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Invariant Measures in Abstract Topological Dynamics

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In this expository paper I mention a number of problem areas from abstract topological dynamics and I indicate how the presence of an invariant measure helps in solving the problem. The topics discussed include results about maximal equicontinuous factors, disjointness of minimal flows and the relationship between disjointness and common factors.

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§ 0. Introduction

Abstract Topological Dynamics deals with problems that can be traced back to classical dynamics, but the framework in which this is done is that of an arbitrary topological group or semigroup (representing the set of possible values of *time*) acting by means of continuous transformations on a topological space (representing the set of possible states of some fictitious physical system). Its foundations were laid essentially by Poincaré in his famous Memoir which won the prize offered by King Oscar II of Sweden (January 21, 1889) and which dealt with the stability of our solar system. One of his fundamental new ideas was not to solve the equations of motion but to study qualitative aspects of the motions defined by the equations — among others by considering geometrical aspects of the "picture" of trajectories in phase space. This Memoir, however, contains also a result that may be considered as the first theorem of Ergodic Theory, a branch of mathematics that, at that time, still had to be born. In Ergodic Theory one considers (semi)groups (again representing time; mostly $\mathbb{R}$ or $\mathbb{Z}^+$) acting on a measure space by means of measure preserving mappings (here the measurable sets represent *events* and their measure the probability of an event).

I cannot go here into the different developments of these two branches of mathematics: Topological Dynamics and Ergodic Theory, but let me state simply that the last two decades have witnessed a growing interaction between them. In this paper I present some aspects of the role that invariant measures can play in abstract Topological Dynamics: I mention a number of problems from Topological Dynamics and I show how invariant measures contribute to their solution. Neither in the choice of the problems, nor in the treatment of each problem I pursue completeness. In particular, I hardly say anything about what can be done without invariant measures (a lot!). Note also that my point of view is topological, not measure theoretic. The topics that I treat are

1. Recurrence.
2. Minimality.
3. Weak mixing and equicontinuous factors.
4. Disjointness of minimal flows.
5. Disjointness and common factors.

Now let me present some necessary definitions. The symbol T will always denote an arbitrary topological group; its unit element will be denoted by e (in examples T will sometimes be $\mathbb{R}$ or $\mathbb{Z}$, and then we write 0 instead of e, of course). A *T-flow* or a *flow under T* (or just simply: a *flow*) is a pair $<X,\pi>$ where X is a Hausdorff space and π is a continuous *action* of T on X. By this we mean that $\pi:T\times X\to X$ is a continuous mapping such that

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$$ex = x,\ t(sx) = (ts)x \quad (t,s \in T; x \in X). \tag{1}$$

Here and in the sequel we write tx for $\pi(t,x)$ and, in the same vain, if $S \subseteq T$ and $A \subseteq X$, $SA := \{tx : t \in S \ \&\ x \in A\}$. In particular, $tA := \{tx : x \in A\}$ $(t \in T)$ and $Sx := \{tx : t \in S\}$ $(x \in X)$. The conditions in (1) together with continuity of π imply that for every $t \in T$ the mapping $\pi^t : x \mapsto tx : X \to X$ is a homeomorphism of X onto itself and that $t \mapsto \pi^t$ is a homomorphism of the group T into the group of all autohomeomorphisms of X. In the case that $T = \mathbb{Z}$ one usually uses a different notation : now $\pi^n = (\pi^1)^n$ for all $n \in \mathbb{Z}$ (the n-th iterate of π^1) so the action is completely determined by one single homeomorphism $f := \pi^1$ of X, and we denote the flow by (X,f) rather than $<X,\pi>$ (often such a pair (X,f) is called a *discrete flow*).

Let $<X,\pi>$ be any flow (under T). If $x \in X$ then the subset Tx of X is called *the orbit of* x (in X, under T). A subset A of X is said to be *invariant* whenever $TA = A$. Clearly, every orbit is invariant and each invariant subset of X is a union of orbits. As each π^t is a homeomorphism it follows that the closure of an invariant set is again invariant. In particular, each *orbit closure* (i.e., closure of an orbit) is invariant

By a *measure* μ on a topological space X I shall always mean a countably additive, non-negative function defined on the σ-algebra of Borel subsets of X; if $\mu(X) = 1$ then μ is called a *probability measure.* All measures will assumed to be regular in the sense that $\mu(A) = \sup\{\mu(C) : C \subseteq A \ \&\ C \text{ compact}\}$ for each Borel set A. Then the union of all open null-sets is again an open null-set; its complement is called the *support* of μ (notation: Supp μ). Note that if X and Y are topological spaces and $f : X \to Y$ is a measurable mapping (in this paper, this will always mean: measurable with respect to the σ-algebras of Borel sets) then every {probability} measure μ on X gives rise to a {probability} measure $f\mu$ on Y, defined by $f\mu(A) := \mu(f^{\leftarrow}[A])$ for each Borel subset A of Y. If $<X,\pi>$ is a flow and μ is a measure on X then we write for simplicity $t\mu := \hat{\pi}^t\mu \, (t \in T)$. A measure μ is called *invariant* whenever $t\mu = \mu$ for all $t \in T$.

Not every flow on a compact space has an invariant measure. For example, let T be the free group with two generators s and t, and let T act on the unit circle $\mathbb{S}^1 := \{z \in \mathbb{C} : |z| = 1\}$ as follows:

$$\left.\begin{aligned} s(e^{2\pi i\theta}) &:= e^{2\pi i(\theta+\alpha)} \quad \text{for some } \alpha \in \mathbb{R} \setminus \mathbb{Q} \\ t(e^{2\pi i\theta}) &:= e^{2\pi i\theta^2} \end{aligned}\right\} \tag{2}$$

Then for every $\mu \in \mathfrak{M}(\mathbb{S}^1)$ one has $\lim_{n\to\infty} t^n\mu = \delta_1$, the pointmass at 1, while δ_1 is not invariant under the rotation s. So no $\mu \in \mathfrak{M}(X)$ is invariant under T.

However, there are topological groups (the so-called *amenable* groups) which have the property that *every* continuous action of the group on a compact Hausdorff space admits an invariant probability measure. All groups from the following classes have this property:

(i) Abelian groups;
(ii) Compact groups;
(iii) Solvable groups;
(iv) Compact extensions of solvable groups.

(In (iv) we mean groups T that have a solvable normal subgroup S such that T/S is compact.) For proofs, see e.g. [9], Section III.3. So in particular when $T = \mathbb{R}$ or $\mathbb{Z}$ then each compact orbit closure in any flow carries an invariant probability measure.

1. Recurrence

Let T be $\mathbb{R}$ or $\mathbb{Z}$. In this case one usually interpretes a flow $<X,\pi>$ under T as follows: X is the set of possible states of some fictitious physical system, and for $x \in X$ and $t \in T$ the state tx is the state that is reached by the physical system at time t when it starts at time 0 in state x. One can imagine that it is an important question whether of the physical system will return in the future to its initial state or, if not, whether it will almost return. The former case corresponds with a so-called *periodic point* in X (i.e., a point x_0 such that $tx_0 = x_0$ for some $t \neq 0$), and for the latter case there are several possible definitions. One of them is the following: a point $x \in X$ is called *positively* {*negatively*} *recurrent* [1] whenever for every nbd U of x the following set of "return times" of x in U,

$$R(x,U):=\{t\in T: tx\in U\},$$

as a subset of $\mathbb{R}$ is not bounded from above { below }.

There are several methods to show that certain flows do have recurrent points. One of the earliest is based on the so-called Poincaré Recurrence Theorem (which can be found in Poincaré's famous prize-winning Memoir, mentioned in the Introduction). Consider an arbitrary measure space $(X,\mathfrak{B},\mu)$ with $\mu(X)=1$ and let $f:X\to X$ be a measurable transformation that leaves μ invariant, i.e. $\mu(f^{\leftarrow}[B])=\mu(B)$ for all $B\in\mathfrak{B}$. Poincaré's Theorem says: *for each $B\in\mathfrak{B}$ with $\mu(B)>0$ the set $R(y,B):=\{n\in\mathbb{N}: f^n(y)\in B\}$ is unbounded for almost every $y\in B$.* For a proof, see e.g. [20], § 1.4.

Now we have the following "stability" result:

COROLLARY. *Let $<X,\pi>$ be a flow under $\mathbb{R}$ or $\mathbb{Z}$ and assume that there exists an invariant probability measure μ. If, in addition, X has a countable base, then almost every point of X is both positively and negatively recurrent.*

Proof. Let $\{B_1,B_2,...\}$ be a base for X. By Poincaré's Theorem, applied to $f:=\pi^1$, there is a null-set N_i in B_i such that for all $x\in B_i\setminus N_i$ the set $R(x,B_i)$ in not bounded from above ($i\in\mathbb{N}$). It follows that $N^+:=\bigcup_{i\in\mathbb{N}}N_i$ is a null-set and that every point of $X\setminus N^+$ is positively recurrent. Similarly (with $f:=\pi^{-1}$) one finds a null-set N^- such that every point of $X\setminus N^-$ is negatively recurrent. Now $(X\setminus N^+)\cap(X\setminus N^-)$ is the desired set. □

REMARKS. 1. If in the above Supp $\mu=X$, i.e., every non-empty open set has positive measure, then every null-set has empty interior. In that case there is a *dense set of points that are both positively and negatively recurrent.* This conclusion can also be shown to hold if the condition that X has a countable base is replaced by either of the following conditions:

(a) X is completely metrizable;

(b) X is locally compact.

In both cases the assumption that each non-empty open set has positive measure implies that the so-called non-wandering set of $<X,\pi>$ equals all of X, and it is known that then the desired conclusion follows from (a) (classical) or from (b) (see e.g. [14]).

2. The conclusion of the Corollary or of Remark 1 applies to the important case of a bounded manifold of constant energy in a flow in $\mathbb{R}^n$, defined by an autonomous Hamiltonian system of differential equations (Liouville's Theorem asserts that such a flow has an invariant measure with full support: see [1], § 16).

3. Here is another example: define $f:[0;1]\to[0;1]$ by

$$f(x):=\begin{cases}2x \text{ for } 0\leqslant x\leqslant ½\\ 2(1-x) \text{ for } ½\leqslant x\leqslant 1\end{cases}$$

(the "tent-map"). It is easy to see that f preserves Lebesgue measure. Now the above methods can be applied with almost no modifications to show that there is a dense set of positively recurrent points. It is easy to show that the homeomorphic deformation $x\mapsto\frac{1}{2}(1-\cos\pi x)$ "distorts" f into the mapping $g:[0;1]\to[0;1]$, given by

$$g(x):=4x(1-x)\qquad(0\leqslant x\leqslant 1).$$

So this much-studied quadratic mapping of the interval has a dense set of positively recurrent points as well. For more about invariant measures for maps of [0;1], see [17], § 7.

1. In many publications a recurrent point is called a point, stable in the sense of Poisson, or: a Poisson-stable point.

2. Minimality

For convenience, in this Section we still assume that $T=\mathbb{R}$ or $\mathbb{Z}$. A notion that seems to be opposed to recurrence is transitivity: if $\langle X,\pi\rangle$ is a flow under T then a point $x\in X$ is called *transitive* whenever it has a dense orbit: $\overline{Tx}=X$. But actually, there is the following relationship with recurrence: if x is a transitive point and if no nbd of x is covered by any continuous image of a bounded interval in T (in the case $T=\mathbb{Z}$ this means: x is not isolated) then x is positively *or* negatively recurrent (the proof is trivial). As to the existence of transitive points, also here invariant measures are helpful. First a definition: an invariant measure for a flow $\langle X,\pi\rangle$ under T is said to be *ergodic* whenever for every invariant Borel set B either $\mu(B)=0$ or $\mu(X\setminus B)=0$. The following result is classical:

Proposition. *Let $\langle X,\pi\rangle$ be a flow where X is a second countable space and assume that there exists an ergodic invariant probability measure μ such that Supp $\mu=X$. Then μ-almost every point of X is transitive.*

Proof. Let $\{B_1,B_2,\cdots\}$ be a base for X. The set of transitive points is equal to $\bigcap_{n=1}^{\infty}\bigcup_{t\in T}\pi^{-t}[B_n]$. For each $n\in\mathbb{N}$ the set $\bigcup_{t\in T}\pi^{-t}[B_n]$ is open, invariant and non-empty, hence its complement has measure 0. □

Remarks. 1. In the example of Remark 3 in Section 1, the invariant measure can quite easily shown to be ergodic (it has even the much stronger property of being mixing). Hence almost all points in $[0;1]$ have a dense orbit (note that, with minor modifications, the above proposition applies to maps that are not necessarily invertible).

2. It can be shown by examples that in the proposition not necessarily all points of X are transitive. For example, let $h:\mathbb{T}^2\to\mathbb{T}^2$ be such that it lifts to $H:\mathbb{R}^2\to\mathbb{R}^2$, where H is a linear mapping with matrix $\begin{pmatrix}2&1\\1&1\end{pmatrix}$. All points of $\mathbb{T}^2$ with rational coordinates are periodic, hence not transitive. Yet Haar measure in $\mathbb{T}^2$ (or, if you want, Lebesgue measure in the unit square) is invariant under h and it is ergodic (cf [20], 1.10.1).

A flow in which *all* points are transitive is called *minimal:* there are no proper closed invariant subsets. More generally, a subset of an arbitrary flow $\langle X,\pi\rangle$ is called *minimal* whenever it is non-empty, closed and invariant, while it includes no proper subsets that are closed and invariant (this definition is also valid if T is an arbitrary group). *A subset A of X is minimal iff $A\neq\phi$ and $A=\overline{Tx}$ for all $x\in A$.* The importance of minimal sets comes not only from the fact they are the "indecomposable" parts of a flow, but also from the fact that points in compact minimal sets exhibit a quite "regular" recurrent behaviour. Indeed, though a recurrent point may be considered as being more or less stable (it will almost return in the future) it is quite unsatisfactory that between the subsequent return times of the point to one of its nbds there can be arbitrarily large gaps. If for every nbd of the point these gaps are of bounded length (depending on the nbd, in general) then the point is called *almost periodic.* Now there is the following connection between compact minimal sets and almost periodic points: *in a flow $\langle X,\pi\rangle$ with X locally compact a point x is almost periodic iff there is a compact minimal subset of X that contains x, iff $\overline{Tx}$ is a compact minimal subset of X.* This is one of the forms of the so-called Birkhoff Recurrence Theorem; cf. [3], 2.5.

Using Zorn's lemma it is easy to show that every *compact* invariant subset of a flow, provided it is non-empty, includes a minimal subset. But there may be many of them. Therefore, the following result is often very useful.

Theorem. *Let $\langle X,\pi\rangle$ be a flow and assume that X is a compact Hausdorff space, and that the flow has a unique invariant probability measure μ. Then Supp μ is the unique minimal subset of X.*

Proof. Let M be any minimal subset of X. As T is abelian, the flow restricted to M has an invariant probability measure ν. Then ν can be identified with a probability measure on all of X with Supp $\nu\subseteq M$. But ν (on X) is also invariant, so by our assumption, $\mu=\nu$. Consequently, Supp $\mu\subseteq M$. But Supp μ is a non-empty closed invariant set, so Supp $\mu=M$. □

A flow $<X,\pi>$ which has a *unique* invariant probability measure is called *uniquely ergodic.* There is an immense literature on the question whether of a given (class of) flow(s) is uniquely ergodic or not. Let me give one reference which deals with unique ergodicity of shift systems: [13]; but let me also mention two examples:

EXAMPLES. 1. Let $f:\mathbb{S}^1 \to \mathbb{S}^1$ be a homeomorphism without periodic points. Then the discrete flow $(\mathbb{S}^1, f)$ is uniquely ergodic. See [5]. In the proof it is shown that if λ is any f-invariant probability measure on $\mathbb{S}^1$ then

$$\gamma:\xi \mapsto \lambda[\xi_\circ;\xi] \text{ (mod 1)} : \mathbb{S}^1 \to \mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1$$

($\xi_\circ \in \mathbb{S}^1$ arbitrarily chosen; $[\xi_\circ;\xi]$ is the counter-clock wise arc from $\xi_\circ$ to ξ) is a continuous mapping such that $\gamma \circ f = \tilde{\alpha} \circ \gamma$, where $\tilde{\alpha}$ is the rotation of $\mathbb{S}^1$ over an angle $\alpha := \gamma(f(\xi_\circ))$. It turns out that $\alpha \notin \mathbb{Q}$. Note that γ is constant on all intervals that constitute the complement of Supp λ. Consequently, if $(\mathbb{S}^1, f)$ if given to be minimal, then the closed invariant set Supp λ equals $\mathbb{S}^1$, hence γ is injective. This gives a very elegant proof of the following classical result: *if* $(\mathbb{S}^1, f)$ *is a minimal discrete flow then there are no periodic points and* $(\mathbb{S}^1, f)$ *is conjugate to* $(\mathbb{S}^1, \tilde{\alpha})$ *for some* $\alpha \notin \mathbb{Q}$. *(For the notion of conjugation (or: isomorphism of flows), see the next section).*

2. Let Γ be a discrete subgroup of the group $SL(2,\mathbb{R})$ such that $X := SL(2,\mathbb{R})/\Gamma$ is compact. Define an action χ of $\mathbb{R}$ on X by

$$\chi^t[g] := [n_t g] \quad (t \in \mathbb{R}, g \in SL_2(\mathbb{R}));$$

here $[\cdot]: SL(2,\mathbb{R}) \to X$ is the quotient mapping and $n_t := \binom{1\,t}{0\,1}$ for $t \in \mathbb{R}$. The flow $<X,\chi>$ under $\mathbb{R}$ is called the *horocycle flow.* It is uniquely ergodic, as was shown by Furstenberg in [7]. As the support of the invariant measure turns out to be all of X, the theorem above showns that *the horocycle flow is minimal,* an old result of Hedlund from 1936. For higher dimensional generalizations, see [18].

3. WEAK MIXING AND EQUICONTINUOUS FACTORS

From now on, T is arbitrary. An important issue in abstract topological dynamics is the study of the structure of minimal flows and their classification. Up to now this program is far from being completed. An indispensable notion is that of a morphism. Let $<X,\pi>$ and $<Y,\sigma>$ be flows, both under the same group T. A *morphism* $\gamma:<X,\pi> \to <Y,\sigma>$ is a continuous mapping $\gamma:X \to Y$ such that $\gamma \circ \pi^t = \sigma^t \circ \gamma$ for all $t \in T$. A morphism γ such that γ is a homeomorphism of X onto Y is called an *isomorphism.* A surjective morphism is also called a *factor mapping,* and in that case $<Y,\sigma>$ is called *a factor of* $<X,\pi>$ and $<X,\pi>$ an *extension* of $<Y,\sigma>$ (in the literature, also γ is often called an extension). The "structure theory" of minimal flows happens to have obtained the following form: one considers several classes of minimal flows, each class defined by some additional property that makes the flows in the class look more simple; then one tries to relate members from different classes with each other, either using morphisms (this Section) or by forming products (the next Section).

The simplest compact minimal flows are the equicontinuous ones. A flow $<X,\pi>$ with X a compact Hausdorff space is said to be *equicontinuous* whenever the group of homeomorphisms $\{\pi^t : t \in T\}$ of X is uniformly equicontinuous on X with respect to the unique uniformity that generates the topology of X. The structure of compact minimal equicontinuous flows in fairly well understood: they can all be obtained as bT/H, where bT denotes the Bohr compactification of T and H is a closed normal subgroup of bT (use [3], 4.6.1).

People have been looking for minimal flows that have non-trivial equicontinuous factors (every flow has a *trivial* equicontinuous factor: a one-point space with the obvious constant action of T). I cannot go here into details about the reasons for this; let me restrict myself to saying that equicontinuous factors pop up everywhere in structure theory. From this point of view the following result is interesting. First a definition: a flow $<X,\pi>$ is called *weakly mixing* whenever the invariant sets in $X \times X$ (under coordinate-wise action of T) are either dense or nowhere dense; equivalently, $<X,\pi>$ is weakly mixing iff for every choice of four non-empty open subsets U_1, U_2, V_1 and V_2 in X there exists $t \in T$ with $U_i \cap tV_i \neq \phi (i=1,2)$. If X is metrizable and a Baire space, then $<X,\pi>$ is weakly mixing iff there is a point with dense orbit in $X \times X$.

THEOREM. *Let $<X,\pi>$ be a minimal flow with X a compact Hausdorff space. If the flow has an invariant measure, then the following are equivalent:*

(i) $<X,\pi>$ is weakly mixing;

(ii) The trivial flow is the only equicontinuous factor of $<X,\pi>$.

Proof. See [19]. □

REMARK. The invariant measure is needed only to prove (ii) ⇒ (i) (the implication (i) ⇒ (ii) is generally valid). This result was known much earlier under the additional hypothesis that X is metrizable. Using the methods of [11] it might be possible to obtain Ellis' non-metric version [4] of Furstenberg's celebrated structure theorem for minimal distal flows from the above result (compare with [8]).

4. Disjointness of minimal flows

When two minimal flows $<X,\pi>$ and $<Y,\sigma>$ (both under the same group T) are related to each other in some sense then this can often be discovered by looking for invariant subsets in the product flow $<X \times Y,\tau>$ (where τ is defined by $\tau^t(x,y) := (\pi^t x, \sigma^t y)$ for $t \in T$ and $(x,y) \in X \times Y$). For example, if $<Y,\sigma> = <X,\pi>$ is non-trivial then the diagonal $\Delta_X := \{(x,x): x \in X\}$ is a proper closed invariant subset of $X \times X$. Similary, if $<Y,\sigma>$ is a factor of $<X,\pi>$, say by means of a morphism γ, then $\{(x,\gamma(x)): x \in X\}$ is a proper closed invariant subset of $X \times Y$ (unless Y is a one-point space). There are more complicated examples supporting the abovementioned idea. This leads to the following definition: two minimal flows $<X,\pi>$ and $<Y,\sigma>$ with X and Y compact Hausdorff spaces are said to be *disjoint* from each other (notation: $<X,\pi> \perp <Y,\sigma>$) whenever their product $<X \times Y,\tau>$ is minimal. There is an equivalent formulation in terms of factors: two compact minimal flows are disjoint iff whenever they are a factor of a third compact minimal flow then their product is also a factor.

Now for various classes $\mathcal{K}$ of compact minimal flows one would like to know which minimal flows are disjoint from all members of $\mathcal{K}$; the class of such flows in denoted by $\mathcal{K}^{\perp}$. In order to give some examples, let me first give some definitions.

Two points x_1 and x_2 in a flow $<X,\pi>$ are called *proximal* (to each other) whenever, in the product flow on $X \times X$, $\overline{T(x_1,x_2)} \cap \Delta_X \neq \phi$. If X is compact then this is equivalent to saying that there is a net $\{t_\alpha\}$ in T such that $t_\alpha x_1$ and $t_\alpha x_2$ approach each other arbitrarily close (their "distance" being measured in terms of the uniformity of X). Two points that are either *equal* or *not proximal* to each other are said to be *distal* to each other. The *flow* $<X,\pi>$ is called *proximal* {*distal*} whenever all pairs of points are proximal {distal}. Let $\mathcal{P}$ {$\mathcal{D}$} denote the class of all proximal {distal} compact minimal flows. Then it can be shown that $\mathcal{P} \subseteq \mathcal{D}^{\perp}$, hence $\mathcal{P}^{\perp} \supseteq \mathcal{D}^{\perp\perp} \supseteq \mathcal{D}$; so for short: $\mathcal{P} \perp \mathcal{D}$.

Let $\mathcal{E}$ and $\mathcal{WM}$ denote the classes of all compact minimal flows that are equicontinuous, respectively, weakly mixing. It is easy to show that $\mathcal{E} \subseteq \mathcal{D}$. Some less trivial inclusions are in the following:

$$\mathcal{P} \subseteq \mathcal{WM} \subseteq \mathcal{WM}^{\perp\perp} \subseteq \mathcal{E}^{\perp} = \mathcal{D}^{\perp}.$$

For this, and much more, consult Chapter VI of [21]. Let me give two brief comments on these inclusions. First, the inclusion $\mathcal{WM} \subseteq \mathcal{E}^{\perp}$ follows (in a non-trivial way) from the fact that a weakly mixing flow can have no non-trivial equicontinuous factor (this is the trivial part of the Theorem mentioned in Section 3). Second, the equality $\mathcal{E}^{\perp} = \mathcal{D}^{\perp}$ is related to the deep result that a non-trivial compact minimal distal flow has a non-trivial equicontinuous factor [4]. These comments may suffice to show that there is a close relationship between the studies of disjointness and of factors of minimal flows.

Now what about invariant measures? Using the Riesz Representation Theorem the set of probability measures on a compact Hausdorff space can be identified with the set

$$\mathcal{M}(X) := \{\mu \in C_u(X)^* : \mu \geqslant 0 \ \& \ \mu(1_X) = 1\}.$$

Here $C_u(X)$ is the Banach space of continuous functions on X with the supremum norm, $C_u(X)^*$ is its dual, 1_X is the constant function with value $1 \in \mathbb{R}$ and, as usually, $\mu \geqslant 0$ means $\mu(f) \geqslant 0$ whenever $f \geqslant 0$. With the weak*-topology, $\mathfrak{M}(X)$ is a compact Hausdorff space, and if $<X,\pi>$ is a flow then the transformations $\hat{\pi}^t$ for $t \in T$ (see the Introduction) define a continuous action $\hat{\pi}$ of T on $\mathfrak{M}(X)$. Thus, for each compact flow $<X,\pi>$ we have a compact flow $<\mathfrak{M}(X), \hat{\pi}>$. Now let me cite the following result from [9]:

THEOREM. *Let* $<Y,\sigma>$ *be a compact minimal flow that has an invariant probability measure. Then* $<Y,\sigma>$ *is disjoint from every compact minimal flow* $<X,\pi>$ *that has the property that* $<\mathfrak{M}(X),\hat{\pi}>$ *is a proximal flow.*

REMARKS. 1. If the group T is *amenable* (for examples, see the Introduction) then in particular any compact minimal flow $<X,\pi>$ for which $<\mathfrak{M}(X),\hat{\pi}>$ is proximal has an invariant probability measure. Hence $<X,\pi>$ is disjoint from itself, which can only happen if X is a one-point space. This shows: *If* T *is amenable then for a non-trivial compact minimal flow* $<X,\pi>$ *the flow* $<\mathfrak{M}(X),\hat{\pi}>$ *is never proximal.*

2. The condition that $<\mathfrak{M}(X),\hat{\pi}>$ is proximal is equivalent to the condition that for every $\nu \in \mathfrak{M}(X)$ there exists a Dirac measure δ_x in the orbit closure $\overline{T\nu}$. (This would also imply the conclusion of Remark 1.) Flows with this property play an important role in the analysis of the so-called *Furstenberg boundary* of T. See Chapter IV of [9].

3. The above theorem has the following extension and partial converse: Let $<Y,\sigma>$ be a compact minimal flow and suppose that Y is *metrizable.* Then the following statements are equivalent.

(i) $\exists \mu \in \mathfrak{M}(Y) : \overline{T\mu}$ is a minimal subset of $<\mathfrak{M}(Y),\hat{\sigma}>$;

(ii) $<Y,\sigma>$ is disjoint from every compact minimal flow $<X,\pi>$ for which $<\mathfrak{M}(X),\hat{\pi}>$ is proximal.

For details, see [11].

5. DISJOINTNESS AND COMMON FACTORS

It is easy to show that if two minimal flows $<X,\pi>$ and $<Y,\sigma>$ have a common factor, say

$$<X,\pi> \overset{\varphi}{\rightarrow} <Z,\rho> \overset{\psi}{\leftarrow} <Y,\sigma>,$$

then $R_{\varphi\psi} := \{(x,y) \in X \times Y : \varphi(x) = \psi(y)\}$ is a closed invariant subset of $X \times Y$, $\neq \phi$ because φ and ψ are surjections. So if $X \times Y$ is minimal (i.e., $<X,\pi> \perp <Y,\sigma>$) then $R_{\varphi\psi} = X \times Y$, hence φ and ψ are constant and, consequently, Z is a one-point space. Let us call two minimal flows *relatively prime* whenever they have no non-trivial common factors. We have shown: if $<X,\pi>$ and $<Y,\sigma>$ are minimal flows then

$$<X,\pi> \perp <Y,\sigma> \;\Rightarrow\; <X,\pi> \text{ and } <Y,\sigma> \text{ are relatively prime.}$$

What about the implication the other way round (where X and Y are, in addition, assumed to be compact)?

In 1968 an example was constructed by A.W. Knapp, showing that in general the converse in false: if T is the symmetric group on four elements then there are subgroups H_1 and H_2 of T such that the canonical T-flows on the coset spaces T/H_1 and T/H_2 are relatively prime, while $(T/H_1) \times (T/H_2)$ is *not* minimal under T. But this didn't solve the original question (posed by Furstenberg in 1967; see [6]) whether of the converse holds in the case $T = \mathbb{Z}$. Also, one may ask under which additional conditions (on $<X,\pi>$ and/or $<Y,\sigma>$) the converse is true for arbitrary T. As to this last problem, we refer to Section VI.4 of [21] or Section 3.19 of [2].

For the case $T = \mathbb{Z}$ quite recently a counterexample was constructed by S. Glasner and B. Weiss in [12]. Let me outline their construction. Recall the description of horocycle flows $<G/\Gamma,\chi>$ with $G = SL(2,\mathbb{R})$ (Section 2 above). Also recall that each horocycle flow $<G/\Gamma,\chi>$ - which is a flow under $\mathbb{R}$ - is uniquely ergodic. From this it follows (using the fact that the so-called geodesic flow on G/Γ provides for each t a conjugation of χ^t either with χ^1 or with χ^{-1}):

FACT 1. *The time-one discrete flow* $(G/\Gamma,\chi^1)$ *is uniquely ergodic and minimal.*

In the papers [15] and [16], M. Ratner obtained the following results. Let Γ,Γ_1 and Γ_2 be discrete subgroups of G such that G/Γ and $G/\Gamma_i (i=1,2)$ are compact. In any one of these three spaces the horocycle flow will be denoted by χ.

FACT 2. *Assume that the discrete flows* $(G/\Gamma_1,\chi^1)$ *and* $(G/\Gamma_2,\chi^1)$ *are isomorphic as measure preserving flows. Then* Γ_1 *and* Γ_2 *are conjugate subgroups of G.*

The condition here means that there is a bijection $\varphi:G/\Gamma_1\rightarrow G/\Gamma_2$ such that φ and φ^{-1} are measurable, $\hat{\varphi}:\mathfrak{M}(G/\Gamma_1)\rightarrow\mathfrak{M}(G/\Gamma_2)$ carries the invariant measure of G/Γ_1 to the invariant measure of G/Γ_2 while $\varphi\circ\chi^1=\chi^1\circ\varphi$ (a.e.).

FACT 3. *Let X denote an arbitrary measure space and let* $h:X\rightarrow X$ *be a measure preserving transformation. If the system* (X,h) *is in the measure-theoretic sense (in the same vain as explained above) a factor of* $(G/\Gamma,\chi^1)$, *then* (X,h) *is measure-theoretically isomorphic to some horocycle transformation flow* $(G/\Gamma_1,\chi^1)$ *with* $\Gamma_1\supseteq\Gamma$.

Using these facts, Glasner and Weiss could show:

PROPOSITION. *Let* G,Γ_1 *and* Γ_2 *be as above and assume, in addition, that*

(a). $\Gamma_1\cap\Gamma_2$ *is of finite index both in* Γ_1 *and* Γ_2.

(b). *For all* $g\in G$, $\Gamma_1\cup g\Gamma_2 g^{-1}$ *generates a non-discrete subgroup of G.*

Then the compact minimal discrete flows $(G/\Gamma_1,\chi^1)$ *and* $(G/\Gamma_2,\chi^1)$ *are relatively prime but not disjoint.*

Proof. For $i=1,2$, there is a natural surjective morphism φ_i of $(G/\Gamma_1\cap\Gamma_2,\chi^1)$ onto $(G/\Gamma_i,\chi^1)$, and by assumption (a) both morphisms have finite fibers. So the induced map of $G/\Gamma_1\cap\Gamma_2$ into $(G/\Gamma_1)\times(G/\Gamma_2)$ is not surjective. Its range is a *closed* invariant subset of $(G/\Gamma_1)\times(G/\Gamma_2)$, hence this product is not minimal. So $(G/\Gamma_1,\chi^1)$ and $(G/\Gamma_2;\chi^1)$ are not disjoint. (NB. It is easy to see that condition (a) implies that $G/\Gamma_1\cap\Gamma_2$ is compact; this is used in showing that the range of the mapping induced by φ_1 and φ_2 is closed.)

Now suppose that (X,h) is a compact minimal flow and that it is a common factor of $(G/\Gamma_1,\chi^1)$ and $(G/\Gamma_2,\chi^1)$. Being a factor of uniquely ergodic systems it is uniquely ergodic itself (one way to see this is to use one of the characterizations of [20], Theorem 6.19). Unicity of its invariant probability measure then implies that (X,h) is a factor of $(G/\Gamma_i,\chi^1)$ for $i=1,2$ also in the measure-theoretic sense. So by Fact 3, (X,h) is isomorphic (in the measure-theoretic sense) to both $(G/\hat{\Gamma}_1,\chi^1)$ and $(G/\hat{\Gamma}_2,\chi^1)$, where for $i=1,2$, $\hat{\Gamma}_i$ is a discrete subgroup of G with $\hat{\Gamma}_i\supseteq\Gamma_i$. These two systems then are mutually isomorphic as measure preserving flows, so Fact 2 implies that $\hat{\Gamma}_1=g\hat{\Gamma}_2 g^{-1}$ for some $g\in G$. In particular,

$$g\Gamma_2 g^{-1}\subseteq g\hat{\Gamma}_2 g^{-1}=\hat{\Gamma}_1 \text{ and } \Gamma_1\subseteq\hat{\Gamma}_1.$$

Since $\hat{\Gamma}_1$ is a discrete subgroup of G, this contradicts condition (b). □

Finally, in their paper Glasner and Weiss give examples of subgroups of G satisfying the conditions of the Proposition (in fact, they present a countable family such that each pair of different members satisfies the conditions (a) and (b). For details, see [12].

REFERENCES

[1] V.I. ARNOLD, *Mathematical methods of classical mechanics*, Springer-Verlag, Berlin, 1978.

[2] I.U. BRONŠTEĬN, *Extensions of minimal transformation groups*, Sijthoff & Noordhoff, Alphen, 1979.

[3] R. ELLIS, *Lectures on topological dynamics*, W.A. Benjamin, Inc., New York, 1969.

[4] R. ELLIS, The Furstenberg structure theorem, *Pacific J. Math.* **76** (1978), 345-349.

[5] H. FURSTENBERG, Strict ergodicity and transformations of the torus, *Amer. J. Math.* **83** (1961),

573-601.

[6] H. FURSTENBERG, Disjointness in ergodic theory, minimal sets, and a problem in diophantine approximation, *Math. Syst. Theory* **1** (1967), 1-49.

[7] H. FURSTENBERG, The unique ergodicity of the horocycle flow, in: A. BECK (ed.), *Recent advances in topological dynamics*, LNM 318, Springer-Verlag, Berlin, 1973, pp. 95-115.

[8] H. FURSTENBERG & S. GLASNER, On the existence of isometric extensions, *Amer. J. Math.* **100** (1978), 1185-1212.

[9] S. GLASNER, *Proximal flows*, LNM 517, Springer-Verlag, Berlin, 1976.

[10] S. GLASNER, Relatively invariant measures, *Pacific J. Math.* **58** (1975), 393-410.

[11] S. GLASNER, Compressibility properties in topological dynamics, *Amer. J. Math.* **97** (1975), 148-171.

[12] S. GLASNER & B. WEISS, Minimal transformations with no common factors need not be disjoint, *Israel J. Math.* **45** (1983), 1-8.

[13] M. KEANE, Generalized Morse sequences, *Z. Wahrscheinlichkeitstheorie u. verw. Gebiete* **10** (1968), 335-353.

[14] R.A. KNIGHT, Central motions, *Ann. Polon. Math.* **44** (1984), 197-200.

[15] M. RATNER, Rigidity of horocycle flows, *Ann. of Math.* **115** (1982), 597-614.

[16] M. RATNER, Factors of the horocycle flow, *Ergod. Th. & Dynam. Sys.* **2** (1982), 465-485.

[17] S.J. VAN STRIEN, Smooth dynamics on the interval (with an emphasis on unimodel maps), Report 87-09, Delft University of Technology, 1987.

[18] W.A. VEECH, Unique ergodicity of horospherical flows, *Amer. J. Math.* **99** (1977), 827-859.

[19] J. DE VRIES & J.C.S.P. VAN DER WOUDE, Invariant measures and the equicontinuous structure relation I, in: J. FLACHSMEYER (ed.), *Proc. Conf. Topology and Measure IV* (Trassenheide, GDR, 1983), Wissenschaftliche Beiträge der Ernst-Moritz-Arndt-Universität Greifswald, Greifswald, 1984, pp. 242-253.

[20] P. WALTERS, *An introduction to ergodic theory*, Springer-Verlag, Berlin, 1982.

[21] J.C.S.P. VAN DER WOUDE, *Topological dynamix*, CWI Tracts No 22, Centre for Mathematics and Computer Science (CWI), Amsterdam, 1986.