[\langle a'(x_n), x_n \rangle]$ but
- (v) $\langle x'(x_n), x_n \rangle \notin U_n^{n-2}[\langle a'(x_n), x_n \rangle]$.

Applying the Baire Category Theorem to the irrationals in \mathbb{R} , we may find a set D of irrational numbers dense (with respect to the Euclidean topology)

in some open interval (u,v) , such that all $x_n \in D$ have a common $a'(x_n) = a'$ and a common positive lower bound ϵ to all coordinate-to-coordinate distances $|x'_i(x_n) - a'_i|$ for $1 \leq i \leq n-1$. Choose a rational point a_n in (u,v) such that (vi) $1/q(a_n) < \epsilon$, and let $a = \langle a', a_n \rangle \in M^n$. Next, choose $x_n \in D$ such that $\langle a', x_n \rangle \in W[a]$, and let $x = \langle x'(x_n), x_n \rangle \in M^n$. Then by (iv) $x \in W^{n-1}[\langle a', x_n \rangle]$ and hence $x \in W^n[a]$. To complete the proof we will show that $x \notin U_n^{n-1}[a]$.

For suppose otherwise, and find a minimal $m \leq n-2$ and $y \in U_n[a]$ with $x \in U_n^m[y]$. Then not all coordinates of y are rational: if $m = 0$ this follows as $y = x$; alternatively if $m > 0$ this follows by (i), since $U_n[y] \not\subseteq U_n[a]$, because given $z \in U_n[y]$ with $x \in U_n^{m-1}[z]$ then $z \notin U_n[a]$ from the minimality of m . Since if y_i is irrational then $x_i = y_i$, while $|x_i - a_i| \geq \epsilon$ and yet by (ii) and (vi) $|y_i - a_i| < 1/q(a_n) < \epsilon$ for $1 \leq i \leq n-1$, we may suppose that y_1, \dots, y_{n-1} are rational and $y_n = x_n$ is irrational. Then by (iii), $U_n[y] \subseteq U_n[\langle a', x_n \rangle]$. It follows that $U_n^{n-2}[y] \subseteq U_n^{n-2}[\langle a', x_n \rangle]$. This is a contradiction, since $x \notin U_n^{n-2}[\langle a', x_n \rangle]$ from (v).

Thus $x \notin U_n^{n-1}[a]$ as required.

COROLLARY. The space M^ω is not n -pretransitive for any non-negative integer n .

PROOF. This follows since n -pretransitivity is closed-hereditary [2], while for each $n \in \mathbb{N}$ the space M^ω contains closed subspaces $M^n \times \{\langle x_{n+1}, x_{n+2}, \dots \rangle\}$ homeomorphic to M^n .

From an earlier non-regular example by the author of a non- n -pretransitive quasi-metrizable space for each $n \in \mathbb{N}$, Jacob Kofner has independently shown that the n^{th} power of the Michael line is not $(n-1)$ -pretransitive. The author would like to thank Jacob Kofner for helpful discussions during the preparation of this paper for publication.

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ADDED IN PROOF. The space M^n is not $(n-1)$ -pretransitive according to the Theorem above, but it is n -pretransitive [2, and J. Kofner, *Products of ordered spaces and transitivity*, this volume]. A modification of the proof of the Theorem above yields the following slightly stronger result concerning the product of n suborderable spaces: the space $\mathbb{R} \times M^{n-1}$ is not n -pretransitive (but is $(n+1)$ -pretransitive).

COVERING PROPERTIES OF LINEARLY ORDERED TOPOLOGICAL
SPACES AND THEIR PRODUCTS

by

Marlene E. Gewand and Scott W. Williams

1. INTRODUCTION

While the Tychonoff theorem asserts that any product of compact spaces is compact, other covering properties, paracompactness and the Lindelöf property in particular, fail to be productive even in finite products. The question of when such properties are productive has been asked many times and particular cases have been answered. A list of papers concerning these questions would be too lengthy to produce here, but a few are given in the references ([11], [12], [15]). These questions continue to be of interest. In this paper we consider the case when one of the factors is a linearly ordered topological space (LOTS). The technique of defining an equivalence relation on a LOTS and then examining the resulting quotient space has proven to be useful in determining properties of the LOTS. We use this technique here to examine the covering properties of LOTS and of products of LOTS with other spaces.

Notations and Definitions

All spaces are assumed to be Hausdorff and regular.

A *linearly ordered topological space* (LOTS) is a linearly ordered set with its interval topology. An *interior gap* of a LOTS X is a Dedekind cut (A/B) of X such that A has no supremum (sup) and B has no infimum (inf). An *end-gap*, left or right, means the absence of an infimum or supremum of the linearly ordered set. The *Dedekind compactification* X^+ of a LOTS X is formed by suitably ordering $X \cup \{g: g \text{ is a gap of } X\}$ in a manner similar to the completion of the rationals; X is dense in the compact space X^+ . For further details on LOTS, their gaps, and their compactifications, we suggest [4]. The lexicographic product of two linearly ordered sets X and Y is denoted $X \text{ lex } Y$.

Intervals in a LOTS X are denoted by $[a,b]$ when closed and by $]a,b[$ when open, and in the latter case a and/or b may be a gap. Other intervals are denoted by $W(a) = \{x \in X: x < a\}$ and $W^*(a) = \{x \in X: a < x\}$. A *convex set* C satisfies " $a,b \in C$ and $a < x < b$ imply $x \in C$ ". Singleton sets are considered to be convex.

A topological space X is a α -*Lindelöf* if and only if every open cover of X has a subcover of cardinality less than or equal to α . A space is *linearly* α -*Lindelöf* if every open cover, linearly ordered by inclusion, has a subcover of cardinality less than or equal to α .

For any topological space X , we define the subspace η^*X by $\eta^*X = \{x \in X: x \text{ does not have a compact neighbourhood in } X\}$. A *scattered-like decomposition* of X is defined inductively by letting $\eta_0X = X$ and, for $0 < \beta$, $\eta_\beta X = \bigcap \{\eta_\alpha^*X: \alpha < \beta\}$. We note that for any space X , there exists a first ordinal γ such that $\eta_\gamma X = \eta_{\gamma+1} X$. We let $\eta X = \eta_\gamma X$.

We follow the notation and definitions of JUHÁSZ [6] in defining the following cardinal functions.

The *Lindelöf degree* of a space X is

$$L(X) = \omega \cdot \min\{\alpha: X \text{ is } \alpha\text{-Lindelöf}\}.$$

The *character* at a point $p \in X$ is

$$\chi(p,X) = \min\{|N|: N \text{ is a neighbourhood base for } p\}.$$

The *character* of a space X is

$$\chi(X) = \sup\{\chi(p,X): p \in X\}.$$

The *density* of a space X is

$$d(X) = \omega \cdot \min\{|S|: S \subseteq X, \bar{S} = X\}.$$

2. PRELIMINARIES

The following two lemmas will be called upon in the next section. They indicate conditions under which a subspace of a LOTS may be viewed in terms of the real line. These results were announced in 1974 [16] and since that time, similar results have appeared. We refer the interested reader to the

recent work of VAN WOUWE [13].

LEMMA 2.1. *Given any countable subspace of a LOTS, there exists an order-preserving homeomorphism onto a subspace of \mathbb{R} .*

PROOF. We denote by P , a countable subspace of a LOTS X , with its subspace topology τ and its restricted linear order.

Let P^* be the set of all $(p, q) \in P \times (]0, 1[\cap \mathbb{Q})$, where \mathbb{Q} is the set of rational numbers, that satisfy one of the following:

- (i) $q = \frac{1}{2}$,
- (ii) $\frac{1}{2} < q$ if p has an immediate successor in P or p is the last element of P ,
- (iii) $q < \frac{1}{2}$ if p has an immediate predecessor in P or p is the first element of P .

Let τ^* be the topology on P^* generated by taking as a subbase the lexicographic order topology on P^* together with sets

$$\{(p, \frac{1}{2})\} \cup W^*((p, \frac{1}{2})) \quad \text{if } \{p\} \cup W^*(p) \in \tau,$$

and

$$\{(p, \frac{1}{2})\} \cup W((p, \frac{1}{2})) \quad \text{if } \{p\} \cup W(p) \in \tau.$$

P^* is order-isomorphic to $\mathbb{Q} \cap]0, 1[$ since it is countable, possesses no end-points, and no adjacent points. Moreover, the map $p \rightarrow (p, \frac{1}{2})$ is an order-homeomorphism from P onto a subspace of P^* . So we consider P as that subspace.

Let $f: (]0, 1[\cap \mathbb{Q}) \rightarrow P^*$ be an order-isomorphism, rewrite $]0, 1[\cap \mathbb{Q}$ as a sequence $\{q_n: n \in \omega\}$, and define for each $r \in \mathbb{R}$

$$i(r, n) = \begin{cases} -1 & \text{if } r < q_n \text{ and } \{f(q_n)\} \cup W^*(f(q_n)) \in \tau^*, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$j(r, n) = \begin{cases} 1 & \text{if } q_n < r \text{ and } \{f(q_n)\} \cup W(f(q_n)) \in \tau^*, \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$g(r) = r + \sum_{n=1}^{\infty} \frac{1}{2^n} i(r,n) + \sum_{n=1}^{\infty} \frac{1}{2^n} j(r,n).$$

Suppose $r, s \in \mathbb{R}$ and $r < s$. Then $i(s,n) = -1$ implies $i(r,n) = -1$, while $j(r,n) = 1$ implies $j(s,n) = 1$; therefore $g: \mathbb{R} \rightarrow \mathbb{R}$ is an order-isomorphism onto its image.

We now show $(g \circ f^{-1}) \upharpoonright P^*$ is an order-homeomorphism onto its image. Suppose $\{f(q_k)\} \cup W^*(f(q_k)) \in \tau^*$; then $\{g(q_k)\} \cup W^*(g(q_k)) \in \mathbb{R}/g(Q \cap]0,1[)$ from the definition of g . Suppose $\{f(q_k)\} \cup W^*(f(q_k)) \notin \tau^*$; then $i(r,k) = 0$ for every $r \in \mathbb{R}$.

Let $\varepsilon > 0$ and choose m so large that $\sum_{n=m}^{\infty} \frac{2}{2^n} < \frac{\varepsilon}{3}$ and $q_k - \frac{\varepsilon}{3} < q_m < q_k$.

We further suppose there is an $s > m$ so large that

- (a) $q_m < q_s < q_k$,
- (b) $i(q_s, n) = -1$ and $i(q_k, n) = 0$ implies $m < n$,
- (c) $j(q_s, n) = 0$ and $j(q_k, n) = 1$ implies $m < n$.

In this case

$$\sum_{n=1}^{\infty} \frac{1}{2^n} j(q_s, n) \leq \sum_{n=1}^{\infty} \frac{1}{2^n} j(q_k, n) \leq \sum_{n=1}^{\infty} \frac{1}{2^n} j(q_s, n) + \frac{\varepsilon}{3}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{2^n} i(q_k, n) - \frac{\varepsilon}{3} \leq \sum_{n=1}^{\infty} \frac{1}{2^n} i(q_s, n) \leq \sum_{n=1}^{\infty} \frac{1}{2^n} i(q_k, n).$$

So $g(q_k) < g(q_s) + \varepsilon$ and hence there are points of $g(]0,1[\cap Q)$ arbitrarily close to $g(q_k)$ from below. Thus $\{g(q_k)\} \cup W^*(g(q_k)) \notin \mathbb{R}/g(Q \cap]0,1[)$.

On the other hand, if there is no such s , then an entire interval of points in $Q \cap]0,1[$ with supremum q_k is translated uniformly. So in this case, $\{g(q_k)\} \cup W^*(g(q_k)) \notin \mathbb{R}/g(Q \cap]0,1[)$. Similarly, $\{f(q)\} \cup W^*(f(q)) \in \tau^*$ if and only if $\{g(q)\} \cup W^*(g(q)) \in \mathbb{R}/g(Q \cap]0,1[)$. Hence it follows that $(g \circ f^{-1}) \upharpoonright P^*: P^* \rightarrow g(Q \cap]0,1[)$ is an order-homeomorphism. Since P is a subspace of P^* , $g(P)$ is a subspace of $g(Q \cap]0,1[)$. \square

LEMMA 2.2. *Suppose X is a closed subspace of a LOTS with its subspace topology σ in relationship to its restricted linear order and suppose (X, σ) is separable. Then there is a subspace $Y \subseteq \mathbb{R} \text{ lex}\{0,1\}$ such that (X, σ) is order-homeomorphic to a subspace of Y .*

PROOF. Let P be a countable dense subspace of X and for each $x \in X - P$, choose a sequence $\{p(x,n): n < \omega\} \subseteq P$ either strictly increasing or strictly

decreasing and converging to x .

Define $h: X \rightarrow \mathbb{R}$ by

$$h(x) = \begin{cases} (g \circ f^{-1})(x) & \text{if } x \in P \\ \lim_{n \rightarrow \infty} (g \circ f^{-1})(p(x, n)) & \text{otherwise} \end{cases}$$

where g and f are defined as in Lemma 2.1.

Now if $r \in]0, 1[\cap (\mathbb{R} - \mathbb{Q})$, then

$$\begin{aligned} g(r) &= \sup_{\mathbb{R}} \{g(q) : q \in (\mathbb{Q} \cap]0, 1[), q < r\} \\ &= \inf_{\mathbb{R}} \{g(q) : q \in (\mathbb{Q} \cap]0, 1[), r < q\}. \end{aligned}$$

Moreover, g performs a translation on $\{0\} \cup W(0)$ and on $\{1\} \cup W^*(1)$. Hence, h is at most two-to-one, and if $h^{-1}(r)$ consists of two points, then those points are adjacent in X and neither may belong to P . Let

$$Y = ((\mathbb{R} \text{ lex } \{0, 1\}) - \{(h(x), 1) : x \in P\} \cup \{(h(x), 1) : |h^{-1}(h(x))| = 1$$

and $(\{x\} \cup W^*(x)) \notin \sigma$ and $(\{x\} \cup W(x)) \notin \sigma$).

Give Y the restricted order and the order topology induced by that order. Define a map $h^*: X \rightarrow Y$ as follows:

$$h^*(x) = \begin{cases} (h(x), 1) & \text{if } |h^{-1}(h(x))| = 1 \text{ and } (\{x\} \cup W^*(x)) \in \sigma \\ (h(x), 1) & \text{if } |h^{-1}(h(x))| = 2 \text{ and } x = \sup_X h^{-1}(h(x)) \\ (h(x), 0) & \text{otherwise.} \end{cases}$$

Then h^* is an order-homeomorphism onto its image as a subspace of Y . \square

The main results of this paper are concerned with product spaces. However as a preliminary result, we would like to characterize the Lindelöf degree of GO-spaces. FABER [2] gives very useful characterizations of paracompact and Lindelöf GO-spaces. The characterizations given here were obtained independently of Faber's work and were announced by the authors in 1975 [3].

From the characterizations of compactness, countable compactness, and paracompactness for LOTS, one may conjecture that every LOTS in which each gap is of countable character is Lindelöf. However the space $]0, 1[\text{ lex }]0, 1[$

has Lindelöf degree c , the power of the continuum, while each of its gaps has countable character. Knowing the character of the gaps does yield a bound on the Lindelöf degree; but to properly characterize the Lindelöf degree we need an additional property.

THEOREM 2.3. *For any GO-spaces X , the following are equivalent:*

- (i) $L(X) \leq \alpha$
- (ii) (a) $\chi(g, X^+) \leq \alpha$ for every $g \in X^+ - X$ and
(b) every cover of X by pairwise disjoint clopen convex sets has cardinality no greater than α .
- (iii) (a) $\chi(g, X^+) \leq \alpha$ for every $g \in X^+ - X$ and
(b') every clopen convex cover of X has a subcover of cardinality no greater than α .
- (iv) X is linearly α -Lindelöf.

PROOF. (i) implies (iii): Suppose $L(X) \leq \alpha$. We only need to show condition (a). Suppose $\{x_\beta: \beta < \gamma\}$ is an increasing sequence in X with $\alpha < \text{cf}(\gamma)$. We wish to show $\{x_\beta: \beta < \gamma\}$ converges in X . The family $\{W(x_\beta): \beta < \gamma\} \cup W^*(\sup\{x_\beta: \beta < \gamma\})$ is an open cover of $A = X - \{\sup\{x_\beta: \beta < \gamma\}\}$ with no subcover of cardinality less than or equal to α . Thus $A \neq X$ and $\{x_\beta: \beta < \gamma\}$ converges in X . Similarly, decreasing sequences in X with cofinality greater than α converge in X . Hence $\chi(g, X^+) \leq \alpha$ for every $g \in X^+ - X$.

(iii) implies (ii) is immediate, as is (i) implies (iv).

(ii) implies (i): Suppose C is an open cover of X . We define a relation R on X as follows: For $x, y \in X$, xRy if and only if there are points $a, b \in X$ such that $x, y \in [a, b]$ and $[a, b]$ can be covered by a subfamily of C of cardinality less than or equal to α . It is easily seen that R is an equivalence relation and we observe that Rx , the equivalence class determined by x , is an interval for each $x \in X$. Furthermore it can be shown that if $\sup(Rx) \in X$, then $\sup(Rx) \in Rx$ and $\sup(Rx) = \sup X$ and similarly for the infimum. Hence each Rx is clopen. Then by condition (ii)(b), $|\{Rx: x \in X\}| \leq \alpha$.

By arguments similar to those used in the proof of Theorem 3.1, we can show that each Rx can be covered by a subfamily of C of cardinality less than or equal to α .

Hence $L(X) \leq \alpha$.

(iv) implies (ii): Let X be a linearly α -Lindelöf GO-space. Suppose $\{x_\beta: \beta < \gamma\}$ is an increasing sequence in X with $\alpha < \text{cf}(\gamma)$. For each $\delta < \gamma$, let $U_\delta = W(x_\delta) \cup W^*(\sup\{x_\beta: \beta < \gamma\})$. Then $\{U_\beta: \beta < \gamma\}$ is an open cover, linearly ordered by inclusion, of $A = X - \{\sup\{x_\beta: \beta < \gamma\}\}$ with no subcover of

cardinality less than or equal to α . Thus $A \neq X$ and $\{x_\beta: \beta < \gamma\}$ converges in X . The situation for decreasing sequences is similar. Hence $\chi(g, X^+) = \alpha$ for each $g \in X^+ - X$.

Now let $C = \{C_\delta: \delta < \beta\}$ be a cover of X by pairwise disjoint clopen convex sets. Then C must be a minimal cover. For each $\delta < \beta$, let $B_\delta = \bigcup\{C_\gamma: \gamma \leq \delta\}$. The family $B = \{B_\delta: \delta < \beta\}$ is linearly ordered by inclusion and since X is linearly α -Lindelöf, B , as a minimal cover, must be of cardinality less or equal to α . Hence $|C| \leq \alpha$. \square

Of course every Lindelöf space is linearly Lindelöf. MISČENKO [8] has constructed a space where the converse of this fails. Theorem 2.3 establishes the converse for GO-spaces.

3. COVERING PROPERTIES OF PRODUCTS WHERE ONE FACTOR IS A LOTS

The first result and some others in this section are improvements upon results of the second author [15].

THEOREM 3.1. *If X is a LOTS and $\chi(g, X^+) \leq \alpha$ for every $g \in X^+ - X$, then $L(X \times Y) \leq 2^\alpha$ for every Lindelöf space Y .*

PROOF. Let C be an open cover of $X \times Y$. Define a relation R on X as follows: For $x, y \in X$, xRy if and only if there are points $a, b \in X$ such that $x, y \in [a, b]$ and $[a, b] \times Y$ can be covered by a subfamily of C of cardinality less than or equal to α . We immediately see that R is an equivalence relation and we observe that Rx , the equivalence class determined by x , is an interval for each $x \in X$. Also it can be shown that if $\sup(Rx) \in X$, then $\sup(Rx) \in Rx$ and $\sup(Rx) = \sup X$ and similarly for the infimum. Hence each Rx is clopen.

We show that for each $x \in X$, $Rx \times Y$ can be covered by a subfamily of C of cardinality less than or equal to α . Consider the case where $\sup(Rx)$ and $\inf(Rx) \in X^+ - X$. The other cases follow from slight modifications to the following argument. Since $\chi(g, X^+) \leq \alpha$ for each $g \in X^+ - X$, there are ordinals γ and δ and sequences $\{x_\beta: \beta < \gamma\}$ and $\{y_\beta: \beta < \delta\}$ such that

- (a) $\sup\{|\gamma|, |\delta|\} \leq \alpha$,
- (b) $x_0 = y_0$,
- (c) $\{x_\beta: \beta < \gamma\}$ is strictly decreasing and cointial with Rx , and
- (d) $\{y_\beta: \beta < \delta\}$ is strictly increasing and cofinal with Rx .

For each $\beta < \gamma$, there is $C_\beta \subseteq C$ such that $|C_\beta| \leq \alpha$ and C_β covers $[x_{\beta+1}, x_\beta] \times Y$. And for each $\beta < \delta$, there is $C^\beta \subseteq C$ such that $|C^\beta| \leq \alpha$ and C^β covers

$[y_\beta, y_{\beta+1}] \times Y$. Then $(\bigcup_{\beta < \gamma} C_\beta) \cup (\bigcup_{\beta < \delta} C_\beta^\beta) \subseteq C$ has cardinality less than or equal to α and covers $Rx \times Y$.

The quotient space X/R has a natural order:

$$Rx < Ry \text{ if and only if } x < y \text{ and } Rx \cap Ry = \emptyset.$$

Moreover the order topology agrees with the quotient topology.

Since $\chi(g, X^+) \leq \alpha$ for each $g \in X^+ - X$, we have $\chi(c, (X/R)^+) \leq \alpha$ for each $c \in (X/R)^+$. So by the theorem of Arhangel'skii^Y that $|Y| \leq 2^{L(Y) \cdot \chi(Y)}$ for each Hausdorff space Y , [6], we have $|X/R| \leq 2^\alpha$.

Thus there are no more than 2^α equivalence classes of X each of which has the property that its product with Y can be covered by a subfamily of C of cardinality less than or equal to α . Hence $L(X \times Y) \leq 2^\alpha$. \square

COROLLARY 3.2. *If X is a LOTS and $\chi(g, X^+) \leq \alpha$ for every $g \in X^+ - X$, then $L(X) \leq 2^\alpha$. \square*

Juhász and Hajnal have shown that the product of two Lindelöf spaces need not have Lindelöf degree less than or equal to 2^ω . However it follows from Theorem 3.1 that if one of the factor spaces is a LOTS, then the Lindelöf degree of the product is controlled.

COROLLARY 3.3. *If X is a LOTS and $L(X) \leq \alpha$, then $L(X \times Y) \leq 2^\alpha$ for every Lindelöf space Y . \square*

In an attempt to improve upon the results of TELGÁRSKY [11] concerning C -scattered spaces, we defined the following relation based upon the scattered-like decomposition of a space.

DEFINITION 3.4. Suppose X is a LOTS. Define a relation R on X as follows: For $x, y \in X$, xRy if and only if there are points $a, b \in X$ such that $x, y \in [a, b]$ and $|\eta[a, b]| = 0$.

This relation is used in the remainder of this paper. The second author has given examples which show that there is no relationship between $\eta(X) \leq \omega$ and X being the countable union of C -scattered spaces.

We observe that the following are true for any LOTS X :

- (i) R is an equivalence relation;
- (ii) Rx is a closed convex set in X for each $x \in X$;
- (iii) $|\eta Rx| = 0$ for each $x \in X$; and

- (iv) If $\inf(Rx) \in X$, then either $\inf(Rx) \in \eta X$ or $\inf(Rx) = \inf X$; and similarly for the supremum.

Furthermore, we define an order on X/R in the natural way and denote the set X/R with the order topology by $(X/R, \leq)$. The following observations are made:

- (v) $R(\eta X)$ is dense in X/R ;
 (vi) the quotient topology is finer than the order topology;
 (vii) neither X/R nor $R(\eta X)$ contains adjacent points.

We can now show the following theorems involving this equivalence relation.

THEOREM 3.5. *Let X be a Lindelöf LOTS and let Y be a Lindelöf space. Then $X \times Y$ is Lindelöf if and only if $X/R \times Y$ is Lindelöf.*

PROOF. Suppose $X \times Y$ is Lindelöf. $X/R \times Y$ can be viewed as a closed continuous image of $X \times Y$ and thus it is Lindelöf.

Conversely suppose $X/R \times Y$ is Lindelöf. Let C be an open cover of $X \times Y$ where, without loss of generality, members of C are of the form $I \times J$ with I open and convex in X and J open in Y . We find a countable open refinement of C covering $X \times Y$.

We consider the case where $\inf(Rx)$ and $\sup(Rx) \in \eta X$ when $x \in \eta X$. Moreover we assume $\inf X$ and $\sup X \notin X$. Slight modifications of the proof for the other cases can easily be made.

We wish to define an open cover \mathcal{B} of $X/R \times Y$.

If $Rx \notin R(\eta X)$, let

$$\mathcal{B}(Rx) = \{R(I) \times J : x \in I \text{ and } I \times J \in C\}.$$

If $Rx \in R(\eta X)$, we define $\mathcal{B}(Rx)$ in the following way: for each $K = (I_1 \times J_1, I_2 \times J_2) \in C \times C$ where $\inf(Rx) \in I_1$ and $\sup(Rx) \in I_2$, we choose an open convex set C_K in X such that

- (a) $\inf I_1 \leq \inf C_K < \inf(Rx) \leq \sup(Rx) < \sup C_K \leq \sup I_2$; and
 (b) $\inf C_K, \sup C_K \in X^+ - X$ and $\inf C_K, \sup C_K$ do not belong to the X^+ interior of any Ry for $y \in X$.

Then we let $\mathcal{B}(Rx) = \{R(C_K) \times (J_1 \cap J_2) : K \text{ as above}\}$.

Let $\mathcal{B} = \cup\{\mathcal{B}(Rx) : Rx \in X/R\}$.

$X/R \times Y$ is assumed to be Lindelöf, so there is a countable subcover \mathcal{U} of \mathcal{B} . For each $U \in \mathcal{U}$, let $U^* = \{(x,y) \in X \times Y : (Rx,y) \in U\}$. Let $\mathcal{U}^* = \{U^* : U \in \mathcal{U}\}$. Then \mathcal{U}^* is a countable open cover of $X \times Y$.

For each $U \in \mathcal{U}$, we choose, when possible, $K(U) = (I_1 \times J_1, I_2 \times J_2)$ such that $U \subseteq (R(C_{K(U)}) \times (J_1 \cap J_2)) \in \mathcal{B}(Rx)$ and $Rx \in R(\eta X)$. We choose, if possible, $x_1 \in I_1 \cap]\inf(Rx), \sup(Rx)[$; otherwise we let $x_1 = \inf(Rx)$. Similarly, choose, if possible, $x_2 \in I_2 \cap]\inf(Rx), \sup(Rx)[$; otherwise let $x_2 = \sup(Rx)$.

We let

$$W_1(U) = \begin{cases} C_{K(U)} \cap]\inf X, x_1[& \text{if } x_1 \neq \inf(Rx), \\ C_{K(U)} \cap]\inf X, x_1] & \text{if } x_1 = \inf(Rx), \end{cases}$$

and we let

$$W_2(U) = \begin{cases} C_{K(U)} \cap]x_2, \sup X[& \text{if } x_2 \neq \sup(Rx), \\ C_{K(U)} \cap [x_2, \sup X[& \text{if } x_2 = \sup(Rx). \end{cases}$$

For each such U and Rx , there is a countable open refinement $\mathcal{W}(U, Rx)$ of \mathcal{C} whose union is $U^* \cap (]\inf(Rx), \sup(Rx)[\times Y)$.

Let $A_1 = \{U^* \cap (W_i(U) \times Y) : U \in \mathcal{U}, i = 1, 2\}$.

Let $A_2 = \mathcal{U} \setminus \{\mathcal{W}(U, Rx) : U^* \cap (]\inf(Rx), \sup(Rx)[\times Y) \text{ is not covered by } A_1\}$.

Now for each $Rx \notin R(\eta X)$ and each $U \in \mathcal{U}$ such that $U \subseteq (R(I) \times J) \in \mathcal{B}(Rx)$, let $\mathcal{V}(U, Rx)$ be a countable open refinement of \mathcal{C} whose union is $U^* \cap (Rx \times Y)$.

Let $A_3 = \mathcal{U} \setminus \{\mathcal{V}(U, Rx) : A \cap U^* \cap (Rx \times Y) = \emptyset \text{ for every } A \in A_1 \cup A_2\}$.

Then we claim that $A_1 \cup A_2 \cup A_3$ is a countable open refinement of \mathcal{C} .

Clearly $A_1 \cup A_2 \cup A_3$ is an open cover of $X \times Y$.

We first show that $A_1 \cup A_2 \cup A_3$ is a refinement of \mathcal{C} . By definition, $A_2 \cup A_3$ refines \mathcal{C} ; so suppose $A \in A_1$. Then there is $U \in \mathcal{U}$ and $i \in \{1, 2\}$ such that $A = U^* \cap (W_i(U) \times Y)$. Without loss of generality, we will assume $i = 1$. Consider $K(U) = (I_1 \times J_1, I_2 \times J_2)$. Let $(x,y) \in A$. Then $(x,y) \in U^*$ implies $(Rx,y) \in U$ and $(x,y) \in W_1(U) \times Y$ implies either $x \in W_1(U) = C_{K(U)} \cap]\inf X, x_1[$ or $x \in C_{K(U)} \cap]\inf X, x_1]$. In either case, $\inf I_1 \leq \inf C_{K(U)} < x \leq x_1 \in I_1$, so $x \in I_1$. And $(Rx,y) \in U \subseteq (R(C_{K(U)}) \times (J_1 \cap J_2))$ implies $y \in J_1$. Hence $(x,y) \in I_1 \times J_1$ and $A \subseteq I_1 \times J_1 \in \mathcal{C}$. In the same way, if $A = U^* \cap (W_2(U) \times Y)$, then $A \subseteq I_2 \times J_2 \in \mathcal{C}$.

It follows from the fact that U^* is countable, and from the way we have defined A_1 , A_2 and A_3 , that $A_1 \cup A_2 \cup A_3$ is countable. \square

A similar proof to this yields:

THEOREM 3.6. *Let X be a paracompact LOTS and let Y be a paracompact space. Then $X \times Y$ is paracompact if and only if $X/R \times Y$ is paracompact.*

The following Michael-inspired results were first announced in 1975 [14].

THEOREM 3.7. *Suppose X is a Lindelöf LOTS and $|\eta X| \leq \omega$. Then $X \times Y$ is Lindelöf for every Lindelöf space Y if and only if $X \times S$ is normal for every $S \subseteq \mathbb{R}$.*

PROOF. Suppose there is a Lindelöf space Y such that $X \times Y$ is not Lindelöf. Then by Theorem 3.5, $X/R \times Y$ is not Lindelöf. Thus $|\eta(X/R)| = \omega$. $R(\eta X)$ is not a G_δ -set in X/R ; for otherwise $X/R - R(\eta X)$ would be an F_σ and hence it would be Lindelöf. Also $(X/R, \leq) - R(\eta X)$ cannot contain adjacent points in its restricted order. So by Lemmas 2.1 and 2.2, we may assume, without loss of generality, that $(X/R, \leq)$ is a subspace of \mathbb{R} and $R(\eta X)$ is the set of rationals in $(X/R, \leq)$.

Let $(X/R, \leq) - R(\eta X)$ have the subspace topology in $(X/R, \leq)$. It follows from the techniques of MICHAEL [7], that $X/R \times (X/R, \leq) - R(\eta X)$ is not normal because the sets

$$A = R(\eta X) \times (X/R, \leq) - R(\eta X)$$

and

$$B = \{(Rx, Rx) : Rx \in X/R - R(\eta X)\}$$

are closed disjoint sets which cannot be separated.

Since $X/R \times (X/R, \leq) - R(\eta X)$ is not normal, then $X \times (X/R, \leq) - R(\eta X)$ is also not normal.

The other implication is easily seen to be true. \square

Again we have the paracompact version of this.

THEOREM 3.8. *Suppose X is paracompact LOTS and $|\eta X| \leq \omega$. Then $X \times Y$ is paracompact for every paracompact space Y if and only if $X \times S$ is normal for every $S \subseteq \mathbb{R}$.*

In 1947, SORGENFREY [10] showed that the product of two Lindelöf LOTS need not be normal. PRZYMUSIŃSKI [9] showed, in 1973, that the product of two Lindelöf GO-spaces need not be collectionwise normal even while being normal, assuming the existence of a Q-set, a consequence of Martin's Axiom. In the next theorem we show conditions under which these properties are preserved in products of LOTS.

THEOREM 3.9. *Let X and Y be paracompact LOTS such that $d(\eta X) + d(\eta Y) = \omega$ (so $|nX| + |nY| \leq 2^\omega$). Then the following are equivalent.*

- (i) $X \times Y$ is paracompact;
- (ii) $X \times Y$ is collectionwise normal;
- (iii) $X/R \times Y/R$ is Lindelöf.

Furthermore, if $2^\omega < 2^{\omega_1}$, then

- (iv) $X \times Y$ is normal,

is equivalent to the above statements.

PROOF. (i) implies (ii) implies (iv) are well-known.

(iii) implies (i): Suppose $X/R \times Y/R$ is Lindelöf. Then $X/R \times Y/R$ is paracompact. By Theorem 3.6, $X \times Y/R$ is paracompact and $X \times Y$ is paracompact.

(ii) implies (iii): Let C be an open cover of $X/R \times Y/R$. Without loss of generality, we may assume that each member of C is of the form $I \times J$ where I and J are open convex sets in X/R and Y/R , respectively. Suppose that no countable subfamily of C covers X/R and Y/R . Since $d(\eta X) + d(\eta Y) = \omega$, X/R and Y/R are separable. Thus, by Lemma 2.2, $(X/R, \leq)$ and $(Y/R, \leq)$ may be considered as subspaces of \mathbb{R} . There is a countable subfamily $C_0 \subseteq C$ such that

$$\begin{aligned} & U\{(I - \{\inf I, \sup I\}) \times (J - \{\inf J, \sup J\}) : I \times J \in C_0\} \\ & = U\{(I - \{\inf I, \sup I\}) \times (J - \{\inf J, \sup J\}) : I \times J \in C\}. \end{aligned}$$

Let $U = \{r \in X/R : \{r\} \text{ is open}\}$ and $V = \{s \in Y/R : \{s\} \text{ is open}\}$. Let $C_1 \subseteq C$ be a countable family covering $(X/R \times V) \cup (U \times Y/R)$.

Let $B_0 = C_0 \cup C_1$. Let $\{(r_\alpha, s_\alpha) : \alpha < \beta\}$ be a well-ordering of $(X/R \times Y/R) - U\{B : B \in B_0\}$.

Let $(u_0, v_0) = (r_0, s_0)$. Suppose B_γ and (u_γ, v_γ) have been defined for all γ , $0 \leq \gamma < \omega_1$, in such a way that $B_\gamma \subseteq C$ is countable and $(u_\gamma, v_\gamma) \notin U\{B : B \in B_\delta, \delta \leq \gamma\}$. We define B_α and (u_α, v_α) in the following way. There is a countable subfamily $B_\alpha \subseteq C$ which covers $U\{(X/R \times \{v_\gamma\}) \cup (\{u_\gamma\} \times Y/R) : \gamma < \alpha\}$. We choose (u_α, v_α) to be (r_σ, s_σ) where $\sigma < \beta$ is the first ordinal such that $(r_\sigma, s_\sigma) \notin U\{B : B \in B_\gamma, \gamma \leq \alpha\}$.

For each $\alpha < \omega_1$, choose $x_\alpha \in \eta X \cap \{x \in X: xRu_\alpha\}$ and $y_\alpha \in \eta Y \cap \{y \in Y: yRv_\alpha\}$. Then $\{(x_\alpha, y_\alpha): \alpha < \omega_1\}$ is a closed discrete subset of $\eta X \times \eta Y$. Furthermore $\eta X \times \eta Y$ is separable. By a theorem from mathematical folklore (perhaps due to F.B. Jones), $\eta X \times \eta Y$ is not collectionwise normal. Thus $X \times Y$ is not collectionwise normal.

To show (iv) implies (iii), we assume $2^\omega < 2^{\omega_1}$. Now suppose $X/R \times Y/R$ is not Lindelöf. From the proof of (ii) implies (iii), we saw that $\eta X \times \eta Y$ has a closed discrete subspace of cardinality ω_1 . Additionally, $\eta X \times \eta Y$ is separable. By JONES' well-known result [5], $\eta X \times \eta Y$ is not normal. Hence $X \times Y$ is not normal. \square

COROLLARY 3.10. *Suppose X and Y are Lindelöf LOTS such that $d(\eta X) + D(\eta Y) = \omega$. Then $X \times Y$ is Lindelöf if and only if $X \times Y$ is paracompact.*

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CONTINUOUS IMAGES OF THE LEXICOGRAPHIC DOUBLE INTERVAL
AND THE PROBLEM OF PROJECTIVE SETS IN GENERAL SPACES

by

A.J. Ostaszewski

1. INTRODUCTION

Let $L = [0,1] \times [0,1]$ be ordered lexicographically so that $\langle x,i \rangle < \langle y,j \rangle$ provided either $x < y$ or $x = y$ and $i < j$. Let $P = [0,1] \times \{0,1\}$. We shall consider L and P as topological spaces, the topology being derived from the lexicographic order. It is known [1] that L is compact and that the closed subspace P is hereditarily Lindelöf. SKULA [8] has shown that the Souslin- \mathcal{F} subsets S of P have the property that with at most countably many exceptions x the *twin*, viz. $\langle x,1-i \rangle$, of an element $\langle x,i \rangle$ of S is also in S . Thus $[0,1] \times \{0\}$ is not Souslin- \mathcal{F} in P . (For definitions of Souslin sets see [6].) Skula reports Kurepa to have asked whether $[0,1] \times \{0\}$ is a projective set in P . We show that the answer is negative, as expected, but only after addressing the implied question of how to define in a general topological space a projective hierarchy analogous to that in metric spaces. Compare [3]. We shall consider three natural definitions, which turn out to be equivalent for P . We employ the techniques of [5] where we had obtained Skula's result independently by an alternative argument which moreover made possible the characterization of analytic and descriptive Borel sets of the lexicographic square L . We recall, for present purposes, that a set A in a Hausdorff space X is said to be *analytic* provided there is a compact-valued mapping K with domain the Baire space $I = \mathbb{N}^{\mathbb{N}}$ (with product topology) such that

$$A = K[I] \equiv \bigcup_{\sigma \in I} K(\sigma)$$

where K is *upper semicontinuous* in the sense that if for some $\sigma \in I$ and some open G we have $K(\sigma) \subseteq G$ then there exists an integer n so that for all τ in $B(\sigma|n) = \{\tau \in I : (\forall i \leq n), \tau(i) = \sigma(i)\}$ we also have $K(\tau) \subseteq G$. If, moreover, $K(\sigma) \cap K(\tau) = \emptyset$ for $\sigma \neq \tau$ we say that A is *descriptive-Borel* (or in the newer terminology of [6] *K-Lusin*).

2. MAIN RESULT

Our analysis of projectivity centers around one theorem and its corollaries. We need one definition.

DEFINITION. We say that a set $A \subseteq P = [0,1] \times \{0,1\}$ is *almost twinned* if the set of "exceptional points of A", namely

$$E(A) = \{x \in [0,1]: (\exists i)\langle x,i \rangle \in A \text{ and } \langle x,1-i \rangle \notin A\},$$

is at most countable; A is *twinned* if $E(A)$ is empty.

THEOREM. *The continuous image of an almost twinned set is itself almost twinned.*

PROOF. Let us agree to denote projection from P onto $[0,1]$ by π and the transposition taking $\langle x,i \rangle$ to $\langle x,1-i \rangle$ by T. Now let A be an almost twinned subset of P and let $f: A \rightarrow P$ be continuous. Clearly, for the purposes of the theorem, we may assume that A is twinned. Put

$$A_n = \{a \in A: |\pi f(a) - \pi f(Ta)| \geq 1/n\}.$$

We claim that A_n is countable. Choose for each a in A_n an open set U_a in $[0,1]$ of diameter less than $1/n$ containing $\pi f(a)$. Then since $f(Ta) \notin U_a \times \{0,1\}$, there is by the continuity of f a half-open interval V_a in $[0,1]$ with πa as the included endpoint such that

$$f[V_a \times \{0,1\} \setminus \{Ta\}] \subseteq U_a \times \{0,1\}.$$

Thus $b \notin A_n$. Now it suffices to invoke the fact that P is hereditarily Lindelöf and our claim is established.

Let

$$(1) \quad A^* = A \setminus \bigcup_{n=1}^{\infty} A_n$$

and consider $b \in E(f[A^*])$. Suppose for example that $b = \langle \beta, 0 \rangle$ and choose $a \in A^*$ with $b = f(a)$. Since $\pi f(a) = \pi f(Ta)$ we have

$$f(a) = \langle \beta, 0 \rangle = f(Ta).$$

Now f is continuous at both a and Ta hence there exists an open interval I_a of $[0,1]$ containing πa such that

$$f[I_a \times \{0,1\}] \subseteq \{\langle x,i \rangle : \langle x,i \rangle < \langle \beta,1 \rangle\}.$$

Then

$$(\sup \pi f[I_a \times \{0,1\}]) = \beta.$$

Clearly we may suppose I_a has rational endpoints. Thus the set of exceptional points b , being determined by the countable family of rational intervals, is itself countable. Thus $E(f[A^*])$ and hence $E(f[A])$ are countable, as required.

The above proof owes much to Roy O. Davies who considerably shortened the author's cumbersome version.

COROLLARY 1. Let $e_0: [0,1] \rightarrow P$ be defined by $e_0(x) = \langle x, 0 \rangle$ and let π be the projection from P to $[0,1]$. If $A \subseteq P$ is almost twinned and $f: A \rightarrow P$ is continuous then there exists $A' \subseteq A$ with $A \setminus A'$ at most countable so that

$$\pi f e_0: \pi[A'] \rightarrow \pi f[A']$$

is continuous in the usual topology of $[0,1]$.

PROOF. In the notation of (1), take

$$A' = A^* \setminus \{a: f(a) = Tf(a)\},$$

and the result is clear.

It follows that a continuous function from an almost twinned set into P may arbitrarily transpose or not transpose the twin images of points (here we ignore the countable set of exceptional points).

COROLLARY 2. If $D \subseteq \mathbb{N}^{\mathbb{N}}$ and $g: D \rightarrow P$ is continuous then $g[D]$ is almost twinned.

PROOF. Let us identify $\mathbb{N}^{\mathbb{N}}$ with the set of irrationals in $[0,1]$ via continued fraction expansion. Regarding now D as a set in $[0,1]$ the function $f: D \times \{0,1\} \rightarrow P$ defined by

$$f(\langle d,i \rangle) = g(d)$$

is continuous and $D \times \{0,1\}$ is twinned. This result embraces Skula's theorem.

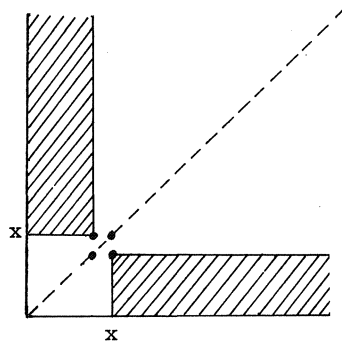
REMARK. Unfortunately Corollary 2 does not generalize to analytic sets in P^2 along expected lines. It is *not* true that with countably many exceptions if a point $\langle x,i,y,j \rangle$ lies in an analytic set then so do the other three points $\langle x,i',y,j' \rangle$ (for $i' = i$ or $1-i$ and $j' = j$ or $1-j$); for example if S and T are arbitrary sets in $[0,1]$, then the set

$$\Delta(S,T) = \{ \langle x,0,x,0 \rangle, \langle x,1,x,1 \rangle : x \in [0,1] \} \cup \{ \langle x,0,x,1 \rangle : x \in S \} \\ \cup \{ \langle x,1,x,0 \rangle : x \in T \},$$

is closed in P^2 . To see this observe first that the diagonal set $\Delta = \{ \langle x,i,x,j \rangle : i,j \in \{0,1\} \text{ and } x \in [0,1] \}$ is closed, secondly that the sets

$$\bigcup_{x \notin S} ([0,x] \times \{0,1\} \setminus \{ \langle x,1 \rangle \}) \times ([x,1] \times \{0,1\} \setminus \{ \langle x,0 \rangle \}), \\ \bigcup_{x \notin T} ([x,1] \times \{0,1\} \setminus \{ \langle x,0 \rangle \}) \times ([0,x] \times \{0,1\} \setminus \{ \langle x,1 \rangle \})$$

are open and finally that subtracting these from Δ gives $\Delta(S,T)$. The general term just displayed in the formulas is illustrated below.



Replace each point of the square by four copies as indicated to obtain P^2 .

However, one can prove the following:

PROPOSITION. *Except on countably many verticals and horizontals if $\langle x, i, y, j \rangle$ lies in an analytic set $A \subseteq P^2$ then necessarily either the pair of points*

$$\langle x, 0, y, 1 \rangle \quad \text{and} \quad \langle x, 1, y, 0 \rangle$$

or the pair of points

$$\langle x, 0, y, 0 \rangle \quad \text{and} \quad \langle x, 1, y, 1 \rangle$$

lie in the set.

SKETCH OF A PROOF. Let $\phi: I \rightarrow [0, 1]^2$ be a continuous injection. Let $A = K[I]$ where K is upper semicontinuous. Define

$$H(\sigma) = K((\sigma_1, \sigma_3, \sigma_5, \dots)) \cap \theta(\phi(\sigma_2, \sigma_4, \sigma_6, \dots) \times \{0, 1\}^2)$$

where $\theta: [0, 1]^2 \times \{0, 1\}^2 \rightarrow P^2$ takes $\langle x, y, i, j \rangle$ to $\langle x, i, y, j \rangle$. Thus H is upper semicontinuous and four-valued at most, ($|H(\sigma)| \leq 4$). Put

$$J_{00} = \{\sigma \in I: \exists x \exists y \{\langle x, 0, y, 0 \rangle, \langle x, 0, y, 1 \rangle\} \supseteq H(\sigma)\}.$$

For $\sigma \in J_{00}$ let $u_{00}(\sigma) = x$ if $\{\langle x, 0, y, 0 \rangle, \langle x, 0, y, 1 \rangle\} \supseteq H(\sigma)$. Then u_{00} is continuous on J_{00} and has a local maximum at all points of J_{00} . Hence, by [5], u_{00} has countable range. This proves the claim.

The example cited before the Proposition shows the result to be the best possible.

3. THE PROBLEM OF PROJECTIVITY

There are two approaches to defining projective sets in the metric context. There is an extrinsic form allowing complementation and projection parallel to an axis that is a complete separable metric space and there is an intrinsic form (KURATOWSKI [2]) allowing complementation and formation of continuous images by functions whose domain and range are in the space in question. In both cases a hierarchy is constructed starting with Borel sets

and closing off under the two operations. Clearly the intrinsic definition generalizes immediately and according to it all projective sets in P are almost twinned by the Main Theorem. Evidently a projective set in P takes the form

$$E \times \{0,1\},$$

modulo a countable set, where E is projective in $[0,1]$.

With regard to the extrinsic definition one should immediately rule out projections, say, from P^2 to P . For an arbitrary set S in $[0,1]$ we observe, as in the last section, that $\Delta([0,1] \setminus S, [0,1])$ is closed, whereas $P^2 \setminus \Delta$ is σ -compact. Consequently

$$\{\langle x, 0, x, 1 \rangle : x \in S\}$$

is a G_δ -set for arbitrary S and has $S \times \{0\}$ as its projection. Thus arbitrary sets would be projective.

For an extrinsic definition we should therefore choose to define $\mathbb{P}_n(X)$ inductively, for any space X , as follows. Let $\mathbb{P}_1(X)$ consist of the analytic subsets of X . If $\mathbb{P}_n(X)$ has been defined for all X , then $\mathbb{P}_{n+1}(X)$ consists of the complements in X of the sets in $\mathbb{P}_n(X)$ in case n is odd, while for even values of n , the sets of $\mathbb{P}_{n+1}(X)$ will be the projections onto X of the sets in $\mathbb{P}_n(X \times I)$, where as before $I = N^N$ is the Baire space.

A third definition also comes to mind. Call a set $H \subseteq X$ projective-u.s.c. if there is a projective set $E \subseteq I$ and an upper semicontinuous compact-valued map K defined on E such that

$$H = K[E] = \bigcup_{\sigma \in E} K(\sigma).$$

For compact spaces X one may show that this third definition is embraced by the second. This follows from a weak kind of LAVRENTIEFF Lemma [7].

EXTENSION LEMMA. *Let X be a compact Hausdorff space and let K be a compact-valued upper semicontinuous mapping defined on a subset E of N^N . Define for τ in N^N*

$$H(\tau) = \{x \in X : (\forall \text{ open } U \ni x)(\tau \in \text{cl}\{\sigma : K(\sigma) \cap U \neq \emptyset\})\}.$$

Then H is a compact-valued upper semicontinuous mapping that agrees with K on E .

PROOF. Evidently $H(\tau)$ is closed for all τ and $H(\tau) = \emptyset$ for $\tau \notin \mathcal{c}E$. To see that $K(\tau) = H(\tau)$ for $\tau \in E$, consider $x \in H(\tau) \setminus K(\tau)$. Choose U, V disjoint open with $x \in U$ and $K(\tau) \subseteq V$. Then for all n large enough if $\sigma \in B(\tau|n)$ we have $K(\sigma) \subseteq V$ so $K(\sigma) \cap U = \emptyset$. To show upper semicontinuity at an arbitrary $\tau \in \mathcal{c}E$, let G be open with $H(\tau) \subseteq G$. Choose V open with $H(\tau) \subseteq V \subseteq \mathcal{c}V \subseteq G$. Suppose there is a sequence $\langle \sigma_n, x_n \rangle$ in $E \times X$ with $\sigma_n \rightarrow \tau$ and

$$x_n \in K(\sigma_n) \setminus V.$$

Let x^* be a point of accumulation of $\{x_n : n = 1, 2, \dots\}$. Clearly, $x^* \in X \setminus V$, but if U is any open set containing x^* we have for any n the existence of an m so large that $\sigma_m \in B(\tau|n)$ and $x_m \in U$ showing $U \cap K(\sigma_m) \neq \emptyset$ i.e. $x^* \in H(\tau)$. So, after all, there exists N so large that for σ in $E \cap B(\tau|N)$

$$K(\sigma) \subseteq V.$$

Hence

$$H(\sigma) \subseteq \mathcal{c}V, \quad \text{for } \sigma \in B(\tau|N).$$

This completes the proof.

We may exemplify the consequences of the lemma by considering a set Y in a compact space X where $Y = K[E]$ and K is upper semicontinuous defined on a set $E \subseteq I$ that is (in the metric sense) the projection of a co-analytic set, say $C \subseteq I^2$. Then we have, writing

$$H = \bigcup_{\sigma \in I} H(\sigma) \times \{\sigma\},$$

where H is obtained from K as in the Lemma, that

$$y \in Y \iff \exists \sigma \exists \tau (\langle x, \sigma, \tau \rangle \in (H \times I) \cap X \times C),$$

so

$$Y = \text{proj}_x [(H \times I) \cap (X \times C)].$$

But H is closed (by upper semicontinuity) and one needs to check (routinely by induction) that if a set Y is in $\mathbb{P}_n(X)$ whilst, say, S is in $\mathbb{P}_n(I)$, then $Y \times S$ is in $\mathbb{P}_n(X \times I)$. This uses the homeomorphism of I^2 with I .

Returning to the case when X is P it should now be clear that sets in $\mathbb{P}_n(X \times I)$ may be characterized, by an argument as in the Proposition of Section 2, as taking the form

$$(E \times \{0,1\}) \cup S,$$

with E in $\mathbb{P}_n([0,1] \times I)$ and with $S \subseteq X \times I$ such that $\text{proj } S$ is countable and, for x in $\text{proj } S$, $\{x\} \times I \cap S$ is in $\mathbb{P}_n(I)$. Combining this with the Main Theorem, Corollary 1 and the argument above shows all three definitions to be coextensive.

REMARK. It is interesting to note that Novikoff's results on projective sets of the second class [4] (derived from an analysis of sieves) carry over to the projective sets as defined in the third u.s.c. definition.

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NEW PROOFS OF A METRIZATION THEOREM FOR ORDERED SPACES

by

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1. INTRODUCTION

In 1977, Bennett and the second author proved that a generalized ordered space X is metrizable if and only if each subspace of X is a p -space in the sense of ARHANGEL'SKII [5], [1,2]. Their proof placed emphasis on special ordered-space-constructions which tend to be quite complicated. Subsequent papers by VAN WOUWE [9] and the first author [7] gave somewhat easier proofs, but the result is still not readily available to non-specialists. The purpose of this paper is to combine the approaches in [5] and [7] to obtain a "soft" proof of the Bennett-Lutzer theorem and to show the result also follows from recent work of Z. BALOGH [3].

2. REVIEW OF KNOWN RESULTS

Originally, p -spaces were studied because of the following fundamental result.

THEOREM 2.1. [1]: *A completely regular space X can be mapped perfectly onto a metric space if and only if X is a paracompact p -space.*

That result can be sharpened if one considers only generalized ordered spaces (= GO -spaces = suborderable spaces [8], [9]).

THEOREM 2.2. [9], [7]: *If a GO -space X is a paracompact p -space, then there is a metrizable GO -space M and a perfect, monotonic mapping $g: X \rightarrow M$ (i.e., if $x_1 \leq x_2$ in X , then $g(x_1) \leq g(x_2)$ in M).*

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Numerous metrization theorems for p -spaces are known; we will use the following result of Bennett.

THEOREM 2.3. [4]: *If a paracompact p -space X has a σ -disjoint base, then X is metrizable.*

GO -spaces having σ -disjoint bases are particularly easy to work with. First, we may assume that members of the σ -disjoint base are order-convex. Second, it is easy to prove

THEOREM 2.4. [4]: *If X is a first-countable GO -space which is the union of a countable family C of subspaces each of which has σ -disjoint base for its relative topology, then X has a σ -disjoint base.*

Theorem 2.4 is particularly useful since no assumptions about members of C are made, i.e., one does not need to know that members of C are closed, open, dense, etc.

3. THE ORDERED SPACE PROOF

If every subspace of X is a p -space in its relative topology, we will say that X is *hereditarily a p -space*.

LEMMA 3.1. *If a GO -space X is hereditarily a p -space, then X is first-countable and paracompact.*

Outline of Proof. If X is not first countable, then for some cardinal κ with $\text{cf}(\kappa) > \omega$, the subspace $T = \{\alpha < \kappa \mid \alpha \text{ is not a limit ordinal}\} \cup \{\kappa\}$ of $[0, \kappa]$ embeds in X . But T cannot be a p -space: consider the compactification of T obtained by taking the closure of T in $[0, \kappa]$. And if X is not paracompact, then some stationary subset S of some uncountable regular cardinal λ embeds in X , and such an S cannot be hereditarily a p -space. Details appear in [5]. \square

Next we give a simple proof of a crucial lemma in [5].

LEMMA 3.2. *Let Z be any linearly ordered set and let Y be an infinite subset of Z . Then there are sets D and E such that*

- (a) $D \cup E = Y$ and $D \cap E = \emptyset$;
- (b) if J is a convex subset of Z such that $|J \cap Y| \geq \omega_0$, then $D \cap J \neq \emptyset \neq E \cap J$.

PROOF. We say that a pair (A, B) of subsets of Y is properly interlaced if: (1) $A \cap B = \emptyset$; and (2) given $a_1 < a_2$ in A , $B \cap]a_1, a_2[\neq \emptyset$ and given $b_1 < b_2$ in B , $A \cap]b_1, b_2[\neq \emptyset$. Since any infinite linearly ordered set contains a sequence which is strictly monotonic, any infinite linearly ordered set contains a properly interlaced pair. Hence the collection $\Psi = \{(A, B) : A \text{ and } B \text{ are properly interlaced sets in } Y\}$ is nonvoid. Partially order Ψ by $(A_1, B_1) \leq (A_2, B_2)$ iff $A_1 \subset A_2$ and $B_1 \subset B_2$. Apply Zorn's lemma to choose a maximal element (A_0, B_0) of Ψ . If some convex subset J of Z has infinite intersection with Y and if $A_0 \cap J = \emptyset$, then $|B_0 \cap J| \leq 1$ so that some convex set $I \subset J$ has infinite intersection with Y and is also disjoint from B_0 . In $I \cap Y$ choose an infinite strictly monotonic sequence $\langle y_n \rangle$, say $y_1 < y_2 < \dots$. Depending upon the relationship between the largest points of $A_0 \cap]\leftarrow, y_1[$ and $B_0 \cap]\leftarrow, y_1[$ (if such points exist), we may add the set $\{y_{2n-1} : n \geq 1\}$ to A_0 and $\{y_{2n} : n \geq 1\}$ to B_0 (or vice versa) to obtain a pair $(A'_0, B'_0) \in \Psi$ which is strictly above (A_0, B_0) in the ordering of Ψ , and that is impossible. Finally, we let $D = A_0$ and $E = Y - D$ to obtain the required sets. \square

LEMMA 3.3. *Let X be any paracompact first countable GO-space. Then there are subsets G, H of X such that*

- (a) $G \cup H = X$ and $G \cap H = \emptyset$;
- (b) G is an open metrizable subspace of X ;
- (c) H is dense in itself (i.e., each $p \in H$ is a limit point of the set $H - \{p\}$);
- (d) there are disjoint dense subsets D and E of H such that if $d_1 < d_2$ are points of D then $[d_1, d_2] \cap D$ is not compact, and if $e_1 < e_2$ are points of E , then $[e_1, e_2] \cap E$ is not compact, and $D \cup E = H$.

PROOF. Define an equivalence relation on X by the rule that $a \sim b$ iff the closed interval between a and b is metrizable. For any $a \in X$, the equivalence class of a , which we denote by $\text{cls}(a)$ is a convex F_σ -subset of X ; hence $\text{cls}(a)$ is paracompact. It follows from the Smirnov metrization theorem [6] that $\text{cls}(a)$ is metrizable and from first-countability of X that $\text{cls}(a)$ is actually closed in X . (This does not mean, however, that $\text{cls}(a)$ has end-points or that $|\text{cls}(a)| > 1$.)

Let $G = \cup \{\text{Int}_X(\text{cls}(a)) : a \in X\}$. Then G is an open metrizable subspace of X . Let $H = X - G$. If some point $p \in H$ were isolated in H , then for some open convex set J in X , $J \cap H = \{p\}$. Then $J - \{p\} \subset G$, so $J - \{p\}$ is metrizable. But X is first countable at p , so J is also metrizable, whence $p \in J \subset G$, contrary to $J \cap H \neq \emptyset$. Therefore the space H is dense in itself.

Next observe that if $p < q$ are points of H such that $[p, q] \cap H$ is finite, then $[p, q] \cap H = \{p, q\}$ and $\text{cls}(p) = [p, q] = \text{cls}(q)$. Therefore the sets

$$N_1 = \{p \in H : \text{for some } q \in H \text{ with } q > p, |[p, q] \cap H| = 2\},$$

$$N_2 = \{q \in H : \text{for some } p \in H \text{ with } p < q, |[p, q] \cap H| = 2\},$$

are disjoint. Further if $p \in N_1$ then for every $x \in]\leftarrow, p[$, the set $]x, p[\cap H$ is infinite and an analogous assertion holds for each $q \in N_2$.

Now apply Lemma 3.2 with $Z = H$ and $Y = H - (N_1 \cup N_2)$ to find disjoint sets D' and E' whose union is Y and which have the property that whenever a convex subset J of X has the property that $J \cap Y$ is infinite, then $J \cap D' \neq \emptyset \neq J \cap E'$. Let $D = D' \cup N_1$ and $E = E' \cup N_2$. Then $D \cap E = \emptyset$.

Suppose $d_1 < d_2$ belong to D . If $|[d_1, d_2] \cap H| < \omega_0$, then $|[d_1, d_2] \cap H| = 2$ so that $d_2 \in N_2 \subset E$ contrary to $d_2 \in D$. Therefore $]d_1, d_2[\cap H$ is infinite, so we may choose $e \in E \cap]d_1, d_2[$. Since e cannot be an isolated point of H , we may assume that each neighbourhood of e in H contains an infinite set of the form $[e, x[\cap H$ where $x > e$. Then each neighbourhood of e meets D so that e is a limit point of $[d_1, d_2] \cap D$ which is not in D , showing that $[d_1, d_2] \cap D$ is not compact. The analogous assertion about E is proved similarly. \square

COROLLARY 3.4. *Suppose X is a GO-space which is hereditarily a p -space. Let H , D and E be the subsets constructed in Lemma 3.3. Then both D and E are metrizable.*

PROOF. We begin by remarking that Lemma 3.1 allows us to carry out the construction in Lemma 3.3. By hypothesis, D is a paracompact p -space; according to Theorem 2.4, there is a monotonic perfect mapping g from D onto some metrizable space M . If $d_1 < d_2$ and $g(d_1) = g(d_2)$ then $[d_1, d_2] \cap D$ would be a closed subset of the compact set $g^{-1}[g(d_1)]$ which is impossible in the light of Lemma 3.3(e). Hence g is 1-1 and therefore a homeomorphism. Hence D is metrizable. Similarly, E is metrizable. \square

THEOREM 3.5. *If a generalized ordered space X is hereditarily a p -space, then X is metrizable.*

PROOF. Let G , H , D and E be the sets found in (3.3). Then G , D and E are metrizable by (3.3(b)) and (3.4) so each has a σ -disjoint base for its relative topology. According to (2.5), so does $X = G \cup D \cup E$. According to (3.1),

X is paracompact so that (2.2) applies to make X metrizable. \square

4. A SECOND PROOF, USING BALOGH'S THEOREM

Z. BALOGH [3] has obtained a general structure theorem for completely regular spaces whose every subspace is a paracompact p -space, namely

THEOREM 4.1. *Suppose every subspace of X is a paracompact p -space. Then either*

- (a) X is metrizable; or
- (b) X contains the one-point compactification of an uncountable discrete space; or
- (c) X contains the Alexandroff double (cf. [6] or [3]) $A(M)$ of a metric space M such that M is not σ -discrete and yet each subset of M is an F_σ -set.

To deduce Theorem 3.5 from Balogh's result, we first prove that if a GO -space X is hereditarily a p -space then X is hereditarily paracompact and first-countable (cf. 3.1). Obviously, then, X cannot contain a one-point compactification of an uncountable discrete space. We claim that X cannot contain $A(M)$, the Alexandroff double of a metric space M as described in 4.1(c). Obviously, such an $A(M)$ is the union of two metrizable subspaces so that, if $A(M)$ were embedded in a GO -space, then $A(M)$ would have a σ -disjoint base (cf. (2.4)). From (2.3), it would follow that $A(M)$ is metrizable and that it is impossible because M is not σ -discrete (cf. [3]).

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PRODUCTS OF ORDERED SPACES AND TRANSITIVITY

by

Jacob Kofner

Recently R. Fox has solved a long standing γ -space problem by exhibiting a γ -space which is not a quasi-metrizable [4]. This was done by first discovering that for each integer $n \geq 0$ there are quasi-metrizable spaces which are not n -(pre)transitive. Whether such spaces exist had been a question posed by P. FLETCHER and W.F. LINDGREN [3]. Nevertheless, it was quite surprising that such spaces are rather usual. In fact, a modification of Fox's construction yielded that (the Michael line) ^{$n+1$} is not n -(pre)transitive [5]. The Michael line is a nice quasi-metrizable suborderable space, obtained from the reals, retopologized by making all irrationals isolated.

It is known that each suborderable space is a 3-transitive [6]. We show here that each quasi-metric suborderable space is 2-transitive and that any finite power of a quasi-metric GO-space with a σ -discrete dense set is 2-transitive. We show further that the n^{th} power of any quasi-metric suborderable space, the non-isolated points of which have a σ -discrete dense set, is $(n+1)$ -transitive.

1. Remember that a binary relation $V \subset X \times X$ is a *neighbournet* on X , provided that each $V\{x\} = \{y \in X: (y,x) \in V\}$ is a neighbourhood of x in X . A decreasing sequence of neighbournets $\langle V_i \rangle$ is *basic* provided that for each $x \in X$, $\langle V_i\{x\} \rangle$ forms a neighbourhood base for x in X . The n^{th} power V^n means the composition $V \circ V \circ \dots \circ V$ of n copies of V , that is $V^n\{x\} = V(V(\dots V\{x\})\dots)$ (n times), $V^0\{x\}$ means $\{x\}$, and V^+ (for any binary relation) means $\bigcap \{V \circ U \mid U \text{ is a neighbournet in } X\}$, or, equivalently, $V^+\{x\} = \bigcap \{V(G) \mid G \text{ is a neighbourhood of } x\}$. We denote $(V^n)^+$ by V^{n+} . Obviously $V^{n+} \subset V^{n+1}$.

A binary relation V is transitive if $y \in V\{x\}$ implies $V\{y\} \subset V\{x\}$. A space X has a basic sequence of transitive neighbournets iff it is non-archimedian quasi-metrizable [7].

DEFINITION 1. Let n^* mean either n or n^+ . A space X is called n^* -transitive if for each neighbourhood V on X there is a transitive neighbourhood $W \subset V^{n^*}$. [2].

2. **PROPOSITION 2.** A space X is 1^+ -transitive if it has a basic sequence of transitive neighbourhoods $\langle U_i \rangle$ such that for each $x \in X$ and each U_i , every sequence of points $x_k \rightarrow x$ has a subsequence y_k with either $U_i\{y_k\} \subset U_i\{y_{k+1}\}$ or with the sequence of sets $U_i\{y_k\} \rightarrow x$ as $k \rightarrow \infty$.

PROOF. Let V be a neighbourhood on X . We shall show that there is a transitive neighbourhood $W \subset V^+$. We assume that each $V\{x\} = U_{i(x)}\{x\}$ for some $i(x)$. Set $W\{x\} = V^+\{x\} - \{y \mid V^+\{y\} \not\subset V^+\{x\}\}$. It follows that W is a transitive reflexive relation on X . It remains to show that each $W\{x\}$ is a neighbourhood of x . If $W\{x\}$ is not a neighbourhood of x , there is a sequence $x_k \rightarrow x$ such that $x_k \in V\{x\} - W\{x\}$ for each k . By definition of $W\{x\}$, $V^+\{x_k\} \not\subset V^+\{x\}$ for each k . By definition of V^+ and first countability of X , we can assume moreover that $V\{x_k\} \not\subset V^+\{x\}$, replacing, if necessary, points x_k by nearby points. Since $U_{i(x)}$ is transitive and k_i are decreasing, $i(x_k) < i(x)$, for otherwise $V\{x_k\} = U_{i(x_k)}\{x_k\} \subset U_{i(x)}\{x_k\} \subset U_{i(x)}\{x\} = V\{x\}$. By choosing a subsequence, if necessary, we can assume that all $i(x_k) = i_0$. Then by the condition of the proposition, there exists a subsequence y_k of x_k such that either $U_{i_0}\{y_k\} \rightarrow x$ as $k \rightarrow \infty$, or $U_{i_0}\{y_k\} \subset U_{i_0}\{y_{k+1}\}$. The former is not possible, since $U_{i_0}\{y_k\} = V\{y_k\} \not\subset V^+\{x\}$, while the latter would imply that $U_{i_0}\{y_k\} = V\{y_k\} \subseteq U_{i_0}\{y_k\} = V\{y_k\}$ for each y_k sufficiently close to x , and thus $V\{y_k\} \subset V^+\{x\}$.

Hence $W\{x\}$ is a neighbourhood of x . \square

THEOREM 3. Each quasi-metric GO-space is 1^+ -transitive.

PROOF. Every quasi-metric GO-space X is non-archimedean quasi-metrizable [6], hence it has a basic sequence of transitive neighbourhoods $\langle U_i \rangle$. We assume that each $U_i\{x\}$ is convex, for otherwise we replace $U_i\{x\}$ by its convex component and still have a transitive neighbourhood. Let us show that such U_i satisfy the conditions of Proposition 2. Let $x_k \rightarrow x$. Suppose also that x_k is, say, increasing. If for a subsequence $y_k, y_k \in U_i\{x\}$ but $U_i\{y_k\} \not\subset U_i\{y_{k+1}\}$ then by transitivity also $y_k \notin U_i\{y_{k+1}\}$ since otherwise $y_k \in U_i\{x\} \subset U_i\{y_{k+1}\}$. It follows that $y_k < U_i\{y_{k+1}\} < x$, hence $U_i\{y_{k+1}\} \rightarrow x$. \square

3. PROPOSITION 4. Each finite power X^n of space X is 1^+ -transitive if X has a basic sequence of transitive neighbourhoods $\langle U_i \rangle$ such that for each $x \in X$ and each U_i every sequence of points $x_k \rightarrow x$ has a subsequence y_k with $U_i\{y_k\} \subset U_i\{y_{k+1}\}$.

PROOF. Apply Proposition 2 to X^n and neighbourhoods $U_i^n\{\langle x_1, \dots, x_n \rangle\} = U_i\{x_1\} \times \dots \times U_i\{x_n\}$. \square

THEOREM 5. Any finite power of a quasi-metric suborderable space with a σ -discrete dense set is 1^+ -transitive.

PROOF. Let X be a quasi-metric space with a dense set $D = \bigcup_{i=1}^{\infty} D_i$, $D_1 \subset D_2 \subset \dots$ are discrete. Let $\langle O_i \rangle$ be a basic sequence of transitive neighbourhoods on X such that each $O_i\{x\}$ is convex (see Proof of Theorem 3). We define another sequence $\langle U_i \rangle$ which satisfies Proposition 4.

First pick a complete ordered set $X^* \supset X$ and for each $x \in X$ set $a_i(x) = \sup(\{d < x \mid d \in D_i\} \cup \{r \leq x \mid r \in X, O_i\{r\} \subset [r, \rightarrow[\cap D_i \neq \emptyset\})$ and similarly $b_i(x) = \inf(\{d > x \mid d \in D_i\} \cup \{\ell \geq x \mid \ell \in X, O_i\{\ell\} \subset]\leftarrow, \ell\}, O_i\{\ell\} \cap D_i \neq \emptyset)$. Obviously, $a_i(x) \leq x \leq b_i(x)$. Set $U_i\{x\} = \{x\} \cup]a_i(x), b_i(x)[\cap X$. Let us show that $U_i\{x\}$ is a neighbourhood of x . Indeed, if for example $x \in d\ell\leftarrow, x[$, then $a_i(x) < x$, for otherwise there is a strictly increasing sequence of points $r_k \rightarrow x$, $r_k \in O_i\{x\}$, with $O_i\{r_k\} \subset [r_k, \rightarrow[$ and some $d_k \in O_i\{r_k\} \cap D_i$. Since D_i is discrete, hence $d_k \neq x$, we can assume that all $d_k \geq x$, hence $x \in O_i\{r_2\}$, and by transitivity of O_i , $O_i\{x\} \subset O_i\{r_2\}$. Since $r_1 \in O_i\{x\}$, it follows that $r_1 \in O_i\{r_2\}$, hence $r_1 \in [r_2, \rightarrow[$, while r_k is strictly increasing - a contradiction. We have shown that the sets U_i are neighbourhoods, and it immediately follows that the U_i are transitive. The sequence $\langle U_i \rangle$ is basic since D is a dense set. It remains to show that $\langle U_i \rangle$ satisfies the condition of Proposition 4. Indeed, let $y_k \rightarrow x$; we may assume that y_k is strictly increasing, and $a_i(x) < y_k$. Then $a_i(y_k) = a_i(y_{k+1})$. Since always $b_i(y_k) \leq b_i(y_{k+1})$ it follows that $]a_i(y_k), b_i(y_k)[\cap X \subset]a_i(y_{k+1}), b_i(y_{k+1})[\cap X \subset V\{y_{k+1}\}$, hence for all $k \geq 2$, $V\{y_k\} \subset V\{y_{k+1}\}$. \square

The proof of Theorem 5 used some ideas of [1].

COROLLARY 6. (the Sorgenfrey line)ⁿ is 1^+ -transitive.

PROOF. The Sorgenfrey line is a separable quasi-metric suborderable space. \square

DEFINITION 7. Let $A \subset X$. A binary relation $V \subset A \times X$ is called a *relative neighbournet* on A in X provided that for each $x \in A$, $V\{x\}$ is a neighbourhood of x in X . A sequence $\langle V_i \rangle$ of relative neighbournets on A in X is *basic* provided that for each $x \in A$, $\langle V_i\{x\} \rangle$ forms a base of neighbourhoods of x in X . Let n^* stand either for n or for n^+ . The set A is called *relatively n^* -transitive* in X provided that for each relative neighbournet V on A in X there is a relative transitive neighbournet $W \subset V^{n^*}$. \square

Notice that for $A \subset Y \subset X$, if A is relatively n^* -transitive in X then A is relatively n^* -transitive in Y and if Y is open, then A is n^* -transitive in X .

The following is an immediate generalization of Proposition 4.

PROPOSITION 8. Each finite power X^n of subspace $X \subset X_0$ is relatively 1^+ -transitive in X_0^n if X has a basic sequence of relative transitive neighbournets $\langle \tilde{U}_i \rangle$ in X_0 such that for each $x \in X$ and each \tilde{U}_i , every sequence of points of X , $x_k \rightarrow x$, has a subsequence y_k with $U_i\{y_k\} \subset U_i\{y_{k+1}\}$. \square

THEOREM 9. Any finite power X^n of a subspace X with a σ -discrete dense set of a quasi-metric suborderable space X_0 is relatively 1^+ -transitive in X_0^n .

PROOF. We define a basic sequence $\langle \tilde{U}_i \rangle$ of relative transitive neighbournets on X in X_0 which satisfy Proposition 8. First consider the suborderable space X with a σ -discrete dense set. By the proof of Theorem 4 there exists a basic sequence of transitive neighbournets $\langle U_i \rangle$ on X which satisfy Proposition 4 and for which all $U_i\{x\}$, $x \in X$, are open and convex in X .

Pick now a complete ordered set $X^* \supset X_0 \supset X$ and let, for $x \in X$, $U_i\{x\} = \{x\} \cup]a_i(x), b_i(x)[\cap X$, where $a_i(x), b_i(x) \in X^*$. Let us define new points $\tilde{a}_i(x), \tilde{b}_i(x) \in X^*$ for $x \in X$ as follows. If $x = a_i(x)$ and there is an increasing sequence $a_i \rightarrow x$ in X_0 but $[a_i, x[\cap X \neq \emptyset$, let $\tilde{a}_i(x) = a_i$. Otherwise $\tilde{a}_i(x) = a_i(x)$. Define points $\tilde{b}_i(x)$ similarly. It follows that the sets \tilde{U}_i defined by $U_i\{x\} = \{x\} \cup]\tilde{a}_i(x), \tilde{b}_i(x)[\cap X_0$ for each $x \in X$ satisfy Proposition 6. \square

The following lemma generalizes some results of [3].

LEMMA 10.

- (a) If A is relatively n^* -transitive in X , and $B \subset A$ is closed in X then B is relatively n^* -transitive in X .

- (b) If $\langle A_\alpha \rangle$ is a locally finite collection of closed sets of X , and each A_α is relatively n^* -transitive in X then $A = \bigcup_\alpha A_\alpha$ is relatively n^* -transitive in X .
- (c) Let A be a closed relatively n^* -transitive set in X , and $B = X - A$ be m^* -transitive. Then X is $(n+m)^*$ -transitive.

PROOF.

- (a) Let V be a relative neighbourhood on B in X . Set $V_0\{x\} = V\{x\}$ for $x \in B$ and $V_0\{x\} = X - B$ for $x \in A - B$. Let $W_0 \subset V_0^{n^*}$ be a relative transitive neighbourhood on A in B . Then $W = W_0 \cap (B \times X) \subset V^{n^*}$.
- (b) Let V be a relative neighbourhood on A in X and $V_\alpha = V \cap (A_\alpha \times X)$. Let $W_\alpha \subset V^{n^*}$ be a relative transitive neighbourhood on A in X . Set for $x \in A$ $W\{x\} = \bigcap \{W_\alpha\{x\} \mid x \in A_\alpha\} - \bigcup \{A_\beta \mid x \notin A_\beta\}$. Then W is a relative transitive neighbourhood on A in X and $W \subset V^{n^*}$.
- (c) Let V be a neighbourhood on X , and $V_A = V \cap (A \times X)$ and $V_B = V \cap B^2$ (remember that B is open). Let $W_A \subset V_A^{n^*}$ be a relative transitive neighbourhood on A in X and $W_B \subset V_B^{m^*}$ be a transitive neighbourhood on B . Set $W = W_B \circ (W_A \cup W_B)$. Then $W \subset V^{m^*} \circ V^{n^*} \subset V^{(m+n)^*}$. \square

PROPOSITION 11. Let $Y \subset X$ and each point $x \in Z = X - Y$ is isolated in X . If Y^n is m^* -transitive in X^n then X^n is $(n \cdot m)^*$ -transitive.

PROOF. Notice first that by Lemma 10(a) and by a remark after Definition 7, Y^i is relatively m^* -transitive in X^i for each $i \leq n$. Let for $i = 0, 1, \dots, n$

$$A_i = \{ \langle x_1, \dots, x_n \rangle \mid |\{j \mid x_j \in Y\}| \leq i \}.$$

Obviously A_i is open, $A_0 = Z^n$ and $A_n = X^n$. The subspace A_0 is discrete, hence 0-transitive. Suppose that A_{i-1} is $(i-1) \cdot m^*$ -transitive and let us show that A_i is $i \cdot m^*$ -transitive. Notice that $A_i - A_{i-1}$ is a disjoint union of $\binom{n}{i}$ many subspaces homeomorphic to $Y^i \times Z^{n-i}$, and each one of these is a discrete union in A_i of $|Z^{n-i}|$ many subspaces homeomorphic to Y^i . Pick one of the last ones, say $Y^i \times \{ \langle x_{i+1}, \dots, x_n \rangle \} = B$, $x_{i+1}, \dots, x_n \in Z$. As we noticed in the beginning of the proof, B is relatively m^* -transitive in $X^i \times \{x_{i+1}, \dots, x_n\}$, hence in X^n , and in A_n (see remark after Definition 7). Since $A_i - A_{i-1}$ is a union of a discrete in A_i collection of relatively n^* -transitive closed subsets, like B , $A_i - A_{i-1}$ is also relatively n^* -transitive in A_i by Lemma 10(b). Since A_{i-1} is $(i-1) m^*$ -transitive, A_i is $(m+(i-1) \cdot m)^*$ -transitive by Lemma 10(c), hence $A_n = X$ is $n \cdot m^*$ -transitive. \square

THEOREM 12. Any n^{th} power of a quasi-metric suborderable space where the non-isolated points have a σ -discrete dense set is n^+ -transitive.

PROOF. This follows from Proposition 11 and Theorem 9. \square

THEOREM 13. Any n^{th} power of a space with countably many non-isolated points is n -transitive.

PROOF. Notice that each countable subset is relatively 1-transitive. Indeed, if V is a relative neighbourhood on $\{x_1, x_2, \dots\}$ in X , then a relative neighbourhood W such that $W\{x_n\} = \bigcap \{V\{x_i\} \mid x_n \in V\{x_i\}\} - \{x_1, x_2, \dots, x_{n-1}\}$ is transitive. The proof now follows from Proposition 9. \square

COROLLARY 14. (R. Fox). (the Michael line) n is n -transitive.

PROOF. The Michael line is a (quasi-metric suborderable) space with a countable set of non-isolated points. \square

REMARK. One cannot omit "quasi-metric" in Theorems 3, 5, 9, 12, replace " 1^+ -transitive" by "1-transitive" in Theorems 3, 5, 9, " n^+ -transitive" by " n -transitive" in Theorem 12 or " n -transitive" by " $(n-1)^+$ -transitive" in Theorem 13 for any n . Indeed, the non-quasi-metrizable Engelking-Lutzer line is not 2-transitive [7]; the (quasi-metric) Sorgenfrey line is not 1-transitive [7]; the product of the real line and the $(n-1)^{\text{th}}$ power of the (quasi-metric) Michael line is not n -transitive, while the n^{th} power of the Michael line is not $(n-1)^+$ -transitive [5,6].

The following questions are of interest in view of Theorem 12. Is the n^{th} power of each quasi-metric GO-space n^+ -transitive? Is the square 2^+ -transitive?

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LOCAL BASES AND PRODUCT PARTIAL ORDERS

by

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0. INTRODUCTION

In [2] SHELDON DAVIS defined and initiated the study of *lob-spaces*: space in which each point has a local base linearly ordered by reverse inclusion. In particular he showed that a number of results on precompactness in q -spaces [6] also hold in lob-spaces, though the two classes are incomparable. This work has since been extended considerably [3].

Though in many ways very well behaved, the class of lob-spaces fails miserably to be closed under even finite products. (Consider, for example, $(\omega_1+1) \times (\omega+1)$.) The present work, therefore, developed out of an attempt - mostly unsuccessful, as we shall see - to generalize Davis's results to a 'small' class of spaces closed under finite products and containing all lob-spaces. The attempt did, however, lead to a surprisingly nice structure theory for the spaces in question, and it is that theory which is described in Section 3 below. Section 1 is devoted to the relevant definitions and conventions; Section 2 contains the topological results, mostly concerning cardinal functions at a point; and in Section 4 the interested reader will find an assortment of examples and discussions of special cases. (Some of the material of Sections 2 and 3 have previously been announced in Peter Nyikos's recent survey, [5].)

1. DEFINITIONS AND CONVENTIONS

Our set-theoretic conventions are the usual ones: ordinals (finite as well as infinite) are von Neumann ordinals, and cardinals are initial ordinals. Infinite cardinals will be denoted by κ and λ , possibly with indices, and occasionally by μ . For any set X , $|X|$ denotes the cardinality of X ; and if κ is any cardinal, $[X]^\kappa = \{A \subseteq X: |A| = \kappa\}$. ($[X]^{<\kappa}, [X]^{<=\kappa}$, etc. are defined in the obvious way.) $\mathcal{P}(X) = \{A: A \subseteq X\}$, and $\mathcal{P}^*(X) = \mathcal{P}(X) \setminus \{\emptyset\}$. A finite

sequence, $\langle \alpha_0, \dots, \alpha_{n-1} \rangle$, of ordinals will be denoted by $\bar{\alpha}$. If $A = \{A_i : i \in I\}$ is an indexed family of sets, and $J \subseteq I$, π_J is the canonical projection map from $\prod A$ to $\prod \{A_j : j \in J\}$. (However, we write π_i for $\pi_{\{i\}}$, even if $i \in \omega$; though this conflicts with our convention that $i = \{0, \dots, i-1\}$, no confusion will arise in context.)

The symbol ' \subset ' denotes *proper* inclusion.

If X is a topological space, and $p \in X$, $\chi(p, X)$, $\psi(p, X)$, and $t(p, X)$ are, respectively, the *character*, *pseudo-character*, and *tightness* of X at p : $\chi(p, X) = \inf\{\kappa \geq \omega : \text{there is a local base at } p \text{ of cardinality } \kappa\}$, $\psi(p, X) = \inf\{\kappa \geq \omega : \text{there is a family of } \kappa \text{ open nbhds of } p \text{ whose intersection is } \{p\}\}$, and $t(p, X) = \inf\{\kappa \geq \omega : \text{whenever } A \subseteq X, \text{ and } p \in \text{c}\mathcal{L}A \setminus A, \text{ there is an } S \in [A]^{\leq \kappa} \text{ such that } p \in \text{c}\mathcal{L}S\}$. (And of course, $\chi(X) = \sup\{\chi(p, X) : p \in X\}$, and similarly for $\psi(X)$ and $t(X)$.)

All topological spaces are assumed to be T_1 .

If $\langle P, \leq \rangle$ and $\langle Q, \leq \rangle$ are partial orders, the *product partial order* on $P \times Q$ is defined by: $\langle p, q \rangle \leq \langle p', q' \rangle$ if $p \leq p'$ and $q \leq q'$. (No confusion will arise from the ambiguous use of ' \leq '.) By abuse of notation we refer to the partial order P , rather than $\langle P, \leq \rangle$. We write $f: P \preceq Q$ if $f: P \rightarrow Q$ is a bijection, and, for all $p, p' \in P$, $p \leq p'$ implies that $f(p) \leq f(p')$.

DEFINITION 1.0. A partial order, P , is a *generalized linear order* iff it is isomorphic to a finite product of linear orders.

It is easy to see that any generalized linear order, P , has a cofinal subset isomorphic to a product of regular cardinals, the cardinals being the cofinalities of the linear factors of P . And if \mathcal{B} is a local base at a point p of a space X , so is any $\mathcal{C} \subseteq \mathcal{B}$ which is cofinal in $\langle \mathcal{B}, \supseteq \rangle$. Finally, for any κ the diagonal, $\{\langle \alpha, \alpha \rangle : \alpha \in \kappa\}$, is cofinal in the product partial order $\kappa \times \kappa$, so we make the following definition.

DEFINITION 1.1. Let X be a space, $p \in X$, $\Omega = \{\kappa_i : i \in n\}$ a finite set of distinct, regular, infinite cardinals, and let $P = \prod \Omega$, a generalized linear order. A local (nbhd) base, \mathcal{B} , at p is a *weak (nbhd) Ω -glob* (= generalized linearly ordered base) at p iff $f: \langle P, \leq \rangle \preceq \langle \mathcal{B}, \supseteq \rangle$ for some $f: P \leftrightarrow \mathcal{B}$. By convention we write in that case $B(\bar{\alpha})$ for $f(\bar{\alpha})$. (The distinction between a base and a nbhd base is that for former consists of *open* nbhds only.) Let \mathcal{B} be a weak (nbhd) Ω -glob at p . For each $i \in n$ and $\bar{\alpha} \in P$ we define $E_{\{i\}}^{\mathcal{B}}(\bar{\alpha}) = \bigcap \{B(\bar{\beta}) : \beta_j = \alpha_j \text{ for all } j \in n \setminus \{i\}\}$; and for $I \subseteq n$, $E_I^{\mathcal{B}}(\bar{\alpha}) = \bigcup \{E_{\{i\}}^{\mathcal{B}}(\bar{\alpha}) : i \in I\}$. (We suppress the superscript \mathcal{B} whenever possible.) \mathcal{B} is *strict* iff

$p \notin \text{int } E_{\{i\}}(\bar{\alpha})$ for all $i \in n$ and $\bar{\alpha} \in P$. Finally, \mathcal{B} is a (nbhd) Ω -glob at p iff $\langle \mathcal{B}, \sup \rangle$ is isomorphic to P . (\mathcal{B} is then automatically strict.)

DEFINITION 1.2. A space X , is *globular* iff each point of X has a local base which is a nbhd glob.

Clearly each lob-space is globular, and the class of globular spaces is closed under formation of finite products. (Nyikos has pointed out in [5] that it is also closed under formation of Pixley-Roy hyperspaces.)

2. GLOBS AND CARDINAL FUNCTIONS

I originally discovered the results of this section for 'blobs': globs for which the generalized linear order was a product of only two cardinals. The extension to the general case was kindly carried out by my brother, David W. Scott. We first show that there is no real need to consider weak globs at all.

(Note: Though the arguments establishing the results of this section and the next are in no wise subtle, several require tedious attention to painfully intricate detail. The beleaguered reader would do well to bear in mind that all are based ultimately on the following principle, so obvious as to be easily overlooked: if $\kappa, \lambda \geq \omega$ are distinct, regular cardinals, every κ -sequence in λ is: (1) bounded if $\kappa < \lambda$; and (2) constant on a cofinal subset of κ if $\kappa > \lambda$. It would probably also be helpful to read Example 4.2 before proceeding much further.)

THEOREM 2.0. (The Equivalence Theorem). *Let X be a space, $p \in X$, and let $\Omega = \{\kappa_i : i \in n\}$, where each κ_i is regular and infinite, and $\kappa_0 < \kappa_1 < \dots < \kappa_{n-1}$. Let \mathcal{B} be a strict weak (nbhd) Ω -glob at p . Then \mathcal{B} contains a (nbhd) Ω -glob at p .*

PROOF. Let $P = \prod \Omega$. Recall that for $i \in n$, $\pi_i : P \rightarrow \kappa_i$ is the canonical projection. If $i < j < n$, say that \mathcal{B} is $\langle i, j \rangle$ -strong iff whenever $\bar{\alpha}, \bar{\beta} \in P$, $\alpha_i < \beta_i$ (i.e., $\pi_i(\bar{\alpha}) < \pi_i(\bar{\beta})$), and $\alpha_j > \beta_j$, then $B(\bar{\alpha})$ and $B(\bar{\beta})$ are not related by inclusion. Clearly \mathcal{B} is a (nbhd) Ω -glob at p iff \mathcal{B} is $\langle i, j \rangle$ -strong for all pairs $\langle i, j \rangle$ such that $i < j < n$. It suffices, therefore, to prove the following assertion:

For any $i < j < n$, and any strict weak (nbhd) Ω -glob, \mathcal{B} , there is a strict weak (nbhd) Ω -glob, $\mathcal{C} \subseteq \mathcal{B}$ such that

- (1) C is $\langle i, j \rangle$ -strong, and
 (2) if B is $\langle k, \ell \rangle$ -strong for some $k < \ell < n$, then so is C .

We begin, therefore, by fixing $i < j < n$. Let $\kappa = \kappa_i$, $\lambda = \kappa_j$, $\Omega_L = \{\mu \in \Omega: \mu < \kappa\}$, $\Omega_M = \{\mu \in \Omega: \kappa < \mu < \lambda\}$, $\Omega_R = \{\mu \in \Omega: \mu > \lambda\}$, $P_L = \Pi\Omega_L$, $P_M = \Pi\Omega_M$, and $P_R = \Pi\Omega_R$; as usual, $\Pi\emptyset = 1$.

For $\xi \in \kappa$ let $V_\xi = \{V \subseteq X: B(\bar{\alpha}) \subseteq V \text{ for some } \bar{\alpha} \in P \text{ such that } \alpha_i = \xi\}$; clearly $V_\xi \subseteq V_\eta$ whenever $\xi \leq \eta < \kappa$. If there are a V and a cofinal $K \subseteq \kappa$ such that $V_\xi = V$ for all $\xi \in K$, fix $\xi_0 \in K$, $\bar{\alpha} \in P_L$, and $\bar{\beta} \in P_M \times \lambda \times P_R$; there are then sequences $\langle \bar{\alpha}^\xi: \xi < K \setminus \xi_0 \rangle$ in P_L and $\langle \bar{\beta}^\xi: \xi \in K \setminus \xi_0 \rangle$ in $P_M \times \lambda \times P_R$ such that the latter is increasing and, for each $\xi \in K \setminus \xi_0$, $B(\bar{\beta}^\xi \hat{\ } \xi_0 \hat{\ } \bar{\beta}^\xi) \subseteq B(\bar{\alpha} \hat{\ } \xi \hat{\ } \bar{\beta})$. (As usual, $\bar{\alpha} \hat{\ } \bar{\beta}$ denotes concatenation of sequences.) $|P_L| < \kappa$, so there is a cofinal $K_0 \subseteq K$ on which the first sequence is constantly $\bar{\eta}$, say, and there is an upper bound, $\bar{\rho}$, for the second, since each factor of $P_M \times \lambda \times P_R$ has cofinality greater than κ . But then $B(\bar{\eta} \hat{\ } \xi_0 \hat{\ } \bar{\rho}) \subseteq B(\bar{\alpha} \hat{\ } \xi \hat{\ } \bar{\beta})$ for all $\xi \in K_0$, which contradicts the strictness of B along the i^{th} (or κ) coordinate. Thus, we may assume that $V_\xi \subset V_\eta$ whenever $\xi < \eta < \kappa$. (The necessary modification of B plainly does not decrease the set of pairs for which B is strong.)

Fix $\xi < \eta < \kappa$, and suppose that there are, cofinally in $P_L \times (P_M \times \lambda \times P_R)$, $\bar{\alpha} \hat{\ } \bar{\beta}$ and $\bar{\rho} \hat{\ } \bar{\sigma}$ such that $\bar{\alpha} \hat{\ } \bar{\beta} \not\leq \bar{\rho} \hat{\ } \bar{\sigma}$, but $B(\bar{\alpha} \hat{\ } \xi \hat{\ } \bar{\beta}) \subseteq B(\bar{\rho} \hat{\ } \eta \hat{\ } \bar{\sigma})$. Clearly, then, $V_\xi = V_\eta$, which is impossible. In particular, for each $\xi \in \kappa$ there must be an $\bar{\alpha}^\xi \in P_L$ and a $\bar{\beta}^\xi \in P_M \times \lambda \times P_R$ such that if $\bar{\alpha}^\xi \leq \bar{\rho}$, $\bar{\sigma} \in P_L$, $\bar{\beta}^\xi \leq \bar{\eta}$, $\bar{v} \in P_M \times \lambda \times P_R$, and $\bar{\rho} \hat{\ } \bar{\eta} \not\leq \bar{\sigma} \hat{\ } \bar{v}$, then $B(\bar{\rho} \hat{\ } \xi \hat{\ } \bar{v}) \not\subseteq B(\bar{\sigma} \hat{\ } (\xi+1) \hat{\ } \bar{v})$. Let $\bar{\beta}$ be an upper bound for $\{\bar{\beta}^\xi: \xi \in \kappa\}$, and let K cofinal in κ and $\bar{\alpha} \in P_L$ be such that $\bar{\alpha}^\xi = \bar{\alpha}$ for each $\xi \in K$. By passing to $\{\bar{\rho} \in P_L: \bar{\alpha} \leq \bar{\rho}\} \times K \times \{\bar{\eta} \in P_M \times \lambda \times P_R: \bar{\beta} \leq \bar{\eta}\}$ we may assume that $B(\bar{\alpha} \hat{\ } \xi \hat{\ } \bar{\beta}) \not\subseteq B(\bar{\rho} \hat{\ } \eta \hat{\ } \bar{\sigma})$ whenever $\bar{\alpha}, \bar{\rho} \in P_L$, $\xi < \eta < \kappa$, $\bar{\beta}, \bar{\alpha} \in P_M \times \lambda \times P_R$, and $\bar{\alpha} \hat{\ } \bar{\beta} \not\leq \bar{\rho} \hat{\ } \bar{\sigma}$. (Again, this is a nice 'rectangular' reduction that does not shrink the set of pairs for which B is strong.) In particular, if $\bar{\alpha}, \bar{\beta} \in P$, $\alpha_i < \beta_i$ and $\alpha_j > \beta_j$, then $B(\bar{\alpha}) \not\subseteq B(\bar{\beta})$.

To finish we must so arrange matters that (under the same hypothesis) $B(\bar{\alpha}) \not\subseteq B(\bar{\beta})$. It is enough, however, for B to have the following property: if $\xi \in \lambda$ and $\bar{\alpha}, \bar{\beta} \in P$ are such that $\alpha_j = \xi + 1$, $\beta_j = \xi$, $\beta_i > 0$, and $\alpha_k = 0$ for all $k \in n \setminus \{j\}$, then $B(\bar{\alpha}) \not\subseteq B(\bar{\beta})$. (This is because $\bar{\alpha}$ is the infimum in P of the set of $\bar{\gamma} \in P$ such that $\gamma_i < \beta_i$ and $\gamma_j > \beta_j$, given that $\beta_j = \xi$ and $\beta_i > 0$.) We cut down the j^{th} (or λ) factor of P to get this property.

For each $\eta \in \lambda$ let $\bar{\alpha}^\eta \in P$ be defined so that $\bar{\alpha}_j^\eta = \eta$ and $\bar{\alpha}_k^\eta = 0$ for $k \in n \setminus \{j\}$. The strictness of B ensures that for each $\xi \in \lambda$ and $\bar{\gamma} \in P_R$ there is a least $\eta(\xi, \bar{\gamma}) \in \lambda$ such that: (1) $\eta(\xi, \bar{\gamma}) > \xi$; and (2) if $\bar{\beta} \in P$, $\beta_i > 0$,

$\beta_j \leq \xi$, and $\pi_R(\bar{\beta}) = \bar{\gamma}$ (where $\pi_R: P \rightarrow P_R$ is the projection), then $B(\bar{\alpha}^{\eta(\xi, \bar{\gamma})}) \not\subseteq B(\bar{\beta})$. If $\Omega_R = \emptyset$, $|P_R| = 1$, so let $\eta_\xi = \eta(\xi, \bar{\gamma})$ for each $\xi \in \lambda$, where $\bar{\gamma}$ is the unique member of P_R . Otherwise, $|P_R| > \lambda$; Let \preceq be the lexicographic order on P_R viewed as $\kappa_{n-1} \times \dots \times \kappa_0$, i.e., so that coordinates on shorter factors 'run' faster. We may then treat $\langle \eta(\xi, \bar{\gamma}) : \bar{\gamma} \in P_R \rangle$ as a $|P_R|$ -sequence as $\bar{\gamma}$ runs over $\langle P_R, \preceq \rangle$. This sequence may not be non-decreasing, but it is not hard to see that it must be bounded in λ by some η_ξ . (Fix all but one coordinate of $\bar{\gamma}$, and let that one run over its factor: the resulting subsequence is non-decreasing, hence bounded. But then the given sequence can be replaced by one indexed by the product of the remaining factors; repeat as necessary.)

In any case we now have, for each $\xi \in \lambda$, an $\eta_\xi \in \lambda$ such that: (1) $\eta_\xi > \xi$; and (2) $\eta(\xi, \bar{\gamma}) \leq \eta_\xi$ for each $\bar{\gamma} \in P_R$. Thus, $B(\bar{\alpha}^{\eta_\xi}) \not\subseteq B(\bar{\beta})$ for any $\bar{\beta} \in P$ such that $\beta_i > 0$ and $\beta_j \leq \xi$. Let $\phi: \lambda \rightarrow \lambda$ be such that if $\zeta < \zeta' < \lambda$, then $\phi(\zeta') \geq \eta_{\phi(\zeta)}$. Then we have $B(\bar{\alpha}^{\phi(\zeta+1)}) \not\subseteq B(\bar{\beta})$ whenever $\beta_i > 0$ and $\beta_j \leq \phi(\zeta)$. Replacing λ by $\text{ran } \phi$, we may assume that $B(\bar{\alpha}^{\xi+1}) \not\subseteq B(\bar{\beta})$ if $\beta_i > 0$ and $\beta_j \leq \xi$, which is the desired result.

This completes the proof. \square

In fact we can do a bit better yet. If \mathcal{B} is a strict weak nbhd glob at p , $\{\text{int } \mathcal{B} : \mathcal{B} \in \mathcal{B}\}$ is evidently a strict weak glob at p and as such contains a glob at p .

COROLLARY 2.1. *Let X , p , and Ω be as in the Equivalence Theorem. If there is a strict weak Ω -glob at p , then there is an Ω -glob at p . \square*

In the sequel I state most results in terms of nbhd globs, since they are somewhat easier to work with than globs; in view of Corollary 2.1, however, the distinction will generally prove unimportant. Appropriate modifications are left to the reader.

Of fundamental importance in any investigation of local cardinal functions in globular spaces is the observation that if p has both an Ω -glob and an Ω' -glob, then $\Omega = \Omega'$, i.e., that there is at most one 'size and shape' for a glob at a point.

THEOREM 2.2. (The Uniqueness Theorem). *Let $\Omega = \{\kappa_i : i \in n\}$ and $\Omega' = \{\lambda_i : i \in m\}$, where each κ_i and λ_i is regular, $\kappa_0 < \dots < \kappa_{n-1}$, and $\lambda_0 < \dots < \lambda_{m-1}$. Suppose \mathcal{B} and \mathcal{B}' are, respectively, a nbhd Ω -glob and a nbhd Ω' -glob at $p \in X$. Then $\Omega = \Omega'$.*

PROOF. Let k be minimal in $n \cap m$ such that $\kappa_k \neq \lambda_k$, and assume that $\kappa_k < \lambda_k$. Let $P = \Pi\Omega$ and $P' = \Pi\Omega'$. For each $\xi \in \kappa_k$ let $\bar{\alpha} \in P$ be defined as follows: $\alpha_i^\xi = 0$ if $i \in n \setminus \{k\}$, and $\alpha_k^\xi = \xi$. Let $\Omega'_L = \{\lambda_i : i < k\}$, $\Omega'_R = \Omega' \setminus \Omega'_L$, $P'_L = \Pi\Omega'_L$, and $P'_R = \Pi\Omega'_R$. For each $\xi \in \kappa_k$ there are $\bar{\beta}^\xi \in P'_L$ and $\bar{\gamma}^\xi \in P'_R$ such that $B'(\bar{\beta}^\xi \wedge \bar{\gamma}^\xi) \subseteq B(\bar{\alpha}^\xi)$ and $\bar{\gamma}^\xi \leq \bar{\gamma}^\eta$ whenever $\xi < \eta < \kappa_k$. Let $\bar{\gamma}$ be an upper bound in P'_R for $\{\bar{\gamma}^\xi : \xi \in \kappa_k\}$, and let $K \subseteq \kappa_k$ and $\bar{\beta} \in P'_L$ be such that: (1) K is cofinal in κ_k ; and (2) $\bar{\beta}^\xi = \bar{\beta}$ for all $\xi \in K$. Then $B'(\bar{\beta} \wedge \bar{\gamma}) \subseteq \bigcap \{B(\bar{\alpha}^\xi) : \xi \in K\}$, which is impossible, since B is strict. Thus, $\kappa_i = \lambda_i$ for all $i \in n \cap m$, and we may as well assume that $n \leq m$ (so that $\Omega \subseteq \Omega'$).

If $n < m$, then $|P| < |P'|$. Fix $\bar{\alpha} \in \Pi(\Omega' \setminus \{\lambda_{m-1}\})$. For each $\xi \in \lambda_{m-1}$ there is a $\bar{\beta}^\xi \in P$ such that $B(\bar{\beta}^\xi) \subseteq B'(\bar{\alpha} \wedge \xi)$. But then there is a $\bar{\beta} \in P$ such that $B(\bar{\beta}) \subseteq B'(\bar{\alpha} \wedge \xi)$ for all ξ in a cofinal subset of λ_{m-1} , which is absurd. Hence $n = m$, and $\Omega = \Omega'$. \square

The Uniqueness Theorem justifies the following definition.

DEFINITION 2.3. Let X be a space which is globular at a point $p \in X$, i.e., such that there is an Ω -glob at p for some finite set, Ω , of regular, infinite cardinals. The *glob-character*, $\gamma\chi(p, X)$, of p in X is defined to be Ω .

Thus, if for example X is a lob-space, $\gamma\chi(p, X) = \{\chi(p, X)\}$ for each $p \in X$.

It is sometimes convenient to write $\gamma\chi(p, X) = \{1\}$ if p is an isolated point of X ; to be consistent we then say that $\chi(p, X) = \psi(p, X) = 1$ also (instead of ω). In particular this convention will simplify the statement of Theorem 2.8 below.

THEOREM 2.4. If $\gamma\chi(p, X) = \Omega$, then $\chi(p, X) = \max \Omega$.

PROOF. Let $\Omega = \{\kappa_i : i \in n\}$, where $\kappa_0 < \dots < \kappa_{n-1}$, so that $\max \Omega = \kappa_{n-1}$; clearly $\chi(p, X) \leq \kappa_{n-1}$. Let B be an Ω -glob at p , and suppose that $\chi(p, X) = \lambda < \kappa_{n-1}$. Then there is a family $B_0 = \{B(\bar{\alpha}^\xi) : \xi \in \lambda\} \subseteq B$ which is a base at p . Let $\eta = \sup\{\alpha_{n-1}^\xi : \xi \in \lambda\}$; then $\{B(\bar{\alpha}) : \alpha_{n-1} = \eta\}$ is an $(\Omega \setminus \{\kappa_{n-1}\})$ -glob at p , which contradicts the Uniqueness Theorem. \square

THEOREM 2.5. If $\gamma\chi(p, X) = \Omega$, then $t(p, X) = \max \Omega$.

PROOF. Again let $\Omega = \{\kappa_0, \dots, \kappa_{n-1}\}$, where $\kappa_0 < \dots < \kappa_{n-1}$, and let B be an Ω -glob at p . For each $\bar{\alpha} \in P = \Pi\Omega$ let $B'(\bar{\alpha}) = B(\alpha_0, \dots, \alpha_{n-2}, \kappa_{n-2} \cdot \alpha_{n-1} + \alpha_{n-2})$, where all arithmetic in the last parameter is ordinal arithmetic. Then

$$\begin{aligned}
& B'(0, \dots, 0, \alpha_{n-2} + 1, \alpha_{n-1}) \cup B'(0, \dots, 0, \alpha_{n-1} + 1) \\
& = B(0, \dots, 0, \alpha_{n-2} + 1, \kappa_{n-2} \cdot \alpha_{n-1} + \alpha_{n-2} + 1) \\
& \cup B(0, \dots, 0, \kappa_{n-2} \cdot (\alpha_{n-1} + 1)) \subseteq B(0, \dots, 0, \kappa_{n-2} \cdot \alpha_{n-1} + \alpha_{n-2} + 1) \\
& \neq B(\alpha_0, \dots, \alpha_{n-2}, \kappa_{n-2} \cdot \alpha_{n-1} + \alpha_{n-2}) = B'(\bar{\alpha}) \quad \text{for any } \bar{\alpha} \in P.
\end{aligned}$$

That is, $B' = \{B'(\bar{\alpha}) : \bar{\alpha} \in P\}$ is an Ω -glob at p with property that $B'(\bar{\alpha}) \not\subseteq B'(0, \dots, 0, \alpha_{n-2} + 1, \alpha_{n-1}) \cup B'(0, \dots, 0, \alpha_{n-1} + 1)$ for any $\bar{\alpha} \in P$.

Let $\Omega_L = \{\kappa_0, \dots, \kappa_{n-3}\}$, $\Omega_R = \Omega \setminus \Omega_L$, $P_L = \Pi \Omega_L$, and $P_R = \Pi \Omega_R$. For each $\bar{\alpha} \in P_L$ and $\bar{\beta} \in P_R$ pick a point $x(\bar{\alpha} \hat{\ } \bar{\beta}) \in B'(\bar{\alpha} \hat{\ } \bar{\beta}) \setminus [B'(\bar{0} \hat{\ } (\beta_{n-2} + 1), \beta_{n-1}) \cup B'(\bar{0} \hat{\ } 0 \hat{\ } (\beta_{n-1} + 1))]$, where $\bar{0} = \langle 0, \dots, 0 \rangle$ and $\bar{\beta} = \langle \beta_{n-2}, \beta_{n-1} \rangle$.

Let $D = \{x(\bar{\alpha}) : \bar{\alpha} \in P\}$. Clearly $p \in \text{cl}D$, and $|D| \leq \kappa_{n-1}$. In fact, for fixed $\bar{\alpha} \in P_L$ the points $x(\bar{\alpha} \hat{\ } \bar{\beta})$, ($\bar{\beta} \in P_R$) are distinct, so $|D| = \kappa_{n-1}$. Now if $A \in [D]^{<\kappa_{n-1}}$, there is an upper bound, η , on $\{\alpha_{n-1} : x(\bar{\alpha}) \in A\}$, whence $A \cap B'(\bar{0} \hat{\ } \eta) = \emptyset$, and $p \notin \text{cl}A$. Thus, $\kappa_{n-1} = \chi(p, X) \geq t(p, X) \geq |D| = \kappa_{n-1}$, and the result follows at once. \square

COROLLARY 2.6. *If X is globular, then $t(X) = \chi(X)$, and indeed $t(p, X) = \chi(p, X)$ for each $p \in X$. \square*

(For example, every sequential globular space is first countable, since sequential spaces have countable tightness.)

THEOREM 2.7. *If $\gamma\chi(p, X) = \Omega$, then $\psi(p, X) \in \Omega$. (In Section 4 we shall see that no better result is possible.)*

PROOF. Let $\mu = \psi(p, X)$, and let B be an Ω -glob at p , where $\Omega = \{\kappa_i : i \in n\}$, and $\kappa_0 < \dots < \kappa_{n-1}$. Clearly $\kappa_0 \leq \mu \leq \kappa_{n-1}$, so suppose that $\kappa_i \leq \mu < \kappa_{i+1}$ for some $i < n-1$. By an easy cardinality argument there is then a $\bar{\beta} \in \Pi\{\kappa_j : i < j < n\}$ such that $\cap\{B(\bar{\alpha} \hat{\ } \bar{\beta}) : \bar{\alpha} \in \Pi\{\kappa_j : j < i\}\} = \{p\}$, whence $\mu \leq \kappa_i$; i.e., $\mu = \kappa_i$, and the result follows. \square

And finally we observe that the glob-character behaves remarkably well upon passage to a subspace.

THEOREM 2.8. *Suppose that $p \in Y \subseteq X$, where $\gamma\chi(p, X) = \Omega$. Then $\gamma\chi(p, Y) \subseteq \Omega \cup \{1\}$.*

PROOF. If p is isolated in Y there is nothing to prove, so assume the contrary. Let \mathcal{B} be an Ω -glob at p in X , where, once again, $\Omega = \{\kappa_i : i \in n\}$ and $P = \Pi\Omega$. For each $B \in \mathcal{B}$ let $B' = B \cap Y$, and let $\mathcal{B}' = \{B' : B \in \mathcal{B}\}$. For each $\bar{\alpha} \in P$ and $i \in n$ let $\bar{\alpha}^i = \langle \alpha_0, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_{n-1} \rangle \in P^i = \Pi(\Omega \setminus \{\kappa_i\})$. For each $i \in n$ let $K_i = \{\bar{\alpha}^i \in P^i : \bar{\alpha} \in P \text{ and } p \in \text{int } E_{\{i\}}^{\mathcal{B}'}(\bar{\alpha})\}$. If no K_i is cofinal in P^i we may assume that each $K_i = \emptyset$ and thence that \mathcal{B}' is a strict weak Ω -glob at p in Y . Otherwise, pick $i \in n$ such that K_i is cofinal in P^i , and let $C = \{\text{int } E_{\{i\}}^{\mathcal{B}'}(\bar{\alpha}) : \bar{\alpha} \in P\}$. C is naturally indexed by P^i and is therefore a weak $(\Omega \setminus \{\kappa_i\})$ -glob at p in Y , not necessarily strict. However, if C is not strict we may repeat the process (finitely many times) until we get a strict weak Ω' -glob at p in Y for some $\Omega' \subseteq \Omega$. \square

There seems to be little more that can be said about the relationship between the glob-character and the familiar local cardinal functions. However, the following result, similar to Theorem 2.7, is sometimes useful.

THEOREM 2.9. *Suppose that $\gamma\chi(p, X) = \Omega$, and that $A \subseteq X \setminus \{p\}$ with $p \in \text{cl}A$. Let $\lambda = |A|$. If A is minimal in the sense that $p \notin \text{cl}A_0$ for any $A_0 \in [A]^{<\lambda}$, then $\lambda \in \Omega$. (There are examples to show that this is the strongest possible statement; see Section 4.) And if $\lambda = \min \Omega$, then A contains a λ -sequence converging to p .*

PROOF. Let $\Omega = \{\kappa_i : i \in n\}$, $\kappa_0 < \dots < \kappa_{n-1}$, $P = \Pi\Omega$, and let \mathcal{B} be an Ω -glob at p . By Theorem 2.8, $\gamma\chi(p, A \cup \{p\}) \subseteq \Omega$ (since p is not isolated in $A \cup \{p\}$); $\gamma\chi(p, A \cup \{p\}) = \Omega'$, say. By hypothesis $t(p, A \cup \{p\}) = \lambda$. But by Theorem 2.5, $t(p, A \cup \{p\}) = \max \Omega'$, so $\lambda \in \Omega' \subseteq \Omega$.

Suppose that $\lambda = \kappa_0$. Let $P_R = \Pi(\Omega \setminus \{\lambda\})$, and for each $\xi \in \lambda$ let $F_\xi = \{B(\xi \hat{\ } \bar{\alpha}) : \bar{\alpha} \in P_R\}$. Every member of $[P_R]^{<\lambda}$ has an upper bound in P_R , so for each $\xi \in \lambda$ we have that $p \notin \text{cl}[A \cap B(\xi \hat{\ } \bar{0}) \setminus F_\xi]$. But $p \in \text{cl}[A \cap B(\xi \hat{\ } \bar{0})]$, so $p \in \text{cl}(A \cap F_\xi)$. For each $\xi \in \lambda$ pick $x_\xi \in A \cap F_\xi$. Now let $\eta \in \lambda$ and $\bar{\alpha} \in P_R$ be arbitrary; if $\eta \leq \xi < \lambda$, then $x_\xi \in A \cap F_\xi \subseteq A \cap F_\eta \subseteq A \cap B(\eta \hat{\ } \bar{\alpha})$, so $\langle x_\xi : \xi \in \lambda \rangle \rightarrow p$. \square

COROLLARY 2.10. *If X is globular at p , and $p \in \text{cl}A$ for some countable $A \subseteq X \setminus \{p\}$, then $\omega \in \gamma\chi(p, X)$, and A contains a sequence converging to p . \square*

COROLLARY 2.11. *If X is globular, and every non-isolated $p \in X$ is the limit of some countable $A \subseteq X \setminus \{p\}$, then every non-isolated point of X is a κ -point of X , i.e., the limit of a non-trivial convergent sequence. \square*

COROLLARY 2.12. *If X is globular, then X is countably compact iff X is sequentially compact.*

PROOF. It suffices to show that if X is countably compact, then X is sequentially compact. Let $\langle x_n : n \in \omega \rangle$ be any sequence in X . If some sub-sequence is constant there is nothing to prove, so we may assume that $x_n \neq x_m$ whenever $n < m < \omega$. Let $A = \{x_n : n \in \omega\}$, and let p be a limit point of A . By replacing A by $A \setminus \{p\}$ if necessary we may assume that $p \notin A$. The result now follows from Corollary 2.10. \square

COROLLARY 2.13. *The product of ω_1 (or fewer) countably compact globular spaces is countably compact.*

PROOF. This follows from the well-known fact that the product of ω_1 sequentially compact spaces is countably compact. \square

Corollaries 2.12 and 2.13 extend results of Davis for lob-spaces [2], as does the next result.

COROLLARY 2.14. *If X is Hausdorff, countably compact, and globular, and $\psi(X) \leq 2^\omega$, then $|X| \leq 2^\omega$.*

PROOF. ARKHANGEL'SKII has proved in [1] the corresponding result for sequentially compact (not necessarily globular) spaces. \square

3. STRUCTURE THEORY

In this section we construct a classification of the 'essentially different' Ω -globs for fixed Ω . Central to the classification is the notion of an *abstract simplicial complex*.

DEFINITION 3.0. Let $n \in \omega \setminus 1$. An *abstract simplicial complex (a.s.c.)* on n is a family $K \subseteq P^*(n)$, ($= P(n) \setminus \{\emptyset\}$) such that

- (1) $[n]^1 \subseteq K$; and
- (2) if $S \subseteq K$, then $P^*(S) \subseteq K$.

Fix a space X and a point $p \in X$ such that $\gamma_X(p, X) = \Omega = \{\kappa_i : i \in n\}$, where $\kappa_0 < \dots < \kappa_{n-1}$. Let $P = \Pi\Omega$, and let B be a nbhd Ω -glob at p .

DEFINITION 3.1. For each $\bar{\alpha} \in P$ and $I \in P^*(n)$, $Q(\bar{\alpha}, I)$ is the assertion that $E_I(\bar{\alpha})$ is not a nbhd of p . Equivalently, $Q(\bar{\alpha}, I)$ holds iff for all $\bar{\beta} \in P$,

$$p \in \mathcal{CL}[B(\bar{\beta}) \setminus E_I(\bar{\alpha})].$$

The first formulation is the right one to work with, but the second has a nice geometric significance in the setting of Example 4.2. Either way it is clear that $Q(\bar{\alpha}, I)$ implies $Q(\bar{\beta}, J)$ whenever $\bar{\alpha} \leq \bar{\beta}$ and $\emptyset \neq J \subseteq I \subseteq n$.

DEFINITION 3.2. $K_B = \{I \in P^*(n) : \exists \bar{\alpha}_I \in P(Q(\bar{\alpha}_I, I))\}$. (We suppress the subscript B whenever possible.)

Evidently K is an a.s.c. on n . Moreover, since K is finite, $\{\bar{\alpha}_I : I \in P^*(n)\}$ has an upper bound, $\bar{\alpha}$, in P . We can therefore replace P by its $\bar{\alpha}$ -tail ($= \{\bar{\beta} \in P : \bar{\alpha} \leq \bar{\beta}\}$) and assume that in fact $Q(\bar{0}, I)$ holds for each $I \in K$.

Originally the main result of this section was to have been that K is an invariant of p and X , independent of B .

THEOREM 3.3. (The Type Theorem). *If B and B' are nbhd Ω -globs at p , then $K_B = K_{B'}$.*

My proof of the Type Theorem was somewhat long and involved. Eric K. van Douwen has since pointed out to me a simpler proof of the following stronger result.

THEOREM 3.4. (Theorem on Cofinal Similarity). *Let B and B' be nbhd Ω -globs at p . Then there is a $P_0 = \Pi\{K_i : i \in n\} \subseteq P$, where each K_i is cofinal in κ_i , such that for any $\bar{\alpha}, \bar{\beta} \in P_0$ with $\alpha_i < \beta_i$ for each $i \in n$, $B(\bar{\alpha}) \supseteq B(\bar{\beta})$ and $B'(\bar{\alpha}) \supseteq B'(\bar{\beta})$. (We might describe B and B' as being 'cofinally similar'.)*

To see that the Type Theorem follows from Theorem 3.4, make the following definition.

DEFINITION 3.5. If B is a nbhd Ω -glob at p , and $I \in P^*(n)$, let

$$E_I^B = \{A \subseteq X : A \supseteq E_I^B(\bar{\alpha}) \text{ for some } \bar{\alpha} \in P\}.$$

The following result is then an immediate corollary of Theorem 3.4.

COROLLARY 3.6. *If B and B' are nbhd Ω -globs at p , then $E_I^B = E_I^{B'}$ for each $I \in P^*(n)$. \square*

PROOF of the Type Theorem from Corollary 3.6. Merely observe that

$$K_{\mathcal{B}} = \{I \in \mathcal{P}^*(n) : \text{some member of } E_I^{\mathcal{B}} \text{ is not a nbhd of } p.\} \square$$

I shall give a slightly modified version of van Douwen's proof of Theorem 3.4. However, I shall also include the main lemmas from my original proof of the Type Theorem, as they seem to be of independent interest: they give a geometrical characterization of those $I \in \mathcal{P}^*(n)$ belonging to $K_{\mathcal{B}}$ in terms of the way the members of \mathcal{B} 'fit together'.

DEFINITION 3.7. If $\phi_0, \phi_1: P \rightarrow P$, we write $\phi_1 \geq^* \phi_0$ just in case for each $\bar{\alpha} \in P$ there is a $\bar{\beta} \in P$ such that $\bar{\beta} \geq \bar{\alpha}$ and $\phi_1(\bar{\alpha}) \geq \phi_0(\bar{\beta})$.

LEMMA 3.8. Let $\phi: P \rightarrow P$ be arbitrary. Then there are functions $\psi_i: \kappa_i \rightarrow \kappa_i$ ($i \in n$) such that $\psi = \prod\{\psi_i: i \in n\} \geq^* \phi$. (That is, $\psi(\bar{\alpha}) = \langle \psi_0(\alpha_0), \dots, \psi_{n-1}(\alpha_{n-1}) \rangle$ for each $\alpha \in P$.) Moreover, each ψ_i may be taken to be strictly monotone.

PROOF. Let $P_L = \prod \Omega_L$, where $\Omega_L = \omega \setminus \{\kappa_{n-1}\}$. (The result is trivial if $n = 1$.) Fix $\bar{\alpha} \in P_L$ and consider the κ_{n-1} -sequence $\langle \pi(\phi_0(\bar{\alpha} \hat{\ } \xi)) : \xi \in \kappa_{n-1} \rangle$, where $\pi: P \rightarrow P_L$ is the projection; plainly it is constant on some cofinal $K(\bar{\alpha}) \subseteq \kappa_{n-1}$, say with value $\psi(\bar{\alpha})$. For each $\bar{\alpha} \in P_L$ and $\xi \in \kappa_{n-1}$ let $\zeta(\bar{\alpha}, \xi) = \inf(K(\bar{\alpha}) \setminus \xi)$, so that $\bar{\alpha} \hat{\ } \zeta(\bar{\alpha}, \xi) \geq \bar{\alpha} \hat{\ } \xi$, and $\pi(\phi_0(\bar{\alpha} \hat{\ } \zeta(\bar{\alpha}, \xi))) = \psi(\bar{\alpha})$. Now, $|P_L| < \kappa_{n-1}$, so it is possible to define a function $\psi_{n-1}: \kappa_{n-1} \rightarrow \kappa_{n-1}$ by setting $\psi_{n-1}(\xi) = \sup\{\pi_{n-1}(\phi_0(\bar{\alpha} \hat{\ } \zeta(\bar{\alpha}, \xi))) : \bar{\alpha} \in P_L\}$. But then for any $\bar{\alpha} \in P_L$ and $\xi \in \kappa_{n-1}$, $\psi(\bar{\alpha}) \hat{\ } \psi_{n-1}(\xi) \geq \phi_0(\bar{\alpha} \hat{\ } \zeta(\bar{\alpha}, \xi))$, and $\bar{\alpha} \hat{\ } \zeta(\bar{\alpha}, \xi) \geq \bar{\alpha} \hat{\ } \xi$, so the function $\phi_1 = \psi \times \psi_{n-1} \geq^* \phi_0$. (Plainly we may also ensure that $\psi_{n-1}(\xi) > \sup\{\xi, \sup \text{ran } \psi_{n-1} \upharpoonright \xi\}$.)

The result now follows by an easy (downward) induction. \square

PROOF OF THEOREM 3.4. For each $\bar{\alpha} \in P$ there is a $\phi(\bar{\alpha}) \in P$ such that $\phi(\bar{\alpha}) \geq \bar{\alpha}$, $B(\bar{\alpha}) \supseteq B'(\phi(\bar{\alpha}))$, and $B'(\bar{\alpha}) \supseteq B(\phi(\bar{\alpha}))$. Apply Lemma 3.8 to ϕ to get $\psi = \prod\{\psi_i: i \in n\} \geq^* \phi$, where each ψ_i is strictly monotone. It is easy to see that for each $i \in n$ there is a cofinal $K_i \subseteq \kappa_i$ such that $(\xi, \psi_i(\xi)) \cap K_i = \emptyset$ for each $\xi \in K_i$. (As usual, $(\xi, \psi_i(\xi)) = \{\eta \in \kappa_i: \xi < \eta < \psi_i(\xi)\}$.) Let $K = \prod\{K_i: i \in n\}$, obviously a cofinal subset of P .

Suppose that $\bar{\alpha}, \bar{\beta} \in K$ are such that $\alpha_i < \beta_i$ for each $i \in n$. Then for each $i \in n$ we have $\beta_i \geq \psi_i(\alpha_i)$, whence $\bar{\beta} \geq \psi(\bar{\alpha})$. But $\psi \geq^* \phi$, so there is a $\gamma \in P$ such that $\bar{\gamma} \geq \bar{\alpha}$ and $\psi(\bar{\alpha}) \geq \phi(\bar{\gamma})$; clearly, then,

$B(\bar{\alpha}) \supseteq B(\bar{\gamma}) \supseteq B'(\phi(\bar{\gamma})) \supseteq B'(\psi(\bar{\alpha})) \supseteq B'(\bar{\beta})$, and, similarly, $B'(\bar{\alpha}) \supseteq B(\bar{\beta})$, as required. \square

We have now justified the following definition.

DEFINITION 3.9. If $\gamma\chi(p, X) = \Omega$, and B is any nbhd Ω -glob at p , we define $K(p, X) = K_B$, the *type* of p .

The remainder of this section contains essentially my original proof of the Type Theorem, using the following property of globs (Definition 3.11).

DEFINITION 3.10. For each $\bar{\alpha} \in P$, $i \in n$, and $\xi \in \kappa_i$, define $\bar{\alpha}[i \rightarrow \xi] \in P$ by

$$\bar{\alpha}[i \rightarrow \xi](j) = \begin{cases} \alpha_j, & j \in n \setminus \{i\} \\ \xi, & j = i. \end{cases}$$

DEFINITION 3.11. Let B be a nbhd Ω -glob at p . For each $I \in P^*(n)$, B is *I-obese* iff for each $\bar{\alpha} \in P$, $B(\bar{\alpha}) \not\subseteq \bigcup \{B(\bar{\alpha}[i \rightarrow \alpha_i + 1]) : i \in I\}$.

LEMMA 3.12. Let B be a nbhd Ω -glob at p , let $K = K_B$, and let $I \in P^*(n)$. If $I \notin K$, then no nbhd Ω -glob at p is *I-obese*.

PROOF. If $I \notin K$, then $Q(\bar{\alpha}, I)$ fails for each $\bar{\alpha} \in P$; i.e., $E = \{E_I(\bar{\alpha}) : \bar{\alpha} \in P\}$ is a family of nbhds of p . Moreover, $E_I(\bar{\alpha}) \subseteq B(\bar{\alpha})$ for each $\bar{\alpha} \in P$, so E is a nbhd base at p .

Let C be any nbhd Ω -glob at p . For $i \in I$ let $\Omega_i = \Omega \setminus \{\kappa_i\}$ and $P_i = \Pi\Omega_i$.

Fix $i \in I$. For each $\bar{\eta} \in P_i$ let η' be the unique element of P such that $\eta'_i = 0$, and $\eta'_j = \eta_j$ if $j \in n \setminus \{i\}$; and let $E'_{\{i\}}(\eta) = E'_{\{i\}}(\eta')$. Abusing the notation somewhat we write $\bar{\eta}' = \bar{\eta} \hat{\ } 0$ even if $i \neq n-1$, since i is understood. (More generally, so long as i is fixed we write $\bar{\eta} \hat{\ } \xi$ for the $\bar{\alpha} \in P$ such that $\alpha_i = \xi$, and $\alpha_j = \eta_j$ for $j \in n \setminus \{i\}$.) Let $\bar{\alpha} \in P_i$. For each $\xi \in \kappa_i$ there is an $\bar{\eta}^\xi \in P_i$ such that $E'_{\{i\}}(\bar{\eta}^\xi) \subseteq C(\bar{\alpha} \hat{\ } \xi)$. And now it is not hard to see that there are a cofinal $K \in \kappa_i$ and an $\bar{\eta} \in P_i$ such that for each $\xi \in K$ and $j \in n \setminus \{i\}$, $\eta_j = \eta_j^\xi$ if $j < i$, and $\eta_j \geq \eta_j^\xi$ if $j > i$. Thus $E'_{\{i\}}(\bar{\eta}) \subseteq C(\bar{\alpha} \hat{\ } \xi)$ for each $\xi \in K$, whence it is clear that $E'_{\{i\}}(\bar{\eta}) \subseteq C(\bar{\alpha} \hat{\ } \xi)$ for all $\xi \in \kappa_i$. Denote this $\bar{\eta}$ by $\bar{\mu}(\bar{\alpha}, i)$.

Now let $\bar{\alpha} \in P$. For each $i \in I$ let $\pi^i : P \rightarrow P_i$ be the natural projection. Clearly $C(\bar{\alpha}) \supseteq \bigcup \{E'_{\{i\}}(\bar{\mu}(\pi^i(\bar{\alpha}), i)) : i \in I\}$. Let $\bar{v}(\bar{\alpha}) \in P$ be such that $\bar{\mu}(\pi^i(\bar{\alpha}), i) \leq \pi^i(\bar{v}(\bar{\alpha}))$ for each $i \in I$; then $C(\bar{\alpha}) \supseteq E_I(\bar{v}(\bar{\alpha}))$.

Finally, choose $\bar{\alpha} \in P$ so that $C(\bar{0}) \supseteq E_I(\bar{v}(\bar{0})) \supseteq C(\bar{\alpha})$. For each $i \in I$ let $\bar{\beta}^i \in P$ be such that $\beta_1^i = \alpha_i + 1$, and $\beta_j^i = 0$ if $j \in n \setminus \{i\}$. The definition of $\bar{v}(\bar{0})$ then ensures that for each $i \in I$, $C(\bar{\beta}^i) \supseteq E_{\{i\}}(\bar{u}(\bar{0}, i))$, so $U\{C(\bar{\beta}^i) : i \in I\} \supseteq U\{E_{\{i\}}(\bar{u}(\bar{0}, i)) : i \in I\} \supseteq E_I(\bar{v}(\bar{0})) \supseteq C(\bar{\alpha})$. Thus, C is not I -obese (at $\bar{\alpha}$). \square

LEMMA 3.13. *Let B and K be as in Lemma 3.12. If $I \in K$, then B contains an I -obese nbhd Ω -glob at p .*

PROOF. Let $P = \Pi\Omega$. By passing to a tail of P we may assume that B satisfies $Q(\bar{0}, I)$. For each $\bar{\alpha} \in P$ let $B'(\bar{\alpha}) = B(\bar{\alpha}) \setminus E_I(\bar{0})$; then $p \in \text{cl}B'(\bar{\alpha})$, and, in particular, $B'(\bar{\alpha}) \neq \emptyset$. Since $\cap\{B'(\bar{0}[i \rightarrow \xi]) : \xi \in \kappa_i\} = \emptyset$ for each $i \in I$, whereas $B'(\bar{\alpha}) \neq \emptyset$ for $\bar{\alpha} \in P$, there is a function $\phi : P \rightarrow P$ such that for each $\bar{\alpha} \in P$, $B'(\bar{\alpha}) \not\subseteq U\{B'(\bar{0}[i \rightarrow \pi_i(\phi(\bar{\alpha}))]) : i \in I\}$. Clearly any $\phi_0 \geq^* \phi$ (in the notation of Definition 3.7) also has this property, so by Lemma 3.8 we may assume that $\phi = \Pi\{\psi_i : i \in n\}$ for some strictly monotone functions $\psi_i : \kappa_i \rightarrow \kappa_i$ ($i \in n$).

As in the proof of Theorem 3.4, for each $i \in n$ let $K_i \subseteq \kappa_i$ be cofinal and such that $(\xi, \psi_i(\xi)) \cap K_i = \emptyset$ for each $\xi \in K_i$, and let $K = \Pi\{K_i : i \in n\}$. Suppose that $\bar{\alpha}, \bar{\beta} \in K$, where $\alpha_i < \beta_i$ for each $i \in n$. Then

$$\begin{aligned} B'(\bar{\alpha}) &\not\subseteq U\{B'(\bar{0}[i \rightarrow \pi_i(\phi(\bar{\alpha}))]) : i \in I\} \\ &= U\{B'(\bar{0}[i \rightarrow \psi_i(\alpha_i)]) : i \in I\} \supseteq U\{B'(\bar{0}[i \rightarrow \beta_i]) : i \in I\}, \end{aligned}$$

and restricting B to K produces an I -obese nbhd Ω -glob at p . \square

The Type Theorem is of course an immediate consequence of Lemmas 3.12 and 3.13. Indeed, we can say a little more.

COROLLARY 3.14. *Let B be an nbhd Ω -glob at p , and let $K = K(p, X)$. Then there are cofinal $K_i \subseteq \kappa_i$, ($i \in n$) such that if $K = \Pi\{K_i : i \in n\}$, and $B_0 = \{B(\bar{\alpha}) : \bar{\alpha} \in K\}$, then for each $I \in P^*(n)$, B_0 is I -obese iff $I \in K$. \square*

4. EXAMPLES AND SPECIAL CASES

It is no trick at all to construct a space containing one point with arbitrary, specified glob-character; what may be less clear is that there are non-trivial globular spaces.

EXAMPLE 4.0. Every LOTS (= linearly ordered topological space) is globular, and hence so is every GO-space (= subspace of some LOTS) and every subspace of a finite product of GO-spaces. (In fact it is clear that for any $p \in L$, where L is a LOTS, $|\gamma\chi(p, L)| \leq 2$.) \square

The following example is the prototype of a glob (and the source of all my intuition).

DEFINITION 4.1. For each cardinal $\kappa \geq \omega$, P_κ is the space obtained from $\kappa + 1$ (with order topology) by isolating each point of κ .

EXAMPLE 4.2. Let $\Omega = \{\kappa_i : i \in n\}$ be a set of regular cardinals such that $\omega \leq \kappa_0 < \dots < \kappa_{n-1}$. Let $X = \prod\{P_{\kappa_i} : i \in n\}$, and let $p = \langle \kappa_0, \dots, \kappa_{n-1} \rangle \in X$. Let $P = \prod\Omega$, and set $B(\bar{\alpha}) = \prod\{[\alpha_i, \kappa_i] : i \in n\}$ for each $\bar{\alpha} \in P$. ($[\alpha_i, \kappa_i] = \{\beta \in \kappa_i + 1 : \beta \geq \alpha_i\}$.) Then $\mathcal{B} = \{B(\bar{\alpha}) : \bar{\alpha} \in P\}$ is an Ω -glob at p .

For any $i \in n$ and $\bar{\alpha} \in P$, $E_{\{i\}}(\bar{\alpha}) = \{\bar{x} \in B(\bar{\alpha}) : x_i = \kappa_i\}$. Let $A_i = \bigcap\{E_{\{j\}}(\bar{0}) : j \in n \setminus \{i\}\}$, (so that A_i is homeomorphic to P_{κ_i}), and let $Y_i = (X \setminus E_n(\bar{\alpha})) \cup A_i$. Then $\psi(p, Y_i) = \kappa_i$. (Cf. Theorem 2.7). Note also that in X each A_i is minimal in the sense of Theorem 2.9.

For each $I \in \mathcal{P}^*(n)$ let $S_I = \{\bar{x} \in X : \text{for each } i \in n, x_i = \kappa_i \text{ iff } i \in I\}$. Let K be an a.s.c. on n , and let $Z = \{p\} \cup \bigcup\{S_I : I \in K\}$. If $\mathcal{B}' = \{B \cap Z : B \in \mathcal{B}\}$, then \mathcal{B}' is an Ω -glob at p in Z , and $K_{\mathcal{B}'} = K$. (Intuitively, an Ω -glob is I -obese iff there are enough points in the space to 'fill out' S_I .) \square

As noted in the Introduction, most of Davis's interesting results for lob-spaces do not appear to generalize readily to globular spaces. (I have not tried very hard to find counterexamples to all the *results*, but counterexamples to the *proofs* abound.) The difficulty is that these results all depend on the following lemma, whose natural generalization to globs is false.

LEMMA 4.3. [2]. Suppose that $\gamma\chi(p, X) = \{\kappa\}$ for some regular $\kappa \geq \omega$. If $A \subseteq P(X \setminus \{p\})$ is such that: (1) $x \in \text{cl} \cup A$; but (2) $x \notin \text{cl} A$ for each $A \in A$, then there are an $A' \subseteq A$ and a 1-1 choice function, y , on A' such that $p \in \text{cl} \text{ran } y$. \square

EXAMPLE 4.4. Let $X = (P_\omega \times P_{\omega_1}) \setminus [(\{\omega\} \times \omega_1) \cup (\omega \times \{\omega_1\})]$, and let $p = \langle \omega, \omega_1 \rangle$. Clearly $\gamma\chi(p, X) = \{\omega, \omega_1\}$. For each $n \in \omega$ let $A_n = \{n\} \times \omega_1$, a closed subset of X , and let $A = \{A_n : n \in \omega\}$; then $p \in \text{cl} \cup A$, but $p \notin \text{cl} A$. And for any $A' \subseteq A$ and any (1-1) choice function, y , on A' , $\text{ran } y$ is a closed, discrete subset of X . \square

EXAMPLE 4.5. Let $\omega \leq \kappa_0 < \dots < \kappa_{n-1}$, where the κ_i 's are regular. For $i \in n$ let X_i be a space containing a point, p_i , such that $\gamma\chi(p_i, X_i) = \{\kappa_i\}$. Let X be the quotient of the discrete union of the X_i 's obtained by identifying $\{p_i : i \in n\}$ to a single point, p . Then $\gamma\chi(p, X) = \{\kappa_i : i \in n\}$, and $K(p, X) = [n]^1$.

Clearly Lemma 4.3 *does* extend to the setting of Example 4.5. Unfortunately, I have not been able to show that its analogue holds *whenever* $K(p, X) = [n]^1$, where $n = |\gamma\chi(p, X)|$, except in the case $n = 2$.

QUESTION 4.6. Is $K(p, X) = [n]^1$, where $n = |\gamma\chi(p, X)|$, a sufficient condition for the analogue of Lemma 4.3 to hold at p ?

Essentially the same question may be asked as follows.

DEFINITION 4.7. Let $\gamma\chi(p, X) = \Omega = \{\kappa_i : i \in n\}$. An Ω -glob, B , at p is *ectomorphic* iff there are families $A_i = \{A_i(\alpha) : \alpha \in \kappa_i\}$, ($i \in n$) such that $B(\bar{\alpha}) = \cup\{A_i(\alpha_i) : i \in n\}$ for each $\bar{\alpha} \in \Pi\Omega$, and $\cap A_i = \{p\}$ for each $i \in n$.

It is easy to show that if B is ectomorphic, then $K_B = [n]^1$.

QUESTION 4.8. If $K(p, X) = [n]^1$ for some $n \in \omega$, is there an ectomorphic (nbhd) glob at p ?

(The answer is 'yes' if $n = 2$.)

An affirmative answer to Question 4.8 would of course imply an affirmative answer to Question 4.6.

On seeing an early draft of this paper van Douwen also suggested the following interesting examples.

DEFINITION 4.9. With the usual notation, a nbhd Ω -glob, B , at p is said to be *well-built* iff for each $\bar{\alpha} \in P$, $B(\bar{\alpha}) = \cap\{B(\bar{0}[i \rightarrow \alpha_i]) : i \in n\}$.

All the foregoing examples of globs are well-built, but the following example is not (in an essential way).

EXAMPLE 4.10. Let Ω , n , and P be as usual, with $n \geq 2$. Let p be any point not in P , and let $X = P \cup \{p\}$, topologized as follows: points of P are isolated, and there is an Ω -glob, B , at p defined by setting $B(\bar{\alpha}) = \{p\} \cup \{\bar{\beta} \in P : \exists i \in n (\beta_i \geq \alpha_i)\}$ for each $\bar{\alpha} \in P$. Clearly X is globular and T_4 . However, X admits no well-built nbhd glob at p .

To see this, note that for any $\bar{\alpha} \in P$, $\bar{\alpha} \in \cap B(\bar{\alpha})$, where $B(\bar{\alpha}) = \{\cap\{B(\bar{\alpha}[i \rightarrow \beta_i]) : i \in n\} : \bar{\beta} \geq \bar{\alpha}\}$. Thus, $B(\bar{\alpha})$ cannot be a nbhd base at p . Now apply the following lemma (due to van Douwen).

LEMMA 4.11. *Let B be a nbhd Ω -glob at p in X . (X here is any space.) For $\bar{\alpha} \in P$ define $B(\bar{\alpha})$ to be $\{\cap\{B(\bar{\alpha}[i \rightarrow \beta_i]) : i \in n\} : \bar{\beta} \geq \bar{\alpha}\}$. Then the following are equivalent:*

- (i) *there is a well-built nbhd Ω -glob at p ;*
- (ii) *$\{\bar{\alpha} \in P : B(\bar{\alpha}) \text{ is a nbhd base at } p\}$ is cofinal in P ; and*
- (iii) *$B(\bar{\alpha})$ is a nbhd base at p for some $\bar{\alpha} \in P$.*

PROOF. (i) \Rightarrow (ii). Let C be a well-built nbhd Ω -glob at p , and suppose that $\bar{\alpha} \in P$ is such that $B(\bar{\beta})$ is not a nbhd base at p for any $\bar{\beta} \geq \bar{\alpha}$. Let $K = \Pi\{K_i : i \in n\}$ be as in Theorem 3.4; we may assume that $\bar{\alpha} \leq \bar{\mu}$, where $\bar{\mu}$ is the \leq -minimum of K . Fix $\bar{\beta} \in K$ so that $\beta_i > \mu_i$ for each $i \in n$.

Since $\bar{\beta} \geq \bar{\alpha}$, there is a $\bar{\gamma} \in P$ such that $B \notin C(\bar{\gamma})$ for any $B \in B(\bar{\beta})$, and we may as well assume that $\bar{\beta} \leq \bar{\gamma} \in K$. Choose $\bar{\delta} \in K$ so that $\delta_i > \gamma_i$ for each $i \in n$. Then

$$\begin{aligned} C(\bar{\gamma}) &= \cap\{C(\bar{0}[i \rightarrow \gamma_i]) : i \in n\} \supseteq \cap\{C(\bar{\mu}[i \rightarrow \gamma_i]) : i \in n\} \\ &\supseteq \cap\{B(\bar{\beta}[i \rightarrow \delta_i]) : i \in n\}, \end{aligned}$$

an element of $B(\bar{\beta})$. This contradiction implies the desired result.

(ii) \Rightarrow (iii). This is trivial.

(iii) \Rightarrow (i). Suppose that $B(\bar{\alpha})$ is a nbhd base at p . Let $K = \{\bar{\beta} \in P : \bar{\beta} \geq \bar{\alpha}\}$. For $\bar{\beta} \in K$ define $C(\bar{\beta}) = \cap\{B(\bar{\alpha}[i \rightarrow \beta_i]) : i \in n\}$, and let $C = \{C(\bar{\beta}) : \bar{\beta} \in K\}$. Since $K \cong P$, C is clearly a nbhd Ω -glob at p , and the following computation shows that C is well-built. Fix $\bar{\beta} \in K$. Then

$$\begin{aligned} \cap_{i \in n} C(\bar{\alpha}[i \rightarrow \beta_i]) &= \cap_{i \in n} \cap_{j \in n} B(\bar{\alpha}[j \rightarrow (\bar{\alpha}[i \rightarrow \beta_i])_j]) \\ &= \cap_{i \in n} \cap_{j \in n} B(\bar{\alpha}[j \rightarrow \left\{ \begin{array}{ll} \alpha_j, & j \neq i \\ \beta_i, & j = i \end{array} \right\}]) \\ &= \cap_{i \in n} B(\bar{\alpha}[i \rightarrow \beta_i]) \\ &= C(\bar{\beta}). \quad \square \end{aligned}$$

Finally, there is a highly non-trivial globular space.

EXAMPLE 4.12. In [4] JÓNSSON constructed a compact, zero-dimensional linearly ordered topological space, X , with a dense subset, D , such that if $x, y \in D$, $z \in X \setminus D$, and $x \neq y$, then $\gamma_X(x, X)$, $\gamma_X(y, X)$, and $\gamma_X(z, X)$ are mutually distinct. (In fact, points of D have linearly ordered local bases of distinct, uncountable cofinalities, while points of $X \setminus D$ either have countable character or do not have linearly ordered local bases at all.)

I have modified Jónsson's construction somewhat to produce the following example. Though no longer zero-dimensional, it is, I think easier to visualize.

Let $\kappa_0 = \omega$, and, given ω_n for some $n \in \kappa$, let $\kappa_{n+1} = \omega_{\kappa_n}$. Let $\kappa = \sup\{\kappa_n : n \in \omega\}$. Let $F_0 = {}^1\kappa_0$, and, given F_n for some $n \in \omega$, let

$$F_{n+1} = \{f \in {}^{(n+2)}\kappa_{n+1} : f \upharpoonright (n+1) \in F_n \wedge f(n+1) < \lambda_n(f \upharpoonright (n+1))\},$$

where $\lambda_n : F_n \rightarrow \{\mu \in \kappa_{n+1} \setminus \kappa_n : \text{cf } \mu = \mu\}$ is any (fixed) injection. For each $n \in \omega$ let

$$A_n = \{a \in {}^\omega(\kappa+1) : a \upharpoonright (n+1) \in F_n \wedge \forall m \in \omega \setminus (n+1) (a(m) = \kappa)\},$$

and let $A = \bigcup\{A_n : n \in \omega\}$. For distinct $x = \langle x_i : i \in \omega \rangle$, $y = \langle y_i : i \in \omega \rangle \in A$, if $n = \inf\{i \in \omega : x_i \neq y_i\}$, write $x < y$ iff either: (1) $x_n < y_n$, and n is even; or (2) $x_n > y_n$, and n is odd. Then $\langle A, \langle \rangle \rangle$ is a linear order.

(The easiest way to understand $\langle A, \langle \rangle \rangle$ is to understand its suborders $\langle \bigcup\{A_i : i \in n\}, \langle \rangle \rangle$, of which it is essentially the direct limit. A_0 is just an increasing sequence. Assuming that $\lambda_0(\langle n \rangle) = \omega_{n+1}$ for each $n \in \omega$, $A_0 \cup A_1$ can be visualized as in Figure 1 below. Similarly, each point of A_1 is the limit from *below* of a transfinite sequence of elements of A_2 , each element of A_2 is the limit from *above* of a transfinite sequence of elements from A_3 , and so on.)

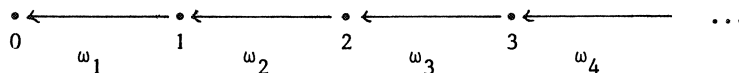


Figure 1

Now view $\langle A, \langle \rangle \rangle$ as a LOTS. It is clear that for each $n \in \omega$ and $x \in A_n$, $\chi(x, A) = \lambda_n(x \uparrow (n+1))$, so that distinct points of A have different characters and *a fortiori* different glob-characters. Let A^+ be the Dedekind compactification of A . It is not hard to see that the points of $A^+ \setminus A$ can be identified naturally with $\{\sigma \in {}^\omega \kappa : \forall n \in \omega (\sigma \uparrow (n+1) \in F_n)\} \cup \{\infty\}$, where $\infty = \langle \kappa, \kappa, \dots \rangle$; the definition of $\langle \rangle$ extends verbatim to a definition of the ordering of A^+ . Moreover, $\chi(x, A^+) = \omega$ for each $x \in A^+ \setminus A$. (Indeed, ∞ is the limit from below (above resp.) of $\{x^n : n \text{ is even (odd, resp.)}\}$, where x^n is the unique member of A_n such that $x^n \uparrow (n+1) = x \uparrow (n+1)$.)

Finally, far from being zero-dimensional, A^+ is connected, since A has no isolated points.

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ON A THEOREM OF Đ. KUREPA

by

Stevo Todorčević

We present a proof of the following theorem of D. Kurepa [8; Th. 8.1].

THEOREM. *For every regular cardinal $\kappa \geq \aleph_0$ there exists a κ -metrizable, non-linearly orderable topological space.*

1. INTRODUCTION

Let $\kappa = \aleph_\alpha$ be a regular cardinal ($\alpha \geq 0$). Call a topological space X a κ -metrizable space or a D_α -space iff there exist $\rho: X^2 \rightarrow \omega_\alpha \cup \{\omega_\alpha\}$ and $\phi: \omega_\alpha \rightarrow \omega_\alpha$ such that:

- (a) $\rho(x, y) = \omega_\alpha$ iff $x = y$;
- (b) $\rho(x, y) = \rho(y, x)$;
- (c) $\rho(x, y), \rho(y, z) > \phi(\xi)$ implies $\rho(x, z) > \xi$;
- (d) the sets $B_\xi(x) = \{y \in X \mid \rho(x, y) > \xi\}$, $x \in X$, $\xi < \omega_\alpha$ form a basis of X .

This definition was given by KUREPA [2] in 1934 using the name pseudo-distancial spaces. The class of all D_0 -spaces is just the class of all metrizable spaces by [6]. The class of all pseudo-distancial spaces was extensively considered by Kurepa, Fréchet, Doss, Colmez, Appert, Papić and others in the year's 40's and 50's. We refer the reader to [9; § 12] and especially to [7] for references until 1963. This class has also the name "spaces with linearly ordered basis of uniformity" (see [9; § 12, Th. 17]). We use the name from [12] where another equivalent definition is given.

Editor's note. Interested readers of this paper may wish to consult [M. Husek, Linearly Uniformizable Spaces, Report 119, Vrije Universiteit, Amsterdam, February 1980] in which the author also proves Kurepa's Theorem A (above) using a simplified version of Kurepa's original argument. In addition, that paper contains theorems which give necessary and sufficient conditions for orderability of any κ -metrizable space.

In this paper we present a proof of the following theorem of D. KUREPA [8; Th. 8.1].

THEOREM A. *For every regular cardinal $\kappa > \aleph_0$ there exists a κ -metrizable space which is not linearly orderable.*

Theorem A is a positive answer to Problème 8.2.1 from [7] after a general theorem about the linear orderability of pseudo-distancial spaces and R-spaces. (R-spaces, called also non-archimedean spaces, were defined by D. KUREPA [4] (see also [3 and 5]) and extensively considered by him and his student P. Papić in 1950's and 1960's; for references see [7 and 8] and [9; §12].) For example, a consequence of this theorem is

THEOREM B. (KUREPA [8 ; Th. 9.5(i)]). *If $\kappa > \aleph_0$ is a regular cardinal then every dense-in-itself κ -metrizable space is linearly orderable.*

This theorem of Kurepa is rediscovered in [1] and [11] ([11; Th. 6] is a special case of it). Let us also mention that in [1 ; p. 38], [11; Question p. 203] and [10; Problem 2.5] the authors ask whether every κ -metrizable space is linearly orderable (for κ regular $> \aleph_0$). The answer is negative by Kurepa's Theorem A.

2. THE CONSTRUCTION

Let $\kappa > \aleph_0$ be a fixed regular cardinal and let $\Omega = \{\delta < \kappa \mid \text{cf}(\delta) = \omega\}$. Let $\eta_\delta = \langle \eta_\delta(n) \mid n < \omega \rangle$ be a strictly increasing sequence of ordinals cofinal with δ , for each $\delta \in \Omega$. For $x \in {}^\kappa 2$ we define $\text{supp}(x) = \{\alpha < \kappa \mid x(\alpha) = 1\}$. For $\delta \in \Omega$ we define $p_\delta \in {}^\kappa 2$ by $\text{supp}(p_\delta) = \{\eta_\delta(n) \mid n < \omega\}$. Now, for each $S \subseteq \Omega$ we define $X(S) = \{p_\delta \mid \delta \in S\} \cup \{x \in {}^\kappa 2 \mid \text{supp}(x) \text{ is finite}\}$. Define $\rho: X(S) \times X(S) \rightarrow \kappa \cup \{\kappa\}$ by $\rho(x,x) = \kappa$ and $\rho(x,y) = \min\{\alpha \mid x(\alpha) \neq y(\alpha)\}$ for $x,y \in X(S)$, $x \neq y$. Then ρ is a " κ -metric" on $X(S)$ in the sense of Section 1 - it is enough to put $\phi = \text{id}$. We consider $X(S)$ as a topological space with the topology introduced by ρ . Now Theorem A follows from the next result.

THEOREM C. *$X(S)$ is linearly orderable iff S is non-stationary in κ .*

PROOF. Assume first that S is non-stationary in κ . Then the fact that $X(S)$ is linearly orderable can be deduced from Theorem 8.2.1(2) of [7]. Namely, using a club disjoint from S we can inductively refine the ramified basis of $X(S)$ to get another ramified basis T of $X(S)$ with the property that if

$B \in T$ has the limit height then B has infinitely many immediate successors in T . (Using this observation the reader can easily find a linear ordering of $X(S)$ which generates the topology on $X(S)$.)

Assume now that S is a stationary subset of Ω . We prove that $X(S)$ is not linearly orderable. Assume the contrary, i.e., that $X(S)$ is a LOTS under the ordering $<$. Since each p_δ , $\delta \in S$, is isolated in $X(S)$ we can define q_δ to be $\max\{x \in X(S) \mid x < p_\delta\}$ for $\delta \in S$. We need the following fact.

CLAIM. If $\langle x_\alpha \mid \alpha < \kappa \rangle$ is a convergent sequence in $X(S)$ then $\{\delta \in S \mid p_\delta \in \{x_\alpha \mid \alpha < \kappa\}\}$ is not stationary in κ .

PROOF. Assume that $S' \subseteq S$ is stationary and that $\langle p_\delta \mid \delta \in S' \rangle$ is a convergent sequence in $X(S)$ and then find a contradiction using the Pressing Down Lemma (PDL).

Now we are ready to consider the following two cases.

CASE 1. $\{\delta \in S \mid \text{supp}(q_\delta) \text{ is infinite}\}$ is stationary in κ .

For each $\delta \in S' := \{\delta \in S \mid \text{supp}(q_\delta) \text{ is infinite}\}$ there exist unique $f(\delta) \in S$ such that $q_\delta = p_{f(\delta)}$. Without loss of generality (using PDL) we can assume $f(\delta) > \delta$ for each $\delta \in S'$. Hence $\text{supp}(q_\delta) \cap \delta$ is finite for each $\delta \in S'$. Using PDL we can find stationary $S'' \subseteq S'$ and finite $F \subseteq \kappa$ such that $\text{supp}(q_\delta) \cap \delta = F$ for each $\delta \in S''$. Define $x \in {}^\kappa 2$ by $\text{supp}(x) = F$. Clearly, $x \in X(S)$ and $\langle q_\delta \mid \delta \in S'' \rangle$ converges to x . Since $X(S)$ is a LOTS by $<$ this implies that $\langle p_\delta \mid \delta \in S'' \rangle$ also converges to x contradicting the Claim.

CASE 2. $\{\delta \in S \mid \text{supp}(q_\delta) \text{ is finite}\}$ is stationary.

Using PDL we can find stationary $S'' \subseteq \{\delta \in S \mid \text{supp}(q_\delta) \text{ is finite}\}$ and finite $F \subseteq \kappa$ such that $\text{supp}(q_\delta) \cap \delta = F$ for each $\delta \in S''$. The rest is as in the Case 1.

This completes the proof of Theorem C.

REMARK. Further applications of the above construction are given in [13].

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CARDINAL FUNCTIONS ON LINEARLY ORDERED
TOPOLOGICAL SPACES

by

Stevo Todorčević

0.0 In what follows X denotes an infinite LOTS and $O(X)$, $(K(X))$ denotes the set of all open (convex) subsets of X . A collection $T \subseteq P(X)$ ($= \{Y \mid Y \subseteq X\}$) is a *tree* if: (1) $\emptyset \notin T$; (2) $u, v \in T \rightarrow (unv = \emptyset \vee u \subseteq v \vee v \subseteq u)$; (3) $\hat{u} = \{v \in T \mid v \not\subseteq u\}$ is well-ordered by \supseteq . If T is a tree and $u \in T$ then T^u denotes the tree $\{v \in T \mid v \subseteq u\}$. The notation is as in [1].

0.1 DEFINITION. $p_0(X) = \sup\{|Y| \mid Y \text{ is well-ordered or conversely well-ordered subset of } X\}$; $p(X) = \min\{\kappa \mid \kappa > |Y| \text{ for every well-ordered or conversely well-ordered subset } Y \text{ of } X\}$.

It is easy to see that if $b \subseteq K(X)$ is a chain then $|b| \leq p_0(X) \leq c(X)$.

1.0 PROPOSITION. If $T \subseteq O(X) \cap K(X)$ is a tree then $|T| \leq \min\{c(X)^+, c(X^2)\}$.

PROOF. Let $T_\alpha = \{u \in T \mid \text{tp}(\hat{u}, \supseteq) = \alpha\}$. Then $T = \bigcup\{T_\alpha \mid \alpha < \gamma T\}$ where $\gamma T = \min\{\alpha \mid T_\alpha = \emptyset\}$. By 0.1, $\gamma T \leq c(X)^+$. So $|T| \leq \sum\{|T_\alpha| \mid \alpha < \gamma T\} \leq c(X) \cdot c(X)^+ = c(X)^+$ since T_α is a disjoint subfamily of $O(X)$. For $u \in T_\alpha$ define $\text{succ}(u) = \{v \in T_{\alpha+1} \mid v \subseteq u\}$. Let $T' = \{u \in T \mid |\text{succ}(u)| \geq 2\}$, $T'' = \{\hat{u} \cup \{u\} \mid u \in T'\}$, $R = T - T''$ and $R_0 =$ the set of all \supseteq -minimal elements of R . Then $T = T'' \cup \bigcup\{T^u \mid u \in R_0\}$, so $|T| \leq |T''| + \sum\{|T^u| \mid u \in R_0\} \leq |T'| \cdot c(X) + c(X) \cdot c(X)$ by 0.1. For every $u \in T'$ choose $v_0(u), v_1(u) \in \text{succ}(u)$, $v_0(u) \neq v_1(u)$. It is easy to check that $u \neq u'$ implies $(v_0(u) \times v_1(u)) \cap (v_0(u') \times v_1(u')) = \emptyset$. So $|T'| \leq c(X^2)$ and $|T| \leq c(X^2) \cdot c(X) + c(X) = c(X^2)$. \square

Editor's Note: The relationships between cardinal functions on linearly ordered spaces have been rediscovered many times by other authors, e.g., [Bennett and Lutzer, Separability, the countable chain condition and the Lindelöf property in linearly ordered spaces, Proc. Amer. Math. Soc. 23 (1969), 664-667] and [van Emde Boas, Krconenberg, van der Slot, and Verbeek, Cardinal functions on ordered spaces, Math. Centre Report ZN 33/70, Amsterdam, 1970].

1.1 PROPOSITION. $\text{hd}(X) \leq \min\{c(X)^+, c(X^2)\}$.

PROOF. Let U_α , $\alpha < \alpha_0$ be a strictly decreasing sequence from $0(X)$. It is enough to prove $|\alpha_0| \leq \{c(X)^+, c(X^2)\}$. For $\alpha < \alpha_0$ let K_α be the family of all convex components of the open set U_α . Let $T = \cup\{K_\alpha \mid \alpha < \alpha_0\}$. Then $T \subseteq 0(X) \cap K(X)$ is a tree. Fix $x_\alpha \in U_\alpha - U_{\alpha+1}$ for every $\alpha < \alpha_0 - 1$. So there exist $u_\alpha \in K_\alpha$ such that $x_\alpha \in u_\alpha$. Clearly $u_\alpha \neq u_\beta$ for $\alpha \neq \beta$, $\alpha, \beta < \alpha_0 - 1$. Hence $|\alpha_0| \leq |T| + \aleph_0 \leq \min\{c(X)^+, c(X^2)\}$ by 1.0. \square

1.2 PROPOSITION. $\text{hl}(X) = c(X)$.

PROOF. Let U_α , $\alpha < \alpha_0$ be a strictly increasing sequence from $0(X)$. It is enough to prove $|\alpha_0| \leq c(X)$. Again let K_α be the family of all convex components of U_α and let $P = \cup\{K_\alpha \mid \alpha < \alpha_0\}$. Then (P, \subseteq) is a well founded poset and so there exists $R_0 =$ the set of all minimal elements in (P, \subseteq) . Clearly R_0 is a disjoint subfamily of $0(X)$. For every $u \in R_0$ choose a maximal chain $b(u)$ of (P, \subseteq) such that $u \in b(u)$. Then $P = \cup\{b(u) \mid u \in R_0\}$ and so $|P| \leq \sum\{|b(u)| \mid u \in R_0\} \leq c(X) \cdot c(X)$ by 0.1. As in 1.1 we can prove $|\alpha_0| \leq |P| + \aleph_0$, so the proof is complete. \square

1.3 PROPOSITION. $|X| \leq 2^{\mathcal{P}(X)}$.

PROOF. Let $\mathcal{T}_2(X)$ be the set of all binary trees T (i.e. $|\text{succ}(u)| \leq 2$, for every $u \in T$) such that $X \in T \subseteq K(X)$. Define \leq on $\mathcal{T}_2(X)$ by: $T \leq T'$ iff T is a \subseteq -final part of T' . Clearly in $(\mathcal{T}_2(X), \leq)$ every chain has an upper bound, so there exists a maximal element T of $(\mathcal{T}_2(X), \leq)$. By 0.1, $\gamma T \leq p(X)$. By the maximality of T we have $\{x\} \in T$ for every $x \in X$. So $|X| \leq |T| \leq 2^{\mathcal{P}(X)}$ since T is a binary tree. \square

2.0 REMARK. The relations $\psi(X) = \chi(X) \leq p_0(X) \leq c(X) \neq \text{hc}(X) = \text{hl}(X) \leq c(X^2) = d(X) = \text{hd}(X) \leq c(X)^+$ immediately follow from 1.1 and 1.2 and $|X| \leq 2^{c(X)}$ follows from 1.3 since $2^{\mathcal{P}(X)} \leq 2^{p_0(X)} \leq 2^{c(X)}$.

2.1 REMARK. The inequality $d(X) \leq c(X)^+$ was first proved in [2; §12.C]. The function $c(X)$ (for X a topological space) was first defined in the same paper. The identities $\text{hd}(X) = d(X)$ and $\text{hl}(X) = c(X)$ were proved in [3; Th. 11 and 12] (see also [4]). The inequality $d(X) \leq c(X^2)$ was proved in [5] (see also [6]) and $|X| \leq 2^{p_0(X)}$ was proved in [7] but this easily follows from an earlier result of Hausdorff on the existence of an $\eta_{\xi+1}$ set of powers 2^{\aleph_η} (see [8]). The definition of $p_0(X)$ and another proof of this relation

were given in [2].

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POSED PROBLEMS

Workshop participants, and others, were invited to submit problems on ordered spaces for discussion and for inclusion in the Workshop proceedings. Problems marked with an asterisk have been (at least partially) solved, sometimes in papers included in this volume, and the proceedings of the 1981 Workshop will contain a discussion of the status of the problems [3]. The name of the poser of the problem is included in parentheses.

*1. (van Douwen, attributed to E. Michael). Suppose X is a compact Hausdorff space which admits a continuous mapping $s: 2^X \rightarrow X$, where 2^X is the Vietoris hyperspace of nonempty closed subsets of X , such that $s(F) \in F$ for each $F \in 2^X$. Must X be orderable? (Yes; [5].)

2. (Purisch). Suppose X is a separable, compact, zero-dimensional monotonically normal space. Must X be orderable?

*3. (Lutzer). Find ways of showing that a given GO -space is *not* orderable. The "classical" approach is to discover a theorem that is true for every orderable space and then to observe that the theorem fails for the given GO -space; hence the GO -space is not orderable. Two such theorems are Lutzer's result that a LOTS with a G_δ -diagonal must be metrizable, and van Wouwe's theorem that a LOTS with a σ -discrete dense subset must be a paracompact p -space, but many examples cannot be decided by these results. (Cf. [2], [9], [6].)

*4. (van Douwen and Lutzer). Is it true that every GO -space has a dense orderable subspace? (Yes; [8].)

5. (Williams). Suppose X_0 and X_1 are co-absolute LOTS. Does it follow that X_1 must contain a dense subspace D_1 such that D_0 and D_1 are homeomorphic? The answer is "yes" if both X_i are connected [8].

6. (Williams). Assume the Continuum Hypothesis and suppose that X is a paracompact, locally compact, non-compact LOTS. Must $\beta X - X$ have a dense, orderable subspace? [8]

7. (Williams). Assume that X^{ω} , the product of countably many copies of the $T_{3,5}$ space X , contains a dense, orderable subspace. Does it follow that X also contains a dense, orderable subspace? [8] (Cf. "Added in proof", below.)

8. (Lutzer). Suppose that X is a perfect (= closed sets are G_δ) suborderable space. Does there exist a perfect orderable space Y in which X embeds as a dense subspace? The answer is "yes" if X has countable cellularity.
9. (Meyer). Suppose that T is a suborderable topology on X . Is T the join of two orderable topologies on X , i.e., do there exist orderable topologies S_1 and S_2 on X such that the collection $S_1 \cup S_2$ is a subbase for T ? The answer is "yes" for the Sorgenfrey line and other partial results are discussed in [4].
- *10. (van Douwen). For $i = 1, 2$ and for any space X , define T_i -psw(X) to be the least cardinal κ such that there is a topology S on X such that (X, S) is a T_i -space having weight κ and $S \subset T$. Assuming that (X, T) is orderable, is it true that T_1 -psw(X) = T_2 -psw(X)? (Yes; cf. [3, Th. 22]. This result is due to B. Scott.)
11. (Maurice and van Wouwe). In ZFC, is there an example of a perfect orderable space which does not have a σ -discrete dense subset? Equivalently, is there a perfect orderable space which does not have a dense metrizable subspace? [9]
12. (Bennett and Lutzer). Suppose each (closed) subspace of a (sub)orderable space X has a σ -minimal base for its topology. Must X be quasi-developable? If X is a compact LOTS whose every subspace has a σ -minimal base, must X be metrizable? [1]
- *13. (van Douwen). Suppose X is a hereditarily paracompact GO-space. Can X be embedded in a GO-space having a σ -minimal base? (No; cf. [3].)
14. (van Wouwe). Suppose X is a (sub)orderable space and suppose each subspace of X is a Σ -space in the sense of Nagami. Must X be metrizable? An equivalent question is: suppose X is a Lindelöf suborderable space and every subspace of X is a Σ -space. Must X be hereditarily Lindelöf? [9]
15. (Lutzer). Suppose X is a compact LOTS and that for any subspace Y of X , the space Y^ω is paracompact. Is X metrizable? What if we assume the Continuum Hypothesis?

16. (van Douwen). Suppose X is a compact LOTS having no isolated points. Does there exist a set $B \subset X$ such that both B and $X - B$ meet every nonvoid closed subset of X which, in its relative topology, has no isolated points?
17. (Mardešić and Papić). Suppose that a compact, connected, locally connected Hausdorff space Y is known to be the continuous image of some compact LOTS. Must Y be the continuous image of some compact, *connected* LOTS?
18. (Treybig and Ward). We say that a space X can be approximated by finite trees if there is a collection T of trees, each with only finitely many endpoints, such that: (a) T is directed by inclusion; (b) $\cup T$ is a dense subspace of X ; (c) given any open cover U of X , some $T = T(U) \in T$ has the property that whenever $T \subset S \in T$ and C is a component of $S - T$, then some member of U contains C . Ward [7] has proved that a space X is the continuous image of some compact, connected LOTS if X can be approximated by finite trees; is that condition also a necessary condition?
19. (Mardešić). Suppose Y is a connected, locally connected, compact Hausdorff space. Is it true that given $p, q \in Y$, there is a compact, connected, *orderable* subspace of Y containing both p and q ?
20. (Treybig and Ward). Characterize all spaces Y which are images of the unit interval $[0,1]$ under continuous, *irreducible* mappings, i.e., under a mapping $f: [0,1] \rightarrow Y$ with the property that $f[C] \neq Y$ whenever C is a proper closed subset of $[0,1]$.
- *21. (Treybig). Suppose X is a compact, connected LOTS which is homeomorphic to each of its non-degenerate closed subintervals. Is there an order-reversing homeomorphism $h: X \rightarrow X$? (Consistently, no; cf. [3].)
22. (Treybig). Let X be a compact, connected LOTS and let Y be a Hausdorff space. We say that a continuous surjection $f: X \rightarrow Y$ has finite oscillation at local separating points of Y provided that whenever U is open in Y and $p \in U$ has the property that $U - \{p\}$ is the union of two mutually separated sets R and S , then there is a finite collection G of open subintervals of X which covers the set $f^{-1}[R \cup S]$ and has the property that no member of G meets both $f^{-1}[R]$ and $f^{-1}[S]$. Now suppose that a compact, connected Hausdorff space Y is known to be the continuous image of some compact, connected LOTS, and that no point of Y separates Y . Must Y be the image of some compact, connected LOTS under a mapping which has finite oscillation at local separating

points of Y ?

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ADDED IN PROOF. After completing this problem-list, I received a letter from P. Simon (Prague) announcing a negative solution of Problem 7. D.J.L.

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