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Analysis of the Incompressible Navier-Stokes Equations with a Quasi Free-Surface Condition

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ABSTRACT
Numerical solution of free-surface flows with a free-surface that can be represented by a height-function, is of great practical importance. Dedicated methods have been developed for the efficient solution of steady free-surface potential flow. These methods solve a sequence of sub-problems, corresponding to the flow equations subject to a quasi free-surface condition. For steady free-surface Navier-Stokes flow, such dedicated methods do not exist. In the present report we propose an extension to Navier-Stokes flow of an iterative method which has been applied successfully to steady free-surface potential flow. We then examine the sub-problem corresponding to the incompressible Navier-Stokes equations subject to the quasi free-surface condition. We consider perturbations of a uniform, horizontal flow of finite depth and we show that the initial boundary value problem associated with the incompressible Navier-Stokes equations and the quasi free-surface condition allows stable wave solutions that exhibit a behavior that is typical of surface gravity waves. This indicates that the iterative method proposed is indeed suitable for solving steady free-surface Navier-Stokes flow. Implementation of the iterative method and numerical experiments are treated in a forthcoming report.

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1. Introduction

The numerical solution of flows which are partially bounded by a freely moving boundary, is of great practical importance. The numerical techniques available to solve such free-surface flows, can be categorized into *surface tracking* methods, the most prominent being the marker and cell method [HW65] and the volume of fluid method [HN81], *interface capturing* methods, e.g., [MOS92, KP97], and *surface fitting* methods [FMJ93]. It is generally acknowledged that if the free-boundary is smooth, in particular if the surface can be represented by a so-called height function, surface fitting methods are unsurpassed in accuracy and robustness. Since the free-surfaces occurring in many practical applications, for instance, ship hydrodynamics, are smooth, surface fitting methods have received much attention.

If time-dependent surface fitting methods are considered, there is generally no essential difference in the treatment of the free-surface in potential flow or Navier-Stokes flow. Independent of the flow model, the solution of the flow equations and the geometry of the free-boundary are usually separated. The flow equations are integrated over a small time interval, with the dynamic condition (cf. section 2.2) imposed at the free-surface. Subsequently, the position of the free surface is determined through the kinematic condition, employing the newly computed velocity field.

For surface fitting methods for steady free-surface flows, such a common approach does not exist. Whereas dedicated techniques have been developed for potential flow [Daw77, Cah84, Rav96], methods for Navier-Stokes flow simply continue the aforementioned transient process until a steady state is reached. This process converges slowly, and more efficient methods for Navier-Stokes flow are desirable.

The methods developed for steady free-surface potential flow for their efficiency exploit the fact that during the solution process neither the kinematic nor the dynamic condition needs to be satisfied. Instead of imposing the dynamic condition on the sub-problems, i.e., the flow problems corresponding to a given free-surface position, and using the kinematic condition to determine a new approximation to the free-surface location, any combination of boundary conditions can be imposed on the sub-problems and any operator that locates the free surface can be employed, provided that the sub-problems are well-posed, the resulting iterative process converges and the corresponding converged solution satisfies the dynamic conditions and the *steady* kinematic condition. This permits the construction of iterative approaches that for each sub-problem evaluation provide a more accurate approximation to the steady free-surface position than would be obtained if the usual time-dependent approach were followed. Efforts can then be directed to solving the sub-problems efficiently. For steady free-surface potential flow, several such iterative methods have been proposed, see, e.g., [Cah84, Rav96]. From the perspective of treatment of the free-surface, the important difference between these methods is the choice of the quasi free-surface condition, i.e., the boundary condition which is imposed on the sub-problems, and the operator that is applied to locate the free-surface after each sub-problem evaluation.

The development of numerical methods for free-surface potential flow benefits from the abundant body of literature that is available on free-surface potential flow. For an overview, see [Lig78, Sto92]. It appears that free-surface Navier-Stokes flow has not received as much attention. The question of viscous action on surface gravity waves has been investigated by several authors, see, e.g., [Ehr91] and the references therein. However, usually the Navier-Stokes equations in primitive variables are then abandoned in an early stage and a vorticity formulation is adopted, following [Lam45]. As a consequence, these analyses are of little avail to the development of computational methods for steady free-surface Navier-Stokes flow.

This report presents an examination of an iterative method for solving steady free-surface Navier-Stokes flow. The quasi free-surface condition and the operator that locates the free surface after each sub-problem evaluation, are a generalization to Navier-Stokes flow of those proposed in [Rav96] for potential flow. It will be shown that the incompressible Navier-Stokes equations, subject to this quasi free-surface condition, allow infinitesimally stable wave solutions. These wave solutions exhibit a behavior that is typical of surface gravity waves, which is indicative for the viability of the iterative method. The Navier-Stokes equations in primitive variable formulation are maintained throughout, although it will result that the wave solutions are unaffected by viscosity.

Implementation of the iterative method and numerical experiments are treated in a forthcoming report.

2. Incompressible free-surface flow

In this section we state the equations governing incompressible free-surface flow. First, we briefly discuss the equations describing viscous flow. Subsequently, appropriate interface conditions for free surface flows are obtained and the quasi free-surface condition is introduced.

2.1 Substrate

As the substrate of the free surface, we consider an incompressible, viscous fluid flow, subject to a constant gravitational force. The flow is characterized by the Froude number, Fr, and the Reynolds number, Re. Although we are interested in steady solutions only, for the purpose of analysis we consider the equations describing the aforementioned flow in their time-dependent form. The (non-dimensionalized) fluid velocity and pressure functions are identified by $\mathbf{v}(\mathbf{x}, t)$ and $p(\mathbf{x}, t)$, respectively, with $t \geq 0$ and $\mathbf{x} = x_\alpha \mathbf{e}^{(\alpha)} \in \mathbb{R}^d$ $(d = 2, 3)$. Here $\mathbf{e}^{(\alpha)}$ and x_α respectively denote Cartesian base vectors and coordinates and the summation convention applies to paired super- and subscripts, unless mentioned otherwise. Further, assuming the gravitational force to act in the negative x_d direction, it proves useful to introduce the hydrodynamic pressure, $\varphi(\mathbf{x}, t) \equiv p(\mathbf{x}, t) + \mathrm{Fr}^{-2} x_d$. Incompressibility

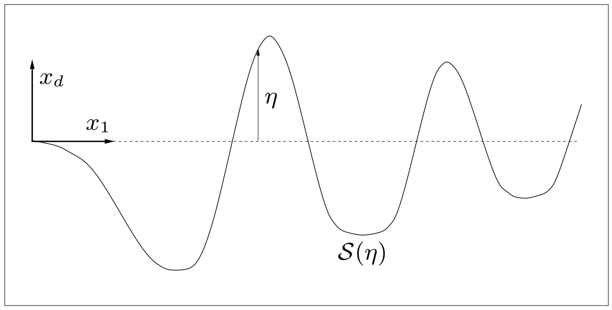

Figure 1: schematic illustration of the free-surface problem.

of the fluid implies that the velocity field is solenoidal:

$$M(\mathbf{v}, \varphi) \equiv \boldsymbol{\nabla} \cdot \mathbf{v} = 0, \tag{2.1a}$$

with $\boldsymbol{\nabla} = (\partial_1, \partial_2, \ldots, \partial_d)$ the usual gradient-operator in $\mathbb{R}^d$. Next, conservation of momentum is expressed by

$$\mathbf{N}(\mathbf{v}, \varphi) \equiv \frac{\partial \mathbf{v}}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{v}\mathbf{v} + \boldsymbol{\nabla}\varphi - \boldsymbol{\nabla} \cdot \boldsymbol{\tau}(\mathbf{v}) = 0, \tag{2.1b}$$

where $\boldsymbol{\tau}(\mathbf{v})$ stands for the viscous stress tensor for a Newtonian fluid, with Cartesian components

$$\tau_{\alpha\beta}(\mathbf{v}) = \mathrm{Re}^{-1} \left(\partial_\alpha v_\beta + \partial_\beta v_\alpha - \lambda (\boldsymbol{\nabla} \cdot \mathbf{v}) \delta_{\alpha\beta} \right). \tag{2.1c}$$

Here $\delta_{\alpha\beta}$ is the Kronecker symbol and λ is Stokes' constant. Clearly, for a solenoidal velocity field, the part multiplied by λ vanishes and, moreover, the viscous term reduces to $\boldsymbol{\nabla} \cdot \boldsymbol{\tau}(\mathbf{v}) = \mathrm{Re}^{-1} \Delta \mathbf{v}$.

2.2 Free-surface conditions

Free-surface flows are essentially two-phase flows, of which the properties of the contiguous bulk-fluids are such that their mutual interaction at the interface can be ignored. For an elaborate discussion of two-phase flows, see, for example, [Ari62] and [Scr60]. The free-surface conditions follow from the general interface conditions and the assumptions that both the density and the viscosity of the adjacent fluid vanish at the interface and, furthermore, that the interface is impermeable. Moreover, here it will be assumed that interfacial stresses can be ignored, which is a valid assumption in many practical applications.

We consider interfaces that can be represented as $\mathcal{S}(\eta) = \{(\mathbf{x}, t) \in \mathbb{R}^d \times \mathbb{R}_+ \mid x_d = \eta(\mathbf{x}_-, t)\}$, where $\mathbf{x}_- = (x_1, \ldots, x_{d-1})$ and $\eta(\mathbf{x}_-, t)$ stands for the vertical distance of the free surface relative to some reference level, usually, the undisturbed free surface; see the illustration in figure 1. The motion of the free surface is governed by a kinematic condition and d dynamic conditions. Impermeability dictates that a fluid particle is confined to the free surface:

$$\frac{\mathrm{d}x_d}{\mathrm{d}t} - \frac{\mathrm{d}\eta(\mathbf{x}_-, t)}{\mathrm{d}t} = 0, \qquad \forall (\mathbf{x}, t) \in \mathcal{S}(\eta).$$

This translates into the kinematic condition

$$K(\mathbf{v}, \eta) \equiv \frac{\partial \eta}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla}(\eta - x_d) = 0, \qquad \forall (\mathbf{x}, t) \in \mathcal{S}(\eta). \tag{2.2a}$$

Continuity of stresses at the interface is expressed by the normal dynamic condition,

$$N(\varphi, \eta) \equiv \varphi - \mathrm{Fr}^{-2} \eta = 0, \qquad \forall (\mathbf{x}, t) \in \mathcal{S}(\eta), \tag{2.2b}$$

and $d-1$ tangential dynamic conditions

$$T^\alpha(\mathbf{v},\eta) \equiv \mathbf{t}_\alpha \cdot \boldsymbol{\tau}(\mathbf{v}) \cdot \mathbf{n} = 0, \qquad \forall(\mathbf{x},t) \in \mathcal{S}(\eta). \tag{2.2c}$$

Here, $\mathbf{t}_\alpha$ ($\alpha = 1,\dots,d-1$) are orthogonal unit tangent vectors to $\mathcal{S}(\eta)$ and $\mathbf{n}$ denotes the unit normal vector to $\mathcal{S}(\eta)$. Notice that (2.2b) implies that the non-dimensionalized pressure vanishes at the free surface. Hence, assuming that viscous contributions to the normal stress are negligible, which is generally appropriate because $\mathbf{n} \cdot \boldsymbol{\tau}(\mathbf{v}) \cdot \mathbf{n} \ll p$, the combined free-surface conditions (2.2) imply that there is no transfer of either mass or momentum through the free-surface. Furthermore note that (2.2c) is naturally satisfied for inviscid fluids. Therefore, equations (2.2c) are often referred to as viscous free-surface conditions.

2.3 Quasi free-surface condition

The free-surface conditions (2.2) introduce an intrinsic coupling between $\mathbf{v}(\mathbf{x},t)$, $\varphi(\mathbf{x},t)$ and the spatial domain on which these are defined, through $\eta(\mathbf{x}_{\text{-}},t)$. It is this coupling that renders the numerical treatment of the steady free-surface problem difficult. However, it is possible to devise a quasi free-surface condition, i.e., a boundary condition that accounts for the movement of the boundary without its explicit deformation. This condition results from a combination of the normal dynamic condition and the kinematic condition. In particular,

$$K(\mathbf{v},\mathrm{Fr}^2\varphi) = 0, \qquad \forall(\mathbf{x},t) \in \mathcal{S}(\tilde{\eta}), \tag{2.3a}$$

for some suitable $\tilde{\eta} \equiv \tilde{\eta}(\mathbf{x}_{\text{-}})$. The location of the free-surface, in correspondence with (2.3a), is then determined by

$$\eta(\mathbf{x}_{\text{-}},t) = \mathrm{Fr}^2\varphi(\mathbf{x},t), \qquad \forall(\mathbf{x},t) \in \mathcal{S}(\tilde{\eta}). \tag{2.3b}$$

It can be verified that if a steady solution to the free-surface problem exists and $\lim_{t\to\infty}\eta(\mathbf{x}_{\text{-}},t) = \tilde{\eta}(\mathbf{x}_{\text{-}})$, then the solution must indeed satisfy (2.3).

Equations (2.3) lend themselves to a straightforward physical interpretation: the difference between $\eta(\mathbf{x}_{\text{-}},t)$ and $\tilde{\eta}(\mathbf{x}_{\text{-}})$ can be translated into an equivalent pressure defect according to

$$p(\mathbf{x},t) = \mathrm{Fr}^{-2}(\eta(\mathbf{x}_{\text{-}},t) - \tilde{\eta}(\mathbf{x}_{\text{-}})), \qquad \forall(\mathbf{x},t) \in \mathcal{S}(\tilde{\eta}).$$

Hence,

$$\eta(\mathbf{x}_{\text{-}},t) = \mathrm{Fr}^2 p(\mathbf{x},t) + \tilde{\eta}(\mathbf{x}_{\text{-}}) = \mathrm{Fr}^2\varphi(\mathbf{x},t), \qquad \forall(\mathbf{x},t) \in \mathcal{S}(\tilde{\eta}).$$

Thus, we obtain (2.3b). Substitution in the kinematic condition (2.2a) then yields (2.3a).

An iterative method based on (2.3), for a given estimation of the position of the free surface solves (2.1), with (2.3a) and (2.2c) imposed at the estimated free surface. Subsequently, equation (2.3b) is used to obtain an improved approximation to the free surface location.

3. Infinitesimal wave solutions

Generally, existence, uniqueness and stability of solutions of the incompressible Navier-Stokes equations are proved by means of variational analyses, see, e.g., [Tem83] and the references therein. Unfortunately, these analyses pose restrictive requirements on the type of auxiliary conditions considered. Furthermore, they provide no insight in the qualitative solution behavior. Therefore, we prefer a different analysis, viz., an infinitesimal analysis, i.e., we consider small perturbations of a basic solution. In particular, we examine wave solutions of the Navier-Stokes equations, subject to the above introduced quasi free-surface condition, corresponding to a perturbed uniform flow.

3.1 Formal solutions

We consider the incompressible Navier-Stokes equations (2.1) on a domain $\Omega = \{\mathbf{x} \in \mathbb{R}^d \mid -\infty < x_1, \ldots, x_{d-1} < +\infty, -1 < x_d < 0\}$ for $t > 0$, subject to suitable initial conditions and the quasi free-surface condition (2.3a) at $x_d = 0$ and an impermeability condition at the bottom:

$$v_d(\mathbf{x}, t) = 0, \qquad x_d = -1. \tag{3.1}$$

It is well known that the incompressible Navier-Stokes equations in d spatial dimensions are elliptic of order $2d$ in space and, therefore, a necessary condition for the initial boundary value problem to be well posed is that d boundary conditions are imposed at all boundaries. Hence, both at $x_d = 0$ and $x_d = -1$, $d-1$ boundary conditions remain to be specified. At $x_d = 0$, these are provided by the viscous dynamic conditions (2.2c). At $x_d = -1$, the conditions depend on the interaction of the fluid with the bottom.

Clearly, a solution to the initial boundary value problem is any pair $(\mathbf{v}, \varphi)(\mathbf{x}, t)$ that satisfies the differential equation (2.1) and the auxiliary conditions. However, we are interested in a specific class of wave solutions only. To keep the analysis tractable, we will restrict ourselves to showing that if (2.3a) and (3.1) are imposed on a subset of solutions of the (linearized) differential equation (2.1), this results in stable wave solutions. Hence, we do not require that these solutions satisfy the viscous dynamic conditions (2.2c). This amounts to ignoring boundary-layer effects on the solutions.

We consider solutions that correspond to a perturbation of magnitude $\mathcal{O}(\epsilon)$, $\epsilon \ll 1$, of a uniform horizontal flow and that can be expressed as formal solutions, i.e., as power series expansions in the parameter ϵ:

$$\begin{pmatrix} \mathbf{v} \\ \varphi \end{pmatrix}(\mathbf{x}, t) = \sum_{\alpha=0}^{\infty} \epsilon^\alpha \begin{pmatrix} \mathbf{v}^{(\alpha)} \\ \varphi^{(\alpha)} \end{pmatrix}(\mathbf{x}, t). \tag{3.2a}$$

We assume that the power series expansion (3.2a) converges uniformly. The generating solution corresponds to a uniform horizontal flow:

$$\begin{pmatrix} \mathbf{v}^{(0)} \\ \varphi^{(0)} \end{pmatrix}(\mathbf{x}, t) = \begin{pmatrix} (v_1^{(0)}, \ldots, v_{d-1}^{(0)}, 0) \\ 0 \end{pmatrix}, \qquad t \geq 0 \tag{3.2b}$$

for constant velocity components $v_1^{(0)}, \ldots, v_{d-1}^{(0)}$. It is easily verified that (3.2b) does indeed solve the initial boundary value problem (2.1), (2.3a), (3.1).

Upon inserting (3.2a) in (2.1), changing the order of operations and collecting identical powers of ϵ, one obtains:

$$\left.\begin{aligned} \sum_{\alpha=0}^{\infty} \epsilon^\alpha \left(\frac{\partial \mathbf{v}^{(\alpha)}}{\partial t} + \boldsymbol{\nabla} \varphi^{(\alpha)} - \mathrm{Re}^{-1} \Delta \mathbf{v}^{(\alpha)} + \sum_{\beta=0}^{\alpha} \mathbf{v}^{(\beta)} \cdot \boldsymbol{\nabla} \mathbf{v}^{(\alpha-\beta)} \right) &= 0 \\ \sum_{\alpha=0}^{\infty} \epsilon^\alpha \left(\boldsymbol{\nabla} \cdot \mathbf{v}^{(\alpha)} \right) &= 0 \end{aligned}\right\} \quad \forall \mathbf{x} \in \Omega,\ t > 0 \tag{3.3}$$

and, similarly, for the boundary conditions,

$$\sum_{\alpha=0}^{\infty} \epsilon^\alpha \left(\frac{\partial \varphi^{(\alpha)}}{\partial t} - \mathrm{Fr}^{-2} v_d^{(\alpha)} + \sum_{\beta=0}^{\alpha} \mathbf{v}^{(\beta)} \cdot \boldsymbol{\nabla} \varphi^{(\alpha-\beta)} \right) = 0, \qquad x_d = 0,\ t > 0 \tag{3.4a}$$

$$\sum_{\alpha=0}^{\infty} \epsilon^\alpha \left(v_d^{(\alpha)} \right) = 0, \qquad x_d = -1,\ t > 0. \tag{3.4b}$$

Because (3.3) and (3.4) must hold for all values of ϵ, it follows that the terms in each of the summations must vanish separately. Hence, with the generating solution according to (3.2b), one obtains the sequence of linear initial boundary value problems

$$\left.\begin{aligned} \underline{\mathbf{N}}(\mathbf{v}^{(\alpha)}, \varphi^{(\alpha)}) &= -\sum_{\beta=1}^{\alpha-1} \mathbf{v}^{(\beta)} \cdot \boldsymbol{\nabla} \mathbf{v}^{(\alpha-\beta)} \\ \boldsymbol{\nabla} \cdot \mathbf{v}^{(\alpha)} &= 0 \end{aligned}\right\} \qquad \forall \mathbf{x} \in \Omega,\ t > 0, \tag{3.5a}$$

with

$$\underline{\mathbf{N}}(\mathbf{v}^{(\alpha)}, \varphi^{(\alpha)}) \equiv \frac{\partial \mathbf{v}^{(\alpha)}}{\partial t} + \mathbf{v}^{(0)} \cdot \boldsymbol{\nabla} \mathbf{v}^{(\alpha)} + \boldsymbol{\nabla} \varphi^{(\alpha)} - \mathrm{Re}^{-1} \Delta \mathbf{v}^{(\alpha)} \tag{3.5b}$$

subject to

$$\frac{\partial \varphi^{(\alpha)}}{\partial t} + \mathbf{v}^{(0)} \cdot \boldsymbol{\nabla} \varphi^{(\alpha)} - \mathrm{Fr}^{-2} v_d^{(\alpha)} = -\sum_{\beta=1}^{\alpha-1} \mathbf{v}^{(\beta)} \cdot \boldsymbol{\nabla} \varphi^{(\alpha-\beta)}, \qquad x_d = 0,\ t > 0, \tag{3.6a}$$

$$v_d^{(\alpha)} = 0, \qquad x_d = -1,\ t > 0, \tag{3.6b}$$

for $\alpha = 1, 2, \ldots$. The non-linearity of the momentum equation (2.1b) and the kinematic condition (2.2a) now appears as a right-hand side term composed of lower-order contributions to the expansion. The homogeneous solutions of the above problems, i.e., the solutions corresponding to a vanishing right-hand side in (3.5) and (3.6), are identical for all values of α and correspond to the infinitesimal solution ($\alpha = 1$).

In the next section we derive general infinitesimal solutions for (3.5). Subsequently, in § 3.3, it will be shown that a subclass of these solutions that satisfies the boundary conditions, is stable and exhibits a behavior that is typical of surface gravity waves.

3.2 General infinitesimal solutions

In this section we derive solutions of (3.5) for $\alpha = 1$. For convenient notation, let $\mathbf{q}(\mathbf{x}, t) = ({v_1}^{(1)}, \ldots, {v_d}^{(1)}, \varphi^{(1)})(\mathbf{x}, t)$. Similar to [Kre70], we introduce new variables, $\mathbf{q}(\mathbf{x}, t) = s(\mathbf{x}, t)\, \mathbf{h}(\mathbf{x}, t)$. However, we suppose that $s(\mathbf{x}, t)$ can be represented by

$$s(\mathbf{x}, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \underline{s}(\boldsymbol{\eta}, \sigma)\, e^{\sigma t + \boldsymbol{\eta} \mathbf{x}}\, \mathrm{d}\sigma\, \mathrm{d}\boldsymbol{\eta} \tag{3.7}$$

and that $\mathbf{h}(\mathbf{x}, t)$ can be Fourier transformed:

$$\mathbf{h}(\mathbf{x}, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{\mathbf{h}}(\boldsymbol{\xi}, \omega)\, e^{\mathrm{i}(\omega t + \boldsymbol{\xi} \mathbf{x})}\, \mathrm{d}\omega\, \mathrm{d}\boldsymbol{\xi}. \tag{3.8}$$

Hence, with $\boldsymbol{\theta} = \boldsymbol{\eta} + \mathrm{i}\boldsymbol{\xi}$, $\tau = \sigma + \mathrm{i}\omega$ and $\hat{\mathbf{q}}(\boldsymbol{\theta}, \tau) = \underline{s}(\boldsymbol{\eta}, \sigma)\, \hat{\mathbf{h}}(\boldsymbol{\xi}, \omega)$, we can write in succinct form

$$\mathbf{q}(\mathbf{x}, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{\mathbf{q}}(\boldsymbol{\theta}, \tau)\, e^{\tau t + \boldsymbol{\theta} \mathbf{x}}\, \mathrm{d}\tau\, \mathrm{d}\boldsymbol{\theta}. \tag{3.9}$$

Inserting the transformed solutions (3.9) into (3.5) and changing the order of operations:

$$\mathbf{P}(\mathbf{q}(\mathbf{x}, t)) \equiv \left(\frac{\mathbf{N}}{M}\right)(\mathbf{q}(\mathbf{x}, t)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{\mathbf{P}}(\boldsymbol{\theta}, \tau) \cdot \hat{\mathbf{q}}(\boldsymbol{\theta}, \tau)\, e^{\tau t + \boldsymbol{\theta} \mathbf{x}}\, \mathrm{d}\tau\, \mathrm{d}\boldsymbol{\theta}, \tag{3.10}$$

with M and $\underline{\mathbf{N}}$ defined by (2.1a) and (3.5b), respectively, and the symbol $\hat{\mathbf{P}}(\boldsymbol{\theta},\tau)$ according to

$$\hat{\mathbf{P}}(\boldsymbol{\theta},\tau) = \begin{pmatrix} H(\boldsymbol{\theta},\tau) & 0 & \dots & 0 & \theta_1 \\ 0 & H(\boldsymbol{\theta},\tau) & \dots & 0 & \theta_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & H(\boldsymbol{\theta},\tau) & \theta_d \\ \theta_1 & \theta_2 & \dots & \theta_d & 0 \end{pmatrix}, \tag{3.11a}$$

where

$$H(\boldsymbol{\theta},\tau) = \tau + \mathbf{v}^{(0)}\cdot\boldsymbol{\theta} - \mathrm{Re}^{-1}\boldsymbol{\theta}\cdot\boldsymbol{\theta}. \tag{3.11b}$$

Herein, $\boldsymbol{\theta}\cdot\boldsymbol{\theta} = \theta_1^2 + \cdots + \theta_d^2$. The values of $\boldsymbol{\theta}$ and τ for which

$$\det(\hat{\mathbf{P}}(\boldsymbol{\theta},\tau)) = -\boldsymbol{\theta}\cdot\boldsymbol{\theta}\,(H(\boldsymbol{\theta},\tau))^{d-1} = 0, \tag{3.12}$$

correspond to nontrivial homogeneous solutions of $\mathbf{P}(\mathbf{q}(\mathbf{x},t))$ and, hence, to solutions of (3.5) for $\alpha = 1$.

In view of the linearity of $\mathbf{P}$, a general homogeneous solution can be expressed as a linear combination of the aforementioned homogeneous solutions. Let $\{\hat{\mathbf{q}}_1(\boldsymbol{\theta},\tau),\dots,\hat{\mathbf{q}}_n(\boldsymbol{\theta},\tau)\}$, $n = \dim(\ker(\hat{\mathbf{P}}(\boldsymbol{\theta},\tau)))$, denote a basis of the kernel of the symbol, e.g., if $\boldsymbol{\theta}\cdot\boldsymbol{\theta} = 0$ and $H(\boldsymbol{\theta},\tau) \neq 0$, then $n = 1$ and

$$\hat{\mathbf{q}}_1(\boldsymbol{\theta},\tau) = (\theta_1,\dots,\theta_d,-H(\boldsymbol{\theta},\tau))^{\mathrm{T}}. \tag{3.13}$$

Then, a general homogeneous solution to (3.5) can be expressed as

$$\mathbf{q}(\mathbf{x},t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{j=1}^{n} w_j(\boldsymbol{\theta},\tau)\,\hat{\mathbf{q}}_j(\boldsymbol{\theta},\tau)\,e^{\tau t+\boldsymbol{\theta}\mathbf{x}}\,\mathrm{d}\tau\,\mathrm{d}\boldsymbol{\theta}, \tag{3.14}$$

for certain weight-functions $w_j(\boldsymbol{\theta},\tau)$.

3.3 Wave solutions

Subsequently, we will exclusively consider a specific subclass of solutions of the form (3.14), viz.,

$$\mathbf{q}(\mathbf{x},t) = \int_{\mathcal{H}}\int_{\mathcal{T}(\boldsymbol{\theta})} w_1(\boldsymbol{\theta},\tau)\,\hat{\mathbf{q}}_1(\boldsymbol{\theta},\tau)\,e^{\tau t+\boldsymbol{\theta}\mathbf{x}}\,\mathrm{d}\tau\,\mathrm{d}\boldsymbol{\theta}, \tag{3.15}$$

with $\mathcal{H} = \{\boldsymbol{\theta}\in\mathbb{C}^d \mid \boldsymbol{\theta}\cdot\boldsymbol{\theta} = 0\}$, $\mathcal{T}(\boldsymbol{\theta}) = \{\tau\in\mathbb{C} \mid H(\boldsymbol{\theta},\tau)\neq 0\}$ and $\hat{\mathbf{q}}_1(\boldsymbol{\theta},\tau)$ according to (3.13). From the previous section, we recall that $\mathcal{H}\times\mathcal{T}$ corresponds to a subclass of solutions of the differential equation (3.5), $\alpha = 1$, for which the kernel of the symbol is spanned by $\hat{\mathbf{q}}_1(\boldsymbol{\theta},\tau)$ only. In the following, it will become evident that this subclass accommodates the wave solutions.

Observe that the viscous contribution to the symbol, i.e., the part multiplied by the inverse of the Reynolds number in (3.11b), vanishes for $\boldsymbol{\theta}\in\mathcal{H}$. This indicates that these solutions are unaffected by viscous effects. Consequently, our results on solution behavior are essentially the same as those obtained in [Lig78] and [Sto92] for potential flow.

Next, it will be shown that the initial boundary value problem (3.5), (3.6), $\alpha = 1$ is well posed for solutions of the form (3.15), in the sense that for all $t > 0$ a solution exists that can be bounded in terms of the initial conditions. Moreover, it results that the solution displays typical wave like behavior in horizontal directions.

The argument used here is similar to that in [Kre70], where for hyperbolic partial differential equations on a half space it is ascertained that an unbounded homogeneous solution of the differential operator can be expressed as a linear combination of l linearly independent normalized solutions, with

l the number of entering characteristics at the boundary. The main theorem of the paper then asserts that the initial boundary value problem is well posed if, in the space of these normalized solutions, only the trivial solution satisfies the homogeneous boundary conditions.

Having established that (3.15) is indeed a homogeneous solution of (3.5), next, we will show that solutions of the form (3.15) that satisfy the boundary conditions can be bounded in terms of the initial conditions.

Let $\tau = \overline{\tau}(\boldsymbol{\theta})$ specify a hyper-surface in $\mathcal{H} \times \mathcal{T}$, such that the specific solution, equation (3.15), solves (3.5) and (3.6) for $\alpha = 1$. Such an expression for τ is generally referred to as a dispersion relation for the initial boundary value problem. We introduce initial conditions $\mathbf{q}_0(\mathbf{x})$:

$$\mathbf{q}(\mathbf{x}, 0) = \mathbf{q}_0(\mathbf{x}), \qquad \forall \mathbf{x} \in \Omega. \tag{3.16}$$

It follows that for all $t > 0$

$$\|\mathbf{q}(\mathbf{x}, t)\|_\Omega = \left\| \int_{\mathcal{H}} w_1(\boldsymbol{\theta}, \overline{\tau}(\boldsymbol{\theta}))\, \hat{\mathbf{q}}_1(\boldsymbol{\theta}, \overline{\tau}(\boldsymbol{\theta}))\, e^{\overline{\tau}(\boldsymbol{\theta})\, t + \boldsymbol{\theta}\mathbf{x}}\, \mathrm{d}\boldsymbol{\theta} \right\|_\Omega \leq \max_{\boldsymbol{\theta} \in \mathcal{H}} e^{\Re(\overline{\tau}(\boldsymbol{\theta}))\, t}\, \|\mathbf{q}_0(\mathbf{x})\|_\Omega, \tag{3.17}$$

Therefore, if the stability condition $\Re(\overline{\tau}(\boldsymbol{\theta})) \leq 0$ is fulfilled for all $\boldsymbol{\theta} \in \mathcal{H}$, then at all times the solution can be bounded in terms of the initial conditions. Subsequently, we will show that this stability condition is indeed fulfilled for solutions (3.15) that satisfy the boundary conditions (3.6), $\alpha = 1$ and for which the initial conditions are bounded on Ω.

In order for the initial conditions to be bounded on Ω, the real part of θ_j must vanish in horizontal directions, i.e., $\Re(\theta_j) = 0$ for $j = 1, \ldots, d-1$. Hence, we substitute $\boldsymbol{\theta} = \boldsymbol{\theta}^+$ or $\boldsymbol{\theta} = \boldsymbol{\theta}^-$, where $\boldsymbol{\theta}^\pm = (\mathrm{i}k_1, \ldots, \mathrm{i}k_{d-1}, \pm\kappa)^{\mathrm{T}}$, with $\kappa = (k_1^2 + \cdots + k_{d-1}^2)^{1/2}$ and $k_j \in \mathbb{R}$. It is easily verified that then $\boldsymbol{\theta}^\pm \in \mathcal{H}$. Thus, denoting by $\boldsymbol{\theta}_{\text{-}} = (\mathrm{i}k_1, \ldots, \mathrm{i}k_{d-1})^{\mathrm{T}}$, $\underline{w}(\boldsymbol{\theta}) = w_1(\boldsymbol{\theta}, \overline{\tau}(\boldsymbol{\theta}))$ and $\underline{H}(\boldsymbol{\theta}) = H(\boldsymbol{\theta}, \overline{\tau}(\boldsymbol{\theta}))$, we find that

$$\begin{aligned} \mathbf{q}(\mathbf{x}, t) = \int_{-\infty}^{\infty} & \underline{w}(\boldsymbol{\theta}^+)\, \left(\boldsymbol{\theta}^+, -\underline{H}(\boldsymbol{\theta}^+)\right)^{\mathrm{T}} \exp(\overline{\tau}(\boldsymbol{\theta}^+)\, t + \boldsymbol{\theta}_{\text{-}} \cdot \mathbf{x}_{\text{-}} + \kappa\, x_d) + \\ & \underline{w}(\boldsymbol{\theta}^-)\, \left(\boldsymbol{\theta}^-, -\underline{H}(\boldsymbol{\theta}^-)\right)^{\mathrm{T}} \exp(\overline{\tau}(\boldsymbol{\theta}^-)\, t + \boldsymbol{\theta}_{\text{-}} \cdot \mathbf{x}_{\text{-}} - \kappa\, x_d)\, \mathrm{d}\boldsymbol{\theta}_{\text{-}} \end{aligned} \tag{3.18}$$

are solutions of the form (3.15) that are bounded on Ω.

Next, equation (3.18) is introduced into the boundary conditions (3.6), $\alpha = 1$. Imposing (3.6b) for all $\mathbf{x}_{\text{-}}$, one obtains a relation for the weight function $\underline{w}(\boldsymbol{\theta})$:

$$\underline{w}(\boldsymbol{\theta}^+)\, \kappa\, e^{-\kappa} - \underline{w}(\boldsymbol{\theta}^-)\, \kappa\, e^{\kappa} = 0. \tag{3.19}$$

Hence, we put $\underline{w}(\boldsymbol{\theta}^+) = \gamma\, e^{\kappa}$ and $\underline{w}(\boldsymbol{\theta}^-) = \gamma\, e^{-\kappa}$, for some arbitrary constant γ. Then, inserting the result in (3.6a), after some minor manipulations, we obtain the dispersion relation:

$$\left(\overline{\tau}(\boldsymbol{\theta}) + \mathrm{i}(v_1^{(0)} k_1 + \cdots + v_{d-1}^{(0)} k_{d-1})\right)^2 = -\mathrm{Fr}^{-2} \kappa \tanh(\kappa) \tag{3.20}$$

Clearly, equation (3.20) implies that $\Re(\overline{\tau}(\boldsymbol{\theta})) = 0$ and it follows that the initial boundary value problem corresponding to (3.5) and (3.6), $\alpha = 1$ is indeed well posed for solutions of the form (3.15).

For $d = 2$, $\mathbf{v}^{(0)} = 0$, the dispersion relation (3.20) is identical to that derived in [Lig78, Sto92] for surface gravity waves in potential flow in a channel of finite, uniform depth. A detailed account of the phenomena to which this dispersion relation gives rise, can be found there. Two particularly important aspects of (3.20) will be recalled here. For stationary surface gravity waves in $\mathbb{R}^2$, we have $\overline{\tau}(\boldsymbol{\theta}) = 0$, $d = 2$ and $k_1 = \kappa$. Hence, with the scaling $|\mathbf{v}^{(0)}| = 1$, the dispersion relation reduces to

$$1 = \mathrm{Fr}^{-2} \kappa^{-1} \tanh(\kappa). \tag{3.21}$$

First, notice that in the sub-critical case, i.e., for $\mathrm{Fr} < 1$, equation (3.21) specifies a unique relation between the Froude number and the wavelength, $\lambda \equiv 2\pi/\kappa$. This implies that a steady solution is

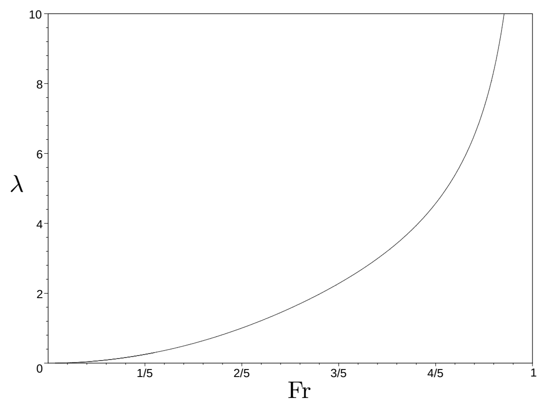

Figure 2: Relation between the wave-length, λ, and the Froude number, Fr, for steady surface gravity waves in a channel of finite, uniform depth.

a purely sinusoidal wave in the horizontal direction, the wavelength of which is determined by the Froude number only. Figure 2 displays the relation between the wavelength of the steady surface gravity wave and the Froude number. Secondly, for supercritical flows, i.e., for $\mathrm{Fr} > 1$, no solution to (3.21) exists. Hence, for supercritical flows, no such sinusoidal wave-like solution can exist.

Summarizing the results of this section, we find that there are wave perturbations of a uniform flow that satisfy the incompressible Navier-Stokes equations. Subject to the quasi free-surface condition (2.3a) and the impermeability condition at the bottom, these waves are stable and obey the usual dispersion relation for surface gravity waves if $\mathrm{Fr} < 1$, whereas for $\mathrm{Fr} > 1$ they are non-existent.

4. Conclusions

Motivated by the demand for more efficient computational methods for steady free-surface Navier-Stokes flow in practical applications, we proposed an extension of an iterative method which has been applied successfully to steady free-surface potential flow. For its efficiency, the method employs a quasi free-surface condition. To investigate the viability of the iterative method, we examined the sub-problem corresponding to the incompressible Navier-Stokes equations subject to this quasi free-surface condition. We considered solutions that can be expanded as perturbations of a uniform horizontal flow. Subsequently, we derived general infinitesimal solutions of the incompressible Navier-Stokes equations. It was then shown that a subclass of these general solutions is stable and displays a behavior that is typical of surface gravity waves, if subject to the quasi free-surface condition. This indicates that the iterative method proposed is indeed suitable for solving steady free-surface Navier-Stokes flow.

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