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J.W. Cohen

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CWI
P.O. Box 94079
1090 GB Amsterdam
The Netherlands

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J.W. Cohen
CWI
P.O. Box 94079, 1090 GB Amsterdam, The Netherlands

ABSTRACT
For the $M/G/1$ fluid model the stationary distribution of the buffer content is investigated for the case that the message length distribution $B(t)$ has a Pareto-type tail, i.e. behaves as $1-\mathrm{O}(t^{-\nu})$ for $t \to \infty$ with $1 < \nu < 2$. This buffer content distribution is closely related to the stationary waiting time distribution $W(t)$ of a stable $M/G/1$ model with service time distribution $B(t)$, in particular when the input rate γ of the messages into the buffer is not less than its output rate $c = 1$. The actual waiting process of this $M/G/1$-model has an imbedded $\mathbf{u}_n$-process which for $\gamma \geq 1$ has the same probabilistic structure as the $\boldsymbol{\omega}_n$-process, the latter one being an imbedded process of the buffer content process. The relations between the stationary distributions $U(t)$ and $W(t)$ are investigated, in particular between their tail probabilities. The results obtained are quite explicit in particular for $\nu = 1\frac{1}{2}$. Further heavy traffic results are obtained. These results lead to a heavy traffic result for the stationary distribution of the $\boldsymbol{\omega}_n$-process and to an asymptotic for the tail probabilities of this distribution.

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1. Introduction

The arrival process for the $M/G/1$-fluid model is defined as follows. Denote by $\mathbf{t}_n$, $n = 1, 2, \ldots$, the epochs at which successive messages arrive,

$$0 < \mathbf{t}_1 < \mathbf{t}_2 < \ldots, \tag{1.1}$$

The sequence $\mathbf{t}_n$, $n = 1, 2, \ldots$, is a Poisson process with rate Λ. The message arriving at $\mathbf{t}_n$ has a duration $\boldsymbol{\tau}_n$. The $\boldsymbol{\tau}_n$, $n = 1, 2, \ldots$, are assumed to be a sequence of i.i.d. nonnegative stochastic variables with distribution $B(\cdot)$ of which the first moment β is finite

$$\beta = \int_0^\infty t \mathrm{d}B(t). \tag{1.2}$$

Denote by $\mathbf{x}_t$ the number of messages simultaneously present. Obviously $\{\mathbf{x}_t, t \geq 0\}$ is the well known $M/G/\infty$ process. It is assumed that the $\mathbf{x}_t$-process is stationary so that

$$\Pr\{\mathbf{x}_t = k\} = \frac{a^k}{k!}\mathrm{e}^{-a}, \quad k = 0, 1, 2, \ldots; \quad t \geq 0, \tag{1.3}$$

where

$$a := \Lambda\beta. \tag{1.4}$$

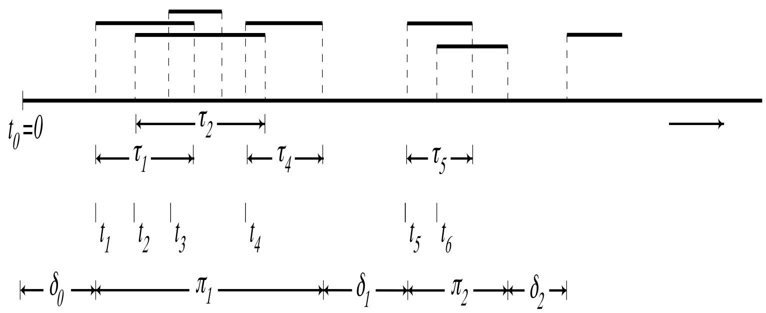

Figure 1

Suppose $\mathbf{x}_t > 0$, then the shortest interval (t_l, t_r) covering point t such that

$$\begin{aligned} x_s &> 0 \quad \text{for} \quad s \in (t_l, t_r), \\ &= 0 \quad \text{for} \quad s = t_l-, \\ &= 0 \quad \text{for} \quad s = t_r+, \end{aligned}$$

will be called an inflow period. So in Fig. 1 $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$ are inflow periods. The intervals between succesive inflow periods are indicated by $\boldsymbol{\delta}_n$, $n = 0, 1, 2, \ldots$; it is assumed that $\mathbf{x}_0 = 0$. The $\boldsymbol{\pi}_n$, $n = 1, 2, \ldots$,are i.i.d. stochastic variables, similarly for $\boldsymbol{\delta}_n$, $n = 0, 1, 2, \ldots$; moreover these sequences are independent, and $\boldsymbol{\delta}_n$ is negative exponentially distributed with

$$\mathrm{E}\{\boldsymbol{\delta}_n\} = \Lambda^{-1}. \tag{1.5}$$

Each message generates with rate γ a workload, i.e. a message of duration $\boldsymbol{\tau}$ generates a total workload $\gamma\boldsymbol{\tau}$, $\gamma > 0$. So the total workload generated during $\boldsymbol{\pi}_1$, see Fig. 1, is $(\boldsymbol{\tau}_1 + \boldsymbol{\tau}_2 + \boldsymbol{\tau}_3 + \boldsymbol{\tau}_4)\gamma$. By $\mathbf{b}_n$ will be denoted the total workload generated in the n-th inflow period $\boldsymbol{\pi}_n$. Obviously the $\mathbf{b}_n$, $n = 1, 2, \ldots$, are i.i.d. stochastic variables.

Denote by $\mathbf{h}_t$ the total workload generated in the interval $[0, t]$. As in [1], it is shown that: for $\mathrm{Re}\,\rho \geq 0$, $t > 0$,

$$\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{h}_t}(\mathbf{x}_t = 0)|\mathbf{x}_0 = 0\} = \mathrm{e}^{-H(\rho,t)}. \tag{1.6}$$

$$\begin{aligned} H(\rho, t) \quad &:= \quad \frac{\Lambda}{2\pi i} \int_{-\mathrm{i}\infty+\varepsilon}^{\mathrm{i}\infty+\varepsilon} \frac{\mathrm{e}^{ut}}{u^2}\{1 - \beta(\gamma\rho + u)\}\mathrm{d}u \quad \text{with} \quad \varepsilon > 0, \\ \beta(\rho) \quad &:= \quad \mathrm{E}\{\mathrm{e}^{-\rho\boldsymbol{\tau}}\}, \end{aligned}$$

the integral being a principal value integral, and $\boldsymbol{\tau}$ is a stochastic variable with distribution $B(\cdot)$.

Denote by $(\mathbf{b}, \boldsymbol{\pi})$ a pair of stochastic variables with the same joint distribution as the pair $(\mathbf{b}_n, \boldsymbol{\pi}_n)$. As in [1] it it shown that: for $\mathrm{Re}\,\rho \geq 0$, $\mathrm{Re}\,s > 0$,

$$[s + \Lambda[1 - \mathrm{E}\{\mathrm{e}^{-\rho\mathbf{b} - s\boldsymbol{\pi}}\}]]^{-1} = \int_0^\infty \mathrm{e}^{-st}\mathrm{e}^{-H(\rho,t)}\mathrm{d}t. \tag{1.7}$$

It is readily shown that, cf. [1],

$$\begin{aligned} H(0, t) \quad &= \quad a \int_0^t \{1 - B(\tau)\}\frac{\mathrm{d}\tau}{\beta}, \qquad (1.8) \\ \frac{1}{\Lambda}H(\rho, t) \quad &= \quad t\{1 - \beta(\gamma\rho)\} + \mathrm{E}\{\boldsymbol{\tau}\mathrm{e}^{-\rho\gamma\boldsymbol{\tau}}\} - \mathrm{E}\{(\boldsymbol{\tau} - t)\mathrm{e}^{-\gamma\rho\boldsymbol{\tau}}(\boldsymbol{\tau} \geq t)\}, \qquad \mathrm{Re}\,\rho \geq 0. \end{aligned}$$

$$\begin{aligned} \mathrm{E}\{\boldsymbol{\pi}\} &= \frac{\mathrm{e}^a - 1}{a}\beta, \\ \mathrm{E}\{\mathbf{b}\} &= \gamma\beta\mathrm{e}^a. \end{aligned} \tag{1.9}$$

From (1.9) we obtain

$$\mathrm{E}\{\mathbf{b}\} \geq \mathrm{E}\{\boldsymbol{\pi}\} \quad \Longleftrightarrow \quad 1 - (1 - a\gamma)\mathrm{e}^a \geq 0. \tag{1.10}$$

Next we describe the service process of the $M/G/1$ fluid model. Each message of the arrival process described above produces a traffic load $\gamma\boldsymbol{\tau}$ which is fed into the buffer with rate γ. The buffer has an infinite capacity for storing the traffic produced by the arrival process. The output rate of the buffer is assumed to be equal to one. Consider the n-th inflow period of the arrival process and let $\boldsymbol{\omega}_n$ be the content of the buffer at the start of the n-th inflow period, see Fig. 2.

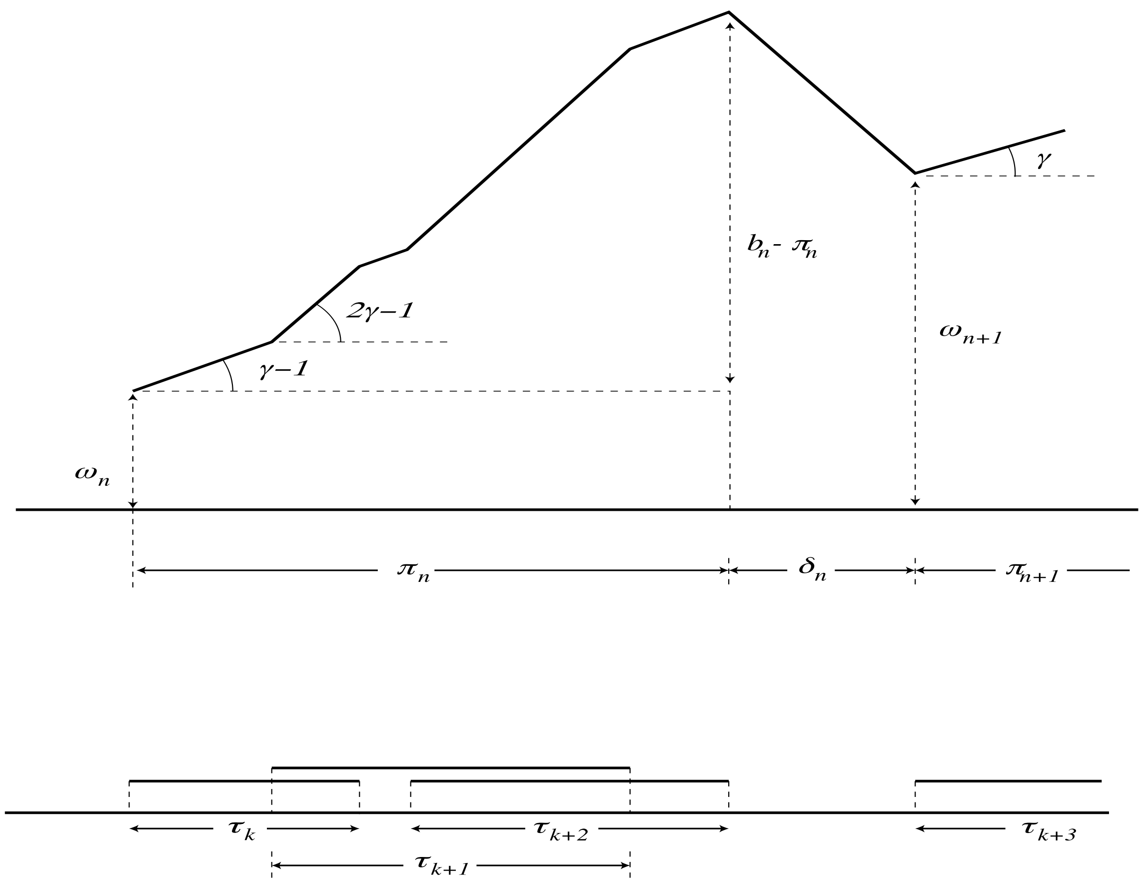

Figure 2

Figure 2 has been drawn for the case that $\gamma > 1$. Hence, since the output rate of the buffer is one, its contents increases at the start of $\boldsymbol{\pi}_n$ with rate $\gamma - 1$, and if at some point t covered by $\boldsymbol{\pi}_n$ the number of messages simultaneously present is $\mathbf{x}_t$ then $\gamma\mathbf{x}_t - 1$ is the net input rate of traffic fed into the buffer. Evidently at the end of the inflow period $\boldsymbol{\pi}_n$ the buffer content has increased by $\mathbf{b}_n - \boldsymbol{\pi}_n$. Obviously, we have for the case $\gamma \geq 1$ that

$$\boldsymbol{\omega}_{n+1} = [\boldsymbol{\omega}_n + \mathbf{b}_n - \boldsymbol{\pi}_n - \boldsymbol{\delta}_n]^+. \tag{1.11}$$

Since $\mathbf{b}_n - \boldsymbol{\pi}_n$ and $\boldsymbol{\delta}_n$ are independent and $\boldsymbol{\delta}_n$ has a negative exponential distribution it is seen that the $\boldsymbol{\omega}_n$-process is the actual waiting process of the $M/G/1$ queueing model.

If however $\gamma < 1$ and $\mathbf{x}_t$ as just defined then the net input rate at time t is equal to $\max(\gamma\mathbf{x}_t - 1, 0)$ and so may be zero. Obviously, for the case $\gamma < 1$ the relation (1.11) does not apply for the buffer content at the start of an inflow period.

The case $\gamma \geq 1$ can be completely analysed by studying the imbedded $M/G/1$ model, see (1.11). For a discussion of the case $\gamma = 1$ see [1].

It is readily shown that the process $\boldsymbol{\omega}_n$, $n = 1, 2, \ldots$, which is a Markovian process is ergodic if and only if

$$\mathrm{E}\{\mathbf{b}\} - \mathrm{E}\{\boldsymbol{\pi}\} < \mathrm{E}\{\boldsymbol{\delta}\},$$

which is equivalent with, cf. (1.9),

$$a\gamma < 1. \tag{1.12}$$

It will always be assumed in the present study that (1.12) holds.

The average increase of the buffer content process at the end of an inflow period is $\mathrm{E}\{\mathbf{b}\} - \mathrm{E}\{\boldsymbol{\pi}\}$ for $\gamma \geq 1$. For $\gamma < 1$ this is also the average increase whenever $\boldsymbol{\omega}_n > \boldsymbol{\pi}_n$. For $a\gamma$ close to one it is evident that the probability of the event $\boldsymbol{\omega}_n > \boldsymbol{\pi}_n$ with n large will be close to one. Consequently, for $a\gamma$ close to one and $\gamma < 1$, the $\boldsymbol{\omega}_n$-process resembles the actual waiting process of an $M/G/1$ queue with arrival rate Λ and service time distribution that of $\mathbf{b} - \boldsymbol{\pi}$. For the approximation of the $\boldsymbol{\omega}_n$-process by the actual waiting process of this $M/G/1$ queue it is a necessary condition that $\mathrm{E}\{\mathbf{b}\} > \mathrm{E}\{\boldsymbol{\pi}\}$. Hence consider (1.10). Denote by a_c the unique zero of $1 - (1 - a\gamma)\mathrm{e}^a$ in $a > 0$ for the case $0 < \gamma < 1$. It is readily seen that

$$a_c < \frac{1}{\gamma}. \tag{1.13}$$

and

$$\begin{aligned} \mathrm{E}\{\mathbf{b}\} - \mathrm{E}\{\boldsymbol{\pi}\} \quad &< 0 \quad \text{for} \quad a < a_c, \\ &= 0 \quad \text{for} \quad a = a_c, \\ &> 0 \quad \text{for} \quad a > a_c. \end{aligned} \tag{1.14}$$

Hence for the average net inflow into the buffer at the end of an inflow period to be positive for $\gamma < 1$ it is necessary that $a > a_c$. Therefore for $\gamma < 1$ the case with a restricted by

$$a_c < a < 1/\gamma, \qquad \gamma < 1, \tag{1.15}$$

is the more interesting one for the analysis of the buffer content process.

It should be noted that $\mathbf{b} - \gamma\boldsymbol{\pi} \geq 0$ with probability one. The event $\mathbf{b} = \gamma\boldsymbol{\pi}$ is equivalent with the event that the inflow period consists of a single message and therefore

$$\Pr\{\mathbf{b} = \gamma\boldsymbol{\pi}\} = \mathrm{E}\{\mathrm{e}^{-\Lambda\boldsymbol{\tau}}\} = \beta(\Lambda). \tag{1.16}$$

In our analysis we shall consider two stochastic sequences, viz. $\{\mathbf{w}_n,\ n = 1, 2, \ldots\}$ and $\mathbf{u}_n,\ n = 1, 2, \ldots$. They are recursively defined by

$$\mathbf{w}_{n+1} = [\mathbf{w}_n + \gamma\boldsymbol{\tau}_n - \boldsymbol{\delta}_n]^+, \qquad n = 1, 2, \ldots, \tag{1.17}$$

$$\mathbf{u}_{n+1} = [\mathbf{u}_n + \mathbf{b}_n - \boldsymbol{\pi}_n - \boldsymbol{\delta}_n]^+, \qquad n = 1, 2, \ldots. \tag{1.18}$$

Obviously, the $\mathbf{w}_n$-process is the actual waiting time process of an $M/G/1$ queue with arrival rate Λ and service time distribution that of $\gamma\boldsymbol{\tau}$. The $\mathbf{u}_n$-process resembles the actual waiting time process of a $GI/G/1$-queue, but it differs from this because $\mathbf{b}_n$ and $\boldsymbol{\pi}_n + \boldsymbol{\delta}_n$ are not independent. The $\mathbf{u}_n$-process is also not the actual waiting process of an $M/G/1$ queue since for $\gamma < 1$ the variable $\mathbf{b}_n - \boldsymbol{\pi}_n$ can be negative with positive probability. However, the $\mathbf{u}_n$-process can be completely analysed, since the analysis of the stochastic sequence

$$\mathbf{u}_{n+1} = [\mathbf{u}_n + \boldsymbol{\xi}_n]^+, \quad n = 1, 2, \ldots.$$

with $\boldsymbol{\xi}_n,\ n = 1, 2, \ldots$, a sequence of i.i.d. stochastic variables is a classical one-dimensional random walk, cf. [7], [8], [9].

For $a\gamma < 1$ the $\mathbf{w}_n$-process posesses a stationary distribution $W(t)$, say, and also the $\mathbf{u}_n$-process has a stationary distribution $U(t)$, say. The main goal of our study is the relation between these stationary distributions for the case that the distribution $B(t)$ has a heavy tail of Pareto type, i.e.

$$1 - B(t) = \mathrm{O}(1/t^\nu) \text{ for } t \to \infty \text{ and } 1 < \nu < 2, \tag{1.19}$$

with the first moment

$$\beta := \int_0^\infty \{1 - B(\tau)\}\mathrm{d}\tau < \infty. \tag{1.20}$$

Next, we review the several sections of the present study, which is based on the results obtained in [1] and [2].

The distribution $W(t)$ stands for the stationary distributions of the $\mathbf{w}_n$-process and $U(t)$ for that of the $\mathbf{u}_n$-process, cf. (1.17) and (1.18); $\mathbf{w}$ and $\mathbf{u}$ will be stochastic variables with distributions $W(t)$ and $U(t)$, respectively.

Starting from the results obtained in [1], a relation between the L.S-transforms $\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\}$ and $\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{u}}\}$ for $\mathrm{Re}\,\rho = 0$ is derived, cf. (2.18), for the case $\gamma > 0$. For $\gamma \geq 1$ this relation simplifies to, see Sections 6 and 9,

$$\frac{\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{u}}\}}{\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\}} = [1 - aG(\rho, -\rho)]\mathrm{e}^a\,,\ \mathrm{Re}\ \rho = 0; \tag{1.21}$$

here $G(\rho, -\rho)$ depends also on γ. For $\gamma \geq 1$ the righthand side is regular for $\mathrm{Re}\,\rho > 0$. Because $\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\}, \mathrm{Re}\,\rho \geq 0$, is known, note that it is the L.S.-transform of the stationary actual waiting time distribution of an M/G/1 queue, it is seen that $\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{u}}\}$ is known for $\mathrm{Re}\,\rho \geq 0$. However, $G(\rho, -\rho)$ is a quite intricate function of ρ. For $\gamma < 1$ the function $G(\rho, -\rho)$ is in general not regular for $\mathrm{Re}\,\rho > 0$, this point is discussed in Section 3. For the case $\gamma < 1$ the determination of $\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{u}}\}$ from (2.18) leads to a Riemann Boundary Value Problem, see end of Section 2.

In Section 4 we introduce a class of message length distributions $B(t)$, which have a heavy tail of the type (1.19),

$$\begin{aligned} &B(t) = 1 - \frac{\sigma^{2-\nu}}{\Gamma(2-\nu)}\delta\int_0^\infty \mathrm{e}^{-\sigma\theta}\frac{\theta}{(\theta+t)^\nu}\mathrm{d}\theta\,,\ t \geq 0,\\ &0 < \delta \leq 1,\ 1 < \nu < 2\,,\ \sigma := \frac{2-\nu}{\nu-1}\frac{\delta}{\beta}\,,\ \delta = 1 - B(0+). \end{aligned} \tag{1.22}$$

This class of distributions with support $[0,\infty)$ has been studied in [2]. The L.S.-transform $\beta(\rho)$ of $B(t)$ is explicitly known and in particular very simple for $\nu = 1\frac{1}{2}$, viz.

$$\frac{1-\beta(\rho)}{\rho\beta} = [1+\sqrt{\beta\rho/\delta}]^{-2}\,,\ \mathrm{Re}\,\rho \geq 0. \tag{1.23}$$

For the applications of the M/G/1-fluid model knowledge concerning the tail probabilities is important. In [2] expressions for the tail probabilities $1-W(t), t\to\infty$, have been derived for the case that the service time distribution of the M/G/1 model with arrival rate Λ is given by (1.22). The asymptotic expression for $1-W(t), t\to\infty$ has been derived from the asymptotic expression for $\mathrm{E}\mathrm{e}^{-\rho\mathbf{w}}\}$ for $|\rho|\to 0, \mathrm{Re}\,\rho\geq 0$. In this derivation the algebraic singularity of $\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\}$ at $\rho = 0$ plays an essential role. In Section 5 the first term of the asymptotic series for $G(\rho,-\rho), |\rho|\to 0, \mathrm{Re}\,\rho\geq 0$ is obtained for the case $\gamma = 1, 1<\nu<2$. This asymptotic result together with that for $\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\}$ yields by using (1.21) an asymptotic result for $\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{u}}\}$. This result then leads to an asymptotic relation for $1-U(t), t\to\infty$. The case $\gamma = 1, \nu = 1\frac{1}{2}$ is a fair example to illustrate the relation between $1-U(t)$ and $1-W(t)$. From (5.15) and (5.23) it results that: for $t\to\infty$ and fixed a $\in (0,1), \nu = 1\frac{1}{2}$,

$$1-U(t)-\{1-W(t)\} = -\frac{2a}{\sqrt{\pi}}[a^{\frac{1}{2}}+1\frac{1}{2}(1-\mathrm{e}^{-a})(1+a^{\frac{1}{2}})](\frac{\beta}{t})^{\frac{1}{2}}\{1+\mathrm{O}(\frac{\beta}{t})\}, \tag{1.24}$$

with, cf. (5.24), for $t\to\infty$, $a\in(0,1)$,

$$1-W(t) = \frac{1}{\sqrt{\delta\pi}}\frac{2a}{1-a}(\frac{\beta}{t})^{\frac{1}{2}}\{1+\mathrm{O}(\frac{\beta}{t})\}. \tag{1.25}$$

Note that for $1-a << 1$, the righthand side of (1.24) is in absolute value small compared to that in the righthand side of (1.25). For further comments concerning this point see end of Section 5.

Section 6 concerns the case $\gamma = 1$, $\nu = 1\frac{1}{2}$, and the main results of this section are: (1.26)

i. the variable $(1-\sqrt{a})^2\delta\mathbf{w}$ converges in distribution for $a\uparrow 1$;

ii. $\lim_{a\uparrow 1}\Pr\{(1-\sqrt{a})^2\delta\mathbf{w}\geq t\} = \frac{2}{\sqrt{\pi}}\mathrm{e}^{t/\beta}\mathrm{Erfc}(\sqrt{t/\beta}),\ t>0$;

iii. for $t\to\infty,\ H\in\{0,1,2,\dots,\}$:
$$\lim_{a\uparrow 1}\Pr\{(1-\sqrt{a})^2\delta\mathbf{w}\geq t\} = \frac{1}{\pi}\sum_{n=0}^{H}(-1)^n\frac{\Gamma(n+\frac{1}{2})}{(t/\beta)^{n+\frac{1}{2}}}+\mathrm{O}((\frac{t}{\beta})^{-H-1\frac{1}{2}});$$

iv. $(1-\sqrt{a})^2\delta\mathbf{u}$ converges in distribution for $a\uparrow 1$ and has the same limiting distribution as $(1-\sqrt{a})^2\delta\mathbf{w}$ for $a\uparrow 1$.

It should be noted that (1.26)ii is a heavy traffic result for the actual waiting time of an M/G/1 queue with traffic load a and service time distribution $B(t)$ given by (4.1) with $\nu = 1\frac{1}{2}$. Similarly (1.26)iv is a heavy traffic result for the $\mathbf{u}$-process, cf. (1.18). Analogous results are derived in Sections 7, 8 and 9. They may be summarized as follows.

With

$$\tilde{\Delta} = [\frac{1-\gamma a}{\gamma a}(2-\nu)]^{\frac{1}{\nu-1}}\frac{2-\nu}{\nu-1}\frac{\delta}{\beta},\ \gamma>0,\ 1<\nu<2,\ 0<\delta\leq 1,\ a\gamma<1, \tag{1.27}$$

holds:

(1.28)

i. $\tilde{\Delta}\mathbf{w}$ and $\tilde{\Delta}\mathbf{u}$ both converge in distribution for $a\gamma \uparrow 1$ to the same limiting distribution,

ii. $\lim\limits_{a\gamma\uparrow 1} \mathrm{E}\{\mathrm{e}^{-r\tilde{\Delta}\mathbf{w}}\} = \lim\limits_{a\gamma\uparrow 1} \mathrm{E}\{\mathrm{e}^{-r\tilde{\Delta}\mathbf{u}}\} = \dfrac{1}{1+r^{\nu-1}}$, Re $r \geq 0$,

iii. for $t \to \infty, H \in \{1,2,\ldots\}$,

$$\begin{aligned}\lim_{a\gamma\uparrow 1} \Pr\{\tilde{\Delta}\mathbf{w} \geq t\} &= \lim_{a\gamma\uparrow 1} \Pr\{\tilde{\Delta}\mathbf{u} \geq t\} = \\ &\sum_{n=1}^{H} (-1)^{n-1} \frac{\Gamma(n(\nu-1)) \sin \pi n(\nu-1)}{t^{n(\nu-1)}} + \mathrm{O}(t^{-(H+1)(\nu-1)}).\end{aligned}$$

The relations (1.24) , . . . , (1.28) are the main results of the present study.

The function $[1+r^{\nu-1}]^{-1}, 1<\nu<2, \mathrm{Re}\, r \geq 0$, is the L.S.-transform of a probability distribution with support $[0,\infty)$ For the first time it has appeared in reliability studies of Kovalenko, see for further details [9].

Finally, we make the following comments. The essential feature of the M/G/1-fluid model is the gradual input into the buffer of the traffic generated by the arrival process. Consider the M/G/1 model with arrival Λ and service time $\gamma\boldsymbol{\tau}$ with $B(t)$ the distribution of $\boldsymbol{\tau}$ and $a\,\gamma < 1$. Then the $\mathbf{w}_n$-process, cf. (1.17), is the actual waiting process of this M/G/1 model. It is readily verified that the $\mathbf{u}_n$=process, cf. (1.18), is an imbedded process of this $\mathbf{w}_n$-process. For $\gamma > 1$ this $\mathbf{u}_n$-process may be also interpreted as the buffer content process, just before the start of the n-th inflow period $\boldsymbol{\pi}_n$, with instantaneous input $\gamma\boldsymbol{\tau}$ instead of gradual input as for the fluid model. Because for the case with instantaneous input the potential output rate of the buffer is timely more efficiently used than with gradual input it follows that $\mathbf{u}_n \leq \mathbf{w}_n$ with probability one if $\mathbf{u}_0 = \mathbf{w}_0$; here $\mathbf{w}_n$ is the buffer content at the start of the n-th inflow period, see Figure 2. With $\boldsymbol{\omega}$ a stochastic variable with distribution the stationary distribution of the $\boldsymbol{\omega}_n$-process with $a\gamma < 1$ it follows that

$$\mathbf{u} \leq \boldsymbol{\omega} \text{ with probability one .} \tag{1.29}$$

Hence

$$\Pr\{\boldsymbol{\omega} \geq t\} \leq \Pr\{\mathbf{u} \geq t\}\,,\, t \geq 0. \tag{1.30}$$

Note that in (1.30) the equality sign holds for all $t \geq 0$ if $\gamma \geq 1$. It has been shown in Section 9 that $\tilde{\Delta}\mathbf{u}$ converges in distribution for $a\gamma \uparrow 1$. Consequently, we have from (1.30) for $\gamma < 1$,

$$\limsup_{a\gamma\uparrow 1} \Pr\{\tilde{\Delta}\boldsymbol{\omega} \geq t\} \leq \lim_{a\gamma\uparrow 1} \Pr\{\tilde{\Delta}\mathbf{u} \geq t\} = \lim_{a\gamma\uparrow 1} \Pr\{\tilde{\Delta}\mathbf{w} \geq t\}. \tag{1.31}$$

Because $\Pr\{\tilde{\Delta}\mathbf{w} \geq t\}$ is monotone in $a\gamma$ we have from (1.31) for $\gamma < 1$,

$$\lim_{a\gamma\uparrow 1} \Pr\{\tilde{\Delta}\boldsymbol{\omega} \geq t\} \leq \lim_{a\gamma\uparrow 1} \Pr\{\tilde{\Delta}\mathbf{w} \geq t\}. \tag{1.32}$$

For $a \in (a_c, 1/\gamma)$ we have $\mathrm{E}\{\mathbf{b}\} > \mathrm{E}\{\boldsymbol{\pi}\}$, cf. (1.14), and the $\boldsymbol{\omega}_n$-process becomes instable for a $\gamma \to 1$, i.e. $\boldsymbol{\omega}_n \to \infty$ with probability one for $a\gamma \to 1$. For all n for which $\boldsymbol{\omega}_n$ is large the relation (1.11) applies also for $\gamma < 1$, and consequently the $\boldsymbol{\omega}_n$-process behaves as the $\mathbf{u}_n$-process with $\gamma < 1$. This leads to the conclusion that in (1.32) "$\leq$" may be replaced by "$=$". Further for $\gamma \geq 1$ the $\mathbf{u}_n$-process and the $\boldsymbol{\omega}_n$-process have the same stochastic structure and so the same have traffic distribution for $a\gamma \uparrow 1$, since it has above been argumented that this also holds for $\gamma < 1$ we conclude: for $a\gamma \to 1, \gamma > 0$,

$$\tilde{\Delta}\boldsymbol{\omega}, \tilde{\Delta}\mathbf{w} \text{ and } \tilde{\Delta}\mathbf{u} \text{ all converge in distribution to the same probability distribution.} \tag{1.33}$$

2. On the relation between $\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\}$ and $\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{u}}\}$

Let $\mathbf{w}$ be a stochastic variable with distribution the stationary distribution of the $\mathbf{w}_n$-process; analogously $\mathbf{u}$ is defined for the $\mathbf{u}_n$-sequence, cf. (1.17) and (1.18). So

$$\Pr\{\mathbf{w} < w\} = W(w), \quad \Pr\{\mathbf{u} < u\} = U(u). \tag{2.1}$$

In this section we derive a relation between the Laplace-Stieltjes transforms $\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\}$ and $\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{u}}\}$ of the distributions $W(w)$ and $U(u)$.

From (1.7) we have: for Re $\rho \geq 0$, Re $s > 0$,

$$\begin{aligned} s + \Lambda[1 - \mathrm{E}\{\mathrm{e}^{-\rho\mathbf{b}-s\boldsymbol{\pi}}\}]^{-1} &= \int_0^\infty \mathrm{e}^{-st}\mathrm{e}^{-H(\rho,t)}\mathrm{d}t \\ &= \int_0^\infty \mathrm{e}^{-[s+\Lambda(1-\beta(\gamma\rho))]t}\mathrm{e}^{-[H(\rho,t)-\Lambda t(1-\beta(\gamma\rho))]}\mathrm{d}t. \end{aligned} \tag{2.2}$$

Partial integration of the righthand side of (2.2) yields: for Re $\rho \geq 0$, Re $s > 0$,

$$\begin{aligned} \frac{s + \Lambda[1 - \beta(\gamma\rho)]}{s + \Lambda[1 - \mathrm{E}\{\mathrm{e}^{-\rho\mathbf{b}-s\boldsymbol{\pi}}\}]} &= [-\mathrm{e}^{-H(\rho,t)-st}]_{t=0}^{\infty} \\ &\quad - \int_0^\infty \Lambda\mathrm{E}\{\mathrm{e}^{-\gamma\rho\boldsymbol{\tau}}(\boldsymbol{\tau} \geq t)\}\mathrm{e}^{-H(\rho,t)}\mathrm{d}t. \end{aligned} \tag{2.3}$$

Note that, cf. (1.8),

$$\begin{aligned} \frac{1}{\Lambda}\frac{\mathrm{d}}{\mathrm{d}t}[H(\rho,t) - \{1 - \beta(\gamma\rho)\}t] &= \frac{1}{\Lambda}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\tau=t}^\infty (\tau - t)\mathrm{e}^{-\gamma\rho\tau}\mathrm{d}B(\tau) \\ &= -\frac{1}{\Lambda}\int_{\tau=t}^\infty \mathrm{e}^{-\gamma\rho\tau}\mathrm{d}B(\tau) = -\frac{1}{\Lambda}\mathrm{E}\{\mathrm{e}^{-\gamma\rho\boldsymbol{\tau}}(\boldsymbol{\tau} \geq t)\}. \end{aligned} \tag{2.4}$$

From (1.8) it is readily seen that

$$\begin{aligned} &H(\rho, 0) = 0 \ \text{ for Re } \rho \geq 0, \\ &|H(\rho,t)| \to \infty \text{ for } t \to \infty, \text{ Re } \rho > 0, \end{aligned} \tag{2.5}$$

and so from (2.4) and (2.5): for Re $\rho \geq 0$, Re $s \geq 0$,

$$\frac{s + \Lambda[1 - \beta(\gamma\rho)]}{s + \Lambda[1 - \mathrm{E}\{\mathrm{e}^{-\rho\mathbf{b}-s\boldsymbol{\pi}}\}]} = 1 - a\int_0^\infty \mathrm{e}^{-s\boldsymbol{\tau}}\mathrm{E}\{\mathrm{e}^{-\gamma\rho\boldsymbol{\tau}}(\boldsymbol{\tau} \geq t)\}\mathrm{e}^{-H(\rho,t)}\frac{\mathrm{d}t}{\beta}. \tag{2.6}$$

For the proof that this relation holds for Re $s \geq 0$, see Appendix A.

Put for Re $\rho \geq 0$, Re $s \geq 0$,

$$G(\rho, s) := \int_0^\infty \mathrm{e}^{-(s+\gamma\rho)t}\mathrm{E}\{\mathrm{e}^{-\gamma\rho(\boldsymbol{\tau}-t)}(\boldsymbol{\tau} \geq t)\}\mathrm{e}^{-H(\rho,t)}\frac{\mathrm{d}t}{\beta}. \tag{2.7}$$

A simple calculation shows by using (a.3) of Appendix A that

$$G(0,0) = \frac{1}{a}(1 - \mathrm{e}^{-a}). \tag{2.8}$$

Let $\boldsymbol{\delta}$ be a stochastic variable with the same distribution as $\boldsymbol{\delta}_n$, so that

$$\mathrm{E}\{\mathrm{e}^{-\rho\boldsymbol{\delta}}\} = \frac{\Lambda}{s+\Lambda},$$

and such that $\boldsymbol{\tau}$ and $\boldsymbol{\delta}$ are independent and similarly $\boldsymbol{\delta}$ is independent of $\mathbf{b}$ and $\boldsymbol{\pi}$, then we may rewrite (2.6) as: for Re $\rho \geq 0$, Re $s \geq 0$,

$$\frac{1 - \mathrm{E}\{\mathrm{e}^{-\gamma\rho\boldsymbol{\tau} - s\boldsymbol{\delta}}\}}{1 - \mathrm{E}\{\mathrm{e}^{-\rho\mathbf{b} - s(\boldsymbol{\pi}+\boldsymbol{\delta})}\}} = 1 - aG(\rho, s). \tag{2.9}$$

In particular it follows from (2.9) by taking $\rho = -s$, with Re $\rho = 0$, that: for Re $\rho = 0$,

$$\frac{1 - \mathrm{E}\{\mathrm{e}^{-\gamma\rho\boldsymbol{\tau} + \rho\boldsymbol{\delta}}\}}{1 - \mathrm{E}\{\mathrm{e}^{-\rho(\mathbf{b} - \boldsymbol{\pi} - \boldsymbol{\delta})}\}} = 1 - aG(\rho, -\rho). \tag{2.10}$$

Denote for the M/G/1 queue with actual waiting time process given by (3.6) by $\tilde{\mathbf{i}}$ the idle time and by $\tilde{\mathbf{n}}$ the number of customers served in a busy period. From [3] p.21 or [4] we then have, note that $a\gamma < 1$: for Re $\rho = 0$,

$$\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\} = \frac{1}{\mathrm{E}\{\tilde{\mathbf{n}}\}} \frac{1 - \mathrm{E}\{\mathrm{e}^{\rho\tilde{\mathbf{i}}}\}}{1 - \mathrm{E}\{\mathrm{e}^{-\gamma\rho\boldsymbol{\tau} + \rho\boldsymbol{\delta}}\}}. \tag{2.11}$$

For the $\mathbf{u}_n$-process defined in (3.7) define

$$\mathbf{n} := \min\{n : \mathbf{u}_{n+1} = 0, n = 1, 2, \ldots | \mathbf{u}_1 = 0\}, \tag{2.12}$$

$$\mathbf{i} := -[\mathbf{u}_{\mathbf{n}} + \mathbf{b}_{\mathbf{n}} - \boldsymbol{\pi}_{\mathbf{n}} - \boldsymbol{\delta}_{\mathbf{n}}]^-,$$

then, cf. [3], p.21 or [4] : for Re $\rho = 0$,

$$\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{u}}\} = \frac{1}{\mathrm{E}\{\mathbf{n}\}} \frac{1 - \mathrm{E}\{\mathrm{e}^{\rho\mathbf{i}}\}}{1 - \mathrm{E}\{\mathrm{e}^{-\rho(\mathbf{b} - \boldsymbol{\pi} - \boldsymbol{\delta})}\}}. \tag{2.13}$$

Note that $a\gamma < 1$ so that by taking $\rho = 0$ in (2.11) and (2.13),

$$\frac{\mathrm{E}\{\tilde{\mathbf{i}}\}}{\mathrm{E}\{\boldsymbol{\delta}\} - \gamma\beta} = \mathrm{E}\{\tilde{\mathbf{n}}\}\ ,\ \frac{\mathrm{E}\{\mathbf{i}\}}{\mathrm{E}\{\boldsymbol{\delta}\} - \mathrm{E}\{\mathbf{b} - \boldsymbol{\pi}\}} = \mathrm{E}\{\mathbf{n}\}. \tag{2.14}$$

Because

$$\mathrm{E}\{\tilde{\mathbf{i}}\} = \mathrm{E}\{\boldsymbol{\delta}\} = \frac{1}{\Lambda},$$

we obtain

$$\mathrm{E}\{\tilde{\mathbf{n}}\} = \frac{1}{1 - a\gamma}, \tag{2.15}$$

and from (1.9) and (2.14) it is seen that

$$\Lambda\mathrm{E}\{\mathbf{i}\} = \mathrm{E}\{\mathbf{n}\}(1 - a\gamma)\mathrm{e}^a. \tag{2.16}$$

From (2.11) and (2.13) we have: for Re $\rho = 0$,

$$\begin{aligned}\frac{\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{u}}\}}{\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\}} &= \frac{1-\mathrm{E}\{\mathrm{e}^{\rho\mathbf{i}}\}}{1-\mathrm{E}\{\mathrm{e}^{\rho\tilde{\mathbf{i}}}\}}\,\frac{\mathrm{E}\{\tilde{\mathbf{n}}\}}{\mathrm{E}\{\mathbf{n}\}}\,\frac{1-\mathrm{E}\{\mathrm{e}^{-\gamma\rho\boldsymbol{\tau}+\rho\boldsymbol{\delta}}\}}{1-\mathrm{E}\{\mathrm{e}^{-\rho(\mathbf{b}-\boldsymbol{\Pi}-\boldsymbol{\delta}})\}} \\ &= \frac{1-\mathrm{E}\{\mathrm{e}^{\rho\mathbf{i}}\}}{1-\mathrm{E}\{\mathrm{e}^{\rho\tilde{\mathbf{i}}}\}}\,\frac{\mathrm{E}\{\tilde{\mathbf{n}}\}}{\mathrm{E}\{\mathbf{n}\}}[1-aG(\rho,-\rho)].\end{aligned} \tag{2.17}$$

From (2.15), (2.16) and (2.17) it is seen that: for Re $\rho=0$,

$$\frac{\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{u}}\}}{\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\}} = \frac{[1-\mathrm{E}\{\mathrm{e}^{\rho\mathbf{i}}\}]/\mathrm{E}\{\mathbf{i}\}}{1-\mathrm{E}\{\mathrm{e}^{\rho\tilde{\mathbf{i}}}\}/\mathrm{E}\{\tilde{\mathbf{i}}\}}\{1-aG(\rho,-\rho)\}\mathrm{e}^{a}. \tag{2.18}$$

Obviously $\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{u}}\}$ and $\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\}$ are both regular for $\mathrm{Re}\,\rho>0$, continuous for $\mathrm{Re}\,\rho>0$; moreover it is wellknown that $\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\}$ has no zeros in $\mathrm{Re}\,\rho\geq 0$, a property which follows immediately from the Pollaczek-Khintchine formula. Hence the lefthand side of (2.18) is regular for $\mathrm{Re}\,\rho>0$, continuous for $\mathrm{Re}\,\rho\geq 0$. Further $\mathbf{i}$ and $\tilde{\mathbf{i}}$ are both nonnegative variables with

$$\mathrm{E}\{\mathrm{e}^{\rho\tilde{\mathbf{i}}}\} = \Lambda/(\Lambda-\rho)\ ,\ \mathrm{Re}\,\rho\leq 0,$$

and so

$$[1-\mathrm{E}\{\mathrm{e}^{\rho\mathbf{i}}\}]/[1-\mathrm{E}\{\mathrm{e}^{\rho\tilde{\mathbf{i}}}] \tag{2.19}$$

is regular for $\mathrm{Re}\,\rho<0$, continuous for $\mathrm{Re}\,\rho\leq 0$. Consequently the relation (2.18) together with the regularity properties of the two quotients formulates a Riemann Boundary Value Problem, cf. [7]. Although this boundary value problem is a fairly standard one we shall not investigate it here because for the case that $B(\cdot)$ has a heavy tail, cf. (1.19), its analusis is quite intricate.

3. On $G(\rho,-\rho)$

From the expression (2.7) for $G(\rho,s)$ and from that for $H(\rho,t)$, cf. (1.8), we obtain by taking $s=-\rho$ with Re $\rho=0$: for Re $\rho=0$,

$$\begin{aligned} G(\rho,-\rho) &= \int_0^\infty \mathrm{e}^{(\gamma-1)\rho t}\mathrm{E}\{\mathrm{e}^{-\gamma\rho(\boldsymbol{\tau}-t)}(\boldsymbol{\tau}\geq t)\}\mathrm{e}^{-H(\rho,t)}\frac{\mathrm{d}t}{\beta} \\ &= \mathrm{e}^{\frac{a}{\beta}\beta^{(1)}(\gamma\rho)}\int_0^\infty \mathrm{e}^{-[-1+\gamma+a\gamma\frac{1-\beta(\gamma\rho)}{\gamma\rho\beta}]\rho t}\mathrm{E}\{\mathrm{e}^{-\gamma\rho(\boldsymbol{\tau}-t)}(\boldsymbol{\tau}\geq t)\}\mathrm{e}^{a\mathrm{E}\{\frac{\boldsymbol{\tau}-t}{\beta}\mathrm{e}^{-\gamma\rho\boldsymbol{\tau}}(\boldsymbol{\tau}\geq t)\}}\frac{\mathrm{d}t}{\beta}.\end{aligned} \tag{3.1}$$

Consider the function

$$f(\rho,\gamma) := -1+\gamma+\,a\,\gamma\frac{1-\beta(\gamma\rho)}{\gamma\rho\beta}\ \text{ for Re }\rho\geq 0,\ \gamma<1. \tag{3.2}$$

From figure 3 it is readily seen that $f(\rho,\gamma)$ has

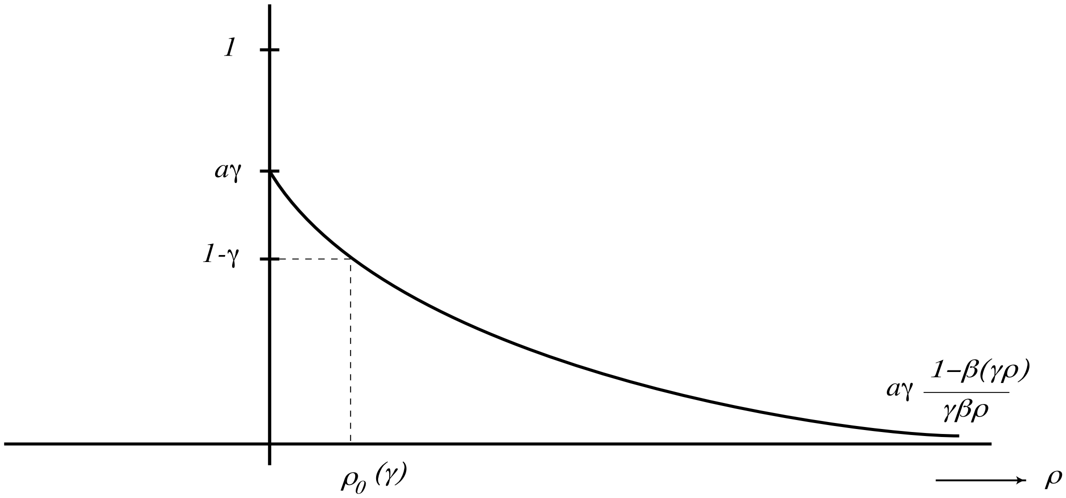

FIGURE 3

a zero $\rho_0(\gamma)$ in $\rho > 0$ if and only if

$$\gamma > \frac{1}{a+1}. \tag{3.3}$$

It is seen that

$$\rho_0(\gamma) \to \infty \text{ for } \gamma \to 1. \tag{3.4}$$

Rewrite (3.1) as follows: for $\mathrm{Re}\,\rho = 0$,

$$\begin{aligned} G(\rho,-\rho) \quad = \quad & \mathrm{e}^{\frac{a}{\beta}\beta^{(1)}(\gamma\rho)} \int\limits_0^\infty \mathrm{e}^{-[-1+\gamma+a\gamma\frac{1-\beta(\gamma\rho)}{\gamma\rho\beta}]\rho t}\, \mathrm{e}^{a\int\limits_t^\infty \frac{1-B(\tau)}{\beta}\mathrm{d}\tau} \times \\ & [1-B(t)-\mathrm{E}\{(1-\mathrm{e}^{-\gamma\rho(\boldsymbol{\tau}-t)})(\boldsymbol{\tau}\geq t)\}]\mathrm{e}^{-a\mathrm{E}\{\frac{\boldsymbol{\tau}-t}{\beta}(1-\mathrm{e}^{-\gamma\rho\boldsymbol{\tau}})(\boldsymbol{\tau}\geq t)\}}\frac{\mathrm{d}t}{\beta}. \end{aligned} \tag{3.5}$$

From $f(\rho,\gamma) > 0$ for $0 \leq \rho < \rho_0(\gamma)$ it is seen that the integral in the righthand side exists for

$$0 \leq \mathrm{Re}\ \rho < \rho_0(\gamma). \tag{3.6}$$

4. THE MESSAGE LENGTH DISTRIBUTION $B(\cdot)$

In this section we describe a class of message length distributions $B(\cdot)$ which have a heavy tail as characterised in (1.19).

The distribution $B(t)$ to be considered is defined by, cf. [2], form. (1.3),

$$1 - B(t) = \frac{\sigma^{2-\nu}}{\Lambda(2-\nu)}\delta\int\limits_0^\infty \mathrm{e}^{-\sigma\theta}\frac{\theta}{(\theta+t)^\nu}\mathrm{d}\theta,\ t \geq 0, \tag{4.1}$$

with the parameters δ, σ, ν satisfying

$$0 < \delta \leq 1,\ \sigma := \frac{2-\nu}{\nu-1}\frac{\delta}{\beta},\ 1 < \nu < 2, \tag{4.2}$$

and $\Gamma(x)$ being the Gamma function. Note that

$$\beta = \int_0^\infty t\mathrm{d}B(t) < \infty, \qquad 1 - B(0+) = \delta. \tag{4.3}$$

From [2], form. (4.7) we have: for $t \geq 0$,

$$1 - B(t) = \frac{2-\nu}{\nu-1}\delta[-\frac{(\sigma t)^{2-\nu}}{\Gamma(3-\nu)} + \frac{\nu-1+st}{\Gamma(3-\nu)}\mathrm{e}^{\sigma t}\Gamma(2-\nu,\sigma t)], \tag{4.4}$$

and, cf. [2], form. (4.11): for $t \to \infty$ and every $H \in \{1, 2\ldots\}$,

$$\begin{aligned} 1 - B(t) = \frac{2-\nu}{\nu-1}\delta[\frac{\nu-1}{\Gamma(3-\nu)}(\sigma t)^{1-\nu} + \\ \frac{\nu-1+\sigma t}{\Gamma(3-\nu)}\{\sum_{m=1}^{H}(-1)^m\frac{\Gamma(\nu-1+m)}{\Gamma(\nu-1)}\frac{1}{(\sigma t)^{m+\nu-1}} + \mathrm{O}((\sigma t)^{-H-\nu}\}], \end{aligned} \tag{4.5}$$

with

$$\Gamma(\alpha, x) := \int_x^\infty \mathrm{e}^{-u}u^{\alpha-1}\mathrm{d}u. \tag{4.6}$$

Further with: for $\mathrm{Re}\,\rho \geq 0$,

$$\beta(\rho) := \int_{0-}^\infty \mathrm{e}^{-\rho t}\mathrm{d}B(t), \ \frac{1-\beta(\rho)}{\rho\beta} = \int_0^\infty \mathrm{e}^{-\rho t}\frac{1-B(t)}{\beta}\mathrm{d}t, \tag{4.7}$$

we have, cf. [2], form. (2.20): for $\mathrm{Re}\,\rho \geq 0$,

$$\frac{1-\beta(\rho)}{\rho\beta} = \frac{\sigma}{\sigma-\rho} + \frac{1}{2-\nu}\frac{\sigma\rho}{(\sigma-\rho)^2} - \frac{1}{2-\nu}\frac{\sigma\rho}{(\sigma-\rho)^2}(\frac{\rho}{\sigma})^{\nu-2}. \tag{4.8}$$

The case with $\nu = 1\frac{1}{2}$ is of special interest, because of its simplicity.

For $\nu = 1\frac{1}{2}$ we have, cf. [2], form. (3.7), (3.9),

$$\begin{aligned} 1 - B(t) &= \frac{2\delta}{\sqrt{\pi}}[-\sigma t + (1+2\sigma t)\mathrm{e}^{\sigma t}\mathrm{Erfc}(\sqrt{\sigma t})], \ t \geq 0, \\ &= \frac{2\delta}{\pi}\sum_{n=1}^{H}(-1)^{n-1}\frac{n\Gamma(n+1/2)}{(\sigma t)^{n+1/2}} + \mathrm{O}((\sigma t)^{-H+1/2}) \text{ for } t \to \infty, \end{aligned} \tag{4.9}$$

$$\frac{1-\beta(\rho)}{\rho\beta} = \frac{1}{[1+\sqrt{\beta\rho/\delta}]^2}, \qquad \mathrm{Re}\,\rho \geq 0,$$

with $H \in \{1, 2, \ldots\}$ and the complementary error function defined by

$$\mathrm{Erfc}(x) := \int_x^\infty \mathrm{e}^{-u^2}\mathrm{d}u.$$

From (4.5) it is readily seen that the distribution $B(t)$ introduced above, cf. (4.1), has a heavy tail of the type (1.19).

Remark 4.1. The power ρ^ν occurring in the relations above is defined by their principal values, i.e. it is positive for $\rho > 0$. □

5. The case $\gamma = 1$

In this section we consider the case with

$$\gamma = 1 \text{ and } a < 1. \tag{5.1}$$

For $\gamma = 1$ the stochastic variable $\mathbf{b} - \boldsymbol{\pi}$ is nonnegative with probability one and, cf. (1.16),

$$0 < \Pr\{\mathbf{b} - \boldsymbol{\tau} = 0\} = \beta(\Lambda) < 1. \tag{5.2}$$

Hence it is seen that for the case $\gamma = 1$ the $\mathbf{u}_n$-processes is the actual waiting time process of an M/G/1 queue with service time distribution that of $\mathbf{b} - \boldsymbol{\tau}$ and arrival rate Λ. For such a queueing model the stationary distribution of the idle time $\mathbf{i}$ has the same distribution as the interarrival time, and so, cf. (2.18), by noting that $\mathbf{i}$ and $\tilde{\mathbf{i}}$ have the same distribution, we have from (2.18) and (3.5): for $\operatorname{Re}\rho \geq 0$,

$$\frac{\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{u}}\}}{\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\}} = \{1 - aG(\rho, -\rho)\}\mathrm{e}^{a}, \tag{5.3}$$

$$\begin{aligned} G(\rho,-\rho) \quad = \quad & \mathrm{e}^{\frac{a}{\beta}\beta^{(1)}(\rho)} \int_0^\infty \mathrm{e}^{-a\frac{1-\beta(\rho)}{\beta\rho}\rho t}\,\mathrm{e}^{a\int_t^\infty \frac{1-B(\tau)}{\beta}\mathrm{d}\tau} \times \\ & [1 - B(t) - \mathrm{E}\{(1 - e^{-\rho(\boldsymbol{\tau}-t)})(\boldsymbol{\tau} \geq t)\}]\mathrm{e}^{-a\mathrm{E}\{\frac{\boldsymbol{\tau}-t}{\beta}(1-\mathrm{e}^{-\rho\boldsymbol{\tau}})(\boldsymbol{\tau}\geq t)\}}\frac{\mathrm{d}t}{\beta}. \end{aligned} \tag{5.4}$$

It remains to show that (5.3) and (5.4) hold for $\operatorname{Re}\rho > 0$, because (2.18), (3.5) and (5.1) imply only that they hold for $\operatorname{Re}\rho = 0$. However the integral of (5.4) exists for $\operatorname{Re}\rho \geq 0$ and is regular in $\operatorname{Re}\rho > 0$, continuous for $\operatorname{Re}\rho \geq 0$. Since both sides of (5.3) are regular in $\operatorname{Re}\rho > 0$, continuous in $\operatorname{Re}\rho \geq 0$ and (5.3) holds for $\operatorname{Re}\rho = 0$ it follows by analytic continuation that (5.3) holds for $\operatorname{Re}\rho \geq 0$.

The relation (5.3) will be the starting point for the derivation of the expression for the tail probabilities of the stationary distribution $U(t)$ of $\mathbf{u}$. Essential in our analysis is the application of the Theorem of Doetsch, cf. [6], vol. II p. 159 or appendix A of [2]. In [2] it has been shown that $[1 - \mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\}]/\rho$ is regular in the domain D^-, see Figure 4, for the case that $B(t)$ is given by (4.1).

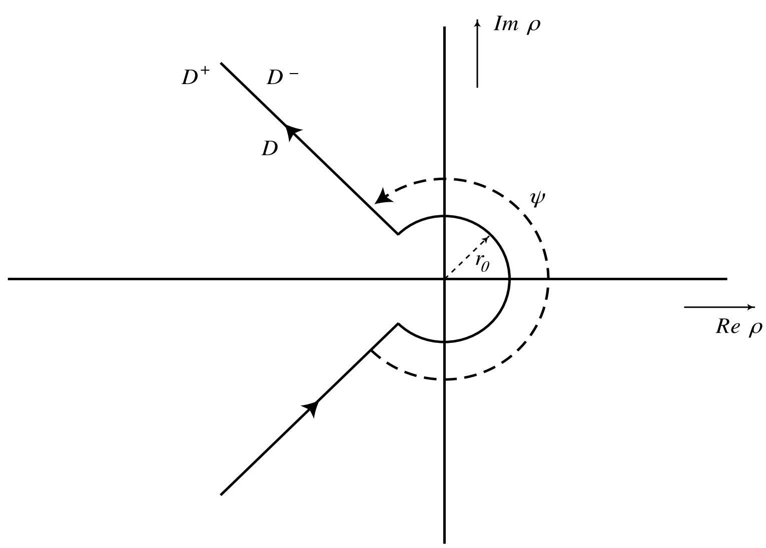

Figure 4

Here D^- is the domain in the ρ-plane to the right of the open contour D, which is formed by the circular arc$\{\rho : \rho = r_0 \mathrm{e}^{\pm \mathbf{i}\psi}\}$ with $0 < r_0 << 1$, $\frac{1}{2}\pi < \psi < \pi$, and the half lines $\{\rho : \rho = r\mathrm{e}^{\pm\mathbf{i}\psi}, r \geq r_0\}$.

In this domain D^- with $0 < r_0 << 1$ the L.S-transform $[1 - \mathrm{E}\{\mathrm{e}^{\rho \mathbf{W}}\}]/\rho$ of $1 - W(t)$ has for $|\rho| \downarrow 0$ a series expansion in powers of ρ. Application of the Theorem of Doetsch to this series expansion leads to the asymptotic expansion of $1 - W(t)$ for $t \to \infty$, see Sections 3 and 6 of [2]. An essential characteristic of this approach is that the terms with integer powers of ρ in the series expansion do not contribute to the asymptotic series of $1 - W(t)$.

From (5.3) we first derive a series expansion in powers of ρ for $\rho \in D^-$ and $|\rho| \downarrow 0$, and then apply the just mentioned theorem of Doetsch. Herefore we have to show that $\{1 - aG(\rho, -\rho)\}\mathrm{e}^a$ has such a series expansion.

To investigate $G(\rho, -\rho)$ for $|\rho| \downarrow 0$, Re $\rho \geq 0$, we first consider the integral $I(\rho)$, cf. (5.4),

$$I(\rho) := \int_0^\infty \mathrm{e}^{-\rho t}(1 - B(t))\mathrm{e}^{a \int_t^\infty \frac{1-B(\tau)}{\beta}\mathrm{d}\tau} \frac{\mathrm{d}t}{\beta}, \text{ Re } \rho \geq 0. \tag{5.5}$$

In appendix C, cf. (c.14) and below, it is shown that for $\rho \to 0$, Re $\rho \geq 0$,

$$I(\rho) = \frac{1}{a}(\mathrm{e}^a - 1) + \frac{1}{2-\nu}(\rho/\sigma)^{\nu-1}\{1 + \mathrm{O}(|\frac{\rho}{\sigma}|)\} + \mathrm{O}(|\frac{\rho}{\sigma}|). \tag{5.6}$$

In appendix B it has been shown, cf. (b.4) that: for Re $\rho \geq 0$,

$$\begin{aligned} &\mathrm{E}\{[1 - \mathrm{e}^{-\rho(\boldsymbol{\tau} - t)}](\boldsymbol{\tau} \geq t)\} = \\ &\frac{\delta}{\Gamma(2-\nu)}\left[\frac{\rho^\nu \sigma^{2-\nu}\mathrm{e}^{\rho t}}{(\sigma-\rho)^2}\Gamma(1-\nu, \rho t) - \frac{\rho\sigma \mathrm{e}^{\sigma t}}{(\sigma-\rho)^2}\Gamma(1-\nu,\sigma t)\right. \\ &\left.-\frac{\rho \mathrm{e}^{\sigma t}}{\sigma - \rho}\Gamma(2-\nu,\sigma t) + \frac{\rho\sigma t \mathrm{e}^{\sigma t}}{\sigma-\rho}\Gamma(1-\nu,\sigma t)\right], \end{aligned} \tag{5.7}$$

and. cf. (b.7); for $0 \leq t < T < \infty, |\rho| \to 0$, Re $\rho \geq 0$,

$$\mathrm{E}\{[1 - \mathrm{e}^{-\rho(\boldsymbol{\tau} - t)}](\boldsymbol{\tau} \geq t)\} = \frac{\delta}{1-\nu}(\rho/\sigma)^\nu\{1 + \mathrm{O}(\rho/\sigma)\} + \mathrm{O}(\rho/\sigma)), \tag{5.8}$$

and, cf. (b.8), for $t >> T$, $|\rho| \to 0$, Re $\rho \geq 0$,

$$\mathrm{E}\{[1 - \mathrm{e}^{-\rho(\boldsymbol{\tau} - t)}](\boldsymbol{\tau} \geq t)\} = \frac{\delta}{1-\nu}[(\frac{\rho}{\sigma})^\nu\{1 + \mathrm{O}(\rho/\sigma)\} + \mathrm{O}(\rho/\sigma)][1 + \frac{1}{(\sigma t)^\nu}(1 + \mathrm{O}(\frac{1}{\sigma t})]. \tag{5.9}$$

Next consider the integral: for Re $\rho \geq 0$,

$$I_1(\rho) := \int_0^\infty \mathrm{e}^{-\rho t}\mathrm{E}\{[1 - \mathrm{e}^{-\rho(\boldsymbol{\tau} - t)}](\boldsymbol{\tau} \geq t)\}\mathrm{e}^{a \int_t^\infty \frac{1-B(\tau)}{\beta}\mathrm{d}\tau}\frac{\mathrm{d}t}{\beta}. \tag{5.10}$$

Because

$$0 < \mathrm{e}^{a \int_t^\infty \frac{1-B(\tau)}{\beta}\mathrm{d}\tau} < \mathrm{e}^a, \tag{5.11}$$

it is readily verified by using (5.8) and (5.9) that: for $|\rho| \to 0$, Re $\rho \geq 0$,

$$I_1(\rho) = \frac{\delta}{1-\nu}(\rho/\sigma)^\nu\{1 + \mathrm{O}(\rho/\sigma)\} + \mathrm{O}(\rho/\sigma). \tag{5.12}$$

Observe that: for Re $\rho \geq 0$,

$$\mathrm{E}\{(\boldsymbol{\tau}-t)\mathrm{e}^{-\rho(\boldsymbol{\tau}-t)}(\boldsymbol{\tau}\geq t)\}=\frac{\mathrm{d}}{\mathrm{d}\rho}\mathrm{E}\{[1-\mathrm{e}^{-\rho(\boldsymbol{\tau}-t)}](\boldsymbol{\tau}\geq t)\}, \tag{5.13}$$

and so the asymptotic behaviour of $\mathrm{E}\{(\boldsymbol{\tau}-t)[1-\mathrm{e}^{-\rho\boldsymbol{\tau}}](\boldsymbol{\tau}\geq t)\}$ for $|\rho|\to 0$, Re $\rho\geq 0$ may be obtained from (5.7) and (5.13), it has been derived in Appendix B, see (b.11).

From the results above we can derive an asymptotic expression for $G(\rho,-\rho)$ for $|\rho|\to 0$, Re $\rho\geq 0$. First note that the last factor in (5.4) goes to one for $|\rho|\to 0$, Re $\rho\geq 0$. From (5.6) and (5.8) it is seen that the contribution of the term $\mathrm{E}\{[1-\mathrm{e}^{-\rho(\boldsymbol{\tau}-t)}](\boldsymbol{\tau}\geq t)\}$ of the integrand to the integral in (5.4) is of $\mathrm{O}(|\rho/\sigma|)$ compared to the contribution of the term $1-B(t)$.

From (5.4), (5.6), (5.8), (b.11) and (b.14), we obtain for $|\rho|\to 0$, Re $\rho\geq 0$,

$$\begin{aligned} G(\rho,-\rho) &= \mathrm{e}^{-a}\{1+\frac{\nu a}{2-\nu}(\frac{\rho}{\sigma})^{\nu-1}\}[\frac{1}{a}(\mathrm{e}^{a}-1)+\frac{1}{2-\nu}(a\frac{1-\beta(\rho)}{\sigma\beta})^{\nu-1}]\times \\ &\quad [1+\frac{\delta\nu}{\nu-1}a\frac{1-\beta(\rho)}{\sigma\beta})^{\nu-1}]\{1+\mathrm{O}(|\frac{\rho}{\sigma}|)\}+\mathrm{O}(|\frac{\rho}{\sigma}|) \\ &= \frac{1}{a}(1-\mathrm{e}^{-a})+C(\frac{\rho}{\sigma}|)^{\nu-1}\{1+\mathrm{O}(|\frac{\rho}{\sigma}|)\}+\mathrm{O}(|\frac{\rho}{\sigma}|), \end{aligned} \tag{5.14}$$

with, cf. (4.2) and (4.8),

$$C:=\frac{1}{2-\nu}[a^{\nu-1}+\nu(1-\mathrm{e}^{-a})(1+a^{\nu-1})]. \tag{5.15}$$

Remark 5.1. Note that $C>0$ and finite for $0<a\leq 1$. Actually, for $a=1$, Re $\rho\geq 0$,

$$|G(\rho,-\rho)|<\infty, \qquad\square \tag{5.16}$$

From (5.3), (5.14) and (5.15) we obtain: for Re $\rho\geq 0, |\rho|\to 0$,

$$\frac{\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{u}}\}}{\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\}}=1-a(\rho\beta)^{\nu-1}C\{1+\mathrm{O}(|\beta\rho|)\}+\mathrm{O}(|\beta\rho|). \tag{5.17}$$

We rewrite (5.17) as: for $|\rho|\to 0$, Re $\rho\geq 0$,

$$\begin{aligned} \frac{1-\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{u}}\}}{\rho} &= \frac{1-\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\}}{\rho}-a(\rho\beta)^{\nu-2}\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\}C\{1+\mathrm{O}(|\beta\rho|)\}+ \\ &+ \frac{1}{\rho}\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\}\mathrm{O}(|\beta\rho|). \end{aligned} \tag{5.18}$$

Next we show that the relation (5.18) also holds for $\rho\in D^{-}$. To do so it is firstly observed that the function $\{1-\beta(\rho)\}/\rho\beta$ and $\beta^{(1)}(\rho)$ can be continued analytically into D^{-}, see herefor the relation (4.8) and (b.12) of appendix B. From (b.4) and (b.9) it is seen that the other functions of ρ occuring in the righthand side of (5.3) can also be continued analytically into D^{-}. Consequently the function $G(\rho,-\rho)$, cf. (5.4), has an analytic continuation into D^{-}, because its integrand has such a continuation. From the derivations which have led to (5.18) and by noting that $\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\}$ has an analytic continuation into D^{-}, cf. [2], form. (3.15) and (5.28), it follows from (5.3) that $\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{u}}\}$ has an analytic continuation in D^{-}. Consequently, we have from (5.18) that: for $|r_0|<<1, |\rho|\to 0$, $\rho\in D^{-}$,

$$\begin{aligned} \frac{1-\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{u}}\}}{\rho} &= \frac{1-\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\}}{\rho}=a(\beta\rho)^{\nu-1}\frac{1-\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\}}{\rho}C\{1+\mathrm{O}(|\beta\rho|)\} \\ &+ \frac{1}{\rho}\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\}\mathrm{O}(|\beta\rho|)-a(\beta\rho)^{\nu-2}\beta C\{1+\mathrm{O}(|\beta\rho|)\}. \end{aligned} \tag{5.19}$$

We consider from now on the case $v = 1\frac{1}{2}$, see also Remark 5.1 below. In [2], cf. form (3.15), it has been shown that: for $|\rho| \to 0$, $\rho \in D^-$ and $\nu = 1\frac{1}{2}$,

$$\begin{aligned} \frac{1-\mathrm{E}\{\mathrm{e}^{-\rho \mathbf{w}}\}}{\rho} &= \frac{1}{2\delta}(1-a)\beta\sqrt{a}\sum_{n=0}^{H}(-1)^n\left[\frac{1}{(1-\sqrt{a})^{n+2}}-\frac{1}{(1+\sqrt{a})^{n+2}}\right](\rho\beta/\delta)^{\frac{1}{2}n-\frac{1}{2}} \\ &+ \mathrm{O}(|\rho\beta/\delta|^{\frac{1}{2}H+\frac{1}{2}}), \end{aligned} \tag{5.20}$$

for every $H\{0,1,2,\ldots\}$, so for $H = 1$:

$$\begin{aligned} \frac{1-\mathrm{E}\{\mathrm{e}^{-\rho \mathbf{w}}\}}{\rho} &= \frac{2}{\delta}\frac{a\beta}{1-a}\left(\frac{\delta}{\rho\beta}\right)^{\frac{1}{2}} - \frac{1-a}{2\delta}\beta\sqrt{a}\left\{\frac{1}{(1-\sqrt{a})^3}-\frac{1}{(1+\sqrt{a})^3}\right\}\left(\frac{\rho\beta}{\delta}\right)^{\frac{1}{2}} \\ &+ \frac{1-a}{(1-\sqrt{a})^5}\beta\mathrm{O}(|\frac{\rho\beta}{\delta}|^{1\frac{1}{2}}). \end{aligned} \tag{5.21}$$

We next substitute (5.21) into the righthand side of (5.19). This leads to: for $|\rho| \to 0$, $\rho \in D^-$,

$$\begin{aligned} \frac{1-\mathrm{E}\{\mathrm{e}^{-\rho \mathbf{u}}\}}{\rho} &- \frac{1-\mathrm{E}\{\mathrm{e}^{-\rho \mathbf{w}}\}}{\rho} = -a(\beta\rho)^{-\frac{1}{2}}\beta C\{1+\mathrm{O}(|\rho\beta|)\} + \\ &[\frac{2}{\sqrt{\delta}}\frac{a^2\beta}{1-a} - \beta\frac{1-a}{2\delta^{1\frac{1}{2}}}a\sqrt{a} + a\frac{1-a}{(1-\sqrt{a})^5}\mathrm{O}(|\rho\beta|)]C\{1+\mathrm{O}(|\rho\beta|\} + \\ &+ \frac{\mathrm{O}(|\beta\rho|)}{\rho}\{1-\frac{2}{\delta}\beta\frac{a\delta^{\frac{1}{2}}}{1-a}(|\rho\beta|)^{\frac{1}{2}}[1+\mathrm{O}(|\rho\beta|)]\}. \end{aligned} \tag{5.22}$$

To the relation (5.22) we apply the theorem of Doetsch [6], vol. II, p. 159, see also appendix A of [2]. As in appendix A of [2], it is shown that this theorem can be applied. Its application yields: for $t \to \infty$,

$$1-U(t)-\{1-W(t)\} = -\frac{a}{\Gamma(\frac{1}{2})}C\left(\frac{\beta}{t}\right)^{\frac{1}{2}} - \frac{2}{\sqrt{\delta}}\frac{a}{1-a}\frac{1}{\Gamma(-1\frac{1}{2})}\left(\frac{\beta}{t}\right)^{1\frac{1}{2}}D\{1+\mathrm{O}\left(\frac{\beta}{t}\right)\} \tag{5.23}$$

for $t \to \infty$, here D is a constant.

To prove (5.23) note that

$$\frac{1-\mathrm{E}\{\mathrm{e}^{-\rho \mathbf{u}}\}}{\rho} = \int_0^\infty \mathrm{e}^{-\rho t}\{1-U(t)\}\mathrm{d}t,\ \frac{1-\mathrm{E}\{\mathrm{e}^{-\rho \mathbf{w}}\}}{\rho} = \int_0^\infty \mathrm{e}^{-\rho t}\{1-W(t)\}\mathrm{d}t.$$

Further, the theorem of Doetsch applies, so that the general term in the righthand side of (5.22) yields a contribution to the asymptotic series in the righthand side of (5.23) according to

$$c_n\rho^{\lambda_n} \Longrightarrow c_n\frac{t^{-\lambda_n-1}}{\Gamma(-\lambda_n)},$$

for arbitrary $\lambda_0 < \lambda_1 < \lambda_2, \ldots$, and with

$$[\Gamma(-\lambda_n)]^{-1} := 0 \qquad \text{for} \qquad \lambda_n = 0,1,2,\ldots.$$

Since $|\frac{1}{\rho}\mathrm{O}(\rho\beta)| = \mathrm{O}(1)$ for $\rho \downarrow 0$ it is readily seen that (5.22) eads to (5.23), and so (5.23) has been proved. □

From [2], form. (3.16), we have: for $t \to \infty$.

$$
\begin{aligned}
1 - W(t) &= \frac{1}{2}(1-a)\frac{\sqrt{2}}{\pi}\sum_{m=0}^{H}(-1)^m\left[\frac{1}{(1-\sqrt{a})^{2m+2}} - \frac{1}{(1+\sqrt{a})^{2m+2}}\right]\frac{\Gamma(m+\frac{1}{2})}{(t\delta)/\beta)^{m+\frac{1}{2}}} \\
&+ \mathrm{O}\left((t\delta/\beta)^{-H-1\frac{1}{2}}\right) \\
&= \frac{2a}{(1-a)}\frac{1}{\sqrt{\delta\pi}}\left(\frac{\beta}{t}\right)^{\frac{1}{2}} - \frac{(1-a)\sqrt{a}}{4\sqrt{\pi\delta^3}}\{\frac{1}{(1-\sqrt{a})^4} - \frac{1}{(1+\sqrt{a})^4}\}\left(\frac{\beta}{t}\right)^{1\frac{1}{2}} \\
&+ \mathrm{O}(\left(\frac{\beta}{t}\right)^{2\frac{1}{2}}).
\end{aligned}
\tag{5.24}
$$

From (5.23) and (5.24) it is seen that the coefficients of the terms in the symptotic series for $1-U(t)$ differ from the corresponding coefficients in the asymptotic series of $1-W(t)$.

In so far it concerns the terms with the factor $(\beta/t)^{\frac{1}{2}}$ it is seen that $1-U(t)$ and $1-W(t)$ differ by

$$
-\frac{a}{\sqrt{\pi}}C\left(\frac{\beta}{t}\right)^{\frac{1}{2}}. \tag{5.25}
$$

Concerning the terms with the factor $(\frac{\beta}{t})^{1\frac{1}{2}}$, $1-U(t)$ and $1-W(t)$ differ by

$$
-\frac{2}{\sqrt{\delta}}\frac{a}{1-a}\frac{1}{\Gamma(-1\frac{1}{2})}\left(\frac{\beta}{t}\right)^{1\frac{1}{2}}D. \tag{5.26}
$$

The term in (5.25) should be compared with the first term in the righthand side of (5.24) and it is then seen that for fixed $a \approx 1$ the term (5.25) can be neglected, with respect to that in (5.24).

Futher it is seen that the ratio of the second term in the righthand side of (5.24) and the term in (5.26) behaves as $(1-\sqrt{a})^{-2}$ for $a \to 1$. Note that C and D are finite for all $a < \infty$. This follows from the fact that $G(\rho,-\rho)$, $\mathrm{Re}\,\rho \geq 0$ is finite for all finite a which implies that the coefficients in the series expansion of $G(\rho,-\rho)$ in powers of $\rho^{\frac{1}{2}}$ are all finite, note that $G(\rho,-\rho)$ is regular for $\mathrm{Re}\,\rho \geq 0$, $\rho \neq 0$.

The relation (5.23) is for fixed $a \in (0,1)$ an asymptotic relation between the tail probabilities $1-U(t)$ and $1-W(t)$ and since $C > 0$ it is seen that $1-U(t) < 1-W(t)$ for $t \to \infty$. This result is plausible because the distribution $W(t)$ concerns a buffer model with instantaneous input of traffic into the buffer whereas $U(t)$ relates to the model with gradual input, note that $\gamma = 1$.

Remark 5.1. By using the asymptotic expression for $1-W(t)$, $t \to \infty$ for the case $1 < \nu < 2$, cf. [2], form. (5.29), the analysis above can be repeated for $1 < \nu < 2$, it is, however, quite intricate and therefore omitted. In the derivation of the asymptotic expression for $\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{u}}\}/\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\}$ the analysis has been restricted to the determination of the first term of the asymptotic series. The determination of further terms, although possible, requires very much algebra. □

6. Heavy traffic analysis for the case $\gamma = 1, \nu = 1\frac{1}{2}$

In this section we shall consider the behaviour of $\mathbf{u}$ for the case that

$$
\gamma = 1, 0 < a < 1,\ a \approx 1,\ \nu = 1\frac{1}{2}. \tag{6.1}
$$

From [2], form. (3.13), we have for $t \geq 0$,

$$
\begin{aligned}
1 - W(t) = \Pr\{\mathbf{w} \geq t\} &= (1+\sqrt{a})(\tfrac{a}{\pi})^{\frac{1}{2}}\mathrm{e}^{(1-\sqrt{a})^2\delta t/\beta}\mathrm{Erfc}((1-\sqrt{a})\sqrt{\delta t/\beta}) \\
&\quad -(1-\sqrt{a})(\frac{a}{\pi})^{\frac{1}{2}}\mathrm{e}^{(1+\sqrt{a})^2\delta t/\beta}\mathrm{Erfc}((1+\sqrt{a})\sqrt{\delta t/\beta}).
\end{aligned}
\tag{6.2}
$$

Put

$$\Delta := (1-\sqrt{a})^2\delta, \tag{6.3}$$

It follows from (6.2) that: for $t \geq 0$,

$$\begin{aligned}\Pr\{(1-\sqrt{a})^2\delta\mathbf{w} \geq t\} \quad &= (1+\sqrt{a})(\frac{a}{\pi})^{\frac{1}{2}}\mathrm{e}^{t/\beta}\mathrm{Erfc}(\sqrt{t/\beta})\\ &-(1-\sqrt{a})(\frac{a}{\pi})^{\frac{1}{2}}e^{(\frac{1+\sqrt{a}}{1-\sqrt{a}})^2\frac{t}{\beta}}\mathrm{Erfc}(\frac{1+\sqrt{a}}{1-\sqrt{a}}\sqrt{t/\beta}).\end{aligned} \tag{6.4}$$

Consider the relation (6.4) for the case $a \approx 1$, cf. (6.1). From [7], vol.2, p. 147, we have for: $\mathrm{Re}\, x > 0, |x| \to \infty$,

$$\mathrm{Erfc}(x) = \int_x^\infty \mathrm{e}^{-u^2}\mathrm{d}u = \frac{1}{2\sqrt{\pi}}\mathrm{e}^{-x^2}[\sum_{n=0}^{H}(-1)^n\frac{\Gamma(\frac{1}{2}+n)}{x^{2n+1}} + \mathrm{O}(|x|^{-2H-3}), \tag{6.5}$$

for every $H \in \{0,1,2,\ldots\}$. Hence for $a \uparrow 1$ and fixed $t \in (0,\infty)$,

$$\begin{aligned}&(1-\sqrt{a})(\frac{a}{\pi})^{\frac{1}{2}}\mathrm{e}^{(\frac{1+\sqrt{a}}{1-\sqrt{a}})^2\frac{t}{\beta}}\mathrm{Erfc}(\frac{1+\sqrt{a}}{1-\sqrt{a}}\sqrt{\frac{t}{\beta}}) =\\ &\frac{1}{4\sqrt{\pi}}(1-\sqrt{a})^2(\frac{\beta}{t})^{\frac{1}{2}}\{1+\mathrm{O}((1-\sqrt{a})^2\frac{\beta}{t})\}.\end{aligned} \tag{6.6}$$

Consequently we obtain from (6.4) and (6.6) that: for fixed $t \in (0,\infty)$,

$$\begin{aligned}&\lim_{a\uparrow 1}\Pr\{(1-\sqrt{a})^2\delta\mathbf{w} \geq t\} = \frac{2}{\sqrt{\pi}}\mathrm{e}^{t/\beta}\mathrm{Erfc}(\sqrt{\frac{t}{\beta}}),\\ &\Pr\{(1-\sqrt{a})^2\delta\mathbf{w} \geq t\} = \frac{2}{\sqrt{\pi}}\mathrm{e}^{t/\beta}\mathrm{Erfc}(\sqrt{\frac{t}{\beta}})+\\ &\frac{1}{4\sqrt{\pi}}(1-\sqrt{a})^2(\frac{\beta}{t})^{\frac{1}{2}}\{1+\mathrm{O}((1-\sqrt{a})^2\frac{\beta}{t})\} \text{ for } a\uparrow 1.\end{aligned} \tag{6.7}$$

The relation (6.7) shows that the stochastic variable $(1-\sqrt{a})^2\delta\mathbf{w}$ converges in distribution for $a \uparrow 1$. The relation (6.7) formulates the heavy traffic distribution for the actual waiting time distribution of the M/G/1 queueing model with service time distribution $B(t)$ as given by (4.1) with $\nu = 1\frac{1}{2}$.

From (5.24) we have for $t \to \infty$ and fixed $a \in (0,1)$,

$$\begin{aligned}\Pr\{(1-\sqrt{a})^2\delta\mathbf{w} \geq t\} \quad &= \frac{1}{2}(1+\sqrt{a})[\frac{\sqrt{a}}{\pi}\sum_{m=0}^{H}(-1)^m[1-[\frac{1-\sqrt{a}}{1+\sqrt{a}}]^{2m+2}]\frac{\Gamma(m+\frac{1}{2})}{(t/\beta)^{m+\frac{1}{2}}}\\ &+\{1-(\frac{1-\sqrt{a}}{1+\sqrt{a}})^{2H+4}\}\mathrm{O}((\frac{t}{\beta})^{-H-1\frac{1}{2}})],\end{aligned} \tag{6.8}$$

with $H \in \{0,1,2\ldots\}$.
Consequently: for fixed $a \in (0,1), a \approx 1$ and $t \to \infty$,

$$\Pr\{(1-\sqrt{a})^2\delta\mathbf{w} \geq t\} = \frac{1}{\pi}\sum_{m=0}^{H}(-1)^m\frac{\Gamma(m+\frac{1}{2})}{(t/\beta)^{m+\frac{1}{2}}} + \mathrm{O}((\frac{t}{\beta})^{-H-1\frac{1}{2}}). \tag{6.9}$$

Note that (6.7) implies by using the asymptotic series for the error function, cf. [7], p. 147, that: for $t \to \infty$ and $H \in \{0,1,2,\ldots\}$,

$$\lim_{a\uparrow 1}\Pr\{(1-\sqrt{a})^2\delta\mathbf{w} \geq t\} = \frac{1}{\pi}\sum_{n=0}^{H}(-1)^n\frac{\Gamma(n+\frac{1}{2})}{(t/\beta)^{n+\frac{1}{2}}} + \mathrm{O}((\frac{t}{\beta})^{-H-1\frac{1}{2}}). \tag{6.10}$$

Next, we turn our attention to the limiting distribution $U(t)$ of the $\mathbf{u}_n$-process, cf. (1.18).
From (2.18) and (6.3) we have for $\operatorname{Re} r \geq 0$,

$$\frac{\mathrm{E}\{\mathrm{e}^{-r\Delta\mathbf{u}}\}}{\mathrm{E}\{\mathrm{e}^{-r\Delta\mathbf{w}}\}} = [1 - aG(r\Delta, -r\Delta)]\mathrm{e}^a. \tag{6.11}$$

Because $\Delta \downarrow 0$ for $a \uparrow 1$ and $G(r\Delta, -r\Delta) \to 1 - \mathrm{e}^{-1}$, cf. (2.8), for $r \neq 0, a \uparrow 1$, we obtain from (6.11) that: for $\operatorname{Re} r \geq 0$,

$$\lim_{a\uparrow 1} \frac{\mathrm{E}\{\mathrm{e}^{-r\Delta\mathbf{u}}\}}{\mathrm{E}\{\mathrm{e}^{-r\Delta\mathbf{w}}\}} = 1. \tag{6.12}$$

From (6.7) and Feller's continuity theorem, cf. [8] vol II. p. 431, it follows that the L.S.-transform $\mathrm{E}\{\mathrm{e}^{-r\Delta\mathbf{w}}\}$ of the distribution of $\Delta\mathbf{w}$ has a limit for $a \uparrow 1, \operatorname{Re} r \geq 0$. Consequently, (6.12) implies that $\mathrm{E}\mathrm{e}^{-r\Delta\mathbf{u}}\}, \operatorname{Re} r \geq 0$, has a limit for $a \uparrow 1$. By applying again Feller's continuity theorem it follows that $\Delta\mathbf{u}$ converges in distribution for $a \uparrow 1$, i.e.

$$\Delta\mathbf{u} \textit{ has the same limiting distribution as } \Delta\mathbf{w}. \tag{6.13}$$

the latter distribution is given by (6.7).
Concerning the tail of the distributions of $\Delta\mathbf{u}$ and $\Delta\mathbf{w}$ we obtain from (5.23) for fixed a, $a \approx 1$ and $t \to \infty$,

$$\begin{aligned} &\Pr\{(1-\sqrt{a})^2\delta\mathbf{u} \geq t\} - \Pr(1-\sqrt{a})^2\delta\mathbf{w} \geq t\} = \\ &= -\frac{1}{\sqrt{\pi}}(1-\sqrt{a})\delta C(\frac{\beta}{t})^{\frac{1}{2}} + \mathrm{O}((1-\sqrt{a})^2\delta\frac{\beta}{t}), \end{aligned} \tag{6.14}$$

with C given by (5.14) and the tail of the distribution of $(1-\sqrt{a})^2\delta\mathbf{w}$ given by (6.9). Note that $C > 0$ for $a = 1, \delta = 1, \nu = 1\frac{1}{2}$, and that the first term in the righthand side of (6.9) is positive. Hence the first term in the asymptotic expression for $\Pr\{(1-\sqrt{a})^2\delta\mathbf{u} \geq t\}$ is somewhat smaller than the coresponding one in the asymptotic series for $\Pr\{(1-\sqrt{a})^2\delta\mathbf{w} \geq t\}$. This is plausible because for the $\mathbf{u}_n$-process the traffic is fed gradually into the buffer, whereas for the $\mathbf{w}_n$-process the traffic of a message is instantaneously fed into the buffer, note that the case $\gamma = 1$ is considered in this section.

7. Heavy traffic analysis for the case $\gamma = 1, 1 < \nu < 2$

In this section we consider the case

$$\gamma = 1,\ 0 < a < 1,\ a \uparrow 1,\ 1 < \nu < 2. \tag{7.1}$$

In [2], section 6, the stationary waiting time distribution $W(t)$ of the M/G/1 queue with traffic load a and service time distribution $B(t)$ as given by (4.1) with $1 < \nu < 2$ has been analyzed. In that section we have not derived an explicit expression for $W(t)$, however, the L.S.-transform $\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\}$ is explicitly available, cf. [2], form. (2.20) and (5.7). Therefore we use here a slightly different approach from that in the preceding section.
Put

$$\mu := 2 - \nu, \tag{7.2}$$

and suppose that μ is rational, say,

$$\mu = \frac{M}{N} \text{ with } M < N \text{ and g.c.d. } (M, N) = 1. \tag{7.3}$$

As in [2], cf. (5.10), (5.11), (5.12) and (5.13), it is shown that: for $\operatorname{Re} \rho \geq 0$,

$$\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\} = \frac{(1-y^N)^2}{A(y)}, \tag{7.4}$$

with

$$
\begin{aligned}
y \;&=\; (\frac{\rho}{\sigma})^{1/N}, \\
A(y) &= (1-y^N)^2 + \frac{a}{1-a} y^{N+M}[y^{N+M} - \frac{M+N}{M} y^M + \frac{N}{M}],
\end{aligned}
\tag{7.5}
$$

with the principal value of $(\rho/\sigma)^{1/N}$ so defined that $y > 0$ for $\rho > 0$.

In [2] appendix B it has been shown that $y = 1$ is a double zero of $A(y)$, of the $2N-2$ other zeros $y_n(a)$ of $A(y)$ there are exactly $N - M$ for which holds

$$
y_n(a) = (\frac{1-a}{a}\frac{M}{N})^{\frac{1}{N-M}} \mathrm{e}^{\frac{2n+1}{N-M}\pi i}, \; n = 1, \ldots, N-M, \tag{7.6}
$$

all the other zeros $y_n(a), n = N-M, \ldots, 2N$, are bounded away from zero for $a \uparrow 1$.
Put

$$
\rho = r\Delta \; , \; \mathrm{Re}\; r \geq 0, \; \Delta > 0, \tag{7.7}
$$

then we show below that a $\Delta \equiv \Delta(a)$ exists such that $\mathrm{E}\{\mathrm{e}^{-r\Delta \mathbf{w}}\}$ has a limit for $a \uparrow 1$.

With, cf. (7.5),

$$
y = (\frac{r\Delta}{\sigma})^{1/N}, \tag{7.8}
$$

we have from (7.5),

$$
A(y) = [1 - (\frac{r\Delta}{\sigma})^N]^2 + \frac{a}{1-a}\frac{N}{M}(\frac{r\Delta}{\sigma})^{\frac{N-M}{N}}[1 - \frac{M+N}{N}(\frac{r\Delta}{\sigma})^{\frac{M}{N}} + \frac{M}{N}(\frac{r\Delta}{\sigma})^{\frac{N+M}{N}}]. \tag{7.9}
$$

Take $\Delta > 0$ such that

$$
\frac{a}{1-a}\frac{N}{M}(\frac{\Delta}{\sigma})^{\frac{N-M}{N}} = 1.
$$

i.e.

$$
\Delta = [\frac{1-a}{a}\mu]^{\frac{1}{1-\mu}}\sigma. \tag{7.10}
$$

It then follows from (7.8) that: for $a \uparrow 1$,

$$
A((\frac{r\Delta}{\sigma})^{1/N}) = 1 + r^{\frac{N-M}{N}}, \tag{7.11}
$$

and hence from (7.4), (7.8), (7.9), (7.10) and (7.11),

$$
\lim_{a\uparrow 1} \mathrm{E}\{\mathrm{e}^{-r\Delta\mathbf{w}}\} = \frac{1}{1 + r^{\frac{N-M}{N}}}, \mathrm{Re}\, r \geq 0. \tag{7.12}
$$

As in [2] remark 6.1, it is shown that the assumption of rationality of μ, cf. (7.3), is not essential and so we have with

$$
\Delta = [\frac{1-a}{a}(2-\nu)]^{\frac{1}{\nu-1}} \frac{2-\nu}{\nu-1}\frac{\delta}{\beta}, \tag{7.13}
$$

that: for $1 < \nu < 2, \mathrm{Re}\, r \geq 0$,

$$
\lim_{a\uparrow 1} \mathrm{E}\{\mathrm{e}^{-r\Delta\mathbf{w}}\} = \frac{1}{1 + r^{\nu-1}}. \tag{7.14}
$$

From (7.14) we have: for $\mathrm{Re}\, r \geq 0, |r| < 1$,

$$[1-\lim_{a\uparrow 1}\mathrm{E}\{\mathrm{e}^{-r\Delta\mathbf{w}}\}]\,/r=\sum_{n=1}^{\infty}(-1)^{n-1}r^{n(\nu-1)-1}. \tag{7.15}$$

To the relation (7.15) we may apply the theorem of DOETSCH, [6] vol. II, p. 159, see appendix A of [2]. We obtain: for $t\to\infty, H\in\{1,2,\dots\}$,

$$\lim_{a\uparrow 1}\Pr\{\Delta\mathbf{w}\geq t\}=\sum_{n=1}^{H}(-1)^{n-1}\frac{t^{-n(\nu-1)}}{\Gamma(1-n(\nu-1))}+\mathrm{O}(t^{-(H+1)(\nu-1)}), \tag{7.16}$$

note that in the righthand of (7.16) all those terms with $n(\nu-1)$ a nonnegative integer should be deleted.

By using

$$\Gamma(-[n(\nu-1)-1])=\frac{\pi}{\Gamma(n(\nu-1))\sin\pi n(\nu-1)}, \tag{7.17}$$

we obtain: for $t\to\infty, H\in\{1,2,\dots\}$,

$$\lim_{a\uparrow 1}\Pr\{\Delta\mathbf{w}\geq t\}=\frac{1}{\pi}\sum_{n=1}^{H}(-1)^{n-1}\frac{\Gamma(n(\nu-1))\sin\pi n(\nu-1)}{t^{n(\nu-1)}}+\mathrm{O}(t^{-(H+1)(\nu-1)}). \tag{7.18}$$

As in section 6, cf. (6.13), it is shown that: for $a\uparrow 1$,

$\Delta\mathbf{u}$ *and* $\Delta\mathbf{w}$ *have the same limiting distribution.* (7.19)

8. HEAVY TRAFFIC ANALYSIS FOR THE CASE $\gamma<1, 1<\nu<2$.

In this section we derive a heavy traffic result for the stationary distribution $U(t)$ of the $\mathbf{u}_n$-process, cf. (1.18), for the case

$$0<\gamma<1,\ 0<a<1,\ a\uparrow 1,\ 1<\nu<2,\ a\gamma<1. \tag{8.1}$$

Again we start from the relation, i.e. for $\mathrm{Re}\, r=0$,

$$\frac{\mathrm{E}\{\mathrm{e}^{-r\tilde{\Delta}\mathbf{u}}\}}{\mathrm{E}\{\mathrm{e}^{-r\tilde{\Delta}\mathbf{w}}\}}=\frac{[1-\mathrm{E}\{\mathrm{e}^{r\tilde{\Delta}\mathbf{i}}\}])/\mathrm{E}\{\tilde{\Delta}\mathbf{i}\}}{[1-\mathrm{E}\{\mathrm{e}^{r\tilde{\Delta}\tilde{\mathbf{i}}}\}])/\mathrm{E}\{\tilde{\Delta}\tilde{\mathbf{i}}\}}\{1-G(r\Delta,-r\Delta)\}\mathrm{e}^{a}, \tag{8.2}$$

with $G(\rho,-\rho)$ given by (2.7) and, cf. (7.13),

$$\tilde{\Delta}:=[\frac{1-\gamma a}{\gamma a}(2-\nu)]^{\frac{1}{\nu-1}}\frac{2-\nu}{\nu-1}\frac{\delta}{\gamma\beta}. \tag{8.3}$$

Because $\mathbf{i}$ has a negative exponential distribution with mean Λ^{-1} it follows that: for $\mathrm{Re}\, r=0$,

$$\lim_{a\uparrow 1}\frac{1-\mathrm{E}\{\mathrm{e}^{r\tilde{\Delta}\mathbf{i}}\}}{\mathrm{E}\{\tilde{\Delta}\mathbf{i}\}}=1. \tag{8.4}$$

The joint distribution of $(\mathbf{b}_n,\boldsymbol{\pi}_n)$, cf. (1.8), is a true probability distribution for every finite $a>1$ similarly for that of $\boldsymbol{\delta}_n$. Consequently, since $\mathbf{u}_n\geq 0$ with probability one we have from (2.12),

$$\begin{aligned}\lim_{a\uparrow 1}\tilde{\Delta}\mathbf{i}\ &=\lim_{a\uparrow 1}-[\tilde{\Delta}\mathbf{u}_n+\tilde{\Delta}\mathbf{b}_n-\tilde{\Delta}\boldsymbol{\pi}_n-\tilde{\Delta}\boldsymbol{\delta}_n]^-\\ &=\lim_{a\uparrow 1}\{-[\tilde{\Delta}\mathbf{u}_n]^-\}=0.\end{aligned} \tag{8.5}$$

Hence from (2.8), (8.2), (8.4) and (8.8) we have: for $\mathrm{Re}\, r\geq 0$,

$$\lim_{a\uparrow 1}\frac{\mathrm{E}\{\mathrm{e}^{-r\tilde{\Delta}\mathbf{u}}\}}{\mathrm{E}\{\mathrm{e}^{-r\tilde{\Delta}\mathbf{w}}\}}=1. \tag{8.6}$$

In (8.6) $\mathbf{w}$ is a stochastic variable with distribution $W(t)$, which is the stationary distribution of the actual waiting time of an M/G/1 queue with arrival rate Λ and service time $\gamma\boldsymbol{\tau}$ and $B(t)$, as given by (4.1) for $1<\nu<2$, the distribution of $\boldsymbol{\tau}$. Hence $\mathrm{E}\{\mathrm{e}^{-\gamma\rho\boldsymbol{\tau}}\}$ is given by (4.8) with ρ replaced by $\gamma\rho$. Section 7 concerns the case $\gamma=1$, and so far the present case, cf. (8.1), we have to replace in (7.5) ρ by $\gamma\rho$ in the derivation of the expression of $\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\}$. Hence for the present case (7.8) becomes

$$y=(\frac{r\gamma\Delta}{\delta})^{1/N}. \tag{8.7}$$

For the present M/G/1 model the traffic load is

$$a\gamma=\Lambda\beta\gamma, \tag{8.8}$$

and so in (7.9),

a as to be replaced by $a\gamma$,

Δ as to be replaced by $\Delta\gamma$.

As in section 7 we obtain that: for $\mathrm{Re}\,\rho\geq 0$,

$$\lim_{a\gamma\uparrow 1}\mathrm{E}\{\mathrm{e}^{-r\tilde{\Delta}\mathbf{u}}\}=\lim_{a\gamma\uparrow 1}\}\mathrm{E}\{\mathrm{e}^{-r\tilde{\Delta}\mathbf{w}}\}=\frac{1}{1+r^{\nu-1}}, \tag{8.9}$$

with $\tilde{\Delta}$ given by (8.3). Actually, it remains to show that (8.9) applies for $\mathrm{Re}\,\rho>0$. But this follows easily by analytic continuation since the principal value of $r^{\nu-1}$ is defined to positive for $r>0$ and because $\tilde{\Delta}\mathbf{u}$ and $\tilde{\Delta}\mathbf{w}$ are nonnegative with probability one.

Hence as in section 6 it follows from (8.9) by using Feller's continuity theorem that: for $a\gamma\uparrow 1,\ \gamma<1$

$\tilde{\Delta}\mathbf{u}$ *and* $\tilde{\Delta}\mathbf{w}$ *have the same limiting distribution.* (8.10)

and hence they have the same tail probabilities. These are given by the righthand side of (7.16).

Remark 8.1. Note that (7.12) and (8.3) show that

$\Delta=\tilde{\Delta}$ for $\gamma=1$. □

9. Heavy traffic analysis for the case $\gamma>1, 1<\nu<2$.

In this section we consider the case

$$\gamma>1,\ a\gamma<1,\ 1<\nu<2. \tag{9.1}$$

For this case we have that $\mathbf{b}-\boldsymbol{\pi}\geq 0$ with probability one since $\mathbf{b}\geq\gamma\boldsymbol{\pi}$ with probability one, cf. (1.16). Hence the $\mathbf{u}_n$-process, cf. (1.18), is for the case $\gamma>1$ again the actual waiting process of an M/G/1 queue with arrival rate Λ^{-1} and service time distribution the distribution of $\mathbf{b}-\boldsymbol{\pi}$. By using the same argumentation as in section 7 but with Δ replaced by $\tilde{\Delta}$, cf. (7.10) and (8.3), we obtain again that: for $a\gamma\uparrow 1$

$\tilde{\Delta}\mathbf{u}$ *and* $\tilde{\Delta}\mathbf{w}$ *have the same limiting distribution,* (9.2)

and the tail probabilities, i.e. for $t\to\infty$,

$$\lim_{a\gamma\uparrow 1}\Pr\{\tilde{\Delta}\mathbf{w}\geq t\}$$

are given by the righthand side of (7.16).

Appendix A

In this appendix we show that the relation (2.6) holds for $\mathrm{Re}\,\rho \geq 0, \mathrm{Re}\,s > 0$. The validity of (2.6) for $\mathrm{Re}\,\rho \geq 0$, $\mathrm{Re}\,s > 0$, follows immediately from its derivation from (2.4) via the partial integration. From (1.6) it is seen that: for $\mathrm{Re}\,\rho \geq 0, t \geq 0$,

$$H(0,t) = \frac{\Lambda}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty+\varepsilon}^{\mathrm{i}\infty+\varepsilon} \frac{\mathrm{e}^{ut}}{u^2}\{1-\beta(u)\}\mathrm{d}u = \Lambda \int_0^t \{1-B(\tau)\}\mathrm{d}\tau, \tag{a.1}$$

because

$$\int_0^\infty \mathrm{e}^{-u\tau}\{1-B(\tau)\}\mathrm{d}\tau = \frac{1-\beta(u)}{u},\ \mathrm{Re}\,u \geq 0. \tag{a.2}$$

Hence from (1.6) and (a.1) for $\mathrm{Re}\,\rho \geq 0$,

$$0 \leq |\mathrm{E}\{\mathrm{e}^{-\gamma\rho\boldsymbol{\tau}}(\boldsymbol{\tau} \geq t)\}\mathrm{e}^{-H(\rho,t)}| \leq \{1-B(t)\}e^{-\Lambda\int_0^t\{1-B(\tau)\}\mathrm{d}\tau}.$$

Because

$$\begin{aligned}&\int_0^\infty \{1-B(t)\}e^{-\Lambda\int_0^t\{1-B(\tau)\}\mathrm{d}\tau}\mathrm{d}t = [-\frac{1}{\Lambda}\mathrm{e}^{-\Lambda\int_0^t\{1-B(\tau)\}\mathrm{d}\tau}]_0^\infty \\ &= \frac{1}{\Lambda}\{1-\mathrm{e}^{-\Lambda\beta}\} = \frac{1}{\Lambda}\{1-\mathrm{e}^{-a}\},\end{aligned} \tag{a.3}$$

it follows that the integral in the righthand side of (2.6) exsists for $\mathrm{Re}\,\rho \geq 0$ and $s = 0$, and since this integral is a Laplace transform it exists also for $\mathrm{Re}\,s \geq 0$. The lefthand side of (2.6) exist for $\mathrm{Re}\,\rho \geq 0, \mathrm{Re}\,s \geq 0$ and so continuity implies that (2.6) holds for $\mathrm{Re}\,\rho \geq 0, \mathrm{Re}\,s \geq 0$.

Appendix B

We consider here: for $\mathrm{Re}\,\rho \geq 0$,

$$\begin{aligned}&\mathrm{E}\{[1-\mathrm{e}^{-\rho(\boldsymbol{\tau}-t)})(\boldsymbol{\tau} \geq t)\} = \\ &\int_t^\infty \{1-\mathrm{e}^{-\rho(\tau-t)}\}\mathrm{d}B(\tau) = \rho\int_t^\infty \mathrm{e}^{-\rho(\tau-t)}\{1-B(\tau)\}\mathrm{d}\tau.\end{aligned} \tag{b.1}$$

By using (4.1) we have: for $\mathrm{Re}\,\rho \geq 0$,

$$\rho\int_t^\infty \mathrm{e}^{-\rho(\tau-t)}\{1-B(\tau)\}\mathrm{d}\tau = \frac{\delta\rho\sigma^{2-\nu}}{\Gamma(2-\nu)}\int_0^\infty \theta\mathrm{e}^{-\sigma\theta}\int_0^\infty \mathrm{e}^{-\rho u}(\theta+u+t)^{-\nu}\mathrm{d}u\mathrm{d}\theta. \tag{b.2}$$

we have with $\sigma > \rho > 0$,

$$\int_0^\infty \theta e^{-\sigma\theta}\int_0^\infty e^{-\rho u}(\theta+u+t)^{-\nu}\mathrm{d}u\mathrm{d}\theta = e^{\sigma t}\int_t^\infty (\mathrm{v}-t)e^{-(\sigma-\rho)\mathrm{v}}\int_{\mathrm{w}=\mathrm{v}}^\infty e^{-\rho\mathrm{w}}\mathrm{w}^{-\nu}\mathrm{d}\mathrm{w}\mathrm{d}\mathrm{v} =$$
$$e^{\sigma t}[-\frac{\mathrm{v}-t}{\sigma-\rho}]e^{-(\sigma-\rho)\mathrm{v}}\int_{\mathrm{v}}^\infty e^{-\rho\mathrm{w}}\mathrm{w}^{-\nu}\mathrm{d}\mathrm{w}|_{\mathrm{v}=t}^\infty]+$$
$$e^{\sigma t}\int_t^\infty[-\frac{e^{-(\sigma-\rho)\mathrm{v}}\mathrm{d}\mathrm{v}}{\sigma-\rho}\int_{\mathrm{v}}^\infty e^{-\rho\mathrm{w}}\mathrm{w}^{-\nu}\mathrm{d}\mathrm{w}-\frac{\mathrm{v}-t}{\sigma-\rho}e^{-(\sigma-\rho)\mathrm{v}}e^{-\rho\mathrm{v}}\mathrm{v}^{-\nu}]\mathrm{d}t =$$
$$\frac{e^{\sigma t}}{\sigma-\rho}\int_t^\infty e^{-(\sigma-\rho)\mathrm{v}}\int_{\mathrm{v}}^\infty e^{-\rho\mathrm{w}}\mathrm{w}^{-\nu}\mathrm{d}\mathrm{w}\mathrm{d}\mathrm{v}-\frac{e^{\sigma t}}{\sigma-\rho}\int_t^\infty(\mathrm{v}-t)e^{-\sigma\mathrm{v}}\mathrm{v}^{-\nu}\mathrm{d}v = \tag{b.3}$$
$$-\frac{e^{\sigma t}}{\sigma-\rho}\int_t^\infty(\mathrm{v}-t)e^{-\sigma\mathrm{v}}\mathrm{d}\mathrm{v}+\frac{e^{\sigma t}}{\sigma-\rho}[-\frac{e^{-(\sigma-\rho)\mathrm{v}}}{\sigma-\rho}\int_{\mathrm{v}}^\infty e^{-\rho\mathrm{w}}\mathrm{w}^{-\nu}\mathrm{d}\mathrm{w}|_t^\infty$$
$$-\frac{e^{\sigma t}}{(\sigma-\rho)^2}\int_t^\infty e^{-\sigma\nu}\mathrm{v}^{-\nu}\mathrm{d}\mathrm{v} =$$
$$\frac{e^{\rho t}}{(\sigma-\rho)^2}\int_t^\infty e^{-\rho\mathrm{w}}\mathrm{w}^{-\nu}\mathrm{d}\mathrm{w}-\frac{e^{\sigma t}}{(\sigma-\rho)^2}\int_t^\infty e^{-\sigma\mathrm{v}}\mathrm{v}^{-\nu}\mathrm{d}\mathrm{v}-\frac{e^{\sigma t}}{\sigma-\rho}\int_t^\infty(\mathrm{v}-t)e^{-\sigma\mathrm{v}}\mathrm{v}^{-\nu}\mathrm{d}\mathrm{v} =$$

$$\frac{\rho^{\nu-1}e^{\rho t}}{(\sigma-\rho)^2}\int_{\rho t}^\infty e^{-u}u^{-\nu}\mathrm{d}u-\frac{e^{\sigma t}\sigma^{\nu-1}}{(\sigma-\rho)^2}\int_{\sigma t}^\infty e^{-u}u^{-\nu}\mathrm{d}u-\frac{e^{\sigma t}\sigma^{\nu-1}}{(\sigma-\rho)^2}\int_{\sigma t}^\infty(\frac{u}{\sigma}-t)e^{-u}u^{-\nu}\mathrm{d}u =$$
$$\frac{\rho^{\nu-1}e^{\rho t}}{(\sigma-\rho)^2}\Gamma(1-\nu,\rho t)-\frac{\sigma^{\nu-1}e^{\sigma t}}{(\sigma-\rho)^2}\Gamma(1-\nu,\sigma t)$$
$$-\frac{\sigma^{\nu-2}e^{\sigma t}}{\sigma-\rho}\Gamma(2-\nu,\sigma t)+\frac{\sigma^{\nu-1}te^{\sigma t}}{\sigma-\rho}\Gamma(1-\nu,\sigma t).$$

The lefthand side of (b.2) is regular for $\mathrm{Re}\,\rho \geq 0$. It is readily seen from the last member of (b.3) that $\sigma = \rho$ is not a pole and hence by analytic continuation the relation (b.3) holds for all ρ with $\mathrm{Re}\,\rho \geq 0$.

From (b.1) and (b.3) we obtain for $\mathrm{Re}\,\rho \geq 0$,

$$\mathrm{E}\{[1-e^{-\rho(\boldsymbol{\tau}-t)}](\boldsymbol{\tau}\geq t)\} =$$
$$\frac{\delta}{\Gamma(2-\nu)}[\frac{\rho^\nu\sigma^{2-\nu}e^{\rho t}}{(\sigma-\rho)^2}\Gamma(1-\nu,\rho t)-\frac{\rho\sigma e^{\sigma t}}{(\sigma-\rho)^2}\Gamma(1-\nu,\sigma t) \tag{b.4}$$
$$-\frac{\rho e^{\sigma t}}{\sigma-\rho}\Gamma(2-\nu,\sigma t)+\frac{\rho\sigma te^{\sigma t}}{\sigma-\rho}\Gamma(1-\nu,\sigma t)].$$

From [6], vol 2, p. 135, we have

$$\Gamma(\mu,\sigma t) = \Gamma(\mu)-\sum_{n=0}^\infty\frac{(-1)^n}{n!}\frac{(\sigma t)^{\mu+n}}{\mu+n}\ ,\ t>0, \tag{b.5}$$

and for $t \to \infty$ and every $H \in \{1, 2, \ldots\}$,

$$\Gamma(\mu,\sigma t) = (\sigma t)^{-\mu-1}e^{-\sigma t}[\sum_{m=0}^H(-1)^m\frac{\Gamma(1-\mu+m)}{\Gamma(1-\mu)}\{(\sigma t)^{-m}+\mathrm{O}((\sigma t)^{-H-1})\}]. \tag{b.6}$$

From (b.4) and (b.5) it is seen that: for $0 \leq t < T < \infty$,

$$\mathrm{E}\{[1-\mathrm{e}^{-\rho(\boldsymbol{\tau}-t)}](\boldsymbol{\tau}\geq t)\}=$$
$$\frac{\delta}{\Gamma(2-\nu)}[\Gamma(1-\nu)(\rho/\sigma)^{\nu}\{1+\mathrm{O}(|\frac{\rho}{\sigma}|)\}+\mathrm{O}(|\frac{\rho}{\sigma}|)], \tag{b.7}$$

with $|\rho|\to 0,\ \mathrm{Re}\,\rho\geq 0$,
and: for $t>>T$,

$$\mathrm{E}\{[1-\mathrm{e}^{-\rho(\boldsymbol{\tau}-t)}](\boldsymbol{\tau}\geq t)\}=\frac{\delta}{1-\nu}(\frac{\rho}{\sigma})^{\nu}\{1+\mathrm{O}(|\frac{\rho}{\sigma}|)\}+\mathrm{O}(|\frac{\rho}{\sigma}|)\{1+\frac{1}{(\sigma t)^{\nu-1}}(1+\mathrm{O}(\frac{1}{\sigma t}))\}, \tag{b.8}$$

for $|\rho|\to 0, \mathrm{Re}\,\rho\geq 0$.
To calculate the behaviour of

$$\mathrm{E}\{(\boldsymbol{\tau}-t)[1-\mathrm{e}^{-\rho\boldsymbol{\tau}}](\boldsymbol{\tau}\geq t)\},\ \mathrm{Re}\,\rho\geq 0,$$

note that we have from (5.7) or (b.4): for $\mathrm{Re}\,\rho\geq 0$,

$$\mathrm{E}\{(\boldsymbol{\tau}-t)\mathrm{e}^{-\rho(\boldsymbol{\tau}-t)}(\boldsymbol{\tau}\geq t)\}=\frac{\mathrm{d}}{\mathrm{d}\rho}\mathrm{E}\{(1-\mathrm{e}^{-\rho(\boldsymbol{\tau}-t)})(\boldsymbol{\tau}\geq t)\}=$$
$$\begin{aligned}&\frac{\delta}{\Gamma(2-\nu)}[\{\frac{\nu\rho^{\nu-1}\sigma^{2-\nu}}{(\sigma-\rho)^2}-\frac{2\rho^{\nu}\sigma^{2-\nu}}{(\sigma-\rho)^3}+\frac{\rho^{\nu}t\sigma^{2-\nu}}{(\sigma-\rho)^2}\}\mathrm{e}^{\rho t}\int_{\rho t}^{\infty}\mathrm{e}^{-u}u^{-\nu}\mathrm{d}u-\frac{\sigma^{2-\nu}t^{-\nu}}{(\sigma-\rho)^2}\\&-\frac{\sigma\mathrm{e}^{\sigma t}}{(\sigma-\rho)^2}\Gamma(1-\nu,\sigma t)-\frac{\mathrm{e}^{\sigma t}}{\sigma-\rho}\Gamma(2-\nu,\sigma t)+\frac{\sigma t}{\sigma-\rho}\Gamma(1-\nu,\sigma t)\\&+\frac{2\rho\sigma\mathrm{e}^{\sigma t}}{(\sigma-\rho)^3}\Gamma(1-\nu,\sigma t)+\frac{\rho\mathrm{e}^{\sigma t}}{(\sigma-\rho)^2}\Gamma(2-\nu,\sigma t)-\frac{\rho\sigma t}{(\sigma-\rho)^2}\Gamma(1-\nu,\sigma t)].\end{aligned} \tag{b.9}$$

From (b.9) we obtain: for $\mathrm{Re}\,\rho\geq 0$,

$$\mathrm{E}\{(\boldsymbol{\tau}-t)[1-\mathrm{e}^{-\rho\boldsymbol{\tau}}](\boldsymbol{\tau}\geq t)\}=$$
$$\begin{aligned}&\frac{-\delta\mathrm{e}^{-\rho t}}{\Gamma(2-\nu)}[\{\frac{\nu\rho^{\nu-1}\sigma^{2-\nu}}{(\sigma-\rho)^2}-\frac{2\rho^{\nu}\sigma^{2-\nu}}{(\sigma-\rho)^3}+\frac{\rho^{\nu}t\sigma^{2-\nu}}{(\sigma-\rho)^2}\}\mathrm{e}^{\rho t}\int_{\rho t}^{\infty}\mathrm{e}^{-u}u^{-\nu}\mathrm{d}u+\\&\frac{2\rho\sigma\mathrm{e}^{\sigma t}}{(\sigma-\rho)^3}\Gamma(1-\nu,\sigma t)+\frac{\rho\mathrm{e}^{\sigma t}}{(\sigma-\rho)^2}\Gamma(2-\nu,\sigma t)-\frac{\rho\sigma t}{(\sigma-\rho)^2}\Gamma(1-\nu,\sigma t)].\end{aligned} \tag{b.10}$$

From (b.10) we obtain: for $|\rho|\to 0,\ \mathrm{Re}\,\rho\geq 0$,

$$\mathrm{E}\{(\boldsymbol{\tau}-t)[1-\mathrm{e}^{-\rho\boldsymbol{\tau}}](\boldsymbol{\tau}\geq t)\}=$$
$$\begin{aligned}&-\frac{\delta}{\Gamma(2-\nu)}[\nu\rho^{\nu-1}\sigma^{-\nu}\Gamma(1-\nu)\{1+\mathrm{O}(|\frac{\sigma}{\sigma})\}+\mathrm{O}(|\frac{\rho}{\sigma}|)]=\\&=-\frac{\delta\nu}{\nu-1}[\frac{1}{\sigma}(\frac{\rho}{\sigma})^{\nu-1}\{1+\mathrm{O}(|\frac{\rho}{\sigma}|)\}+\mathrm{O}(|\frac{\rho}{\sigma}|)].\end{aligned} \tag{b.11}$$

Next we derive an asymptotic expression for $\beta^{(1)}(\rho)$ for $\rho\to 0$. From (4.8) we have: for $\mathrm{Re}\,\rho\geq 0$,

$$\begin{aligned}&\beta^{(1)}(\rho)=\frac{\mathrm{d}}{\mathrm{d}\rho}\beta(\rho)=\frac{\mathrm{d}}{\mathrm{d}\rho}\{1-\frac{\rho\beta\sigma}{\sigma-\rho}-\frac{1}{2-\nu}\frac{\beta\sigma\rho^2}{(\sigma-\rho)^2}+\frac{1}{2-\nu}\frac{\beta\sigma\rho^2}{(\sigma-\rho)^2}(\frac{\rho}{\sigma})^{\nu-2}\}=\\&=\frac{-\beta\sigma}{\sigma-\rho}+\frac{\rho\beta\sigma}{(\sigma-\rho)^2}-\frac{2}{2-\nu}\frac{\beta\sigma\rho}{(\sigma-\rho)^2}+\frac{2}{2-\nu}\frac{\beta\sigma\rho^2}{(\sigma-\rho)^3}\\&+\frac{2}{2-\nu}\frac{\beta\sigma\rho}{(\sigma-\rho)^2}(\frac{\rho}{\sigma})^{\nu-2}-\frac{2}{2-\nu}\frac{\beta\sigma\rho^2}{(\sigma-\rho)^3}(\frac{\rho}{\sigma})^{\nu-2}-\frac{\beta\sigma\rho^2}{(\sigma-\rho)^2}\sigma^{2-\nu}\rho^{\nu-3}.\end{aligned} \tag{b.12}$$

Hence: for $|\rho| \to 0, \mathrm{Re}\,\rho \geq 0$,

$$\beta^{(1)}(\rho) = -\beta + \frac{\nu\beta}{2-\nu}(\frac{\rho}{\sigma})^{\nu-1}\{1 + \mathrm{O}(|\frac{\rho}{\sigma}|)\} + \mathrm{O}(|\frac{\rho}{\sigma}|). \tag{b.13}$$

Consequently: for $|\rho| \to 0$, $\mathrm{Re}\,\rho \geq 0$,

$$\mathrm{e}^{\frac{a}{\beta}\beta^{(1)}(\rho)} = \mathrm{e}^{-a}\{1 + \frac{\nu a}{2-\nu}(\frac{\rho}{\sigma})^{\nu-1}(1 + \mathrm{O}(|\frac{\rho}{\sigma}|)) + \mathrm{O}(|\frac{\rho}{\sigma}|)\}. \tag{b.14}$$

Appendix C

On the integral $\int_0^\infty \mathrm{e}^{-\rho t}\frac{1-B(t)}{\beta}\mathrm{e}^{-a\int\limits_0^t \frac{1-B(\tau)}{\beta}\mathrm{d}\tau}\mathrm{d}t$.

The integral

$$I(\rho) := \int\limits_0^\infty \mathrm{e}^{-\rho t}\frac{1-B(t)}{\beta}\mathrm{e}^{a\int\limits_t^\infty \frac{1-B(t)}{\beta}\mathrm{d}\tau}\mathrm{d}t \tag{c.1}$$

converges absolutely for $\mathrm{Re}\,\rho \geq 0$. It follows that

$$\begin{aligned} I(0) &= \frac{1}{a}(\mathrm{e}^a - 1), \\ I(\rho) &= \frac{1}{a}\mathrm{e}^a - \frac{\rho}{a}\int\limits_0^\infty \mathrm{e}^{-\rho t + a\int\limits_t^\infty \frac{1-B(\tau)}{\beta}\mathrm{d}\tau}\mathrm{d}t,\ \mathrm{Re}\,\rho > 0. \end{aligned} \tag{c.2}$$

For $B(t)$ given by (4.1) the relation (4.5) describes the asymptotic series for $1 - B(t)$, which we write as: for $t \to \infty$,

$$1 - B(t) = \sum_{n=0}^{H} b_n(\sigma t)^{-(n+\nu)} + \mathrm{O}((\sigma t)^{-(H+1+\nu))}), \tag{c.3}$$

for every $H \in \{0, 1, 2, \ldots, \}$, the coefficients b_n follow from (4.5).

Obviously we have for every $t > 0$,

$$\mathrm{e}^{a\int\limits_t^\infty \frac{1-B(\tau)}{\beta}\mathrm{d}\tau} = \sum_{n=0}^{\infty}\frac{a^n}{n!}[\int\limits_t^\infty \frac{1-B(\tau)}{\beta}\mathrm{d}\tau]^n, \tag{c.4}$$

and this series converges absolutely. Hence the lefthand side of (c.4) possesses an asymptotic expansion for $t \to \infty$. From (c.2) and (c.4) we have: for $\mathrm{Re}\,\rho > 0$,

$$\begin{aligned} I(\rho) \quad &= \frac{1}{a}\mathrm{e}^{-a} - \frac{\rho}{a}\int\limits_0^\infty \mathrm{e}^{-\rho t}\mathrm{d}t - \rho\int\limits_0^\infty \mathrm{e}^{\rho t}\{\int\limits_t^\infty \frac{1-B(\tau)}{\beta}\mathrm{d}\tau\}\mathrm{d}t \\ &-\rho\sum_{n=2}^{\infty}\frac{a^{n-1}}{n!}\int\limits_0^\infty \mathrm{e}^{-\rho t}\{\int\limits_t^\infty \frac{1-B(\tau)}{\beta}\mathrm{d}\tau\}^n\mathrm{d}t. \end{aligned} \tag{c.5}$$

From (4.5) we have: for $t \to \infty$,

$$\begin{aligned} 1 - B(t) \quad &= \frac{2-\nu}{\nu-1}\delta\frac{-1}{\Gamma(3-\nu)}\frac{\Gamma(\nu)}{\Gamma(\nu-1)}\frac{1}{(\sigma t)^\nu}\{1 + \mathrm{O}(\frac{1}{\sigma t})\} \\ &= \frac{-\delta}{\Gamma(2-\nu)}\frac{1}{(\sigma t)^\nu}\}1 + \mathrm{O}(\frac{1}{\sigma t})\}. \end{aligned} \tag{c.6}$$

Hence: for $t \to \infty$,

$$\int_t^\infty \frac{1-B(\tau)}{\beta}\mathrm{d}\tau = \frac{\delta}{(\nu-1)\Gamma(2-\nu)}\frac{1}{\sigma\beta}\frac{1}{(\sigma t)^{\nu-1}}\{1+\mathrm{O}(\frac{1}{\sigma t})\}. \tag{c.7}$$

Hence for $\mathrm{Re}\,\rho \geq 0$ and T sufficiently large,

$$\begin{aligned} I(\rho) \quad &= \frac{1}{a}(\mathrm{e}^a-1) - \rho\int_0^T \mathrm{e}^{-\rho t}\{\int_t^\infty \frac{1-B(\tau)}{\beta}\mathrm{d}\tau\}\mathrm{d}t \\ &-\rho\sum_{n=2}^\infty \frac{a^{n-1}}{n!}\int_0^T \mathrm{e}^{-\rho t}\{\int_t^\infty \frac{1-B(\tau)}{\beta}\mathrm{d}\tau\}^n\mathrm{d}t+ \\ &+\rho\frac{\delta}{(\nu-1)\Gamma(2-\nu)}\frac{1}{\sigma\beta}\int_T^\infty \mathrm{e}^{-\rho t}\frac{\mathrm{d}t}{(\sigma t)^{\nu-1}}\{1+\mathrm{O}(\frac{1}{\sigma t})\}+ \\ &+\rho\sum_{n=2}^\infty[\frac{-\delta}{(\nu-1)\Gamma(2-\nu)}\frac{1}{\sigma\beta}]^n\frac{a^{n-1}}{n!}\int_T^\infty \mathrm{e}^{-\rho t}\frac{\mathrm{d}t}{(\sigma t)^{n(\nu-1)}}\{1+\mathrm{O}(\frac{1}{(\sigma t)^n})\}. \end{aligned} \tag{c.8}$$

For our asymptotic analysis in Section 5 we need the asymptotic behaviour of $I(\rho)$ for $\rho \downarrow 0$. Note that the first two integrals in the righthand side of (c.8) are regular functions of ρ for all finite ρ.

Consider therefore, with fixed $T > 0$,

$$J(\rho,\mu) := \int_T^\infty \mathrm{e}^{-\rho t}t^\mu \mathrm{d}t. \tag{c.9}$$

As in [6] vol. I, p. 467, it is shown that for $\mu < 0$ with μ not a negative integer that

$$J(\rho,\mu) = \frac{\Gamma(\mu+1)}{\rho^{\mu+1}} + g(\rho), \tag{c.10}$$

with $g(\rho)$ an entire function of ρ, for every finite T. So we have

$$\int_T^\infty \mathrm{e}^{-\rho t}\frac{\mathrm{d}t}{(\sigma t)^{\nu-1}} = \frac{1}{\sigma^{\nu-1}}\frac{\Gamma(2-\nu)}{\rho^{2-\nu}} + g(\rho). \tag{c.11}$$

Hence it is seen that for finite but large $T, I(\rho)$ for $\mathrm{Re}\,\rho \geq 0$ may be written as

$$\begin{aligned} I(\rho) \quad &= \frac{1}{a}(\mathrm{e}^a-1)+h(\rho) \\ &+\frac{1}{\sigma\beta}\frac{\delta}{(\nu-1)\Gamma(2-\nu)}\frac{\Gamma(2-\nu)}{(\rho/\sigma)^{1-\nu}}\{1+\mathrm{O}(|\rho/\sigma|)\} \\ &= \frac{1}{a}(\mathrm{e}^a-1)+h(\rho)+\frac{1}{2-\nu}(\frac{\rho}{\sigma})^{\nu-1}\{1+\mathrm{O}(|\rho/\sigma|)\}, \end{aligned} \tag{c.12}$$

with $h(\rho)$ an entire function of ρ. Because $1 < \nu < 2$ we obtain from (c.2) that

$$h(0) = 0. \tag{c.13}$$

Hence: for $|\rho| \downarrow 0,\ |\arg\,\rho| \leq \frac{1}{2}\pi$,

$$I(\rho) = \frac{1}{a}(\mathrm{e}^a-1) - \frac{1}{2-\nu}(\rho/\sigma)^{\nu-1}\{1+\mathrm{O}(|\rho/\sigma|)\} + \mathrm{O}(|\rho/\sigma|). \tag{c.14}$$

because $h(\rho)$ is an entire function.

From (c.3) and (c.5) it is seen that $I(\rho)$ for Re $\rho \geq 0$ is the sum of an entire function of ρ and a function of ρ^{n-1}.

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