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A Proof of the Relativized, Non-Metric Form of Furstenberg's Structure Theorem

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In this note we present a rather direct proof of the result mentioned in the title. It combines methods of McMahon, McMahon and Wu, Glasner and Furstenberg. No use is made of auxiliary topologies (τ-topologies, or Furstenberg topologies), but instead the notion of a relatively invariant measure plays a central role.

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1. Introduction

In this paper we assume that the reader is familiar with the basic facts from abstract Topological Dynamics, as. e.g. exposed in the first chapter of [14]. Save a few exceptions, notation and terminology will be as in [14]. By a *flow* we shall mean a topological transformation group (ttg) with a compact Hausdorff phase space, and instead of the term "homomorphism of compact ttg's" we shall use *"morphism of flows"*. For the convenience of the reader we recapitulate a few definitions. All flows are assumed to have phase group T, except in Section 3, where we also consider subgroups of T (in that case, we add phrases like "... under H" or "H-...."). A set B in a flow $\mathscr{X}$ (resp. the flow $\mathscr{X}$ itself) is said to be *minimal* whenever for every point $x \in A$ (resp. $x \in X$) the equality $\overline{Tx} = A$ (resp. $\overline{Tx} = X$) holds. An *extension* is a surjective morphism of flows. If $\phi: \mathscr{X} \to \mathscr{Z}$ is an extension then

$$R_\phi \quad (\text{or } R(\phi)) := \{(x_1, x_2) \in X \times X : \phi(x_1) = \phi(x_2)\}$$

is a non-empty closed invariant set in the flow $\mathscr{X} \times \mathscr{X}$ (i.e., the space $X \times X$ on which T acts coordinate-wise); the subflow of $\mathscr{X} \times \mathscr{X}$ on R_ϕ is denoted by $\mathscr{R}_\phi$. Note that an extension $\phi: \mathscr{X} \to \mathscr{Z}$ is an isomorphism (i.e., injective, hence a homeomorphism) iff $R_\phi = \Delta_X := \{(x,x) : x \in X\}$.

An extension $\phi: \mathscr{X} \to \mathscr{Z}$ is said to be *equicontinuous* (in [14] the term "almost periodic" is used) whenever the action of T on X is "equicontinuous relative Z", i.e., uniformly equicontinuous on fibers of ϕ and uniformly so with respect to the fibers; more precisely:

$$\forall \alpha \in \mathfrak{U}\ \exists \beta \in \mathfrak{U}_X : T\beta \cap R_\phi \subseteq \alpha.$$

Here $\mathfrak{U}_X$ denotes the (unique) uniform structure generating the topology of X, and $T\beta$ has to be understood in the flow $\mathscr{X} \times \mathscr{X}$ (each $\beta \in \mathfrak{U}_X$ is a subset of $X \times X$). We say that an extension $\phi: \mathscr{X} \to \mathscr{Z}$ is *weakly mixing* whenever the flow $\mathscr{R}_\phi$ is *ergodic*, i.e., TU is dense in R_ϕ for every non-empty open subset U of R_ϕ (in its relative topology in $X \times X$, i.e., $U = V \cap R_\phi$ with V open in $X \times X$). It follows in a straigthforward way from these definitions that an extension $\phi: \mathscr{X} \to \mathscr{Z}$ that is both weakly mixing and equicontinuous has $R_\phi = \Delta_X$, i.e., is an isomorphism.

A pair of points $(x_1, x_2) \in X \times X$, where $\mathscr{X}$ is a flow, is said to be *proximal* (to each other) whenever $\overline{T(x_1,x_2)} \cap \Delta_X \neq \varnothing$ (here $T(x_1,x_2)$ has to be understood in $\mathscr{X} \times \mathscr{X}$); it is called a *distal pair* whenever $x_1 = x_2$ or (x_1,x_2) is not proximal. An extension $\phi: \mathscr{X} \to \mathscr{Z}$ is called *proximal* (resp., *distal*) whenever each pair of points in a fiber of ϕ is proximal (resp. distal), i.e., each $(x_1,x_2) \in R_\phi$ is a proximal (resp., distal) pair.

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By a *strictly-I-extension* we mean a "transfinite composition" of equicontinuous extensions. For the technical definition we refer to [13], p. 800/801 (where this notion is called "I-extension") or to [14], definition 4.1 with in condition (b) the words "either proximal or" omitted. The general form of Furstenberg's Structure Theorem (FST) now reads:

THEOREM (FST). *Every distal extension of minimal flows is a strictly-I-extension.*

In this general form the result was proved in [11]. For earlier versions, see [6], [3], [7] and [5]. By using an extension of the techniques of [5], McMahon and Wu in [11] reduce the general FST to one of the key results of [4]. The proof of this result in [4] depends heavily on Ellis' algebraic approach, including various τ-topologies. We shall use the construction of [11] in a different way and prove that every extension of minimal flows that is both distal and weakly mixing is an isomorphism: the "relativization" of the key result of [5]. Then an application of a construction from [8] reduces the proof of FST to the fact that an open RIM extension θ that has no non-trivial equicontinuous factor is weakly mixing. We outline a proof of this fact, which is a simplification and clarification of the proof in [10] (cf. also [14], VII 3.11).

2. RELATIVELY INVARIANT MEASURES

An extension $\theta:\mathscr{X}\to\mathscr{Y}$ is called a *RIM-extension* whenever there exists a continuous function $\lambda:Y\to M_1(X)$ (called a *section* for θ) such that (writing λ_y for $\lambda(y)$):

(a) $\forall (t,y)\in T\times Y$: $\lambda_{ty}=t\lambda_y$,

(b) $\forall y\in Y$: $\operatorname{Supp}\lambda_y\subseteq\theta^{\leftarrow}[y]$.

Here $M_1(X)$ is the space of all probability measures on X and for $\mu\in M_1(X)$ and $t\in T$, $t\mu$ is defined by $(t\mu)(f):=\mu(f^t)$, where $f^t(x):=f(tx)$ for $f\in C(X)$ and $x\in X$. We quote the following result from [8].

(2.1) THEOREM [8]. *Let $\theta:\mathscr{X}\to\mathscr{Y}$ be an extension of minimal flows. Then there is a commuting diagram of the form*

$$\begin{array}{ccc} \mathscr{X}' & \xrightarrow{\sigma} & \mathscr{X} \\ \theta' \downarrow & & \downarrow \theta \\ \mathscr{Y}' & \xrightarrow{\tau} & \mathscr{Y} \end{array}$$

where σ and τ are proximal extensions and θ' is a RIM-extension of minimal flows. The flow $\mathscr{X}'$ is obtainted as subflow of $\mathscr{X}\times\mathscr{Y}'$, namely on a minimal subset X' of the set

$$R_{\theta\tau} := \{(x,y)\in X\times Y' : \theta(x) = \tau(y')\},$$

and σ and θ' are the restrictions to X' of the canonical projections of $X\times Y'$ onto X and Y', respectively.

(2.2) REMARKS. 1. It is easy to verify that if in the above θ is a distal extension then θ' is distal as well, hence θ' is an *open* mapping ([1], p. 142), i.e. θ' is an open RIM-extension. (By [14], VII. 1.10 one may always assume that θ' is open, but that proof is much less elementary than the above).

2. If θ is distal then (as σ is proximal) $R_{\theta\tau}$ is minimal ("relativize" the proof of [9], II. 1.3), hence $X'=R_{\theta\tau}$.

(2.3) We shall use sections of RIM-extensions to construct continuous invariant fibre-wise pseudometrics (CIFP's). Let $\theta:\mathscr{X}\to\mathscr{Y}$ be an extension of flows. A *CIFP for θ* is a continuous mapping $\rho:R_\theta\to\mathbb{R}^+$ such that

(a) $\forall y\in Y: \rho|_{\theta^{\leftarrow}[y]\times\theta^{\leftarrow}[y]}$ is a pseudo-metric on $\theta^{\leftarrow}[y]$,

(b) $\forall (x_1,x_2)\in R_\theta\ \forall t\in T: \rho(tx_1,tx_2)=\rho(x_1,x_2)$.

If ρ is a CIFP for θ then $D(\rho):=\{(x_1,x_2)\in R_\theta:\rho(x_1,x_2)=0\}$ is a non-empty closed invariant (in the flow $\mathscr{X}\times\mathscr{X}$) equivalence relation on X.

(2.4) LEMMA. *Let $\theta:\mathscr{X}\to\mathscr{Y}$ be a RIM-extension and let λ be a section for θ. For every closed invariant subset N of R_θ, put $N[x]:=\{x'\in\mathscr{X}:(x,x')\in N\}$ $(x\in X)$ and*

$$\rho_N(x_1,x_2):=\lambda_{\theta(x_1)}((N[x_1]\setminus N[x_2])\cup(N[x_2]\setminus N[x_1]))$$

for $(x_1,x_2)\in R_\theta$. Then ρ_N is a CIFP for θ.

PROOF. It is elementary to prove that ρ_N fulfills (a) and (b) of (2.3). That ρ_N is continuous is implicitly proved on p. 228 of [10] (alternative proof: "relativize" p. 128/120 of [1]). $\square$

We say that an extension $\theta:\mathscr{X}\to\mathscr{Y}$ *has no proper equicontinuous factors* whenever in every factorization $\theta=\theta_2\circ\theta_1$ with θ_1 an extension and θ_2 an equicontinuous extension we have that θ_2 is an isomorphism.

(2.5) LEMMA. *Let $\theta:\mathscr{X}\to\mathscr{Y}$ be an extension without proper equicontinuous factors. Then every CIFP for θ is zero on R_θ.*

PROOF. $D(\rho)$ is a closed invariant equivalence relation on X so one can form the flow $\mathscr{X}/D(\rho)$ on the quotient space $X/D(\rho)$ such that the quotient map $\pi:\mathscr{X}\to\mathscr{X}/D(\rho)$ is an extension of flows. It is not hard to see that $\bar\rho(\pi x_1,\pi x_2):=\rho(x_1,x_2)$ $((x_1,x_2)\in R_\theta)$ defines a CIFP $\bar\rho$ on $R_{\bar\theta}=(\pi\times\pi)[R_\theta]$ such that $D(\bar\rho)=\Delta_{X/D(\rho)}$. By a straightforward compactness argument one shows that for every open $\alpha\in\mathscr{U}_{X/D(\rho)}$ there exists $\epsilon>0$ such that

$$U_\epsilon:=\{(\bar x_1,\bar x_2)\in R_{\bar\theta}:\bar\rho(\bar x_1,\bar x_2)\leqslant\epsilon\}\subseteq\alpha.$$

As U_ϵ is a nbd of $\Delta_{X/D(\rho)}$ in $R_{\bar\theta}$ there is $\beta\in\mathscr{U}_{X/D(\rho)}$ such that $\beta\cap R_{\bar\theta}\subseteq U_\epsilon$. But $TU=U$, hence $T\beta\cap R_{\bar\theta}\subseteq U\subseteq\alpha$. So $\bar\theta$ is equicontinuous, hence $\bar\theta$ is an isomorphism, i.e., $R_{\bar\theta}=\Delta_{X/D(\rho)}$. Hence $R_\theta=(\pi\times\pi)^{\leftarrow}[R_{\bar\theta}]=(\pi\times\pi)^{\leftarrow}[\Delta_{X/D(\rho)}]=D(\rho)$. $\square$

(2.6) LEMMA. *Let $\theta:\mathscr{X}\to\mathscr{Y}$ be an open RIM-extension of minimal flows. If every CIFP for θ is zero on R_θ then θ is weakly mixing.*

PROOF. let U_1 and U_2 be open subsets of X such that $(U_1\times U_2)\cap R_\theta\neq\varnothing$. We have show that $N:=\overline{T(U_1\times U_2)\cap R_\theta}$ equals R_θ, i.e., that $(V_1\times V_2)\cap N\neq\varnothing$ for every pair of open sets V_1 and V_2 in X such that $(V_1\times V_2)\cap R_\theta\neq\varnothing$. We may and shall assume that $\theta[U_1]=\theta[U_2]$ and $\theta[V_1]=\theta[V_2]$ [[for $i=1,2$, replace U_i by $U_i\cap\theta^{\leftarrow}[U]$, where $U:=\theta[U_1]\cap\theta[U_2]\neq\varnothing$ is open (because θ is open); similar for V_i]]. As $\mathscr{X}$ is minimal there exists $t_0\in T$ with $W:=t_0U_2\cap V_2\neq\varnothing$; select $w\in W$ such that $w\in\operatorname{Supp}\lambda_{\theta(w)}$. [[Note that the set of points $x\in X$ with $x\in\operatorname{Supp}\lambda_{\theta(x)}$ is dense: pick any $x_0\in X$, $x'\in\operatorname{Supp}\lambda_{\theta(x_0)}\subseteq\theta^{\leftarrow}[\theta x_0]$; then $\theta(x')=\theta(x_0)$, so $x'\in\operatorname{Supp}\lambda_{\theta(x')}$, hence $tx'\in\operatorname{Supp}t\lambda_{\theta(x')}=\operatorname{Supp}\lambda_{\theta(tx')}$, and Tx' is dense.]] By the choice of U_i and V_i $(i=1,2)$ there are $x_1\in V_1$ and $x_2\in t_0U_1$ with $\theta(x_1)=\theta(w)=\theta(x_2)$. Then

$$\varnothing\neq(\{x_2\}\times W)\cap R_\theta\subseteq t_0(U_1\times U_2)\cap R_\theta\subseteq N, \tag{1}$$

hence $W\cap\theta^{\leftarrow}[\theta x_2]\subseteq N[x_2]$, and therefore

$$(W\cap\theta^{\leftarrow}[\theta x_2])\setminus N[x_1]\subseteq N[x_2]\setminus N[x_1]. \tag{2}$$

By (2.4) and the hypothesis of the lemma it follows that the right-hand side of (2) has measure zero under $\lambda_{\theta(x_2)}$. As the left-hand side of (2) is open in $\theta^{\leftarrow}[\theta x_2]$ and the support of $\lambda_{\theta(x_2)}$ is included in $\theta^{\leftarrow}[\theta x_2]$ it follows that the left-hand side of (2) is disjoint from $\operatorname{Supp}\lambda_{\theta(x_2)}$, so $W\cap\operatorname{Supp}\lambda_{\theta(x_2)}\subseteq N[x_1]$. As $w\in W\cap\operatorname{Supp}\lambda_{\theta(x_2)}$ [[for $\theta(x_2)=\theta(w)$]] it follows that $(x_1,w)\in N$. But also $(x_1,w)\in V_1\times V_2$, so $(V_1\times V_2)\cap N\neq\varnothing$. $\square$

(2.7) COROLLARY [10]. *Let $\theta:\mathscr{X}\to\mathscr{Y}$ be an open RIM-extension of minimal flows. If θ has no proper equicontinuous factors then θ is weakly mixing.*

PROOF. Clear from (2.5) and (2.6). □

REMARK. It is one of the central problems in Topological Dynamics for which type of extensions of minimal flows the conclusion of (2.7) holds. See [14], VII. 3.17. The following theorem is a special case of this more general result. Its proof was motivated by Lemma 5.1 of [7].

(2.8) LEMMA. *Let $\theta:\mathscr{X}\to\mathscr{Y}$ be a distal extension of minimal flows. If θ has no proper equicontinuous factors then θ is weakly mixing.*

PROOF. Consider the diagram in (2.1). Then θ' is an open RIM-extension (cf. (2.2)1) and $(\sigma\times\sigma)[R(\theta')]=R_\theta$ (an easy consequence of (2.2) 2). It is sufficient to show that every CIFP ρ' for θ' factorises as $\rho'=\rho\circ(\sigma\times\sigma)$ with $\rho:R_\theta\to\mathbb{R}^+$: then ρ is a CIFP for θ, hence zero on R_θ by (2.5), so ρ' is zero on $R_{\theta'}$, and (since this holds for every CIFP for θ') by (2.6), θ' is weakly mixing. As $\sigma\times\sigma:\mathscr{R}(\theta')\to\mathscr{R}(\theta)$ is an extension of flows it follows that $\mathscr{R}(\theta)$ is ergodic, so θ is weakly mixing.

It remains to show that if $(x'_1,x'_2)\in R(\theta')$ and $(z'_1,z'_2)\in R(\theta')$ with $(\sigma\times\sigma)\ (x'_1,x'_2)=(\sigma\times\sigma)(z'_1,z'_2)$ then $\rho'(x'_1,x'_2)=\rho'(z'_1,z'_2)$ (this value then unambiguously defines $\rho(\sigma x'_1,\sigma x'_2)$). For $i=1,2$, let $x_i:=\sigma(x'_i)=\sigma(z'_i)$, $y':=\theta'(x'_1)=\theta'(x'_2)$ and $w':=\theta'(z'_1)=\theta'(z'_2)$; then $x'_i=(x_i,y')$, $z'_i=(x_i,w')$ and $(y',w')\in R_\tau$, so (y',w') is a proximal pair. Hence there are a net $\{t_\alpha\}_\alpha$ in T and a point $y''\in Y'$ such that $t_\alpha y'\rightsquigarrow y''$ and $t_\alpha w'\rightsquigarrow y''$. It follows that for $i=1,2$, assuming that $t_\alpha x_i\rightsquigarrow\bar{x}_i$ in X, $t_\alpha x'_i\rightsquigarrow(\bar{x}_i,y'')$ and $t_\alpha z'_i\rightsquigarrow(\bar{x}_i,y'')$. So

$$\rho'(x'_1,x'_2)=\rho'(t_\alpha x'_1,t_\alpha x'_2)\rightsquigarrow\rho'((\bar{x}_1,y''),(\bar{x}_2,y''))=:d,$$

hence $\rho'(x'_1,x'_2)=d$. Similarly, $\rho'(z'_1,z'_2)=d$ and, indeed, $\rho'(x'_1,x'_2)=\rho'(z'_1,z'_2)$. □

REMARK. The remarks in (2.2) hold also for so-called RIC-extensions. So the proof and the conclusion of (2.8) are valid in that case as well. Details will be published elsewhere.

3. THE CONSTRUCTION OF ELLIS/MCMAHON & WU

Aim of this section is to outline a proof of

(3.1) THEOREM. *Let $\theta:\mathscr{X}\to\mathscr{Y}$ be a distal extension of minimal flows. If θ is weakly mixing then θ is an isomorphism.*

Clearly, if we combine this theorem with (2.8) then we get: *a distal extension of minimal flows that has no proper equicontinuous factors is an isomorphism.* As by a transfinite construction one easily shows that every distal extension $\phi:\mathscr{X}\to\mathscr{Z}$ of minimal flows factorizes as $\phi=\psi_\infty\circ\theta_\infty$ with ψ_∞ a strictly-I-extension and θ_∞ an extension without proper equicontinuous factors which is, in addition, distal (for $R(\theta_\infty)\subseteq R_\phi$), the general FST easily follows.

In order to prove (3.1) we quote from [11] the following result.

(3.2) LEMMA. *let $\theta:\mathscr{X}\to\mathscr{Y}$ be an open extension of minimal flows and let ρ be a continuous pseudo-metric on $\mathscr{X}$. Then for every countable subgroup H of T there exists a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{\theta} & Y \\ \sigma_H\downarrow & & \tau_H\downarrow \\ X^*_H & \xrightarrow{\theta_H} & Y_H \end{array}$$

of continuous surjections with the following properties:
1. X_H^* *and* Y_H *are compact metric spaces,*
2. $\forall y \in Y: \sigma_H[\theta^{\leftarrow} y] = \theta_H^{\leftarrow}[\tau_H y]$
3. $(\sigma_H \times \sigma_H)[R_\theta] = R(\theta_H)$,
4. *There exist continuous actions of* H *on* X_H^* *and* Y_H *such that* σ_H, θ_H *and* τ_H *are extensions of* H-*flows.* □

REMARK. Condition 3 is not in [11], but follows easily from equality 2. We do *not* need that $\sigma_H \times \sigma_H$ maps $Q(\theta)$ onto $Q(\theta_H)$ for suitable H. Note that (3.2) can be applied when θ is distal (for distal implies open).

(3.3) LEMMA. *Let* $\theta: \mathscr{X} \to \mathscr{Y}$ *be an open weakly mixing extension of minimal flows. Then for every countable subgroup* H *of* T *there exists a countable subgroup* K *of* T *with* $K \supseteq H$ *such that* $\theta_K: X_K^* \to Y_K$ *is a weakly mixing extension of* K-*minimal* K-*flows.*

PROOF. Apply the method of [5] in the context of the construction of (3.2). (To justify the method one has to observe that for an ascending sequence $\{H_n\}_{n \in \mathbb{N}}$ of countable subgroups of T and $K := \cup_n H_n$ one can obtain not only X_K^* as inverse limit of the spaces X_{H_n}, but also $R(\theta_K)$ as the inverse limit of the spaces $R(\theta_{H_n})$.) □

(3.4) LEMMA. *Let* $\theta: \mathscr{X} \to \mathscr{Y}$ *be a distal extension of minimal flows. Then for every countable subgroup* K *of* T *such that* X_K^* *and* Y_K *are* K-*minmal,* $\theta_K: X_K^* \to Y_K$ *is a distal extension of minimal* K-*flows.*

PROOF. Let K be a countable subgroup of T, $K \supseteq H$, such that X_K^* (hence Y_K) is minimal under K. In [11] only point-distality (and openness) of θ_K could be obtained. We obtain distality, as follows. In Y there exists a K-minimal closed subset Y'; let $X' := \theta^{\leftarrow}[Y']$ and $\theta' := \theta|_{X'}: X' \to Y'$. Then θ' is a distal extension of K-flows, and as Y' is minimal under K' the set $R(\theta')$ is a union of K-minimal subsets ⟦ apply [9], II. 1.2 to $(x'_1, x'_2) \mapsto \theta'(x'_1) = \theta'(x'_2): R(\theta') \to Y'$⟧. As Y_K is minimal under K, τ_K maps Y' onto Y_K, so by condition 2 of (3.2), $\sigma_K \times \sigma_k$ maps $R(\theta')$ onto $R(\theta_K)$ and, consequently, $R(\theta_K)$ is a union of K-minimal sets. It follows easily that the (minimal!) orbit closure of any point in $R(\theta_K)$ is either included in or disjoint from $\Delta_{X_K^*}$, hence such a point is a K-distal pair. □

(3.5) COROLLARY. *Let* $\theta: \mathscr{X} \to \mathscr{Y}$ *be a distal and weakly mixing extension of minimal flows. Then for every countable subgroup* H *of* T *there exists a countable subgroup* K *of* T *with* $K \supseteq H$ *and* θ_K *an isomorphism.*

PROOF. By (3.2), (3.3) and (3.4) there is a countable subgroup $K \supseteq H$ such that $\theta_K: X_K^* \to Y_K$ is both a weakly mixing and a distal extension of *metric* K-minimal flows. But in the metrizable case it is easy and well-known that this implies that θ_K is an isomorphism ⟦ briefly: $R(\theta_K)$ has a point with dense orbit (because $R(\theta_K)$ is second countable), and it is a union of minimal sets (use [9], II.1.2) ⟧. □

(3.6) LEMMA. *Let* $\theta: \mathscr{X} \to \mathscr{Y}$ *be an extension of minimal flows. If* θ *is not an isomorphism than in (3.2) the pseudo-metric* ρ *can be chosen such that* θ_H *is not an isomorphism for any countable subgroup* H *of* T.

PROOF. Cf. [11], the proof of remark (i) on p. 286. □

(3.7) PROOF OF THEOREM (3.1): clear from (3.5) and (3.6). □

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