BOOTSTRAPPING U-QUANTILES

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ABSTRACT

The asymptotic consistency of the bootstrap approximation for generalized quantiles of U-statistic structure (U-quantiles for short) is established. The same method of proof also yields the asymptotic accuracy of the bootstrap approximation in this case. Applications to location and spread estimators, such as the classical sample quantile, the Hodges-Lehmann estimator of location and a spread estimator proposed by Bickel and Lehmann are given.

1. INTRODUCTION

Let $X_1, X_2, \ldots$ be independent random variables defined on a common probability space $(\Omega, \mathcal{A}, P)$, having common unknown distribution function (df) F. Let $h(x_1, \ldots, x_m)$ be a kernel of degree m (i.e. a real-valued measurable function symmetric in its m arguments) and let

$$H_F(y) = P(h(X_1, \ldots, X_m) \leq y), \qquad y \in \mathbb{R} \tag{1}$$

denote the df of the random variable $h(X_1, \ldots, X_m)$. Define, for each $n \geq m$ and real y,

$$H_n(y) = \binom{n}{m}^{-1} \sum_{1 \leq i_1 < \ldots < i_m \leq n} \cdots \sum I(h(X_{i_1}, \ldots, X_{i_m}) \leq y) \tag{2}$$

the empirical df of U-statistic structure.

Let, for $0 < p < 1$, $\xi_p = H_F^{-1}(p)$, denote the p-th quantile corresponding to H_F, and let $\hat{\xi}_{pn} = H_n^{-1}(p)$ denote its empirical counterpart. Generalized

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quantiles of the form $\hat{\xi}_{pn} = H_n^{-1}(p)$ are called U-quantiles. Choudhury and Serfling (1988) note that $\hat{\xi}_{pn} \to \xi_p$, a.s. $[P]$, as $n \to \infty$, and, in addition, that, as $n \to \infty$,

$$n^{\frac{1}{2}}(\hat{\xi}_{pn} - \xi_p) \xrightarrow{d} N(0, \sigma^2) \tag{3}$$

where

$$\sigma^2 = m^2 \zeta_p h_F^{-2}(\xi_p) \tag{4}$$

with

$$\zeta_p = Var(g_p(X_1)) > 0 \tag{5}$$

and

$$g_p(X_1) = E(I(h(X_1, \ldots, X_m) \leq \xi_p) \mid X_1) - p \tag{6}$$

provided H_F has density h_F positive at ξ_p.

In applications one often wishes to establish a confidence interval for $\xi_p = H_F^{-1}(p)$ and a studentized version of (3) is required. A strongly consistent estimator of the asymptotic variance σ^2 is proposed by Choudhury and Serfling (1988). It requires $0(n^{2m-1})$ computational steps. They also propose a strongly consistent but less efficient estimator requiring only $0(n)$ computational steps.

The aim of this paper is to employ bootstrap methods for the construction of a confidence interval for $\xi_p = H_F^{-1}(p)$. In Section 2 we establish a bootstrap analog of (3), under a slightly more stringent smoothness condition on H_F and in Section 3 we establish the asymptotic accuracy of this bootstrap approximation. Applications to certain estimators of location and spread, such as the classical sample quantile, the Hodges-Lehmann estimator of location and a spread estimator proposed in Bickel and Lehmann (1979) are discussed in Section 4.

2. CONSISTENCY OF THE BOOTSTRAP FOR U-QUANTILES

Let $\hat{F}_n$ denote the empirical df based on $X_1, \ldots, X_n$. Define $\hat{\xi}^*_{pn} = H_n^{*-1}(p), 0 < p < 1$, where H_n^* denotes the empirical df of U-statistic structure based on the bootstrap sample $X_1^*, \ldots, X_n^*$. Here and elsewhere $X_1^*, \ldots, X_n^*$

denotes a random sample of size n from $\hat{F}_n$, conditionally given $X_1, \ldots, X_n$.

Our first main result is as follows :

Theorem 2.1. *Suppose that H_F is continuously differentiable (with density h_F) on a neighborhood of ξ_p with $h_F(\xi_p) > 0$. Then, for almost every sample sequence $X_1, X_2, \ldots$*

$$n^{\frac{1}{2}}(\hat{\xi}^*_{pn} - \hat{\xi}_{pn}) \xrightarrow{d} N(0, \sigma^2) \tag{7}$$

with σ^2 as in (4).

Proof. With $\hat{P}_n$ the probability measure corresponding to $\hat{F}_n$, we have

$$\begin{aligned}
&\hat{P}_n(n^{\frac{1}{2}}(\hat{\xi}^*_{pn} - \hat{\xi}_{pn}) \leq x) \\
&= \hat{P}_n(n^{\frac{1}{2}}(H^{*-1}_n(p) - H^{-1}_n(p)) \leq x) \\
&= \hat{P}_n(H^{*-1}_n(p) \leq H^{-1}_n(p) + xn^{-\frac{1}{2}}) \\
&= \hat{P}_n(H^*_n(H^{-1}_n(p) + xn^{-\frac{1}{2}}) \geq p) \\
&= \hat{P}_n(W^*_n \geq -\bar{D}_n)
\end{aligned} \tag{8}$$

where

$$W^*_n = n^{\frac{1}{2}}\{H^*_n(H^{-1}_n(p) + xn^{-\frac{1}{2}}) - \bar{H}_n(H^{-1}_n(p) + xn^{-\frac{1}{2}})\} \tag{9}$$

with, for each $n \geq m$ and real y,

$$\bar{H}_n(y) = n^{-m} \sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} I(h(X_{i_1}, \ldots, X_{i_m}) \leq y) \tag{10}$$

the empirical df of von Mises structure, and

$$\bar{D}_n = n^{\frac{1}{2}}\{\bar{H}_n(H^{-1}_n(p) + xn^{-\frac{1}{2}}) - p\}. \tag{11}$$

We first consider $\bar{D}_n$. Note that

$$\bar{D}_n = \sum_{i=1}^{3} \bar{D}_{in} \tag{12}$$

where

$$\begin{aligned}
\bar{D}_{1n} = n^{\frac{1}{2}}\{&\bar{H}_n(H^{-1}_n(p) + xn^{-\frac{1}{2}}) - H_F(H^{-1}_n(p) + xn^{-\frac{1}{2}})\} \\
&-n^{\frac{1}{2}}\{\bar{H}_n(H^{-1}_n(p)) - H_F(H^{-1}_n(p))\}
\end{aligned} \tag{13}$$

$$\bar{D}_{2n} = n^{\frac{1}{2}}\{H_F(H^{-1}_n(p) + xn^{-\frac{1}{2}}) - H_F(H^{-1}_n(p))\} \tag{14}$$

and

$$\bar{D}_{3n} = n^{\frac{1}{2}}\{\bar{H}_n(H_n^{-1}(p)) - p\}. \tag{15}$$

To treat $\bar{D}_{1n}$ note first that $\bar{D}_{1n} = D_{1n} + O(n^{-\frac{1}{2}})$ a.s. $[P]$, as $n \to \infty$, where D_{1n} is obtained from $\bar{D}_{1n}$ by replacing $\bar{H}_n$ by H_n, with H_n as in (2). Suppose without loss of generality that $x > 0$. Clearly, for n sufficiently large,

$$|D_{1n}| \leq \sup_{\substack{|t-s| \leq xn^{-\frac{1}{2}} \\ s,t \in J}} |U_n(t) - U_n(s)| \quad \text{a.s. } [P] \tag{16}$$

where J is the neighborhood of ξ_p on which H_F is continuously differentiable, and

$$U_n(t) = n^{\frac{1}{2}}(H_n(t) - H_F(t)), \quad t \in I\!R \tag{17}$$

denotes the empirical process of U-statistic structure.

Similarly as in Silverman (1983) it is easy to see that

$$|D_{1n}| \leq (n!)^{-1} \sum_{\alpha} \sup_{\substack{|t-s| \leq xn^{-\frac{1}{2}} \\ t,s \in J}} |U^{\alpha}_{[\frac{n}{m}]}(t) - U^{\alpha}_{[\frac{n}{m}]}(s)| \tag{18}$$

where, for any given permutation α of $\{1, 2, \ldots, n\}$ $U^{\alpha}_{[\frac{n}{m}]}(t)$ denotes the empirical process based on the $[\frac{n}{m}]$ independent random variables $h(X_{\alpha(mj+1)}, \ldots, X_{\alpha(mj+m)}), j = 0, 1, \ldots, [\frac{n}{m}] - 1$, all with common df H_F. With impunity we may replace at stage $n+1$ any of the $n.n!$ permutations α of $\{1, \ldots, n+1\}$ which do not extend those of $\{1, \ldots, n\}$ by one of the $n!$ permutations which do extend those of $\{1, \ldots, n\}$. Application of relation (2.13) of Stute (1982) to each of the resulting $n!$ terms appearing on the r.h.s. of (18) directly yields that $D_{1n} = O(n^{-\frac{1}{4}}(ln\, n)^{\frac{1}{2}})$ a.s. $[P]$, as $n \to \infty$, hence,

$$\bar{D}_{1n} = O(n^{-\frac{1}{4}}(ln\, n)^{\frac{1}{2}}) \quad \text{a.s. } [P], \quad \text{as } n \to \infty. \tag{19}$$

Here we have used the smoothness of H_F as well as the inequality (18).

Next we consider $\bar{D}_{2n}$. Using again the smoothness assumption on H_F and employing the a.s. $[P]$ convergence of $\hat{\xi}_{pn}$ to ξ_p, as $n \to \infty$, we easily obtain from the mean value theorem that

$$\bar{D}_{2n} \to x h_F(\xi_p) \quad \text{a.s. } [P], \quad \text{as } n \to \infty. \tag{20}$$

Finally note that $\bar{D}_{3n} = O(n^{-\frac{1}{2}})$. We can conclude that

$$\bar{D}_n \to x h_F(\xi_p) \text{a.s. } [P], \quad \text{as } n \to \infty. \tag{21}$$

Next we consider the limit behaviour of W_n^*, $n = m, m+1, \ldots$ (cf (9)), conditionally given $\hat{F}_n$. Obviously, given $\hat{F}_n, W_n^*$ is a normalized U-statistic of degree m, with bounded kernel, depending on n, of the form

$$h_n(x_1, \ldots, x_m) = I(h(x_1, \ldots, x_m) \leq \hat{\xi}_{pn} + xn^{-\frac{1}{2}}) - \bar{H}_n(\hat{\xi}_{pn} + xn^{-\frac{1}{2}}). \tag{22}$$

Of course $E_{\hat{F}_n} W_n^* = E_{\hat{F}_n} h_n(X_1^*, \ldots, X_m^*) = 0$, a.s. $[P]$, whereas it is easily checked that

$$Var_{\hat{F}_n}(W_n^*) \sim m^2 E_{\hat{F}_n} g_n^2(X_1^*) \quad \text{a.s. } [P], \text{ as } n \to \infty, \tag{23}$$

where

$$g_n(X_1^*) = E_{\hat{F}_n}(h_n(X_1^*, \ldots, X_m^*)|X_1^*). \tag{24}$$

A simple argument involving the strong law for U-statistics with estimated parameters (Theorem 2.9 of Iverson and Randles (1989)) directly yields that

$$E_{\hat{F}_n} g_n^2(X_1^*) \to \zeta_p \quad \text{a.s. } [P], \text{ as } n \to \infty, \tag{25}$$

with ζ_p as in (5).

At this point we apply the Berry-Esseen bound for U-statistics of degree m of van Zwet (1984) to find that

$$\begin{aligned} &\sup_y |\hat{P}_n(W_n^* \leq y) - \Phi(ym^{-1}\zeta_p^{-\frac{1}{2}})| \\ &\qquad = O\{(\frac{E_{\hat{F}_n}|g_n(X_1^*)|^3}{(E_{\hat{F}_n} g_n^2(X_1^*))^{\frac{3}{2}}} + \frac{E_{\hat{F}_n} h_n^2(X_1^*, \ldots, X_m^*)}{E_{\hat{F}_n} g_n^2(X_1^*)})n^{-\frac{1}{2}}\}. \end{aligned} \tag{26}$$

Note that, in contrast to Corollary 4.1 of van Zwet (1984), the asymptotic variance instead of the exact variance of W_n^* is employed. It is easy to see that this does not affect the bound (26). The different standardization will give rise to an additional term of type

$$\frac{E_{\hat{F}_n} h_n^2(X_1^*, \ldots, X_m^*)}{E_{\hat{F}_n} g_n^2(X_1^*)} n^{-\frac{1}{2}} \tag{27}$$

which is already present in van Zwet's bound. Because h_n is bounded by 1, for all n, and combining (25) with the fact that $\zeta_p > 0$, we easily see that the moments appearing on the r.h.s. of (26) are $O(1)$ a.s. $[P]$, as $n \to \infty$. Hence the r.h.s. of (26) is $O(n^{-\frac{1}{2}})$ a.s. $[P]$, as $n \to \infty$.

From (8), (21) and (26) we obtain

$$\begin{aligned} \hat{P}_n(n^{\frac{1}{2}}(\hat{\xi}_{pn}^* - \hat{\xi}_{pn}) \leq x) & \\ &= 1 - \Phi(-\bar{D}_n m^{-1} \zeta_p^{-\frac{1}{2}}) + O(n^{-\frac{1}{2}}) \\ &= \Phi(x\sigma^{-1}) + o(1) \end{aligned} \tag{28}$$

a.s. $[P]$, as $n \to \infty$. This completes the proof of theorem. □

For the special case $m = 1, h(x) = x, p = \frac{1}{2}$, the classical sample median, our result reduces to Proposition 5.1 of Bickel and Freedman (1981). An insightful proof of their proposition is given in an unpublished note by Sheehy and Wellner (1988). Our proof is in part inspired by their argument.

3. ACCURACY OF THE BOOTSTRAP FOR U-QUANTILES

From (3) and (7) we know that the bootstrap approximation for a normalized U-quantile is asymptotically consistent. In this section we investigate the a.s. rate at which the difference between the bootstrap approximation and the exact distribution of a normalized U-quantile tends to zero, as the sample size gets large.

Theorem 3.1. *Suppose that the assumptions of Theorem 2.1 are satisfied. Suppose, in addition, that h_F satisfies a Lipschitz condition of order $\geq \frac{1}{2}$ on a neighborhood of ξ_p. Then*

$$\sup_x |\hat{P}_n(n^{\frac{1}{2}}(\hat{\xi}_{pn}^* - \hat{\xi}_{pn}) \leq x) - P(n^{\frac{1}{2}}(\hat{\xi}_{pn} - \xi_p) \leq x)| = O(n^{-\frac{1}{4}}(ln\, n)^{\frac{3}{4}}) \tag{29}$$

a.s.$[P]$, *as* $n \to \infty$.

For the special case $m = 1, h(x) = x$, the classical p-th sample quantile, Singh (1981) obtained a slightly better a.s. rate : the factor $(ln\, n)^{\frac{3}{4}}$ in (29) is replaced by $(\ln\ln n)^{\frac{1}{2}}$ in this case. Whenever the same improvement holds true for U-quantiles appears to be an interesting open problem.

Proof. First note that

$$\sup_x |\hat{P}_n(n^{\frac{1}{2}}(\hat{\xi}^*_{pn} - \hat{\xi}_{pn}) \le x) - P(n^{\frac{1}{2}}(\hat{\xi}_{pn} - \xi_p) \le x)| \le \sum_{i=1}^{3} I_{in} \tag{30}$$

where, for some constant $K > 0$,

$$I_{1n} = \sup_{|x| \le K(\ln n)^{\frac{1}{2}}} |\hat{P}_n(n^{\frac{1}{2}}(\hat{\xi}^*_{pn} - \hat{\xi}_{pn}) \le x) - \Phi(x\sigma^{-1})| \tag{31}$$

and

$$I_{2n} = \sup_{|x| > K(\ln n)^{\frac{1}{2}}} |\hat{P}_n(n^{\frac{1}{2}}(\hat{\xi}^*_{pn} - \hat{\xi}_{pn}) \le x) - \Phi(x\sigma^{-1})| \tag{32}$$

and

$$I_{3n} = \sup_x |P(n^{\frac{1}{2}}(\hat{\xi}_{pn} - \xi_p) \le x) - \Phi(x\sigma^{-1})|.$$

We first consider I_{1n}. Going through the proof of Theorem 2.1 we easily verify that

$$\sup_{|x| \le K(\ln n)^{\frac{1}{2}}} |\bar{D}_n - xh_F(\xi_p)| = 0(n^{-\frac{1}{4}}(\ln n)^{\frac{3}{4}}) \quad \text{a.s. } [P], \text{ as } n \to \infty. \tag{33}$$

Here we have used (see (18)) that

$$\sup_{|x| \le K(\ln n)^{\frac{1}{2}}} |D_{1n}| \le (n!)^{-1} \sum_{\alpha} \sup_{\substack{|t-s| \le Kn^{-\frac{1}{2}}(\ln n)^{\frac{1}{2}} \\ t,s \in J}} |U^{\alpha}_{[\frac{n}{m}]}(t) - U^{\alpha}_{[\frac{n}{m}]}(s)|$$
$$= 0(n^{-\frac{1}{4}}(\ln n)^{\frac{3}{4}}) \quad \text{a.s. } [P], \; n \to \infty$$

by application of relation (2.13) in Stute (1982). Also (20) is replaced by the stronger assertion that

$$\sup_{|x| \le K(\ln n)^{\frac{1}{2}}} |\bar{D}_{2n} - xh_F(\xi_p)| = 0(n^{-\frac{1}{4}}(\ln n)^{\frac{3}{4}}) \quad \text{a.s. } [P], \text{ as } n \to \infty.$$

For this we used Lemma 3.1 of Choudhury and Serfling (1988) and the Lipschitz condition on h_F. Combining (33) with (28) directly yields

$$I_{1n} = 0(n^{-\frac{1}{4}}(ln\, n)^{\frac{3}{4}}) \quad \text{a.s. } [P], \text{ as } n \to \infty. \tag{34}$$

For the quantity I_{2n} we have

$$\begin{aligned} I_{2n} \leq \hat{P}_n(\hat{\xi}^*_{pn} - \hat{\xi}_{pn} > Kn^{-\frac{1}{2}}(ln\, n)^{\frac{1}{2}}) \\ +\hat{P}_n(\hat{\xi}^*_{pn} - \hat{\xi}_{pn} < -Kn^{-\frac{1}{2}}(ln\, n)^{\frac{1}{2}}) \\ +2(1 - \Phi(K(ln\, n)^{\frac{1}{2}}\sigma^{-1}). \end{aligned} \tag{35}$$

The third term is $0(n^{-\frac{1}{2}})$ by taking K large enough. It remains to estimate the two other terms. Since the argument is the same for both, we only deal with the first term of the r.h.s. of (35). Similarly as in (8) we write

$$\begin{aligned} &\hat{P}_n(\hat{\xi}^*_{pn} - \hat{\xi}_{pn} > Kn^{-\frac{1}{2}}(ln\, n)^{\frac{1}{2}}) \\ &= \hat{P}_n(H^*_n(\hat{\xi}_{pn} + Kn^{-\frac{1}{2}}(ln\, n)^{\frac{1}{2}}) - \bar{H}_n(\hat{\xi}_{pn} + Kn^{-\frac{1}{2}}(ln\, n)^{\frac{1}{2}}) \\ &\qquad < p - \bar{H}_n(\hat{\xi}_{pn} + Kn^{-\frac{1}{2}}(ln\, n)^{\frac{1}{2}})) \end{aligned} \tag{36}$$

Application of Lemma 3.1 of Choudhury and Serfling (1988) directly yields that for all n sufficiently large,

$$p - \bar{H}_n(\hat{\xi}_{pn} + Kn^{-\frac{1}{2}}(lnn)^{\frac{1}{2}}) \leq p - \bar{H}_n(\xi_p + \frac{K}{2}n^{-\frac{1}{2}}(ln\, n)^{\frac{1}{2}}) \tag{37}$$

a.s. $[P]$, provided we take K large enough. A simple argument involving Corollary 2.1 of Helmers, Janssen and Serfling (1988) and the a.s. closeness of $\bar{H}_n$ and H_n gives us (with C_m as in the corollary)

$$\begin{aligned} &p - \bar{H}_n(\xi_p + \frac{K}{2}n^{-\frac{1}{2}}(lnn)^{\frac{1}{2}}) \\ &\leq p - H_F(\xi_p + \frac{K}{2}n^{-\frac{1}{2}}(ln\, n)^{\frac{1}{2}}) + C_m n^{-\frac{1}{2}}(ln\, n)^{\frac{1}{2}} + 0(n^{-1}) \end{aligned} \tag{38}$$

a.s. $[P]$. The smoothness assumption of the theorem directly implies that

$$\begin{aligned} &p - H_F(\xi_p + \frac{K}{2}n^{-\frac{1}{2}}(ln\, n)^{\frac{1}{2}}) \\ &= -\frac{K}{2}h_F(\xi_p)n^{-\frac{1}{2}}(ln\, n)^{\frac{1}{2}}(1 + o(1)) \quad \text{a.s. } [P], \text{ as } n \to \infty. \end{aligned} \tag{39}$$

Together (37), (38) and (39) yield that $p - \bar{H}_n(\hat{\xi}_{pn} + Kn^{-\frac{1}{2}}(ln\, n)^{\frac{1}{2}}) < 0$, for all n sufficiently large, a.s. $[P]$, provided we take K large enough.

We can now apply an exponential bound for U-statistics with bounded kernels of Hoeffding (1963) (see also Serfling (1980), p. 201) to find that

$$\begin{aligned}&\hat{P}_n(H_n^*(\hat{\xi}_{pn}+Kn^{-\frac{1}{2}}(ln\,n)^{\frac{1}{2}})-\bar{H}_n(\hat{\xi}_{pn}+Kn^{-\frac{1}{2}}(ln\,n)^{\frac{1}{2}})\\&\qquad\qquad< p-\bar{H}_n(\hat{\xi}_{pn}+Kn^{-\frac{1}{2}}(ln\,n)^{\frac{1}{2}}))\\&\le exp\{-\tfrac{1}{8}[\tfrac{n}{m}]n^{-1}ln\,nK^2h_F^2(\xi_p)\}\\&=0(n^{-\frac{1}{2}})\quad \text{a.s. } [P],\quad \text{as } n\to\infty,\end{aligned}\tag{40}$$

provided K is taken sufficiently large. This together with (35) and (36) implies that

$$I_{2n}=0(n^{-\frac{1}{2}})\quad \text{a.s. } [P],\quad \text{as } n\to\infty.\tag{41}$$

Hence I_{2n} is of negligible order for our purposes. It remains to consider I_{3n}. Clearly, as $n\to\infty$,

$$I_{3n}=\sup_x|P(n^{\frac{1}{2}}(\hat{\xi}_{pn}-\xi_p)\le x)-\Phi(x\sigma^{-1})|=O(n^{-\frac{1}{2}}),\tag{42}$$

i.e. the Berry-Esseen bound for U-quantiles is valid. To check (42) is an easy matter in view of the classical proof of a Berry-Esseen bound for ordinary sample quantiles (see, e.g. Serfling (1980), p.81-84). We have to apply instead of the Lemma on p.75 of Serfling (1980), the exponential bound of Hoeffding (1963) for U-statistics with bounded kernels. Also a Berry-Esseen bound for U-statistics is needed.

Combining (34), (41) and (42) with (30), we find that the theorem is proved. □

4. APPLICATIONS

In this section we indicate briefly applications of our results to the problem of obtaining confidence intervals for $\xi_p=H_F^{-1}(p)$. Let $u_{\frac{\alpha}{2}}=\Phi^{-1}(1-\frac{\alpha}{2})$. The normal approximation (3) yields an approximate two-sided confidence interval

$$(\hat{\xi}_{pn}-n^{-\frac{1}{2}}\hat{\sigma}_n u_{\frac{\alpha}{2}}\ ,\ \hat{\xi}_{pn}+n^{-\frac{1}{2}}\hat{\sigma}_n u_{\frac{\alpha}{2}})\tag{43}$$

for ξ_p. Here $\hat{\sigma}_n^2$ denotes a consistent estimator (e.g., the one proposed by Choudhury and Serfling (1988)) of the asymptotic variance σ^2. Clearly, the error rates corresponding to the upper and lower confidence limits in (43) will depend on the rate at which $\hat{\sigma}_n^2$ approaches σ^2.

A bootstrap based confidence interval for ξ_p is given by

$$(\hat{\xi}_{pn}-n^{-\frac{1}{2}}c^*_{n,1-\frac{\alpha}{2}}\ ,\ \hat{\xi}_{pn}-n^{-\frac{1}{2}}c^*_{n,\frac{\alpha}{2}})\tag{44}$$

where $c^*_{n,\frac{\alpha}{2}}$ and $c^*_{n,1-\frac{\alpha}{2}}$ denote the $\frac{\alpha}{2}$-th and $(1-\frac{\alpha}{2})$-th percentile of the (simulated) bootstrap approximation. It is easily verified that the upper and lower confidence limits in (44) have error rates equal to $\frac{\alpha}{2} + 0(n^{-\frac{1}{4}}(ln\, n)^{\frac{3}{4}})$.

We discuss a few specific examples of U-quantiles. In the first of these we take $m = 1$, $h(x) = x$ and obtain the classical p-th sample quantile $\hat{\xi}_{pn} = \hat{F}_n^{-1}(p)$, $0 < p < 1$. Our second example is obtained by taking $p = \frac{1}{2}$, $m = 2$, $h(x_1, x_2) = (x_1 + x_2)/2$. In this case $\hat{\xi}_{\frac{1}{2}n} = H_n^{-1}(\frac{1}{2})$ becomes the well-known Hodges-Lehmann location estimator. In the third and final example we take $p = \frac{1}{2}$, $m = 2$, $h(x_1, x_2) = |x_1 - x_2|$. In this case $\hat{\xi}_{\frac{1}{2}n} = H_n^{-1}(\frac{1}{2})$ reduces to an estimator of spread proposed by Bickel and Lehmann (1979).

A further investigation into the relative merits of the normal and bootstrap based confidence intervals (43) and (44) for U-quantiles appears to be worthwhile. The authors hope to report on these matters elsewhere.

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