



Centrum voor Wiskunde en Informatica

REPORTRAPPORT

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Modelling, Analysis and Simulation (MAS)

MAS-R0024 September 30, 2000

Report MAS-R0024
ISSN 1386-3703

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CWI is the National Research Institute for Mathematics and Computer Science. CWI is part of the Stichting Mathematisch Centrum (SMC), the Dutch foundation for promotion of mathematics and computer science and their applications.

SMC is sponsored by the Netherlands Organization for Scientific Research (NWO). CWI is a member of ERCIM, the European Research Consortium for Informatics and Mathematics.

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ABSTRACT

In this article we present a new approach to the numerical valuation of derivative securities. The method is based on our previous work where we formulated the theory of pricing in terms of tradables. The basic idea is to fit a finite difference scheme to exact solutions of the pricing PDE. This can be done in a very elegant way, due to the fact that in our tradable based formulation there appear no drift terms in the PDE. We construct a mixed scheme based on this idea and apply it to price various types of arithmetic Asian options, as well as plain vanilla options (both european and american style) on stocks paying known cash dividends. We find prices which are accurate to $\sim 0.1\%$ in about 10ms on a Pentium 233MHz computer and to $\sim 0.001\%$ in a second. The scheme can also be used for market conform pricing, by fitting it to observed option prices.

2000 Mathematics Subject Classification: 91B24, 60H10, 58J35, 58J70.

Keywords and Phrases: contingent claim pricing, scale-invariance, homogeneity, partial differential equation.

Note: Work carried out under project MAS3.1 "Mathematical finance". The research of C.D.D. Neumann was partially supported by the SWON-program "Financial derivatives".

1. INTRODUCTION

One of the most popular methods for pricing (exotic) derivative securities is to make use of finite difference schemes to solve the PDE associated with the pricing problem. In a world where prices are driven by Wiener processes such PDE's are usually of generalized diffusion type

$$(\partial_t + \alpha(x, t) + \beta(x, t)\partial_x + \gamma(x, t)\partial_x^2) V(x, t) = 0$$

In such an equation the terms containing α, β and γ will be called the constant, drift and diffusion term respectively. In the traditional formulation of option pricing theory, the PDE to be solved generally contains not only diffusion terms but also non-trivial constant and drift terms. The most obvious example is of course the Black-Scholes PDE:

$$\left(\partial_t - r + rS\partial_S + \frac{1}{2}\sigma^2 S^2 \partial_S^2 \right) V(S, t) = 0$$

It is well known that exactly the presence of drift terms in the PDE can lead to severe numerical problems, unless they are handled with much care.

In this article we present a new approach to the numerical valuation of derivative securities. The approach is based on our previous work [HN99a, HN99b, HN00a] where we introduced a new formalism for contingent claim pricing. The core of this formalism is the idea that problems should be formulated in terms of *tradable objects only*. (Note that we use a broad definition of the term tradable: every quantity that can be represented by a self-financing portfolio is considered to be a tradable). From a numerical point of view, the formalism has the distinct advantage that pricing PDE's never contain drift nor constant terms. This makes it a very natural starting point for the construction of finite difference schemes for pricing derivative securities.

Our key idea is to require that a given set of solutions of the underlying PDE should also be an exact solution of the numerical scheme. We consider the case where there are two underlying tradables in the pricing problem, leading to a one dimensional PDE. It then turns out that we only need three exact solutions to fix a fully implicit (or explicit) scheme. By combining the fully implicit and fully

explicit schemes into a Crank-Nicolson type scheme, we obtain a very powerful mixed scheme, still fitted to the exact solutions. Details on convergence and stability properties will be treated in future work.

So how do we fix the scheme? A very natural choice is to use the two tradables themselves as exact solutions of the PDE. So we need only one more non-trivial solution of the PDE. Such a solution is in general not too hard to obtain, as we show in this article. Note that instead of using exact solutions we could also use observed market prices to fix the scheme, generalizing in a natural way the concept of implied trees of Ref. [DK94]. The latter idea will be explored in future work.

In this article we present results on the performance of our scheme when applied to the pricing of a well known problematic case: the arithmetic Asian option, and furthermore results on the pricing of plain vanilla options on stocks paying cash dividends, which are related to arithmetic Asians by a symmetry operation (see Ref. [HN00a]). Note however that these applications only serve as an example of the power of the our scheme. The scheme itself can be used in a much broader class of pricing problems.

Let us recall some other approaches to the problem of pricing arithmetic Asians which can be found in the literature. To start with, there are the (semi-)analytical approaches. The first result in this direction was given in Ref. [GY93], who derived a closed form expression for the Laplace-transform of the price of a continuously sampled average price call. In our paper [HN00a] we provided a similar solution for the price of an arithmetic average strike put. Exploiting the various symmetries underlying the pricing of contingent claims, we furthermore showed that unseasoned arithmetic average price and strike options are closely related objects, which can be transformed into each other by a suitable substitution of parameters. However, to find actual prices, one needs to calculate an inverse Laplace transform. This turns out to be a hard problem, which requires a substantial amount of numerical work [FMW98] and makes the method less suitable for practical purposes.

For this reason, people usually revert to approximative or numerical methods instead. These methods can be roughly classified in three categories. The first category consists of analytical approximations. In Ref. [TW91] the first two moments of the distribution of the arithmetic average are fitted to a lognormal distribution. This leads to a Black-Scholes type pricing formula. Alternatively, in Refs. [Cur94, RS95, Tho99] quite accurate upper- and lower bounds on prices are derived. These methods can be very fast. Disadvantages are that they have a fixed accuracy and they can not easily be adapted to handle discretely sampled Asians and early exercise features.

The second category consists of methods which use the path integral or Feynman-Kac formulation of the problem as a starting point. A path integral can easily be evaluated using a Monte Carlo approach [KV92]. Just generate a large number of paths and take averages over the payoffs associated with every individual path. This method is versatile, simple to use and easy to implement. Main disadvantages are that convergence is slow, and the incorporation of early exercise features is notably hard. To speed up the calculation, one can make use of a proper control-variate to reduce the variance. In the case of arithmetic Asians, the obvious choice of a control variate is a geometric Asian for which closed form formulas do exist. Another disadvantage of Monte Carlo simulation is that paths are sampled discretely. Therefore the method inherently makes a discretization error when calculating continuously sampled Asians.

The third category consists of methods which numerically solve the PDE associated to the pricing problem. This approach is flexible and early exercise features can be incorporated in a straightforward manner. For arithmetic Asians there exist various forms of this PDE (e.g. Refs. [Ing87, RS95, Zha99, HN99b]). In principle all these forms can be related to each other by suitable changes of variables. However, to the best of our knowledge, our formulation of pricing theory in terms of tradables is the first and only one which leads in a natural and straightforward way to a PDE without drift and constant terms. Other approaches lead to PDE's which are plagued by the presence of drift terms. The main problem associated to the appearance of a drift term is that for certain parameter ranges it tends to dominate the diffusion term. The PDE then becomes of hyperbolic type instead of parabolic type. If such a problem is attacked using standard techniques for solving parabolic

(diffusion) equations, one finds spurious oscillations in the numerical solution. Similar problems are well known in the field of computational fluid dynamics, where they are known as boundary layer problems. It is this field in which methods were developed to circumvent the problem. One approach is to make use of so-called flux-delimiters. Their use in the pricing of arithmetic Asians was introduced in Ref. [ZFV97]. Unfortunately, their scheme is rather complicated to implement as it uses a non-uniform grid. Another approach is to use so-called fitted schemes [Duf00, Mor96]. Here the idea is to construct a finite difference scheme in such a way that stationary (time-independent) solutions of the underlying PDE are exact solutions of the numerical scheme.

Of course, boundary layer problems can be trivially avoided if drift terms are absent. The elegance of our idea lies in the fact that it is based on a formulation which accomplishes just this. This is a consequence of using the proper coordinates: tradables.

The outline of this article is as follows. Section 2 gives some background and motivation. In section 3 the construction of our finite difference scheme is described. In section 4 we apply the scheme to the PDE for Asian type options as well as vanilla options on stock paying cash dividend. Section 5 provides numerical results for both type of options. In section 6 we conclude and give an outlook.

2. MOTIVATION

In previous work [HN99a, HN99b, HN00a], we have shown that in a market with two tradables, say S and P , where price processes are driven by Brownian motions, the pricing PDE can be reduced to the form

$$\left(-\partial_\tau + \frac{1}{2}\sigma(x, \tau)^2 x^2 \partial_x^2\right) V(x, \tau) = 0$$

where P was chosen as numeraire, τ is time to maturity, $x \equiv S/P$ and the price is $PV(x, \tau)$. Note the absence of a drift term. Due to this fact, the PDE has the trivial solutions x and 1 , corresponding to the basic tradables S and P respectively. So the pricing problem is fixed by knowledge of $\sigma(x, \tau)$. Alternatively, suppose we know one more solution of this equation, one that has a non-vanishing second derivative with respect to x (gamma), call it $R(x, \tau)$. Then we can write

$$\frac{1}{2}\sigma(x, \tau)^2 x^2 = \frac{\partial_\tau R(x, \tau)}{\partial_x^2 R(x, \tau)}$$

In other words, the knowledge of the price of one sufficiently well-behaved derivative security also fixes the model. Let us consider as an example the Black-Scholes context. It is easy to construct a tradable with payoff x^2 at time $\tau = 0$ (We will call it a quadratic tradable. It is a special case of a power tradable. See Ref. [HN99b]). It is given by

$$x^2 e^{\sigma^2 \tau}$$

Plugging this into the formula we find

$$\sigma(x, t) = \sigma$$

as expected. Note that if we use observed option prices of, say, call options, taking derivatives w.r.t. expiration time and strike, we recover the concept of local volatility, the basic idea behind implied trees [DK94].

3. FINITE DIFFERENCES

We can use the same line of thought to construct very efficient finite difference schemes. The idea is to construct a scheme in such a way that it has the underlying tradables as well as one given derived tradable as exact solutions. We will now show that these conditions uniquely fix a fully implicit scheme. Indeed, a fully implicit scheme is defined by the relations

$$a\delta_x^+ \delta_x^- V(x, \tau) + b\delta_x^0 V(x, \tau) + cV(x, \tau) = d\delta_\tau^- V(x, \tau)$$

where

$$\delta_x^+ \delta_x^- V(x, \tau) \equiv \frac{V(x + \delta x, \tau) - 2V(x, \tau) + V(x - \delta x, \tau)}{\delta x^2}$$

$$\delta_x^0 V(x, \tau) \equiv \frac{V(x + \delta x, \tau) - V(x - \delta x, \tau)}{2\delta x}$$

$$\delta_\tau^- V(x, \tau) \equiv \frac{V(x, \tau) - V(x, \tau - \delta\tau)}{\delta\tau}$$

and a, b, c, d are functions of x and τ . Now demanding that $V = x$ and $V = 1$ are solutions of the scheme corresponds to setting $b = 0, c = 0$. Next, demanding that a given tradable $R(x, \tau)$ is also a solution leads to

$$a\delta_x^+ \delta_x^- R(x, \tau) = d\delta_\tau^- R(x, \tau)$$

Therefore, the fully implicit scheme, implied by the tradables is given by

$$\Theta^-(x, \tau) \delta_x^+ \delta_x^- V(x, \tau) = \Gamma(x, \tau) \delta_\tau^- V(x, \tau) \quad (3.1)$$

where

$$\Theta^-(x, \tau) \equiv \delta_\tau^- R(x, \tau)$$

$$\Gamma(x, \tau) \equiv \delta_x^+ \delta_x^- R(x, \tau)$$

A remarkably simple scheme, by virtue of the fact that in our formulation of the pricing problem there are no drift terms in the PDE. Of course we can also derive an explicit scheme along the same lines. We find

$$\Theta^+(x, \tau) \delta_x^+ \delta_x^- V(x, \tau) = \Gamma(x, \tau) \delta_\tau^+ V(x, \tau) \quad (3.2)$$

where now

$$\delta_\tau^+ V(x, \tau) \equiv \frac{V(x, \tau + \delta\tau) - V(x, \tau)}{\delta\tau}$$

$$\Theta^+(x, \tau) \equiv \delta_\tau^+ R(x, \tau)$$

To further improve the accuracy of the scheme, we have combined the two schemes into a Crank-Nicolson type of scheme, i.e. we take the sum of Eqs. 3.1, 3.2, taking care of a proper time shift in one of them.

4. APPLICATION TO A SPECIFIC PDE

A very interesting application of the scheme is in finding solutions of the PDE

$$(-\partial_\tau + \frac{1}{2}\sigma^2(x - \phi(\tau))^2 \partial_x^2) V(x, \tau) = 0 \quad (4.1)$$

where $\phi(\tau)$ is a deterministic function. As was shown in our articles [HN99b, HN00a], this PDE can be used to price arithmetic Asian options as well as plain vanilla options on stocks paying cash dividends.

Let us briefly recall how the PDE is related to such options. To price an arithmetic average strike put (unseasoned) with a payoff defined by

$$\left(\int_0^\tau w(u) \frac{S(u)}{P(u)} du - kS \right)^+ \quad (4.2)$$

we set

$$\phi(\tau) = \int_0^\tau w(u) du$$

and then solve the PDE with boundary condition

$$V(x, 0) = (x - k)^+$$

The price then follows from

$$\text{Price} = SV(\phi(\tau), \tau)$$

The PDE can also be used to price average price options (using time reversal duality) and seasoned Asians in general. Details can be found in Ref. [HN00a]. To price a vanilla option (say a call) with strike K on a stock paying a stream of cash dividends with density $w(\tau)$, expressed in units of the bond, we set

$$\phi(\tau) = - \int_0^\tau w(u) du$$

(notice the sign) and then solve the PDE with boundary condition

$$V(x, 0) = (x - K)^+$$

The price then follows from

$$\text{Price} = PV(S/P + \phi(\tau), \tau)$$

Note that if dividend payments are discrete (or for the Asian, sampling is discrete), $w(\tau)$ becomes a weighted sum of Dirac delta-functions, and $\phi(\tau)$ becomes a piecewise constant function, making jumps at dividend (sample) dates.

Now the key question is: can we find exact solutions to the PDE, simple enough, which can be used to set up our implied mixed scheme? Fortunately, the answer is yes. Again the idea is to use a quadratic tradable, equalling x^2 at $\tau = 0$. This is a good choice for two reasons. Firstly it is the simplest non-trivial tradable. Secondly it has an x -independent gamma, which is convenient for numerical reasons. Assuming that the solution remains a quadratic polynomial in x we try a solution of the form

$$R(x, \tau) = A(\tau)x^2 + B(\tau)x + C(\tau)$$

Plugging this into Eq. 4.1, we find the following relations for the functions A, B, C :

$$\partial_\tau A(\tau) = \sigma^2 A(\tau)$$

$$\partial_\tau B(\tau) = -2\sigma^2 \phi(\tau) A(\tau)$$

$$\partial_\tau C(\tau) = \sigma^2 \phi(\tau)^2 A(\tau)$$

and they have the following solution

$$A(\tau) = e^{\sigma^2 \tau}$$

$$B(\tau) = -2\sigma^2 \int_0^\tau \phi(u) e^{\sigma^2 u} du$$

$$C(\tau) = \sigma^2 \int_0^\tau \phi(u)^2 e^{\sigma^2 u} du$$

These equations can easily be solved exactly, both in the case of discrete sampling (where ϕ is piecewise constant) and continuous sampling. For example, if we have a continuous exponential weight function $w(\tau) = e^{-r\tau}/T$, we have

$$\phi(\tau) = \frac{1 - e^{-r\tau}}{rT}$$

and

$$B(\tau) = \frac{2}{T(r - \sigma^2)} - \frac{2e^{\sigma^2 \tau}}{rT} \left(1 + \frac{e^{-r\tau} \sigma^2}{r - \sigma^2} \right)$$

$$C(\tau) = -\frac{2}{T^2(r - \sigma^2)(2r - \sigma^2)} + \frac{e^{\sigma^2 \tau}}{r^2 T^2} \left(1 + \frac{2e^{-r\tau} \sigma^2}{r - \sigma^2} - \frac{e^{-2r\tau} \sigma^2}{2r - \sigma^2} \right)$$

5. RESULTS

We have implemented the mixed scheme in C++. The numerical tests were performed on a Pentium 233MHz machine. We have considered various Asian options as well as options on stocks paying cash dividends. For each option, we have compared results for three different grid sizes. The results coming from the largest grid are so much more accurate than the ones that we could find in the literature, that we have used this as benchmark. We claim that these results are accurate to at least six significant digits. The results compare well with numbers recently obtained in Ref. [Zha99]. The gridsizes that we have used are summarized in the table.

Name	$N_x N_\tau$
“Exact”	10^8
1 sec	10^6
10 ms	10^4

The time labels of course indicate the execution speed of the algorithm on our computer. The total number of points on the grid were divided over the x and τ direction in such a way that $N_\tau/N_x \sim \sigma^2 T$. In all cases, care was taken that non-differentiable points on the payoff (at the strike) fell on gridpoints. The boundary conditions in the τ -direction were placed at zero and twice the strike. They are taken to be time-independent (since gamma is very small there). So for a payoff of the form

$$V(x, 0) = (x - k)^+$$

we also imposed

$$V(0, \tau) = 0, \quad V(2k, \tau) = k$$

Note that the scheme can be improved on this point: for high volatility it might be necessary to place the bounds further away from the strike, while for low volatility the scheme can be improved by moving them closer. This leads to a speedup of at least a factor 5 for the lowest volatilities.

The first instrument we consider is a standard continuously sampled average strike Asian put option. This corresponds to the choice of weight function $w(\tau) = e^{-r\tau}/T$ (see Eq. 4.2 with $k = 1$). We use the parameters values used in Ref. [RS95]: initial stockprice $S = 100$ and expiry time is 1 year. Note that except for the lowest volatility setting, the 1 second results are practically equal to the exact ones. As was mentioned before, the results can be improved by using tighter boundary conditions for low volatility.

σ	r	Exact	1 sec	10 ms
0.1	0.05	1.24546	1.24546	1.24617
	0.09	.699292	.699311	.701360
	0.15	.251676	.251712	.255256
0.2	0.05	3.40481	3.40481	3.40500
	0.09	2.62202	2.62202	2.62224
	0.15	1.71019	1.71021	1.71193
0.3	0.05	5.62603	5.62602	5.62592
	0.09	4.73955	4.73954	4.73940
	0.15	3.60981	3.60981	3.60985

Table 1: Continuously sampled average strike put

To see how the scheme performs in a discrete sampling setting, we next consider a discretely sampled average strike Asian put. Parameters settings are $S = 100$, $\sigma = 0.2$, $r = 0.1$ and expiry time is 1 year. There are N sample dates, all with weight $1/N$ in terms of the money value of the stock and taken at the beginning of each of the N equal length time periods in which the year is divided. This corresponds to a weight function

$$w(\tau) = \sum_{i=1}^N \frac{e^{-r\tau}}{N} \delta\left(\tau - \frac{i}{N}\right)$$

Note that for $N = 1$ the option is essentially a plain vanilla put with strike 100. For $N \rightarrow \infty$, the option becomes a continuously sampled Asian. It must be noted that in most occasions sample dates do not correspond to gridpoints in the time direction. In some cases there are many more gridpoints than there are samples, in other cases the reverse is true. Obviously, the scheme handles both cases very well. This is due to the fact that the scheme is fitted to an exact solution of the PDE.

N	Exact	1 sec	10 ms
1	3.75342	3.75342	3.75334
2	3.12047	3.12047	3.12062
4	2.79055	2.79055	2.79106
8	2.62151	2.62151	2.62217
16	2.53578	2.53578	2.53645
32	2.49257	2.49257	2.49324
64	2.47088	2.47088	2.47153
128	2.46001	2.46001	2.46065
256	2.45456	2.45457	2.45521
512	2.45184	2.45185	2.45249
1024	2.45048	2.45048	2.45112
∞	2.44912	2.44912	2.44976

Table 2: Discretely sampled average strike put

Since prices of average price Asians can be related to prices of average strike Asians by T(ime-reversal) duality [HN00a], it is not really necessary to give results for average price Asians. But in order to allow the reader to compare our result to those in the literature, we will give the results anyway. We consider standard continuously sampled average price calls with parameter settings $S = 100$, $r = 0.09$ and expiry in 1 year.

σ	K	Exact	1 sec	10 ms
0.05	95	8.80884	8.80884	8.80895
	100	4.30823	4.30825	4.31013
	105	.958384	.958385	.958493
0.1	95	8.91185	8.91189	8.91538
	100	4.91512	4.91515	4.91802
	105	2.07006	2.07006	2.07022
0.3	90	14.9840	14.9841	14.9951
	100	8.82876	8.82880	8.83275
	110	4.69671	4.69670	4.69640
0.5	90	18.1886	18.1888	18.2055
	100	13.0281	13.0282	13.0357
	110	9.12429	9.12429	9.12454

Table 3: Continuously sampled average price call

As a final example we consider plain vanilla options on a stock paying cash dividend. We consider a stock with initial value $S = 100$ and volatility $\sigma = 0.1$. The stock will pay a cash dividend of 5 after half a year. Interest rate $r = 0.05$. We consider a put and a call option on this stock expiring in 1 year. In this case we not only look at European style but also at American style options. To calculate the latter we take the maximum of the early exercise payoff and the calculated option value at each timestep. For a call, the early exercise payoff takes the form

$$(x - \phi(\tau) - Ke^{r\tau})^+$$

in our coordinates. (Note that we can also use this technique to value American style average strike Asians). The results show that convergence is slower in the American case. This can be explained by the fact that the early exercise feature is only enforced at gridpoints, in effect approximating the American with a Bermudan option. Therefore the value is systematically underestimated. Of course this is a general problem when evaluating American options with finite difference schemes. Nevertheless, the accuracy of our scheme is still of order 0.5% for such options in the 10 ms gridsize.

Call	European			American		
K	Exact	1 sec	10 ms	Exact	1 sec	10 ms
95	6.63807	6.63808	6.63910	8.00342	8.00405	7.97394
100	3.89199	3.89199	3.89207	4.55708	4.55755	4.52858
105	2.05220	2.05220	2.05248	2.28620	2.28646	2.27272

Put	European			American		
K	Exact	1 sec	10 ms	Exact	1 sec	10 ms
95	1.88141	1.88142	1.88244	2.26310	2.26192	2.25389
100	3.89148	3.89148	3.89156	4.72489	4.72226	4.70623
105	6.80784	6.80784	6.80812	8.22660	8.22212	8.20011

Table 4: Vanilla options with cash dividends

6. CONCLUSIONS AND OUTLOOK

In this article we introduced a very simple and natural finite-difference scheme for pricing derivative securities. The scheme is easy to implement and very flexible. No special tricks need to be used to reach a high accuracy. The scheme follows in a straightforward manner from our formulation of the pricing problem in terms of tradables. The key idea is that one can often find *exact* solutions to the governing PDE. These solutions can be used to tune the finite-difference scheme in a very simple manner. Problems usually associated to the presence of drift terms, e.g. boundary layers, are trivially avoided. Numerical computations for arithmetic Asian options show that convergence is fast and neither the solution nor the greeks are plagued by oscillations. Also, financial instruments often depend on values of underlying on discrete points in time, and this can lead to discretization errors in a finite difference scheme when these points do not match with gridpoints. Our algorithm compensates for this problem by making use of exact solutions.

Although this article focussed on adapting a finite-difference scheme to exact solutions of a particular pricing PDE, it does not take too much effort to extend our method in such a way that observed prices are used to construct an implied finite difference scheme, extending the idea of implied trees. We will come back to this in a future article.

ACKNOWLEDGEMENTS

We would like to thank Daniel Duffy and Piet Hemker for extensive discussions on boundary layer problems, fitting schemes, and singular perturbation methods.

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